Quasisymmetric Koebe uniformization of metric surfaces with uniformly relatively separated boundary

by

Jonathan Rehmert

B.S., College of the Ozarks, 2017

M.S., Kansas State University, 2018

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2022

Abstract

We study the problem of determining when metric surfaces can be mapped quasisymmetrically onto a circle domain with uniformly relatively separated boundary components. Mario Bonk[1] completely characterized this for domains in $\hat{\mathbb{C}}$. He proved that if the boundary components of a domain in $\hat{\mathbb{C}}$ are uniformly relatively separated uniform quasicircles then the domain is quasisymmetric to a circle domain. However, Merenkov and Wildrick[2] showed the existence of a metric surface whose boundary components are uniformly relatively separated uniform quasicircles which fails to be quasisymmetric to a circle domain. They offered an alternative characterization for metric surfaces using properties which are not invariant under quasisymmetries, and they expressed interest in replacing these with properties which are.

In this dissertation, we introduce what we call the 2-transboundary Loewner property. This first appeared in Bonk's work[1]. It is an analog of the Loewner property of Heinonen and Koskela[3] in terms of Schramm's[4] transboundary modulus. Using recent quasiconformal uniformization results of Rajala[5] and Ikonen[6], we prove that under some mild assumptions, a metric surface is quasisymmetric to a circle domain with uniformly relatively separated boundary components if and only if it is 2-transboundary Loewner. Since the 2-transboundary Loewner property is invariant under quasisymmetries, this answers the question posed by Merenkov and Wildrick. It is also a natural generalization of Bonk's result to metric surfaces, as it is equivalent to his theorem for domains in $\hat{\mathbb{C}}$. Applying our results, we give new examples of metric surfaces which we show are quasisymmetric to circle domains.

Quasisymmetric Koebe uniformization of metric surfaces with uniformly relatively separated boundary

by

Jonathan Rehmert

B.S., College of the Ozarks, 2017

M.S., Kansas State University, 2018

A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2022

Approved by:

Major Professor
Hrant Hakobyan

Copyright

Abstract

We study the problem of determining when metric surfaces can be mapped quasisymmetrically onto a circle domain with uniformly relatively separated boundary components. Mario Bonk[1] completely characterized this for domains in $\hat{\mathbb{C}}$. He proved that if the boundary components of a domain in $\hat{\mathbb{C}}$ are uniformly relatively separated uniform quasicircles then the domain is quasisymmetric to a circle domain. However, Merenkov and Wildrick[2] showed the existence of a metric surface whose boundary components are uniformly relatively separated uniform quasicircles which fails to be quasisymmetric to a circle domain. They offered an alternative characterization for metric surfaces using properties which are not invariant under quasisymmetries, and they expressed interest in replacing these with properties which are.

In this dissertation, we introduce what we call the 2-transboundary Loewner property. This first appeared in Bonk's work[1]. It is an analog of the Loewner property of Heinonen and Koskela[3] in terms of Schramm's[4] transboundary modulus. Using recent quasiconformal uniformization results of Rajala[5] and Ikonen[6], we prove that under some mild assumptions, a metric surface is quasisymmetric to a circle domain with uniformly relatively separated boundary components if and only if it is 2-transboundary Loewner. Since the 2-transboundary Loewner property is invariant under quasisymmetries, this answers the question posed by Merenkov and Wildrick. It is also a natural generalization of Bonk's result to metric surfaces, as it is equivalent to his theorem for domains in $\hat{\mathbb{C}}$. Applying our results, we give new examples of metric surfaces which we show are quasisymmetric to circle domains.

Table of Contents

List of Figures

Nomenclature

$\mathbb{N}$ The set of non-negative integers

$\mathbb{R}$ The set of real numbers

$\mathbb{C}$ The complex plane

$\hat{\mathbb{C}}$ The Riemann sphere

$\mathbb{D}$ The unit disk

$\#(S)$ The number of elements in a set S

$\overline{X}$ The completion of a metric space X

$B(x, r)$ Open ball of radius r centered at x

$B[x, r]$ Closed ball of radius r centered at x

$A(x, r, R)$ Open annulus of inner radius r and outer radius R centered at x

$A[x, r, R]$ Closed annulus of inner radius r and outer radius R centered at x

∂X The metric boundary of X

$\partial_0 X$ The set of connected components of ∂X

$\operatorname{diam}(A)$ The diameter of the set A

$\Delta(E, F)$ The relative distance between E and F

$\mathcal{H}^s(A)$ The s-dimensional Hausdorff measure of A

$\dim_{\mathcal{H}}(A)$ The Hausdorff dimension of A

$\ell(\gamma)$ The length of a curve γ

s_γ The length function of a curve γ

γ_s The arc length parameterization of a curve γ

$\int_\gamma \rho \, ds$ The path integral of a non-negative Borel function, ρ, over a curve γ.

$\ell_\rho(\gamma)$ The ρ-length of a curve γ

$\rho \wedge \Gamma$ The Borel function ρ is admissible for the family of curves Γ

$\mathrm{mod}(\Gamma)$ The 2-modulus of a family of curves Γ

$A(\rho)$ Shorthand for $\int_X \rho^2 d\mu$

$\Gamma_1 < \Gamma_2$ The family of curves Γ_1 minorizes the family of curves Γ_2

$\mathbb{I}_A$ The indicator (or characteristic) function of the set A

$\Gamma(E, F; \Omega)$ The family of curves connecting E and F in Ω

$f(\Gamma)$ The collection of curves which are the composition of some curve in Γ with f

$X_\mathcal{K}$ The $\mathcal{K}$-quotient of X

$\pi_\mathcal{K}$ The quotient map from X to $X_\mathcal{K}$

$\ell_\rho^\mathcal{K}(\gamma)$ The ρ-length of a curve γ relative $\mathcal{K}$

$(\rho; \{\rho_i\}_{i\in I})$ A transboundary mass distribution with Borel ρ and weights ρ_i

$\ell_P^\mathcal{K}(\gamma)$ For a transboundary mass distribution, P, the P-length of γ relative $\mathcal{K}$

$P \wedge_\mathcal{K} \Gamma$ The transboundary mass distribution P is admissible to the family of curves Γ relative $\mathcal{K}$

ρ_P The Borel function in $X_\mathcal{K}$ corresponding to the transboundary mass distribution P

$\mathrm{mod}_{\mathcal{K}}(\Gamma)$ The transboundary modulus of a family of curves Γ relative $\mathcal{K}$

$A_{\mathcal{K}}(P)$ Shorthand for $\int_{X_{\mathcal{K}}} \rho_P^2 \, d\mu_{\mathcal{K}}$

Acknowledgments

I would like to acknowledge some of my office mates, Huy, Jacob, and Stucky, for being a continual resounding board for my ideas. I'm grateful to Elizabeth Hale for pointing me to helpful references and giving me useful insights. I would like to thank Gabriel Necoechea for his technical assistance. Many thanks to Matthew Naeger for bringing important topological facts to my attention. I greatly appreciate Dimitrios Ntalampekos for pointing out errors that I made. Last, I would like to thank my advisor, Hrant Hakobyan, for being an endless wellspring of patience throughout the writing process.

Dedication

This is dedicated to everyone who invested in my life, pushed me to improve, and supported me on my journey. Chief among these is my wife, Alissa, without whom I would not have accomplished much.

Preface

This document includes a nomenclature section which is written only as a reference for the various notation conventions used. The intended use is to consult it when one comes across an unfamiliar symbol. There is also an index at the end which is primarily for quickly finding definitions of terminology. The page numbers in the index only give the page on which the term was defined, and it is not a list of every time that term is used. The intended use is to consult it when one sees a term and wishes to recall precisely how it was defined.

Chapter 1

Introduction

The quasisymmetric uniformization problem is one of the central problems of contemporary geometric function theory and analysis on metric spaces. In his 2006 address to the International Congress of Mathematicians, Mario Bonk ([7], p. 1353) formulated the problem as follows.

"**Quasisymmetric Uniformization Problem.** Suppose X is a metric space homeomorphic to some 'standard' metric space Y. When is X quasisymmetrically equivalent to Y?"

This question was answered for $Y = \mathbb{S}^1$ by Tukia and Väisälä[8]; they showed X is quasisymmetric to Y if and only if X is doubling and linearly locally connected. Here, doubling means that every ball of radius r can be covered by N balls of radius $r/2$ where N is some fixed constant. Linear local connectivity rules out cusp-like behavior by requiring any two points inside a ball of radius r can be connected inside the ball of radius λr and any two points outside the ball can be connected outside the ball of radius r/λ for some fixed $\lambda \geq 1$. See Section 1.4 for a precise definition.

When $Y = \hat{\mathbb{C}}$, the quasisymmetric uniformization problem was given a partial answer by Bonk and Kleiner[9]. They showed that X is quasisymmetric to Y if X is linearly locally connected and Ahlfors 2-regular. Ahlfors 2-regularity requires the Hausdorff measure of every ball to be comparable to the square of its radius. Rajala[5] gave an alternative proof of the characterization which required only that the Hausdorff measure of every ball is bounded above by a constant multiple of its radius squared.

If Y is a domain in $\hat{\mathbb{C}}$ whose complementary components are closed disks or points (called a circle domain), which is finitely connected, then Merenkov and Wildrick[2] gave the following characterization. X is quasisymmetric to Y if X is Ahlfors 2-regular, linearly locally connected, and $\overline{X}$ is compact. More details on the latter results are given in Section 1.5.

This dissertation investigates this question with Y being a countably connected circle domain in $\hat{\mathbb{C}}$. A consequence of the main result (Theorem 1.5.7) is as follows.

Theorem 1.0.1. *Suppose (X, d) is a metric space homeomorphic to a domain in $\hat{\mathbb{C}}$ with $\partial_0 X$ countable. Suppose X is Ahlfors 2-regular and linearly locally connected. Then X is quasisymmetric to a circle domain with uniformly relatively separated bounding circles if and only if*

- *$\overline{X}$ is compact and*
- *X is 2-transboundary Loewner.*

The term above, 2-transboundary Loewner, is introduced in this work. It is analogous to the Loewner property of Heinonen and Koskela[3] in terms of Schramm's[4] transboundary modulus. Intuitively, it requires that the space X support rich families of transboundary curves: curves going through the holes in the space. See Chapter 3 for a precise formulation.

1.1 Motivating Remarks

A natural question to ask is of what relevance this question has to mathematics as a whole. To answer this, we will outline some of the topics where the theory of quasisymmetric mappings proved useful.

In 1968, Mostow[10] proved that if two compact hyperbolic manifolds with dimension $n \geq 3$ are homeomorphic, then they are isometric. The homeomorphism lifts to a homeomorphism between their universal covering spaces, which can be extended to infinity. This creates a homeomorphism from the sphere of dimension $n-1$ to itself. The assumptions on the manifolds will imply that this map is in fact a quasisymmetry. It is the behavior of quasisymmetries in dimension 2 or above that gives the result; moreover, the fact that quasisymmetries are not well-behaved in dimension 1 explains why the result is not true for 2 dimensional manifolds.

A natural generalization of this topic arises in geometric group theory. Cannon [11] conjectured that every hyperbolic group whose boundary is homeomorphic to the 2-sphere is isomorphic to a Kleinian group. Due to a result of Sullivan [12], this is equivalent to the following statement. If G is a hyperbolic group with boundary homeomorphic to the 2-sphere, then the boundary (equipped with a proper metric) is quasisymmetric to the 2-sphere. Thus it becomes very natural to ask when a space is quasisymmetric to the sphere.

It is also natural to ask when a space is quasisymmetric to the Sierpiński carpet because of a result of Kapovich and Kleiner [13]. For certain hyperbolic groups with one dimensional boundary, they classified the boundary as homeomorphic to either a Menger curve, Sierpiński carpet, or $\mathbb{S}^1$.

The details of the results just discussed are beyond the scope of this document and won't be elaborated; they are listed here to convince the reader that the problems discussed in this document are of interest to many mathematicians. With that being said, more motivations for the subject will be discussed in detail in the sections that follow. Section 1.2 outlines the theory of conformal uniformization, and Section 1.3 connects conformal maps to quasisymmetric maps by introducing quasiconformal maps. The introduction ends with Section 1.5 giving an overview of many results pertaining to the quasisymmetric uniformization problem; all the definitions needed to state these results are given in Section 1.4.

1.2 Conformal Maps

Let $\Omega, \Omega' \subset \mathbb{C}$ be domains. A function $f : \Omega \to \Omega'$ is called a conformal map if it is a holomorphic homeomorphism. The reader may be familiar with conformal maps as maps which preserve angles. This definition fits this description; indeed, having non-zero derivative means that at that point, f is locally a non-zero $\mathbb{C}$-linear map, and all of these maps are angle preserving. However, emphasis should be placed on the fact that we require f to be injective. Under this definition, $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$ is not a conformal map, despite the fact that it is angle preserving. For the entirety of this document, when we say a map is conformal, we mean to imply it is also a homeomorphism.

Our discussion of conformal uniformization begins with the following theorem first formulated by Riemann in 1851.

Theorem 1.2.1 (Riemann Mapping Theorem). *Let $\Omega \subset \mathbb{C}$ be a simply connected domain with $\Omega \neq \mathbb{C}$. Fix $z_0 \in \Omega$. There is a unique conformal map $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$.*

We are less concerned with the uniqueness of the map than with its existence. A related result in the classical theory is a classification of simply connected Riemann surfaces.

Theorem 1.2.2 (Koebe-Poincaré Uniformization Theorem). *Let X be a simply connected Riemann surface. Then X is conformal to either $\mathbb{D}$, $\mathbb{C}$, or $\hat{\mathbb{C}}$.*

For domains which fail to be simply connected, a natural model space to consider are circle domains.

Definition 1.2.3. *We say a domain $\Omega \subset \mathbb{C}$ or $\Omega \subset \hat{\mathbb{C}}$ is a* ***circle domain*** *in case every connected component of $\partial\Omega$ is a circle or a point.*

For example, an annulus is a circle domain. The complement of the middle-thirds Cantor set in the plane is also a circle domain, showing that circle domains can have uncountably many boundary components. For a domain in the sphere, we say it is finitely connected if it has finitely many boundary components. We say a domain in the plane is finitely connected if its image under stereographic projection is finitely connected.

Theorem 1.2.4 (Koebe[14]). *Let $\Omega \subset \mathbb{C}$ be a finitely connected domain. Then there is a conformal map $f : \Omega \to \Omega'$ where $\Omega' \subset \mathbb{C}$ is a circle domain.*

It took 75 years for this result to be extended to the countable case. We say a domain in the sphere is countably connected if its boundary has countably many connected components. We say a domain in the plane is countably connected if the corresponding domain in the sphere is.

Theorem 1.2.5 (He-Schramm[15]). *Let $\Omega \subset \mathbb{C}$ be a countably connected domain. Then there is a conformal map $f : \Omega \to \Omega'$ where $\Omega' \subset \mathbb{C}$ is a circle domain.*

As we have seen already, there are circle domains which are neither finitely connected nor countably connected. We say a domain in the sphere is uncountably connected if its boundary has uncountably many connected components. We say a domain in the plane is uncountably connected if the corresponding domain in the sphere is. Every domain in the plane or the sphere fits into these connectivity categories. One of the main motivations for the topic we're discussing is the conjecture formulated by Koebe in 1908.

Conjecture 1.2.6 (Koebe[16]). *Every domain in the plane is conformal to a circle domain.*

Koebe's conjecture is still open. One avenue of difficulty for proving the conjecture is that uncountably connected circle domains can only have countably many non-trivial boundary components; whereas, general domains may have uncountably many non-trivial boundary components. This is evident in the fact that one can obtain a conformal uniformization by switching the model space to what follows.

Definition 1.2.7. *We say $\Omega \subset \mathbb{C}$ is a **slit domain** if every connected component of the complement is either a point or a closed, vertical line segment. These line segments are referred to as slits.*

The requirement that the slits be vertical is for simplicity; one could require only that the slits are all parallel. However, these are conformally equivalent to having vertical slits via rotation. Unlike circle domains, slit domains can have uncountably many non-trivial boundary components. For example, if $\mathcal{C}$ denotes the middle-thirds Cantor set, then the complement of $\mathcal{C} \times [0, 1]$ is a slit domain. Slit domains make a nice model space for conformal uniformization because of the following result.

Theorem 1.2.8 (de Possel[17], Grötzsch[18]). *Every domain in the plane is conformal to a slit domain.*

1.3 Quasiconformal Maps

Let $\Omega \subset \mathbb{R}^2$ be open and let $u, v : \Omega \to \mathbb{R}^2$ be differentiable. Let $f(z) = w$ where $z = x + iy$ and $w = u(x, y) + iv(x, y)$. Define

$$f_z = \frac{1}{2}(f_x - if_y) = \frac{1}{2}((u_x + iv_x) - i(u_y + iv_y)) = \frac{1}{2}((u_x + v_y) + i(v_x - u_y))$$
$$f_{\overline{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}((u_x + iv_x) + i(u_y + iv_y)) = \frac{1}{2}((u_x - v_y) + i(v_x + u_y))$$

The Cauchy-Riemann equations can be formulated as $f_{\overline{z}} = 0$. If f is an orientation-preserving diffeomorphism, we can say that locally, f sends circles to ellipses whose eccentricity we call the dilatation , and it is given by

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}.$$

Notice that if f is conformal, then $f_z \neq 0$ and $f_{\overline{z}} = 0$, and $D_f \equiv 1$. In general, $D_f \geq 1$ at each point. The diffeomorphism f is called K-quasiconformal if $D_f(z_0) \leq K$ for all $z_0 \in \Omega$. Intuitively, this definition states that quasiconformal maps send infinitesimal circles to infinitesimal ellipses of bounded eccentricity.

This is, however, the classical formulation of analytic quasiconformality, and we will need it to be formulated for general metric spaces. The idea is the same: we want the ratio of the maximum stretch of the image of a circle to the minimum to be bounded uniformly as the radius of the circle goes to zero.

Definition 1.3.1. *Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ a homeomorphism, and*

$K \geq 1$. *We say f is (metrically)* ***K-quasiconformal*** *in case for all $x_0 \in X$*

$$D_f(x_0) := \limsup_{r \to 0^+} \frac{\sup\limits_{d_X(x,x_0) \leq r} d_Y(f(x), f(x_0))}{\inf\limits_{d_X(x,x_0) \geq r} d_Y(f(x), f(x_0))} \leq K.$$

If a map is 1-quasiconformal, we say it is ***conformal****. We say a map is quasiconformal if it is K-quasiconformal for some K.*

In the case that X and Y are both planar domains, and f is orientation-preserving diffeomorphism, the metric definition coincides with the classical definition. Indeed, as the radius of a circle goes to zero, the image looks like an ellipse; thus, the major axis will be the maximum distance achieved within the disk and the minor axis will be the minimum distance achieved without. It is also worth observing that $D_f \geq 1$ since

$$D_f(x_0) = \limsup_{r \to 0^+} \frac{\sup\limits_{d_X(x,x_0) \leq r} d_Y(f(x), f(x_0))}{\inf\limits_{d_X(x,x_0) \geq r} d_Y(f(x), f(x_0))} \geq \limsup_{r \to 0^+} \frac{\sup\limits_{d_X(x,x_0) = r} d_Y(f(x), f(x_0))}{\inf\limits_{d_X(x,x_0) = r} d_Y(f(x), f(x_0))} \geq 1.$$

Thus we can say that f is conformal if and only if $D_f \equiv 1$. We will establish more properties of quasiconformal maps in Chapter 2. However, one thing we will establish here is their relationship to quasisymmetric mappings.

Definition 1.3.2 ([8]). *Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ a homeomorphism, and $\eta : [0, \infty) \to [0, \infty)$ a homeomorphism. We say f is* ***η-quasisymmetric*** *if for all pairwise distinct $a, b, c \in X$, we have*

$$\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))} \leq \eta\left(\frac{d_X(a,b)}{d_X(a,c)}\right).$$

We say f is quasisymmetric if it is η-quasisymmetric for some η. We say X and Y are quasisymmetric if there is a quasisymmetric map $f : X \to Y$.

Quasisymmetric mappings are the main focus of this dissertation, and many of their properties will be detailed in Chapter 2.

Quasisymmetries exhibit the property of quasiconformality, but on a more global level. The dilatation of a quasisymmetry is not only uniformly bounded, but the limiting ratio can be bounded independent of r.

Proposition 1.3.3. *Quasisymmetric maps are quasiconformal. More precisely, an η-quasisymmetry is an $\eta(1)$-quasiconformality.*

1.4 Terminology

The goal of this section is to define all the terms used in the statements of the theorems in Section 1.5. Their properties will be established in greater detail in Chapter 2.

Let (X, d) be a metric space. Given $x \in X$ and $r > 0$ we denote the open ball of radius r centered at x by

$$B(x, r) := \{y \in X \mid d(x, y) < r\},$$

and the closed ball is denoted by

$$B[x, r] := \{y \in X \mid d(x, y) \leq r\}.$$

Given $x \in X$ and $R > r \geq 0$ we denote the open annulus of inner radius r and outer radius R centered at x by

$$A(x, r, R) := \{y \in X \mid r < d(x, y) < R\},$$

and the closed annulus is denoted by

$$A[x, r, R] := \{y \in X \mid r \leq d(x, y) \leq R\}.$$

We denote the completion of (X, d) by $(\overline{X}, d)$. We denote the metric boundary of X by $\partial X := \overline{X} \setminus X$, and may just refer to it as the boundary of X . The set of connected components of ∂X will be denoted $\partial_0 X$.

Given $A \subset X$ we define its diameter as

$$\operatorname{diam}(A) := \sup\{d(x, y) \mid x, y \in A\}.$$

Note that $\operatorname{diam}(A) = 0$ if and only if A is a singleton. We do allow for $\operatorname{diam}(A) = \infty$, in which case we say A is unbounded . Given another subset $B \subset X$ we define the distance between A and B to be

$$d(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}.$$

It should be noted here that this is not a metric on the subsets of X. However, if A, B are closed, and at least one of them is compact, then $d(A, B) = 0$ if and only if $A \cap B \neq \emptyset$. The following concept is useful when discussing quasisymmetric mappings as it has controlled distortion under quasisymmetries (see Proposition 2.1.9).

Definition 1.4.1 ([19])**.** *Let (X, d) be a metric space and $E, F \subset X$ be closed and disjoint. Define their* ***relative distance*** *by*

$$\Delta(E, F) := \frac{d(E, F)}{\min(\operatorname{diam}(E), \operatorname{diam}(F))}.$$

If E or F is a singleton, we define their relative distance to be ∞. If E and F are unbounded, we define the relative distance to be 0.

This definition is well-formulated as $d(E, F) < \infty$, and if $d(E, F) = 0$ for closed, disjoint sets, then neither can have 0 diameter.

Definition 1.4.2. *Let (X, d) be a metric space and $\mathcal{E} = \{E_i\}_{i \in I}$ be a collection of pairwise disjoint, closed subsets of X. We say $\mathcal{E}$ is* ***uniformly relatively separated*** *if there exists some $\alpha > 0$ such that*

$$\Delta(E_i, E_j) \geq \alpha$$

for all $i, j \in I, i \neq j$.

Often, we will be concerned with the relative distance between connected sets. We say $E \subset X$ is a continuum if it is compact and connected, and we say E is non-degenerate if it contains at least two distinct points. Thus for disjoint, non-degenerate continua, their relative distance is always non-zero and finite. The following property quantifies connectivity properties of a space and is invariant under quasisymmetries (see Proposition 2.1.7).

Definition 1.4.3 ([9]). *Let (X, d) be a metric space. We say (X, d) is **linearly locally connected** if there exists a $\lambda \geq 1$ such that for all $a \in X$ and $r > 0$ the following properties hold:*

(i) for each $x, y \in B(a, r)$ there exists a continuum $E \subset B(a, \lambda r)$ with $x, y \in E$,

(ii) for each $x, y \in X \setminus B(a, r)$, there exists a continuum $E \subset X \setminus B(a, \frac{r}{\lambda})$ with $x, y \in E$.

There is a similar property which essentially demands that both properties hold with the same continuum.

Definition 1.4.4. *Let (X, d) be a metric space. We say (X, d) is **annularly linearly locally connected** if there exists a $\lambda \geq 1$ such that for all $a \in X$ and $R > r \geq 0$ the following property holds: for each $x, y \in A(a, r, R)$ there exists a continuum $E \subset A(a, \frac{r}{\lambda}, \lambda R)$ with $x, y \in E$.*

Annular linear local connectivity is stronger than linear local connectivity. For example, $\mathbb{S}^1$ is linearly locally connected with $\lambda = 1$, as every ball is connected, and the complement of every ball is connected. However, it is not annularly linearly locally connected as for all $x \in \mathbb{S}^1$ and $0 < r < R < \operatorname{diam}(\mathbb{S}^1)$, $A(x, r, R)$ is disconnected. In order to obtain a continua connecting the two connected components, one would need $\lambda > \operatorname{diam}(\mathbb{S}^1)/R$, which goes to ∞ as $R \to 0$.

Now we recall the definition of Hausdorff measure.

Definition 1.4.5 ([20]). *Let (X, d) be a metric space and $A \subset X$. For all $s \geq 0$ and $\delta > 0$, we write*

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s \mid A \subset \cup_{i=1}^{\infty} A_i,\ \operatorname{diam}(A_i) \leq \delta \textit{ for all } i\}.$$

Notice that $\mathcal{H}^s_\delta$ decreases in δ, so we can define

$$\mathcal{H}^s(A) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

We call this quantity the ***s-dimensional Hausdorff measure of A****.*

We will assume the reader is familiar with the basic properties of the Hausdorff measure. The following proposition won't be proved, but is used to define the dimension.

Proposition 1.4.6. *Let (X, d) be a metric space and $A \subset X$. There exists a unique $s \in [0, \infty]$ such that for all $t > s$,*

$$\mathcal{H}^t(A) = 0,$$

and for all $0 \leq t < s$,

$$\mathcal{H}^t(A) = \infty.$$

We define the ***Hausdorff dimension of A*** *to be s, denoted $\dim_{\mathcal{H}}(A) := s$.*

If we ever refer to the Hausdorff measure of A without specifying the dimension, we mean the $\dim_{\mathcal{H}}(A)$-dimensional Hausdorff measure of A. When a metric space (X, d) is equipped with a Borel measure μ, we call the triple (X, d, μ) a metric measure space . Many of the properties of metric measure spaces can be defined on metric spaces by using the Hausdorff measure.

Definition 1.4.7. *Let (X, d, μ) be a metric measure space. We say μ is* ***doubling*** *if there exists a constant $C \geq 1$ such that for every $x \in X$ and $r > 0$,*

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Also, we require $0 < \mu(B(x, r)) < \infty$ for all balls.

Definition 1.4.8. *Let (X, d, μ) be a metric measure space. Let $s > 0$, we say (X, d, μ) is* ***Ahlfors s-regular*** *in case there exists a $C \geq 1$ such that for every $x \in X$ and $0 < r <$*

diam(X),

$$\frac{1}{C}r^s \leq \mu(B(x,r)) \leq Cr^s.$$

We say (X,d) is Ahlfors s-regular if $(X,d,\mathcal{H}^{\dim_{\mathcal{H}}(X)})$ is. If we say a space is Ahlfors regular, we mean it is Ahlfors s-regular for some $s>0$.

Ahlfors regular measures are doubling: $\mu(B(x,2r)) \leq 2^s Cr^s \leq 2^s C^2 \mu(B(x,r))$. Notice that a necessary condition for (X,d) to be Ahlfors s-regular is that $\dim_{\mathcal{H}}(X) = s$; although, it is not sufficient. Indeed look at the planar set $(x, x\sin(1/x))$ for $-1 \leq x \leq 1$ equipped with the euclidean metric; it has Hausdorff dimension 1, but every ball around $(0,0)$ has infinite $\mathcal{H}^1$ measure, so it isn't Ahlfors 1-regular. We will point out that if X is locally compact and Ahlfors s-regular for some Borel μ, then $\mathcal{H}^s$ is comparable to μ[21]. Thus (X,d,μ) being Ahlfors regular implies (X,d) is Ahlfors regular. Also note that quasisymmetries can change the Hausdorff dimension (see Section 2.2), and thus they can also map a space which is Ahlfors s-regular to one that is not.

The notion of a metric doubling measure, introduced by David and Semmes[22], arises in the theory as a way to quasisymmetrically deform metrics. While we will not use this concept directly, we define a weakened version of it below for reference, as it appears in Section 1.5.

Definition 1.4.9 ([23])**.** *Let (X,d,μ) be a metric measure space. For convenience, define for $x,y \in X$,*

$$B_{xy} := B(x,d(x,y)) \cup B(y,d(x,y)).$$

A finite sequence of points $x_0,...,x_m \in X$ is called a δ-chain from x to y if $x_0 = x, x_m = y$, and $d(x_j,x_{j-1}) \leq \delta$ for all $j=1,...,m$. Now fix $s>0$ and define

$$q^{\delta}_{\mu,s}(x,y) := \inf\left\{ \sum_{j=1}^{m} \mu(B_{x_j x_{j-1}})^{1/s} \mid (x_j)_{j=0}^m \text{ is a } \delta\text{-chain from } x \text{ to } y \right\}.$$

The μ-length is defined to be

$$q_{\mu,s}(x,y) := \limsup_{\delta \to 0} q^{\delta}_{\mu,s}(x,y).$$

*The μ-length can be infinite. We say μ is a **weak metric doubling measure of dimension s** on (X,d) if μ is doubling, and if there exists a $0 < c \leq 1$ such that for all $x, y \in X$,*

$$c\mu(B_{xy})^{1/s} \leq q_{\mu,s}(x,y).$$

1.5 Literature Review and Main Result

The goal of this section is to give the current state of affairs regarding the problem of quasisymmetric embeddability of metric surfaces into the sphere. The first case we will consider is when our space is homeomorphic to the sphere.

Theorem 1.5.1 (Bonk-Kleiner[9]). *Let (X,d) be an Ahlfors 2-regular metric space homeomorphic to $\hat{\mathbb{C}}$. Then X is quasisymmetric to $\hat{\mathbb{C}}$ if and only if X is linearly locally connected.*

The Bonk-Kleiner Theorem was motivated by questions in Geometric Group Theory (see discussion at the beginning of this chapter), and it has been applied there[24]. This also yielded progress in Complex Dynamics[25]. Interest in this result has generated alternative proofs[5;26;27] which demonstrate how deep the result is from multiple perspectives.

We've already alluded to the fact that linear local connectivity is a quasisymmetrically invariant property. However, we also observed that Ahlfors 2-regularity is not quasisymmetrically invariant; indeed, there are quasisymmetric images of the sphere which fail to be Ahlfors 2-regular. Bonk and Kleiner[9] did give a complete characterization, however, in terms of discrete modulus. The following theorem also captures non-Ahlfors 2-regular examples by replacing Ahlfors regularity with the quasisymmetrically invariant weak metric doubling measure.

Theorem 1.5.2 (Lohvansuu-Rajala-Rasimus[23]). *Let (X,d) be a metric space homeomorphic* *to $\hat{\mathbb{C}}$. Then X is quasisymmetric to $\hat{\mathbb{C}}$ if and only if it is linearly locally connected and carries a weak metric doubling measure of dimension 2.*

When considering cases which are not simply connected, one has to adjust the model space. The model space chosen here is motivated by Geometric Group Theory, as work by Kapovich and Kleiner[13] generated interest in classifying quasisymmetric images of the Sierpiński Carpet. So we are interested in equivalence to model spaces which are connected subsets of the sphere whose boundary components are circles or points which aren't necessarily domains. Quasisymmetric images of such sets into the sphere have countably many non-degenerate complementary components which are disjoint Jordan regions.

Theorem 1.5.3 (Bonk[1])**.** *Let $\{S_i\}$ be a countable collection of uniformly relatively separated uniform quasicircles in $\hat{\mathbb{C}}$ bounding disjoint Jordan regions. Then there is a quasisymmetric map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $f(S_i)$ is a round circle for all i.*

It should be noted that this theorem applies not only to domains, but also to sets like the Sierpiński Carpet.

Regarding more general metric spaces, we restrict our discussion to uniformizing with respect to circle domains. We will first consider finitely connected domains.

Theorem 1.5.4 (Merenkov-Wildrick[2])**.** *Let (X, d) be a metric space that's Ahlfors 2-regular and homeomorphic to a finitely connected circle domain $\Omega \subset \hat{\mathbb{C}}$. Then X is quasisymmetric to Ω if and only if X is linearly locally connected and $\overline{X}$ is compact.*

Similar to the Bonk-Kleiner theorem (1.5.1), this has been generalized to include examples which aren't Ahlfors 2-regular.

Theorem 1.5.5 (Rajala-Rasimus[28])**.** *Let (X, d) be a metric space homeomorphic to a finitely connected circle domain $\Omega \subset \hat{\mathbb{C}}$. Then X is quasisymmetric to Ω if and only if X is linearly locally connected, $\overline{X}$ is compact, and X carries a weak metric doubling measure of dimension 2.*

It should be noted that Theorems 1.5.4 and 1.5.5 do not extend directly to infinitely many boundary components, because the key estimates depend on the number of boundary components. As such, the countably connected case lacks a complete characterization; with the following theorem being the most general statement so far.

Theorem 1.5.6 (Merenkov-Wildrick[2]). *Let (X, d) be a metric space homeomorphic to a countably connected circle domain $\Omega \subset \mathbb{S}^2$. Suppose that the collection of boundary components of X has finite rank. Suppose that*

(a) X is Ahlfors 2-regular,

(b) The following condition holds

$$\sum_{k=0}^{\infty} n_k 2^{-2k} < \infty$$

where

$$n_k := \sup_{\substack{x \in X \\ 0 < r < 2\mathrm{diam}(X)}} \#\{E \in \partial_0 X \mid E \cap B(x, r) \neq \emptyset, 2^{-k} r < \mathrm{diam}(E) \leq 2^{-k+1} r\}.$$

Then X is quasisymmetric to a uniformly relatively separated circle domain if and only if

(1) The boundary components of X are uniformly relatively separated,

(2) $\overline{X}$ is compact,

(3) X is annularly linearly locally connected.

See Merenkov and Wildrick[2] for a definition of rank. While most properties given here are quasisymmetric invariants, conditions (a) and (b) above are not invariant under quasisymmetries, and the authors expressed interest in rectifying that. This was the primary motivation for the following result, where the only property which is not invariant under quasisymmetries is requiring $\mathcal{H}^2$ locally finite.

Theorem 1.5.7. *Suppose (X, d) is a metric space homeomorphic to a domain in $\hat{\mathbb{C}}$ with $\mathcal{H}^2$ locally finite and $\partial_0 X$ countable. Suppose X is locally reciprocal. Then X is quasisymmetric to a circle domain with uniformly relatively separated bounding circles if and only if*

- *$\overline{X}$ is compact,*

- *X is (metric) doubling,*

- *X is LLC-1, and*

- *X is 2-transboundary Loewner.*

By LLC-1, we mean condition (i) of linear local connectivity. We prove this by first using local reciprocality and the machinery of Rajala[5] and Ikonen[6] to obtain a quasiconformal map to a circle domain. We then use the 2-transboundary Loewner property to say that this map is, in fact, quasisymmetric. The rest of the properties are used in the finer details of the argument. In Section 4.4, we establish a class of domains where Theorem 1.5.7 applies, and the other theorems discussed in this section do not.

One might be interested on how these questions have been addressed for uncountably connected circle domains. We will end this section by pointing out that answering this question is equivalent to answering the question for countably connected circle domains.

Proposition 1.5.8. *Let (X, d) be a metric space with uncountably many boundary components which is homeomorphic to an uncountably connected circle domain in the sphere. Let $\tilde{X} = X \cup C$ where $C = \{c \in \partial X \mid \{c\} \in \partial_0 X\}$. X is quasisymmetric to an uncountably connected circle domain if and only if $\tilde{X}$ is quasisymmetric to a countably connected circle domain.*

Proof. $\Rightarrow$) Let $\Omega \subset \hat{\mathbb{C}}$ be an uncountably connected circle domain, and suppose $f : X \to \Omega$ is a quasisymmetry. Then by Remark 2.1.6 f extends as a quasisymmetry to the boundary. Thus the extension of f is defined on C, and $f(C)$ must be the union of degenerate complementary components of Ω. So $f(\tilde{X})$ must be a circle domain with no degenerate complementary components, and hence must be countably connected.

$\Leftarrow$) Let $\tilde{\Omega} \subset \hat{\mathbb{C}}$ be a countably connected circle domain, and suppose $f : \tilde{X} \to \tilde{\Omega}$ is a quasisymmetry. Then $\tilde{X}$ must have countably many boundary components, and so C must be uncountable. Also, since X is homeomorphic to a circle domain, $f(X)$ must be homeomorphic to a circle domain. Hence, $f(X)$ is an open, connected subset of $\hat{\mathbb{C}}$ whose complementary components include the complementary components of $\tilde{\Omega}$, as well as $f(C)$, which is a totally disconnected, uncountable set. $f(X)$ must be an uncountably connected circle domain. $\square$

The outline of the rest of the document is as follows. Chapter 2 will discuss the established theory of quasisymmetric mappings and quasiconformal mappings, as well as modulus and the Loewner property. Chapter 3 contains new information which will be needed for the results; including transboundary modulus, as defined for general metric spaces, and various transboundary analogs of the Loewner property. Chapter 4 will be the proof of the main theorem, and it contains some applications of the main result to particular examples.

Chapter 2

Quasisymmetric Mappings, Quasiconformal Mappings, and Modulus of Curves

The goal of this chapter is to review the theory of quasisymmetric and quasiconformal mappings. Specifically, their various definitions and equivalence, as well as a few properties. Modulus of curve families will also be introduced, as it is used in defining quasiconformal maps and establishing their properties. Many of the arguments in this chapter are well-known, and are given for completeness.

