SIMPLIFIED AND APPROXIMATE SOLUTION FOR GRIDWORKS WITH FIXED EDGES

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INTRODUCTION

The exact analysis of gridworks by the mathematical theory of elasticity is rather complicated and impractical in comparison with the practical methods of analysis for the other parts of the whole building structure. A slab-beam-column structure can be analyzed by an engineer who has only an undergraduate background, but a grid-floor structure, although it may be only a small part of the whole structure, will usually take a lot of time for analysis or may even puzzle him. Therefore, engineers have been forced to devise approximate means of solution for these problems.

Since orthogonal gridworks loaded normally to their planes are popular in practice in bridge and floor structures, especially in long span and heavy service loading structures, the method presented in this report is aimed at this type of structure where the grid members in each direction are of constant size and spacing, and are fixed at their end points. The behavior of the structures discussed in this report is limited to the elastic range.

This method is based on the physical behavior of the structure. Deflection is induced as the structure is loaded. The applied load at each joint can be replaced by two unknown components acting separately upon the orthogonal beams at that joint. The equations for the joint deflections of all transverse and longitudinal beams caused by the unknown components of load will be derived. Since deflections derived for both transverse and longitudinal beams intersecting at each joint should be compatible, we obtain a series of simultaneous equations in terms of unknown components; say equation set A. From equilibrium we can also get another series of equations by summing up the unknown components which must be equal to the applied load at each joint; say equation set B. Therefore, we have two unknowns and two equations at each joint and the unknown components can then be found. By applying the appropriate components to individual beams, we find the moments and deflections at the joints in all of the beams.

The torsional moments may cause the most significant stresses in the beams, therefore they should be taken into account in the solution of grid type problems. The method presented in this report is to use the slopes of the beams at the joints as the twist angle for the orthogonal beam at the joint. From the torsional moment equations, $T=G \cdot J \cdot \theta$, where θ is the angle of twist per unit length of the beam, we can obtain the torsional moment.

Since the deflection will be affected by the torsional moment, we then use these torsional moments to modify the original deflection equations at all of the joints to find a modified equation set A. With the previous equation set B and the new equation set A' we obtain a new set of torsional moments. Continuing this procedure until the torsional moments from the last two iterations approach each other to the desired accuracy, the final result is taken to be the solution for the problem.

CHAPTER ONE

LITERATURE REVIEW

Beam gridworks are practical structures of interest. Many papers previously written have dealt with the solution of statically indeterminate problems which arise in the determination of the deflections and internal forces which occur for various external loadings.

1. I. Martin and J. Hermandez (1) expressed the flexural and torsional moments in terms of the rotation of the joints in both directions and of the deflection of the joints. For each joint there are three unknowns: two rotations and one deflection and three equations of equilibrium can be established. The summation of all moments acting in each direction on the joint must be zero and the summation of vertical forces must also be zero at the joint. Thus, for each joint we can derive three equations and solve for the three unknowns.

A section of a gridwork, which includes joints A, B, C, D, and E, is shown in Figure 1-1. The equilibrium of joint B in the direction of axis ABC will be studied and all angular deformations in planes parallel to axis ABC will be referred to as rotations and all angular deformations in planes parallel to axis DBE will be referred to as gyrations. All bars are assumed to have uniform section.

The equilibrium equations are derived as follows:

(A). For the transverse direction:

After loading the joint or joints the physical behavior of the structure will be as shown in Figure 1-2. The slope-deflection equations are then derived as:

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$$M_{BA}=\frac{K_{ab}}{2}\left(\theta_A+2\theta_B+\frac{3(\Delta A-\Delta B)}{L_{ab}}\right)\pm M_{Fab} \quad \ldots \quad (1\text{-}1)$$

$$M_{AB}=\frac{K_{ab}}{2}\left(2\theta_A+\theta_B+\frac{3(\Delta A-\Delta B)}{L_{ab}}\right)\pm M_{Fab} \quad \ldots \quad (1\text{-}1a)$$

$$M_{BC}=\frac{K_{bc}}{2}\left(\theta_C+2\theta_B-\frac{3(\Delta C-\Delta B)}{L_{bc}}\right)\pm M_{Fbc} \quad \ldots \quad (1\text{-}2)$$

$$M_{CB}=\frac{K_{bc}}{2}\left(2\theta_C+\theta_B-\frac{3(\Delta C-\Delta B)}{L_{bc}}\right)\pm M_{Fbc} \quad \ldots \quad (1\text{-}2a)$$

where K_{ab}, K_{bc} are the flexural stiffness factors for bars AB, BC;

$$K_{ab}=\frac{4E_{ab}I_{ab}}{L_{ab}},\quad K_{bc}=\frac{4E_{bc}I_{bc}}{L_{bc}}.$$

A rotation θB will also induce torsional end moments $T_{BD\theta B}$ and $T_{DB\theta B}$ in bar BD, and $T_{BE\theta B}$ and $T_{EB\theta B}$ in bar BE.

$$T_{BD\theta B}=K_{Tbd}\,\theta B \quad \ldots \quad (1\text{-}3)$$

$$T_{DB\theta B}=-K_{Tbd}\,\theta B \quad \ldots \quad (1\text{-}3a)$$

$$T_{BE\theta B}=K_{Tbe}\,\theta B \quad \ldots \quad (1\text{-}4)$$

$$T_{EB\theta B}=-K_{Tbe}\,\theta B \quad \ldots \quad (1\text{-}4a)$$

where K_{Tbd}, K_{Tbe} are the torsional stiffness factor of bars BD, BE;

$K_{Tbd}=\dfrac{G_{bd}J_{bd}}{L_{bd}}$, $K_{Tbe}=\dfrac{G_{be}J_{be}}{L_{be}}$, G is the modulus of elasticity in shear,

J for a circular cross section is the polar moment of inertia, for a rectangular cross section it is equal to βbd^3, where b is the width and d is the depth of the beam, β is a coefficient depending on cross-sectional properties.

At the same time, a rotation θD and θE also induces torsional end moments $T_{BD\theta D}$ and $T_{DB\theta D}$ in bar BD, $T_{BE\theta E}$ and $T_{EB\theta E}$ in bar BE.

$$T_{BD\theta D}=-K_{Tbd}\theta D \qquad (1-5)$$

$$T_{DB\theta D}=K_{Tbd}\theta D \qquad (1-5a)$$

$$T_{BE\theta E}=-K_{Tbe}\theta E \qquad (1-6)$$

$$T_{EB\theta E}=K_{Tbe}\theta E \qquad (1-6a)$$

The final torsional end moments T_{BD} and T_{BE} are:

$$T_{BD}=T_{BD\theta B}+T_{BD\theta D}+T_{FBD} \qquad (1-7)$$

$$T_{BE}=T_{BE\theta B}+T_{BE\theta E}+T_{FBE} \qquad (1-8)$$

In the case where no external moments are applied at joint B, the summation of internal moments at joint B must be zero.

$$M_{BA}+M_{BC}+T_{BD}+T_{BE}=0 \qquad (A)$$

(B). For the longitudinal direction:

Equations similar to the equations of Section (A) can be established for Section (B) if the bars which previously resisted the flexural bending moments are now interchanged with those which resisted the torsional moments and vice-versa. Similarly, the rotations are interchanged with the gyrations. Therefore, the final equilibrium equation can be set up as follows:

$$M_{BD}+M_{BE}+T_{AB}+T_{BC}=0 \qquad (B)$$

(C). The summation of vertical forces must be zero also.

$$V_{BA}+V_{BC}+V_{BD}+V_{BE}+\frac{M_{AB}+M_{BA}}{L_{ab}}+\frac{M_{BC}+M_{CB}}{L_{bc}}+\frac{M_{BE}+M_{EB}}{L_{be}}+\frac{M_{BD}+M_{DB}}{L_{bd}}+P_B=0 \quad \ldots\ldots\ldots \quad (C)$$

Where V_{BA}, V_{BC}, V_{BD} and V_{BE} are the shears, at joint B, of bars AB, BC, BD and BE and P_B is the vertical load applied at B.

In the same way the equations at each joint can be found. We can then find the rotations and deflections at each joint, and then the flexural and torsional moments.

2. W. W. Ewell, S. Okubo and J. I. Abrams (2) have also presented a technique to find the deflections in gridworks and slabs. The method employs an auxiliary force system for controlling the vertical displacements of the joints and a moment and torque distribution process for the transmission of the displacement effects.

A part of a gridwork, which includes joints A, B, C, D and E as shown in Figure 1-1 will be studied here.

When joint B is displaced a distance ΔB, fixed-end moments of $6EI\Delta B/L_{ab}^2$ are induced at points A, B and C along beam ABC. For equal spacing of the beams, the moment can be written as $6EI\Delta B/L^2$, where it is assumed that $L_{ab}=L_{bc}=L$. Fixed-end moments of $6EI\Delta B/L_{db}^2$ are also induced at points D, B, and E along the beam DBE. These moments can be written as $\frac{6}{k^2}EI\Delta B/L^2$, where $L_{db}=L_{be}=KL$.

The stiffnesses and distribution factors for the beam elements of the grid are formed by assuming that an applied moment (M), in the plane of beam ABC, is applied to the rigidly connected system shown in

Figure 1-3. The tangent to the elastic curve will then rotate through an angle θ. If torsional fixity is realized at points D and E, the beam DBE will twist through the same angle, θ, at point B. Similarly, a moment applied in the plane DBE will produce rotation of the tangent to the elastic curve in that plane and consequent twist in beam ABC.

In order to resist the applied moment, M, bending moments M_{BA} and M_{BC} will be developed in beam ABC, and torsional moments T_{BD} and T_{BE} will be introduced in beam DBE.

When the rotation through the angle θ has taken place,

$$M=M_{BA}+M_{BC}+T_{BD}+T_{BE} \quad \ldots \quad (1\text{-}9)$$

For a unit rotation, θ, in beam ABC, from the slope-deflection equations we can find:

$$M_{BA}=\frac{4EI}{L} \text{ and } M_{BC}=\frac{4EI}{L} \quad \ldots \quad (1\text{-}10)$$

For a unit twist, θ, in the beam DBE,

$$T_{BD}=\frac{GJ}{L_1}=T_{BE} \quad \ldots \quad (1\text{-}10a)$$

with the expressions for moment in the longitudinal and transverse beams known, distribution factors can be written for any joint. For example:

$$M_{BA}=\frac{4EI/L}{4EI/L+4EI/L+GJ/L_1+GJ/L_1} \quad \ldots \quad (1\text{-}11)$$

A technique similar to moment distribution then can be used to analyse the stresses in the whole structure caused by the displacement AB, using the stiffnesses and distribution factors [illegible]. The

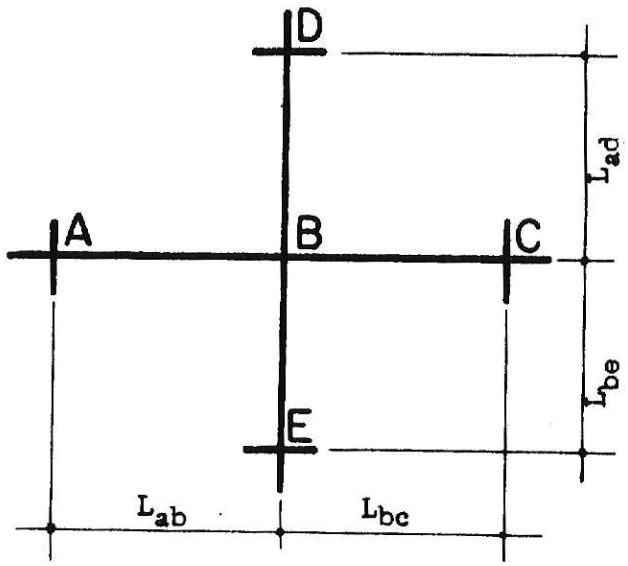


Fig. 1-1 Section of gridwork

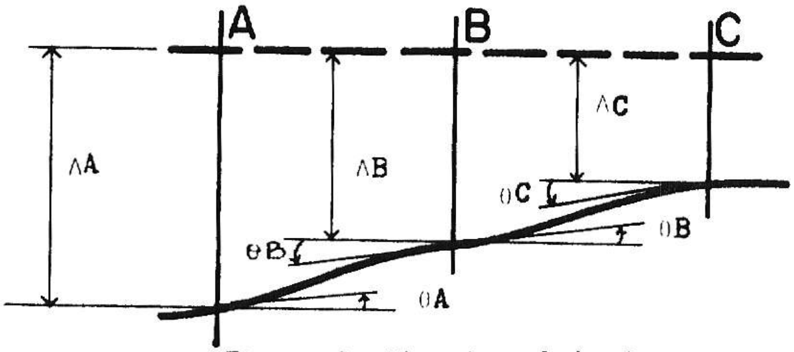


Fig. 1-2 Structure behavior

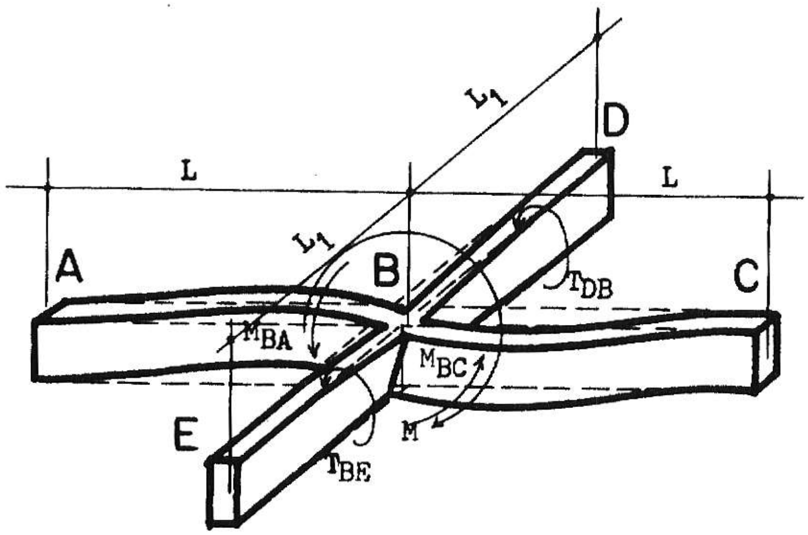


Fig. 1-3 Bending and twisting of a rotated rigid joint

fixed-end moments along beam ABC and along beam DBE will be taken into account separately.

Figures 1-4 and 1-5 show the first distribution and carry-over at joint B. The same procedure will be followed at each joint of the whole grid structure.

When the distribution process is complete, final bending moment values at the beam ends are used to determine the reactions in terms of the displacement, ΔB. Figure 1-6 shows an assumed deflection pattern due to the displacement, ΔB, at joint B. The notation should be self evident. For instance, $R_{A\Delta B}$ is the reaction at joint A caused by a displacement, ΔB, at joint B. It should be remembered at this point that all reactions are expressed as coefficients of $\Delta B EI/L^2$.

The complete reaction at joint B, with all intersection points of the grid displaced through unknown Δ values, will be:

$$R_B = R_{B\Delta A} + R_{B\Delta B} + R_{B\Delta C} + R_{B\Delta D} + R_{B\Delta E} + \quad \dots \dots \quad (1\text{-}12)$$

Since $R_{A\Delta B}$, $R_{B\Delta B}$, $R_{C\Delta B}$, $R_{D\Delta B}$, $R_{E\Delta B}$ ..... are reactions caused by ΔB, they can be derived from the end moments obtained by the distribution process, shown in Figure 1-4 and 1-5. The equations of static equilibrium are used to determine these reactions after the beams are subdivided into their components lengths as follows:

$$R_{B\Delta B} = \left(\sum_{i=1}^{4} R_{Bi} \right) = \frac{M_{AB}+M_{BA}}{L} + \frac{M_{BE}+M_{EB}}{KL} + \frac{M_{BC}+M_{CB}}{L} + \frac{M_{DB}+M_{BD}}{KL} + k_B EI\, B/_{L}3 \quad \dots \quad (1\text{-}13)$$

where M_{AB}, M_{BA}, M_{BE}, M_{EB}, M_{BC}, M_{CB}, M_{DB}, M_{BD}, shown in Figure 1-7, are end moments on beams caused by ΔB, and are derived in terms of $EI\Delta B/_{L}2$.

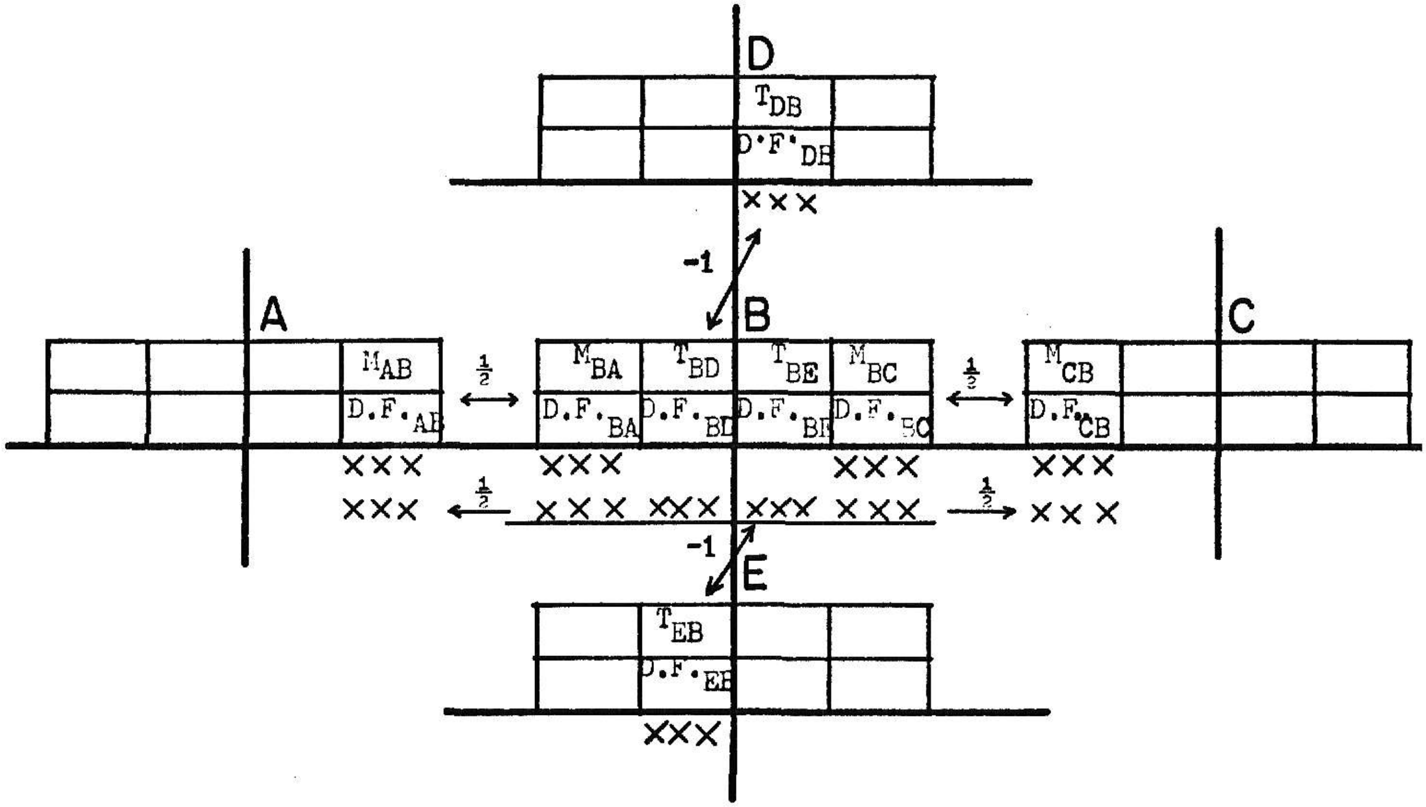


Fig. 1-4 Procedure for the first distribution of the fixed-end moment along beam ABC at joint B.

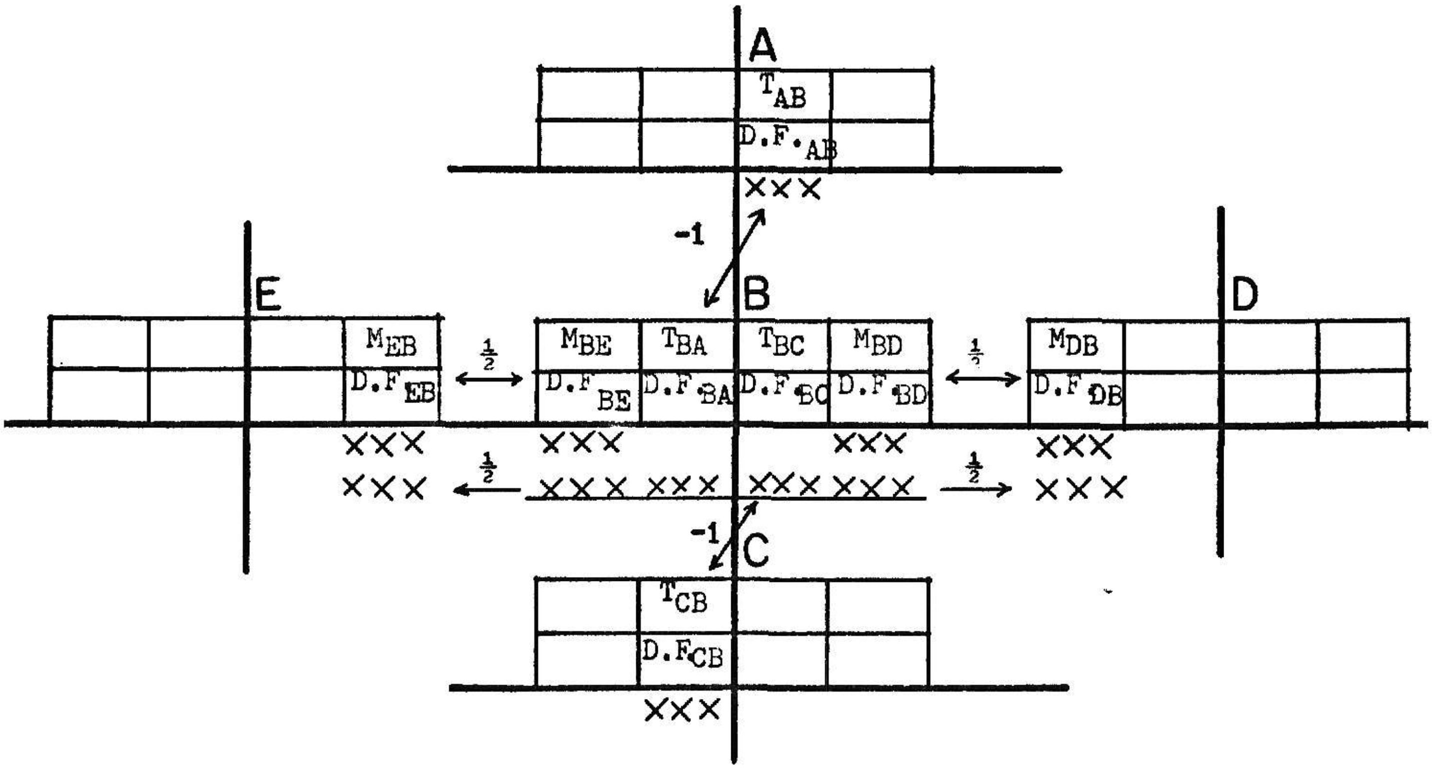


Fig. 1-5 Procedure for the first distribution of the fixed-end moment along beam DBE at joint B.

Following the same procedure at each joint in the grid we obtain:

$$\left.\begin{aligned} R_{A\Delta B} &= \left(\sum_{i=1}^{4} R_{Ai}\right) = k_A EI\ B/_L 3 \\ R_{C\Delta B} &= \left(\sum_{i=1}^{4} R_{Ci}\right) = k_C EI\ B/_L 3 \\ &\vdots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1\text{-}13a)$$

By the theorem of reciprocity, the reaction $k_A EI\ B/_L 3$ at joint A caused by deflection ΔB at joint B, is equal to the reaction at joint B caused by an equal deflection ΔA at joint A. If similar reasoning is employed at each of the other grid points, Equation 1-14 can be written as:

$$\left.\begin{aligned} R_{B\Delta B} &= k_B EI\Delta B/_L 3 \\ R_{B\Delta A} &= k_A EI\Delta A/_L 3 \\ R_{B\Delta C} &= k_C EI\Delta C/_L 3 \\ &\vdots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1\text{-}13b)$$

The total reaction at joint B can be derived as:

$$R_B = k_A EI\Delta A/_L 3 + k_B EI\Delta B/_L 3 + k_C EI\Delta C/_L 3 = \ldots\ldots \quad \ldots\ldots (1\text{-}12a)$$

Similar equations can be derived by using a similar distribution process at each joint on the grid. These reactions are then equated to the load P that exists at the intersection, or else equated to zero if there is no load at the joint. Now we have n joints, n unknown

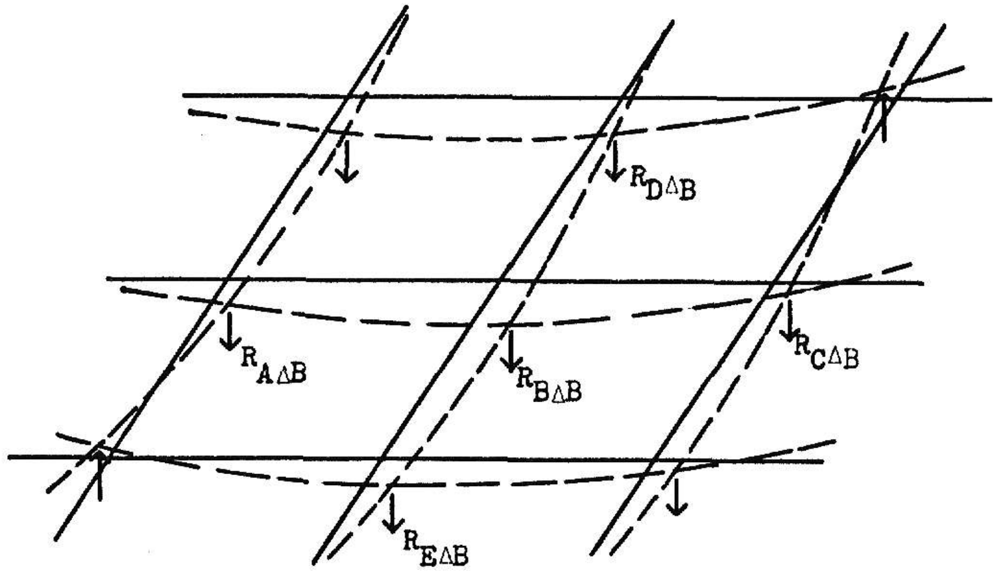


Fig. 1-6 Assumed deflection pattern due to displacement; ΔB

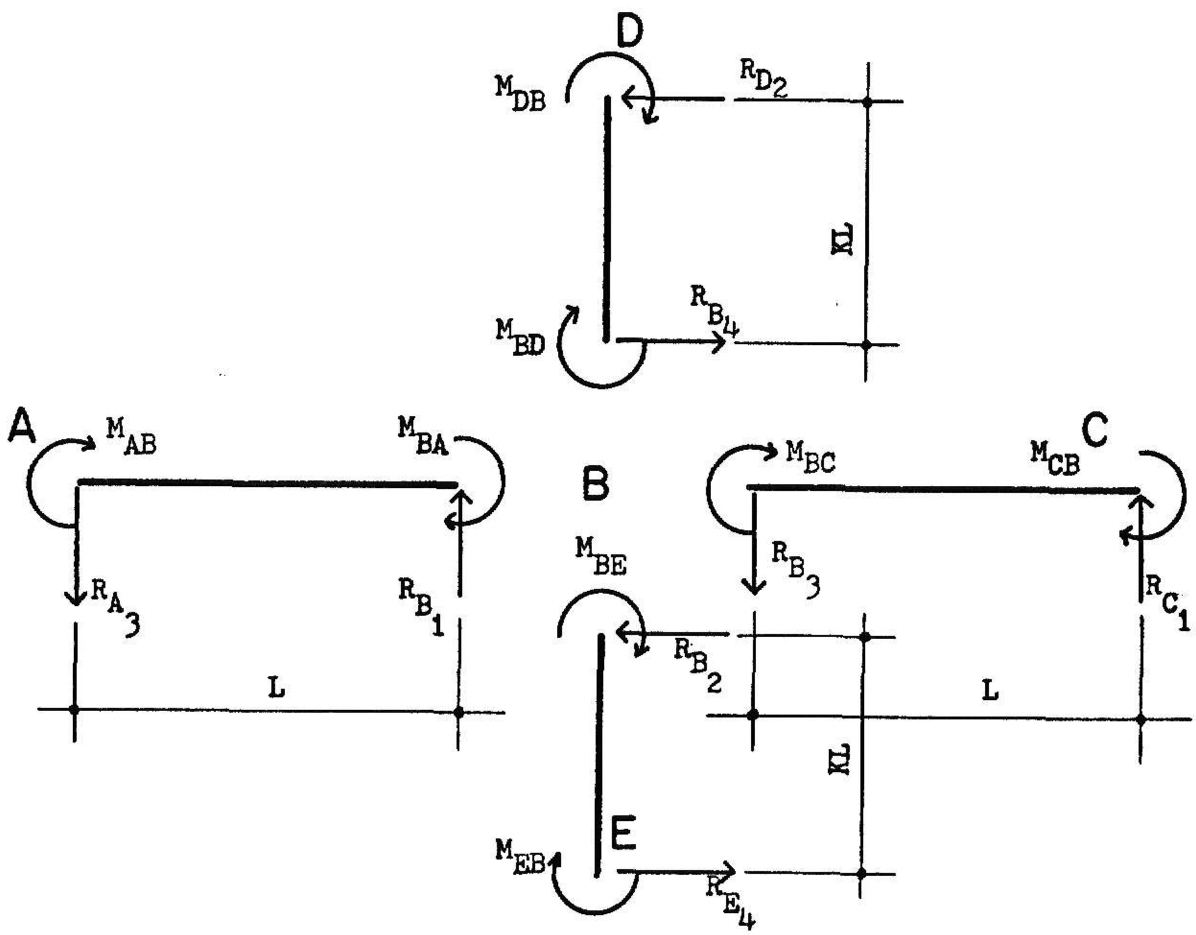


Fig. 1-7 Reactions on beams caused by their end moments

displacements and n equations. Those equations can then be solved for the unknowns.

With the deflection values known at each joint, the moments and torques expressed in the distribution process in terms of the various Δ's can be accumulated by simple multiplication and addition.

3. M. Hetényi (4) assumed that the individual beams comprising the gridwork deflect without rotation at their intersections with other beams. This implies infinite torsional stiffness at each joint. Under such conditions each beam element of the grid will act separately as a beam restrained against rotation at both ends, as shown in Figure 1-8.

The flexibility of each beam element can be characterized by the amount of load necessary to produce a unit relative defection between the two ends of the beam. This load under the given conditions, for a beam of length L and flexural rigidity EI_0, will be equal to $12EI_0/L^3$, as shown in Figure 1-9.

Let us now consider a gridwork consisting of two parallel main girders supported on a series of cantilever cross beams, as shown in Figure 1-10.

The flexibility K_F of each cross beam is $12EI_0/L^3$. If the crossbeam are sufficiently closely spaced, their resistance can be replaced by distributed reactive forces acting along the main girders. The intensity of this distributed reaction R_p will be proportional at every point to the relative displacement of the ends of the crossbeams Δy at that point, $R_p = K_{F\Delta y}$, the proportionality factor per unit length of the main girder being $K_F = 12EI_0/CL^3$, where c is the spacing of the crossbeams.

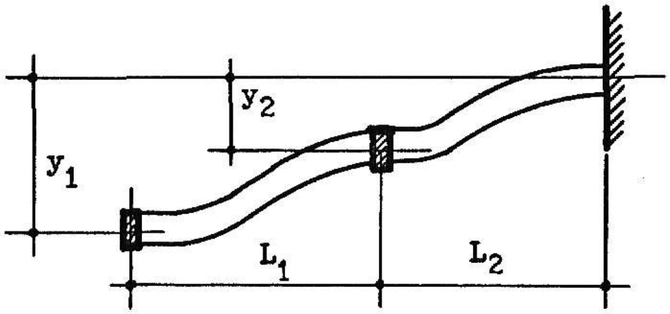
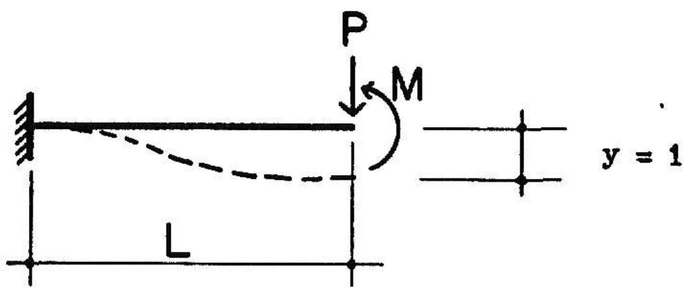


Figure 1-8 Beam element behavior assuming restrained ends



$$y = \frac{PL^3}{12EI_0} \quad \text{So:} \quad P = \frac{12EI_0}{L^3}\, y = \frac{12EI_0}{L^3} \qquad \text{(where } y = 1\text{)}$$

Fig. 1-9 Beam fixed at one end, free to deflect vertically but not rotate at the other end.

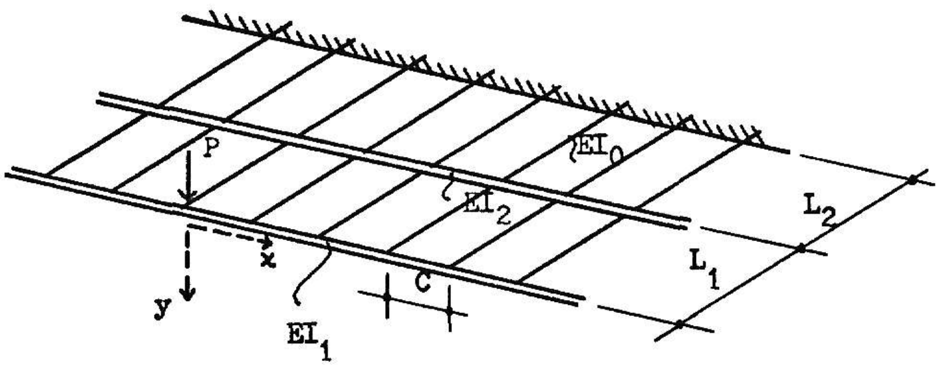


Fig. 1-10 Cantilever gridwork

The deflection of the outer main girder of flexural rigidity EI_1 will be denoted by y_1 and the deflection of the main girder of flexural rigidity EI_2 by y_2 as shown in Figure 1-8.