2.1 Properties of Quasisymmetries

Here we will discuss basic properties of quasisymmetric maps and their pertinence to the quasisymmetric equivalence of spaces (see Definition 1.3.2). First we will observe that if $f : X \to Y$ is η_f-quasisymmetric and $g : Y \to Z$ is η_g-quasisymmetric, then $g \circ f : X \to Z$ is $\eta_g \circ \eta_f$-quasisymmetric. This is because η_g must be increasing, so that for all pairwise distinct $a, b, c \in X$,

$$\frac{d_Z(g(f(a)), g(f(b)))}{d_Z(g(f(a)), g(f(c)))} \le \eta_g\left(\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))}\right) \le \eta_g\left(\eta_f\left(\frac{d_X(a,b)}{d_X(a,c)}\right)\right).$$

This implies transitivity of quasisymmetry as a relation between spaces (if X is quasisymmetric to Y and Y is quasisymmetric to Z then X is quasisymmetric to Z). This relation is reflexive, since the identity map is quasisymmetric. Symmetry follows from the following proposition, so that it is an equivalence relation.

Proposition 2.1.1. *The inverse of a quasisymmetric map is quasisymmetric. More precisely, if (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is an η-quasisymmetry, then $f^{-1} : Y \to X$ is an ν-quasisymmetry, where $\nu(t) = \frac{1}{\eta^{-1}(1/t)}$ for $t > 0$ and $\nu(0) = 0$.*

Proof. Notice that η and η^{-1} must be increasing functions. Pick any pairwise distinct $a, b, c \in Y$. Then there are pairwise distinct $a', b', c' \in X$ with $f(a') = a, f(b') = b, f(c') = c$. We have

$$\begin{aligned}
\frac{d_Y(f(a'), f(c'))}{d_Y(f(a'), f(b'))} &\le \eta\left(\frac{d_X(a', c')}{d_X(a', b')}\right)\\
\frac{d_Y(a,c)}{d_Y(a,b)} &\le \eta\left(\frac{d_X(f^{-1}(a), f^{-1}(c))}{d_X(f^{-1}(a), f^{-1}(b))}\right)\\
\eta^{-1}\left(\frac{d_Y(a,c)}{d_Y(a,b)}\right) &\le \frac{d_X(f^{-1}(a), f^{-1}(c))}{d_X(f^{-1}(a), f^{-1}(b))}\\
\eta^{-1}\left(\frac{d_Y(a,c)}{d_Y(a,b)}\right)^{-1} &\ge \frac{d_X(f^{-1}(a), f^{-1}(b))}{d_X(f^{-1}(a), f^{-1}(c))}.
\end{aligned}$$

In other words,

$$\frac{d_X(f^{-1}(a), f^{-1}(b))}{d_X(f^{-1}(a), f^{-1}(c))} \le \eta^{-1}\left(\left(\frac{d_Y(a,b)}{d_Y(a,c)}\right)^{-1}\right)^{-1} = \nu\left(\frac{d_Y(a,b)}{d_Y(a,c)}\right).$$

□

We will now give some necessary conditions for spaces to be quasisymmetrically equivalent. We say a metric space, (X, d), is bounded in case $\mathrm{diam}(X) < \infty$.

Proposition 2.1.2. *A bounded metric space is never quasisymmetric to an unbounded one.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ an η-quasisymmetry. Suppose by way of contradiction that one of the spaces is bounded and the other is unbounded; without loss of generality, suppose X is bounded and Y is unbounded (apply same argument to f^{-1} if reversed). Fix distinct $a, c \in X$, and call $\delta_X := d_X(a, c) > 0$ and $\delta_Y := d_Y(f(a), f(c)) > 0$. Since Y is unbounded, for every $n \in \mathbb{N}$ with $n > \delta_Y$, we have $B(f(a), n) \neq Y$, which means there exists some $b_n \in Y$ with $d_Y(f(a), b_n) > n$ and $b_n \neq f(c)$. Then we can say for all $n > \max(\delta_Y, \delta_Y \eta(\mathrm{diam}(X)/\delta_X))$,

$$\frac{n}{\delta_Y} < \frac{d_Y(f(a), b_n)}{d_Y(f(a), f(c))} \leq \eta\left(\frac{d_X(a, f^{-1}(b_n))}{d_X(a, c)}\right) \leq \eta\left(\frac{\mathrm{diam}(X)}{\delta_X}\right);$$

which is a contradiction. □

Proposition 2.1.3 ([19])**.** *Let (X, d_X) and (Y, d_Y) be metric spaces with $f : X \to Y$ a quasisymmetry. Let $A \subset B \subset X$ be such that $0 < \mathrm{diam}(A) \leq \mathrm{diam}(B) < \infty$. Then the following holds*

$$\left(2\eta\left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(A)}\right)\right)^{-1} \leq \frac{\mathrm{diam}(f(A))}{\mathrm{diam}(f(B))} \leq \eta\left(\frac{2\mathrm{diam}(A)}{\mathrm{diam}(B)}\right).$$

Proof. First note that $\mathrm{diam}(f(B)) \geq \mathrm{diam}(f(A)) > 0$ because f is a homeomorphism. Also note that by Proposition 2.1.2, we must have $\mathrm{diam}(f(B)) < \infty$. Let $(b_n)_{n\in\mathbb{N}}$ and $(b'_n)_{n\in\mathbb{N}}$ be sequences of points in B such that $\frac{1}{2}\mathrm{diam}(B) \leq d_X(b_n, b'_n)$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} d_X(b_n, b'_n) = \mathrm{diam}(B).$$

For all $a \in A$ we have

$$d_X(b_n, b'_n) \leq d_X(b_n, a) + d_X(b'_n, a).$$

Hence we must have $d_X(b_n, a) \geq \frac{1}{2}d_X(b_n, b'_n)$ or $d_X(b'_n, a) \geq \frac{1}{2}d_X(b_n, b'_n)$. Without loss of generality, suppose $d_X(b_n, a) \geq \frac{1}{2}d_X(b_n, b'_n)$. Pick any other $a' \in A$ and use quasisymmetry to say

$$\begin{aligned} d_Y(f(a), f(a')) &\le \eta\left(\frac{d_X(a,a')}{d_X(b_n,a)}\right) d_Y(f(b_n), f(a)) \\ &\le \eta\left(\frac{2d_X(a,a')}{d_X(b_n,b'_n)}\right) \operatorname{diam}(f(B)) \\ &\le \eta\left(\frac{2\operatorname{diam}(A)}{d_X(b_n,b'_n)}\right) \operatorname{diam}(f(B)). \end{aligned}$$

Taking a limit on both sides gives

$$d_Y(f(a), f(a')) \le \eta\left(\frac{2\operatorname{diam}(A)}{\operatorname{diam}(B)}\right) \operatorname{diam}(f(B)),$$

and then taking a supremum over all $a, a' \in A$ yields

$$\operatorname{diam}(f(A)) \le \eta\left(\frac{2\operatorname{diam}(A)}{\operatorname{diam}(B)}\right) \operatorname{diam}(f(B)).$$

This gives one side of the inequality, to obtain the other, apply the upper inequality to f^{-1} (see Proposition 2.1.1). Let $\nu(t) = \frac{1}{\eta^{-1}(1/t)}$,

$$\begin{aligned} \frac{\operatorname{diam}(f^{-1}(f(A)))}{\operatorname{diam}(f^{-1}(f(B)))} &\le \nu\left(\frac{2\operatorname{diam}(f(A))}{\operatorname{diam}(f(B))}\right) \\ \frac{\operatorname{diam}(A)}{\operatorname{diam}(B)} &\le \eta^{-1}\left(\frac{\operatorname{diam}(f(B))}{2\operatorname{diam}(f(A))}\right)^{-1} \\ \eta\left(\frac{\operatorname{diam}(B)}{\operatorname{diam}(A)}\right) &\ge \frac{\operatorname{diam}(f(B))}{2\operatorname{diam}(f(A))} \\ \left(2\eta\left(\frac{\operatorname{diam}(B)}{\operatorname{diam}(A)}\right)\right)^{-1} &\le \frac{\operatorname{diam}(f(A))}{\operatorname{diam}(f(B))}. \end{aligned}$$

□

Corollary 2.1.4. *Quasisymmetric maps send Cauchy sequences to Cauchy sequences. More precisely, if (X, d_X) and (Y, d_Y) are metric spaces, $f : X \to Y$ a quasisymmetry, and $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence, then $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof. Let $B = \{x_n \mid n \in \mathbb{N}\}$ and for each $N \in \mathbb{N}$, let $A_N = \{x_n \mid n > N\}$. If $\operatorname{diam}(A_N) = 0$

for any $N \in \mathbb{N}$, then the sequence is eventually constant, which means its image is Cauchy. So suppose $\mathrm{diam}(A_N) > 0$ for all $N \in \mathbb{N}$. Observe that the sequence being Cauchy implies $\lim_{N\to\infty} \mathrm{diam}(A_N) = 0$; moreover, if $\lim_{N\to\infty} \mathrm{diam}(f(A_N)) = 0$ then $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Now, $\mathrm{diam}(B) < \infty$ because, if it wasn't, then $\mathrm{diam}(A_N) = \infty$ for all $N \in \mathbb{N}$. So we can apply Proposition 2.1.3 on $A_N \subset B$ to say, for all $N \in \mathbb{N}$,

$$\frac{\mathrm{diam}(f(A_N))}{\mathrm{diam}(f(B))} \leq \eta\left(\frac{2\mathrm{diam}(A_N)}{\mathrm{diam}(B)}\right).$$

Observe that by Proposition 2.1.2, $\mathrm{diam}(f(B)) < \infty$. Now take a limit,

$$\begin{aligned}
\lim_{N\to\infty} \frac{\mathrm{diam}(f(A_N))}{\mathrm{diam}(f(B))} &\leq \lim_{N\to\infty} \eta\left(\frac{2\mathrm{diam}(A_N)}{\mathrm{diam}(B)}\right) \\
\frac{\lim_{N\to\infty} \mathrm{diam}(f(A_N))}{\mathrm{diam}(f(B))} &\leq \eta\left(\frac{2\lim_{N\to\infty} \mathrm{diam}(A_N)}{\mathrm{diam}(B)}\right) \\
\frac{\lim_{N\to\infty} \mathrm{diam}(f(A_N))}{\mathrm{diam}(f(B))} &\leq 0 \\
\lim_{N\to\infty} \mathrm{diam}(f(A_N)) &= 0.
\end{aligned}$$

□

Corollary 2.1.5. *Quasisymmetric maps always extend quasisymmetrically to the completion; that is, if (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ a quasisymmetry, then there is a quasisymmetry $\overline{f} : \overline{X} \to \overline{Y}$ with $\overline{f}(x) = f(x)$ for all $x \in X$.*

Proof. For each $x \in \partial X$ there is a Cauchy sequence $(x_n) \subset X$ converging to it. By Corollary 2.1.4, $y_n := (f(x_n))$ is a Cauchy sequence in Y, which means it converges to some $y \in \overline{Y}$. Define $\overline{f}(x) = y$ for all $x \in \partial X$ and $\overline{f}(x) = f(x)$ for all $x \in X$. Then $\overline{f} : \overline{X} \to \overline{Y}$ is a well-defined continuous map. It is surjective, as every Cauchy sequence in Y has a corresponding Cauchy sequence in X whose limit point is mapped to the limit point in $\overline{Y}$. Suppose $\overline{f}$ fails to be injective: let $a \neq b \in \overline{X}$ satisfy $\overline{f}(a) = \overline{f}(b)$. Let $(a_n), (b_n) \subset X$ be Cauchy sequences converging to a, b respectively. Since $\overline{f}(a) = \overline{f}(b)$, we must have $(f(a_n))$ and $(f(b_n))$ converge to the same point in $\overline{Y}$. Consider the sequence $(f(a_0), f(b_0), f(a_1), f(b_1), ...) \subset Y$ and notice that this sequence must converge to $\overline{f}(a) = \overline{f}(b)$; hence, it must be Cauchy. Since f^{-1} sends Cauchy sequences to Cauchy sequences, by Corollary 2.1.4, we must have $(a_0, b_0, a_1, b_1, ...)$ is Cauchy. However, it isn't because eventually, $d_X(a_n, b_n) > \frac{1}{2} d_X(a, b)$. So the contradiction gives us injectivity. We conclude $\overline{f}$ is invertible, and the symmetry of the construction allows us to conclude $\overline{f}$ is a homeomorphism.

To see that $\overline{f}$ is quasisymmetric, pick any distinct $a, b, c \in \overline{X}$ with corresponding Cauchy sequences $(a_n), (b_n), (c_n) \subset X$ converging to them. By passing to subsequences if necessary, assume a_n, b_m, c_k are pairwise distinct for all n, m, k. Then

$$\frac{d_Y(\overline{f}(a), \overline{f}(b))}{d_Y(\overline{f}(a), \overline{f}(c))} = \lim_{n\to\infty} \frac{d_Y(f(a_n), f(b_n))}{d_Y(f(a_n), f(c_n))} \leq \lim_{n\to\infty} \eta\left(\frac{d_X(a_n, b_n)}{d_X(a_n, c_n)}\right) = \eta\left(\frac{d_X(a, b)}{d_X(a, c)}\right).$$

□

Remark 2.1.6. *If $(X, d_X), (Y, d_Y)$ are metric spaces and $f : X \to Y$ a quasisymmetry, then $\overline{f}$ (as in Corollary 2.1.5) quasisymmetrically maps ∂X onto ∂Y. This gives rise to a bijection $F : \partial_0 X \to \partial_0 Y$ that satisfies the following claim: if $C \in \partial_0 X$ then $\overline{f}$ quasisymmetrically maps C onto $F(C)$.*

The fact that the boundaries of metric spaces being quasisymmetric is necessary for the spaces to be quasisymmetric is useful. For example, a Jordan region in the plane is quasisymmetric to a disk only if the Jordan curve is quasisymmetric to a circle. Moreover, if one has multiple boundary components, then each component must have a quasisymmetric counterpart. We will often abuse notation and just refer to $\overline{f}$ as f.

For the remainder of this section, we establish the quasisymmetric invariance of some geometric properties of metric spaces.

Proposition 2.1.7. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ a quasisymmetry. If X is linearly locally connected, then Y is linearly locally connected.*

Proof. Let $\beta \geq 1$ be the constant given by the linear local connectivity of X. We claim Y is linearly locally connected with $\lambda := \eta(\beta)$. Note $\eta(\beta) \geq \eta(1) \geq 1$.

(i) Pick any $y \in Y$ and $r > 0$. Let $a, b \in B(y, r)$. We must find a continuum $E \subset B(y, \lambda r)$ containing a, b. Let $x := f^{-1}(y)$, $c := f^{-1}(a)$, $d := f^{-1}(b)$, and $\delta := \sup_{f(v) \in B(y,r)} d_X(v, x)$. Notice that $c, d \in B(x, \delta)$, and thus there is a continuum $F \subset B(x, \beta\delta)$ containing c and d. Let $E := f(F)$. Notice that E contains a, b and is contained in $f(B(x, \beta\delta))$. We claim that $f(B(x, \beta\delta)) \subset B[y, \lambda r]$. To show this, let $(v_n) \subset f^{-1}(B(y, r))$ be a sequence such that $d_X(v_n, x) \to \delta$. Pick any $u \in B(x, \beta\delta)$. Use quasisymmetry of f to say

$$\frac{d_Y(f(u), y)}{d_Y(f(v_n), y)} \leq \eta\left(\frac{d_X(u, x)}{d_X(v_n, x)}\right) < \eta\left(\frac{\beta\delta}{d_X(v_n, x)}\right).$$

So for all $u \in B(x, \beta\delta)$ and all n we have

$$d_Y(f(u), y) < d_Y(f(v_n), y)\eta\left(\frac{\beta\delta}{d_X(v_n, x)}\right) < r\eta\left(\frac{\beta\delta}{d_X(v_n, x)}\right).$$

Take a limit of both sides in n to obtain

$$d_Y(f(u), y) \leq r\eta\left(\frac{\beta\delta}{\delta}\right) = r\eta(\beta).$$

Thus $E \subset f(B(x, \beta\delta)) \subset B[y, \lambda r]$ and therefore $E \subset B(y, \lambda r)$.

(ii) Pick any $y \in Y$ and $r > 0$. Let $a, b \in Y \setminus B(y, r)$. We must find a continuum $E \subset Y \setminus B(y, \frac{r}{\lambda})$ containing a, b. Let $x := f^{-1}(y)$, $c := f^{-1}(a)$, $d := f^{-1}(b)$, and $\delta := \inf_{f(v) \in Y \setminus B(y,r)} d_X(v, x) > 0$. Then notice $c, d \in X \setminus B(x, \delta)$, so by linear local connectivity, there is a continuum $F \subset X \setminus B(x, \frac{\delta}{\beta})$ containing c and d. Let $E := f(F)$ and notice that $a, b \in E \subset f(X \setminus B(x, \frac{\delta}{\beta})) = Y \setminus f(B(x, \frac{\delta}{\beta}))$. Now we wish to show that $Y \setminus f(B(x, \frac{\delta}{\beta})) \subset Y \setminus B(y, \frac{r}{\lambda})$. To this end, let $(v_n) \subset f^{-1}(Y \setminus B(y, r))$ be a sequence satisfying $d_X(v_n, x) \to \delta$, and pick any $u \in X \setminus B(x, \frac{\delta}{\beta})$. Quasisymmetry gives

$$\frac{r}{d_Y(f(u), y)} \leq \frac{d_Y(f(v_n), y)}{d_Y(f(u), y)} \leq \eta\left(\frac{d_X(v_n, x)}{d_X(u, x)}\right) \leq \eta\left(\frac{\beta d_X(v_n, x)}{\delta}\right) \to \eta(\beta).$$

We arrive at the conclusion

$$d_Y(f(u), y) \geq \frac{r}{\lambda}.$$

Thus $E \subset f(X \setminus B(x, \frac{\delta}{\beta})) \subset Y \setminus B(y, \frac{r}{\lambda})$. $\square$

It's worth noting that the same argument shows that annular linear local connectivity is a quasisymmetric invariant. For domains in the plane, notice that satisfaction of condition (ii) of linear local connectivity means that the space doesn't have any outward pointing sharp cusps. Also satisfaction of condition (i) of linear local connectivity means that the space doesn't have any inward pointing sharp cusps. So this result is one way of saying that quasisymmetries cannot create cusps.

Now we've already seen that quasisymmetries take sets of diameter 0 or ∞ to sets of diameter 0 or ∞ respectively. The same is true of relative distance, though the following proposition is needed to establish that.

Proposition 2.1.8. *Let $(X, d_X), (Y, d_Y)$ be metric spaces with $f : X \to Y$ a quasisymmetry. Let $A, B \subset X$ be closed sets satisfying $d_X(A, B) = 0$, then $d_Y(f(A), f(B)) = 0$.*

Proof. If A, B intersect, then the claim is obvious. Suppose that they are disjoint, and notice then that neither diameter can be finite. There exists sequences $(a_n) \subset A$ and $(b_n) \subset B$ such that $d_X(a_n, b_n) \to 0$. Now, we claim there is a point $b \in B$, $b \neq b_n$ for all sufficiently large n, satisfying the following claim. There exists an $\epsilon > 0$ such that $d_X(b, a_n) > \epsilon$ and $d_Y(f(b), f(a_n)) > \epsilon$ for all sufficiently large n. If the first inequality is false, then that would mean a subsequence of (a_n) converges to b, but A is closed, so we would have $b \in A$ violating disjointness. Similarly, the second inequality failing means $f(A)$ and $f(B)$ fail to be disjoint. Obtain ϵ by taking half the minimum distance between b and the two sequences. Alright, now that we are armed with appropriate points, we can say for sufficiently large n,

$$\frac{d_Y(f(a_n), f(b_n))}{\epsilon} \leq \frac{d_Y(f(a_n), f(b_n))}{d_Y(f(a_n), f(b))} \leq \eta\left(\frac{d_X(a_n, b_n)}{d_X(a_n, b)}\right) \leq \eta\left(\frac{d_X(a_n, b_n)}{\epsilon}\right) \to 0.$$

Thus, $d_Y(f(a_n), f(b_n)) \to 0$. $\square$

Proposition 2.1.9. *Relative distance is η-quasi-preserved under quasisymmetries; meaning, if $(X, d_X), (Y, d_Y)$ are metric spaces, $f : X \to Y$ a quasisymmetry, and $A, B \subset X$ disjoint, closed sets, then*

$$\frac{1}{2\eta(\Delta(A,B)^{-1})} \leq \Delta(f(A), f(B)) \leq \eta(2\Delta(A,B))$$

Proof. First note that if A and B are unbounded, then $\Delta(A, B) = 0$, but Proposition 2.1.2 implies $f(A)$ and $f(B)$ are unbounded, so $\Delta(f(A), f(B)) = 0$ too. Similarly, if either A or B is a singleton, then $f(A)$ or $f(B)$ is as well and $\Delta(A, B) = \Delta(f(A), f(B)) = \infty$. Thus the conclusion holds in these cases. Moving forward, assume $\infty > \min(\operatorname{diam}(A), \operatorname{diam}(B)) > 0$. This will imply $0 < \min(\operatorname{diam}(f(A)), \operatorname{diam}(f(B))) < \infty$. The final problematic case to address, is that it is possible for $d_X(A, B) = 0$, even for bounded A. In this case, the claim holds if and only if $d_Y(f(A), f(B)) = 0$, which is the case by Proposition 2.1.8. We will assume now that $d_X(A, B) > 0$ and $d_Y(f(A), f(B)) > 0$.

Suppose, without loss of generality, that $\operatorname{diam}(f(A)) \leq \operatorname{diam}(f(B))$. Let (a_n) be a sequence of points in A and (b_n) be a sequence of points in B with $d_X(a_n, b_n) \to d_X(A, B)$. Fix any $\epsilon > 0$. We claim that, for all n, there is some $a'_n \in A$ with $d_X(a_n, a'_n) \geq \operatorname{diam}(A)/2 - \epsilon$. To see this, note that

$$\operatorname{diam}(A) = \sup_{a,a' \in A} d_X(a, a') \leq \sup_{a,a' \in A} (d_X(a, a_n) + d_X(a', a_n)) \leq 2 \sup_{a \in A} d_X(a, a_n).$$

Now for all n,

$$\Delta(f(A), f(B)) = \frac{d_Y(f(A), f(B))}{\operatorname{diam}(f(A))} \leq \frac{d_Y(f(a_n), f(b_n))}{d_Y(f(a_n), f(a'_n))} \leq \eta\left(\frac{d_X(a_n, b_n)}{d_X(a_n, a'_n)}\right) \leq \eta\left(\frac{d_X(a_n, b_n)}{\operatorname{diam}(A)/2 - \epsilon}\right).$$

Take $n \to \infty$ and $\epsilon \to 0$ to say

$$\Delta(f(A), f(B)) \leq \eta\left(2\frac{d_X(A, B)}{\operatorname{diam}(A)}\right) \leq \eta(2\Delta(A, B)).$$

Now apply the same argument to f^{-1}. Let $\nu(t) = \frac{1}{\eta^{-1}(1/t)}$. Then

$$\Delta(A, B) = \Delta(f^{-1}(f(A)), f^{-1}(f(B))) \leq \nu(2\Delta(f(A), f(B)))$$

Doing some inequality flips gives us

$$\begin{aligned}
\frac{1}{\Delta(A,B)} &\geq \eta^{-1}\left(\frac{\operatorname{diam}(f(A))}{2d_Y(f(A), f(B))}\right) \\
\eta\left(\frac{1}{\Delta(A,B)}\right) &\geq \frac{\operatorname{diam}(f(A))}{2d_Y(f(A), f(B))} \\
\frac{1}{2\eta(\Delta(A,B)^{-1})} &\leq \frac{d_Y(f(A), f(B))}{\operatorname{diam}(f(A))} = \Delta(f(A), f(B)).
\end{aligned}$$

□

Corollary 2.1.10. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ a quasisymmetry. Let $\{E_i\}_{i \in I}$ be a collection of pairwise disjoint, closed subsets of X which is uniformly relatively separated. Then $\{f(E_i)\}_{i \in I}$ is uniformly relatively separated.*

Proof. Let $\alpha > 0$ satisfy $\Delta(E_i, E_j) \geq \alpha$ for all $i \neq j$. Then for all $i \neq j$,

$$\Delta(f(E_i), f(E_j)) \geq \frac{1}{2\eta(\Delta(E_i, E_j)^{-1})} \geq \frac{1}{2\eta(\alpha^{-1})}.$$

□

We will end this section by considering when weaker requirements on functions give quasisymmetry. The following condition will characterize this well.

Definition 2.1.11. *Let (X, d) be a metric space. We say X is (metric)* ***doubling*** *in case there is some constant $N_0 \in \mathbb{N}$ with the following property. For all $x \in X$ and $r > 0$, there exists $S \subset X$ with $\#(S) \leq N_0$ such that*

$$B(x, r) \subset \cup_{y \in S} B(y, r/2).$$

Proposition 2.1.12 (Heinonen[19] (Theorem 10.18)). *Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a quasisymmetry. If X is doubling then Y is doubling.*

Definition 2.1.13. *Let (X, d_X) and (Y, d_Y) be doubling metric spaces and $f : X \to Y$ a homeomorphism. We say f is* ***weakly quasisymmetric*** *if there is some $H > 0$ such that for all pairwise distinct $a, b, c \in X$*

$$d_X(a, b) \leq d_X(a, c) \Rightarrow d_Y(f(a), f(b)) \leq H d_Y(f(a), f(c)).$$

Quasisymmetries are weak quasisymmetries with $H = \eta(1)$. The converse is true in doubling metric spaces.

Lemma 2.1.14 (Heinonen[19] (Theorem 10.19)). *Let (X, d_X) and (Y, d_Y) be doubling metric spaces and $f : X \to Y$ a homeomorphism. If f is H-weakly quasisymmetric, then f is η-quasisymmetric, where η depends only on H and the doubling constant.*

2.2 Examples of Quasisymmetries

In this section, we'll give some examples of quasisymmetric maps. We'll also establish that some of the properties we've discussed are not quasisymmetrically invariant through counterexamples.

Example 2.2.1. *Isometric and bi-Lipschitz maps are quasisymmetric.*

Proof. Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ a homeomorphism.

- If f is an isometry , then $d_Y(f(a), f(b)) = d_X(a, b)$ for all $a, b \in X$. Thus for all distinct $a, b, c \in X$
$$\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))} = \frac{d_X(a, b)}{d_X(a, c)},$$
and f is quasisymmetric with $\eta(t) = t$.

- If f is bi-Lipschitz , then there is some $L > 0$ such that $\frac{1}{L} d_X(a,b) \leq d_Y(f(a), f(b)) \leq L d_X(a,b)$ for all $a, b \in X$. Thus for all distinct $a, b, c \in X$

$$\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))} \leq \frac{L d_X(a,b)}{(1/L) d_X(a,c)},$$

and f is quasisymmetric with $\eta(t) = L^2 t$.

□

Proposition 2.2.2. *Let (X, d) be a metric space. For any $0 < \alpha \leq 1$, (X, d^α) is also a metric space. Moreover, the Hausdorff dimension of (X, d^α) is $\frac{\dim_{\mathcal{H}}(X)}{\alpha}$.*

Proof. The fact that (X, d^α) is a metric space follows from the fact that for any $s, t \geq 0$, $(s+t)^\alpha \leq s^\alpha + t^\alpha$. Thus, for any $a, b, c \in X$,

$$d^\alpha(a,c) = (d(a,c))^\alpha \leq (d(a,b) + d(b,c))^\alpha \leq d(a,b)^\alpha + d(b,c)^\alpha = d^\alpha(a,b) + d^\alpha(b,c).$$

To see the change in Hausdorff dimension, note that for any $A \subset X$, $\mathrm{diam}(A)$ (with respect to d^α) is $\mathrm{diam}(A)^\alpha$ (with respect to d). □

We say we're snowflaking the metric d when we replace it with d^α for $0 < \alpha < 1$. This is because if we take X to be the unit circle and d the planar metric, then (X, d^α) is bi-Lipschitz to the von Koch snowflake with the planar metric for $\alpha = \frac{\log(3)}{\log(4)}$.[29] Indeed, snowflaking often generates fractal spaces.

Example 2.2.3. *Snowflaking is a quasisymmetric process. That is, the identity map $id : (X, d) \to (X, d^\alpha)$ is a quasisymmetry.*

Proof. Pick any distinct $a, b, c \in X$.

$$\frac{d^\alpha(id(a), id(b))}{d^\alpha(id(a), id(c))} = \left(\frac{d(a,b)}{d(a,c)}\right)^\alpha.$$

So it is quasisymmetric with $\eta(t) = t^\alpha$. □

This example illustrates that quasisymmetries do not preserve Hausdorff dimension; consequently, they also don't preserve Ahlfors regularity. Snowflaking the sphere is an example of a quasisymmetric image of the sphere that isn't Ahlfors 2-regular. If one defines f on $\hat{\mathbb{C}}$ as a snowflaking map on the northern hemisphere and lets f be the identity on the southern hemisphere, the resulting image won't be Ahlfors s-regular for any s.

Any quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is quasisymmetric[30]. Likewise, any quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ or $f : \mathbb{D} \to \mathbb{D}$ will be quasisymmetric. We will elaborate on this in Section 2.5, where conditions under which one can conclude that a quasiconformal map is quasisymmetric are given.

2.3 Modulus

An interval is a non-empty, connected subset of $\mathbb{R}$ (note this includes singletons). Given a metric space (X, d), a curve in X is a continuous function $\gamma : I \to X$ where I is an interval. We call the curve open, closed, or compact if I is open, closed, or compact respectively. If $I' \subset I$ is an interval, we call the restriction of γ to I' a subcurve of γ. We will often abuse notation and denote $\gamma(I)$ as γ; moreover, we will abuse language and refer to $\gamma(I)$ as a curve. If γ is a singleton, we say the curve is constant .

Definition 2.3.1 ([31])**.** *Let (X, d) be a metric space and $I \subset \mathbb{R}$ a compact interval. Then $I = [a, b]$ for some $a \leq b$. Given a curve $\gamma : I \to X$, we define its **length** to be*

$$\ell(\gamma) := \sup_{a=t_0<t_1<\ldots<t_n=b} \sum_{k=1}^{n} d(\gamma(t_{k-1}), \gamma(t_k)).$$

If $a = b$ we define the length to be 0. Note the length may be infinite. For non-compact I, we define the length to be

$$\ell(\gamma) := \sup_{\gamma' \in CS(\gamma)} \ell(\gamma'),$$

where $CS(\gamma)$ is the collection of all compact subcurves of γ.

The following result we will not prove here, but we will state it for reference.

Proposition 2.3.2 (Proposition 5.1.11[31]). *Let (X, d) be a metric space, $I \subset \mathbb{R}$ an interval, and $\gamma : I \to X$ a curve. Then*

$$\operatorname{diam}(\gamma(I)) \leq \mathcal{H}^1(\gamma(I)) \leq \ell(\gamma).$$

Moreover, if γ is injective, then $\mathcal{H}^1(\gamma(I)) = \ell(\gamma)$.

One thing this illustrates in particular, is that length of a curve doesn't depend on the parameterization of the curve if it's injective. Also, recall that in our abuse of notation, we may refer to $\mathcal{H}^1(\gamma(I))$ as $\mathcal{H}^1(\gamma)$.

We say a curve γ is rectifiable in case $\ell(\gamma) < \infty$. We say γ is locally rectifiable if every compact subcurve of γ is rectifiable. For every rectifiable curve, $\gamma : [a, b] \to X$, we can define its associated length function $s_\gamma : [a, b] \to [0, \ell(\gamma)]$ by $s_\gamma(t) := \ell(\gamma|_{[a,t]})$. The length function is increasing and continuous[31]. The arc length parameterization of a rectifiable curve $\gamma : [a, b] \to X$, is another curve $\gamma_s : [0, \ell(\gamma)] \to X$ which is the unique curve satisfying $\gamma(t) = \gamma_s(s_\gamma(t))$ for all $t \in [a, b]$.

Definition 2.3.3 ([31]). *Let (X, d) be a metric space, I a compact interval, and $\gamma : I \to X$ a rectifiable curve. Let $\rho : X \to [0, \infty]$ be a Borel function; this will imply $\rho \circ \gamma_s$ is measurable. The* ***path integral of ρ over γ*** *is defined to be*

$$\int_\gamma \rho \; ds := \int_0^{\ell(\gamma)} \rho(\gamma_s(t)) \; dt.$$

For I non-compact and γ locally rectifiable, we define the path integral of ρ over γ to be

$$\int_\gamma \rho \; ds := \sup_{\gamma' \in CS(\gamma)} \int_{\gamma'} \rho \; ds,$$

where $CS(\gamma)$ is the collection of all compact subcurves of γ. For convenience, we will sometimes refer to the path integral of ρ over γ as the ***ρ-length of γ*** *and denote it*

$$\ell_\rho(\gamma) := \int_\gamma \rho \; ds.$$

Observe that path integrals over a constant curve are 0. Also observe that if $\rho \equiv 1$ then $\ell_\rho(\gamma) = \ell(\gamma)$. Path integrals are not defined for curves which are not locally rectifiable. We now introduce an important tool: modulus of curve families. While modulus is frequently defined with a parameter p, it suits our purposes to just define it for $p = 2$.

Definition 2.3.4 ([31]). *Let (X, d, μ) be a metric measure space with (X, d) separable and μ a locally finite Borel regular measure. Let Γ be a collection of curves in X, and $\rho : X \to [0, \infty]$ a Borel function. We say **ρ is admissible for Γ** , denoted $\rho \wedge \Gamma$, in case for all locally rectifiable $\gamma \in \Gamma$,*

$$\ell_\rho(\gamma) \geq 1.$$

*The **modulus of Γ** is defined as*

$$\operatorname{mod}(\Gamma) := \inf_{\rho \wedge \Gamma} \int_X \rho^2 d\mu.$$

In general, modulus takes on values in $[0, \infty]$. We draw attention to the fact that if there are no admissible functions for Γ, then $\operatorname{mod}(\Gamma) = \infty$. This happens, for example, if Γ contains a constant curve. We also draw attention to the fact that if Γ has no locally rectifiable curves, then the zero function is admissible for Γ and $\operatorname{mod}(\Gamma) = 0$.

It will be convenient to define the following short hand notation when the ambient measure space is clear,

$$A(\rho) := \int_X \rho^2 d\mu.$$

For Γ_1, Γ_2 families of curves in X, we say Γ_1 minorizes Γ_2, denoted $\Gamma_1 < \Gamma_2$, if every curve in Γ_2 contains a subcurve in Γ_1. The following properties of modulus are standard, but we include their proofs to parallel a later discussion on transboundary modulus (Proposition 3.1.2).

Proposition 2.3.5. *Let (X, d, μ) be a metric measure space with (X, d) separable and μ a locally finite Borel regular measure. For each $k \in \mathbb{N}$, let Γ_k be a collection of curves in X.*

(i) $\operatorname{mod}(\emptyset) = 0$.

(ii) *If* $\Gamma_1 \subset \Gamma_2$, *then* $\operatorname{mod}(\Gamma_1) \leq \operatorname{mod}(\Gamma_2)$. *(Monotonicity)*

(iii) *If* $\Gamma_1 < \Gamma_2$, *then* $\operatorname{mod}(\Gamma_1) \geq \operatorname{mod}(\Gamma_2)$. *(Overflowing)*

(iv) $\operatorname{mod}(\cup_{k\in\mathbb{N}}\Gamma_k) \leq \sum_{k\in\mathbb{N}} \operatorname{mod}(\Gamma_k)$. *(Subadditivity)*

(v) *If there are pairwise disjoint Borel sets,* $B_k \subset X$, *such that* $\gamma_k \subset B_k$ *for all* $\gamma_k \in \Gamma_k$, *then* $\operatorname{mod}(\cup_{k\in\mathbb{N}}\Gamma_k) = \sum_{k\in\mathbb{N}} \operatorname{mod}(\Gamma_k)$.

Proof. (i) Notice that the zero function is admissible for $\emptyset$, which will attain the infimizing quantity of 0.

(ii) Pick any $\rho \wedge \Gamma_2$. Then for all $\gamma \in \Gamma_1$, we have $\gamma \in \Gamma_2$ and hence $\ell_\rho(\gamma) \geq 1$. Thus, $\rho \wedge \Gamma_1$. Thus the set of admissible functions for Γ_2 is a subset of the set of admissible functions for Γ_1. The infimum of a subset is larger than the infimum of the superset.

(iii) Pick any $\rho \wedge \Gamma_1$. Every curve $\gamma_2 \in \Gamma_2$ has a subcurve $\gamma_1 \in \Gamma_1$, and by virtue of being a subcurve, $\ell_\rho(\gamma_2) \geq \ell_\rho(\gamma_1) \geq 1$. Thus $\rho \wedge \Gamma_2$, and infimizing over the superset gives a smaller quantity.

(iv) Let $\Gamma = \cup_{k\in\mathbb{N}}\Gamma_k$. Observe that the claim holds if the right hand side is infinite, so suppose that it is finite. Fix $\epsilon > 0$. For each k, choose $\rho_k \wedge \Gamma_k$ such that

$$\operatorname{mod}(\Gamma_k) + \epsilon 2^{-k} \geq A(\rho_k).$$

Let $\rho = \sqrt{\sum_{k\in\mathbb{N}} \rho_k^2}$. Notice that $\rho \wedge \Gamma$, because for each $\gamma \in \Gamma$, there is some $N \in \mathbb{N}$ with $\gamma \in \Gamma_N$, and thus

$$\int_\gamma \rho \, ds = \int_\gamma \sqrt{\sum_{k\in\mathbb{N}} \rho_k^2} \, ds \geq \int_\gamma \sqrt{\rho_N^2} \, ds = \int_\gamma \rho_N \, ds \geq 1.$$

That means

$$\operatorname{mod}(\Gamma) \leq \int_X \rho^2 d\mu = \int_X \sum_{k\in\mathbb{N}} \rho_k^2 d\mu = \sum_{k\in\mathbb{N}} A(\rho_k) \leq \sum_{k\in\mathbb{N}} (\operatorname{mod}(\Gamma_k) + \epsilon 2^{-k}) = 2\epsilon + \sum_{k\in\mathbb{N}} \operatorname{mod}(\Gamma_k).$$

Taking the limit as $\epsilon \to 0$ gives the result.