On the assumption that the external loading on the girders consists of concentrated forces, the only distributed loading will be the pressure R_{P1} and R_{P2} defined according to the flexural theory of beams as,

$$EI_1 \frac{d^4 y_1}{dx^4} = -R_{P1} \quad \ldots\ldots\ldots\ldots \quad (1\text{-}14)$$

$$EI_2 \frac{d^4 y_2}{dx^4} = -(R_{P2} - R_{P1}) \quad \ldots\ldots\ldots\ldots \quad (1\text{-}15)$$

where

$$R_{P1} = K_{F1}(y_1 - y_2), \qquad K_{F1} = 12EI_0/L_1^{\,3}$$

$$R_{P2} = K_{F2}y_2, \qquad K_{F2} = 12EI_0/L_2^{\,3}$$

By solving the differential equations we can find the resulting stresses.

4. E. Ma (3) represented a gridwork supported on all sides, as shown in Figure 1-11. He assumed that the interconnections of the gridworks are pin connected such that they will transmit tensions and compressions only.

Let $[\alpha]$ and $[\beta]$ be the influence matrices of the transverse and longitudinal beams, respectively, $[F]$ and $[f]$ their internal force matrices and $[\delta]$ and $[\Delta]$ their corresponding deflection matrices, then

$$\underset{m}{m[\alpha]} \cdot \underset{n}{m[F]} = \underset{n}{m[\delta]} \quad \ldots\ldots \quad (1\text{-}16)$$

and

$$\underset{n}{n[\beta]} \cdot \underset{m}{n[f]} = \underset{m}{n[\Delta]} \quad \ldots\ldots \quad (1\text{-}17)$$

The deflection relation between the transverse and the longitudinal beams is such that

$$[\delta] = [\Delta]^T \quad \ldots\ldots \quad (1\text{-}18)$$

where $[\Delta]^T$ is the transpose of $[\Delta]$.

The force relation of the systems yields

$$[F] + [f]^T = [G] \quad \ldots\ldots \quad (1\text{-}19)$$

where $[f]^T$ is the transpose of $[f]$ and $[G]$ is the matrix of external

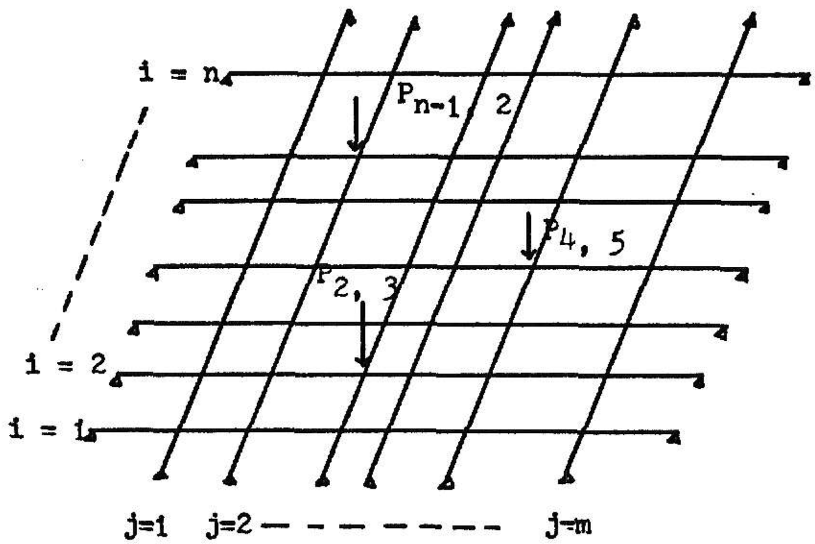


Fig. 1-11 Gridwork with n transverse and m longitudinal beams

forces acting at the joints of the gridwork.

Substituting Equations (1-16) and (1-17) into Equation (1-18) yields

$$[\alpha][F] = [f]^T[\beta]^T \quad \text{. . . . .} \quad (1\text{-}20)$$

From Equation (1-19)

$$[F] = [G] - [f]^T$$

substituting into Equation (1-20) yields

$$[\alpha][G] - [\alpha][f]^T = [f]^T[\beta]^T$$

$$[\alpha][f]^T + [f]^T[\beta]^T = [\alpha][G] \quad \text{. . . . .} \quad (1\text{-}21)$$

where $[\beta]^T$ is the transpose of $[\beta]$.

From Equation (1-21) it is possible to find [f] and then, by substitution into Equation (1-19), [F] can be found.

CHAPTER TWO

DERIVATION OF EQUATIONS

1. Basic formulas

The formulas for the deflections and moments for beams which are fixed at both ends and loaded with a concentrated load and an applied moment at any point, as shown in Figure 2-1, are:

(A). Due to concentrated load

(a). Deflection:

$$\delta_i = \frac{Pb^2x^2}{6EIl^3}(3al-3ax-bx), \text{ where } x \leq a \quad \ldots\ldots\ldots\ldots \quad (2\text{-}1)$$

and

$$\delta_i = \frac{Pa^2(l-x)^2}{6EIl^3}[3bl-3b(l-x)-a(l-x)] \text{ , where } x > a \quad \ldots \quad (2\text{-}1a)$$

(b). Fixed-end Moment:

$$M_A = Pab^2/l^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}2)$$

$$M_B = Pa^2b/l^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}2a)$$

(c). Intermediate Moment between ends:

$$M_i = \frac{Pb^2}{l^3}(3a+b)x - Pab^2/l^2, \text{ where } x \leq a \quad \ldots\ldots\ldots\ldots \quad (2\text{-}3)$$

and

$$M_i = \frac{Pa^2}{l^3}(3b+a)(l-x) - Pa^2b/l^2, \text{ where } x > a \quad \ldots\ldots \quad (2\text{-}3a)$$

(d). Reaction:

$$R_A = \frac{Pb^2}{l^3}(3a+b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}4)$$

$$R_B = \frac{Pa^2}{l^3}(a+3b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}4a)$$

(B). Due to applied moment

(a). Deflection:

$$\Delta_1 = \frac{x^2}{6EI}\left\{2M_A + M_A\frac{(1-x)}{1} + M_B\frac{x}{1} - \frac{Tx}{1}\right\}, \text{ where } x \leq a \quad \ldots (2\text{-}5)$$

and

$$\Delta_1 = \frac{(1-x)^2}{6EI}\left\{2M_B + M_B\frac{x}{1} + M_A\frac{(1-x)}{1} + \frac{T(1-x)}{1}\right\}, \text{ where } x > a \quad \ldots (2\text{-}5a)$$

where M_A and M_B are Fixed-end Moments at ends A and B as shown below:

$$M_A = \frac{Tb}{1^3}(-b^2 + 2a^2 + ab) \quad \ldots \ldots \ldots (2\text{-}6)$$

$$M_B = \frac{Ta}{1^3}(a^2 - 2b^2 - ab) \quad \ldots \ldots \ldots (2\text{-}6a)$$

If there is more than one concentrated load or applied moment acting on the beam, we can obtain the results by superimposing the values which are caused by the loads separately.

2. General Equation

(A). Deflection

(a). Due to concentrated loads:

If the beam contains n equally spaced points, the deflection of the points caused by all point loads can be derived as follows:

$$\{Y_j^P \mid_1^{n-1}\} = \delta_{j1} + \delta_{j2} + \delta_{j3} + \ldots + \delta_{j\,n-1} \quad \ldots \ldots (2\text{-}7)$$

where δ_{j1}, δ_{j2}, δ_{j3}, . . . $\delta_{j\,n-1}$ are the deflections at point j caused by transverse loads P_1, P_2, P_3, . . . P_{n-1} respectively.

From Equations (2-1) and (2-1a) it can be shown that:

$$\delta_{ji} = \frac{P_i(n-i)^2(j)^2 l^3}{6EIn^6}\,[3ni-3ij-(n-i)j]\ , \text{ where } j \le i \quad \ldots\ldots\ (2\text{-}7a)$$

and

$$\delta_{ji} = \frac{P_i(i)^2(n-j)^2 l^3}{6EIn^6}\,[3n(n-i)-3(n-i)(n-j)-(n-j)i]\ , \text{ where } j > i \quad \ldots\ (2\text{-}7b)$$

then

$$\{Y_j^P \mid {}_1^{n-1}\} = \sum_{i=1}^{n-1} \delta_{ji} = \sum_{i=1}^{n-1} P_i\,[a_{ji}] \quad \ldots\ldots\ldots\ldots\ (2\text{-}8)$$

where $[a_{ji}] = \dfrac{(n-i)^2(j)^2 l^3}{6EIn^6}\,[3ni-3ij-(n-i)j]$, when $j \le i$

and

$[a_{ji}] = \dfrac{(i)^2(n-j)^2 l^3}{6EIn^6}\,[3n(n-i)-3(n-i)(n-j)-(n-j)i]$, when $j > i$

(b). Due to applied moments:

$$\{Y_j^T \mid {}_1^{n-1}\} = \Delta_{j1} + \Delta_{j2} + \Delta_{j3} + \ldots + \Delta_{j\,n-1} \quad \ldots\ldots\ldots\ (2\text{-}6a)$$

where $\Delta_{j1}, \Delta_{j2}, \Delta_{j3}, \ldots \Delta_{j\,n-1}$ are the deflections at point j caused by torques $T_1, T_2, T_3, \ldots T_{n-1}$, respectively.

From Equations (2-5) and (2-5a) yields

$$\Delta_{ji} = \frac{(i)^2 l^2}{6EIn^2}\left[2M_A + M_A\frac{(n-i)}{n} + M_B\frac{i}{n} - \frac{T_i(i)}{n}\right],\ \text{when } j \le i \quad \ldots\ (2\text{-}9)$$

and

$$\Delta_{ji} = \frac{(nl-il)^2}{6EIn^2}\left[2M_B + M_B\frac{i}{n} + M_A\frac{(n-i)}{n} + T_i\frac{(n-i)}{n}\right],\ \text{when } j > i \quad \ldots\ (2\text{-}9a)$$

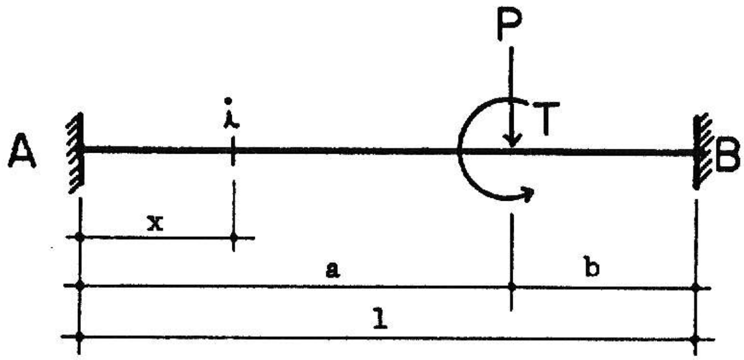


Fig. 2-1 Fixed ended beam

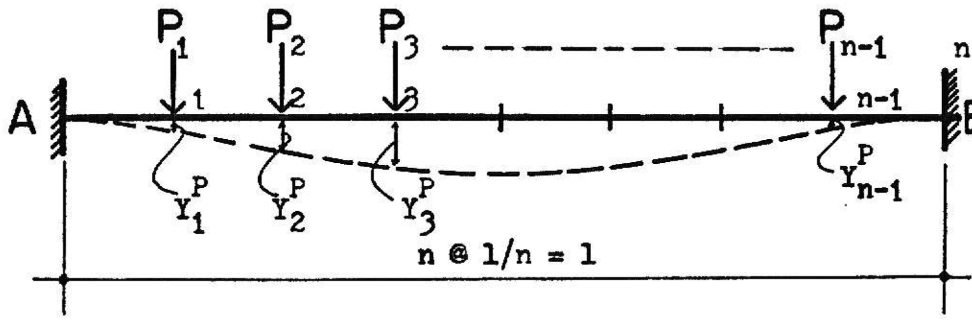


Fig. 2-2 Fixed ended beam transversely loaded at n equally spaced points

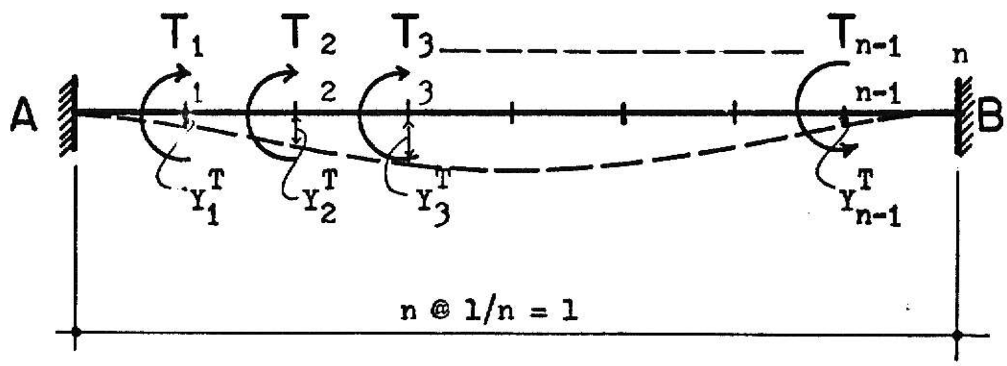


Fig. 2-3 Fixed ended beam with moments applied at n equally spaced points

where

$$M_A = \frac{T_i(i)}{n^3}\left(-n^3 + 4n^2 i - 3ni^2\right) = T_i\,[c_i] \quad \ldots\ldots\ldots\ldots\ldots \quad (2\text{-}6b)$$

$$M_B = \frac{T_i(i)}{n^3}\left(3ni^2 - 2n^2 i\right) = T_i\,[d_i] \quad \ldots\ldots\ldots\ldots\ldots \quad (2\text{-}6c)$$

Substituting into Equations (2-9) and (2-9a) yields

$$\Delta_{ji} = \frac{T_i(i)^2 l^2}{6EIn^2}\left[2c_i + c_i\frac{(n-i)}{n} + d_i\frac{i}{n} - \frac{i}{n}\right] \quad \ldots\ldots\ldots\ldots \quad (2\text{-}9b)$$

$$\Delta_{ji} = \frac{T_i(nl-il)^2}{6EIn^2}\left[2d_i + d_i\frac{i}{n} + c_i\frac{(n-i)}{n} + \frac{(n-i)}{n}\right] \quad \ldots\ldots\ldots\ldots \quad (2\text{-}9c)$$

Then

$$\{Y_j^T \mid {}_1^{n-1}\} = \sum_{i=1}^{n-1} \Delta_{ji} = \sum_{i=1}^{n-1} T_i\,[b_{ji}] \quad \ldots\ldots\ldots\ldots\ldots \quad (2\text{-}10)$$

where

$$[b_{ji}] = \frac{(i)^2 l^2}{6EIn^2}\left[2c_i + c_i\frac{(n-i)}{n} + d_i\frac{i}{n} - \frac{i}{n}\right], \text{ when } j \le i$$

and

$$[b_{ji}] = \frac{(nl-il)^2}{6EIn^2}\left[2d_i + d_i\frac{i}{n} + c_i\frac{(n-i)}{n} + \frac{(n-i)}{n}\right], \text{ when } j > i$$

(c). The final deflection at each point will be

$$\{Y_j \mid {}_1^{n-1}\} = \{Y_j^P \mid {}_1^{n-1}\} + \{Y_j^T \mid {}_1^{n-1}\}$$

$$= \sum_{i=1}^{n-1}\left[\delta_{ji} + \Delta_{ji}\right]$$

$$= \sum_{i=1}^{n-1}\left[P_i(a_{ji}) + T_i(b_{ji})\right] \quad \ldots\ldots\ldots\ldots \quad (2\text{-}11)$$

(B). Fixed-end Moment

(a). Due to concentrated load

The loading as shown in Figure 2-2 induces Fixed-end Moments at ends A and B which are:

From Equation (2-2) and (2-2a)

$$M_A^P = \sum_{i=1}^{n-1} P_i(\frac{i}{n} l) \left\{\frac{(n-i)}{n} l\right\}^2 \Big/ l^2$$

$$= \sum_{i=1}^{n-1} P_i(i)(n-i)^2 l \Big/ n^3 \quad \ldots\ldots\ldots\ldots \text{(2-12)}$$

$$M_A^P = \sum_{i=1}^{n-1} P_i(\frac{i}{n} l)^2 \left\{\frac{(n-i)}{n} l\right\} \Big/ l^2$$

$$= \sum_{i=1}^{n-1} P_i(i)^2(n-i) l \Big/ n^3 \quad \ldots\ldots\ldots\ldots \text{(2-12a)}$$

(b). Due to applied moment

The Equations (2-6b) and (2-6c) will be used to express the Fixed-end Moments, M_A^T, M_B^T, caused by the loading as shown in Figure 2-3.

(c). The final Fixed-end Moments then are

$$M_A = M_A^P + M_A^T \quad \ldots\ldots\ldots\ldots \text{(2-13)}$$

and

$$M_B = M_B^P + M_B^T \quad \ldots\ldots\ldots\ldots \text{(2-14)}$$

(C). Intermediate Moment between ends

The loading as shown in Figure 2-2 will also induce moments at intermediate points, which can be expressed as:

$$M_j = m_{j1} + m_{j2} + m_{j3} + \ldots + m_{j\,n-1} \quad \ldots\ldots\ldots\ldots \text{(2-15)}$$

where $m_{j1}, m_{j2}, m_{j3}, \ldots m_{j\,n-1}$ are bending moments at point j caused by $P_1, P_2, P_3, \ldots P_{n-1}$.

From Equations (2-3) and (2-3a) we obtain

$$m_{ji} = P_i(n-i)^2 l \left\{ \frac{[3i+(n-i)]j}{n^4} - \frac{i}{n^3} \right\} \text{, when } j \leq i \ \ldots\ldots \ (2\text{-}16)$$

and

$$m_{ji} = P_i(i)^2 l \left\{ \frac{[3(n-i)+i](n-j)}{n^4} - \frac{(n-i)}{n^3} \right\} \text{, when } j > i \ \ldots\ldots \ (2\text{-}16a)$$

Then

$$\{M_j \mid_1^{n-1}\} = \sum_{i=1}^{n-1} m_{ji} \ \ldots\ldots\ldots\ldots\ldots\ldots \ (2\text{-}15a)$$

(D). Slope

The slopes of the elastic deflection curve of the beam can be found by using the finite-difference method. We thus write the slope as $\tan\theta = \frac{dy}{dx} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}$.

At point j, Δy should be equal to the difference between the deflection at points $j + \frac{\Delta x}{2}$ and $j - \frac{\Delta x}{2}$.

By using x_1 and x_2 instead of $\frac{j}{n}l$ in Equations (2-1), (2-1a), (2-5), (2-5a) we obtain:

(a).

$$\delta_{j \pm \frac{\Delta x}{2},\, i} = \frac{P_i(n-i)^2 x^2}{6EIn^3}[3il - 3ix - (n-i)x] \text{, when } x \leq \frac{i}{n} l \ \ldots \ (2\text{-}17)$$

and

$$\delta_{j \pm \frac{\Delta x}{2},\, i} = \frac{P_i(i)^2(l-x)^2}{6EIn^3}[3(n-i)l - 3(n-i)(l-x) - i(l-x)] \text{,}$$

$$\text{when } x > \frac{i}{n} l \ \ldots\ldots\ldots\ldots \ (2\text{-}17a)$$

(b).
$$\Delta_{j\pm\frac{\Delta x}{2},\, i} = \frac{x^2}{6EI}\left\{2M_A + M_A\frac{(1-x)}{1} + M_B\frac{x}{1} - \frac{T_i x}{1}\right\},\ \text{when}$$

$$x \le \frac{i}{n}1 \quad \ldots\ldots\ldots\ldots\ldots \quad (2\text{-}17b)$$

and

$$\Delta_{j\pm\frac{\Delta x}{2},\, i} = \frac{(1-x)^2}{6EI}\left\{2M_B + M_B\frac{x}{1} + M_A\frac{(1-x)}{1} + \frac{T_i(1-x)}{1}\right\},\ \text{when}$$

$$x > \frac{i}{n}1 \quad \ldots\ldots\ldots\ldots\ldots \quad (2\text{-}17c)$$

Then substituting x_1 and x_2 respectively for x in Equation (2-17) or (2-17a), and in Equation (2-17b) or (2-17c), where

$x_1 = \frac{j}{n}1 - \frac{\Delta x}{2}$, $x_2 = \frac{j}{n}1 + \frac{\Delta x}{2}$, we obtain:

$$\left\{Y_{j\pm\frac{\Delta x}{2}} \,\Big|\, {}^{n-1}_{1}\right\} = \sum_{i=1}^{n-1}\left(\delta_{j\pm\frac{\Delta x}{2},\, i} + \Delta_{j\pm\frac{\Delta x}{2},\, i}\right)$$

then

$$\tan\theta = \frac{Y_{j+\frac{\Delta x}{2}} - Y_{j-\frac{\Delta x}{2}}}{\Delta x} = \theta\ ,\ \text{where}\quad \theta \to 0 \quad \ldots \quad (2\text{-}18)$$

(E). Torsional Moment

If the beam is twisted through an angle θ, then the beam will induce a torsional moment

$T = G\cdot J\cdot\theta/1$ . . . . (2-19)

where G is the modulus of elasticity in shear and J, for a circular cross section, is the polar moment of inertia, while for a rectangular cross section J is equal to βbd^3, where b is the width and d is the depth of the beam and β is a coefficient depending on cross-sectional properties.

(F). Reaction

The reaction at ends A and B are:

$$R_A = \sum_{i=1}^{n-1} P_i(n-i)^2(3i+(n-i))/_{n}3 \quad \ldots \ldots \quad (2\text{-}20)$$

$$R_B = \sum_{i=1}^{n-1} P_i(i)^2(i+3(n-i))/_{n}3 \quad \ldots \ldots \quad (2\text{-}20a)$$

3. Method of Analysis

The method of analysis of gridworks using the equations derived above will be presented in the following.

(a). Assume a joint load $P_{(x,y)}$ to be replaced by $P_{x(x,y)}$ and $P_{y(x,y)}$ which are acting separately on beams parallel to the X-axis and the Y-axis as shown in Figure 2-4.

(b). From Equation (2-8) we can find the deflections $Y^{P_x}_{(x,y)}$ and $Y^{P_y}_{(x,y)}$ for all n joints of the gridwork, then from

$$Y^{P_x}_{(x,y)} = Y^{P_y}_{(x,y)} \quad \ldots \ldots \quad (2\text{-}A)$$

n equations in $P_x(x,y)$ and $P_y(x,y)$ can be found.

Summing vertical forces at each grid point;

$$\left.\begin{array}{l} P_x(x,y) + P_y(x,y) = P(x,y), \text{ where } P(x,y) \text{ exists} \\ \text{and } P_x(x,y) + P_y(x,y) = 0, \text{ where } P(x,y) \text{ does not exist} \end{array}\right\} \ldots (2\text{-}B)$$

This also yield n equations in $P_x(x,y)$ and $P_y(x,y)$.

(c). From formulas (2-A) and (2-B), there are 2n equations with 2n unknowns, and they can be solved for the values of $P_x(x,y)$ and $P_y(x,y)$.

By placing these loads on their corresponding joints and using Equations (2-13), (2-14), (2-15a), (2-18), and (2-19), we can find the flexural and torsional moments in all transverse and longitudinal beams.

(d). Substituting the torsional moments into Equation (2-10)

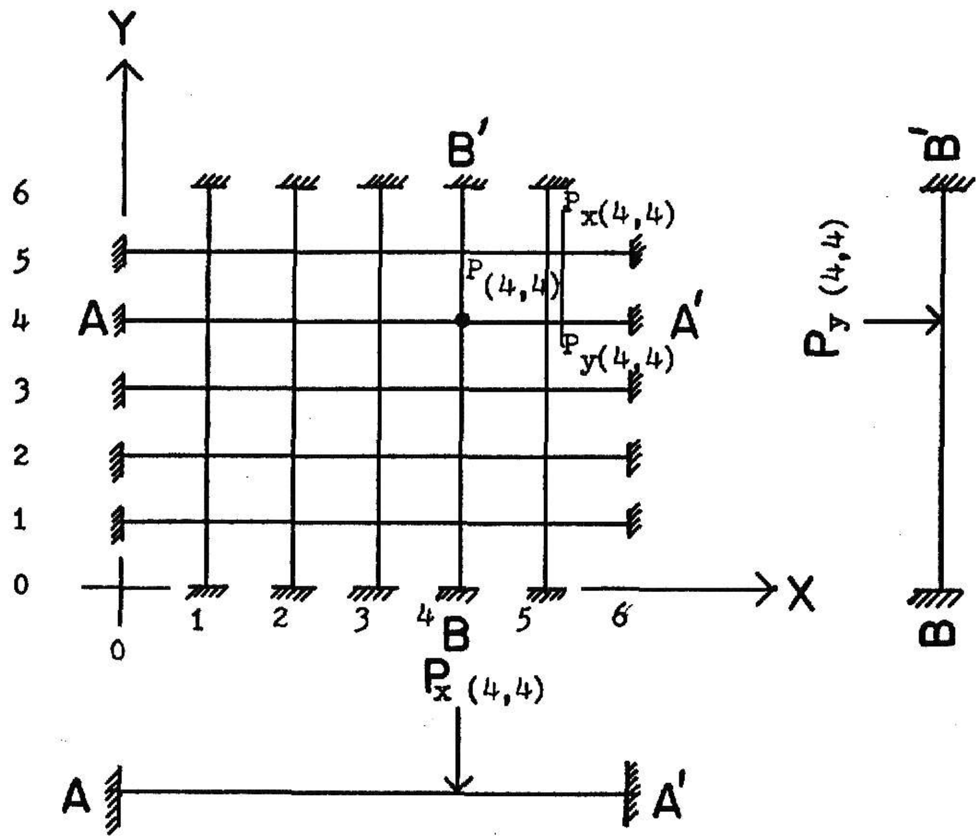


Fig 2-4 Component loads on gridwork beams

we find $Y^{T_x}_{(x,y)}$ and $Y^{T_y}_{(x,y)}$.

Formula (2-A) then can be modified as follows:

$$Y^{P_x}_{(x,y)} + Y^{T_x}_{(x,y)} = Y^{P_y}_{(x,y)} + Y^{T_y}_{(x,y)}$$

Since T_x and T_y are known then

$$Y^{P_x}_{(x,y)} + Y^{P_y}_{(x,y)} = Y^{T_y}_{(x,y)} - Y^{T_x}_{(x,y)} = \left\{ a^T_{xy} \right\} \quad . \; . \; . \; . \; . \; (2\text{-}A')$$

Using formula (2-A') instead of formula (2-A) in procedure (c), new values of $P_x(x,y)$ and $P_y(x,y)$ can be found. Following the same procedure repeatedly the results of stresses can be obtained to the desired accuracy.

CHAPTER THREE

SIMPLIFIED EQUATIONS FOR SIMPLE SPECIAL CASES

The equations presented in this chapter are aimed at the simple types of gridwork which are widely used in practical work. In these equations, the deflections of the beams caused by the torsional moments which are induced in the orthogonal beams are neglected. Simultaneously, the grid members are assumed to be of constant E in both directions; of constant size and spacing in each direction, and fixed at their far ends.

(1). Case one

In this case two orthogonal beams with applied load P on their

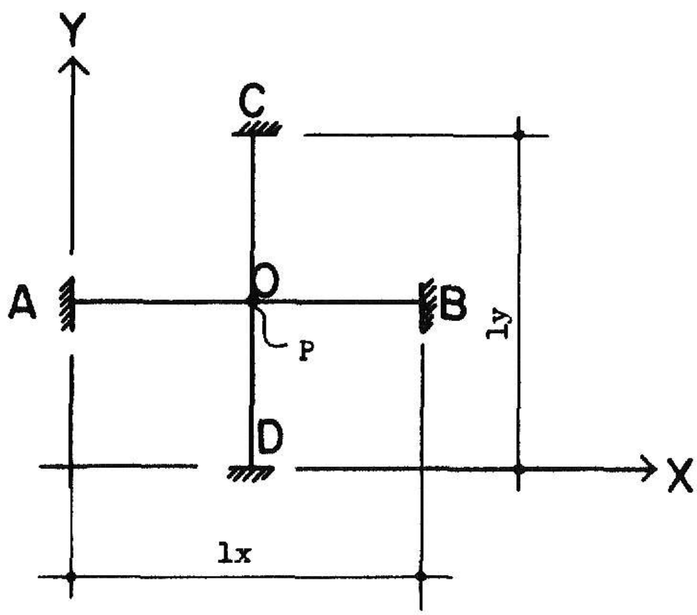


Fig. 3-1 Gridwork of two orthogonal beams

intersection as shown in Figure 3-1 will be studied.

(a). Equations

Assume the applied load P at joint O to be replaced by P_x

and P_y, where P_x and P_y are the loads carried by the beams parallel to the X-axis and Y-axis, respectively, then

$$P_x + P_y = P \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3\text{-}1)$$

The deflections at joint 0 for bars AB and CD are:

$$Y_0^{AB} = \frac{P_x(l_x)^3}{192EI_{AB}}, \qquad Y_0^{CD} = \frac{P_y(l_y)^3}{192EI_{CD}}$$

where I_{AB}, I_{CD} are the moment of inertia of beam AB and CD, respectively.

From $Y_0^{AB} = Y_0^{CD}$, yields

$$P_x = P_y\left(\frac{l_y}{l_x}\right)^3\left(\frac{I_{AB}}{I_{CD}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3\text{-}2)$$

Substituting Equation (3-2) into Equation (3-1), yields

$$P_x = P\,\frac{(l_y)^3(I_{AB})}{(l_x)^3(I_{CD})+(l_y)^3(I_{AB})} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}3)$$

$$P_y = P\,\frac{(l_x)^3(I_{CD})}{(l_x)^3(I_{CD})+(l_y)^3(I_{AB})} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}3a)$$

By using Equation (2-12), the Fixed-end Moments in AB and CD are found to be:

$$M_{AB} = \frac{1}{8}P_x l_x = \frac{P(l_y)^3(I_{AB})(l_x)}{8\left[(l_x)^3(I_{CD})+(l_y)^3(I_{AB})\right]} \quad \ldots\ldots\ldots \quad (3\text{-}1\text{-}A)$$

$$M_{CD} = \frac{1}{8}P_y l_y = \frac{P(l_x)^3(I_{CD})(l_y)}{8\left[(l_x)^3(I_{CD})+(l_y)^3(I_{AB})\right]} \quad \ldots\ldots\ldots \quad (3\text{-}1\text{-}B)$$

By using Equation (2-15a), the Moments at joint 0 for AB and CD are found to be:

$$M_0^{AB} = \frac{1}{8} P_x l_x = M_{AB} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3\text{-}1\text{-}A_1)$$

$$M_0^{CD} = \frac{1}{8} P_y l_y = M_{CD} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3\text{-}1\text{-}B_1)$$

There are no torsional moments induced in this type of gridwork.

(b). Numerical Example

Consider a gridwork fixed at each support, loaded with a 36 kip concentrated load at joint 0. The stiffnesses of the two beams are constant and equal and the dimensions of the gridwork are shown in Figure 3-2.

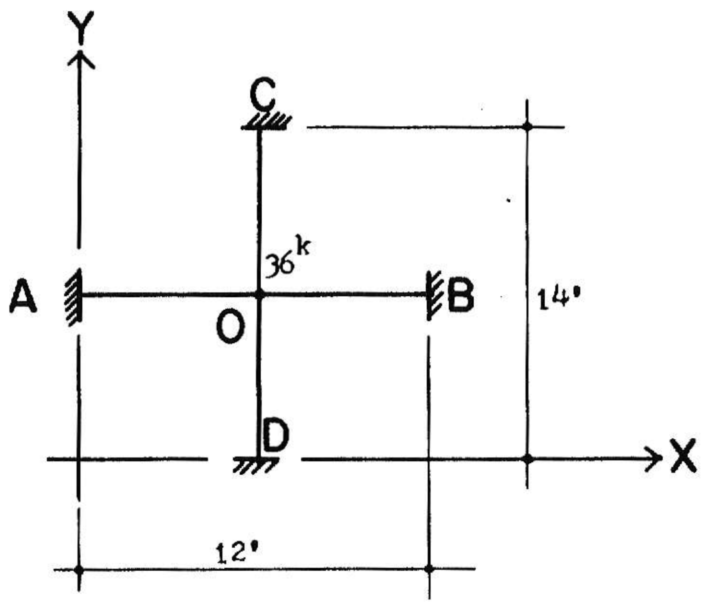


Fig. 3-2 Calculation example for case one

From Equations (3-1-A) to (3-1-D),

$$M_{AB} = \frac{36(14)^3(12)}{8\left[(12)^3+(14)^3\right]} = 33.2^{\text{k-ft}}$$

$$M_{CD} = \frac{36(12)^3(14)}{8\left[(12)^3+(14)^3\right]} = 24.3 \text{ k-ft}$$

$$M_0^{AB} = M_{AB} = 33.2 \text{ k-ft}$$

$$M_0^{CD} = M_{CD} = 24.3 \text{ k-ft}$$

$$R_A = R_B = \frac{P_x}{2} = \frac{1}{2} \cdot 36 \cdot \frac{(14)^3}{(12)^3+(14)^3} = 11.0 \text{ kips}$$

$$R_C = R_D = \frac{P_y}{2} = \frac{1}{2} \cdot 36 \cdot \frac{(12)^3}{(12)^3+(14)^3} = 7.0 \text{ kips}$$

(2). Case two

The gridwork as shown in Figure 3-3 will be studied in this case.

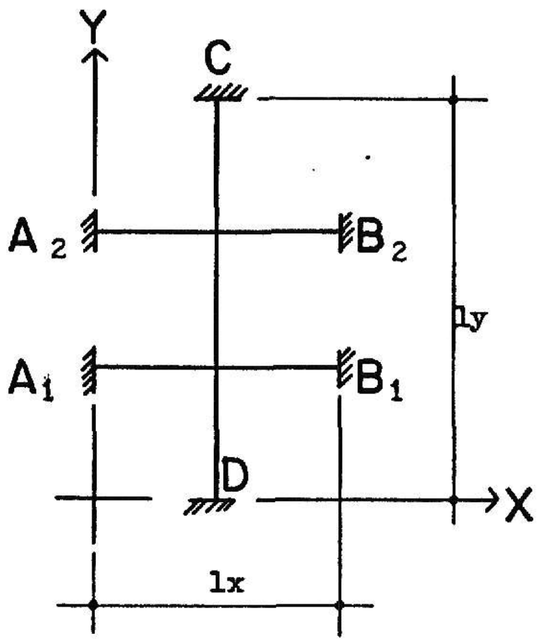


Fig. 3-3 Gridwork with one longitudinal beam and two transverse beams.