(v) Let $\Gamma = \cup_{k\in\mathbb{N}}\Gamma_k$. By (iv) we need only show

$$\operatorname{mod}(\Gamma) \geq \sum_{k\in\mathbb{N}} \operatorname{mod}(\Gamma_k).$$

Let $\rho \wedge \Gamma$, and for each k, let $\rho_k = \rho\mathbb{I}_{B_k}$. Notice that ρ_k are all Borel because B_k are. Moreover, for each $\gamma_k \in \Gamma_k$, since $\gamma_k \subset B_k$,

$$\int_{\gamma_k} \rho_k \, ds = \int_{\gamma_k} \rho \, ds \geq 1.$$

Hence $\rho_k \wedge \Gamma_k$ for all k. Since the B_k are pairwise disjoint, we can say

$$\sum_{k\in\mathbb{N}} \operatorname{mod}(\Gamma_k) \leq \sum_{k\in\mathbb{N}} \int_X \rho_k^2 d\mu = \sum_{k\in\mathbb{N}} \int_{B_k} \rho^2 d\mu = \int_{\cup B_k} \rho^2 d\mu \leq \int_X \rho^2 d\mu.$$

Thus by infimizing over admissible functions for Γ, we obtain the result. $\square$

Proposition 2.3.6 ([31])**.** *Let (X, d, μ) be a metric measure space with (X, d) separable and μ locally finite. Let $\Omega \subset X$ be a Borel set and Γ a collection of curves in Ω. If there is some $L > 0$ such that, for all $\gamma \in \Gamma$, we have $\ell(\gamma) \geq L$ then*

$$\operatorname{mod}(\Gamma) \leq L^{-2}\mu(\Omega).$$

Proof. Let $\rho = L^{-1}\mathbb{I}_\Omega$. Then for all $\gamma \in \Gamma$,

$$\int_\gamma \rho \, ds = L^{-1} \int_\gamma 1 \, ds = L^{-1}\ell(\gamma) \geq 1.$$

Thus

$$\operatorname{mod}(\Gamma) \leq \int_X \rho^2 d\mu = L^{-2}\mu(\Omega).$$

$\square$

For X, Y metric spaces, $f : X \to Y$ a continuous function, and Γ a collection of curves in X, we define $f(\Gamma) = \{f \circ \gamma \mid \gamma \in \Gamma\}$ which is a collection of curves in Y.

Proposition 2.3.7. *Let $\Omega, \Omega' \subset \mathbb{R}^2$ be domains and $f : \Omega \to \Omega'$ a (classically) conformal map. If Γ is a family of curves in Ω, then* $\operatorname{mod}(f(\Gamma)) = \operatorname{mod}(\Gamma)$.

Proof. Pick any $\rho \wedge f(\Gamma)$. Let $\rho' = (\rho \circ f)|f'|$. We claim $\rho' \wedge \Gamma$, as for all $\gamma \in \Gamma$,

$$\int_\gamma \rho' \, ds = \int_{f(\gamma)} (\rho'(f^{-1}))|(f^{-1})'| \, ds = \int_{f(\gamma)} \rho|f'(f^{-1})||(f^{-1})'| \, ds = \int_{f(\gamma)} \rho \, ds \geq 1.$$

Thus we can say

$$\operatorname{mod}(\Gamma) \leq \int_\Omega (\rho')^2 dA = \int_{\Omega'} (\rho'(f^{-1}))^2 J(f^{-1}) \, dA = \int_{\Omega'} \rho^2 |f'(f^{-1})|^2 |(f^{-1})'|^2 dA = \int_{\Omega'} \rho^2 dA.$$

This works for any ρ, thus by infimizing, we get $\operatorname{mod}(\Gamma) \leq \operatorname{mod}(f(\Gamma))$. Applying the same argument to f^{-1} gives equality. $\square$

Quite often, we will be concerned with the modulus of curves connecting continua in a metric space. Given a metric space (X, d), disjoint, closed, and connected $E, F \subset X$, and $\Omega \subset X$, we define the family of curves connecting E and F in Ω as

$$\Gamma(E, F; \Omega) := \{\gamma \mid \gamma \text{ is a curve}, \gamma \cup E \cup F \text{ is connected}, \gamma \subset \Omega\}.$$

Bear in mind that this set could be empty. When looking at the modulus of this family, one can restrict the discussion from X to Ω. This is because for all $\rho \wedge \Gamma(E, F; \Omega)$, we have $\rho \mathbb{I}_\Omega \wedge \Gamma(E, F; \Omega)$, and thus the infimum of $A(\rho)$ is the same as the infimum of $\int_\Omega \rho^2 d\mu$. When it is clear that all the curves are living in some subset Ω, we may use $A(\rho)$ to refer to integrating over Ω instead of X. The following examples illustrate why these particular families are of interest when it comes to modulus.

Example 2.3.8. *Consider $\mathbb{R}^2$ with the standard metric and Lebesgue measure. Let $R \subset \mathbb{R}^2$ be defined as $R = [0, a] \times [0, b]$ for $a, b > 0$. Let $E = \{0\} \times [0, b]$ and $F = \{a\} \times [0, b]$, then*

$$\operatorname{mod}(\Gamma(E, F; R)) = \frac{b}{a}.$$

Proof. Let $\rho \wedge \Gamma(E, F; R)$. By considering the horizontal line segments in $\Gamma(E, F; R)$, we can say for all $y_0 \in [0, b]$,

$$\int_0^a \rho(x, y_0)\ dx \geq 1,$$

and by integrating both sides with respect to y,

$$\int_0^b \int_0^a \rho(x, y) dx\ dy \geq b.$$

Now apply Cauchy-Schwarz to the area integral:

$$\int_R \rho^2 dA = \int_0^b \int_0^a \rho(x, y)^2 dx\ dy \geq \frac{1}{ab} \left(\int_0^b \int_0^a \rho(x, y) dx\ dy \right)^2 \geq \frac{b^2}{ab} = \frac{b}{a}.$$

This shows that $\operatorname{mod}\,(\Gamma(E, F; R)) \geq \frac{b}{a}$. To see equality, notice that the curves all have length at least a, and so by Proposition 2.3.6, we have the $\operatorname{mod}(\Gamma(E, F; R)) \leq a^{-2}(ab) = \frac{b}{a}$. $\square$

Similarly, if one looks at the connecting curves between the horizontal sides of the rectangle, one can see that it is $\frac{a}{b}$. This gives rise to a more general fact which is useful for computing lower bounds on the modulus when in the plane. It is sometimes formulated in terms the modulus of connecting curves and separating curves of a quadrilateral. Here, we formulate it in terms of curves connecting different parts of the boundary.

Proposition 2.3.9 (Duality of Modulus). *Let $Q \subset \mathbb{R}^2$ be homeomorphic to $[0, 1]^2$. Let $A_L, A_R, A_B, A_T \subset Q$ correspond to $\{0\} \times [0, 1], \{1\} \times [0, 1], [0, 1] \times \{0\}, [0, 1] \times \{1\}$ respectively. Then*

$$\operatorname{mod}(A_L, A_R; Q)\ \operatorname{mod}\,(A_B, A_T; Q) = 1.$$

Proof. Let C be the four points in the boundary of Q corresponding to the vertices of $[0, 1]^2$. Use the Riemann Mapping Theorem (Theorem 1.2.1) to get a conformal map from the interior of Q to $\mathbb{D}$. This map will extend to be a homeomorphism of the boundary, and thus will send C to four points on the unit circle; the collection of these four points will be called C'. Let $R = [0, x] \times [0, 1]$. The Riemann Mapping Theorem gives a conformal map from the interior of R to $\mathbb{D}$ which also extends as a homeomorphism of the boundary. By composing with a Möbius transformation, and selecting the proper point to be mapped to 0, we can guarantee three of the vertices of R get mapped to C'; indeed, with an appropriate choice of x, we can get all four vertices to map onto C' (this fact is not obvious, and we don't include the details here; see Lehto-Virtanen[32] Chapter 2). This gives a map from Q to R which is conformal on the interior and sends A_L, A_R to the left and right sides of R respectively. The conformal invariance of modulus (Proposition 2.3.7) reveals that

$$\operatorname{mod}(A_L, A_R; Q) = \operatorname{mod}(\{0\} \times [0,1], \{x\} \times [0,1]; R) = \frac{1}{x}.$$

The last equality is applying Example 2.3.8. Similarly, $\operatorname{mod}(A_B, A_T; Q) = x$. We obtain the result. $\square$

Example 2.3.10. *Consider $\mathbb{R}^2$ with the standard metric and Lebesgue measure. For all $0 < R' < R < \infty$, let $E = B[0, R']$ and $F = \mathbb{R}^2 \setminus B(0, R)$. For fixed $\theta_1 \in (0, 2\pi]$, let $A = \{(r, \theta) \mid R' \leq r \leq R,\ 0 \leq \theta \leq \theta_1\}$, a polar rectangle . We have*

$$\operatorname{mod}(\Gamma(E, F; A)) = \frac{\theta_1}{\log(R/R')}.$$

Proof. Suppose $\rho \wedge \Gamma(E, F; A)$. For fixed $\theta_0 \in [0, \theta_1)$, consider the path $\gamma(r) = (r, \theta_0)$, we can say

$$\int_{R'}^{R} \rho(r, \theta_0)\ dr = \int_\gamma \rho\ ds \geq 1,$$

and can integrate both sides to say

$$\int_0^{\theta_1} \int_{R'}^{R} \rho(r, \theta)\ dr\ d\theta \geq \theta_1.$$

Now use Cauchy-Schwarz to say

$$\begin{aligned}\int_A \rho^2 dA = \int_0^{\theta_1} \int_{R'}^{R} \rho(r,\theta)^2 r\ dr\ d\theta &\geq \left(\int_0^{\theta_1} \int_{R'}^{R} \rho(r, \theta)\ dr\ d\theta\right)^2 \left(\int_0^{\theta_1} \int_{R'}^{R} r^{-1}\ dr\ d\theta\right)^{-1} \\ &\geq (\theta_1)^2 \frac{1}{\theta_1 \log(R/R')} = \frac{\theta_1}{\log(R/R')}.\end{aligned}$$

And by infimizing, we get a lower bound on the modulus. To see the upper bound, let $\rho(r,\theta) = \frac{1}{\log(R/R')r}$. Pick any $\gamma \in \Gamma(E,F;A)$. Fix $\theta_0 \in [0,\theta_1)$. Notice that $\ell(\gamma) \geq R - R'$, and for any (r,θ), we have $\rho(r,\theta) = \rho(r,\theta_0)$. This means $\rho(\gamma_s(t)) = \rho(|\gamma_s(t)|,\theta_0)$. Moreover, $|\gamma_s(t)| \geq R' + t$. Thus

$$\int_\gamma \rho \, ds = \int_0^{\ell(\gamma)} \rho(\gamma_s(t)) \, dt = \int_0^{\ell(\gamma)} \rho((|\gamma_s(t)|),\theta_0) \, dt \geq \int_0^{R-R'} \rho(R'+t,\theta_0) \, dt = \frac{1}{\log(R/R')} \int_{R'}^{R} \frac{1}{r} dr = 1.$$

Thus,

$$\operatorname{mod}(\Gamma(E,F;A)) \leq \int_0^{\theta_1} \int_{R'}^{R} \frac{r}{\log(R/R')^2 r^2} dr \, d\theta = \frac{\theta_1}{\log(R/R')}.$$

□

In general metric spaces, computing the modulus exactly is frequently infeasible. However, the following result gives an upper bound on the modulus of an annulus. It, unsurprisingly, bears resemblance to the planar case.

Proposition 2.3.11 ([31]). *Let (X,d,μ) be a metric measure space with (X,d) separable and μ a locally finite Borel regular measure. Fix some $x_0 \in X$. Suppose there exists constants $C_0, R_0 > 0$ such that*

$$\mu(B(x_0,r)) \leq C_0 r^2$$

for all $0 < r < R_0$. Then

$$\operatorname{mod}(\Gamma(B[x_0,r], X \setminus B(x_0,R); X)) \leq \frac{128\log(2)C_0}{\log(R/r)}$$

whenever $0 < 2r < R < R_0$.

Proof. Fix r, R as in the statement. For all $x \in A[x_0,r,R]$, let

$$\rho(x) = \frac{4\log(2)}{\log(R/r)d(x,x_0)}$$

and let $\rho(x) = 0$ otherwise. We claim that $\rho \wedge \Gamma(B[x_0,r], X \setminus B(x_0,R); X)$. To see this, let k be the smallest integer such that $2^k r \geq R$, and notice $k \geq 2$. Pick any $\gamma \in \Gamma(B[x_0,r], X \setminus$

$B(x_0,R);X)$. Since for all $x \in \gamma$, $\rho(x) \geq 4\log(2)(\log(R/r)R)^{-1}$, we have $\ell_\rho(\gamma) = \infty$ if γ isn't rectifiable. Suppose γ is rectifiable. For all $j \in \{1,...,k-1\}$, there is a subcurve γ_j of γ lying in $A(x_0, 2^{j-1}r, 2^j r)$ with $\ell(\gamma_j) \geq 2^{j-1}r$; this follows from Proposition 2.3.2. We now show admissibility:

$$\int_\gamma \rho\ ds \geq \sum_{j=1}^{k-1}\int_{\gamma_j}\rho\ ds \geq 4\frac{\log(2)}{\log(R/r)}\sum_{j=1}^{k-1}\int_{\gamma_j}\frac{1}{2^j r}\ ds = 4\frac{\log(2)}{\log(R/r)}\sum_{j=1}^{k-1}\frac{\ell(\gamma_j)}{2^j r} \geq \frac{2\log(2)}{\log(R/r)}(k-1),$$

but recall $2^k r \geq R$, so $k \geq \frac{\log(R/r)}{\log(2)}$. Thus,

$$\frac{2\log(2)}{\log(R/r)}(k-1) \geq \frac{\log(2)}{\log(R/r)}k \geq 1.$$

Now using k as above, we can say

$$\begin{aligned} A(\rho) &\leq \int_{A[x_0,r,R]}\rho^2 d\mu \leq \sum_{j=1}^{k}\int_{A[x_0,2^{j-1}r,2^j r]}\rho^2 d\mu \\ &\leq \left(\frac{4\log(2)}{\log(R/r)}\right)^2\sum_{j=1}^{k}\frac{\mu(B[x_0,2^j r])}{4^{j-1}r^2} \\ &\leq \left(\frac{4\log(2)}{\log(R/r)}\right)^2\sum_{j=1}^{k}\frac{C_0 4^j r^2}{4^{j-1}r^2} \\ &= \frac{64k\log(2)^2 C_0}{\log(R/r)^2}. \end{aligned}$$

Recall that k was the smallest integer satisfying $2^k r \geq R$, thus $k < 1 + \frac{\log(R/r)}{\log(2)} < 2\frac{\log(R/r)}{\log(2)}$. And so

$$\operatorname{mod}(\Gamma(B[x_0,r], X \setminus B(x_0,R);X)) < \frac{128\log(2)C_0}{\log(R/r)}.$$

$\square$

Corollary 2.3.12 ([31]). *Let (X,d,μ) be a metric measure space with (X,d) separable and μ a locally finite Borel regular measure. Fix some $x_0 \in X$. Suppose there exists constants $C_0, R_0 > 0$ such that*

$$\mu(B(x_0,r)) \leq C_0 r^2$$

for all $0 < r < R_0$. *Suppose* Γ *is a family of curves satisfying every* $\gamma \in \Gamma$ *passes through* x_0 *and is non-constant. Then* $\operatorname{mod}(\Gamma) = 0$.

Proof. For $n \in \mathbb{N}$, $n \geq 1$, let $\Gamma_n = \{\gamma \in \Gamma \mid \gamma \not\subseteq B(x_0, \frac{1}{n})\}$. Since every $\gamma \in \Gamma$ is non-constant, then γ contains some $x \in X$ with $d(x, x_0) > 0$. This means that $\gamma \not\subseteq B(x_0, d(x, x_0))$, hence $\gamma \in \Gamma_n$ for sufficiently large n. So for all $N \in \mathbb{N}$, $\Gamma \subset \cup_{n \geq N} \Gamma_n$. This means, by subadditivity,

$$\operatorname{mod}(\Gamma) \leq \sum_{n=N}^{\infty} \operatorname{mod}(\Gamma_n).$$

Thus it suffices to show $\operatorname{mod}(\Gamma_N) = 0$ for all N sufficiently large. Fix N such that $\frac{1}{N} < R_0$. Let $n > 2N$. Then for all $\gamma \in \Gamma_N$, we have that γ contains a subcurve, γ', with

$$\gamma' \in \Gamma(B[x_0, \frac{1}{n}], X \setminus B(x_0, \frac{1}{N}); X) =: \Gamma_N^n.$$

By the overflowing and monotonicity properties of modulus, we have $\operatorname{mod}(\Gamma_N) \leq \operatorname{mod}(\Gamma_N^n)$ for all $n > 2N$. But, by Proposition 2.3.11, for some constant $C > 0$, we have

$$\operatorname{mod}(\Gamma_N^n) \leq \frac{C}{\log(n/N)} \to 0$$

as $n \to \infty$. Thus $\operatorname{mod}(\Gamma_N) = 0$. □

You may recall that Ahlfors regularity plays an important role in quasisymmetric uniformization. However, as these results just illustrated, one can say some interesting things about spaces which only satisfy the upper bound; therefore, we give the following definition.

Definition 2.3.13 (cf Definition 1.4.8)**.** *Let* (X, d, μ) *be a metric measure space. We say* X *is* ***upper Ahlfors 2-regular*** *if there is some* $C \geq 1$ *such that, for all* $x \in X$ *and* $r > 0$, *we have*

$$\mu(B(x, r)) \leq Cr^2.$$

Upper Ahlfors 2-regularity doesn't give strong information about the Hausdorff dimension. For example, if one looks at the unit ball $B \subset \mathbb{R}^n$, $n \geq 2$, equipped with the standard metric and measure, it is upper Ahlfors 2-regular: for $r < 1$, $\mu(B(x,r) \cap B) \leq Cr^n \leq Cr^2$, and for $r \geq 1$ $\mu(B(x,r) \cap B) \leq \mu(B) \leq \mu(B)r^2$. However, the mass distribution principle does imply that $\dim_{\mathcal{H}}(X) \geq 2$ if X is upper Ahlfors 2-regular for some Borel μ[20].

Corollary 2.3.14. *Let (X, d, μ) be a metric measure space with (X, d) separable and μ a locally finite Borel regular measure. Suppose X is upper Ahlfors 2-regular with constant C_0. Let $E, F \subset X$ be disjoint continua with $\Delta(E, F) > 2$. Then,*

$$\operatorname{mod}(\Gamma(E, F; X)) \leq \frac{128 \log(2) C_0}{\log(\Delta(E, F))}.$$

Proof. Without loss of generality, suppose $\min(\operatorname{diam}(E), \operatorname{diam}(F)) = \operatorname{diam}(E)$. By Corollary 2.3.12, we can assume $\operatorname{diam}(E) > 0$. Notice for all $x \in E$, we have $E \subset B[x, \operatorname{diam}(E)]$. For all $y \in F$ we have

$$d(x, y) \geq d(E, F) > 2\operatorname{diam}(E),$$

and so $F \subset X \setminus B[x, 2\operatorname{diam}(E)]$ for all $x \in E$. Thus, for each $x \in E$, we can say that $\Gamma(E, F; X)$ is minorized by $\Gamma(B[x, \operatorname{diam}(E)], X \setminus B(x, d(E, F)), X)$. Thus by Proposition 2.3.11, there is a constant $C > 0$ such that

$$\operatorname{mod}(\Gamma(E, F; X)) \leq \frac{C}{\log(\Delta(E, F))}.$$

□

2.4 Loewner Spaces

A natural question to ask is if one can bound $\operatorname{mod}(\Gamma(E, F; X))$ from below by a function of the relative distance. This is not possible in general, however. Consider $X \subset \mathbb{R}^2$ defined by $X = \mathbb{R}^2 \setminus (\{0\} \times [-1, 1])$, a slit domain, equipped with the standard Euclidean metric and Lebesgue measure. For $n \geq 1$, let $E_n = \{-\frac{1}{n}\} \times [-\frac{1}{n}, \frac{1}{n}]$ and $F_n = \{\frac{1}{n}\} \times [-\frac{1}{n}, \frac{1}{n}]$. Notice $\Delta(E_n, F_n) = 1$ for all $n \geq 1$. However, we claim that $\operatorname{mod}(\Gamma(E_n, F_n; X)) \to 0$ as $n \to \infty$. Indeed, for $n \geq 3$, one can define the polar rectangle $A_n = \{(r, \theta) \mid \frac{2}{n} < r < 1,\ \frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$, and since $E_n \subset B(0, \frac{2}{n})$, we can say that $\Gamma(B(0, \frac{2}{n}), \mathbb{R}^2 \setminus B(0,1); A)$ minorizes $\Gamma(E_n, F_n; X)$. So, referring to Example 2.3.10, we can say $\operatorname{mod}(\Gamma(E_n, F_n; X)) \leq \frac{\pi}{\log(n/2)} \to 0$.

In some spaces, however, we can bound $\operatorname{mod}(\Gamma(E, F; X))$ from below by a function of the relative distance. For example, this can be done in the plane[33]. If this lower bound exists, it gives deep insight into the geometry of the space.

Definition 2.4.1 (Heinonen-Koskela[3])**.** *Let (X, d, μ) be a metric measure space with (X, d) separable and μ a locally finite Borel regular measure. We say X is a **Loewner space** (or X is Loewner) , in case there is a decreasing function $\Psi : (0, \infty) \to (0, \infty)$ such that for all disjoint continua $E, F \subset X$ we have*

$$\operatorname{mod}(\Gamma(E, F; X)) \geq \Psi(\Delta(E, F)).$$

If X is Loewner and upper Ahlfors 2-regular, then by Corollary 2.3.14, we must have $\Psi(t) \to 0$ as $t \to \infty$. This is good since, if E or F is a singleton, the modulus is 0 (Corollary 2.3.12) and the relative distance is ∞.

Loewner[33] observed that disjoint, non-degenerate continua in $\mathbb{R}^2$ have positive capacity, implying that the plane is Loewner; which is why the property is named after him. We will state without proof that $\mathbb{R}^2, \mathbb{D}$, and $\hat{\mathbb{C}}$ are Loewner. Heinonen and Koskela[3] were the first to define the Loewner property as it appears here. They showed that upper Ahlfors 2-regular, Loewner spaces were Ahlfors 2-regular and linearly locally connected, among other properties. They introduced it because it is a sufficient condition to conclude that a metrically quasiconformal map is quasisymmetric; we go into more details on this result in Section 2.5. We conclude by remarking that removing countably many points doesn't change whether or not a space is Loewner.

Proposition 2.4.2. *Let (X, d, μ) be a metric measure space with (X, d) separable and μ a locally finite Borel regular measure. Suppose X is upper Ahlfors 2-regular. Let $S \subset X$ be a countable set. If X is Loewner then $X \setminus S$ is Loewner.*

Proof. Let $E, F \subset X \setminus S$ be disjoint, non-degenerate continua. We claim that

$$\operatorname{mod}(\Gamma(E,F;X) \setminus \Gamma(E,F;X \setminus S)) = 0.$$

Let $S = \{s_i\}_{i \in I}$ where I is countable. Let

$$\Gamma_i = \{\gamma \in \Gamma(E,F;X) \mid s_i \in \gamma\}.$$

By Proposition 2.3.12, we have $\operatorname{mod}(\Gamma_i) = 0$ for all $i \in I$. Thus, we can use subadditivity

$$\operatorname{mod}(\Gamma(E,F;X) \setminus \Gamma(E,F;X \setminus S)) = \operatorname{mod}\Big(\cup_{i \in I} \Gamma_i\Big) \leq \sum_{i \in I} \operatorname{mod}(\Gamma_i) = 0.$$

Now, by monotonicity, we have $\operatorname{mod}(\Gamma(E,F;X \setminus S)) \leq \operatorname{mod}(\Gamma(E,F;X))$, and by subadditivity

$$\operatorname{mod}\,(\Gamma(E,F;X)) \leq \operatorname{mod}\,(\Gamma(E,F;X \setminus S)) + \operatorname{mod}\,(\Gamma(E,F;X) \setminus \Gamma(E,F;X \setminus S)) = \operatorname{mod}\,(\Gamma(E,F;X \setminus S)).$$

Thus we conclude

$$\operatorname{mod}(\Gamma(E,F;X)) = \operatorname{mod}(\Gamma(E,F;X \setminus S)).$$

Thus, if X is Loewner, this equality gives that $X \setminus S$ is Loewner. $\square$

2.5 Definitions of Quasiconformality

In the first chapter, we gave a definition for quasiconformality (Definition 1.3.1), which we referred to as metric quasiconformality. We showed that quasisymmetric maps are metrically quasiconformal. Indeed, quasisymmetries have globally bounded metric dilatations. Thus, it is very useful to know under what conditions quasiconformal maps are quasisymmetric. One may reasonably be skeptical that such an equivalence would be widespread, but Heinonen and Koskela[30] proved that they are equivalent when mapping between Euclidean spaces.

They famously showed that the lim sup in the definition of metric quasiconformality can be replaced with lim inf in the Euclidean setting. They went on to prove the following, much more general result.

Theorem 2.5.1 (Heinonen-Koskela[3]). *Let $(X, d_X), (Y, d_Y)$ be separable metric spaces. Suppose $(X, d_X, \mathcal{H}^2)$ and $(Y, d_Y, \mathcal{H}^2)$ are Ahlfors 2-regular, X is Loewner, and Y is linearly locally connected. Let $f : X \to Y$ be metrically quasiconformal. Then the following hold.*

(1) If X and Y are bounded, then f is quasisymmetric.

(2) If X and Y are unbounded and f sends bounded sets to bounded sets, then f is quasisymmetric.

There are other definitions of quasiconformality. This section is to discuss what they are and how they relate; although we will restrict our discussion to 2 dimensions: these definitions have natural extensions for general p-modulus, Ahlfors s-regularity, etc.. If X and Y are planar domains and f is conformal in the classical sense, then $\mathrm{mod}(f(\Gamma)) = \mathrm{mod}(\Gamma)$ for all curve families Γ (Proposition 2.3.7). Indeed, the converse turns out to be true as well[34] for orientation preserving homeomorphisms. Thus, an alternative definition of a conformal map is a homeomorphism that preserves modulus. This gives rise to the following definition of quasiconformality, which was first given by Ahlfors[34] in the planar setting.

Definition 2.5.2. *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be a metric measure spaces with $(X, d_X), (Y, d_Y)$ separable and μ_X, μ_Y locally finite Borel regular measures. For $K \geq 1$, we say a homeomorphism $f : X \to Y$ is **(geometrically) K-quasiconformal** in case for all families of curves Γ in X, we have*

$$\frac{1}{K} \,\mathrm{mod}\,(\Gamma) \leq \mathrm{mod}(f(\Gamma)) \leq K \,\mathrm{mod}\,(\Gamma).$$

The geometric definition is the one we will find most useful for our purposes, since modulus is the main tool used in this work. The following proposition illustrates one instance of this utility.

Proposition 2.5.3. *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be a metric measure spaces with $(X, d_X), (Y, d_Y)$ separable and μ_X, μ_Y locally finite Borel regular measures. Suppose $f : X \to Y$ is quasisymmetric and geometrically quasiconformal. Then if X is Loewner, Y is Loewner.*

Proof. Pick any disjoint continua $E, F \subset Y$. We must find some decreasing function $\Psi : (0, \infty) \to (0, \infty)$ such that

$$\operatorname{mod}(\Gamma(E, F; Y)) \geq \Psi(\Delta(E, F)).$$

Notice that $E' := f^{-1}(E)$ and $F' := f^{-1}(F)$ are disjoint continua, and so, because X is Loewner, there exists a decreasing function $\Psi' : (0, \infty) \to (0, \infty)$ such that

$$\operatorname{mod}(\Gamma(E', F'; X)) \geq \Psi'(\Delta(E', F')).$$

Let f^{-1} be ν-quasisymmetric. Now we apply Definition 2.5.2 and Proposition 2.1.9 (on f^{-1}) to say

$$\operatorname{mod}(\Gamma(E, F; Y)) \geq K^{-1} \operatorname{mod}\,(\Gamma(E', F'; X)) \geq K^{-1}\Psi'(\Delta(E', F')) \geq K^{-1}\Psi'(\nu(2\Delta(E, F))).$$

Thus, let $\Psi(t) = K^{-1}\Psi'(\nu(2t))$. It is decreasing since Ψ' decreases and ν increases. $\square$

This proposition illustrates one reason why we're concerned about when quasisymmetric maps are geometrically quasiconformal. Since quasisymmetric maps are always metrically quasiconformal, one might hope that quasisymmetric maps are always geometrically quasiconformal. However, one needs geometric conditions on the space to draw this conclusion.

Theorem 2.5.4 (Tyson[21])**.** *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be separable, locally compact, and connected metric measure spaces with locally finite Borel measures μ_X and μ_Y. Suppose (X, d_X, μ_X) and (Y, d_Y, μ_Y) are Ahlfors 2-regular. If $f : X \to Y$ is quasisymmetric then f is geometrically quasiconformal.*

There's another definition of quasiconformality that will be mentioned. When we discussed the classical notion of analytic quasiconformality in Chapter 1, we said that the diffeomorphic image of infinitesimal balls were ellipses whose eccentricity was given by $\frac{|f_z|+|f_{\overline{z}}|}{|f_z|-|f_{\overline{z}}|} \leq K$; this can be rewritten as $|Df|^2 \leq KJ(f)$ where Df is the derivative and $J(f)$ is the Jacobian. That inequality, when paired with the requirement that f be absolutely continuous on almost every line, gives another definition of quasiconformality[34]. The following definition generalizes this by appealing to the theory Newton-Sobolev class of functions to discuss notions of derivative and Jacobian on metric measure spaces. We will not define the Newton-Sobolev class here, as we aren't going to use it directly.

Definition 2.5.5 ([35])**.** *Let (X, d_X, μ_X) be a metric measure space with (X, d_X) separable and μ_X a locally finite Borel regular measure. Let (Y, d_Y, μ_Y) be a metric measure space. Let $f : X \to Y$ be a homeomorphism. Let $g : X \to \mathbb{R}$ be a Borel function. Define Γ_g to be the collection of all rectifiable curves $\gamma : [a, b] \to X$ such that*

$$\int_\gamma g \; ds \geq d_Y(f(\gamma(a)), f(\gamma(b))).$$

Let Γ be the set of all rectifiable curves $\gamma : [a, b] \to X$. We say g is an ***upper gradient*** *of f if $\Gamma_g = \Gamma$. We say g is a* ***weak upper gradient*** *of f if $\operatorname{mod}(\Gamma \setminus \Gamma_g) = 0$. We say g is a minimal weak upper gradient if for all weak upper gradients, h, we have $h \geq g$ μ_X-almost everywhere.*

Let $K \geq 1$ be given. We say f is ***(analytically) K-quasiconformal*** *in case $f \in N^{1,2}_{loc}(X : Y)$ (the Newton-Sobolev class, see[36]), and*

$$g_f(x)^2 \leq KJ_f(x)$$

for μ_X-almost every $x \in X$, where g_f is the minimal weak upper gradient and

$$J_f(x) := \limsup_{r\to 0} \frac{\mu_Y(f(B(x, r)))}{\mu_X(B(x, r))}.$$

The following theorem shows that the analytic and geometric definitions of quasiconformality are equivalent in remarkable generality: there are practically no assumptions on the spaces.

Theorem 2.5.6 (Williams[35]). *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be a metric measure spaces with $(X, d_X), (Y, d_Y)$ separable and μ_X, μ_Y locally finite Borel regular measures. Let $f : X \to Y$ be a homeomorphism. Then the following are equivalent with the same constant $K \geq 1$.*

- *$f \in N^{1,2}_{loc}(X : Y)$, and for μ_X-almost every $x \in X$,*

$$g_f(x)^2 \leq K J_f(x).$$

- *For every collection of curves, Γ, in X, we have*

$$\mathrm{mod}(\Gamma) \leq K \ \mathrm{mod}\ (f(\Gamma)).$$

Now, it is of interest to know under what conditions all these definitions are equivalent. Indeed, it would be rather counter-intuitive to call them alternative definitions if they weren't equivalent in reasonable generality. The following geometric condition will give equivalence.

Definition 2.5.7 (Heinonen-Koskela-Shanmugalingam-Tyson[36]). *Let (X, d, μ) be a metric measure space with (X, d) separable and μ a locally finite Borel regular measure. We say X is said to be of* ***locally 2-bounded geometry*** *if X is path connected, locally compact, and satisfies the following conditions. There is a constant $C \geq 1$ such that every point in X has a neighborhood U with*

$$\mu(B_R) \leq CR^2$$

whenever $B_R \subset U$ is a ball of radius R. There is a constant $\lambda \leq 1$ and decreasing function $\Psi : (0, \infty) \to (0, \infty)$ such that every point $x_0 \in X$ has a neighborhood U with the following property. For all $B(x, R) \subset U$, if $E, F \subset B(x, \lambda R)$ are disjoint and non-degenerate continua, then

$$\mathrm{mod}(\Gamma(E, F; X)) \geq \Psi(\Delta(E, F)).$$

If X is locally upper Ahlfors 2-regular and locally Loewner, then X will be of locally 2-bounded geometry. We say f is locally η-quasisymmetric in case every point has a neighborhood where f is η-quasisymmetric.

Theorem 2.5.8 (Heinonen-Koskela-Shanmugalingam-Tyson[36]). *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be a metric measure spaces with $(X, d_X), (Y, d_Y)$ separable and μ_X, μ_Y locally finite Borel regular measures. Let $f : X \to Y$ be a homeomorphism. Suppose X and Y are of locally 2-bounded geometry. Then the following are equivalent.*

- *f is metrically quasiconformal.*
- *f is geometrically quasiconformal.*
- *f is analytically quasiconformal.*
- *f is locally quasisymmetric.*

Moreover, if f satisfies one (all) of these conditions, then f is absolutely continuous in measure, and $\mathrm{mod}(\Gamma) = 0$, where Γ is the collection of curves, γ, in X where f fails to be absolutely continuous along γ.

Corollary 2.5.9. *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be metric measure spaces with $(X, d_X), (Y, d_Y)$ separable and μ_X, μ_Y locally finite Borel regular measures. Let $f : X \to Y$ be a homeomorphism. Suppose X and Y are path connected, locally compact, upper Ahlfors 2-regular, and locally Loewner. Then f the conclusions of Theorem 2.5.8 hold.*

As an example, domains in $\mathbb{R}^2$ are always upper Ahlfors 2-regular and locally Loewner. Thus this implies that all definitions of quasiconformality are equivalent for homeomorphisms between planar domains. The same can be said for domains in $\hat{\mathbb{C}}$.

For the remainder of this dissertation, we will chiefly use the geometric definition of quasiconformality. Past this point, if a map is ever referred to as quasiconformal without specifying the definition, geometric quasiconformality is what is meant to be implied.

Chapter 3

Transboundary Modulus and Transboundary Loewner Properties

The main tool our results will use is that of transboundary modulus. In this chapter, we define it in the context of metric spaces homeomorphic to planar domains. Since it has only been previously defined for Euclidean settings (see[4],[1],[37],[38]), we will need to establish the theory in this broader context.

3.1 Transboundary Modulus

Allow us to start with an important terminological clarification. Whenever we say a set is countable, we mean that it is either countably infinite, finite, or empty. So a countable set may have finitely many elements.

As the last few sections of Chapter 2 have demonstrated, modulus is a useful tool in building geometric arguments between metric measure spaces. This is largely due to its quasi-invariance under quasiconformal maps. This section, we introduce a similar tool which is also quasi-invariant under quasiconformal maps, and thus provides additional utility.

Transboundary modulus was first introduced as transboundary extremal length by Schramm[4] and was used for giving an alternative proof to the countable case of Koebe's conjecture (Theorem 1.2.5). He defined it for domains in the sphere; the definition given here will be more general.

Let (X,d) be a metric space and $\mathcal{K} = \{K_i\}_{i\in I}$ be a countable collection of pairwise disjoint continua in X. Let $K = \cup_{i\in I} K_i$ and $D = X \setminus K$. Suppose that D is homeomorphic to a domain in $\hat{\mathbb{C}}$. Define an equivalence relation on X as follows. For all $a \in X$, $a \sim a$; if $a, b \in K_i$ for some $i \in I$, then $a \sim b$. We will consider the quotient space, $X_{\mathcal{K}} := X/\sim$, equipped with the quotient topology. We will call $X_{\mathcal{K}}$ the $\mathcal{K}$-quotient of X . Let $\pi_{\mathcal{K}} : X \to X_{\mathcal{K}}$ be the quotient map. Notice $\pi_{\mathcal{K}}$ is injective on D; thus, $\pi_{\mathcal{K}}|_D$ is a homeomorphism. Let $D_{\mathcal{K}} := \pi_{\mathcal{K}}(D)$. For each $i \in I$, $\pi_{\mathcal{K}}(K_i)$ is a singleton in $X_{\mathcal{K}}$; define $k_i = \pi_{\mathcal{K}}(K_i)$. Let $k = \cup_{i\in I} k_i$. Suppose we equip (X,d) with a Borel measure μ. We can define a Borel measure $\mu_{\mathcal{K}}$ on $X_{\mathcal{K}}$ by

$$\mu_{\mathcal{K}}(E) = \mu(\pi_{\mathcal{K}}^{-1}(D_{\mathcal{K}} \cap E)) + \sum_{i\in I} \delta_{k_i}(E) = \mu(\pi_{\mathcal{K}}^{-1}(E \setminus k)) + \sum_{i\in I} \delta_{k_i}(E \cap k)$$

for measurable $E \subset X_{\mathcal{K}}$. Notice that we only need μ to be defined on D for $\mu_{\mathcal{K}}$ to be well-defined.