(a). Equations

Assume the applied load, P, at joint (x,y) to be replaced by $P_x(x,y)$ and $P_y(x,y)$, where P_x and P_y are the loads carried by the beams parallel to the X-axis and Y-axis, respectively, then

$$\left.\begin{aligned} P_x(1,1) + P_y(1,1) &= P(1,1) \\ P_x(1,2) + P_y(1,2) &= P(1,2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3\text{-}4)$$

The deflections at joint(x,y) for beams A_1B_1, A_2B_2 and CD are:

$$Y^{A_1B_1}_{(1,1)} = \frac{P_{x(1,1)}(l_x)^3}{192EI_{AB}} \quad , \qquad Y^{A_2B_2}_{(1,2)} = \frac{P_{x(1,2)}(l_x)^3}{192EI_{AB}}$$

and using Equation (2-8) yields

$$Y^{CD}_{(1,1)} = P_y(1,1)\frac{16(l_y)^3}{6EI_{CD}\cdot 3^6} + P_y(1,2)\frac{11(l_y)^3}{6EI_{CD}\cdot 3^6}$$

$$Y^{CD}_{(1,2)} = P_y(1,1)\frac{11(l_y)^3}{6EI_{CD}\cdot 3^6} + P_y(1,2)\frac{16(l_y)^3}{6EI_{CD}\cdot 3^6}$$

From

$$Y^{A_1B_1}_{(1,1)} - Y^{CD}_{(1,1)} = 0 \quad \text{and} \quad Y^{A_2B_2}_{(1,2)} - Y^{CD}_{(1,2)} = 0 \quad ,$$

$$\left.\begin{aligned} &P_x(1,1)\frac{(l_x)^3}{32I_{AB}} - P_y(1,1)\frac{16(l_y)^3}{3^6.I_{CD}} - P_y(1,2)\frac{11(l_y)^3}{3^6.I_{CD}} = 0 \\ &\text{and} \\ &P_x(1,2)\frac{(l_x)^3}{32I_{AB}} - P_y(1,1)\frac{11(l_y)^3}{3^6.I_{CD}} - P_y(1,2)\frac{16(l_y)^3}{3^6.I_{CD}} = 0 \end{aligned}\right\} \ldots (3\text{-}5)$$

Setting $\frac{l_y}{l_x} = a$, $\frac{I_{AB}}{I_{CD}} = b$ and substituting into Equation (3-5) yields

$$\left.\begin{aligned} &P_x(1,1)\frac{3^6}{32} - P_y(1,1)\cdot 16a^3b - P_y(1,2)\cdot 11a^3b = 0 \\ &P_x(1,2)\frac{3^6}{32} - P_y(1,1)\cdot 11a^3b - P_y(1,2)\cdot 16a^3b = 0 \end{aligned}\right\} \quad \ldots\ldots (3\text{-}6)$$

(1). Applied load at joint(1,1)

Equation (3-4), yields

$$\left.\begin{aligned} &P_x(1,1)+P_y(1,1)=P(1,1) \\ &P_x(1,2)+P_y(1,2)=0 \end{aligned}\right\} \quad \ldots\ldots (3\text{-}4a)$$

Solving Equations (3-4a) and (3-6)

$$\left.\begin{aligned} &P_x(1,1)=c_{11}^{11}\ P(1,1) \\ &P_x(1,2)=c_{12}^{11}\ P(1,1) \\ &P_y(1,1)=d_{11}^{11}\ P(1,1) \\ &P_y(1,2)=d_{12}^{11}\ P(1,1) \end{aligned}\right\} \quad \ldots\ldots (3\text{-}7a)$$

where c_{11}^{11}, c_{12}^{11}, d_{11}^{11}, d_{12}^{11} are influence coefficients indicating the distribution of the applied load P(1,1) between the transverse and longitudinal beams.

Through the use of a computer program, these coefficients were calculated as shown in Figure 3-4.

(2). Applied load at joint (1,2)

From the same procedures shown in Section (1).

$$\left.\begin{aligned} &P_x(1,1)=c_{11}^{12}\ P(1,2) \\ &P_x(1,2)=c_{12}^{12}\ P(1,2) \end{aligned}\right\} \quad \ldots\ldots (3\text{-}7b)$$

$P_y(1,1)=d_{11}^{12}\ P(1,2)$

$P_y(1,2)=d_{12}^{12}\ P(1,2)$

where c_{11}^{12}, c_{12}^{12}, d_{11}^{12}, d_{12}^{12} are influence coefficients indicating the distribution of the applied load P(1,2) between the transverse and longitudinal beams. From the symmetric geometry of the gridwork,

$c_{11}^{12}=c_{12}^{11}$, $c_{12}^{12}=c_{11}^{11}$, $d_{11}^{12}=d_{12}^{11}$, $d_{12}^{12}=d_{11}^{11}$

(3). By superposition, the results for the component loads $P_x(x,y)$ and $P_y(x,y)$ will be:

$$\left.\begin{aligned}
P_x(1,1)&=c_{11}^{11}P(1,1)+c_{11}^{12}P(1,2)\\
P_x(1,2)&=c_{12}^{11}P(1,1)+c_{12}^{12}P(1,2)\\
P_y(1,1)&=d_{11}^{11}P(1,1)+d_{11}^{12}P(1,2)\\
P_y(1,2)&=d_{12}^{11}P(1,1)+d_{12}^{12}P(1,2)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}2\text{-}A)$$

The stress Equations then can be derived in terms of these component loads $P_x(x,y)$ and $P_y(x,y)$.

From Equation (2-12), the Fixed-end Moments for beams A_1B_1, A_2B_2, and CD are found to be:

$$M_{A_1B_1}=\frac{l_x}{8}\ [P_x(1,1)] \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}2\text{-}B_1)$$

$$M_{A_2B_2}=\frac{l_x}{8}\ [P_x(1,2)] \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}2\text{-}B_2)$$

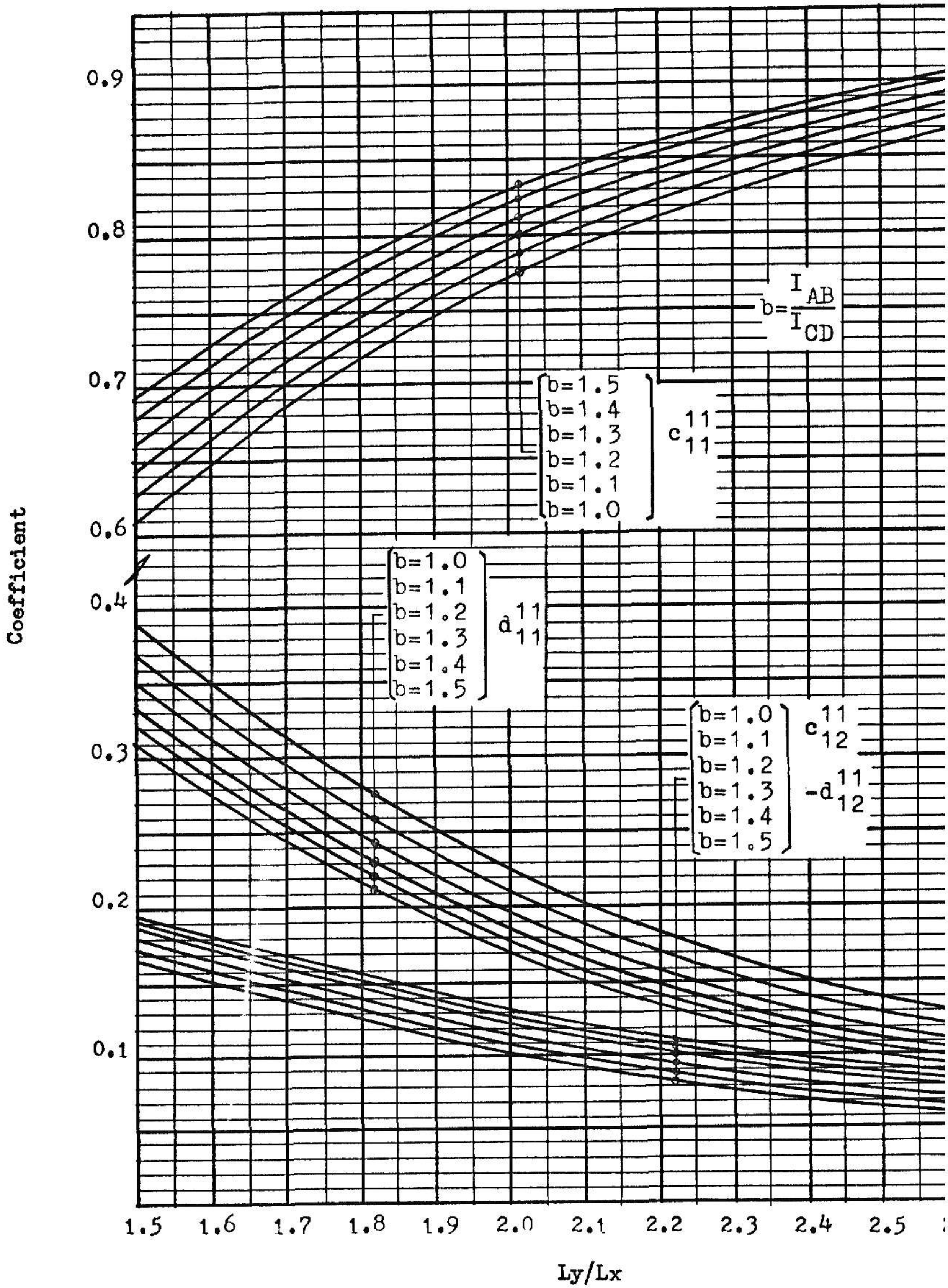


Fig. 3-4 Influence coefficients of P_x and P_y caused by P(1,1), (c

$$M_{CD} = \frac{2}{27} l_y \quad [2P_y(1,1)+P_y(1,2)] \quad \ldots\ldots\ldots\ldots (3\text{-}2\text{-}B_3)$$

$$M_{DC} = \frac{2}{27} l_y \quad [P_y(1,1)+2P_y(1,2)] \quad \ldots\ldots\ldots\ldots (3\text{-}2\text{-}B_4)$$

From Equation (2-15a), the moments at the intermediate joints are:

$$M^{A_1B_1}_{(1,1)} = M_{A_1B_1} \quad \ldots\ldots\ldots\ldots (3\text{-}2\text{-}C_1)$$

$$M^{A_2B_2}_{(1,2)} = M_{A_2B_2} \quad \ldots\ldots\ldots\ldots (3\text{-}2\text{-}C_2)$$

$$M^{CD}_{(1,1)} = \frac{l_y}{81} \left[8P_y(1,1)+P_y(1,2)\right] \quad \ldots\ldots\ldots\ldots (3\text{-}2\text{-}C_3)$$

$$M^{CD}_{(1,2)} = \frac{l_y}{81} \left[P_y(1,1)+8P_y(1,2)\right] \quad \ldots\ldots\ldots\ldots (3\text{-}2\text{-}C_4)$$

In this type of gridwork, the beams A_1B_1 and A_2B_2 will have induced torsional moments.

By using Equation (2-17) or (2-17a),

$$\left.\begin{aligned} Y^{CD}_{(1,1+\Delta y)} &= P_y(1,1)\cdot C_y + P_y(1,2)\cdot D_y \\ Y^{CD}_{(1,1-\Delta y)} &= P_y(1,1)\cdot E_y + P_y(1,2)\cdot F_y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3\text{-}8)$$

$$\left.\begin{aligned} Y^{CD}_{(1,2+\Delta y)} &= P_y(1,1)\cdot D_y + P_y(1,2)\cdot C_y \\ Y^{CD}_{(1,2-\Delta y)} &= P_y(1,1)\cdot F_y + P_y(1,2)\cdot E_y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3\text{-}8a)$$

where
$$C_y = \frac{6l_y(l_y-y_2)^2-7(l_y-y_2)^3}{6EI_{CD}\cdot 27}$$

$$D_y = \frac{6l_y y_2^2-7y_2^3}{6EI_{CD}\cdot 27}$$

$$E_y = \frac{12l_y y_1^2-20y_1^3}{6EI_{CD}\cdot 27}$$

$$F_y = \frac{6l_y y_1^2-y_1^3}{6EI_{CD}\cdot 27}$$

and $y_1 = \frac{l_y}{3} - \Delta y$, $y_2 = \frac{l_y}{3} + \Delta y$

in which Δy is any arbitrary value. Then

$$\theta^x_{(1,1)} = \frac{Y^{CD}_{(1,1+\Delta y)} - Y^{CD}_{(1,1-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots (3\text{-}9)$$

$$\theta^x_{(1,2)} = \frac{Y^{CD}_{(1,2+\Delta y)} - Y^{CD}_{(1,2-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots (3\text{-}9a)$$

By substituting Equations (3-8) and (3-8a) into Equations (3-9), (3-9a), respectively $\theta^x_{(1,1)}$ and $\theta^x_{(1,2)}$, can be found. The torsional moments then can be derived as follows:

$$T^x_{(1,1)} = \frac{2\cdot G\cdot J_{AB}\cdot\theta^x_{(1,1)}}{l_x} \quad \ldots\ldots\ldots\ldots (3\text{-}2\text{-}D_1)$$

$$T^x_{(1,2)} = \frac{2\cdot G\cdot J_{AB}\cdot\theta^x_{(1,2)}}{l_x} \quad \ldots\ldots\ldots\ldots (3\text{-}2\text{-}D_2)$$

Where $T^x_{(1,1)}$ and $T^x_{(1,2)}$ are the torsional moments for the beams parallel to the X-axis in the (x,y) position.

From Equations (2-20) and (2-20a), yields

$$R_{A_1} = R_{B_1} = \frac{1}{2} P_x\,(1,1) \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}2\text{-}E_1)$$

$$R_{A_2} = R_{B_2} = \frac{1}{2} P_x(1,2) \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}2\text{-}E_2)$$

$$R_D = \frac{20}{27} P_y(1,1) + \frac{7}{27} P_y(1,2) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}2\text{-}E_3)$$

$$R_C = \frac{7}{27} P_y(1,1) + \frac{20}{27} P_y(1,2) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}2\text{-}E_4)$$

(b). Numerical Example

Consider a gridwork as shown in Figure 3-5 which is loaded with two 36 Kip concentrated loads at the beam intersections. The factors I, E, G, and J are assumed to be the same for all beams.

From Figure 3-4 we find

$c^{11}_{11} = 0.613$ $\quad c^{12}_{11} = 0.187$ $\quad c^{11}_{12} = 0.187$ $\quad c^{12}_{12} = 0.613$

$d^{11}_{11} = 0.387$ $\quad d^{12}_{11} = -0.187$ $\quad d^{11}_{12} = -0.187$ $\quad d^{12}_{12} = 0.387$

substituting into Equation (3-2-A), yields

$$P_x(1,1) = (0.613+0.187)\cdot 36 = 28.8 \text{ Kips}$$

$$P_x(1,2) = (0.187+0.613)\cdot 36 = 28.8 \text{ Kips}$$

$$P_y(1,1) = (0.387-0.187)\cdot 36 = 7.2 \text{ Kips}$$

$$P_y(1,2) = (-0.187+0.387)\cdot 36 = 7.2 \text{ Kips}$$

Then substituting Equations ($3\text{-}2\text{-}B_1$) into ($3\text{-}2\text{-}C_4$), yields

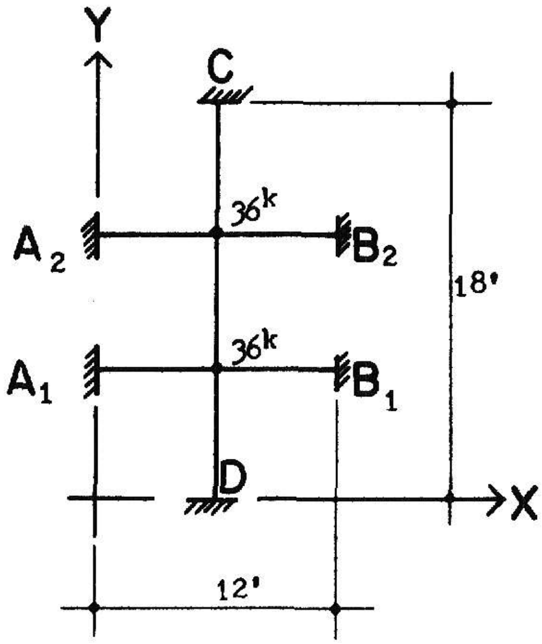


Fig. 3-5 Calculation example for case two

$$M_{A_1B_1} = \frac{1}{8}(28.8)\cdot 12 = 43.2^{\text{ k-ft}} = M^{A_1B_1}_{(1,1)}$$

$$M_{A_2B_2} = \frac{1}{8}(28.8)\cdot 12 = 43.2^{\text{ k-ft}} = M^{A_2B_2}_{(1,2)}$$

$$M_{CD} = \frac{2}{27}(18)(3X7.2) = 28.8^{\text{ k-ft}} = M_{DC}$$

$$M^{CD}_{(1,1)} = \frac{18}{81}(9X7.2) = 14.4^{\text{ k-ft}} = M^{CD}_{(1,2)}$$

Substituting $P_x(x,y)$ and $P_y(x,y)$ into Equations (3-8) and (3-8a), and assuming $\Delta y=1ft$, yields

$$y_1=5^{ft}, \quad y_2=7^{ft}$$

$$C_y = \frac{6X18X(18-7)^2-7(18-7)^3}{6X27} = 23.2$$

$$D_y = \frac{6X18X7^2-7X7^3}{6X27} = 17.8$$

$$E_y = \frac{12X18X5^2-7X5^3}{6X27} = 17.9$$

$$F_y = \frac{6X18X5^2-7X5^3}{6X27} = 11.3$$

$$Y^{CD}_{(1,1+\Delta y)} = 7.2(23.2+17.8) = 285 = Y^{CD}_{(1,2+\Delta y)}$$

$$Y^{CD}_{(1,1-\Delta y)} = 7.2(17.9+11.3) = 210 = Y^{CD}_{(1,2-\Delta y)}$$

$$\theta^{x}_{(1,1)} = \frac{285-210}{2X1} = 37.5$$

$$T^{x}_{(1,1)} = \frac{2X1X1X37.5}{12} = 6.25\ ^{k\text{-}ft} = T^{x}_{(1,2)}$$

$$R_{A_1} = R_{B_1} = R_{A_2} = R_{B_2} = \frac{1}{2}(28.8) = 14.4\ ^{kips}$$

$$R_C = R_D = \left(\frac{20}{27}+\frac{7}{27}\right) X\ 7.2 = 7.2^{kips}$$

end.

(3). Case three

In this case there are two beams intersecting in both the longitudinal and transverse directions as shown in Figure 3-6.

(a). Equations

Assume the applied load P at joint (x,y) to be replaced by $P_x(x,y)$ and $P_y(x,y)$, where P_x and P_y are the loads carried by the beams parallel to the X-axis and Y-axis, respectively, then

$$\left.\begin{aligned} P_x(1,1)+P_y(1,1)&=P(1,1)\\ P_x(2,1)+P_y(2,1)&=P(2,1)\\ P_x(1,2)+P_y(1,2)&=P(1,2)\\ P_x(2,2)+P_y(2,2)&=P(2,2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}10)$$

By using Equation (2-8), the deflections at joint (x,y) for beams A_1B_1, A_2B_2, C_1D_1 and C_2D_2 are:

$$Y^{A_1B_1}_{(1,1)} = P_x(1,1)\cdot A_x + P_x(2,1)\cdot B_x$$

$$Y^{A_1B_1}_{(2,1)} = P_x(1,1)\cdot B_x + P_x(2,1)\cdot A_x$$

$$Y^{A_2A_2}_{(1,2)} = P_x(1,2)\cdot A_x + P_x(2,2)\cdot B_x$$

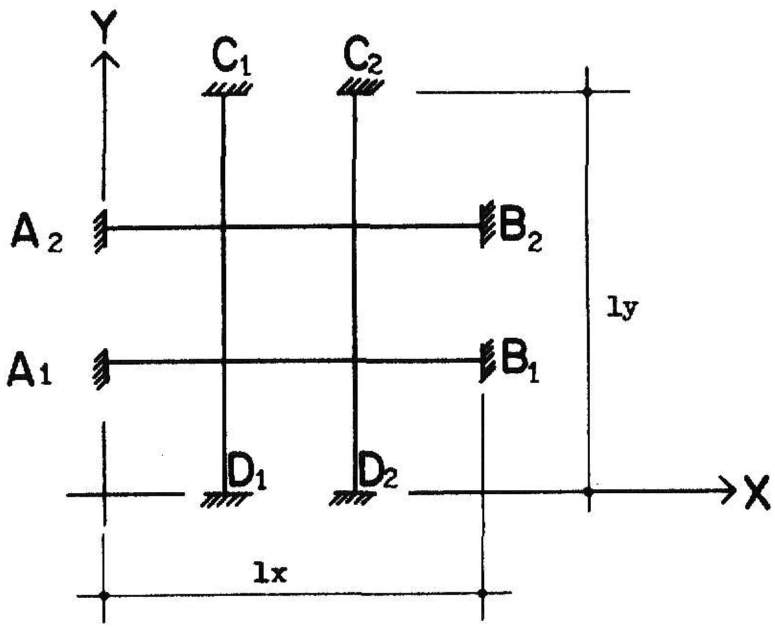


Fig. 3-6 Gridwork with two longitudinal and transverse beams

$$Y^{A_2B_2}_{(2,2)} = P_x(1,2)\cdot B_x + P_x(2,2)\cdot A_x$$

where $A_x = \dfrac{16(l_x)^3}{6EI_{AB}\cdot 3^6}$, $\qquad B_x = \dfrac{11(l_x)^3}{6EI_{AB}\cdot 3^6}$

$$Y^{C_1D_1}_{(1,1)} = P_y(1,1)\cdot A_y + P_y(1,2)\cdot B_y$$

$$Y^{C_1D_1}_{(1,2)} = P_y(1,1)\cdot B_y + P_y(1,2)\cdot A_y$$

$$Y^{C_2D_2}_{(2,1)} = P_y(2,1)\cdot A_y + P_y(2,2)\cdot B_y$$

$$Y^{C_2D_2}_{(2,2)} = P_y(2,1)\cdot B_y + P_y(2,2)\cdot A_y$$

where $A_y = \dfrac{16(l_y)^3}{6EI_{CD}\cdot 3^6}$, $\qquad B_y = \dfrac{11(l_y)^3}{6EI_{CD}\cdot 3^6}$

Recognizing that

$$Y^{A_1B_1}_{(1,1)} = Y^{C_1D_1}_{(1,1)} \quad , \quad Y^{A_1B_1}_{(2,1)} = Y^{C_2D_2}_{(2,1)}$$

$$Y^{A_2B_2}_{(1,2)} = Y^{C_1D_1}_{(1,2)} \quad , \quad Y^{A_2B_2}_{(2,2)} = Y^{C_2D_2}_{(2,2)}$$, yields

$$\left.\begin{aligned}
&P_x(1,1)\cdot A_x + P_x(2,1)\cdot B_x - P_y(1,1)\cdot A_y - P_y(1,2)\cdot B_y = 0 \\
&P_x(1,1)\cdot B_x + P_x(2,1)\cdot A_x - P_y(2,1)\cdot A_y - P_y(2,2)\cdot B_y = 0 \\
&P_x(1,2)\cdot A_x + P_x(2,2)\cdot B_x - P_y(1,1)\cdot B_y - P_y(1,2)\cdot A_y = 0 \\
&P_x(1,2)\cdot B_x + P_x(2,2)\cdot A_x - P_y(2,1)\cdot B_y - P_y(2,2)\cdot A_y = 0
\end{aligned}\right\} \quad . . (3\text{-}11)$$

Setting $\dfrac{l_y}{l_x} = a$, $\dfrac{I_{AB}}{I_{CD}} = b$ and substituting into Equation

(3-11), yields

$$\left.\begin{aligned}
&P_x(1,1).16 + P_x(2,1).11 - P_y(1,1).16a^3b - P_y(1,2).11a^3b = 0\\
&P_x(1,1).11 + P_x(2,1).16 - P_y(2,1).16a^3b - P_y(2,2).11a^3b = 0\\
&P_x(1,2).16 + P_x(2,2).11 - P_y(1,1).11a^3b - P_y(1,2).16a^3b = 0\\
&P_x(1,2).11 + P_x(2,2).16 - P_y(2,1).11a^3b - P_y(2,2).16a^3b = 0
\end{aligned}\right\} \quad \ldots (3\text{-}11a)$$

(1). Applied load at joint (1,1)

Equation (3-10), yields

$$\left.\begin{aligned}
&P_x(1,1) + P_y(1,1) = P(1,1)\\
&P_x(2,1) + P_y(2,1) = 0\\
&P_x(1,2) + P_y(1,2) = 0\\
&P_x(2,2) + P_y(2,2) = 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3\text{-}10a)$$

By solving Equations (3-10a) and (3-11a),

$$\left.\begin{aligned}
&P_x(x,y) = c_{xy}^{11}P(1,1)\\
&P_y(x,y) = d_{xy}^{11}P(1,1)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3\text{-}12a)$$

where c_{xy}^{11}, d_{xy}^{11} are influence coefficients indicating the distribution of the applied load P(1,1) between the transverse and longitudinal beams.

Through the use of a computer program, these coefficients were calculated as shown in Figure 3-7.

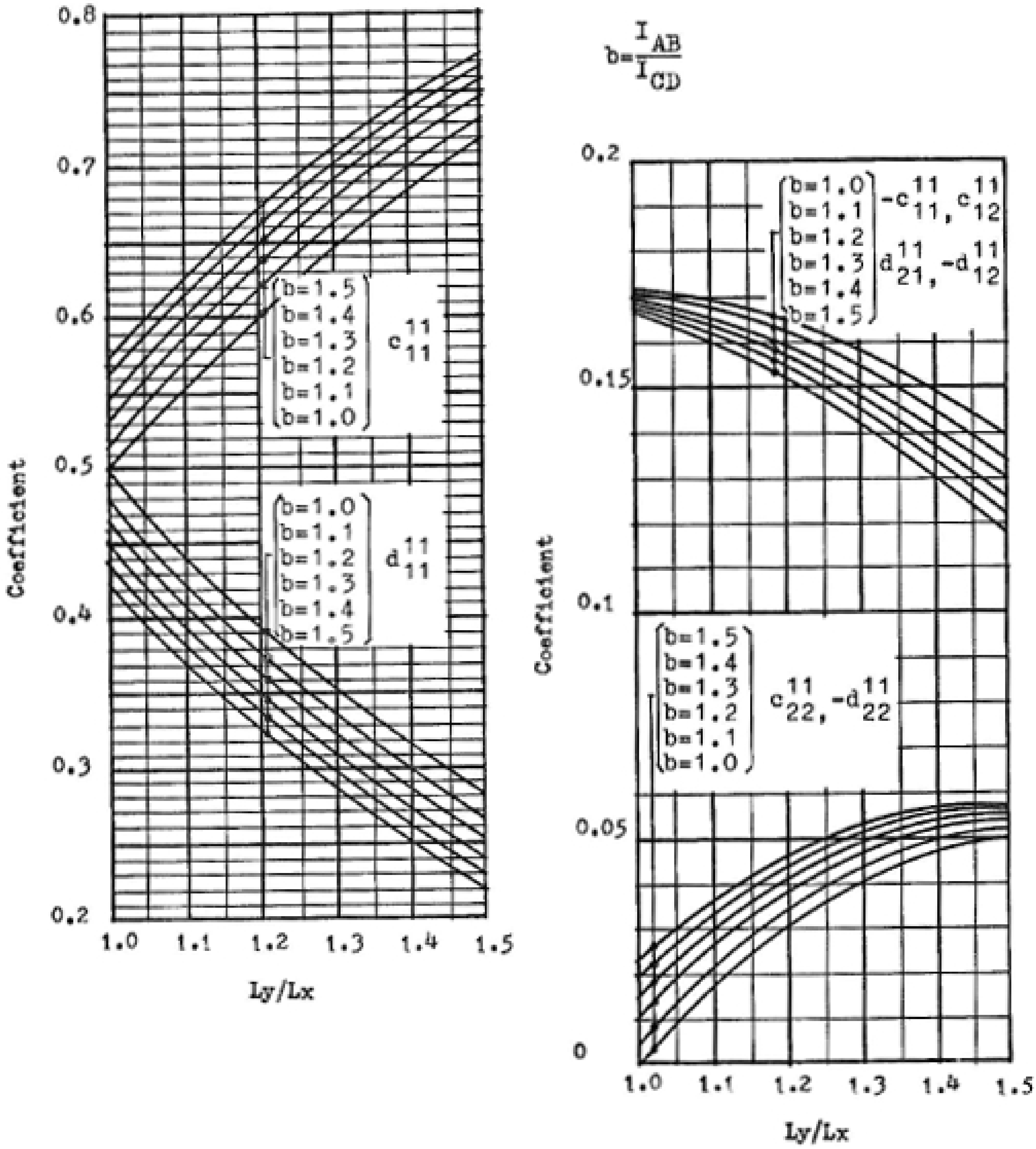


Fig. 3-7 Influence Coefficients of Px and Py caused by P(1,1), (case 3)

(2) Applied load at joint (1,2)

By the same process shown in Section (1),

$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{12}P(1,2) \\ P_y(x,y) &= d_{xy}^{12}P(1,2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3\text{-}12b)$$

where c_{xy}^{12}, d_{xy}^{12} are influence coefficients indicating the distribution of the applied load P(1,2) between the transverse and longitudinal beams. From the symmetric geometry of the gridwork,

$$c_{11}^{12} = c_{12}^{11} \ , \quad c_{12}^{12} = c_{11}^{11} \ , \quad c_{21}^{12} = c_{22}^{11} \ , \quad c_{22}^{12} = c_{21}^{11}$$

$$d_{11}^{12} = d_{12}^{11} \ , \quad d_{12}^{12} = d_{11}^{11} \ , \quad d_{21}^{12} = d_{22}^{11} \ , \quad d_{22}^{12} = d_{21}^{11}$$

(3) Applied load at joint (2,1) and (2,2) also can be found as follows:

$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{21}P(2,1) \\ P_y(x,y) &= d_{xy}^{21}P(2,1) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3\text{-}12c)$$

$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{22}P(2,2) \\ P_y(x,y) &= d_{xy}^{22}P(2,2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3\text{-}12d)$$

where

$$c_{11}^{21} = c_{21}^{11} \ , \quad c_{12}^{21} = c_{22}^{11} \ , \quad c_{21}^{21} = c_{11}^{11} \ , \quad c_{22}^{21} = c_{12}^{11} \ ,$$

$$d_{11}^{21} = d_{21}^{11} \ , \quad d_{12}^{21} = d_{22}^{11} \ , \quad d_{21}^{21} = d_{11}^{11} \ , \quad d_{22}^{21} = d_{12}^{11} \ ,$$

$$c_{11}^{22} = c_{22}^{11} \quad , \quad c_{12}^{22} = c_{21}^{11} \quad , \quad c_{21}^{22} = c_{12}^{11} \, , \quad c_{22}^{22} = c_{11}^{11} \quad ,$$

$$d_{11}^{22} = d_{22}^{11} \quad , \quad d_{12}^{22} = d_{21}^{11} \quad , \quad d_{21}^{22} = d_{12}^{11} \, , \quad d_{22}^{22} = d_{11}^{11} \quad ,$$

(4) By superposition, the results of the component loads $P_x(x,y)$ and $P_y(x,y)$ will be:

$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{11}P(1,1) + c_{xy}^{12}P(1,2) + c_{xy}^{21}P(2,1) + c_{xy}^{22}P(2,2) \\ P_y(x,y) &= d_{xy}^{11}P(1,1) + d_{xy}^{12}P(1,2) + d_{xy}^{21}P(2,1) + d_{xy}^{22}P(2,2) \end{aligned}\right\} \quad \ldots \text{(3-3-A)}$$

The stress equations then can be derived in terms of those component loads $P_x(x,y)$ and $P_y(x,y)$.