By a curve in $X_{\mathcal{K}}$, we mean a continuous function $\gamma : J \to X_{\mathcal{K}}$ where $J \subset \mathbb{R}$ is an interval. We may also call γ a transboundary curve . We will often abuse notation and write $\gamma(J) = \gamma$. If $\gamma \subset D_{\mathcal{K}}$, then there is a curve $\gamma' \subset D$ with $\gamma = \pi_{\mathcal{K}}(\gamma')$. Sometimes we will abuse notation and write $\gamma = \gamma'$ if it's clear the curve is in $D_{\mathcal{K}}$. Since $D_{\mathcal{K}}$ is homeomorphic to a domain, it must be open in $X_{\mathcal{K}}$. Hence $\gamma^{-1}(D_{\mathcal{K}})$ must be a relatively open subset of J; thus, it is a union of relatively open subintervals. Let $\{J_m\}_{m\in I_\gamma}$ be the collection of relatively open subintervals of J such that

$$\cup_{m\in I_\gamma} J_m = \gamma^{-1}(D_{\mathcal{K}}).$$

Notice that I_γ is countable. It's empty if and only if γ is a constant curve in k: $\gamma(t) = k_i$ for some $i \in I$. If I_γ has one element, then $\gamma \subset D_{\mathcal{K}}$. For each $m \in I_\gamma$, define the curve

$\gamma_m : J_m \to D$ so that

$$\gamma_m(t) = \pi_{\mathcal{K}}^{-1}(\gamma(t)).$$

Notice that

$$\cup_{m \in I_\gamma} \pi_{\mathcal{K}}(\gamma_m) = \gamma \setminus k.$$

If γ is non-constant, then γ_m is non-constant for all $m \in I_\gamma$. We say γ is locally rectifiable relative $\mathcal{K}$ in case γ_m is locally rectifiable for all $m \in I_\gamma$. Given a Borel $\rho : D \to [0, \infty]$, we will say the ρ-length of γ relative $\mathcal{K}$ is

$$\ell_\rho^{\mathcal{K}}(\gamma) := \sum_{m \in I_\gamma} \ell_\rho(\gamma_m).$$

For a curve γ in X, we will often use the following notational shortcuts for $\{\gamma_m\}$ corresponding to $\pi_{\mathcal{K}}(\gamma)$.

$$\ell(\gamma \setminus K) = \sum_{m \in I_{\pi_{\mathcal{K}}(\gamma)}} \ell(\gamma_m)$$

$$\ell_\rho(\gamma \setminus K) = \sum_{m \in I_{\pi_{\mathcal{K}}(\gamma)}} \ell_\rho(\gamma_m)$$

$$\int_{\gamma \setminus K} \rho \, ds = \sum_{m \in I_{\pi_{\mathcal{K}}(\gamma)}} \int_{\gamma_m} \rho \, ds$$

Definition 3.1.1. *Let (X, d) be a metric space and μ a locally finite Borel measure on X. Let $\mathcal{K} = \{K_i\}_{i \in I}$ be a countable collection of pairwise disjoint continua in X, and let $K = \cup_{i \in I} K_i$. Suppose $X \setminus K$ is homeomorphic to a domain in $\hat{\mathbb{C}}$. Let $k_i = \pi_{\mathcal{K}}(K_i)$ and $k = \pi_{\mathcal{K}}(K)$. Let Γ be a family of curves in $X_{\mathcal{K}}$. By a* ***transboundary mass distribution,*** *we mean a tuple, $P = (\rho; \{\rho_i\}_{i \in I})$, where $\rho : X \setminus K \to [0, \infty]$ is a Borel function, and $\rho_i \geq 0$ is a real number for each $i \in I$ (we will call ρ_i the weight corresponding to K_i). We say* ***P is admissible for Γ relative*** *$\mathcal{K}$, denoted $P \wedge_{\mathcal{K}} \Gamma$, in case for all $\gamma \in \Gamma$ which are locally* *rectifiable relative $\mathcal{K}$, we have*

$$\ell_P^{\mathcal{K}}(\gamma) := \ell_\rho^{\mathcal{K}}(\gamma) + \sum_{k_i \in \gamma} \rho_i \geq 1.$$

Let $\rho_P : X_{\mathcal{K}} \to [0, \infty]$ be a Borel function defined as $\rho_P(x) = \rho((\pi_{\mathcal{K}})^{-1}(x))$ for $x \in X_{\mathcal{K}} \setminus k$ and $\rho_P(k_i) = \rho_i$. We then define the ***transboundary modulus of Γ*** *to be*

$$\operatorname{mod}_{\mathcal{K}}(\Gamma) := \inf_{P \wedge_{\mathcal{K}} \Gamma} \int_{X_{\mathcal{K}}} \rho_P^2 \, d\mu_{\mathcal{K}} = \inf_{P \wedge_{\mathcal{K}} \Gamma} \left(\int_{X \setminus K} \rho^2 d\mu + \sum_{i \in I} \rho_i^2 \right).$$

If Γ is a family of curves in X, we say $P \wedge_{\mathcal{K}} \Gamma$ if $P \wedge_{\mathcal{K}} (\pi_{\mathcal{K}}(\Gamma))$, and we define

$$\operatorname{mod}_{\mathcal{K}}(\Gamma) := \operatorname{mod}_{\mathcal{K}}(\pi_{\mathcal{K}}(\Gamma)).$$

In general, transboundary modulus takes on values in $[0, \infty]$. We point out that if $\mathcal{K}$ is empty, or if every $\gamma \in \Gamma$ is disjoint with K, then this definition coincides with the definition of modulus. Much like modulus, if there are no admissible mass distributions for Γ, then $\operatorname{mod}_{\mathcal{K}}(\Gamma) = \infty$. This happens, for example, if Γ contains a constant curve outside of K. We also draw attention to the fact that if Γ has no curves which are locally rectifiable relative $\mathcal{K}$, then the zero distribution is admissible for Γ and $\operatorname{mod}_{\mathcal{K}}(\Gamma) = 0$.

We will use the following notation when convenient,

$$A_{\mathcal{K}}(P) := \int_{X_{\mathcal{K}}} \rho_P^2 \, d\mu_{\mathcal{K}} = \int_{X \setminus K} \rho^2 d\mu + \sum_{i \in I} \rho_i^2.$$

Proposition 3.1.2. *Let (X, d, μ) be a metric measure space with μ a locally finite Borel regular measure. Fix a countable collection of pairwise disjoint continua $\mathcal{K} = \{K_i\}_{i \in I}$, let $K = \cup_{i \in I} K_i$, and suppose $X \setminus K$ is homeomorphic to a domain in $\hat{\mathbb{C}}$. For each $n \in \mathbb{N}$, let Γ_n be a collection of curves in X.*

(i) $\operatorname{mod}_{\mathcal{K}}(\emptyset) = 0$.

(ii) *If $\Gamma_1 \subset \Gamma_2$, then* $\operatorname{mod}_{\mathcal{K}}(\Gamma_1) \leq \operatorname{mod}_{\mathcal{K}}(\Gamma_2)$. *(Monotonicity)*

(iii) *If* $\Gamma_1 < \Gamma_2$*, then* $\mathrm{mod}_{\mathcal{K}}(\Gamma_1) \geq \mathrm{mod}_{\mathcal{K}}(\Gamma_2)$. *(Overflowing)*

(iv) $\mathrm{mod}_{\mathcal{K}}(\cup_{n\in\mathbb{N}}\Gamma_n) \leq \sum_{n\in\mathbb{N}} \mathrm{mod}_{\mathcal{K}}(\Gamma_n)$. *(Subadditivity)*

(v) *Suppose there are pairwise disjoint Borel sets,* $B_n \subset X$*, such that for all* $i \in I$ *there is at most one* $n \in \mathbb{N}$ *with* $K_i \cap B_n \neq \emptyset$*. If, for all* $n \in \mathbb{N}$*,* $\gamma_n \subset B_n$ *for all* $\gamma_n \in \Gamma_n$*, then* $\mathrm{mod}_{\mathcal{K}}(\cup_{n\in\mathbb{N}}\Gamma_n) = \sum_{n\in\mathbb{N}} \mathrm{mod}_{\mathcal{K}}(\Gamma_n)$.

Proof. (i) Notice that the zero distribution is admissible for $\emptyset$, which will attain the infimizing quantity of 0.

(ii) Pick any $P \wedge_{\mathcal{K}} \Gamma_2$. Then for all $\gamma \in \Gamma_1$, we have $\gamma \in \Gamma_2$ and hence $\ell_P^{\mathcal{K}}(\gamma) \geq 1$. Thus, $P \wedge_{\mathcal{K}} \Gamma_1$. Thus the set of admissible distributions for Γ_2 is a subset of the set of admissible distributions for Γ_1. The infimum of a subset is larger than the infimum of the superset.

(iii) Pick any $P \wedge_{\mathcal{K}} \Gamma_1$. Every curve $\gamma_2 \in \Gamma_2$ has a subcurve $\gamma_1 \in \Gamma_1$. Since every K_i intersecting γ_1 also intersects γ_2, we have $\ell_P^{\mathcal{K}}(\gamma_2) \geq \ell_P^{\mathcal{K}}(\gamma_1) \geq 1$. Thus $P \wedge_{\mathcal{K}} \Gamma_2$, and infimizing over the superset gives a smaller quantity.

(iv) Let $\Gamma = \cup_{n\in\mathbb{N}}\Gamma_n$. Observe that the claim holds if the right hand side is infinite, so suppose that it is finite. Fix $\epsilon > 0$. For each n, choose $P_n = (\rho_n; \{\rho_{i,n}\}_{i\in I}) \wedge_{\mathcal{K}} \Gamma_n$ such that

$$\mathrm{mod}_{\mathcal{K}}(\Gamma_n) + \epsilon 2^{-n} \geq A_{\mathcal{K}}(P_n).$$

Let $\rho = \sqrt{\sum_{n\in\mathbb{N}} \rho_n^2}$. For each $i \in I$, define $\rho_i = \sqrt{\sum_{n\in\mathbb{N}} \rho_{i,n}^2}$. Define $P = (\rho; \{\rho_i\}_{i\in I})$. Notice that for all $n \in \mathbb{N}$, $\rho \geq \rho_n$ and $\rho_i \geq \rho_{i,n}$. Thus, we have $\ell_P^{\mathcal{K}}(\gamma) \geq \ell_{P_n}^{\mathcal{K}}(\gamma)$ for all $n \in \mathbb{N}$ and any curve γ. Given any $\gamma \in \Gamma$, we have $\gamma \in \Gamma_n$ for some $n \in \mathbb{N}$, and $\ell_P^{\mathcal{K}}(\gamma) \geq \ell_{P_n}^{\mathcal{K}}(\gamma) \geq 1$. Thus $P \wedge_{\mathcal{K}} \Gamma$. That means

$$\mathrm{mod}\,_{\mathcal{K}}(\Gamma) \leq \int_{X\setminus K} \rho^2 d\mu + \sum_{i\in I} \rho_i^2 = \int_X \sum_{n\in\mathbb{N}} \rho_n^2 d\mu + \sum_{i\in I} \sum_{n\in\mathbb{N}} \rho_{i,n}^2 = \sum_{n\in\mathbb{N}} A_{\mathcal{K}}(P_n) \leq 2\epsilon + \sum_{n\in\mathbb{N}} \mathrm{mod}\,_{\mathcal{K}}(\Gamma_n).$$

Taking the limit as $\epsilon \to 0$ gives the result.

(v) Let $\Gamma = \cup_{n\in\mathbb{N}}\Gamma_n$. By ($iv$) we need only show

$$\operatorname{mod}_{\mathcal{K}}(\Gamma) \geq \sum_{n\in\mathbb{N}} \operatorname{mod}_{\mathcal{K}}(\Gamma_n).$$

Let $P = (\rho; \{\rho_i\}_{i\in I}) \wedge_{\mathcal{K}} \Gamma$. For each n, let $\rho_n = \rho \mathbb{I}_{B_n}$. Notice that ρ_n are all Borel because B_n are all Borel. Let $\rho_{i,n} = \rho_i$ if $K_i \cap B_n \neq \emptyset$ and set $\rho_{i,n} = 0$ otherwise. For each $n \in \mathbb{N}$, define $P_n = (\rho_n; \{\rho_{i,n}\}_{i\in I})$. For each $\gamma_n \in \Gamma_n$, since $\gamma_n \subset B_n$,

$$\ell^{\mathcal{K}}_{P_n}(\gamma_n) = \ell^{\mathcal{K}}_{\rho_n}(\gamma_n) + \sum_{\gamma_n\cap K_i\neq\emptyset} \rho_{i,n} = \ell^{\mathcal{K}}_{\rho}(\gamma_n) + \sum_{\gamma_n\cap K_i\neq\emptyset} \rho_i = \ell^{\mathcal{K}}_{P}(\gamma_n) \geq 1.$$

Hence $P_n \wedge_{\mathcal{K}} \Gamma_n$ for all $n \in \mathbb{N}$. Since the B_n are pairwise disjoint and every K_i intersects at most one, we can say $\pi_{\mathcal{K}}(B_n)$ are disjoint.

$$\sum_{n\in\mathbb{N}} \operatorname{mod}_{\mathcal{K}}(\Gamma_n) \leq \sum_{n\in\mathbb{N}} \int_{X_{\mathcal{K}}} \rho^2_{P_n}\, d\mu_{\mathcal{K}} = \sum_{n\in\mathbb{N}} \int_{\pi_{\mathcal{K}}(B_n)} \rho^2_{P_n}\, d\mu_{\mathcal{K}} = \int_{\cup_{n\in\mathbb{N}}\pi_{\mathcal{K}}(B_n)} \rho^2_P\, d\mu_{\mathcal{K}} \leq \int_{X_{\mathcal{K}}} \rho^2_P\, d\mu_{\mathcal{K}}.$$

Thus by infimizing over admissible distributions for Γ, we obtain the result. $\square$

We remark that one can make corresponding statements about families of curves in $X_{\mathcal{K}}$ by pulling them back to X. For example, if Γ_1, Γ_2 are collections of curves in $X_{\mathcal{K}}$ with $\Gamma_1 \subset \Gamma_2$, then one can find curve families Γ'_1, Γ'_2 in X with $\Gamma'_1 \subset \Gamma'_2$ and $\pi_{\mathcal{K}}(\Gamma'_1) = \Gamma_1$ and $\pi_{\mathcal{K}}(\Gamma'_2) = \Gamma_2$, so that $\operatorname{mod}_{\mathcal{K}}(\Gamma_1) \leq \operatorname{mod}_{\mathcal{K}}(\Gamma_2)$.

These statements resemble statements made about modulus. A natural question to ask is if, like modulus, transboundary modulus is preserved under conformal maps. It turns out that maps distort transboundary modulus exactly as much as they distort modulus, as the following result shows.

Lemma 3.1.3. *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be a metric measure spaces with μ_X, μ_Y locally finite Borel regular measures. Pick any countable collection of pairwise disjoint continua, $\mathcal{K} = \{K_i\}_{i\in I}$ in X and $\mathcal{J} = \{J_1\}_{i\in I}$ in Y. Let $k_i = \pi_{\mathcal{K}}(K_i)$, $K = \cup_{i\in I}K_i$ and $j_i = \pi_{\mathcal{J}}(J_i)$, $J = \cup_{i\in I}J_i$. Suppose $X \setminus K$ and $Y \setminus J$ are homeomorphic to domains in $\hat{\mathbb{C}}$. Suppose there* *is a homeomorphism* $f : X_{\mathcal{K}} \to Y_{\mathcal{J}}$ *with* $f(k_i) = j_i$ *for all* $i \in I$*. Suppose* $\pi_{\mathcal{J}}^{-1} \circ f \circ \pi_{\mathcal{K}} : X \setminus K \to Y \setminus J$ *is geometrically* H*-quasiconformal for* $H \geq 1$*. Let* Γ *be any family of curves in* $X_{\mathcal{K}}$*. Then*

$$H^{-1} \operatorname{mod}_{\mathcal{K}}(\Gamma) \leq \operatorname{mod}_{\mathcal{J}}(f(\Gamma)) \leq H \operatorname{mod}_{\mathcal{K}}(\Gamma).$$

Proof. Throughout this proof, we will mildly abuse notation and just call $(\pi_{\mathcal{J}}^{-1} \circ f \circ \pi_{\mathcal{K}})|_{X \setminus K}$ the much shorter name, f. First we use Theorem 2.5.6 to say that f is analytically quasiconformal on $X \setminus K$. Williams[35] showed that if γ is any absolutely continuous curve in $X \setminus K$, f is absolutely continuous on γ, and ρ is a non-negative Borel function on $Y \setminus J$ with $A(\rho) < \infty$, then we have that

$$\int_{\gamma} (\rho \circ f) g_f \, ds \geq \int_{f(\gamma)} \rho \, ds.$$

Every compact rectifiable curve has an absolutely continuous arc-length parameterization (Proposition 5.1.8[31]). Thus every locally rectifiable γ in $X \setminus K$ satisfies

$$\int_{\gamma} (\rho \circ f) g_f \, ds \geq \int_{f(\gamma)} \rho \, ds,$$

with f absolutely continuous on every compact subcurve of γ and $A(\rho) < \infty$. Let Γ_f be the collection of curves in $X \setminus K$ on which f fails to be absolutely continuous. Shanmugalingam (Proposition 3.1[39]) showed that if $f \in N^{1,2}_{\mathrm{loc}}(X \setminus K : Y \setminus J)$, then $\operatorname{mod}(\Gamma_f) = 0$. Now, let $\Gamma_f^{\mathcal{K}}$ be the family of curves in $X_{\mathcal{K}}$ which contain a subcurve in $\pi_{\mathcal{K}}(\Gamma_f)$. We can use the overflowing property to say

$$\operatorname{mod}_{\mathcal{K}}(\Gamma_f^{\mathcal{K}}) \leq \operatorname{mod}_{\mathcal{K}}(\pi_{\mathcal{K}}(\Gamma_f)) = \operatorname{mod}_{\mathcal{K}}(\Gamma_f) = \operatorname{mod}(\Gamma_f) = 0.$$

Notice that monotonicity gives $\operatorname{mod}_{\mathcal{K}}(\Gamma \setminus \Gamma_f^{\mathcal{K}}) \leq \operatorname{mod}_{\mathcal{K}}(\Gamma)$, and subadditivity gives

$$\operatorname{mod}_{\mathcal{K}}(\Gamma) \leq \operatorname{mod}_{\mathcal{K}}(\Gamma \setminus \Gamma_f^{\mathcal{K}}) + \operatorname{mod}_{\mathcal{K}}(\Gamma_f^{\mathcal{K}}) = \operatorname{mod}_{\mathcal{K}}(\Gamma \setminus \Gamma_f^{\mathcal{K}}).$$

So we have equality: $\operatorname{mod}_{\mathcal{K}}(\Gamma \setminus \Gamma_f^{\mathcal{K}}) = \operatorname{mod}_{\mathcal{K}}(\Gamma)$.

Now, pick any $P = (\rho; \{\rho_i\}_{i\in I}) \wedge_{\mathcal{J}} f(\Gamma)$. Suppose, for the moment, that $\operatorname{mod}_{\mathcal{J}}(f(\Gamma)) < \infty$. Then we can suppose that $A(\rho) < \infty$. Let $\rho' = (\rho \circ f)g_f$ and $P' = (\rho'; \{\rho_i\}_{i\in I})$. Pick any locally rectifiable $\gamma \in \Gamma \setminus \Gamma_f^{\mathcal{K}}$. Since $X \setminus K$ is homeomorphic to a domain in $\hat{\mathbb{C}}$, we can construct countably many curves γ_m in $X \setminus K$ such that

$$\ell_{\rho'}^{\mathcal{K}}(\gamma) = \sum_m \ell_{\rho'}(\gamma_m),$$

where γ_m are locally rectifiable (see the discussion preceding Definition 3.1.1). Since $\gamma \notin \Gamma_f^{\mathcal{K}}$, we can say f is absolutely continuous on any compact subcurve of γ_m for all m. Thus, for all m, we can say $\ell_{\rho'}(\gamma_m) \geq \ell_\rho(f(\gamma_m))$. Thus, we can say $\ell_{\rho'}^{\mathcal{K}}(\gamma) \geq \ell_\rho^{\mathcal{J}}(f(\gamma))$, and since the weights are the same, $\ell_{P'}^{\mathcal{K}}(\gamma) \geq \ell_P^{\mathcal{J}}(f(\gamma)) \geq 1$. So $P' \wedge_{\mathcal{K}} \Gamma \setminus \Gamma_f^{\mathcal{K}}$. Thus we can apply analytic quasiconformality to say

$$\begin{aligned}
\operatorname{mod}_{\mathcal{K}}(\Gamma) = \operatorname{mod}_{\mathcal{K}}(\Gamma \setminus \Gamma_f^{\mathcal{K}}) &\leq \int_{X\setminus K} (\rho \circ f)^2 g_f^2 \, d\mu_X + \sum_{i\in I} \rho_i^2 \\
&\leq H\left(\int_{X\setminus K} (\rho \circ f)^2 J_f \, d\mu_X + \sum_{i\in I} \rho_i^2 \right) \\
&\leq H\left(\int_{Y\setminus J} \rho^2 \, d\mu_Y + \sum_{i\in I} \rho_i^2 \right).
\end{aligned}$$

By infimizing over P, we obtain that $\operatorname{mod}_{\mathcal{K}}(\Gamma) \leq H \operatorname{mod}_{\mathcal{J}}(f(\Gamma))$. Note that this inequality still holds if $\operatorname{mod}_{\mathcal{J}}(f(\Gamma)) = \infty$. Apply the same argument to f^{-1} to get the other inequality.

□

The conditions on the statement may seem like obscure circumstances; after all, we usually work with a metric space X directly and not $X \setminus K$. However, these conditions can be met by considering $\overline{X}$ as the ambient space and $\mathcal{K} = \partial_0 X$. Then any f quasiconformal on X can have this lemma applied, provided it gives rise to a homeomorphism on $\overline{X}_{\mathcal{K}}$.

Similar lemmas on the quasiconformal quasi-invariance of transboundary modulus were given by Bonk[1] as well as Hakobyan and Li[37]. Their proofs assumed $\mathcal{K}$ was finite; however, this assumption isn't a significant ingredient in the argument, since the weights on the constructed transboundary mass distributions are unchanged. However, their proofs were for domains in $\hat{\mathbb{C}}$ or $\mathbb{R}^2$, and this is where the generality of Lemma 3.1.3 is significant. A statement for general metric spaces (as argued here) wouldn't be possible without the work of Williams[35] showing general equivalence of the analytic and geometric definitions of quasiconformality.

We will end this section with some example computations of transboundary modulus, and a discussion of its properties. Like modulus, transboundary modulus has nice properties for upper Ahlfors 2-regular spaces.

Proposition 3.1.4. *Let (X, d, μ) be a metric measure space with μ a locally finite Borel regular measure. Fix some $x_0 \in X$. Suppose there exists constants $C_0, R_0 > 0$ such that*

$$\mu(B(x_0, r)) \leq C_0 r^2$$

for all $0 < r < R_0$. Let $\mathcal{K}$ be any countable family of disjoint continua in X, none of which contain x_0, and suppose $X \setminus K$ is homeomorphic to a domain in $\hat{\mathbb{C}}$. Then if Γ is a family of curves satisfying the property that every $\gamma \in \Gamma$ passes through x_0 and is non-constant, then

$$\operatorname{mod}_{\mathcal{K}}(\Gamma) = \operatorname{mod}_{(\mathcal{K} \cup \{\{x_0\}\})}(\Gamma) = 0.$$

Proof. First, notice that there is some $r > 0$ such that $B(x_0, r) \cap K = \emptyset$. Notice that every curve in Γ contains a non-constant subcurve in $B(x_0, r)$ that goes through x_0, call the family of these subcurves Γ'. Then

$$\operatorname{mod}_{\mathcal{K}}(\Gamma) \leq \operatorname{mod}_{\mathcal{K}}(\Gamma') = \operatorname{mod}(\Gamma') = 0$$

by Proposition 2.3.12. Now, let $\mathcal{J} = \mathcal{K} \cup \{\{x_0\}\}$ and let $\Gamma_n = \{\gamma \in \Gamma \mid \gamma \not\subseteq B(x_0, 1/n)\}$ and notice that $\Gamma = \cup_n \Gamma_n$ since no curves are constant. Thus, it suffices to show, for sufficiently large n, that $\operatorname{mod}_{\mathcal{J}}(\Gamma_n) = 0$. Take $N > n > 1/r$. Then every $\gamma \in \Gamma_n$ contains a subcurve in

$\Gamma_{n,N} := \Gamma(B[x_0, 1/N], X \setminus B(x_0, 1/n); X)$ and so

$$\mathrm{mod}_{\mathcal{J}}(\Gamma_n) \leq \mathrm{mod}_{\mathcal{J}}(\Gamma_{n,N}) = \mathrm{mod}(\Gamma_{n,N}) \leq \frac{C}{\log(N/n)}$$

for some constant C by Proposition 2.3.11. Taking $N \to \infty$ shows then that $\mathrm{mod}_{\mathcal{J}}(\Gamma_n) = 0$. □

We remark that for modulus, one stipulates the curves be non-constant because, for $\gamma(t) = x_0$, $\mathrm{mod}(\{\gamma\}) = \infty$. However, for transboundary modulus, if $\mathcal{K}$ contains $\{x_0\}$ we have $\mathrm{mod}_{\mathcal{K}}(\{\gamma\}) = 1$ as the only admissible distributions give weight 1 to $\{x_0\}$. We wrap up our discussion of singletons in the following result.

Corollary 3.1.5. *Let (X, d, μ) be a metric measure space with μ locally finite. Suppose X is upper Ahlfors 2-regular. Let $\mathcal{K}, \mathcal{J}$ be countable collections of disjoint continua in X with $\mathcal{K} \subset \mathcal{J}$. Suppose $X \setminus K$ and $X \setminus J$ are homeomorphic to domains in $\hat{\mathbb{C}}$. Suppose that $\mathcal{J} \setminus \mathcal{K}$ consists only of singletons which are all isolated in J. Then for any family of non-constant curves Γ in X,*

$$\mathrm{mod}_{\mathcal{K}}(\Gamma) = \mathrm{mod}_{\mathcal{J}}(\Gamma).$$

Proof. Let $\mathcal{J} \setminus \mathcal{K} = \{\{s_i\}\}_{i \in I}$. Let Γ_i be the family of non-constant curves in X which go through s_i. Define $\Gamma' = \Gamma \setminus \cup_{i \in I} \Gamma_i$. Since none of the curves in Γ' intersect anything in $\mathcal{J} \setminus \mathcal{K}$, we have

$$\mathrm{mod}_{\mathcal{J}}(\Gamma') = \mathrm{mod}_{\mathcal{K}}(\Gamma').$$

Thus it suffices to show that $\mathrm{mod}_{\mathcal{J}}(\Gamma) = \mathrm{mod}_{\mathcal{J}}(\Gamma')$ and $\mathrm{mod}_{\mathcal{K}}(\Gamma) = \mathrm{mod}_{\mathcal{K}}(\Gamma')$. Since $\Gamma' \subset \Gamma$, monotonicity gives one direction on both equalities. By Proposition 3.1.4, we have

$$\mathrm{mod}_{\mathcal{K}}(\Gamma_i) = 0.$$

Subadditivity gives

$$\mathrm{mod}_{\mathcal{K}}(\cup_{i \in I} \Gamma_i) \leq \sum_{i \in I} \mathrm{mod}_{\mathcal{K}}(\Gamma_i) = 0.$$

Hence

$$\operatorname{mod}_{\mathcal{K}}(\Gamma) \leq \operatorname{mod}_{\mathcal{K}}(\Gamma') + \operatorname{mod}_{\mathcal{K}}(\Gamma \setminus \Gamma') \leq \operatorname{mod}_{\mathcal{K}}(\Gamma') + \operatorname{mod}_{\mathcal{K}}(\cup_{i\in I}\Gamma_i) = \operatorname{mod}_{\mathcal{K}}(\Gamma').$$

For $\mathcal{J}$, notice that because each s_i is isolated, we can say that $X \setminus J_i$ is homeomorphic to a domain where $J_i = J \setminus \{s_i\}$. Hence by Proposition 3.1.4, we have

$$\operatorname{mod}_{\mathcal{J}}(\Gamma_i) = 0.$$

Subadditivity gives

$$\operatorname{mod}_{\mathcal{J}}(\cup_{i\in I}\Gamma_i) \leq \sum_{i\in I} \operatorname{mod}_{\mathcal{J}}(\Gamma_i) = 0.$$

Hence

$$\operatorname{mod}_{\mathcal{J}}(\Gamma) \leq \operatorname{mod}_{\mathcal{J}}(\Gamma') + \operatorname{mod}_{\mathcal{J}}(\Gamma \setminus \Gamma') \leq \operatorname{mod}_{\mathcal{J}}(\Gamma') + \operatorname{mod}_{\mathcal{J}}(\cup_{i\in I}\Gamma_i) = \operatorname{mod}_{\mathcal{J}}(\Gamma').$$

□

Now we will focus our attention to explicit examples in the planar case.

Example 3.1.6. *Let $a, b > 0$ and $R = (0, a) \times (0, b) \subset \mathbb{R}^2$ with the standard metric and area measure. Let $\mathcal{K}$ be a countable collection of points and pairwise disjoint, closed squares in R with sides parallel to the coordinate axes, and suppose $R \setminus K$ is a domain. Let $E = \{0\} \times [0, b]$ and $F = \{a\} \times [0, b]$. Then*

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E, F; \overline{R})) = \frac{b}{a}.$$

Proof. Pick any $P = (\rho; \{\rho_i\}) \wedge_{\mathcal{K}} \Gamma(E, F; \overline{R})$. For each $K_i \in \mathcal{K}$ call its side length s_i ($s_i = 0$ if K_i is a point). By considering horizontal paths, $\gamma(t) = (t, y_0)$, we can say for each $y_0 \in [0, b]$,

$$\int_0^a \rho(x, y_0)\mathbb{I}_{R\setminus K}(x, y_0)\ dx + \sum_i \rho_i \mathbb{I}_{\pi_2(K_i)}(y_0) = \int_{\gamma\setminus K} \rho\ ds + \sum_{K_i\cap\gamma\neq\emptyset} \rho_i \geq 1.$$

Now integrate both sides with respect to y to say

$$b \leq \int_0^b \int_0^a \rho(x,y)\mathbb{I}_{R\setminus K}(x,y)\ dx\ dy + \sum_i \rho_i \int_0^b \mathbb{I}_{\pi_2(K_i)}(y)\ dy = \int_{R\setminus K} \rho\ dA + \sum_i \rho_i s_i.$$

Let $P' = (\rho; \{\rho_i s_i\})$, and notice that the above quantity is $A_{\mathcal{K}}(\sqrt{P'})$. Recall that Cauchy-Schwarz gives us

$$\Big(\int_{R_{\mathcal{K}}} \rho_{P'}\ d\mu_{\mathcal{K}}\Big)^2 \leq \Big(\int_{R_{\mathcal{K}}} \rho_P^2\ d\mu_{\mathcal{K}}\Big)\Big(\int_{R_{\mathcal{K}}} \rho_{(1;\{s_i\})}\ d\mu_{\mathcal{K}}\Big) = \Big(\int_{R_{\mathcal{K}}} \rho_P^2\ d\mu_{\mathcal{K}}\Big)\Big(A(R\setminus K)+\sum_i s_i^2\Big),$$

but since each K_i is a square (or point), we can say

$$A(R \setminus K) + \sum_i s_i^2 = ab.$$

So we conclude that

$$\int_{R_{\mathcal{K}}} \rho_P^2\ d\mu_{\mathcal{K}} \geq \frac{1}{ab}\Big(\int_{R_{\mathcal{K}}} \rho_{P'}\ d\mu_{\mathcal{K}}\Big)^2 \geq \frac{b}{a}.$$

Infimize over all P to get the lower bound.

To see the upper bound, let $\rho = a^{-1}$ and let $\rho_i = a^{-1}s_i$. Then for all $\gamma \in \Gamma(E, F; \overline{R})$, we have

$$\int_{\gamma\setminus K} \rho\ ds + \sum_{\gamma\cap K_i\neq\emptyset} \rho_i = a^{-1}(\ell(\gamma \setminus K) + \sum_{\gamma\cap K_i\neq\emptyset} s_i) \geq 1.$$

So we can say

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E, F; \overline{R})) \leq \int_{R\setminus K} \rho^2\ d\mu + \sum_i \rho_i^2 = \frac{1}{a^2}(A(R \setminus K) + \sum_i s_i^2) = \frac{b}{a}.$$

$\square$

From here, it's not hard to see that the transboundary modulus of curves connecting the horizontal sides with respect to a collection of squares is a/b. Thus, in this example, one can see that transboundary modulus also has a duality property. Just like with modulus, we can establish general duality of transboundary modulus using this special case; however, we will need a result stronger than the Riemann Mapping Theorem.

Theorem 3.1.7 (Schramm[4] (Theorem 7.1))**.** *Let $Q \subset \mathbb{R}^2$ be homeomorphic to $\overline{\mathbb{D}}$. Let $C \subset Q$ correspond to $\partial\mathbb{D}$, and let $Q_0, Q_1 \subset C$ be disjoint continua. Let $\mathcal{K}$ be a countable collection of pairwise disjoint continua in the interior of Q such that $Q \setminus (K \cup C)$ is a domain. Then there is a conformal map $f : Q \setminus (K \cup C) \to \Omega$ where $\Omega \subset \mathbb{R}^2$ is a domain satisfying*

$$\Omega = R \setminus \cup_{i \in I} J_i$$

where $R = (0, a) \times (0, b)$, for some $a, b > 0$, and J_i are all points or pairwise disjoint, closed squares with sides parallel to the coordinate axes. Moreover, f extends as a homeomorphism $f : Q \setminus K \to \Omega \cup \partial R$ with $f(Q_0)$ and $f(Q_1)$ corresponding to the left and right sides of ∂R. We also have that f extends as a homeomorphism from $Q_{\partial_0(Q\setminus(K\cup C))}$ to $\overline{\Omega}_{\partial_0\Omega}$.

Proposition 3.1.8 (Duality of Transboundary Modulus)**.** *Let $Q \subset \mathbb{R}^2$ be homeomorphic to $[0, 1]^2$. Let C correspond to $\partial[0, 1]^2$. Let $A_L, A_R, A_B, A_T \subset C$ correspond to $\{0\} \times [0, 1], \{1\} \times [0, 1], [0, 1] \times \{0\}, [0, 1] \times \{1\}$ respectively. Let $\mathcal{K}$ be any countable collection of pairwise disjoint continua in the interior of Q, and suppose $Q \setminus (K \cup C)$ is a domain. Then*

$$\mathrm{mod}_{\mathcal{K}}(A_L, A_R; Q)\ \mathrm{mod}\ _{\mathcal{K}}(A_B, A_T; Q) = 1.$$

Proof. Use Theorem 3.1.7 to conformally map the interior of $Q \setminus K$ onto a subdomain of a rectangle, $R = (0, a) \times (0, b)$, whose relative complement consists of closed squares and points, which also sends A_L and A_R to the left and right sides of R respectively. Apply Lemma 3.1.3 and Example 3.1.6 to say that

$$\mathrm{mod}_{\mathcal{K}}(\Gamma(A_L, A_R; Q)) = \frac{b}{a}.$$

Similarly, $\Gamma(A_B, A_T; Q)$ gets mapped to the family of curves connecting the horizontal sides of the rectangle and $\mathrm{mod}_{\mathcal{K}}(\Gamma(A_B, A_T; Q)) = a/b$. □

One may be interested in analyzing what happens to the transboundary modulus when one varies $\mathcal{K}$. One can say rather trivial things on this topic; for example, if $\mathcal{J} \subset \mathcal{K}$ are all the continua which intersect any curve in Γ, then $\operatorname{mod}_{\mathcal{J}}(\Gamma) = \operatorname{mod}_{\mathcal{K}}(\Gamma)$. One also can refer to our discussion on adding or removing singletons. However, more general statements are inaccessible, as the following example illustrates.