From Equation (2-12), the Fixed-end Moments for beams A_1B_1, A_2B_2, C_1D_1, and C_2D_2 are determined as:

$$M_{A_1B_1} = \frac{2}{27}(l_x) \quad \left[2P_x(1,1) + P_x(2,1)\right] \quad \ldots\ldots\ldots \text{(3-3-B}_1\text{)}$$

$$M_{B_1A_1} = \frac{2}{27}(l_x) \quad \left[P_x(1,1) + 2P_x(2,1)\right] \quad \ldots\ldots\ldots \text{(3-3-B}_2\text{)}$$

$$M_{A_2B_2} = \frac{2}{27}(l_x) \quad \left[2P_x(1,2) + P_x(2,2)\right] \quad \ldots\ldots\ldots \text{(3-3-B}_3\text{)}$$

$$M_{B_2A_2} = \frac{2}{27}(l_x) \quad \left[P_x(1,2) + 2P_x(2,2)\right] \quad \ldots\ldots\ldots \text{(3-3-B}_4\text{)}$$

$$M_{C_1D_1} = \frac{2}{27}(l_y) \quad \left[2P_y(1,1) + P_y(1,2)\right] \quad \ldots\ldots\ldots \text{(3-3-B}_5\text{)}$$

$$M_{D_1C_1} = \frac{2}{27}(l_y) \quad \left[P_y(1,1) + 2P_y(1,2)\right] \quad \ldots\ldots\ldots \text{(3-3-B}_6\text{)}$$

$$M_{C_2D_2} = \frac{2}{27}(l_y) \quad \left[2P_y(2,1) + P_y(2,2)\right] \quad \ldots\ldots\ldots \text{(3-3-B}_7\text{)}$$

$$M_{D_2C_2} = \frac{2}{27}(l_y) \quad \left[P_y(2,1) + 2P_y(2,2)\right] \quad \ldots\ldots\ldots \text{(3-3-B}_8\text{)}$$

and from Equation (2-15a) the Moments at the intermediate joints are:

$$M^{A_1B_1}_{(1,1)} = \frac{l_x}{81}\,[8P_x(1,1) + P_x(2,1)] \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}3\text{-}C_1)$$

$$M^{A_1B_1}_{(2,1)} = \frac{l_x}{81}\,[P_x(1,1) + 8P_x(2,1)] \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}3\text{-}C_2)$$

$$M^{A_2B_2}_{(1,2)} = \frac{l_x}{81}\,[8P_x(1,2) + P_x(2,2)] \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}3\text{-}C_3)$$

$$M^{A_2B_2}_{(2,2)} = \frac{l_x}{81}\,[P_x(1,2) + 8P_x(2,2)] \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}3\text{-}C_4)$$

$$M^{C_1D_1}_{(1,1)} = \frac{l_y}{81}\,[8P_y(1,1) + P_y(1,2)] \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}3\text{-}C_5)$$

$$M^{C_1D_1}_{(1,2)} = \frac{l_y}{81}\,[P_y(1,1) + 8P_y(1,2)] \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}3\text{-}C_6)$$

$$M^{C_2D_2}_{(2,1)} = \frac{l_y}{81}\,[8P_y(2,1) + P_y(2,2)] \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}3\text{-}C_7)$$

$$M^{C_2D_2}_{(2,2)} = \frac{l_y}{81}\,[P_y(2,1) + 8P_y(2,2)] \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}3\text{-}C_8)$$

Using Equation (2-17) or (2-17a),

$$\left.\begin{aligned} Y^{C_1D_1}_{(1,1+\Delta y)} &= P_y(1,1)\cdot C_y + P_y(1,2)\cdot D_y \\ Y^{C_1D_1}_{(1,1-\Delta y)} &= P_y(1,1)\cdot E_y + P_y(1,2)\cdot F_y \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}13a)$$

$$\left.\begin{aligned} Y^{C_2D_2}_{(2,1+\Delta y)} &= P_y(2,1)\cdot C_y + P_y(2,2)\cdot D_y \\ Y^{C_2D_2}_{(2,1-\Delta y)} &= P_y(2,1)\cdot E_y + P_y(2,2)\cdot F_y \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \qquad (3\text{-}13b)$$

$$\left.\begin{aligned} Y^{C_1D_1}_{(1,2+\Delta y)} &= P_y(1,1)\cdot D_y + P_y(1,2)\cdot C_y \\ Y^{C_1D_1}_{(1,2-\Delta y)} &= P_y(1,1)\cdot F_y + P_y(1,2)\cdot E_y \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}13c)$$

$$\left.\begin{aligned} Y^{C_2D_2}_{(2,2+\Delta y)} &= P_y(2,1)\cdot D_y + P_y(2,2)\cdot C_y \\ Y^{C_2D_2}_{(2,2-\Delta y)} &= P_y(2,1)\cdot F_y + P_y(2,2)\cdot E_y \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}13d)$$

where

$$C_y = \frac{6\,l_y(l_y - y_2)^2 - 7(l_y - y_2)^3}{6EI_{CD}\cdot 27}$$

$$D_y = \frac{6\,l_y y_2^2 - 7y_2^3}{6EI_{CD}\cdot 27}$$

$$E_y = \frac{12\,l_y y_1^2 - 20\,y_1^3}{6EI_{CD}\cdot 27}$$

$$F_y = \frac{6\,l_y y_1^2 - 7y_1^3}{6EI_{CD}\cdot 27}$$

in which $y_1 = \frac{l_y}{3} - \Delta y$, $y_2 = \frac{l_y}{3} + \Delta y$, Δy is any arbitrary value.

and

$$\left.\begin{aligned} Y^{A_1B_1}_{(1+\Delta x,\,1)} &= P_x(1,1)\cdot C_x + P_x(2,1)\cdot D_x \\ Y^{A_1B_1}_{(1-\Delta x,\,1)} &= P_x(1,1)\cdot E_x + P_x(2,1)\cdot F_x \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}14a)$$

$$\left.\begin{aligned} Y^{A_2B_2}_{(1+\Delta x,2)} &= P_x(1,2)\cdot C_x + P_x(2,2)\cdot D_x \\ Y^{A_2B_2}_{(1-\Delta x,2)} &= P_x(1,2)\cdot E_x + P_x(2,2)\cdot F_x \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}14b)$$

$$\left.\begin{aligned} Y^{A_1B_1}_{(2+\Delta x,1)} &= P_x(1,1)\cdot D_x + P_x(2,1)\cdot C_x \\ Y^{A_1B_1}_{(2-\Delta x,1)} &= P_x(1,1)\cdot F_x + P_x(2,1)\cdot E_x \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3\text{-}14c)$$

$$\left.\begin{aligned} Y^{A_2B_2}_{(2+\Delta x,2)} &= P_x(1,2)\cdot D_x + P_x(2,2)\cdot C_x \\ Y^{A_2B_2}_{(2-\Delta x,2)} &= P_x(1,2)\cdot F_x + P_x(2,2)\cdot E_x \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3\text{-}14d)$$

where $C_x = \dfrac{6{}^1x({}^1x - x_2)^2 - 7({}^1x - x_2)^3}{6EI_{AB}\cdot 27}$

$$D_x = \frac{6{}^1x\, x_2^2 - 7x_2^3}{6EI_{AB}\cdot 27}$$

$$E_x = \frac{12{}^1x\, x_1^2 - 20x_1^3}{6EI_{AB}\cdot 27}$$

$$F_x = \frac{6{}^1x\, x_1^2 - 7x_1^3}{6EI_{AB}\cdot 27}$$

in which $x_1 = \dfrac{{}^1x}{3} - \Delta x$, $\quad x_2 = \dfrac{{}^1x}{3} + \Delta x$, where Δx is any arbitrary value.

Then

$$\theta^x_{(1,1)} = \frac{Y^{C_1D_1}_{(1,1+\Delta y)} - Y^{C_1D_1}_{(1,1-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots (3\text{-}15a)$$

$$\theta^x_{(2,1)} = \frac{Y^{C_2D_2}_{(2,1+\Delta y)} - Y^{C_2D_2}_{(2,1-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots (3\text{-}15b)$$

$$\theta^{x}_{(1,2)} = \frac{Y^{C_1D_1}_{(1,2+\Delta y)} - Y^{C_1D_1}_{(1,2-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}15c)$$

$$\theta^{x}_{(2,2)} = \frac{Y^{C_2D_2}_{(2,2+\Delta y)} - Y^{C_2D_2}_{(2,2-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}15d)$$

and

$$\theta^{y}_{(1,1)} = \frac{Y^{A_1B_1}_{(1+\Delta x,1)} - Y^{A_1B_1}_{(1-\Delta x,1)}}{2\Delta x} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}16a)$$

$$\theta^{y}_{(1,2)} = \frac{Y^{A_2B_2}_{(1+\Delta x,2)} - Y^{A_2B_2}_{(1-\Delta x,2)}}{2\Delta x} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}16b)$$

$$\theta^{y}_{(2,1)} = \frac{Y^{A_1B_1}_{(2+\Delta x,1)} - Y^{A_1B_1}_{(2-\Delta x,1)}}{2\Delta x} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}16c)$$

$$\theta^{y}_{(2,2)} = \frac{Y^{A_2B_2}_{(2+\Delta x,2)} - Y^{A_2B_2}_{(2-\Delta x,2)}}{2\Delta x} \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}16d)$$

Substituting Equations (3-13a) to (3-13d) into Equations (3-15a) to (3-15d), and substituting Equations (3-14a) to (3-14d) into Equations (3-16a) to (3-16d), $\theta^{x}_{(x,y)}$ and $\theta^{y}_{(x,y)}$, respectively can be found.

The Torsional Moments then can be derived as follows:

$$T^{x}_{(1,1)} = \frac{3\cdot G\cdot J_{AB}\cdot \theta^{x}_{(1,1)}}{l_x} \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}D_1)$$

$$T^{x}_{(2,1)} = \frac{3\cdot G\cdot J_{AB}\cdot\left[\theta^{x}_{(2,1)} - \theta^{x}_{(1,1)}\right]}{l_x} \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}D_2)$$

$$T^{x}_{(3,1)} = \frac{3\cdot G\cdot J_{AB}\cdot \theta^{x}_{(2,1)}}{l_x} \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}D_3)$$

$$T^{x}_{(1,2)} = \frac{3\cdot G\cdot J_{AB}\cdot \theta^{x}_{(1,2)}}{l_x} \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}D_4)$$

$$T^{x}_{(2,2)} = \frac{3 \cdot G \cdot J_{AB} \cdot [\theta^{x}_{(2,2)} - \theta^{x}_{(1,2)}]}{l_x} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}D_5)$$

$$T^{x}_{(3,2)} = \frac{3 \cdot G \cdot J_{AB} \cdot \theta^{x}_{(2,2)}}{l_x} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}D_6)$$

$$T^{y}_{(1,1)} = \frac{3 \cdot G \cdot J_{CD} \cdot \theta^{y}_{(1,1)}}{l_y} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}D_7)$$

$$T^{y}_{(1,2)} = \frac{3 \cdot G \cdot J_{CD} \cdot [\theta^{y}_{(1,}}{l_y} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}D_8)$$

$$T^{y}_{(1,3)} = \frac{3 \cdot G \cdot J_{CD} \cdot \theta_{(}}{l_y} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}D_9)$$

$$T^{y}_{(2,1)} = \frac{3 \cdot G \cdot J_{CD} \cdot \theta^{y}_{(2,1)}}{l_y} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}D_{10})$$

$$T^{y}_{(2,2)} = \frac{3 \cdot G \cdot J_{CD} \cdot [\theta^{y}_{(2,2)} - \theta^{y}_{(2,1)}]}{l_y} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}D_{11})$$

$$T^{y}_{(2,3)} = \frac{3 \cdot G \cdot J_{CD} \cdot \theta^{y}_{(2,2)}}{l_y} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}D_{12})$$

Where $T^{x}_{(x,y)}$ and $T^{y}_{(x,y)}$ are the Torsional Moments for the beams parallel to the X-axis and the Y-axis at the (x,y) position.

From Equations (2-20) and (2-20a), yields

$$R_{A_1} = \frac{20}{27} P_x(1,1) + \frac{7}{27} P_x(2,1) \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}E_1)$$

$$R_{B_1} = \frac{7}{27} P_x(1,1) + \frac{20}{27} P_x(2,1) \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}E_2)$$

$$R_{A_2} = \frac{20}{27} P_x(1,2) + \frac{7}{27} P_x(2,2) \quad \cdots\cdots\cdots\cdots \quad (3\text{-}3\text{-}E_3)$$

$$R_{E_2} = \frac{7}{27} P_x(1,2) + \frac{20}{27} P_x(2,2) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}E_4)$$

$$R_{D_1} = \frac{20}{27} P_y(1,1) + \frac{7}{27} P_y(1,2) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}E_5)$$

$$R_{C_1} = \frac{7}{27} P_y(1,1) + \frac{20}{27} P_y(1,2) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}E_6)$$

$$R_{D_2} = \frac{20}{27} P_y(2,1) + \frac{7}{27} P_y(2,2) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}E_7)$$

$$R_{C_2} = \frac{7}{27} P_y(2,1) + \frac{20}{27} P_y(2,2) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}3\text{-}E_8)$$

(b) Numerical Example

The gridwork loaded with four 36 kip concentrated loads at four intersections as shown in Figure 3-8 will be considered in this example. The factors I, E, G and J are assumed to be the same for all beams.

From Figure (3-7),

$$c_{11}^{11} = 0.5 = d_{12}^{12} = c_{21}^{21} = c_{22}^{22}, \qquad d_{11}^{11} = 0.5 = d_{12}^{12} = d_{21}^{21} = d_{22}^{22},$$

$$c_{21}^{11} = -0.172 = c_{22}^{12} = c_{11}^{21} = c_{12}^{22}, \qquad d_{21}^{11} = 0.172 = d_{22}^{12} = d_{11}^{21} = d_{12}^{22},$$

$$c_{12}^{11} = 0.172 = c_{11}^{12} = c_{22}^{21} = c_{21}^{22}, \qquad d_{12}^{11} = -0.172 = d_{11}^{12} = d_{22}^{21} = d_{21}^{22},$$

$$c_{22}^{11} = 0 = c_{21}^{12} = c_{12}^{21} = c_{11}^{22}, \qquad d_{22}^{11} = 0 = d_{21}^{12} = d_{12}^{21} = d_{11}^{22},$$

Substituting into Equations (3-3-A),

$$P_x(1,1) = (0.5+0.172-0.172+0)\text{x}36 = 18^{\text{ kips}} = P_y(1,1)$$

$$P_x(1,2) = (0.172+0.5+0-0.172)\text{x}36 = 18^{\text{ kips}} = P_y(1,2)$$

$$P_x(2,1) = (-0.172+0+0.5+0.172)\times 36 = 18 \text{ kips} = P_y(2,1)$$

$$P_x(2,2) = (0-0.172+0.172+0.5)\times 36 = 18 \text{ kips} = P_y(2,2)$$

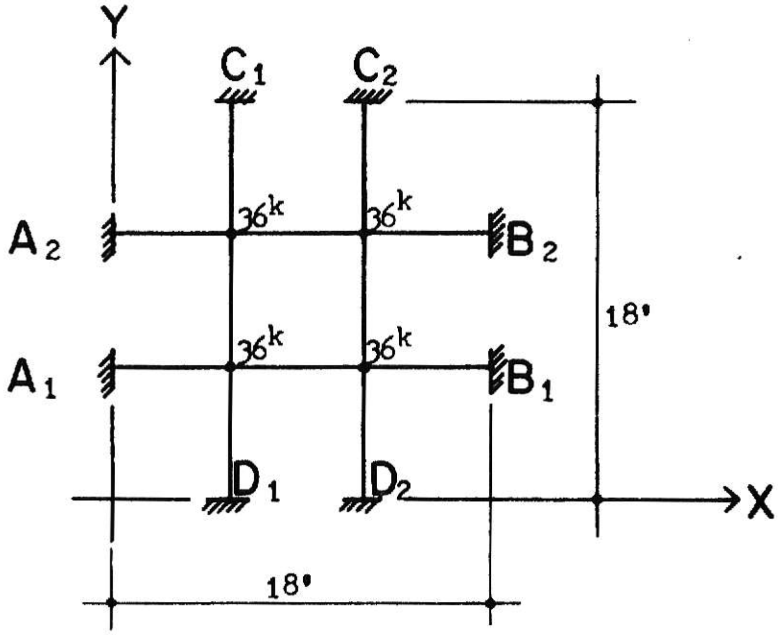


Fig. 3-8 Calculation example for case three.

Then substituting into Equations (3-3-B_1) to (3-3-C_8),

$$M_{A_1B_1} = \frac{2}{27}(18)(3\times18) = 72 \text{ kip-ft} = M_{B_1A_1} = M_{A_2B_2} = M_{B_2A_2}$$

$$= M_{C_1D_1} = M_{D_1C_1} = M_{C_2D_2} = M_{D_2C_2}.$$

$$M^{A_1B_1}_{(1,1)} = \frac{18}{81}(9\times18) = 36 \text{ kip-ft} = M^{A_1B_1}_{(2,1)} = M^{A_2B_2}_{(1,2)} = M^{A_2B_2}_{(2,2)}$$

$$= M^{C_1D_1}_{(1,1)} = M^{C_1D_1}_{(1,2)} = M^{C_2D_2}_{(2,1)} = M^{C_2D_2}_{(2,2)}.$$

By substituting $P_x(x,y)$, $P_y(x,y)$ into Equations (3-13a) to (3-14d) and assuming $\Delta x = \Delta y = 1^{ft}$,

$$x_1 = 5^{ft} = y_1, \quad x_2 = 7^{ft} = y_2$$

$$C_x = \frac{6x18(18-7)^2-7(18-7)^3}{6x27} = 23.2 = C_y$$

$$D_x = \frac{6x18x7^2-7x7^3}{6x27} = 17.8 = D_y$$

$$E_x = \frac{12x18x5^2-20x5^3}{6x27} = 17.9 = E_y$$

$$F_x = \frac{6x18x5^2-7X5^3}{6x27} = 11.3 = F_y$$

$$Y^{A_1 B_1}_{(1+\Delta x,1)} = (23.2+17.8)x18 = 738$$

$$Y^{A_1 B_1}_{(1-\Delta x,1)} = (17.9+11.3)x18 = 526$$

$$Q^x_{(1,1)} = \frac{738-526}{2x1} = 106$$

$$T^x_{(1,1)} = \frac{3x1x1x106}{18} = 17.7^{\,k\text{-}ft} = T^x_{(3,1)} = T^x_{(3,2)} = T^x_{(1,2)}$$

$$= T^y_{(1,1)} = T^y_{(2,1)} = T^y_{(1,3)} = T^y_{(2,3)}$$

$$T^x_{(2,1)} = T^x_{(2,2)} = T^y_{(1,2)} = T^y_{(2,2)} = 0$$

$$R_{A_1} = R_{B_1} = R_{A_2} = R_{B_2} = \left(\frac{20+7}{27}\right) x\ 18 = 18^{\ kips}$$

$$R_{C_1} = R_{C_2} = R_{D_1} = R_{D_2} = \left(\frac{20+7}{27}\right) x\ 18 = 18^{\ kips}$$

end

(4). Case four

Consider the case shown in Figure 3-9, which contains two longitudinal beams and three transverse beams.