Example 3.1.9. *Let $R = (0,a)\times(0,b) \subset \mathbb{R}^2$ with the standard metric and area measure. Let $\mathcal{K} = \{K_i\}$ be a countable collection of disjoint continua in the interior of R, and suppose $R\backslash K$ is a domain. For each i, let $w_i = \operatorname{diam}(\pi_1(K_i))$ and $h_i = \operatorname{diam}(\pi_2(K_i))$. Let $E = \{0\} \times [0,b]$ and $F = \{a\} \times [0,b]$ We have*

$$b^2(A(R \setminus K) + \sum_i h_i^2)^{-1} \leq \operatorname{mod}_{\mathcal{K}}(\Gamma(E,F;\overline{R})) \leq a^{-2}(A(R \setminus K) + \sum_i w_i^2).$$

Proof. Let $\rho = 1/a$. Let $\rho_i = w_i/a$. We claim that $P = (\rho; \{\rho_i\}) \wedge_{\mathcal{K}} \Gamma(E,F;\overline{R})$. This is true because π_1 is 1-Lipschitz, and so for any $\gamma \in \Gamma(E,F;\overline{R})$,

$$\ell(\gamma \setminus K) \geq \ell(\pi_1(\gamma \setminus K)) \geq a - \sum_{K_i \cap \gamma \neq \emptyset} w_i.$$

Thus we can say

$$\int_{\gamma\setminus K} \rho \, ds + \sum_{\gamma \cap K_i \neq \emptyset} \rho_i = a^{-1}(\ell(\gamma \setminus K) + \sum_{\gamma \cap K_i \neq \emptyset} w_i) \geq 1.$$

So, we obtain the upper bound by applying this P,

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E,F;\overline{R})) \leq \int_{R\setminus K} \rho^2 \, dA + \sum_i \rho_i^2 = a^{-2}(A(R \setminus K) + \sum_i w_i^2).$$

To obtain the lower bound, similarly use $(1/b; h_i/b)$ on the curves connecting the horizontal sides and use duality. $\square$

Notice that if each K_i is a square with sides parallel to the coordinate axes, then $A(R \setminus K) + \sum w_i^2 = A(R\setminus K) + \sum h_i^2 = ab$, and the estimates compute the transboundary modulus:

b/a. Also, if one widens one of the squares to a horizontal rectangle, then $A(R\setminus K)+\sum h_i^2 < ab$; so

$$\operatorname{mod}_{\mathcal{K}}(f(\Gamma(E,F;R))) > \frac{b}{a}.$$

Similarly if one lengthens the heights of one of the squares to a vertical rectangle then $A(R \setminus K) + \sum w_i^2 < ab$ and

$$\operatorname{mod}_{\mathcal{K}}(f(\Gamma(E,F;R))) < \frac{b}{a}.$$

Thus we can see that making each K_i "bigger" doesn't necessarily increase nor decrease the transboundary modulus. Also, adding more continua to $\mathcal{K}$ or taking some away may increase or decrease the transboundary modulus (or leave it the same in the case of squares).

3.2 Transboundary Loewner Property

As we've seen, transboundary modulus has many similarities to modulus, and its properties shown here have largely paralleled those of modulus. Therefore, it seems natural to direct our attention to the transboundary analog of the Loewner property discussed in Chapter 2. While this has appeared in the literature implicitly ([1],[38]), the first to name it and explicitly define it was Hakobyan and Li[37]. The definition given below differs slightly from that appearance.

Definition 3.2.1. *Let (X, d, μ) be a metric measure space with μ locally finite. Suppose X is homeomorphic to a domain in $\hat{\mathbb{C}}$, ∂X is compact, and that $\partial_0 X$ is countable. X is* ***transboundary Loewner*** *in case there is a decreasing function $\Psi : (0,\infty) \to (0,\infty)$ satisfying the following property. For all disjoint, non-degenerate continua, $E, F \subset X$,*

$$\operatorname{mod}_{\partial_0 X}(\Gamma(E,F;\overline{X})) \geq \Psi(\Delta(E,F)).$$

If X is Loewner, then it will be transboundary Loewner, as

$$\operatorname{mod}_{\partial_0 X}(\Gamma(E,F;\overline{X})) \geq \operatorname{mod}_{\partial_0 X}(\Gamma(E,F;X)) = \operatorname{mod}(\Gamma(E,F;X)).$$

Proposition 3.2.2. *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be metric measure spaces with μ_X and μ_Y locally finite. Suppose X and Y are homeomorphic to domains in $\hat{\mathbb{C}}$, $\partial X, \partial Y$ are compact, and $\partial_0 X, \partial_0 Y$ are countable. Suppose $f : X \to Y$ is quasisymmetric and geometrically quasiconformal. If X is transboundary Loewner, then Y is transboundary Loewner.*

Proof. Pick any non-degenerate, disjoint continua $A, B \subset Y$. Let $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$, noticing that they are disjoint, non-degenerate continua in X. Since X is transboundary Loewner,

$$\operatorname{mod}_{\partial_0 X}(\Gamma(A', B'; \overline{X})) \geq \Psi'(\Delta(A', B'))$$

for some decreasing function Ψ'. Since f is quasisymmetric, f extends to a homeomorphism of the completions (see Remark 2.1.6). This means that f gives rise to a homeomorphism $f : (\overline{X})_{\partial_0 X} \to (\overline{Y})_{\partial_0 Y}$, and thus we can apply Lemma 3.1.3 on $\overline{X}$ and $\overline{Y}$. Letting f^{-1} be ν-quasisymmetric and f be H-quasiconformal, and recalling Proposition 2.1.9, we conclude

$$\operatorname{mod}_{\partial_0 Y}(\Gamma(A, B; \overline{Y})) \geq H^{-1} \operatorname{mod}_{\partial_0 X}(\Gamma(A', B'; \overline{X})) \geq H^{-1}\Psi'(\Delta(A', B')) \geq H^{-1}\Psi'(\nu(2\Delta(A, B))).$$

$\Psi(t) := H^{-1}\ \Psi'(\nu(2t))$ is decreasing since Ψ' is decreasing and ν is increasing, so Y is transboundary Loewner. □

In Section 3.3, we will show that circle domains are transboundary Loewner. This fact, combined with the quasisymmetric invariance of the transboundary Loewner property, means that the property is necessary for quasisymmetric equivalence to a circle domain. Before we establish examples of transboundary Loewner spaces, however, we wish to establish some non-examples of transboundary Loewner spaces. In particular, Merenkov and Wildrick[2] gave an insightful example of a "nice" metric space which fails to be quasisymmetrically equivalent to a circle domain. We will show that this space fails to be transboundary Loewner.

Bonk[1] showed that, for domains in the sphere, the boundary consisting of uniformly relatively separated uniform quasicircles was sufficient to conclude quasisymmetric equivalence to a circle domain. Merenkov and Wildrick[2] showed that this condition was not sufficient for general metric spaces. They did so by constructing a counterexample. Here we will define a class of examples which contain the one given by Merenkov and Wildrick. It will show that the boundary consisting of uniformly relatively separated uniform quasicircles is insufficient to conclude transboundary Loewner.

For each $n \in \mathbb{N}$, let

$$D_n = \{[i2^{-n}, (i+1)2^{-n}] \times [j2^{-n}, (j+1)2^{-n}] : i, j \in \mathbb{N}, 0 \leq i, j \leq 2^n - 1\};$$

that is, the collection of all dyadic squares of generation n. Let $D = \cup_{n\in\mathbb{N}} D_n$. For $Q \in D$, let $c(Q)$ denote the center of Q. Let $L = (\ell_n)_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying $0 \leq \ell_n < 2^{-n}$. For each $n \in \mathbb{N}$ and $Q \in D_n$, let

$$s_Q(L) = \left(\{0\} \times \left[-\frac{\ell_n}{2}, \frac{\ell_n}{2}\right]\right) + c(Q),$$

and

$$s_n(L) = \{s_Q(L) \mid Q \in D_n\}.$$

Notice that $s_n(L)$ is a collection of 4^n disjoint vertical slits, each with length ℓ_n and centered at a dyadic square of generation n. Notice also, that $\Delta(s_Q(L), s_{Q'}(L)) \geq 1$ for all $Q, Q' \in D$. Let

$$S_k(L) := (0,1)^2 \setminus (\bigcup_{n=0}^{k} \bigcup_{Q\in D_n} s_Q(L)).$$

Notice that $S_k(L)$ is a slit domain (with the exception of the bounding square). Equip $S_k(L)$ with its internal path metric. Notice then that each component of $\partial S_k(L)$ is a 1-quasicircle.

Let $E = \{0\} \times [0,1]$ and $F = \{1\} \times [0,1]$. Let $\mathcal{S}_k(L) = \cup_{n=0}^{k} s_n(L)$. We would like a bound on $\mathrm{mod}_{\mathcal{S}_k(L)}(\Gamma(E, F; \overline{S_k(L)}))$. By Corollary 3.1.5, we can ignore slits of length zero. Hakobyan and Li[37] showed that

$$\lim_{k\to\infty} \mathrm{mod}_{\mathcal{S}_k(L)}(\Gamma(E, F; \overline{S_k(L)})) = 0$$

if $\sum_{n=0}^{\infty} (2^n \ell_n)^2 = \infty$. Let Γ_k be the collection of curves γ in $S_k(L)$ with non-zero horizontal

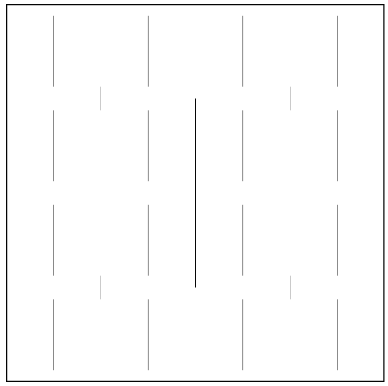

Figure 3.1: $S_3\left(\frac{1}{2}, \frac{1}{16}, \frac{3}{16}\right)$

movement: $\operatorname{diam}(\pi_1(\gamma)) > 0$. It follows from the estimates of Hakobyan and Li[37] that

$$\lim_{k\to\infty} \operatorname{mod}_{\mathcal{S}_k(L)}(\Gamma_k) = 0$$

if $\sum_{n=0}^{\infty}(2^n\ell_n)^2 = \infty$. We will now use this to construct a slit domain which is not transboundary Loewner. Define the set

$$S(L) = \bigcap_{k=0}^{\infty}\left((2^{-k}S_k(L)) \cup (0, 2^{-k-1})^2 \cup ((0,1)^2 \setminus (0, 2^{-k})^2)\right).$$

In other words, for each k, scale $S_k(L)$ down to a square of side length 2^{-k}, and place it in the lowest, leftmost dyadic square. Then fill in the open bottom left quarter of that square, and repeat for $k+1$. Equip $S(L)$ with its inner metric, and notice that it is homeomorphic to a domain in the plane.

Example 3.2.3. *$S(L)$ is not transboundary Loewner for $\sum_{n=0}^{\infty}(2^n\ell_n)^2 = \infty$.*

Proof. Let E_k and F_k be the left and right sides of the dyadic square $Q_k := (2^{-k}, 2^{-k+1})^2$ respectively. Let $f : Q_k \cap S(L) \to S_k(L)$ be the conformal map (onto its image) obtained through translation and dilation. Let $E_k' = E_k + (3^{-100k}, 0)$ and $F_k' = F_k - (3^{-100k}, 0)$. The transboundary modulus of curves going through the endpoints of E_k' and F_k' is 0 (Proposition 3.1.4). Every other transboundary curve, γ, connecting E_k' and F_k' satisfies $f(\gamma)$ contains a

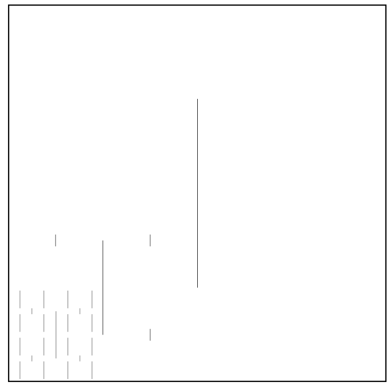

Figure 3.2: $S\left(\frac{1}{2}, \frac{1}{16}, \frac{3}{16}, \dots\right)$

subcurve in Γ_k. Hence

$$\operatorname{mod}_{\partial_0 S(L)}(\Gamma(E'_k, F'_k; S(L))) \leq \operatorname{mod}_{\mathcal{S}_k(L)}(\Gamma_k).$$

Notice that $\Delta(E'_k, F'_k) \leq 1$. Thus for all decreasing functions Ψ, we can find some $k \in \mathbb{N}$ such that $\operatorname{mod}_{\mathcal{S}_k(L)}(\Gamma_k) < \Psi(1)$. It then follows that

$$\operatorname{mod}_{\partial_0 S(L)}(\Gamma(E'_k, F'_k; S(L))) < \Psi(1) \leq \Psi(\Delta(E'_k, F'_k)).$$

Thus $S(L)$ is not transboundary Loewner.

□

Merenkov and Wildrick[2] took $\ell_n = 1/2^{n+1}$ for all n. It turns out that if $\sum_{n=0}^{\infty}(2^n \ell_n)^2 < \infty$, then one can apply the techniques of Merenkov-Wildrick[2] to say $S(L)$ will be quasisymmetric to a circle domain. See Hakobyan-Li[37] for more details. Example 3.3.7 and Proposition 3.2.2 imply that it will then be transboundary Loewner.

3.3 Transboundary Loewner Property in $\mathbb{R}^2$ and $\hat{\mathbb{C}}$

Because the transboundary Loewner property is a quasisymmetric invariant, it can be used to classify when spaces are quasisymmetrically equivalent. It is, therefore, of interest to generate some examples of model spaces which are transboundary Loewner. Our goal will be to derive conditions on $\mathcal{K}$, a collection of pairwise disjoint continua in some Loewner Ω, which are sufficient to conclude $\Omega \backslash K$ is transboundary Loewner. Intuitively, these conditions will prohibit the continua in $\mathcal{K}$ from being too "thin". In particular, we will show that circle domains are transboundary Loewner, which was first shown by Merenkov[38] (it follows from Proposition 5.3).

Definition 3.3.1 (Schramm[4]). *Let $A \subset \mathbb{R}^2$ be Borel and $1 \geq \tau > 0$. We say A is* ***τ-fat*** *if, for all $x \in A$ and $r > 0$ with $A \not\subseteq B(x,r)$, we have*

$$\mathcal{H}^2(A \cap B(x,r)) \geq \tau \mathcal{H}^2(B(x,r)).$$

It's not hard to see that disks are 1/4-fat. Rectangles are $(2\pi c)^{-1}$-fat where c is the ratio of the larger side length to the shorter one. Similarly, ellipses are fat with τ depending on the eccentricity.

For a general estimate on transboundary modulus, we will need the continua through which the curves go to be fat. This, however, won't be enough. We will also need the following property.

Definition 3.3.2. *Let $A \subset \mathbb{R}^2$ and $\lambda \geq 1$. We say A is* ***λ-quasiround*** *if, there is some $x \in A$ and $r > 0$ with*

$$B(x,r) \subset A \subset B(x, \lambda r).$$

A set is λ-quasiround for some λ if and only if it is bounded and has non-empty interior. What's important is that we keep track of how thick an annulus containing the boundary must be. Disks are λ-quasiround for any $\lambda > 1$. Like with fatness, rectangles and ellipses are quasiround with constants depending on their eccentricity. However, in general quasiround sets are not fat: take a Jordan domain with an outward pointing cusp; moreover, not all bounded fat sets are uniformly quasiround. That is, for sufficiently small τ and for all $\lambda \geq 1$, there is a bounded τ-fat set which is not λ-quasiround. Take, for instance, $\mathbb{D} \setminus \alpha\mathbb{Z} \times \mathbb{R}$ for small α. With more effort, one can come up with a family of Jordan domains which are τ-fat but not uniformly quasiround.

Proposition 3.3.3 ([1],[38],[37])**.** *Let $E \subset \mathbb{R}^2$ be a continuum, $\lambda \geq 1$, and $\tau > 0$. Let $\mathcal{K} = \{K_i\}_{i\in I}$ be a collection of pairwise disjoint, τ-fat subsets of the plane satisfying $K_i \cap E \neq \emptyset$ and*

$$\lambda \mathrm{diam}(K_i) \geq \mathrm{diam}(E)$$

for all $i \in I$. Then $\#(\mathcal{K}) \leq (\lambda^2 + 6\lambda + 1)/\tau$.

Proof. Fix any point $e \in E$. Since E is a compact, $\mathrm{diam}(E) < \infty$. Notice $E \subset B[e, \mathrm{diam}(E)]$, and thus $K_i \cap B[e, \mathrm{diam}(E)] \neq \emptyset$ for all $i \in I$. Define

$$I_1 = \{i \in I \mid K_i \not\subseteq B(e, (1 + \lambda^{-1})\mathrm{diam}(E))\}.$$

Define $I_2 = I \setminus I_1$. For $i \in I_1$, we can say there is some $x_i \in \partial B(e, \mathrm{diam}(E)) \cap K_i$. Notice that

$$B(x_i, \lambda^{-1}\mathrm{diam}(E)) \subset B(e, (1 + \lambda^{-1})\mathrm{diam}(E));$$

indeed, for $y \in B(x_i, \lambda^{-1}\mathrm{diam}(E))$, we can say

$$|y - e| \leq |y - x_i| + |x_i - e| \leq \lambda^{-1}\mathrm{diam}(E) + \mathrm{diam}(E).$$

Therefore, for $i \in I_1$, we can say $B(x_i, \lambda^{-1}\mathrm{diam}(E))$ does not contain K_i. We can use fatness to say

$$\mathcal{H}^2(K_i \cap B(x_i, \lambda^{-1}\mathrm{diam}(E))) \geq \tau\mathcal{H}^2(B(x_i, \lambda^{-1}\mathrm{diam}(E))).$$

Now, for each $i \in I_1$, we can say $B(x_i, \lambda^{-1}\mathrm{diam}(E)) \subset A[e, (1-\lambda^{-1})\mathrm{diam}(E), (1+\lambda^{-1})\mathrm{diam}(E)]$. Since each K_i is disjoint, we can say

$$\begin{aligned}\pi(((1 + \lambda^{-1})\mathrm{diam}(E))^2 - ((1 - \lambda^{-1})\mathrm{diam}(E))^2) &= \mathcal{H}^2(A[e, (1 - \lambda^{-1})\mathrm{diam}(E), (1 + \lambda^{-1})\mathrm{diam}(E)]) \\ &\geq \mathcal{H}^2(\cup_{i\in I_1} B(x_i, \lambda^{-1}\mathrm{diam}(E))) \\ &\geq \mathcal{H}^2(\cup_{i\in I_1} K_i \cap B(x_i, \lambda^{-1}\mathrm{diam}(E)))\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in I_1} \mathcal{H}^2(K_i \cap B(x_i, \lambda^{-1}\mathrm{diam}(E))) \\
&\geq \tau \sum_{i\in I_1} \mathcal{H}^2(B(x_i, \lambda^{-1}\mathrm{diam}(E))) \\
&= \tau \sum_{i\in I_1} \pi(\lambda^{-1}\mathrm{diam}(E))^2 \\
&= \pi\tau\lambda^{-2}\mathrm{diam}(E)^2\#(I_1).
\end{aligned}
$$

Thus

$$
\#(I_1) \leq \tau^{-1}\lambda^2((1+\lambda^{-1})^2 - (1-\lambda^{-1})^2) = \tau^{-1}\lambda^2(4\lambda^{-1}) = \frac{4\lambda}{\tau}.
$$

To estimate $\#(I_2)$, notice that, by compactness of K_i there exists $x_i, y_i \in K_i$ with

$$
d(x_i, y_i) = \mathrm{diam}(K_i) \geq \lambda^{-1}\mathrm{diam}(E).
$$

Thus, $B(x_i, \lambda^{-1}\mathrm{diam}(E))$ does not contain K_i. Use fatness to say

$$
\mathcal{H}^2(B(x_i, \lambda^{-1}\mathrm{diam}(E)) \cap K_i) \geq \tau\mathcal{H}^2(B(x_i, \lambda^{-1}\mathrm{diam}(E))).
$$

For $i \in I_2$, we have $K_i \subset B(e, (1+\lambda^{-1})\mathrm{diam}(E))$. Therefore, using disjointness of K_i,

$$
\begin{aligned}
\pi((1+\lambda^{-1})\mathrm{diam}(E))^2 &= \mathcal{H}^2(B(e, (1+\lambda^{-1})\mathrm{diam}(E))) \\
&\geq \mathcal{H}^2(\cup_{i\in I_2} K_i) \\
&= \sum_{i\in I_2} \mathcal{H}^2(K_i) \\
&\geq \sum_{i\in I_2} \mathcal{H}^2(K_i \cap B(x_i, \lambda^{-1}\mathrm{diam}(E))) \\
&\geq \tau \sum_{i\in I_2} \mathcal{H}^2(B(x_i, \lambda^{-1}\mathrm{diam}(E))) \\
&= \tau \sum_{i\in I_2} \pi(\lambda^{-1}\mathrm{diam}(E))^2 \\
&= \pi\tau\lambda^{-2}\mathrm{diam}(E)^2\#(I_2).
\end{aligned}
$$

Hence,

$$\#(I_2) \leq \tau^{-1}\lambda^2(1+\lambda^{-1})^2 = \tau^{-1}(\lambda+1)^2.$$

Thus we have

$$\#(\mathcal{K}) = \#(I) = \#(I_1) + \#(I_2) \leq \tau^{-1}(4\lambda + (\lambda+1)^2) = \tau^{-1}(\lambda^2 + 6\lambda + 1).$$

$\square$

The following result, first shown by Bojarski[40], is well known. However, it is usually formulated for finite sums, and we'll need it for infinite sums; we give an argument here which mimics a proof given by Merenkov[38].

Lemma 3.3.4 (Bojarski's Lemma)**.** *Let $\{B(x_i, r_i)\}_{i\in I}$ be a countable collection of pairwise disjoint open balls in $\mathbb{R}^2$ and $\{a_i\}_{i\in I}$ a countable collection of non-negative real numbers. Let $\lambda \geq 1$ be given. There exists a constant, c_λ, depending only on λ, such that*

$$\int_{\mathbb{R}^2} \left(\sum_{i\in I} a_i \mathbb{I}_{B(x_i,\lambda r_i)} \right)^2 dA \leq c_\lambda \int_{\mathbb{R}^2} \left(\sum_{i\in I} a_i \mathbb{I}_{B(x_i,r_i)} \right)^2 dA = c_\lambda \pi \sum_{i\in I} a_i^2 r_i^2.$$

Proof. Let $\phi \in L^2(\mathbb{R}^2)$. Define the uncentered maximal operator of ϕ by

$$M(\phi)(x,y) = \sup_{(x,y)\in B(z,r)} \frac{1}{\pi r^2} \int_{B(z,r)} |\phi| \; dA.$$

Notice that, in particular, for $(x,y) \in B(x_i, r_i)$, we have

$$M(\phi)(x,y) \geq \frac{1}{\pi \lambda^2 r_i^2} \int_{B(x_i,\lambda r_i)} |\phi| \; dA.$$

Hence,

$$\int_{B(x_i,r_i)} M(\phi) \; dA \geq \frac{1}{\lambda^2} \int_{B(x_i,\lambda r_i)} |\phi| \; dA.$$

Recall that M is a bounded operator (see Duoandikoetxea[41] Theorem 2.5 and following remarks): there exists a constant, C, such that

$$||M(\phi)||_2 \leq C||\phi||_2$$

for all $\phi \in L^2(\mathbb{R}^2)$.

$$\begin{aligned}
\int_{\mathbb{R}^2} \sum_{i\in I} a_i \mathbb{I}_{B(x_i,\lambda r_i)} |\phi| \; dA &= \sum_{i\in I} a_i \int_{\mathbb{R}^2} \mathbb{I}_{B(x_i,\lambda r_i)} |\phi| \; dA && \text{(Monotone Convergence Theorem)} \\
&= \sum_{i\in I} a_i \int_{B(x_i,\lambda r_i)} |\phi| \; dA \\
&\leq \sum_{i\in I} a_i \lambda^2 \int_{B(x_i,r_i)} M(\phi) \; dA \\
&\leq \lambda^2 \int_{\mathbb{R}^2} \sum_{i\in I} a_i \mathbb{I}_{B(x_i,r_i)} M(\phi) \; dA && \text{(Disjointness)} \\
&\leq \lambda^2 ||M(\phi)||_2 \left|\left| \sum_{i\in I} a_i \mathbb{I}_{B(x_i,r_i)} \right|\right|_2 && \text{(Cauchy-Schwarz)} \\
&\leq C\lambda^2 ||\phi||_2 \left|\left| \sum_{i\in I} a_i \mathbb{I}_{B(x_i,r_i)} \right|\right|_2
\end{aligned}$$

for all $\phi \in L^2(\mathbb{R}^2)$. With no loss of generality, suppose $I \subset \mathbb{N}$ and let $I_n = \{1, ..., n\} \cap I$. Let $\phi_n = \sum_{i\in I_n} a_i \mathbb{I}_{B(x_i,\lambda r_i)}$. Notice $\phi_n \in L^2(\mathbb{R}^2)$ for all n and $\phi_n \leq \phi_m$ for $n \leq m$. Suppose $||\phi_n||_2 > 0$ for sufficiently large n (otherwise, the result is trivial). We use the above inequality to say

$$\begin{aligned}
||\phi_n||_2^2 &\leq \int_{\mathbb{R}^2} \sum_{i\in I} a_i \mathbb{I}_{B(x_i,\lambda r_i)} |\phi_n| \; dA \leq C\lambda^2 ||\phi_n||_2 \left|\left| \sum_{i\in I} a_i \mathbb{I}_{B(x_i,r_i)} \right|\right|_2 \\
||\phi_n||_2 &\leq C\lambda^2 \left|\left| \sum_{i\in I} a_i \mathbb{I}_{B(x_i,r_i)} \right|\right|_2 \\
\lim_{n\to\infty} ||\phi_n||_2 &\leq C\lambda^2 \left|\left| \sum_{i\in I} a_i \mathbb{I}_{B(x_i,r_i)} \right|\right|_2 .
\end{aligned}$$

Once again, we use the monotone convergence theorem to conclude

$$\Bigg|\Bigg|\sum_{i\in I} a_i \mathbb{I}_{B(x_i,\lambda r_i)}\Bigg|\Bigg|_2 = \lim_{n\to\infty} ||\phi_n||_2 \le C\lambda^2 \Bigg|\Bigg|\sum_{i\in I} a_i \mathbb{I}_{B(x_i,r_i)}\Bigg|\Bigg|_2.$$

Squaring both sides gives the desired conclusion. □

The following lemma is similar to a lemma of Bonk[1], although he assumed uniform relative separation, and there is no such assumption here. It is also similar to a lemma of Merenkov[38], though he showed it only for circle domains.

Lemma 3.3.5 ([37]). *Let $\Omega \subset \mathbb{R}^2$ be Borel and Γ a collection of curves in Ω. Fix $\tau > 0$ and $\lambda \ge 1$. Let $\mathcal{K} = \{K_i\}_{i\in I}$ be a countable collection of pairwise disjoint, τ-fat, λ-quasiround continua in Ω. Suppose $\Omega \setminus K$ is a domain. Then there are constants c_1 and c_2 depending only on λ and τ such that*

$$\mathrm{mod}_{\mathcal{K}}(\Gamma) \ge \min(c_1, c_2 \,\mathrm{mod}\,(\Gamma)).$$

Proof. Let $c = (1 + 12\lambda + 4\lambda^2)/\tau$. Let $c_1 := 1/(8c^2)$. Suppose $\mathrm{mod}_{\mathcal{K}}(\Gamma) \le 1/(8c^2)$; as otherwise, we have nothing to show. For $\epsilon < 1/(8c^2)$, let $P = (\rho; \{\rho_i\}_{i\in I}) \wedge_{\mathcal{K}} \Gamma$ be such that

$$A_{\mathcal{K}}(P) \le \mathrm{mod}_{\mathcal{K}}(\Gamma) + \epsilon \le \frac{1}{4c^2}.$$

Since each K_i is λ-quasiround, we can find $x_i \in K_i$ and $r_i > 0$ such that

$$B(x_i, r_i) \subset K_i \subset B(x_i, \lambda r_i).$$

For brevity, let $B_i := B(x, 2\lambda r_i)$. Define $g : \Omega \to [0, \infty)$ as

$$g = 2\Bigg(\rho\, \mathbb{I}_{\Omega\setminus K} + \sum_{i\in I} \frac{\rho_i}{\lambda r_i} \mathbb{I}_{B_i\cap\Omega}\Bigg).$$

We claim $g \wedge \Gamma$. To see this, pick any $\gamma \in \Gamma$ and define

$$I_\gamma := \{i \in I \mid \gamma \cap K_i \neq \emptyset,\ 2\lambda \mathrm{diam}(K_i) \geq \mathrm{diam}(\gamma)\}.$$

Now if $i \in I \setminus I_\gamma$, then we either have that $\gamma \cap K_i = \emptyset$ or

$$\mathrm{diam}(\gamma) > 2\lambda \mathrm{diam}(K_i) \geq 4\lambda r_i = \mathrm{diam}(B_i);$$

the latter implies that γ is not contained in B_i. Therefore,

$$\begin{aligned}
\ell_g(\gamma) &= 2\left(\int_{\gamma\setminus K} \rho\, ds + \sum_{i\in I} \frac{\rho_i}{\lambda r_i}\ell(\gamma \cap B_i)\right) \\
&\geq 2\left(\int_{\gamma\setminus K} \rho\, ds + \sum_{\gamma\cap K_i\neq\emptyset, i\in I\setminus I_\gamma} \frac{\rho_i}{\lambda r_i}\ell(\gamma \cap B_i)\right) \\
&\geq 2\left(\int_{\gamma\setminus K} \rho\, ds + \sum_{\gamma\cap K_i\neq\emptyset, i\in I\setminus I_\gamma} \rho_i\right) \\
&= 2\left(\ell_P^{\mathcal{K}}(\gamma) - \sum_{i\in I_\gamma} \rho_i\right)
\end{aligned}$$

Observe that $\rho_i \leq \sqrt{A_{\mathcal{K}}(P)} \leq 1/(2c)$; also, we can use Proposition 3.3.3 to say $\#(I_\gamma) \leq c$. Hence

$$\sum_{i\in I_\gamma} \rho_i \leq \frac{\#(I_\gamma)}{2c} \leq \frac{1}{2}.$$

This gives us admissibility:

$$\ell_g(\gamma) \geq 2\left(\ell_P^{\mathcal{K}}(\gamma) - \sum_{i\in I_\gamma} \rho_i\right) \geq 2\left(\ell_P^{\mathcal{K}}(\gamma) - \frac{1}{2}\right) \geq 1.$$

We will now use g to estimate the modulus. Since $B(x_i, r_i)$ are pairwise disjoint, we can use Bojarski's Lemma (Lemma 3.3.4) to say

$$\begin{aligned}
\operatorname{mod}(\Gamma) &\leq \int_\Omega g^2 \, dA = 4 \int_\Omega \Big(\rho \, \mathbb{I}_{\Omega\setminus K} + \sum_{i\in I} \frac{\rho_i}{\lambda r_i} \mathbb{I}_{B_i\cap\Omega}\Big)^2 dA \\
&\leq 8 \int_\Omega \Big(\rho \, \mathbb{I}_{\Omega\setminus K}\Big)^2 + \Big(\sum_{i\in I} \frac{\rho_i}{\lambda r_i} \mathbb{I}_{B_i\cap\Omega}\Big)^2 dA \\
&= 8\Big(\int_{\Omega\setminus K} \rho^2 \, dA + \int_\Omega \Big(\sum_{i\in I} \frac{\rho_i}{\lambda r_i} \mathbb{I}_{B_i\cap\Omega}\Big)^2 \, dA\Big) \\
&\leq 8\Big(\int_{\Omega\setminus K} \rho^2 \, dA + \int_{\mathbb{R}^2} \Big(\sum_{i\in I} \frac{\rho_i}{\lambda r_i} \mathbb{I}_{B_i}\Big)^2 \, dA\Big) \\
&\leq 8\Big(\int_{\Omega\setminus K} \rho^2 \, dA + c_{2\lambda}\pi \sum_{i\in I} \frac{\rho_i^2}{\lambda^2 r_i^2} r_i^2\Big) \\
&= 8\Big(\int_{\Omega\setminus K} \rho^2 \, dA + \frac{c_{2\lambda}\pi}{\lambda^2} \sum_{i\in I} \rho_i^2\Big) \\
&\leq 8 \max(1, \frac{c_{2\lambda}\pi}{\lambda^2})\Big(\int_{\Omega\setminus K} \rho^2 \, dA + \sum_{i\in I} \rho_i^2\Big) \\
&\leq 8 \max(1, \frac{c_{2\lambda}\pi}{\lambda^2})(\operatorname{mod}_{\mathcal{K}}(\Gamma) + \epsilon)
\end{aligned}$$

By letting $c_2 := (8 \max(1, \frac{c_{2\lambda}\pi}{\lambda^2}))^{-1}$ and taking $\epsilon \to 0$, we obtain the result. □

Corollary 3.3.6. *Let $\Omega \subset \mathbb{R}^2$ be a Borel set with countably many compact boundary components and $\mathcal{K} = \{K_i\}_{i\in I}$ a countable collection of pairwise disjoint continua in Ω which are τ-fat and λ-quasiround for some $\tau > 0$, $\lambda \geq 1$. Let $K = \cup_{i\in I} K_i$, and suppose $\Omega \setminus K$ is a domain. If Ω is Loewner then $\Omega \setminus K$ is transboundary Loewner.*

Proof. Let Ω be Loewner with decreasing function Ψ. Let

$$\Psi'(t) = \min(c_1, c_2\Psi(t)),$$

where c_1, c_2 are the constants from Lemma 3.3.5. Then for any disjoint, non-degenerate continua $E, F \subset \Omega \setminus K$, by Lemma 3.3.5 we have

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E, F; \Omega)) \geq \min(c_1, c_2 \operatorname{mod}(\Gamma(E, F; \Omega))) \geq \min(c_1, c_2\Psi(\Delta(E, F))) = \Psi'(\Delta(E, F)).$$

Let $\mathcal{J}$ be the set of all connected components of $\partial\Omega$. To show $\Omega \setminus K$ is transboundary Loewner, we need to show

$$\operatorname{mod}_{\mathcal{J}\cup\mathcal{K}}(\Gamma(E, F; \overline{\Omega \setminus K})) \geq \Psi'(\Delta(E, F)).$$

Let $K' = \partial(\overline{\Omega} \setminus K)$, the boundary of the interior of K. Observe that $\pi_{\mathcal{K}}(K') = \pi_{\mathcal{K}}(K)$, and thus $\pi_{\mathcal{K}}(\Omega) = \pi_{\mathcal{K}}((\Omega \setminus K) \cup K')$.

$$\begin{aligned}
\operatorname{mod}_{\mathcal{J}\cup\mathcal{K}}(\Gamma(E, F; \overline{\Omega \setminus K})) &\geq \operatorname{mod}_{\mathcal{J}\cup\mathcal{K}}(\Gamma(E, F; (\Omega \setminus K) \cup K')) \\
&= \operatorname{mod}_{\mathcal{K}}(\Gamma(E, F; (\Omega \setminus K) \cup K')) \\
&= \operatorname{mod}_{\mathcal{K}}(\Gamma(E, F; \Omega)) \geq \Psi'(\Delta(E, F)).
\end{aligned}$$

□

Example 3.3.7. *Countably connected circle domains in $\mathbb{R}^2$ and $\hat{\mathbb{C}}$ are transboundary Loewner.*

Proof. Let X be a countably connected circle domain in $\mathbb{R}^2$. Let $\mathcal{K}$ be the collection of non-degenerate, bounded complementary components of X and $\Omega := X \cup K$. Then $\mathcal{K}$ is a collection of closed disks, which are uniformly fat, uniformly quasiround continua. If Ω is Loewner, then we can apply Corollary 3.3.6 to say X is transboundary Loewner. If none of the boundary components of X are points, then Ω is either the plane or a disk, both of which are Loewner. If some of the boundary components are points, then Ω is the plane or the disk with countably many points removed. By Proposition 2.4.2, we have that Ω is Loewner.

Now let X be a countably connected circle domain in $\mathbb{R}^2$. To see this, note that the sphere is Loewner. If X is the sphere, then X is transboundary Loewner. If X is the sphere with only countably many points removed, then X is Loewner by Proposition 2.4.2, which implies X is transboundary Loewner. Suppose that X has at least one non-trivial complementary component: D. Rotate the sphere so that the center of D is the north pole; such a rotation is conformal and quasisymmetric. Use stereographic projection to map X into the plane.

Recall that stereographic projection is conformal and it sends circles not touching the north pole to circles. Thus the image will be a bounded circle domain in the plane. Notice $\hat{\mathbb{C}} \setminus D$ is Loewner, and that its image under stereographic projection is bounded and linearly locally connected. By Theorem 2.5.1, we have that stereographic projection is quasisymmetric as a map from $\hat{\mathbb{C}} \setminus D$, and thus its restriction to X is quasisymmetric. By Proposition 3.2.2, the circle domain in the sphere is transboundary Loewner since the image in the plane is. □

One can make similar statements for countably connected square domains. Let us establish a larger class of uniformly fat, uniformly quasiround shapes. A subset of $\hat{\mathbb{C}}$ or $\mathbb{R}^2$ is called an open η-quasidisk , if it is the quasisymmetric image of $\mathbb{D}$ for some η-quasisymmetry. A closed quasidisk is a quasisymmetric image of $\overline{\mathbb{D}}$. An η-quasicircle is an η-quasisymmetric image of $\partial\mathbb{D}$. We say a collection of quasicircles or quasidisks is uniform if they are all η-quasisymmetric images of a circle or disk for the same η. If one has a Jordan curve which is a quasicircle, its interior will be a quasidisk. The converse is true since quasisymmetries extend to the boundary. We will exclude a vast amount of theory of quasicircles; restricting our discussion to the following properties.

Proposition 3.3.8 (Bonk[1] (Proposition 4.3))**.** *An η-quasidisk is λ-quasiround with λ depending only on η.*

Proposition 3.3.9 (Schramm[4] (Corollary 2.3))**.** *An η-quasidisk is τ-fat with τ depending only on η.*

It becomes immediate then, that one can use Corollary 3.3.6 when $\mathcal{K}$ is a collection of uniform quasidisks. On the topic of quasidisks, they give more examples of spaces which are Loewner.

Proposition 3.3.10 (Bonk[1] (Proposition 7.5))**.** *Let $\{D_i\}_{i=1}^n$ be a finite collection of pairwise disjoint, closed η-quasidisks in $\hat{\mathbb{C}}$. Suppose $\Delta(D_i, D_j) \geq \alpha$ for all $i \neq j$. Then*

$$\hat{\mathbb{C}} \setminus \cup_{i=1}^n D_i$$

is Loewner with Ψ depending only on $n, \eta,$ and α.

We remark that this result holds true in the plane as well. Indeed, much like the discussion in Example 3.3.7, we can put the north pole in the interior of D_1 by rotation, and then say that stereographic projection on $\hat{\mathbb{C}} \setminus D_1$ is quasisymmetric. Thus the image, which will be a bounded domain whose boundary components are uniform quasicircles, will be Loewner with Ψ still depending only on n, η, and α (see Proposition 2.1.9). This process can be reversed for bounded, finitely connected domains in the plane whose boundary components are uniformly relatively separated uniform quasicircles bounding disjoint Jordan regions. If unbounded, there is some sufficiently large disk in which the domain will be Loewner and the relative distance between boundary quasicircles isn't decreased. Then $\mathrm{mod}(\Gamma(E, F; \mathbb{R}^2))$ is greater than the modulus connecting E and F in the disk, which is bounded below.

3.4 N-transboundary Loewner property

A standard technique to show that geometrically quasiconformal maps are quasisymmetric is to reach a contradiction; the failure of quasisymmetry allows one to construct continua which have small relative distance in the domain of the function and large relative distance in the image. So if the domain is Loewner, and the modulus in the image goes to 0, then the contradiction is complete. If the domain is transboundary Loewner, and the transboundary modulus in the image goes to 0, then the contradiction is complete. However, in the context we are concerned with, the transboundary modulus is too big: it doesn't go to 0 in the image. Also, the modulus is too small: the domains fail to be Loewner. Neither of these will give a contradiction; thus, a new quasisymmetrically quasi-invariant quantity is necessary for this argument to work. This was precisely the observation of Bonk[1] when he was uniformizing circle carpets in the plane. His idea was to use a mixed modulus; where one doesn't go through all the boundary components, so as to be smaller than transboundary modulus, but one still goes through some, so as to be larger than modulus. We give a name for spaces which are Loewner with respect to this mixed modulus, and we define it here.

Definition 3.4.1. *Let (X, d, μ) be a metric measure space with μ locally finite and $N \in \mathbb{N}$. Suppose X is homeomorphic to a domain in $\hat{\mathbb{C}}$, ∂X is compact, and $\partial_0 X$ is countable. We say*

*X is **N-transboundary Loewner** in case there is an decreasing function $\Psi : (0,\infty) \to (0,\infty)$ such that the following holds for all $\mathcal{K} \subset \partial_0 X$ with $\#(\mathcal{K}) \leq N$. For all disjoint, non-degenerate continua $E, F \subset X$,*

$$\mathrm{mod}_{\partial_0 X}(\Gamma(E,F;\overline{X}\setminus K)) \geq \Psi(\Delta(E,F)).$$

If X is Loewner, then it will be N-transboundary Loewner for all N:

$$\mathrm{mod}_{\partial_0 X}(\Gamma(E,F;\overline{X}\setminus K)) \geq \mathrm{mod}_{\partial_0 X}(\Gamma(E,F;X)) = \mathrm{mod}(\Gamma(E,F;X)).$$

Conversely, if $\partial_0 X$ is finite and $N \geq \#(\partial_0 X)$, then X being N-transboundary Loewner gives X is Loewner by selecting $\mathcal{K} = \partial_0 X$. Notice that being 0-transboundary Loewner is identical to be transboundary Loewner, as $\overline{X}\setminus K = \overline{X}$. In fact, if X is N-transboundary Loewner for any N, then X is transboundary Loewner, as one can always take $\mathcal{K} = \emptyset$. More generally, for all $M, N \in \mathbb{N}$ with $M \geq N$, if X is M-transboundary Loewner, then X is N-transboundary Loewner.

Proposition 3.4.2. *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be metric measure spaces with μ_X and μ_Y locally finite. Suppose X and Y are homeomorphic to domains in $\hat{\mathbb{C}}$, $\partial X, \partial Y$ are compact, and $\partial_0 X, \partial_0 Y$ are countable. Suppose $f : X \to Y$ is quasisymmetric and geometrically quasiconformal. If X is N-transboundary Loewner, then Y is N-transboundary Loewner.*

Proof. Pick any non-degenerate, disjoint continua $A, B \subset Y$ and any $\mathcal{K} \subset \partial_0 Y$ with $\#(\mathcal{K}) \leq N$. Let $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$, noticing that they are disjoint, non-degenerate continua in X. Since f is quasisymmetric, f extends to a homeomorphism of the completions (see Remark 2.1.6). This means that f gives rise to a homeomorphism $f : (\overline{X})_{\partial_0 X} \to (\overline{Y})_{\partial_0 Y}$. Let $\mathcal{K}' \subset \partial_0 X$ correspond to $\mathcal{K}$, and notice $\#(\mathcal{K}') \leq N$. Since f is N-transboundary Loewner,

$$\mathrm{mod}_{\partial_0 X}(\Gamma(A',B';\overline{X}\setminus K')) \geq \Psi'(\Delta(A',B'))$$

where Ψ' is the function obtained from the transboundary Loewner property of X. Now apply

Lemma 3.1.3 on $\overline{X}$ and $\overline{Y}$. Letting f^{-1} be ν-quasisymmetric and f be H-quasiconformal, and recalling Proposition 2.1.9, we conclude

$$\begin{aligned}\operatorname{mod}_{\partial_0 Y}(\Gamma(A,B;\overline{Y}\setminus K)) &\geq H^{-1} \operatorname{mod}_{\partial_0 X}(\Gamma(A',B';\overline{X}\setminus K')) \\ &\geq H^{-1}\Psi'(\Delta(A',B')) \\ &\geq H^{-1}\Psi'(\nu(2\Delta(A,B))).\end{aligned}$$

$\Psi(t) := H^{-1}\ \Psi'(\nu(2t))$ is decreasing since Ψ' is decreasing and ν is increasing, so Y is N-transboundary Loewner. □

Let us establish some examples (and non-examples) of N-transboundary Loewner spaces.

Example 3.4.3. *There are domains which are transboundary Loewner and not 1-transboundary Loewner.*

Proof. We construct a domain in $\mathbb{R}^2$ as follows. Define the following polar rectangle for $n \in \mathbb{N}$, $n \geq 1$:

$$A_n := \{(r,\theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi - \frac{1}{n}\} + (6n, 0).$$

These polar rectangles are pairwise disjoint and uniformly separated. Let $\Omega = \mathbb{R}^2 \setminus \cup_{n=1}^{\infty} A_n$, and notice Ω is a domain. Since the plane is Loewner, and A_n are uniformly fat and quasiround, we can say, by Corollary 3.3.6, that Ω is transboundary Loewner.

To see that it fails to be 1-transboundary Loewner, let $E_n = \partial B((6n,0),0.9)$ and let $F_n = \partial B((6n,0),2.1)$. Notice

$$\Delta(E_n, F_n) = \frac{1.2}{1.8} = \frac{2}{3}.$$

Now

$$\Gamma(B[(6n,0),1], \mathbb{R}^2 \setminus B((6n,0),2); A[(6n,0),1,2] \setminus A_n) < \Gamma(E_n, F_n; \mathbb{R}^2 \setminus A_n).$$

By the overflowing property and Example 2.3.10, we have

$$\text{mod}_{\,\partial_0\Omega}(\Gamma(E_n,F_n;\mathbb{R}^2\backslash A_n)) \leq \text{ mod }(\Gamma(B[(6n,0),1],\mathbb{R}^2\backslash B((6n,0),2);A[(6n,0),1,2]\backslash A_n)) = \frac{1/n}{\log(2)}.$$

For any decreasing function, Ψ, pick $n > (\Psi(2/3)\log(2))^{-1}$, then

$$\text{mod}_{\partial_0\Omega}(\Gamma(E_n,F_n;\mathbb{R}^2\setminus A_n)) < \Psi(2/3) = \Psi(\Delta(E_n,F_n)).$$

Hence it cannot be 1-transboundary Loewner. $\square$

In the previous example, the complementary components failed to be uniform quasidisks. If we require our complementary components to be uniform quasidisks, we establish a large class of N-transboundary Loewner spaces.

Proposition 3.4.4 (Bonk[1]). *Let $\mathcal{K}$ be countable collection of uniformly relatively separated, closed, η-quasidisks in $\mathbb{R}^2$, and suppose $\mathbb{R}^2\setminus K$ is a domain. Then $\mathbb{R}^2\setminus K$ is N-transboundary Loewner for all $N\in\mathbb{N}$ with decreasing function Ψ depending only on η, the relative separation, and N.*

Proof. Pick any disjoint, non-degenerate continua $E,F\subset\mathbb{R}^2\setminus K$ and any $\mathcal{J}\subset\mathcal{K}$ with $\#(\mathcal{J})\leq N$. η-quasidisks will be uniformly fat and quasiround, so we can use Lemma 3.3.5 to say

$$\text{mod}_{\mathcal{K}}(\Gamma(E,F;\mathbb{R}^2\setminus J)) \geq \min(c_1, c_2 \text{ mod }(\Gamma(E,F;\mathbb{R}^2\setminus J))),$$

where c_1 and c_2 depend only on η. We use Proposition 3.3.10 and following remarks to conclude that $\mathbb{R}^2\setminus J$ is Loewner with Ψ depending only on η, the relative separation, and N. Since η and the relative separation are fixed, we have the same Ψ for all $\mathcal{J}$ of size N. $\square$

The same proof applies to bounded domains in the plane whose boundary components are uniform quasicircles which are uniformly relatively separated and whose union is closed. It also applies to quasidisks in the sphere satisfying the same assumptions by noting stereographic projection is quasisymmetric on bounded subsets of the plane. This can be generalized for $N = 1$. Much like how we don't need uniform relative separation to conclude transboundary Loewner, we don't need it for 1-transboundary Loewner.

Proposition 3.4.5. *Let $\mathcal{K}$ be countable collection of closed, η-quasidisks in $\mathbb{R}^2$, and suppose $\mathbb{R}^2 \setminus K$ is a domain. Then $\mathbb{R}^2 \setminus K$ is 1-transboundary Loewner with decreasing function Ψ depending only on η.*

Proof. Pick any disjoint, non-degenerate continua $E, F \subset \mathbb{R}^2 \setminus K$ and any η-quasidisk, $K_i \in \mathcal{K}$. Use Proposition 3.3.10 to say $\mathbb{R}^2 \setminus K_i$ is Loewner with Ψ depending only on η. Then use the fact that uniform quasidisks are uniformly fat and uniformly quasiround to use Proposition 3.3.5:

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E, F; \mathbb{R}^2 \setminus K_i)) \geq \min(c_1, c_2 \operatorname{mod}(\Gamma(E, F; \mathbb{R}^2 \setminus K_i))) \geq \min(c_1, c_2 \Psi(\Delta(E, F))),$$

where c_1, c_2 and Ψ depend only on η. Thus the same bound will hold for any choice of K_i. $\square$

Once again, this argument will also apply to bounded domains in the plane whose boundary components are uniform quasicircles, as well as quasidisks in the sphere. This proposition suggests that 1-transboundary Loewner does not imply 2-transboundary Loewner.

Example 3.4.6. *There are domains which are 1-transboundary Loewner and not 2-transboundary Loewner.*

Proof. For $n \in \mathbb{N}$, $n \geq 2$, let $S_n = [3n, 3n+1] \times [0, 1]$ and $T_n = [3n+1+\frac{1}{n}, 3n+2+\frac{1}{n}] \times [0, 1]$. Let $\Omega = \mathbb{R}^2 \setminus (\cup_n S_n \cup T_n)$. Notice that Ω is a domain. Since S_n and T_n are all squares, they are uniform quasicircles. By Proposition 3.4.5, this domain must be 1-transboundary Loewner.

To see that it fails to be 2-transboundary Loewner, let $E_n = \{3n + 1 + \frac{1}{2n}\} \times [\frac{1}{8}, \frac{3}{8}]$ and $F_n = \{3n + 1 + \frac{1}{2n}\} \times [\frac{5}{8}, \frac{7}{8}]$. Then $\Delta(E_n, F_n) = 1$ for all n. However, every curve connecting E_n and F_n which avoids S_n and T_n contains a subcurve connecting the vertical sides of one of three rectangles: $[3n+1, 3n+1+\frac{1}{n}] \times [0, \frac{1}{8}]$, $[3n+1, 3n+1+\frac{1}{n}] \times [\frac{3}{8}, \frac{5}{8}]$, $[3n+1, 3n+1+\frac{1}{n}] \times [\frac{7}{8}, 1]$.

Thus by the overflowing property and subadditivity, we can conclude

$$\operatorname{mod}_{\partial_0\Omega}(\Gamma(E_n, F_n; \mathbb{R}^2 \setminus (S_n \cup T_n))) \leq \frac{1/n}{1/8} + \frac{1/n}{1/4} + \frac{1/n}{1/8} = \frac{20}{n}.$$

Thus, for any decreasing function Ψ, we can find an n so that $\frac{20}{n} < \Psi(1)$, and we conclude

$$\operatorname{mod}_{\partial_0\Omega}(\Gamma(E_n, F_n; \mathbb{R}^2 \setminus (S_n \cup T_n))) < \Psi(1) = \Psi(\Delta(E_n, F_n)).$$

□

The previous examples suggest that uniform relative separation of the boundary components is important for a space to be 2-transboundary Loewner. Indeed, it is necessary for circle domains.

Example 3.4.7. *For countably connected circle domains, the following are equivalent.*

(1) They are N-transboundary Loewner for all $N \in \mathbb{N}$.

(2) They are 2-transboundary Loewner.

(3) The bounding circles are uniformly relatively separated.

Proof. Note that (1) ⇒ (2) follows from the definitions.

(3) ⇒ (1) : If the circles are uniformly relatively separated, and none of them are points, then by Proposition 3.4.4, we have that the domain is N-transboundary Loewner for all N. If some of the boundary components are points, then Proposition 2.4.2, we have that Proposition 3.3.10 still holds with countably many points removed. So the complement of N complementary components is still uniformly Loewner, and Lemma 3.3.5 can still be used. We deduce N-transboundary Loewner.

(2) ⇒ (3) : We'll prove the contrapositive. Suppose that the circles fail to be uniformly relatively separated. It will suffice to show that the domain fails to be 2-transboundary Loewner. Let $\mathcal{K}$ be the collection of complementary components of the circle domain. Let $C_n, D_n \in \mathcal{K}$, $\operatorname{diam}(C_n) \leq \operatorname{diam}(D_n)$, be closed disks such that $\Delta(C_n, D_n) < \frac{1}{n}$. For simplicity,

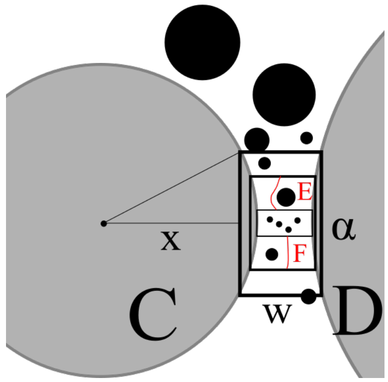

Figure 3.3: *C_n and D_n are in gray and E_n and F_n are in red*

let us assume that the centers of C_n and D_n are in the x-axis, and that the center of C_n is left of the center of D_n. We'll say a rectangle, R, is α-good if its sides are parallel to the coordinate axes, the two left vertices are in the circle bounding C_n, the two right vertices are in the circle bounding D_n, the x-axis is the perpendicular bisector of the left and right sides, the centers of C_n or D_n are not in R, and the lengths of the left and right sides are α. Let

$$\alpha = \mathrm{diam}(C_n)\sqrt{1-(1-\Delta(C_n,D_n))^2}$$

Let R_n^1 be the $\alpha/4$-good rectangle, R_n^2 be the $3\alpha/4$-good rectangle, and R_n^3 be the α-good rectangle. Let us compute the widths of these rectangles. To see this, note first that R_n^3 has the largest width, call it w. Consider the triangle created by the center of C_n, the top left corner of R_n^3, and some point $(x,0)$ intersecting the left side of R_n^3. Then the Pythagorean Theorem dictates

$$x^2 = (\mathrm{diam}(C_n)/2)^2 - (\alpha/2)^2 = (\mathrm{diam}(C_n)/2)^2(1-\Delta(C_n,D_n))^2.$$

Since C_n has the smaller diameter, we can conclude

$$\begin{aligned} w &\leq \mathrm{diam}(C_n) - 2x + d(C_n,D_n) \\ &= \mathrm{diam}(C_n) - \mathrm{diam}(C_n)(1-\Delta(C_n,D_n)) + d(C_n,D_n) \\ &= \mathrm{diam}(C_n)(1-(1-\Delta(C_n,D_n)) + \Delta(C_n,D_n)) \end{aligned}$$

$$= \operatorname{diam}(C_n)(2\Delta(C_n, D_n))$$

Let E_n be any continuum contained in the circle domain connecting the top edge of R_n^2 and the top edge of R_n^1. Let F_n be any continuum connecting the bottom edge of R_n^2 and the bottom edge of R_n^1. Notice that $\min(\operatorname{diam}(E_n), \operatorname{diam}(F_n)) \geq \alpha/4$, and their distance is at least the height of R_n^1. Thus

$$\Delta(E_n, F_n) \leq 1.$$

Let L_i^T and L_i^B be the top and bottom sides of R_n^i respectively. Let $\Gamma_1 := \Gamma(L_1^T, L_1^B; R_n^1 \setminus (C_n \cup D_n))$, $\Gamma_2 := \Gamma(L_2^T, L_3^T; R_n^3 \setminus (C_n \cup D_n))$, and $\Gamma_3 = \Gamma(L_2^B, L_3^B; R_n^3 \setminus (C_n \cup D_n))$. Observe that

$$\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 < \Gamma(E_n, F_n; \mathbb{R}^2 \setminus (C_n \cup D_n)).$$

Hence we can use overflowing and subadditivity to get

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E_n, F_n; \mathbb{R}^2 \setminus (C_n \cup D_n))) \leq \operatorname{mod}_{\mathcal{K}}(\Gamma_1) + \operatorname{mod}_{\mathcal{K}}(\Gamma_2) + \operatorname{mod}_{\mathcal{K}}(\Gamma_3).$$

Let

$$\rho = \frac{4}{\alpha}\mathbb{I}_{R_n^1}.$$

If $K_i \cap R_n^1 = \emptyset$, set $\rho_i = 0$. Also, if $K_i = C_n$ or $K_i = D_n$, set $\rho_i = 0$. Otherwise, let

$$\rho_i = \frac{4h_i}{\alpha},$$

where $h_i = \operatorname{diam}(\pi_2(K_i \cap R_n^1))$. We claim $(\rho; \{\rho_i\}) \wedge_{\mathcal{K}} \Gamma_1$, as for all $\gamma \in \Gamma_1$,

$$\int_{\gamma \setminus K} \rho\, ds + \sum_{K_i \cap \gamma \neq \emptyset} \rho_i = \frac{4}{\alpha}(\ell(\gamma \setminus K) + \sum_{K_i \cap \gamma \neq \emptyset} h_i) \geq \frac{4}{\alpha}(\mathcal{H}^1(\pi_2(\gamma \setminus K)) + \mathcal{H}^1(\pi_2(\gamma \cap K))) \geq 1.$$

Now use this to estimate the modulus:

$$\begin{aligned}
\operatorname{mod}_{\mathcal{K}}(\Gamma_1) &\leq \int_{R_n^1 \setminus K} \rho^2 \, dA + \sum_{K_i \cap R_n^1 \neq \emptyset} \rho_i^2 \\
&\leq \frac{16}{\alpha^2}\left(A(R_n^1 \setminus K) + \frac{4}{\pi} \sum_{K_i \cap R_n^1 \neq \emptyset} \pi \left(\frac{h_i}{2}\right)^2 \right) \\
&\leq \frac{16}{\alpha^2}\left(A(R_n^1 \setminus K) + \frac{4}{\pi} \sum_{K_i \in \mathcal{K}} A(K_i \cap R_n^1) \right) \\
&\leq \frac{64}{\pi\alpha^2} A(R_n^1) \leq \frac{16}{\pi\alpha} w = \frac{4}{\pi} \frac{w}{\alpha/4}.
\end{aligned}$$

We remark that a similar argument will show

$$\begin{aligned}
\operatorname{mod}_{\mathcal{K}}(\Gamma_2) &\leq \frac{4}{\pi} \frac{w}{\alpha/8} = \frac{32}{\pi} \frac{w}{\alpha}, \text{ and} \\
\operatorname{mod}_{\mathcal{K}}(\Gamma_3) &\leq \frac{4}{\pi} \frac{w}{\alpha/8} = \frac{32}{\pi} \frac{w}{\alpha}.
\end{aligned}$$

Thus the sum can be estimated as follows.

$$\begin{aligned}
\operatorname{mod}_{\mathcal{K}}(\Gamma(E_n, F_n; \mathbb{R}^2 \setminus (C_n \cup D_n))) &\leq \frac{80}{\pi} \frac{w}{\alpha} \\
&\leq \frac{80}{\pi} \frac{2\operatorname{diam}(C_n)\Delta(C_n, D_n)}{\operatorname{diam}(C_n)\sqrt{1 - (1 - \Delta(C_n, D_n))^2}} \\
&= \frac{160}{\pi} \frac{\Delta(C_n, D_n)}{\sqrt{2\Delta(C_n, D_n) - \Delta(C_n, D_n)^2}} \\
&= \frac{160}{\pi} \sqrt{\frac{\Delta(C_n, D_n)}{2 - \Delta(C_n, D_n)}} < \frac{160}{\pi} \sqrt{\frac{1}{2n-1}}.
\end{aligned}$$

For any decreasing function, Ψ, we can find an n so that

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E_n, F_n; \mathbb{R}^2 \setminus (C_n \cup D_n))) \leq \frac{160}{\pi} \sqrt{\frac{1}{2n-1}} < \Psi(1) \leq \Psi(\Delta(E_n, F_n)).$$

Hence, it cannot be 2-transboundary Loewner. $\square$

The argument just presented assumed that C_n and D_n were disks. For a bounded circle domain, the failure to be uniformly separated may occur for D_n being the unbounded complementary component, for all sufficiently large n. We remark that a similar argument will show that it fails to be 2-transboundary Loewner. Then, since circle domains in the sphere are quasisymmetric to bounded planar circle domains, we conclude the statement is true for spherical circle domains.

Chapter 4

Quasisymmetric Koebe Uniformization of Metric Surfaces

In this chapter, we establish results characterizing when metric spaces are quasisymmetric to uniformly relatively separated circle domains. Our characterization is complete for metric surfaces: homeomorphic images of a domain in $\hat{\mathbb{C}}$ with locally finite $\mathcal{H}^2$ measure; in that, all other properties assumed are invariant under (geometrically quasiconformal) quasisymmetries.

4.1 Transboundary modulus in circle domains

We are going to need a few facts about the geometry of circle domains as they are our model space for the quasisymmetric uniformization.

Proposition 4.1.1. *Let $z \in \mathbb{R}^2$ and $R > r > 0$. Let $\mathcal{D}$ be a collection of pairwise disjoint, closed disks in $\mathbb{R}^2$ intersecting both $B[z, r]$ and $\mathbb{R}^2 \setminus B(z, R)$. If*

$$\frac{R}{r} \geq 14,$$

then $\#(\mathcal{D}) \leq 2$.

Proof. Let $D = B[z', r']$ be a closed disk intersecting $B[z, r]$ and $\mathbb{R}^2 \setminus B(z, R)$. Notice $r' \geq (R - r)/2$. If $z' = z$, then we have that $B(z, R) \subset D$, and any disk intersecting $B[x, r]$ is not disjoint with D. So suppose $z \neq z'$, and let L be the ray starting at z and going through z'. Every circle centered around z intersects L exactly once. Let C be the circle of radius $(R + r)/2$ centered at z. Let c be the intersection point of L with C. We claim $c \in B[z', r' - (R - r)/2]$. To see this, we'll break it down into cases.

If $z' \in B[z, r]$, then $r' \geq R - |z' - z|$, and

$$|c - z'| = (R - r)/2 + (r - |z - z'|) = (R + r)/2 - |z' - z| \leq (R + r)/2 + r' - R = r' - (R - r)/2.$$

If $z' \in A[z, r, R]$ then $r' \geq |z' - c| + (R - r)/2$, and

$$|c - z'| \leq r' - (R - r)/2.$$

Finally, if $z' \in \mathbb{R}^2 \setminus B(z, R)$ then $r' \geq |z - z'| - r$ and

$$|c - z'| = (R - r)/2 + (|z - z'| - R) \leq (R - r)/2 + (r + r' - R) = r' - (R - r)/2.$$

Pick any $w \in B[c, (R - r)/2]$. Then

$$|w - z'| \leq |w - c| + |c - z'| \leq (R - r)/2 + r' - (R - r)/2 = r'.$$

Thus, $B[c, (R - r)/2] \subset D$. Let C' be the circle of radius $(R + r)/4$ centered at z, and let d be one of the two points satisfying $d \in C' \cap \partial B(c, (R - r)/2)$, and consider the triangle spanned by d, c, and z. Let θ be the angle corresponding to the vertex z. Law of Cosines tells us

$$\begin{aligned}
\left(\frac{R - r}{2}\right)^2 &= \left(\frac{R + r}{2}\right)^2 + \left(\frac{R + r}{4}\right)^2 - 2\frac{R + r}{2}\frac{R + r}{4}\cos(\theta) \\
\cos(\theta) &= \frac{5}{4} - \left(\frac{R - r}{R + r}\right)^2 = \frac{5}{4} - \left(\frac{R/r - 1}{R/r + 1}\right)^2
\end{aligned}$$

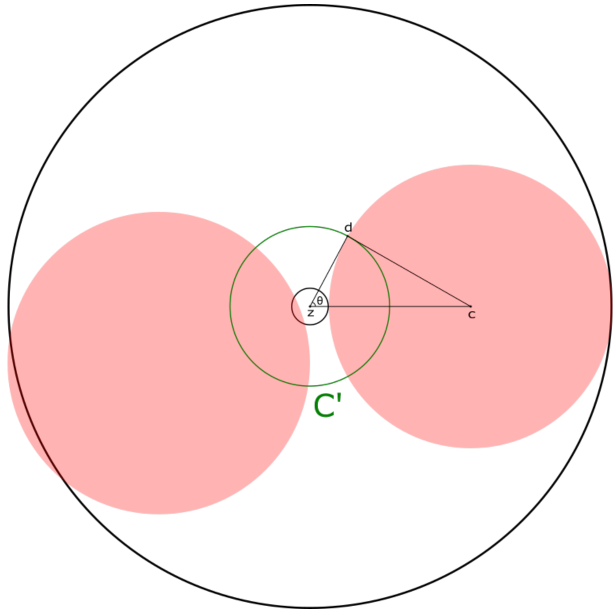

Figure 4.1:

Notice that $\frac{x-1}{x+1}$ is an increasing function on $x \geq 1$. So

$$\frac{R}{r} \geq 14 > \frac{2+\sqrt{3}}{2-\sqrt{3}} \Rightarrow \frac{R/r-1}{R/r+1} > \frac{\sqrt{3}}{2}.$$

Thus

$$\frac{5}{4} - \left(\frac{R/r-1}{R/r+1}\right)^2 < \frac{1}{2}.$$

Since $\arccos(x)$ is a decreasing function, we have

$$\theta > \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

Now we conclude

$$\mathcal{H}^1(C' \cap D) \geq \mathcal{H}^1(C' \cap B[c, (R-r)/2]) = 2\theta\frac{R+r}{4} > \frac{\pi(R+r)}{6}.$$

This is true for all $D \in \mathcal{D}$. By disjointness,

$$\mathcal{H}^1(C') \geq \mathcal{H}^1\bigg(\bigcup_{D\in\mathcal{D}} (C' \cap D)\bigg) = \sum_{D\in\mathcal{D}} \mathcal{H}^1(C' \cap D) > \frac{\pi}{6}(R+r)\#(\mathcal{D}).$$

Hence

$$\#(\mathcal{D}) < \frac{6}{\pi(R+r)}\left(2\pi\frac{R+r}{4}\right) = 3.$$

□

A similar statement will carry over to the sphere as stereographic projection is conformal and quasisymmetric on bounded subsets of the plane, and it sends circles in the plane to circles in the sphere. Quasisymmetries will distort the thickness of an annulus by η and conformalities will preserve all the angles.

Lemma 4.1.2 ([1])**.** *Let $\Omega \subset \mathbb{R}^2$ be a circle domain. If Ω is bounded, let K_0 be the bounding circle of its unbounded complementary component. Let $\mathcal{K}$ be the collection of the bounded complementary components of Ω and let $K_0 \in \mathcal{K}$. Let $\Omega' = \Omega \cup K$. There is a decreasing function, $\Phi : (0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty} \Phi(t) = 0$ such that the following holds. If $E, F \subset \Omega$ are disjoint, non-degenerate continua with $\Delta(E,F) > \exp(\log(14)^{27})$, then there exists $\mathcal{J} \subset \mathcal{K}$ with $\#(\mathcal{J}) \leq 2$, such that*

$$\mathrm{mod}_{\mathcal{K}}(\Gamma(E,F;\Omega' \setminus J)) \leq \Phi(\Delta(E,F)).$$

Proof. We introduce the following notation for this proof. For an annulus, $A := A(x,r,R)$ (or $A[x,r,R]$), and a set C, we define

$$R_C^A = \sup_{y\in A\cap C} |x-y|$$

and

$$r_C^A = \inf_{y\in A\cap C} |x-y|.$$

We will also use

$$w_A(C) := \log\left(\frac{R_C^A}{r_C^A}\right),$$

which we will refer to as the width of the set. If C is disjoint with A, we'll say the width is 0.

Suppose, without loss of generality, that $\min(\operatorname{diam}(E), \operatorname{diam}(F)) = \operatorname{diam}(E)$, and pick any $x \in E$. Let $r_0 = \operatorname{diam}(E)$ and $R_0 = d(E, F)$. Then $E \subset B[x, r_0]$ and $F \subset \mathbb{R}^2 \setminus B(x, R_0)$. Moreover, since $\Delta(E, F) > 1$, we have $r_0 < R_0$. Let $A := A[x, r_0, R_0]$ and notice that curves connecting E and F contain a subcurve connecting the bounding circles of A.

We claim there is a subannulus, $A' := A[x, r', R'] \subset A$, with $R'/r' \geq 14$ and the following property:

$$\#\{K_i \in \mathcal{K} \mid w_{A'}(K_i) > \log(R'/r')^{1/3}\} \leq 2.$$

If we have

$$w_A(K_i) \leq \log(R_0/r_0)^{1/3}$$

for all K_i, then we're done as $A' = A$. Take $\mathcal{J} = \emptyset$. If not, let $D \in \mathcal{K}$ satisfy

$$w_A(D) > \log(R_0/r_0)^{1/3},$$

and let $A_1 = A[x, r_D^A, R_D^A] \subset A$. Notice $w_{A_1}(D) = w_A(D)$ and

$$\frac{R_D^A}{r_D^A} > \exp(\log(R_0/r_0)^{1/3}) > 14.$$

Now if

$$w_{A_1}(K_i) \leq \log(R_D^A/r_D^A)^{1/3}$$

for all $K_i \neq D$, then we're done as $A' = A_1$. Take $\mathcal{J} = \{D\}$. If not, there exists $D' \in \mathcal{K}$, $D' \neq D$, with

$$w_{A_1}(D') > \log(R_D^A/r_D^A)^{1/3},$$

and let $A_2 = A[x, r^{A_1}_{D'}, R^{A_1}_{D'}] \subset A_1$. We claim $A' = A_2$ works. First notice

$$\frac{R^{A_1}_{D'}}{r^{A_1}_{D'}} > \exp(\log(R^A_D/r^A_D)^{1/3}) > \exp(\log(R_0/r_0)^{1/9}) > 14.$$

Now if there is some $D'' \in \mathcal{K}$, $D'' \neq D'$, $D'' \neq D$, with

$$w_{A'}(D'') > \log(R'/r')^{1/3},$$

then

$$\frac{R^{A'}_{D''}}{r^{A'}_{D''}} > \exp(\log(R'/r')^{1/3}) > \exp(\log(R_0/r_0)^{1/27}) > 14.$$

D touches both bounding circles of A_1 and D' touches both bounding circles of A_2. Thus we have D, D', D'' all touch both bounding circles of $A[x, r^{A'}_{D''}, R^{A'}_{D''}] \subset A_2 \subset A_1$ which contradicts Proposition 4.1.1 if they're all disks. If one of them is K_0, then $(\mathbb{R}^2 \setminus \Omega') \cup K_0$ contains a closed disk touching both bounding circles of the annulus, and this disk is disjoint with the other 2, so it still contradicts Proposition 4.1.1. Take $\mathcal{J} = \{D, D'\}$.

Let $\{C_n\}$ be a countable collection of continua such that $C := (\cup_n C_n) \cap A'$ is connected. Pick any $\epsilon > 0$. By virtue of being connected, if $R^{A'}_{C_n} < R^{A'}_C$ then there exists a n' with $R^{A'}_{C_n} < R^{A'}_{C_{n'}}$ and $r^{A'}_{C_{n'}} \leq R^{A'}_{C_n}(1+\epsilon)$. Define $G^\epsilon(n)$ to be the n' value which maximizes $R^{A'}_{C_{n'}}$; such a value exists because its non-existence implies C is disconnected. Similarly, if $r^{A'}_{C_n} > r^{A'}_C$ then there exists an n' with $r^{A'}_{C_n} > r^{A'}_{C_{n'}}$ and $R^{A'}_{C_{n'}}(1+\epsilon) \geq r^{A'}_{C_n}$. Define $g^\epsilon(n)$ to be the n' value which minimizes $r^{A'}_{C_{n'}}$; such a value exists because its non-existence implies C is disconnected. Fix any $\delta > 0$ and take positive numbers $\{\epsilon_j\}_{j\in\mathbb{Z}}$ so that $\sum_{j\in\mathbb{Z}}(\log(1+\epsilon_j)) < \delta$. If there exists some C_n not attaining the width of the union, one can define $n_j = G^{\epsilon_j} \circ \ldots \circ G^{\epsilon_1}(n)$ for integer $j > 0$, $n_j = g^{\epsilon_j} \circ \ldots \circ g^{\epsilon_{-1}}(n)$ for integer $j < 0$ and $n_0 = n$ with $\epsilon_0 = 0$, stopping at some finite point if we ever have $r^{A'}_{C_{n_j}} = r^{A'}_C$ or $R^{A'}_{C_{n_j}} = R^{A'}_C$. Since C is connected, we must have

$$w_{A'}(\cup_{j\in\mathbb{Z}} C_{n_j}) = w_{A'}(C).$$

Thus we have

$$\begin{aligned}
\sum_n w_{A'}(C_n) &\geq \sum_{j\in\mathbb{Z}} w_{A'}(C_{n_j}) \\
&= \lim_{j_0\to\infty} w_{A'}(C_{n_0}) + \sum_{j=1}^{j_0} \log\left(\frac{R^{A'}_{C_{n_j}}}{r^{A'}_{C_{n_j}}}\right) + \sum_{j=-1}^{-j_0} \log\left(\frac{R^{A'}_{C_{n_j}}}{r^{A'}_{C_{n_j}}}\right) \\
&= \lim_{j_0\to\infty} w_{A'}(C_{n_0}) + \log\left(\prod_{j=1}^{j_0} \frac{R^{A'}_{C_{n_j}}}{r^{A'}_{C_{n_j}}}\right) + \log\left(\prod_{j=-1}^{-j_0} \frac{R^{A'}_{C_{n_j}}}{r^{A'}_{C_{n_j}}}\right) \\
&\geq \lim_{j_0\to\infty} w_{A'}(C_{n_0}) + \log\left(\prod_{j=1}^{j_0} \frac{R^{A'}_{C_{n_j}}}{R^{A'}_{C_{n_{j-1}}}(1+\epsilon_j)}\right) + \log\left(\prod_{j=-1}^{-j_0} \frac{r^{A'}_{C_{n_{j+1}}}}{(1+\epsilon_j)r^{A'}_{C_{n_j}}}\right) \\
&= \lim_{j_0\to\infty} w_{A'}(C_{n_0}) + \log\left(\frac{R^{A'}_{C_{n_{j_0}}}}{R^{A'}_{C_{n_0}}}\right) - \log\left(\prod_{j=1}^{j_0}(1+\epsilon_j)\right) + \log\left(\frac{r^{A'}_{C_{n_0}}}{r^{A'}_{C_{n_{-j_0}}}}\right) - \log\left(\prod_{j=-1}^{-j_0}(1+\epsilon_j)\right) \\
&= \lim_{j_0\to\infty} \log\left(\frac{R^{A'}_{C_{n_{j_0}}}}{r^{A'}_{C_{n_{-j_0}}}}\right) - \sum_{|j|\leq j_0} \log(1+\epsilon_j) \\
&\geq \lim_{j_0\to\infty} \log\left(\frac{R^{A'}_{C_{n_{j_0}}}}{r^{A'}_{C_{n_{-j_0}}}}\right) - \delta \\
&= w_{A'}(\cup_{j\in\mathbb{Z}} C_{n_j}) - \delta \\
&= w_{A'}(C) - \delta.
\end{aligned}$$

Take $\delta \to 0$ to say that the sum of the widths exceeds the width of the union.

Now let $E' = B[x, r']$ and $F' = \mathbb{R}^2 \setminus B(x, R')$. Consider $\Gamma := \Gamma(E', F'; \Omega' \setminus J)$. Let $P = (\rho; \{\rho_i\}_{i\in I})$ be transboundary mass distribution with

$$\rho(z) = \frac{1}{\log(R'/r')|x-z|}\mathbb{I}_{A'},$$

$\rho_i = 0$ for $K_i \in \mathcal{J}$ or K_i disjoint with A', and

$$\rho_i = \frac{w_{A'}(K_i)}{\log(R'/r')}$$

for $K_i \in \mathcal{K} \setminus \mathcal{J}$. Notice that, for any curve γ' in $A' \setminus K$, we have

$$\int_{\gamma'} \rho \, ds \geq \log(R'/r')^{-1} w_{A'}(\gamma').$$

Pick any $\gamma \in \Gamma$.

Let $\{\gamma_m\}$ be a collection of subcurves of γ each corresponding to a connected component of $\gamma^{-1}(A' \setminus K)$ (see discussion preceding Definition 3.1.1). Notice then that

$$\bigcup_m \gamma_m \cup \bigcup_{\gamma \cap K_i \neq \emptyset} K_i$$

is connected, and we can say the sum of the widths exceeds the width of the union, which is $\log(R'/r')$. Thus,

$$\int_{\gamma \setminus K} \rho \, ds + \sum_{\gamma \cap K_i \neq \emptyset} \rho_i = \log(R'/r')^{-1} (\sum_m w_{A'}(\gamma_m) + \sum_{\gamma \cap K_i \neq \emptyset} w_{A'}(K_i)) \geq \log(R'/r')^{-1} (\log(R'/r')) = 1.$$

Thus $P \wedge_{\mathcal{K}} \Gamma$. We will now use this to get an upper bound.

Notice, for all $K_i \in \mathcal{K}$ intersecting A' and $K_i \neq K_0$, we have that $K_i \cap A'$ contains a disk of radius $(R_{K_i}^{A'} - r_{K_i}^{A'})/2$. Moreover,

$$\int_{A' \cap K_i} \rho^2 \, d\mathcal{H}^2 \geq \log(R'/r')^{-2} \frac{\mathcal{H}^2(A' \cap K_i)}{(R_{K_i}^{A'})^2} \geq \pi \log(R'/r')^{-2} \frac{(R_{K_i}^{A'} - r_{K_i}^{A'})^2}{4(R_{K_i}^{A'})^2}.$$

For K_0, if it intersects A', let $K_0' = (\mathbb{R}^2 \setminus \Omega') \cup K_0$. Notice that $R_{K_0}^{A'} = R_{K_0'}^{A'}$ and $r_{K_0}^{A'} = r_{K_0'}^{A'}$. The above statement holds for K_0'.

Let

$$\mathcal{K}_1 = \{K_i \in \mathcal{K} \setminus \mathcal{J} \mid K_i \cap A' \neq \emptyset, w_{A'}(K_i) \leq \log(2)\}.$$

If $K_0 \in \mathcal{K}_1$, replace it with K_0'. For $K_i \in \mathcal{K}_1$, we have $R_{K_i}^{A'} \leq 2 r_{K_1}^{A'}$, and

$$\log\left(\frac{R_{K_i}^{A'}}{r_{K_i}^{A'}}\right) = \log\left(1 + \frac{R_{K_i}^{A'} - r_{K_i}^{A'}}{r_{K_i}^{A'}}\right) \leq \frac{R_{K_i}^{A'} - r_{K_i}^{A'}}{r_{K_i}^{A'}} \leq 2 \frac{R_{K_i}^{A'} - r_{K_i}^{A'}}{R_{K_i}^{A'}}.$$

Thus we can say

$$\sum_{K_i \in \mathcal{K}_1} \rho_i^2 \leq \log(R'/r')^{-2} \sum_{K_i \in \mathcal{K}_1} 4\left(\frac{R_{K_i}^{A'} - r_{K_i}^{A'}}{R_{K_i}^{A'}}\right)^2 \leq \frac{16}{\pi} \sum_{K_i \in \mathcal{K}_1} \int_{A' \cap K_i} \rho^2 \, d\mathcal{H}^2 \leq \frac{16}{\pi} \int_{A'} \rho^2 \, d\mathcal{H}^2 = \frac{32}{\log(R'/r')}.$$

Let

$$\mathcal{K}_2 = \{K_i \in \mathcal{K} \setminus \mathcal{J} \mid w_{A'}(K_i) > \log(2)\}.$$

If $K_0 \in \mathcal{K}_2$, replace it with K_0'. For all $K_i \in \mathcal{K}_2$, we have

$$\int_{A' \cap K_i} \rho^2 \, d\mathcal{H}^2 \geq \pi \log(R'/r')^{-2} \frac{(R_{K_i}^{A'} - r_{K_i}^{A'})^2}{4(R_{K_i}^{A'})^2} = \frac{\pi}{4} \log(R'/r')^{-2} \left(1 - \frac{r_{K_i}^{A'}}{R_{K_i}^{A'}}\right)^2 > \frac{\pi}{16} \log(R'/r')^{-2}.$$

Thus we can say

$$\frac{2\pi}{\log(R'/r')} = \int_{A'} \rho^2 \, d\mathcal{H}^2 \geq \sum_{K_i \in \mathcal{K}_2} \int_{A' \cap K_i} \rho^2 \, d\mathcal{H}^2 > \#(\mathcal{K}_2) \frac{\pi}{16} \log(R'/r')^{-2}.$$

Recall that for $K_i \notin \mathcal{J}$, we have $w_{A'}(K_i) \leq \log(R'/r')^{1/3}$. Use this to say

$$\sum_{K_i \in \mathcal{K}_2} \rho_i^2 \leq \#(\mathcal{K}_2) \log(R'/r')^{-2} \log(R'/r')^{2/3} < \frac{16}{\pi} \log(R'/r')^{2/3} \frac{2\pi}{\log(R'/r')} = \frac{32}{\log(R'/r')^{1/3}}.$$

Now we apply the modulus estimate.

$$\int_{\Omega} \rho^2 \, d\mathcal{H}^2 + \sum_{K_i \in \mathcal{K}} \rho_i^2 \leq \int_{A'} \rho^2 \, d\mathcal{H}^2 + \sum_{K_i \in \mathcal{K}_1} \rho_i^2 + \sum_{K_i \in \mathcal{K}_2} \rho_i^2 \leq \frac{2\pi}{\log(R'/r')} + \frac{32}{\log(R'/r')} + \frac{32}{\log(R'/r')^{1/3}};$$

hence

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E', F'; \Omega' \setminus J)) \leq \frac{1}{\log(R'/r')^{1/3}} \left(\frac{2\pi + 32}{\log(14)^{2/3}} + 32\right).$$

Recall $\log(R'/r') > \log(R_0/r_0)^{1/9} = \log(\Delta(E,F))^{1/9}$. Now, since A' separates E and F, we can say every curve in $\Gamma(E, F; \Omega' \setminus J)$ contains a subcurve in Γ, and hence

$$\operatorname{mod}_{\mathcal{K}}(\Gamma(E, F; \Omega' \setminus J)) \leq \frac{1}{\log(\Delta(E,F))^{1/27}} \left(\frac{2\pi + 32}{\log(14)^{2/3}} + 32\right) =: \Phi(\Delta(E,F)).$$

Observe that Φ decreases and $\lim_{t\to\infty}\Phi(t)=0$. $\square$

We remark that this result still holds in the sphere; since any spherical circle domain is quasisymmetric to a bounded planar circle domain. That is, for any E and F with sufficiently large relative distance, there are at most two complementary disks such that the transboundary modulus of curves avoiding those disks and connecting E and F is less than a function of their relative distance which goes to 0 as the relative distance goes to infinity. In fact, Bonk[1] proved this for finitely connected circle domains in the sphere; the proof just presented mimics his proof.

The final geometric fact about circle domains which we'll need concerns their linear local connectivity.

Proposition 4.1.3. *Circle domains are linearly locally connected with $\lambda = 1$.*

Proof. Let Ω be a circle domain and $x \in \Omega$ and $r > 0$. It suffices to show $B(x,r)\cap\Omega$ and $\Omega\backslash B(x,r)$ are connected. Notice that if $\mathbb{R}^2\backslash\Omega$ contains a continuum, E, then that continuum must be contained in a single disk, because the disks are disjoint and closed. If $B(x,r)\cap\Omega$ is disconnected, then there is a continuum, E, in $B(x,r)\setminus\Omega$ disconnecting it. Since E must be contained in a single disk, it means that a single disk is disconnecting $B(x,r)\cap\Omega$. In other words, there is a closed disk, D, such that $B(x,r)\setminus D$ is disconnected. This cannot happen. A similar argument will show $\Omega\setminus B(x,r)$ can't be disconnected because a single disk would have to disconnect it. $\square$

4.2 Quasisymmetric and quasiconformal equivalence

The following definition takes inspiration from the duality of modulus (Proposition 2.3.9).

Definition 4.2.1 ([5]). *Let (X,d,μ) be a metric measure space with μ locally finite. Suppose X is homeomorphic to a domain in $\hat{\mathbb{C}}$ and $\kappa \geq 1$. We say X is **κ-reciprocal** in case for all $Q \subset X$ homeomorphic to $[0,1]^2$ with $A_L, A_R, A_B, A_T \subset Q$ corresponding to $\{0\}\times[0,1]$, $\{1\}\times$*

$[0,1], [0,1]\times\{0\}, [0,1]\times\{1\}$ *respectively, we have*

$$\frac{1}{\kappa} \leq \operatorname{mod}(\Gamma(A_L, A_R; Q)) \operatorname{mod}(\Gamma(A_B, A_T; Q)) \leq \kappa.$$

Moreover, we require that for all $x \in X$ and $R > 0$ with $X \setminus B(x,R) \neq \emptyset$ we have

$$\lim_{r\to 0} \operatorname{mod}(\Gamma(B[x,r], X\setminus B(x,R); X)) = 0.$$

We say X is reciprocal in case it is κ-reciprocal for some κ.

The plane is reciprocal with $\kappa = 1$. Reciprocality is a (geometrically) quasiconformal invariant. Its use below in quasiconformal uniformization of metric spaces homeomorphic to $\mathbb{R}^2$ gives a complete characterization.

Theorem 4.2.2 (Rajala[5])**.** *Let (X,d) be a metric space with $\mathcal{H}^2$ locally finite. Suppose furthermore that X is homeomorphic to $\mathbb{R}^2$. Then X is (geometrically) quasiconformal to a domain in $\mathbb{R}^2$ if and only if X is reciprocal.*

Theorem 4.2.3 (Rajala[5])**.** *Let (X,d) be a metric space with $\mathcal{H}^2$ locally finite. Suppose furthermore that X is homeomorphic to $\mathbb{R}^2$. If X is upper Ahlfors 2-regular, then X is reciprocal.*

This characterization was later generalized to include spaces not necessarily homeomorphic to the plane. We say X is locally reciprocal in case for every $x \in X$, there is a neighborhood of x which is reciprocal. If X is homeomorphic to a domain in $\hat{\mathbb{C}}$, by Theorem 4.2.2, this is equivalent to stating every point has some neighborhood which is quasiconformal to a disk.

Theorem 4.2.4 (Ikonen[6])**.** *Let (X,d) be a metric space with $\mathcal{H}^2$ locally finite. Suppose furthermore that X is homeomorphic to a domain in $\hat{\mathbb{C}}$. If X is locally reciprocal, then X is (geometrically) $\frac{\pi}{2}$-quasiconformal to a Riemannian surface.*

We are also going to need quasisymmetric maps to be geometrically quasiconformal.

Lemma 4.2.5 (Ikonen[6] (Lemma 6.5))**.** *Let (X,d) be a metric space with $\mathcal{H}^2$ locally finite. Suppose X is locally reciprocal and homeomorphic to a domain in $\hat{\mathbb{C}}$. If $f:\mathbb{D}\to V$ is η-quasisymmetric, for some $V\subset X$, then f is (geometrically) K-quasiconformal with K depending only on η.*

Lemma 4.2.6. *Let (X,d) be a metric space with $\mathcal{H}^2$ locally finite. Suppose X is locally reciprocal and $\Omega\subset\hat{\mathbb{C}}$ is a domain. If $f:\Omega\to X$ is η-quasisymmetric, then f is (geometrically) K-quasiconformal with K depending only on η.*

Proof. Use the fact that X is locally reciprocal and Theorem 4.2.4 to say that X is $\pi/2$-quasiconformal to a Riemannian surface. Koebe ([42], Theorem 11C) proved that any Riemannian surface homeomorphic to a domain in $\hat{\mathbb{C}}$ is conformal to a domain in $\hat{\mathbb{C}}$. Thus we obtain a $\pi/2$-quasiconformal homeomorphism $g:X\to\Omega'$ for some domain $\Omega'\subset\hat{\mathbb{C}}$. Thus $g\circ f:\Omega\to\Omega'$ is a homeomorphism. By Lemma 4.2.5, f is locally geometrically K'-quasiconformal where K' depends only on η. So $g\circ f$ is locally geometrically $K'\pi/2$-quasiconformal. It is well known for homeomorphisms between domains in $\hat{\mathbb{C}}$ that local geometric quasiconformality is equivalent to global geometric quasiconformality (see Ahlfors[34], for example). Hence $g\circ f$ is geometrically $K'\pi/2$-quasiconformal. Since g^{-1} is geometrically $\pi/2$-quasiconformal, we get that f is geometrically $K'\pi^2/4$-quasiconformal. Let $K:=K'\pi^2/4$. □

The final lemma we'll need is for quasiconformal maps to extend to the quotient spaces, so that transboundary modulus can be preserved. A sufficient condition comes from linear local connectivity. We say a metric space is LLC-1 if it satisfies property (i) of linear local connectivity (see Definition 1.4.3). Notice that LLC-1 is a quasisymmetric invariant (see Proposition 2.1.7).

Lemma 4.2.7 (Merenkov-Wildrick[2])**.** *Let (X,d) be a metric space with $\overline{X}$ compact. Suppose there is a homeomorphism $f:X\to\Omega$ where $\Omega\subset\hat{\mathbb{C}}$ is a domain. If X is LLC-1, then f extends to a homeomorphism $f:\overline{X}_{\partial_0 X}\to\overline{\Omega}_{\partial_0\Omega}$.*

Any homeomorphism will extend to the end compactifications of X and Ω. What Merenkov and Wildrick[2] showed was that if X is LLC-1, then $\partial_0 X$ can be identified with the ends of X homeomorphically. They also showed that for any domain $\Omega \subset \hat{\mathbb{C}}$, $\partial_0\Omega$ is homeomorphic to the ends compactification of Ω. Thus, f extending to the end compactifications implies f gives rise to a bijection, $f_0 : \partial_0 X \to \partial_0\Omega$, satisfying (x_n) converges to a point in $K_i \in \partial_0 X$ if and only if $(f(x_n))$ converges to a point in $f_0(K_i) \in \partial_0\Omega$.

4.3 Proof of main result

In this section we prove Theorem 4.3.2, which is the main result of this work.

The 2-transboundary Loewner property will enable upgrading a quasiconformal map to a quasisymmetric one. The following argument is of standard type for such results. In particular, it's similar to an argument by Bonk[1]; though it's adapted for a more general context.

Lemma 4.3.1. *Let (X, d, μ) be a metric measure space with μ locally finite. Suppose X is metric doubling, $\overline{X}$ is compact, and $\partial_0 X$ is countable. Suppose there is a geometrically Q-quasiconformal map $f : X \to \Omega$ where $\Omega \subset \hat{\mathbb{C}}$ is a circle domain, which extends as a homeomorphism $f : \overline{X}_{\partial_0 X} \to \overline{\Omega}_{\partial_0\Omega}$. If X is 2-transboundary Loewner with decreasing function Ψ, then f is η-quasisymmetric where η depends only on Q, Ψ,* diam(X)*, and the doubling constant.*

Proof. Since $\overline{X}$ is compact, $\mathrm{diam}(X) < \infty$. There exists $a_0, a_\infty \in X$ such that $d(a_0, a_\infty) > 3\mathrm{diam}(X)/4$. Notice then that $B(a_0, \mathrm{diam}(X)/4) \cap B(a_\infty, \mathrm{diam}(X)/4) = \emptyset$. Since X is homeomorphic to a domain in $\hat{\mathbb{C}}$, X must be connected. Hence $B(a_0, \mathrm{diam}(X)/4) \cup B(a_\infty, \mathrm{diam}(X)/4) \neq X$. Let $a_1 \in X \setminus (B(a_0, \mathrm{diam}(X)/4) \cup B(a_\infty, \mathrm{diam}(X)/4))$. Let $\delta = \min(\mathrm{diam}(X)/4, 1)$. Notice that $d(a_0, a_1), d(a_1, a_\infty), d(a_0, a_\infty) \geq \delta$. Compose f with a Möbius transformation so that $f(a_0) = 0$, $f(a_1) = 1$, and $f(a_\infty) = \infty$. Since Möbius transformations are conformal, the composition is still Q-quasiconformal.

We first want to show that this map is weakly quasisymmetric. Suppose, by way of contradiction, that f fails to be weakly quasisymmetric. Then for all $H > 0$ there exists pairwise distinct $a, b, c \in X$ such that $d(a, b) \leq d(b, c)$ and $|a' - b'| > H|b' - c'|$ where

$a' = f(a), b' = f(b), c' = f(c)$. $b', c' \in B(b', 1.5|b' - c'|)$. Since Ω is linearly locally connected with $\lambda = 1$ (Proposition 4.1.3), we have that there exists a continua $E' \subset \Omega$ connecting b' and c' such that

$$\operatorname{diam}(E') \leq 3|b' - c'|.$$

Since $\delta \leq 1$ we have $B(0, \delta/2), B(1, \delta/2), B(\infty, \delta/2)$ are pairwise disjoint: b' must fail to be in two of them. Moreover, $B(a_0, \delta/2), B(a_1, \delta/2), B(a_\infty, \delta/2)$ are pairwise disjoint, so a must fail to be in two of them. Thus there exists a $u' \in \{0, 1, \infty\}$ such that $|u' - b'| \geq \delta/2$ and $d(a_{u'}, a) \geq \delta/2$.

Since $|a' - b'| \leq 2$, we have

$$\frac{\delta}{4}|a' - b'| \leq \frac{1}{2}\delta \leq |u' - b'|.$$

So $u' \notin B(b', \delta/4|a' - b'|)$. Also, since $\delta \leq 1$, $a' \notin B(b', \delta/4|a' - b'|)$. We can use linear local connectivity (Proposition 4.1.3) to say there is an $F' \subset \Omega$ connecting u' and a' such that

$$F' \cap B[b', \frac{\delta}{4}|a' - b'|] = \emptyset.$$

If $H \geq 24/\delta$ then we have

$$\frac{\delta}{4}|a' - b'| > 6|b' - c'|.$$

So we have $E' \subset B(b', 3|b' - c'|) \subset B(b', \delta/4|a' - b'|)$, and hence

$$d(E', F') \geq \frac{\delta}{4}|a' - b'| - 3|b' - c'| > \frac{\delta}{4}|a' - b'| - \frac{\delta}{8}|a' - b'| > \frac{H\delta}{8}|b' - c'|.$$

Since we have $\min(\operatorname{diam}(E'), \operatorname{diam}(F')) \leq 3|b' - c'|$, we can say

$$\Delta(E', F') \geq \frac{\delta}{24}H.$$

Now let $E = f^{-1}(E')$ and $F = f^{-1}(F')$. Notice E and F are disjoint continua and

$b, c \in E$, $a, a_{u'} \in F$. Observe that

$$\operatorname{diam}(F) \geq d(a, a_{u'}) \geq \frac{\delta}{2} \geq \frac{\delta}{2} \frac{\operatorname{diam}(E)}{\operatorname{diam}(X)}.$$

Since $\delta \leq \operatorname{diam}(X)$, we have $2\operatorname{diam}(X)/\delta \geq 1$. Thus

$$d(E, F) \leq d(a, b) \leq d(b, c) \leq \operatorname{diam}(E) \leq \frac{2\operatorname{diam}(X)}{\delta} \min(\operatorname{diam}(E), \operatorname{diam}(F)).$$

Thus

$$\Delta(E, F) \leq \frac{2\operatorname{diam}(X)}{\delta}.$$

Take H large enough to apply Lemma 4.1.2 in Ω. Letting $\mathcal{K}'$ and Φ be as in the lemma, we obtain $\mathcal{J}' \subset \mathcal{K}'$ with $\#(\mathcal{J}') \leq 2$, such that

$$\operatorname{mod}_{\partial_0\Omega}(\Gamma(E', F'; \overline{\Omega} \setminus J')) = \operatorname{mod}_{\mathcal{K}'}(\Gamma(E', F'; \hat{\mathbb{C}} \setminus J')) \leq \Phi(\Delta(E', F')) \leq \Phi(H(\delta/24)).$$

Since $f : \overline{X}_{\partial_0 X} \to \overline{\Omega}_{\partial_0\Omega}$ is a homeomorphism, we can define $\mathcal{J}$ as the set of connected components of $J := \pi^{-1}_{\partial_0 X}(f^{-1}(\pi_{\partial_0\Omega}(J' \cap \overline{\Omega})))$. Notice $\#(\mathcal{J}) \leq 2$. Since X is 2-transboundary Loewner, say with decreasing function Ψ, we can say

$$\operatorname{mod}_{\partial_0 X}(\Gamma(E, F; \overline{X} \setminus J)) \geq \Psi(\Delta(E, F)) \geq \Psi(2\operatorname{diam}(X)/\delta).$$

Now use Lemma 3.1.3 to say

$$\operatorname{mod}_{\partial_0 X}(\Gamma(E, F; \overline{X} \setminus J)) \leq Q \operatorname{mod}_{\partial_0\Omega}(\Gamma(E', F'; \overline{\Omega} \setminus J')),$$

and thus, for all sufficiently large H,

$$0 < \Psi\left(\frac{2\operatorname{diam}(X)}{\delta}\right) \leq Q\Phi\left(H\frac{\delta}{24}\right).$$

The above inequality must fail for sufficiently large H because $\lim_{t\to\infty} \Phi(t) = 0$; hence, we have a contradiction and f must be weakly quasisymmetric. In fact, since the constant from Proposition 4.1.2 and the function Φ are entirely independent of X, Ω, and f, we can say H depends only on Q, Ψ, and $\operatorname{diam}(X)$.

Notice any subset of $\hat{\mathbb{C}}$ is doubling. Since X is doubling, we can apply Lemma 2.1.14. We have that f must be η-quasisymmetric where η depends on H and the doubling constant. $\square$

We will now use Ikonen[6] to obtain a quasiconformal map through local reciprocality. We will also need to assume LLC-1 to extend the map to the quotient spaces.

Theorem 4.3.2. *Suppose (X, d) is a metric space with $\mathcal{H}^2$ locally finite. Let $\overline{X}$ be compact and $\partial_0 X$ countable. Suppose X is (metric) doubling, LLC-1, homeomorphic to a domain in $\hat{\mathbb{C}}$, and locally reciprocal. If X is 2-transboundary Loewner with decreasing function Ψ, then X is η-quasisymmetric to a circle domain in $\hat{\mathbb{C}}$, where η depends only on Ψ, $\operatorname{diam}(X)$, and the doubling constant.*

Proof. Use the fact that X is locally reciprocal and Theorem 4.2.4 to say that X is $\pi/2$-quasiconformal to a Riemannian surface. Koebe ([42], Theorem 11C) proved that any Riemannian surface homeomorphic to a domain in $\hat{\mathbb{C}}$ is conformal to a domain in $\hat{\mathbb{C}}$. Since X is LLC-1 and $\overline{X}$ is compact, by Lemma 4.2.7, this domain must have countably many boundary components. Thus we use Theorem 1.2.5 to say it is conformal to a circle domain. Thus we obtain a (geometrically) $\pi/2$-quasiconformal map $f : X \to \Omega$ where $\Omega \subset \hat{\mathbb{C}}$ is a countably connected circle domain. Moreover, by use of Lemma 4.2.7, we can say f extends to be a homeomorphism $f : \overline{X}_{\partial_0 X} \to \overline{\Omega}_{\partial_0 \Omega}$. By Lemma 4.3.1, we have that f is η-quasisymmetric with η depending only on Ψ, $\operatorname{diam}(X)$, and the doubling constant. $\square$

Uniformization results for metric surfaces with finitely many boundary components given by Rajala-Rasimus[28] and Merenkov-Wildrick[2] have η depend on the number of boundary components; it's worth pointing out that this result has no such dependency. However, there is a hidden dependency on the relative distance between boundary components, which is revealed in the following necessary conditions for a metric space to be 2-transboundary Loewner.

Corollary 4.3.3. *Suppose (X,d) is a metric space with $\mathcal{H}^2$ locally finite. Let $\overline{X}$ be compact and $\partial_0 X$ countable. Suppose X is metric doubling, LLC-1, homeomorphic to a domain in $\hat{\mathbb{C}}$, and locally reciprocal. If X is 2-transboundary Loewner, then X is N-transboundary Loewner for all N and $\partial_0 X$ consists of uniformly relatively separated uniform quasicircles or points.*

Proof. By Theorem 4.3.2, we have a quasisymmetric and geometrically quasiconformal map to a circle domain in $\hat{\mathbb{C}}$. This will imply that $\partial_0 X$ consists of uniform quasicircles or points. Moreover, by Proposition 3.4.2, we have that the circle domain is 2-transboundary Loewner. Use Example 3.4.7 to say the circle domain is N-transboundary Loewner for all N and that its bounding circles are uniformly relatively separated. Use Proposition 3.4.2 and Corollary 2.1.10 to deduce X is N-transboundary Loewner for all N and $\partial_0 X$ is uniformly relatively separated. □

Thus, in this context, 2-transboundary Loewner is equivalent to being N-transboundary Loewner for all N. So if $\partial_0 X$ is finite, we have X is 2-transboundary Loewner if and only if it is Loewner, although the function making X Loewner will depend on the number of boundary components. This also reduces greatly in the case of domains in the sphere, so that we can say the following.

Corollary 4.3.4. *Let $\Omega \subset \hat{\mathbb{C}}$ be a countably connected domain. Then Ω is 2-transboundary Loewner if and only if its complementary components are uniformly relatively separated uniform quasidisks or points.*

Proof. Every domain in $\hat{\mathbb{C}}$ will have $\mathcal{H}^2$ locally finite, $\overline{\Omega}$ compact, be metric doubling, and locally reciprocal. Also, looking at the discussion following Lemma 4.2.7, we can always extend homeomorphisms between spherical domains to a homeomorphism between the quotient spaces, so we don't need LLC-1. If Ω is 2-transboundary Loewner, by Corollary 4.3.3, we have that the complementary components are uniformly relatively separated uniform quasidisks or points. The converse is given by Proposition 3.4.4. □

Merenkov and Wildrick[2] showed that the boundary components of a metric space being uniformly relatively separated uniform quasicircles is insufficient to conclude quasisymmetric equivalence to a circle domain (see Example 3.2.3); despite the fact, which Bonk[1] showed, that it is sufficient for domains in $\hat{\mathbb{C}}$. However, the counterexample they gave fails to be transboundary Loewner, and hence it fails to be 2-transboundary Loewner. Thus, the above result shows that the 2-transboundary Loewner property is, in some sense, an appropriate perspective for generalizing Bonk's sufficient condition to metric spaces.

Corollary 4.3.5. *Suppose (X, d) is a metric space homeomorphic to a domain in $\hat{\mathbb{C}}$ with $\mathcal{H}^2$ locally finite and $\partial_0 X$ countable. Suppose X is locally reciprocal. Then X is quasisymmetric to a circle domain with uniformly relatively separated bounding circles if and only if*

- *$\overline{X}$ is compact,*
- *X is (metric) doubling,*
- *X is LLC-1, and*
- *X is 2-transboundary Loewner.*

Proof. Use Theorem 4.3.2 and Corollary 4.3.3 to conclude that any X satisfying the assumptions is quasisymmetric to a circle domain with uniformly relatively separated boundary. For the other direction, notice that uniformly relatively separated circle domains in $\hat{\mathbb{C}}$ have all of the properties listed (see Example 3.4.7). Use Lemma 4.2.6 to say the quasisymmetry is also geometrically quasiconformal. All of the listed properties are invariant under maps which are quasisymmetric and geometrically quasiconformal (see Corollary 2.1.5, Proposition 2.1.12, Proposition 2.1.7, and Proposition 3.4.2). □

The statement here simplifies if we additionally assume Ahlfors regularity.

Corollary 4.3.6. *Suppose (X, d) is a metric space homeomorphic to a domain in $\hat{\mathbb{C}}$ with $\partial_0 X$ countable. Suppose X is Ahlfors 2-regular. Then X is quasisymmetric to a circle domain with uniformly relatively separated bounding circles if and only if*

- *$\overline{X}$ is compact,*
- *X is LLC-1, and*

- *X is 2-transboundary Loewner.*

Proof. Observe that Ahlfors regularity gives $\mathcal{H}^2$ locally finite. It also gives metric doubling, since it gives the existence of a doubling measure on X. Use Theorem 4.2.3 combined with the fact that X is homeomorphic to a domain to conclude that every point in X has a neighborhood which is reciprocal. Then apply Corollary 4.3.5 to reach the conclusion. $\square$

4.4 Examples and applications

In this section, we establish a class of examples, which we show are 2-transboundary Loewner, that hitherto were not known to be quasisymmetric to circle domains.

For each $n \in \mathbb{N}$, let

$$D_n = \{[i2^{-n}, (i+1)2^{-n}] \times [j2^{-n}, (j+1)2^{-n}] : i, j \in \mathbb{N}, 0 \leq i, j \leq 2^n - 1\};$$

that is, the collection of all dyadic squares of generation n. Let $D = \cup_{n\in\mathbb{N}} D_n$. For $Q \in D$, let $c(Q)$ denote the center of Q. Let $L = (\ell_n)_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying $0 \leq \ell_n \leq 2^{-n-1}$. For each $n \in \mathbb{N}$ and $Q \in D_n$, let

$$C_Q(L) = \left(\{0\} \times \left[-\frac{\ell_n}{2}, \frac{\ell_n}{2}\right] \cup \left[-\frac{\ell_n}{2}, \frac{\ell_n}{2}\right] \times \{0\}\right) + c(Q),$$

and

$$C_n(L) = \{C_Q(L) \mid Q \in D_n\}.$$

Thus $C_n(L)$ is a collection of 4^n disjoint "plus" signs with width and height ℓ_n, centered at the center of dyadic squares of generation n. In fact, $\Delta(C_Q(L), C_{Q'}(L)) \geq 1/2$ for all $Q, Q' \in D$. Now, for each $k \in \mathbb{N}$, define the metric space

$$T_k(L) = (0,1)^2 \setminus (\cup_{n=0}^{k} \cup_{Q\in D_n} C_Q(L)),$$

equipped $T_k(L)$ with its path metric. Notice that $\partial_0 T_k(L)$ consists of uniformly relatively

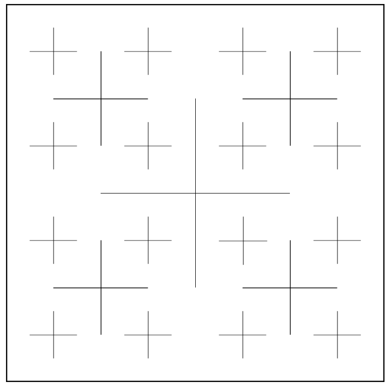

Figure 4.2: $T_3\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$

separated uniform quasicircles. There is a natural projection map $\pi_k : \overline{T_k(L)} \to [0,1]^2$ satisfying $\pi_k(x) = x$ for all $x \in T_k(L)$, and defined on $\partial T_k(L)$ so that the projection is continuous. Notice that $\pi_k|_{T_k(L)}$ is conformal, so the transboundary modulus of any family of curves with the path metric will be the same as with the Euclidean metric (Proposition 3.1.3). Also observe that π_k is 1-Lipschitz. We wish to show that T_k is 2-transboundary Loewner. We proceed with this goal in mind.

Transboundary Modulus of Dyadic Squares in $\mathbf{T_k}$

Let $\mathcal{K}_k = \cup_{n=0}^k C_n(L)$. Let $A_L = \{0\} \times [0,1]$, $A_R = \{1\} \times [0,1]$, $A_B = [0,1] \times \{0\}$, and $A_T = [0,1] \times \{1\}$. Use duality (Proposition 3.1.8) to say

$$\operatorname{mod}_{\mathcal{K}_k}(\Gamma(A_L, A_R; \overline{T_k(L)})) \operatorname{mod}_{\mathcal{K}_k}(\Gamma(A_B, A_T; \overline{T_k(L)})) = 1.$$

However, symmetry will mean that they must be equal. That is, reflecting along the diagonal maps T_k to itself and sends $\Gamma(A_L, A_R; \overline{T_k(L)})$ to $\Gamma(A_B, A_T; \overline{T_k(L)})$. Since this map is conformal, we conclude their transboundary modulus must be the same, which implies

$$\operatorname{mod}_{\mathcal{K}_k}(\Gamma(A_L, A_R; \overline{T_k(L)})) = \operatorname{mod}_{\mathcal{K}_k}(\Gamma(A_B, A_T; \overline{T_k(L)})) = 1.$$

Let Q_L, Q_R, Q_B, Q_T denote the left, right, bottom, and top sides respectively of any $Q \in D$. Let Q° denote the interior of Q. Since Q is symmetric in its interior, we can map curves connecting the vertical sides to curves connecting the horizontal sides through reflection and use duality to say

$$\operatorname{mod}_{\mathcal{K}_k}(\Gamma(Q_L, Q_R; \pi_k^{-1}(Q^\circ))) = \operatorname{mod}_{\mathcal{K}_k}(\Gamma(Q_B, Q_T; \pi_k^{-1}(Q^\circ))) = 1.$$

Chains of Dyadic Squares

We are going to use geometric objects we're calling chains to give estimates on transboundary modulus. Fix $n \in \mathbb{N}$ and $2 \leq j \leq 4^n$. An (n, j)-chain is an ordered tuple of pairwise distinct dyadic squares, $(Q_1, Q_2, ..., Q_j)$, where $Q_i \in D_n$ for all $1 \leq i \leq j$, satisfying Q_i and $Q_{i'}$ intersect at more than one point if and only if $|i - i'| \leq 1$. Thus for every square in the chain there are exactly two other squares which share a side with it, excepting Q_1 and Q_j, which share a side with exactly one other square. For any $k \in \mathbb{N}$, let

$$V := \pi_k^{-1}\bigg(\cup_{i=1}^{j} Q_j \bigg).$$

Let $V_0 \subset V$ correspond the side of Q_1 opposite $Q_1 \cap Q_2$. Let $V_j \subset V$, correspond to the side of Q_j opposite $Q_{j-1} \cap Q_j$. Notice that there is a homeomorphism mapping $\cup_{i=1}^{j} Q_j$ to $[0, 1]^2$ which sends $\pi_k(V_0)$ and $\pi_k(V_j)$ to the left and right sides. Let $V_+, V_- \subset V$ correspond to the other two sides. We would like to estimate $\operatorname{mod}_{\mathcal{K}_k}(\Gamma(V_+, V_-; V))$.

Constructing a Symmetric Transboundary Mass Distribution for Dyadic Squares

To do so, pick any $\epsilon > 0$, and let $Q = [0, 2^{-n}]^2$. Let $P = (\rho; \{\rho_{Q'}\}) \wedge_{\mathcal{K}_k} \Gamma(Q_L, Q_R; \pi_k^{-1}(Q^\circ))$ be such that $A_{\mathcal{K}_k}(P) \leq 1 + \frac{\epsilon}{j}$. Let $r_1 : Q^\circ \to Q^\circ$ be reflection across the horizontal line going through $c(Q)$. Define

$$\rho_1 = \frac{1}{2}(\rho + \rho \circ r_1)$$

on the interior of $Q \setminus K$ and

$$\rho_{1,Q'} = \frac{1}{2}(\rho_{Q'} + \rho_{r_1(Q')})$$

for $Q' \subset Q$ and $C_{Q'}(L) \in \mathcal{K}_k$. Let $P_1 = (\rho_1; \{\rho_{1,Q'}\})$. We claim $P_1 \wedge_{\mathcal{K}_k} \Gamma(Q_L, Q_R; \pi_k^{-1}(Q^\circ))$. Indeed, by conformality of reflection, we have $P(r_1) := (\rho(r_1); \{\rho_{r_1(Q')}\}) \wedge_{\mathcal{K}_k} \Gamma(Q_L, Q_R; \pi_k^{-1}(Q^\circ))$, and thus for all $\gamma \in \Gamma(Q_L, Q_R; \pi_k^{-1}(Q^\circ))$,

$$\ell_{P_1}(\gamma) = \frac{1}{2}\Big(\int_{\gamma\setminus K} \rho \; ds + \sum_{C_{Q'}(L)\cap\gamma\neq\emptyset} \rho_{Q'} \Big) + \frac{1}{2}\Big(\int_{\gamma\setminus K} \rho(r_1) \; ds + \sum_{C_{Q'}(L)\cap\gamma\neq\emptyset} \rho_{r_1(Q')} \Big) \geq 1.$$

By conformality of reflection, we have $A_{\mathcal{K}_k}(P(r_1)) = A_{\mathcal{K}_k}(P)$. Thus, using the fact that $\frac{1}{2}(a+b)^2 \leq a^2 + b^2$, we have

$$\begin{aligned}
A_{\mathcal{K}_k}(P_1) &= \frac{1}{2}\Big(\int_{Q\setminus K} \frac{1}{2}(\rho + \rho\circ r_1)^2 \; d\mathcal{H}^2 + \sum_{C_{Q'}(L)\subset Q} \frac{1}{2}(\rho_{Q'} + \rho_{r_1(Q')})^2 \Big) \\
&\leq \frac{1}{2}\Big(\int_{Q\setminus K} \rho^2 + (\rho\circ r_1)^2 \; d\mathcal{H}^2 + \sum_{C_{Q'}(L)\subset Q} \rho_{Q'}^2 + \rho_{r_1(Q')}^2 \Big) \\
&= \frac{1}{2}(A_{\mathcal{K}_k}(P) + A_{\mathcal{K}_k}(P(r_1))) \\
&= A_{\mathcal{K}_k}(P).
\end{aligned}$$

Thus, in the interior of Q, we can replace P with P_1, which is symmetric across the horizontal line going through $c(Q)$. A similar argument will show that, in the interior of Q, we can replace P_1 with a distribution P_2 which is symmetric across the vertical line going through $c(Q)$, and thus will be symmetric both horizontally and vertically.

Estimate on the Transboundary Modulus of a Chain

Now we want to construct a transboundary mass distribution, P', to estimate mod $_{\mathcal{K}_k}(\Gamma(V_+, V_-; V))$. For each Q_i in the chain, divide it into two closed right triangles, U_i^1, U_i^2, sharing a hypotenuse along a diagonal of Q_i. Ensure that, with the exception of Q_1 and Q_j, each triangle shares a side with a neighboring square in the chain. Notice, then, that each U_i^m has exactly one side contained in $V_+ \cup V_-$, and that the union of these sides is $V_+ \cup V_-$. Let U be the closed right triangle with vertices $(0,0)$, $(0, 2^{-n})$, and $(2^{-n}, 0)$. All of these triangles are congruent to each other; in particular, we can obtain an isometry $r_i^m : U_i^m \to U$ which sends $(V_+ \cup V_-) \cap U_i^m$ to the left side of U. Define P' in the interior of any triangle U_i^m to be $P_2 \circ r_i^m$. On the boundary of U_i^m, define ρ' to be ∞. If $C_{Q'}(L)$ intersects V at more than one point and is not contained in Q_i for some i, define $\rho'_{Q'} = 1$.

To show $P' \wedge_{\mathcal{K}_k} \Gamma(V_+, V_-; V)$, pick $\gamma \in \Gamma(V_+, V_-; V)$. If γ intersects some $C_{Q'}(L) \in \mathcal{K}_k$ intersecting V at more than one point with $Q' \in D_m$, $m < n$, then $\ell_{P'}(\gamma) \geq \rho'_{Q'} = 1$. Suppose that γ doesn't intersect any such boundary components. If γ satisfies its intersection with the boundary of some U_i^m has positive length, then $\ell_{P'}(\gamma) = \infty \geq 1$. Suppose then, that all the length of γ is to be found in the interiors of the triangles U_i^m. Consider its image in U via r_i^m; that is, let

$$\gamma' = \bigcup_{i=1}^{j} r_i^1(\gamma \cap U_i^1) \cup \bigcup_{i=1}^{j} r_i^2(\gamma \cap U_i^2).$$

$\gamma' \setminus K$ will consist of countably many curves connecting the boundary of U to itself, to K, or connecting K to itself. For each of the subcurves, there is a suitable reflection so that the curve obtained from applying all these reflections connects the left and right sides of $Q_0 := [0, 2^{-n}]^2$. Indeed, each such subcurve, γ_0', of γ' will have an associated γ_0, subcurve of γ, which is contained in some U_i^m. Since r_i^m is an isometry, we obtain, for all t,

$$d(\gamma_0(t), V_+ \cup V_-) = d(\gamma_0'(t), Q_0^L \cup Q_0^R),$$

where $\gamma_0'(t) = r_i^m(\gamma_0(t))$. Moreover, if necessary, we can reflect γ_0' over $x = 2^{-n-1}$ to obtain γ_0'' satisfying

$$d(\gamma_0(t), V_-) = d(\gamma_0''(t), Q_0^L)$$

and

$$d(\gamma_0(t), V_+) = d(\gamma_0''(t), Q_0^R),$$

for all t. Additionally consider γ_1', another such subcurve of γ' with corresponding subcurve γ_1 of γ. Take $\gamma : J \to V_{\mathcal{K}_k}$ to not be constant on any non-trivial interval, where J is an interval. We obtain disjoint subintervals $J_0, J_1 \subset J$ with $\gamma_0 : J_0 \to V_{\mathcal{K}_k}$ and $\gamma_1 : J_1 \to V_{\mathcal{K}_k}$. Suppose $J_0 \cup J_1$ is connected, and that $\inf(J_0) \leq \inf(J_1)$. If γ_0' ends in the boundary of U ($\gamma_0(\sup(J_0)) \in \partial U$), then γ_0 and γ_1 must share an endpoint, and this endpoint has two corresponding points in γ_0'' and γ_1'' which must fall on the same vertical line since they share the same distance to Q_0^L and Q_0^R. If γ_0'' and γ_1'' don't share an endpoint, then reflect γ_1'' over

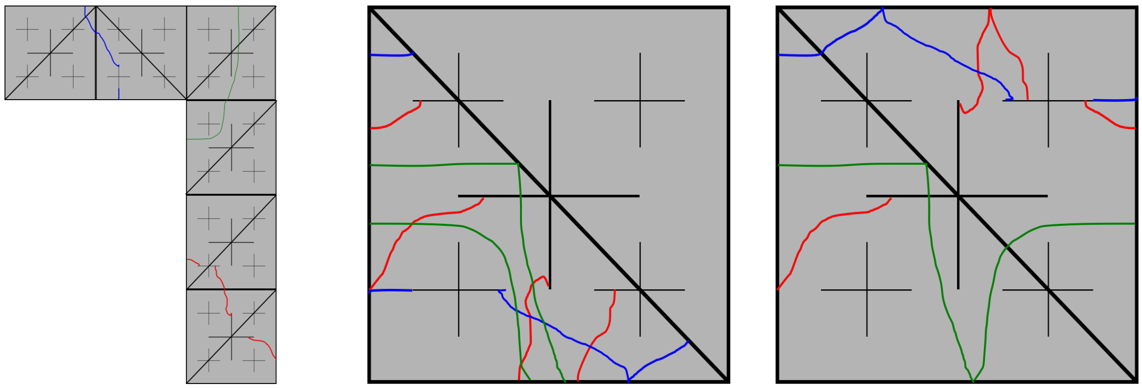

Figure 4.3: *Transboundary paths connecting V_+ and V_- in a chain, their reflection into U (middle), and the adjustment to a horizontal path (right)*

$y = 2^{-n-1}$ and call this curve γ_1'''. γ_1''' and γ_0'' must share an endpoint now, as the symmetry of the U_i^m means that any point falling in two of the triangles, when mapped into two points in U, must have some vertical or horizontal reflection making those two points the same. Now if γ_0' ends in K, then that means γ_0 and γ_1 have an endpoint in the same $C_Q(L)$. Once again, if γ_0'' and γ_1'' don't have corresponding endpoints in the corresponding boundary component, then reflect γ_1'' vertically to obtain γ_1'''. γ_0'' will end at the same boundary component γ_1''' begins in, because of the symmetry. In this way we obtain a transboundary curve γ'''. Notice also that $d(\gamma'''(t), Q_0^L) = d(\gamma(t), V_-)$ and $d(\gamma'''(t), Q_0^R) = d(\gamma(t), V_+)$ for all t. Thus γ''' connects the left and right sides of Q_0. Since P_2 in invariant under horizontal and vertical reflections, and since each r_i^m is conformal we can say $\ell_{P'}(\gamma) = \ell_{P_2}(\gamma''') \geq 1$.

If we use this transboundary mass distribution to estimate the transboundary modulus. Notice that the boundary of U_i^m has 0 area, so the integral of $\rho' = \infty$ over that set is 0. Recall also that if $C_{Q'}(L)$ intersected V at one point, and Q' isn't contained in any Q_i, then we defined $\rho'_{Q'} = 1$; however, since $\ell_{i'} \leq 2^{-i'-1}$ for all i', we can say each Q_i intersects at most one such $C_Q'(L)$. Since P_2 is symmetric vertically and horizontally, we can say its integral over U is exactly half its integral over Q_0. Thus,

$$\begin{aligned}\operatorname{mod}_{\mathcal{K}_k}(\Gamma(V_+, V_-; V)) &\leq A_{\mathcal{K}_k}(P') \\ &= \sum_{i=1}^{j} \int_{(Q_i^\circ)_{\mathcal{K}_k}} \rho_{P'}^2 d\mu_{\mathcal{K}_k} + \sum_{C_{Q'}(L) \not\subseteq V} \rho'_{Q'}\end{aligned}$$

$$\leq \sum_{i=1}^{j}(1+\epsilon/j) + \sum_{C_{Q'}(L)\not\subseteq V} 1$$
$$= (j+\epsilon) + j = 2j+\epsilon$$

Since this holds for all ϵ, we can say

$$\operatorname{mod}_{\mathcal{K}_k}(\Gamma(V_+, V_-; V)) \leq 2j.$$

Claim: $\mathbf{T_k(L)}$ is transboundary Loewner.

Now we will show that $T_k(L)$ is transboundary Loewner. Pick any disjoint, non-degenerate continua $E, F \subset T_k(L)$ with $\operatorname{diam}(E) \leq \operatorname{diam}(F)$. Fix $2^{-100}\operatorname{diam}(E) > \epsilon > 0$ Let γ be a curve connecting E and F in $T_k(L)$ with $\ell(\gamma) \leq d(E,F)+\epsilon$; such a path exists because T_k is equipped with the path metric. Without loss of generality, suppose that γ intersects E and F exactly once, and that $e \in \gamma \cap E$ and $f \in \gamma \cap F$ with $d(e,f) = d(E,F)$.

Constructing the Chain Near E and F

For the moment, consider $E \subset \pi_k(T_k)$ recalling that π_k is 1-Lipschitz. Notice that $E \cap B[e, \operatorname{diam}(E)/2]$ must connect e to $T_k \setminus B(e, \operatorname{diam}(E)/2)$; say $e' \in \partial B(e, \operatorname{diam}(E)/2)$ is in the same connected component of $E \cap B[e, \operatorname{diam}(E)/2]$ as e. Let n be the largest integer satisfying $2^{-n} \geq \operatorname{diam}(E)/8$; this will imply $2^{-n} < \operatorname{diam}(E)/4$. Notice that

$$\max(|\pi_1(e) - \pi_1(e')|, |\pi_2(e) - \pi_2(e')|) \geq \frac{\operatorname{diam}(E)}{2\sqrt{2}} > \frac{\operatorname{diam}(E)}{4} > \frac{1}{2^n}.$$

Hence, there is some dyadic interval $J := [j'2^{-n}, (j'+1)2^{-n}]$ satisfying $E \cap B[e, \operatorname{diam}(E)/2]$ connects the left and right sides of $\pi_1^{-1}(J)$ or the top and bottom sides of $\pi_2^{-1}(J)$. Thus there is a horizontal or vertical (n, j_E)-chain, whose interior doesn't contain e nor e', with $E \cap B[e, \operatorname{diam}(E)/2]$ connecting its longer sides, and $j_E \leq 8$. Call the union of the squares in the chain V^E. A similar analysis on $B[f, \operatorname{diam}(E)/2]$ gives the existence of a vertical or horizontal (n, j_F)-chain with $F \cap B[f, \operatorname{diam}(E)/2]$ connecting its longer sides, and $j_F \leq 8$. Call the the region the chain spans V^F.

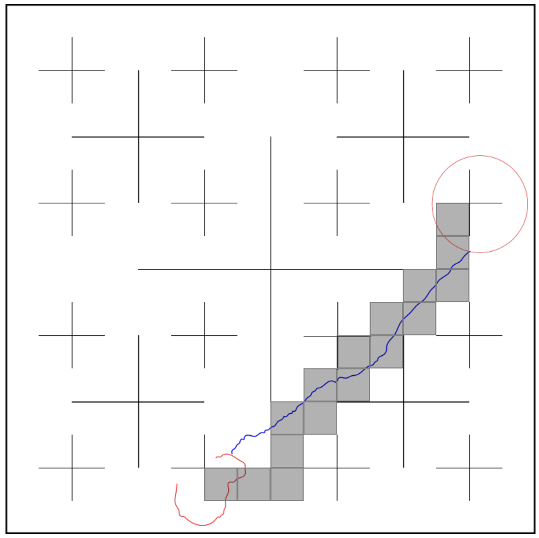

Figure 4.4: *A chain connecting continua (in red) around a path connecting them (in blue)*

Case 1: E and F are Relatively Close

Suppose $\Delta(E,F) \leq 2$. If $V^E \cap V^F \neq \emptyset$, then it contains a chain, V, satisfying $\Gamma(V_0, V_1; V) < \Gamma(E, F; \overline{T_k(L)})$. We use the overflowing property and duality to say

$$\mathrm{mod}_{\mathcal{K}_k}(\Gamma(E,F;\overline{T_k(L)})) \geq \mathrm{mod}_{\mathcal{K}_k}(\Gamma(V_0,V_1;V)) = \frac{1}{\mathrm{mod}_{\mathcal{K}_k}(\Gamma(V_+,V_-;V))} \geq \frac{1}{32}.$$

If $V^E \cap V^F = \emptyset$, then we can form a chain V containing both of them. Indeed, using at most 14 squares (7 perpendicular to V^E and 7 parallel to V^E), we can extend V^E past e to intersect γ. Similarly, we can extend V^F to intersect γ by adding at most 14 squares. If V^E and V^F still don't intersect, then we can use the fact that

$$\ell(\gamma) \leq d(E,F) + \epsilon \leq 2\mathrm{diam}(E) + \epsilon,$$

we can extend V^E to reach V^F using at most 32 squares in D_n. Thus we build a chain, V, containing V^E and V^F, that has at most 76 links. Thus we can argue, much like before,

$$\mathrm{mod}_{\mathcal{K}_k}(\Gamma(E,F;\overline{T_k(L)})) \geq \mathrm{mod}_{\mathcal{K}_k}(\Gamma(V_0,V_1;V)) = \frac{1}{\mathrm{mod}_{\mathcal{K}_k}(\Gamma(V_+,V_-;V))} \geq \frac{1}{152}.$$

Case 2: E and F are Relatively Distant

Now suppose $\Delta(E, F) > 2$. As above, we extend V^E and V^F to intersect γ, and we do so adding at most 14 squares each. Like above, we want a chain, V, containing V^E and V^F, as any such chain will allow us to use the overflowing property to estimate the transboundary modulus. We will do so by looking at all of the squares in D_n which intersect γ. Any subcurve, γ', of γ with $2^{-n-1} \leq \ell(g') < 2^{-n}$ touches at most 4 squares in D_n. We can divide γ into at most $2^{n+1}\ell(\gamma)$ such subcurves which are disjoint. Hence, the number of dyadic squares γ touches is at most

$$2^{n+3}(d(E, F) + \epsilon) \leq \frac{64}{\text{diam}(E)}(d(E, F) + \epsilon) \leq 64\Delta(E, F) + 2^{-94}.$$

Every connected collection of squares in D_n will contain a chain. Combining this with V^E and V^F will yield a chain, V, with at most $64\Delta(E, F) + 45$ links. Moreover, since E and F connect V_+ and V_-, we can say $\Gamma(V_0, V_1; V) < \Gamma(E, F; \overline{T_k(L)})$. Use overflowing, duality, and the estimate on chains to say

$$\text{mod}_{\mathcal{K}_k}(\Gamma(E, F; \overline{T_k(L)})) \geq \text{mod}_{\mathcal{K}_k}(\Gamma(V_0, V_1; V)) = \frac{1}{\text{mod}_{\mathcal{K}_k}(\Gamma(V_+, V_-; V))} \geq \frac{1}{128\Delta(E, F) + 90}.$$

Thus, T_k is transboundary Loewner with function $\Psi(t) = \min(1/152, (128t + 90)^{-1})$.

Claim: $\mathbf{T_k(L)}$ is 1-transboundary Loewner.

We would like to upgrade this statement to 1-transboundary Loewner. If one chooses the bounding square as the boundary component to avoid, it will likely not intersect the chain we constructed above. If it does, the transboundary modulus of curves in the chain avoiding it doesn't change, so the same estimate as above will work. Pick any $C_Q(L) \in \mathcal{K}_k$. Let E, F be as above. If the chain we constructed, V, avoids $C_Q(L)$, then we can use the same estimate. Suppose $C_Q(L)$ intersects V at more than one point.

Case 1: the side length of Q is at most $\mathbf{2^{-n}}$.

If the side length of Q is at most 2^{-n}, then $C_Q(L)$ is contained in one of the links of V, say Q_i. Notice each square in V can be subdivided into 16 squares from D_{n+2}. For Q_i, either the collection of these subsquares which touch V_+ or the collection of these subsquares which touch V_- won't touch $C_Q(L)$, without loss of generality suppose it is V_+. Replace V with all of the squares in D_{n+2} contained in V, which touch V_+. This will still be a chain and it will have at most 7 times the links that V had. Hence, the transboundary modulus of $\Gamma(E, F; \overline{T_k} \setminus C_Q(L))$ will be bounded below by Ψ' where $\Psi' = \Psi/7$.

Case 2: the side length of Q is greater than $\mathbf{2^{-n}}$.

Consider the collection of squares in D_n touching γ as well as those in V^E and V^F. Recall that we built V from this collection. Since the side length of Q is more than 2^{-n}, every square in D_n which intersects $C_Q(L)$ at more than a point has a side contained in it, with at most 8 exceptions: the dyadic squares containing the tips of $C_Q(L)$. We will call those squares the exceptional squares. If there are no exceptional squares in V^E or V^F, then both of these will contain a long side not touching $C_Q(L)$ at more than a point. Divide each square into 4 subsquares in D_{n+1} and replace the square with the two subsquares touching the side not intersecting $C_Q(L)$. Observe that if the short sides intersect $C_Q(L)$, then the transboundary modulus of curves in the chain avoiding $C_Q(L)$ doesn't change. We note that V^E and V^F can always be chosen to avoid given exceptional squares. For non-exceptional squares intersecting γ, divide them into 4 subsquares and throw out all the ones touching $C_Q(L)$. This will still be connected since γ doesn't touch $C_Q(L)$. For exceptional squares intersecting γ, look at the dyadic squares in D_n which are opposite $C_Q(L)$ but adjacent to the exceptional square; such a square exists since $\ell_{i'} \leq 2^{-i'-1}$ for all i'. Subdivide that into 4 squares, and take the two squares neighboring the exceptional square. Add two squares on either side to connect them to the chain. This collection of squares will contain a chain, V, with $\Gamma(E, F; \overline{T_k} \setminus C_Q(L)) < \Gamma(V_0, V_1; V)$. The number of links will, at most, be 16 more than triple the number of squares we had. Thus for $\Psi' = 1/(3/\Psi + 32)$ will bound the transboundary modulus from below. We get that T_k is 1-transboundary Loewner.

Claim: $\mathbf{T_k(L)}$ is 2-transboundary Loewner.

Let us now show that T_k is, in fact, 2-transboundary Loewner. If one of the boundary components chosen to avoid is the bounding square, then the transboundary modulus of curves in the chain doesn't change, and one can use the chain obtained from 1-transboundary

Loewner to avoid the second boundary component chosen. Let E, F be as above and pick any $C_Q(L), C_{Q'}(L) \in \mathcal{K}_k$. Consider the collection of squares touching γ as well as the squares in V^E and V^F. If either $C_Q(L)$ or $C_{Q'}(L)$ fail to intersect this collection, then we can use the same estimate we got from 1-transboundary Loewner, so suppose both of them intersect it at more than one point.

Case 1: the side length of Q or Q' is at most $\mathbf{2^{-n}}$.

If both Q and Q' have side lengths less than or equal to 2^{-n}, we can say they are contained in squares in our collection. If they are contained in the same square, we can remove that square from our chain and add at most 4 more to reconnect the chain avoiding that square. If they are in different squares, we can do that twice and add at most 8 squares to our collection. Thus, we can bound $\mathrm{mod}_{\mathcal{K}_k}(\Gamma(E, F; \overline{T_k} \setminus (C_Q(L) \cup C_{Q'}(L))))$ from below by $\Psi'' = 1/(1/\Psi + 16)$.

If only one of them has side length at most 2^{-n}, then (much like we did with 1-transboundary Loewner), chop each square into 16 subsquares from D_{n+2}; take the ones touching the side of the chain which avoid the small boundary component. Then our collection of squares is increased by at most 7-fold and only intersects at most one boundary component that ought to be avoided: use the construction from 1-transboundary Loewner to avoid that component. We conclude a lower bound with $\Psi'' = 1/(7/\Psi + 16)$.

Case 2: the side length of both Q and Q' are greater than $\mathbf{2^{-n}}$.

The final case to consider is when both Q and Q' have side lengths greater than 2^{-n}. This argument is very similar to the 1-transboundary Loewner case applied twice. By cutting the non-exceptional squares into fourths, one can avoid one of the boundary components, and by cutting into fourths again, we can avoid the other one. This works because γ avoids both $C_Q(L)$ and $C_{Q'}(L)$. This will increase the number of squares by at most a factor of 9. Also, this time, we can have up to 16 exceptional squares. We deal with them the same by moving to an adjacent square and adding two more; though this time we remark that it is important that the interior of the square adjacent to a square exceptional for $C_Q(L)$ won't intersect $C_{Q'}(L)$, so that the smaller squares still avoid both. This is only true because $\ell_{i'} \leq 2^{-i'-1}$ for all i'; in particular, because the boundary components are uniformly relatively separated. In conclusion, we triple and add 16, then triple again and add 16, and conclude a lower bound with $\Psi'' = 1/(9/\Psi + 128)$. We conclude that T_k is 2-transboundary Loewner with Ψ'' entirely independent of k. The spaces T_k are uniformly 2-transboundary Loewner.

Apply the main result to examples built from $\mathbf{T_k}(\mathbf{L})$.

We claim that $\operatorname{diam}(T_k) \leq 4$ for all k. Indeed, any point in T_k can be connected to the outer square via a piecewise linear path, consisting of line segments parallel to the sides of the square, with π_1 injective outside the vertical segments and π_2 injective outside the horizontal segments. For a point, p, with $\pi_k(p) = (x, y)$ such a path can be constructed to have length less than or equal to $\min(x, 1-x) + \min(y, 1-y) \leq 1$. Since any two points on the outer square can be connected by a curve of length 2, we get that for any $p, q \in T_k$, $d(p, q) \leq 4$.

Let $k \geq k'$. Define $\pi_{k,k'} : \overline{T_k(L)} \to \overline{T_{k'}(L)}$ be defined as $\pi_{k,k'}(z) = z$ for $z \in T_k(L)$, and defined on ∂T_k to be continuous. This is a natural projection which will allow us to define an inverse limit of $\left(\overline{T_k(L)}, \pi_{k,k'}\right)$. That is

$$\mathcal{T}(L) := \{(p_0, p_1, ...) \mid p_k \in T_k(L), p_{k+1} = \pi_{k+1,k}(p_k)\}.$$

Since $\pi_{k,k'}$ is 1-Lipschitz, the sequence $(d_{T_k}(p_k, q_k))$ is monotonic. Since it is bounded above by 4, we can say that it converges; hence, we can define a metric on $\mathcal{T}(L)$ as the limit of the metrics of $T_k(L)$.

Example 4.4.1. *$\mathcal{T}(L)$ is quasisymmetric to a round carpet in $\hat{\mathbb{C}}$ provided that $\ell_n \leq 2^{-n-1}$.*

Proof. We use Theorem 4.3.2 to say that T_k is η-quasisymmetric to a circle domain in $\hat{\mathbb{C}}$. However, for all k, T_k has the same doubling constant and upper bound on diameter. Also, each T_k is 2-transboundary Loewner with the same function Ψ. Therefore, the T_k are all quasisymmetric to circle domains with the same η. Keith and Laakso proved (Lemma 2.4.7[43]) that with suitable normalizations, these uniformly quasisymmetric maps will have a subsequence converge to a quasisymmetry $f : \mathcal{T}(L) \to \mathcal{C}$ where $\mathcal{C} \subset \hat{\mathbb{C}}$. The boundary components of $\mathcal{C}$ will all be circles. Moreover, these circles will bound disjoint regions,

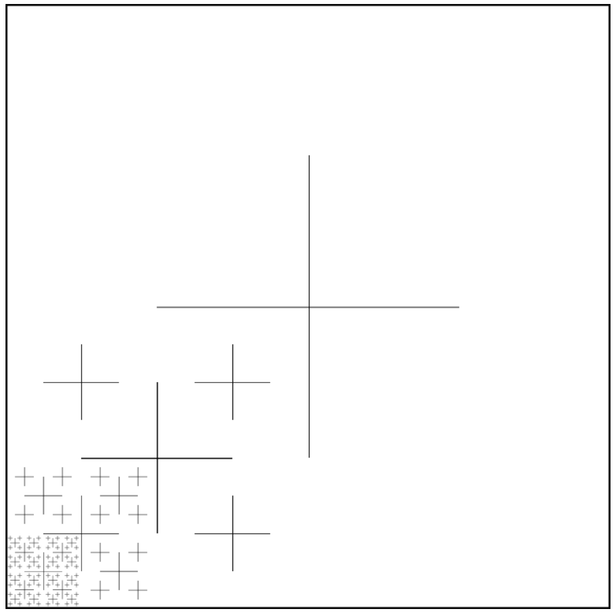

Figure 4.5: $T\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$

they will be dense in $\hat{\mathbb{C}}$, and their diameters will go to 0. By Whyburn's Theorem[44], $\mathcal{C}$ is homeomorphic to the Sierpiński carpet. Thus it is a round carpet. $\square$

We note here that the previous example can be shown through the following alternative method. First one can use a result of Keith-Laakso[43] to say that the conformal dimension of $\mathcal{T}(L)$ is less than 2. Then one can use a result of Haissinsky[45] to show that it can be quasisymmetrically embedded in $\hat{\mathbb{C}}$. Then one can use a result of Bonk[1] (Theorem 1.5.3) to say it is quasisymmetric to a round carpet. However, the following example is inaccessible by this method.

Define the set

$$T(L) = \bigcap_{k=0}^{\infty} \left((2^{-k}T_k(L)) \cup (0, 2^{-k-1})^2 \cup ((0,1)^2 \setminus (0, 2^{-k})^2) \right).$$

In other words, for each k, scale $T_k(L)$ down to a square of side length 2^{-k}, and place it in the lowest, leftmost dyadic square. Then fill in the open bottom left quarter of that square, and repeat for $k+1$. Equip $T(L)$ with its path metric, and notice that it is homeomorphic to a domain in the plane.

Example 4.4.2. *$T(L)$ is 2-transboundary Loewner as long as $\ell_n \leq 2^{-n-1}$.*

Proof. Pick any disjoint continua $E, F \subset T(L)$ and $C_Q(L)$, $C_{Q'}(L)$. In the same way as T_k, we can construct a chain connecting them which avoids $C_Q(L)$ and $C_{Q'}(L)$ with number of links comparable to the relative distance. Symmetry and duality give that the transboundary modulus of curves connecting opposite sides of a dyadic square is 1, and thus the chain will have transboundary modulus of curves connecting opposite sides bounded above by twice the number of links. In the same way as the T_k, we use the overflowing property to get a lower bound on the modulus.

□

This means that $T(L)$ is quasisymmetric to a circle domain, by Corollary 4.3.6. If $\sum_n (2^n \ell_n)^2 = \infty$, then this is an example where Theorem 1.5.6 doesn't apply. In fact, one can use Theorem 1.2.5 to say that it is conformal to a circle domain. Lemma 4.3.1 will imply that this conformal map is quasisymmetric.

Bibliography

[1] Mario Bonk. Uniformization of Sierpiński carpets in the plane. *Invent. Math.*, 186 (3):559–665, 2011. ISSN 0020-9910. doi: 10.1007/s00222-011-0325-8. URL https://doi.org/10.1007/s00222-011-0325-8.

[2] Sergei Merenkov and Kevin Wildrick. Quasisymmetric Koebe uniformization. *Rev. Mat. Iberoam.*, 29(3):859–909, 2013. ISSN 0213-2230. doi: 10.4171/RMI/743. URL https://doi.org/10.4171/RMI/743.

[3] Juha Heinonen and Pekka Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, 181(1):1–61, 1998. ISSN 0001-5962. doi: 10.1007/BF02392747. URL https://doi.org/10.1007/BF02392747.

[4] Oded Schramm. Transboundary extremal length. *J. Anal. Math.*, 66:307–329, 1995. ISSN 0021-7670. doi: 10.1007/BF02788827. URL https://doi.org/10.1007/BF02788827.

[5] Kai Rajala. Uniformization of two-dimensional metric surfaces. *Invent. Math.*, 207 (3):1301–1375, 2017. ISSN 0020-9910. doi: 10.1007/s00222-016-0686-0. URL https://doi.org/10.1007/s00222-016-0686-0.

[6] Toni Ikonen. Uniformization of metric surfaces using isothermal coordinates. *Ann. Fenn. Math.*, 47(1):155–180, 2022. ISSN 2737-0690.

[7] Mario Bonk. Quasiconformal geometry of fractals. In *International Congress of Mathematicians. Vol. II*, pages 1349–1373. Eur. Math. Soc., Zürich, 2006.

[8] P. Tukia and J. Väisälä. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):97–114, 1980. ISSN 0066-1953. doi: 10.5186/aasfm.1980.0531. URL https://doi.org/10.5186/aasfm.1980.0531.

[9] Mario Bonk and Bruce Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.*, 150(1):127–183, 2002. ISSN 0020-9910. doi: 10.1007/s00222-002-0233-z. URL https://doi.org/10.1007/s00222-002-0233-z.

[10] G. D. Mostow. Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. *Inst. Hautes Études Sci. Publ. Math.*, (34):53–104, 1968. ISSN 0073-8301. URL http://www.numdam.org/item?id=PMIHES_1968__34__53_0.

[11] James W. Cannon. The combinatorial Riemann mapping theorem. *Acta Math.*, 173(2): 155–234, 1994. ISSN 0001-5962. doi: 10.1007/BF02398434. URL https://doi.org/10.1007/BF02398434.

[12] Dennis Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 465–496. Princeton Univ. Press, Princeton, N.J., 1981.

[13] Michael Kapovich and Bruce Kleiner. Hyperbolic groups with low-dimensional boundary. *Ann. Sci. École Norm. Sup. (4)*, 33(5):647–669, 2000. ISSN 0012-9593. doi: 10.1016/S0012-9593(00)01049-1. URL https://doi.org/10.1016/S0012-9593(00)01049-1.

[14] Paul Koebe. Abhandlungen zur Theorie der konformen Abbildung. *Math. Z.*, 2(1-2): 198–236, 1918. ISSN 0025-5874. doi: 10.1007/BF01212905. URL https://doi.org/10.1007/BF01212905.

[15] Zheng-Xu He and Oded Schramm. Fixed points, Koebe uniformization and circle packings. *Ann. of Math. (2)*, 137(2):369–406, 1993. ISSN 0003-486X. doi: 10.2307/2946541. URL https://doi.org/10.2307/2946541.

[16] Paul Koebe. Über die Uniformisierung der algebraischen Kurven. I. *Math. Ann.*, 67(2): 145–224, 1909. ISSN 0025-5831. doi: 10.1007/BF01450180. URL https://doi.org/10.1007/BF01450180.

[17] René de Possel. Zum parallelschlitztheorem unendlich vielfach zusammenhängender gebiete. *Nachr. Ges. Wiss. Göttingen*, pages 199–202, 1931. ISSN 0369-6650. URL http://eudml.org/doc/59350.

[18] Herbert Grötzsch. Uber das parallelschlitztheorem der konformen abbildung schlichter bereiche. *Ber. Verh. Sachs. Akad. Wiss. Leipzig*, 84:15–36, 1932. ISSN 0366-0036.

[19] Juha Heinonen. *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001. ISBN 0-387-95104-0. doi: 10.1007/978-1-4613-0131-8. URL https://doi.org/10.1007/978-1-4613-0131-8.

[20] Christopher J. Bishop and Yuval Peres. *Fractals in probability and analysis*, volume 162 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. ISBN 978-1-107-13411-9. doi: 10.1017/9781316460238. URL https://doi.org/10.1017/9781316460238.

[21] Jeremy Tyson. Quasiconformality and quasisymmetry in metric measure spaces. *Ann. Acad. Sci. Fenn. Math.*, 23(2):525–548, 1998. ISSN 1239-629X.

[22] Guy David and Stephen Semmes. Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings. In *Analysis and partial differential equations*, volume 122 of *Lecture Notes in Pure and Appl. Math.*, pages 101–111. Dekker, New York, 1990.

[23] Atte Lohvansuu, Kai Rajala, and Martti Rasimus. Quasispheres and metric doubling measures. *Proc. Amer. Math. Soc.*, 146(7):2973–2984, 2018. ISSN 0002-9939. doi: 10.1090/proc/13971. URL https://doi.org/10.1090/proc/13971.

[24] Mario Bonk and Bruce Kleiner. Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary. *Geom. Topol.*, 9:219–246, 2005. ISSN 1465-3060. doi: 10.2140/gt.2005.9.219. URL https://doi.org/10.2140/gt.2005.9.219.

[25] Peter Haïssinsky and Kevin M. Pilgrim. Coarse expanding conformal dynamics. *Astérisque*, (325):viii+139 pp. (2010), 2009. ISSN 0303-1179.

[26] Dimitrios Ntalampekos and Matthew Romney. Polyhedral approximation of metric surfaces and applications to uniformization. *arXiv preprint*, 2021. URL https://arxiv.org/abs/2107.07422.

[27] Alexander Lytchak and Stefan Wenger. Canonical parameterizations of metric disks. *Duke Math. J.*, 169(4):761–797, 2020. ISSN 0012-7094. doi: 10.1215/00127094-2019-0065. URL https://doi.org/10.1215/00127094-2019-0065.

[28] Kai Rajala and Martti Rasimus. Quasisymmetric Koebe uniformization with weak metric doubling measures. *Illinois J. Math.*, 65(4):749–767, 2021. ISSN 0019-2082. doi: 10.1215/00192082-9501456. URL https://doi.org/10.1215/00192082-9501456.

[29] Jeremy T. Tyson and Jang-Mei Wu. Characterizations of snowflake metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 30(2):313–336, 2005. ISSN 1239-629X.

[30] Juha Heinonen and Pekka Koskela. Definitions of quasiconformality. *Invent. Math.*, 120 (1):61–79, 1995. ISSN 0020-9910. doi: 10.1007/BF01241122. URL https://doi.org/10.1007/BF01241122.

[31] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. *Sobolev spaces on metric measure spaces*, volume 27 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. ISBN 978-1-107-09234-1. doi: 10.1017/CBO9781316135914. URL https://doi.org/10.1017/CBO9781316135914. An approach based on upper gradients.

[32] O. Lehto and K. I. Virtanen. *Quasiconformal mappings in the plane*. Die Grundlehren der mathematischen Wissenschaften, Band 126. Springer-Verlag, New York-Heidelberg, second edition, 1973. Translated from the German by K. W. Lucas.

[33] Charles Loewner. On the conformal capacity in space. *J. Math. Mech.*, 8:411–414, 1959. doi: 10.1512/iumj.1959.8.58029. URL https://doi.org/10.1512/iumj.1959.8.58029.

[34] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. ISBN 0-8218-3644-7. doi: 10.1090/ulect/038. URL https://doi.org/10.1090/ulect/038. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard.

[35] Marshall Williams. Geometric and analytic quasiconformality in metric measure spaces. *Proc. Amer. Math. Soc.*, 140(4):1251–1266, 2012. ISSN 0002-9939. doi: 10.1090/S0002-9939-2011-11035-9. URL https://doi-org.er.lib.k-state.edu/10.1090/S0002-9939-2011-11035-9.

[36] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. Sobolev classes of Banach space-valued functions and quasiconformal mappings. *J. Anal. Math.*, 85:87–139, 2001. ISSN 0021-7670. doi: 10.1007/BF02788076. URL https://doi.org/10.1007/BF02788076.

[37] Hrant Hakobyan and Wenbo Li. Quasisymmetric embeddings of slit sierpiński carpets. *arXiv preprint*, 2021. URL https://arxiv.org/abs/1901.05632.

[38] Sergei Merenkov. Planar relative Schottky sets and quasisymmetric maps. *Proc. Lond. Math. Soc. (3)*, 104(3):455–485, 2012. ISSN 0024-6115. doi: 10.1112/plms/pdr038. URL https://doi.org/10.1112/plms/pdr038.

[39] Nageswari Shanmugalingam. Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana*, 16(2):243–279, 2000. ISSN 0213-2230. doi: 10.4171/RMI/275. URL https://doi.org/10.4171/RMI/275.

[40] B. Bojarski. Remarks on Sobolev imbedding inequalities. In *Complex analysis, Joensuu 1987*, volume 1351 of *Lecture Notes in Math.*, pages 52–68. Springer, Berlin, 1988. doi: 10.1007/BFb0081242. URL https://doi.org/10.1007/BFb0081242.

[41] Javier Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. ISBN 0-8218-2172-5. doi:

10.1090/gsm/029. URL https://doi.org/10.1090/gsm/029. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[42] Lars V. Ahlfors and Leo Sario. *Riemann surfaces*. Princeton Mathematical Series, No. 26. Princeton University Press, Princeton, N.J., 1960.

[43] S. Keith and T. Laakso. Conformal Assouad dimension and modulus. *Geom. Funct. Anal.*, 14(6):1278–1321, 2004. ISSN 1016-443X. doi: 10.1007/s00039-004-0492-5. URL https://doi.org/10.1007/s00039-004-0492-5.

[44] G. T. Whyburn. Topological characterization of the Sierpiński curve. *Fund. Math.*, 45: 320–324, 1958. ISSN 0016-2736. doi: 10.4064/fm-45-1-320-324. URL https://doi.org/10.4064/fm-45-1-320-324.

[45] Peter Haïssinsky. Hyperbolic groups with planar boundaries. *Invent. Math.*, 201(1): 239–307, 2015. ISSN 0020-9910. doi: 10.1007/s00222-014-0552-x. URL https://doi.org/10.1007/s00222-014-0552-x.

Index