(a). Equations

Assume the applied load P at joint (x,y) to be replaced by $P_x(x,y)$ and $P_y(x,y)$, where P_x and P_y are the loads carried by the beams parallel to the X-axis and Y-axis, respectively, then

$$P_x(x,y) + P_y(x,y) = P(x,y) \quad \ldots\ldots\ldots\ldots\ldots \quad (3\text{-}17)$$

Using Equation (2-8), the deflections at joint (x,y) for beams A_1B_1, A_2B_2, A_3B_3, C_1D_1 and C_2D_2 are:

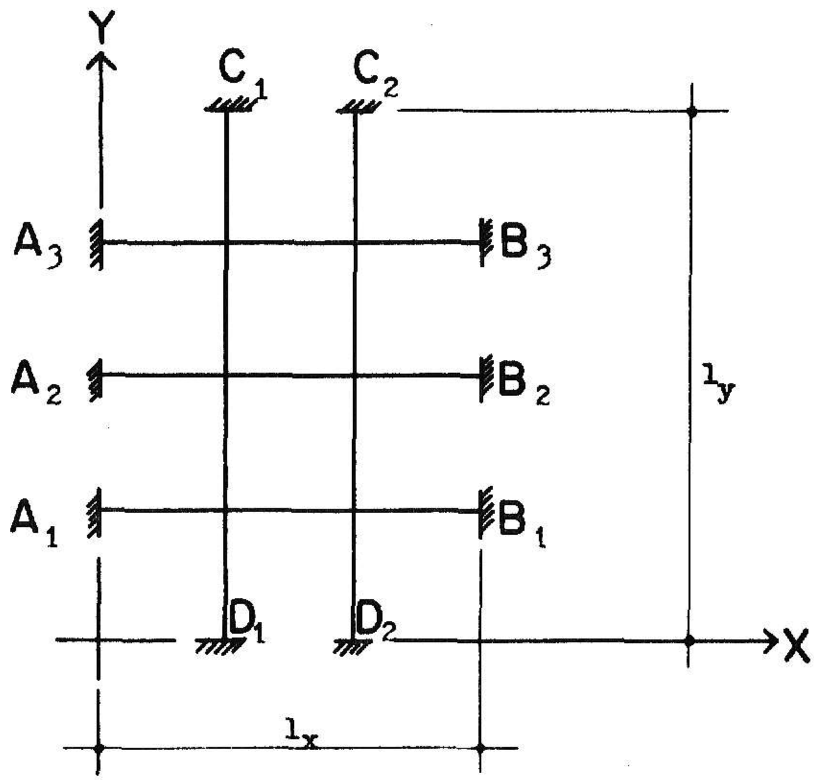


Fig. 3-9 Gridwork with two longitudinal beams and three transverse beams

$$Y^{A_1B_1}_{(1,1)} = P_x(1,1)\cdot A_x + P_x(2,1)\cdot B_x$$

$$Y^{A_1B_1}_{(2,1)} = P_x(1,1)\cdot B_x + P_x(2,1)\cdot A_x$$

$$Y^{A_2B_2}_{(1,2)} = P_x(1,2)\cdot A_x + P_x(2,2)\cdot B_x$$

$$Y^{A_2B_2}_{(2,2)} = P_x(1,2)\cdot B_x + P_x(2,2)\cdot A_x$$

$$Y^{A_3B_3}_{(1,3)} = P_x(1,3)\cdot A_x + P_x(2,3)\cdot B_x$$

$$Y^{A_3B_3}_{(2,3)} = P_x(1,3)\cdot B_x + P_x(2,3)\cdot A_x$$

where $A_x = \dfrac{16(l_x)^3}{6EI_{AB}\cdot 3^6}$, $B_x = \dfrac{11(l_x)^3}{6EI_{AB}\cdot 3^6}$

and

$$Y^{C_1D_1}_{(1,1)} = P_y(1,1)\cdot A_y + P_y(1,2)\cdot B_y + P_y(1,3)\cdot C_y$$

$$Y^{C_1D_1}_{(1,2)} = P_y(1,1)\cdot B_y + P_y(1,2)\cdot 2B_y + P_y(1,3)\cdot B_y$$

$$Y^{C_1D_1}_{(1,3)} = P_y(1,1)\cdot C_y + P_y(1,2)\cdot B_y + P_y(1,3)\cdot A_y$$

$$Y^{C_2D_2}_{(2,1)} = P_y(2,1)\cdot A_y + P_y(2,2)\cdot B_y + P_y(2,3)\cdot C_y$$

$$Y^{C_2D_2}_{(2,2)} = P_y(2,1)\cdot B_y + P_y(2,2)\cdot 2B_y + P_y(2,3)\cdot B_y$$

$$Y^{C_2D_2}_{(2,3)} = P_y(2,1)\cdot C_y + P_y(2,2)\cdot B_y + P_y(2,3)\cdot A_y$$

where $A_y = \frac{54(^1y)^3}{6EI_{CD}\cdot 4^6}$, $B_y = \frac{64(^1y)^3}{6EI_{CD}\cdot 4^6}$, $C_y = \frac{26(^1y)^3}{6EI_{CD}\cdot 4^6}$.

Substituting into

$$Y^{A_1B_1}_{(1,1)} - Y^{C_1D_1}_{(1,1)} = 0 , \qquad Y^{A_1B_1}_{(2,1)} - Y^{C_2D_2}_{(2,1)} = 0 ,$$

$$Y^{A_2B_2}_{(1,2)} - Y^{C_1D_1}_{(1,2)} = 0 , \qquad Y^{A_2B_2}_{(2,2)} - Y^{C_2D_2}_{(2,2)} = 0 ,$$

$$Y^{A_3B_3}_{(1,3)} - Y^{C_1D_1}_{(1,3)} = 0 , \qquad Y^{A_3B_3}_{(2,3)} - Y^{C_2D_2}_{(2,3)} = 0$$

yields

$$\left.\begin{aligned}
&P_x(1,1)\cdot A_x+P_x(2,1)\cdot B_x-P_y(1,1)\cdot A_y-P_y(1,2)\cdot B_y-P_y(1,3)\cdot C_y=0\\
&P_x(1,1)\cdot B_x+P_x(2,1)\cdot A_x-P_y(2,1)\cdot A_y-P_y(2,2)\cdot B_y-P_y(2,3)\cdot C_y=0\\
&P_x(1,2)\cdot A_x+P_x(2,2)\cdot B_x-P_y(1,1)\cdot B_y-P_y(1,2)\cdot 2B_y-P_y(1,3)\cdot B_y=0\\
&P_x(1,2)\cdot B_x+P_x(2,2)\cdot A_x-P_y(2,1)\cdot B_y-P_y(2,2)\cdot 2B_y-P_y(2,3)\cdot B_y=0\\
&P_x(1,3)\cdot A_x+P_x(2,3)\cdot B_x-P_y(1,1)\cdot C_y-P_y(1,2)\cdot B_y-P_y(1,3)\cdot A_y= 0\\
&P_x(1,3)\cdot B_x+P_x(2,3)\cdot A_x-P_y(2,1)\cdot C_y-P_y(2,2)\cdot B_y-P_y(2,3)\cdot A_y=0
\end{aligned}\right\} \quad \ldots \quad (3\text{-}18)$$

Setting $\frac{^1y}{^1x} = a$, $\frac{I_{AB}}{I_{CD}} = b$ and substituting into Equation (3-18), yields

$$\left.\begin{aligned}
&P_x(1,1)\frac{16}{3^6} +P_x(2,1)\frac{11}{3^6} -P_y(1,1)\frac{54a^3b}{4^6} -P_y(1,2)\frac{64a^3b}{4^6} -P_y(1,3)\frac{26a^3b}{4^6} = 0\\
&P_x(1,1)\frac{11}{3^6} +P_x(2,1)\frac{16}{3^6} -P_y(2,1)\frac{54a^3b}{4^6} -P_y(2,2)\frac{64a^3b}{4^6} -P_y(2,3)\frac{26a^3b}{4^6} = 0
\end{aligned}\right|$$

$$\left.\begin{aligned}
&P_x(1,2)\frac{16}{3^6} + P_x(2,2)\frac{11}{3^6} - P_y(1,1)\frac{64a^3b}{4^6} - P_y(1,2)\frac{128a^3b}{4^6} - P_y(1,3)\frac{64a^3b}{4^6} = 0\\
&P_x(1,2)\frac{11}{3^6} + P_x(2,2)\frac{16}{3^6} - P_y(2,1)\frac{64a^3b}{4^6} - P_y(2,2)\frac{128a^3b}{4^6} - P_y(2,3)\frac{64a^3b}{4^6} = 0\\
&P_x(1,3)\frac{16}{3^6} + P_x(2,3)\frac{11}{3^6} - P_y(1,1)\frac{26a^3b}{4^6} - P_y(1,2)\frac{64a^3b}{4^6} - P_y(1,3)\frac{54a^3b}{4^6} = 0\\
&P_x(1,3)\frac{11}{3^6} + P_x(2,3)\frac{16}{3^6} - P_y(2,1)\frac{26a^3b}{4^6} - P_y(2,2)\frac{64a^3b}{4^6} - P_y(2,3)\frac{54a^3b}{4^6} = 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3\text{-}18a)$$

(1). Applied load at joint (1,1)

Equation (3-17), yields

$$\left.\begin{aligned}
&P_x(1,1)+P_y(1,1)=P(1,1)\\
&P_x(2,1)+P_y(2,1)=0\\
&P_x(1,2)+P_y(1,2)=0\\
&P_x(2,2)+P_y(2,2)=0\\
&P_x(1,3)+P_y(1,3)=0\\
&P_x(2,3)+P_y(2,3)=0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3\text{-}17a)$$

Solving Equations (3-17a) and (3-18a),

$$\left.\begin{aligned}
&P_x(x,y) = c^{11}_{xy}P(1,1)\\
&P_y(x,y) = d^{11}_{xy}P(1,1)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3\text{-}19a)$$

Where c^{11}_{xy}, d^{11}_{xy} are influence coefficients indicating the distribution of the applied load P(1,1) between the transverse and longitudinal beams.

Through the use of a computer program, these coefficients were calculated as shown in Figure 3-10.

(2). Applied load at joint (1,2)

From the same process shown in Section 1, we also find

$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{12}P(1,2) \\ P_y(x,y) &= d_{xy}^{12}P(1,2) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}19b)$$

Where c_{xy}^{12}, d_{xy}^{12} are influence coefficients indicating the distribution of the applied load P(1,2) between the transverse and longitudinal beams. These coefficients are shown in Figure 3-11.

(3). Applied load at joint (1,3), (2,1), (2,2), (2,3) then can also be found as follows:

$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{13}P(1,3) \\ P_y(x,y) &= d_{xy}^{13}P(1,3) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}19c)$$

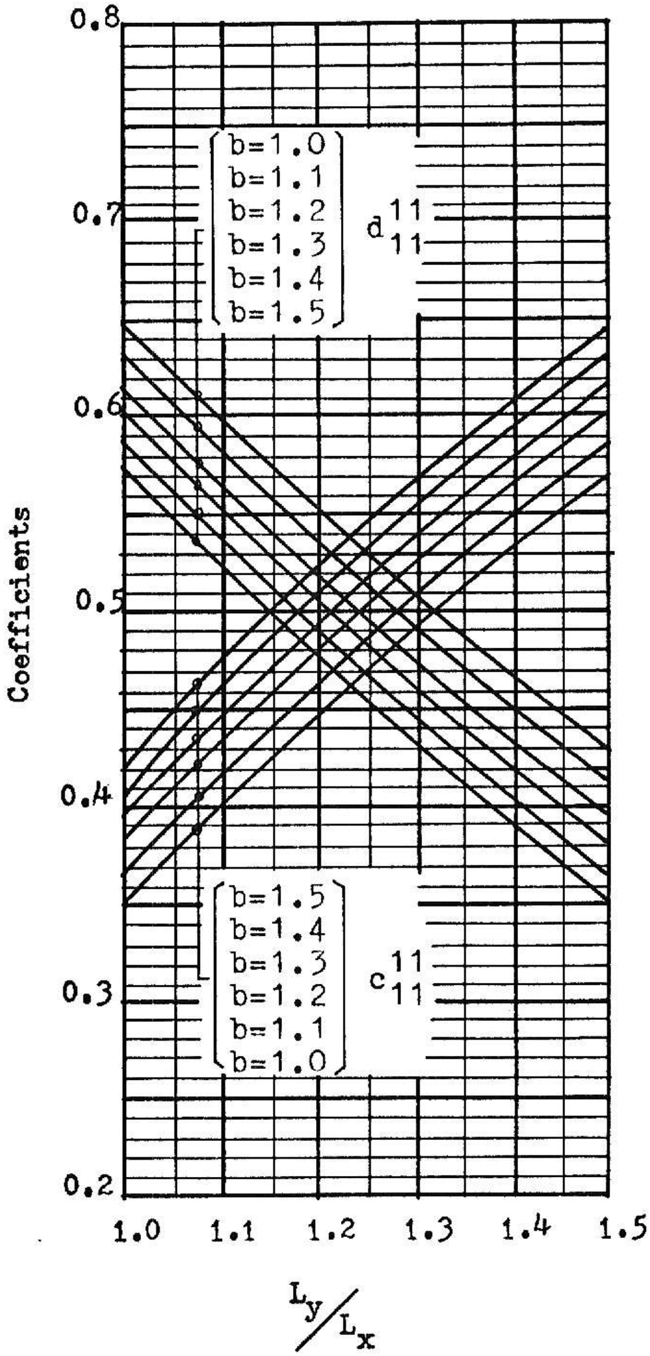
$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{21}P(2,1) \\ P_y(x,y) &= d_{xy}^{21}P(2,1) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}19d)$$

$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{22}P(2,2) \\ P_y(x,y) &= d_{xy}^{22}P(2,2) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}19e)$$

$$\left.\begin{aligned} P_x(x,y) &= c_{xy}^{23}P(2,3) \\ P_y(x,y) &= d_{xy}^{23}P(2,3) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (3\text{-}19f)$$

From the symmetric geometry of the gridwork,

$$c_{11}^{13}=c_{13}^{11},\ c_{21}^{13}=c_{23}^{11},\ c_{12}^{13}=c_{12}^{11},\ c_{22}^{13}=c_{22}^{11},\ c_{13}^{13}=c_{11}^{11},\ c_{23}^{13}=c_{21}^{11},$$



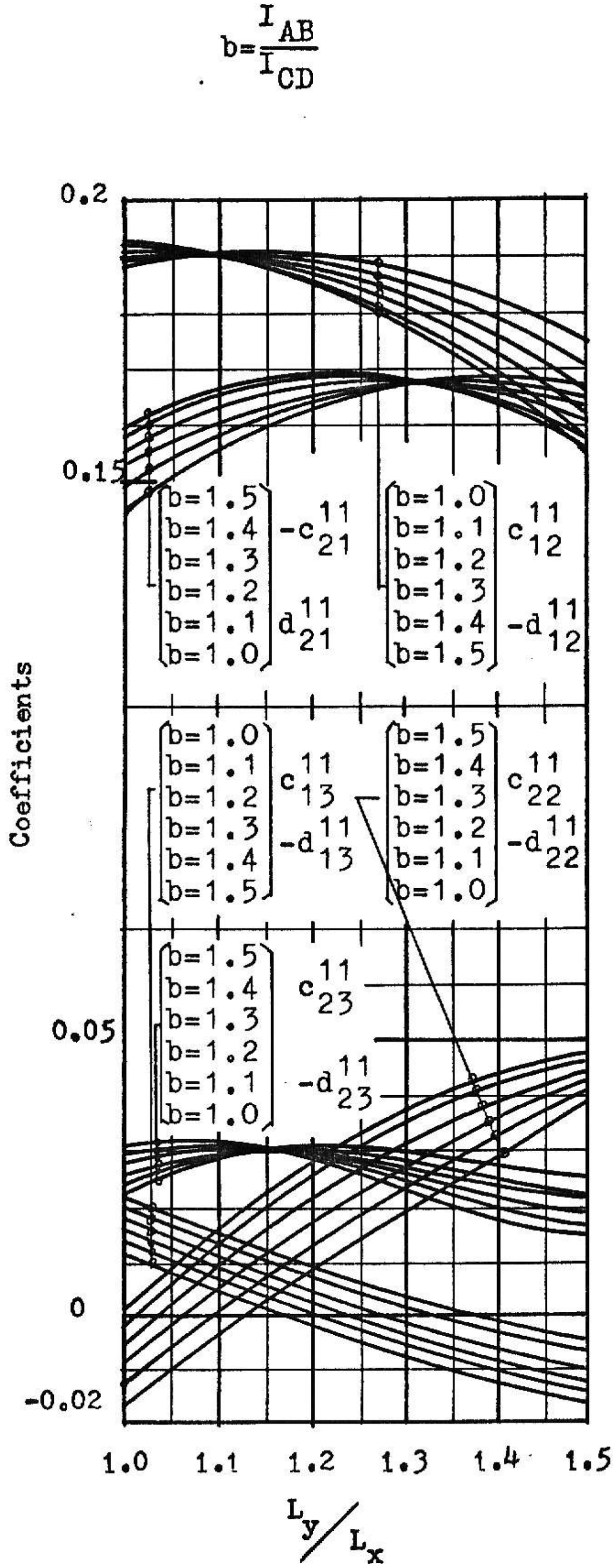


Fig. 3-10 Influence coefficients of P_x and P_y caused by $P(1,1)$, (case 4)

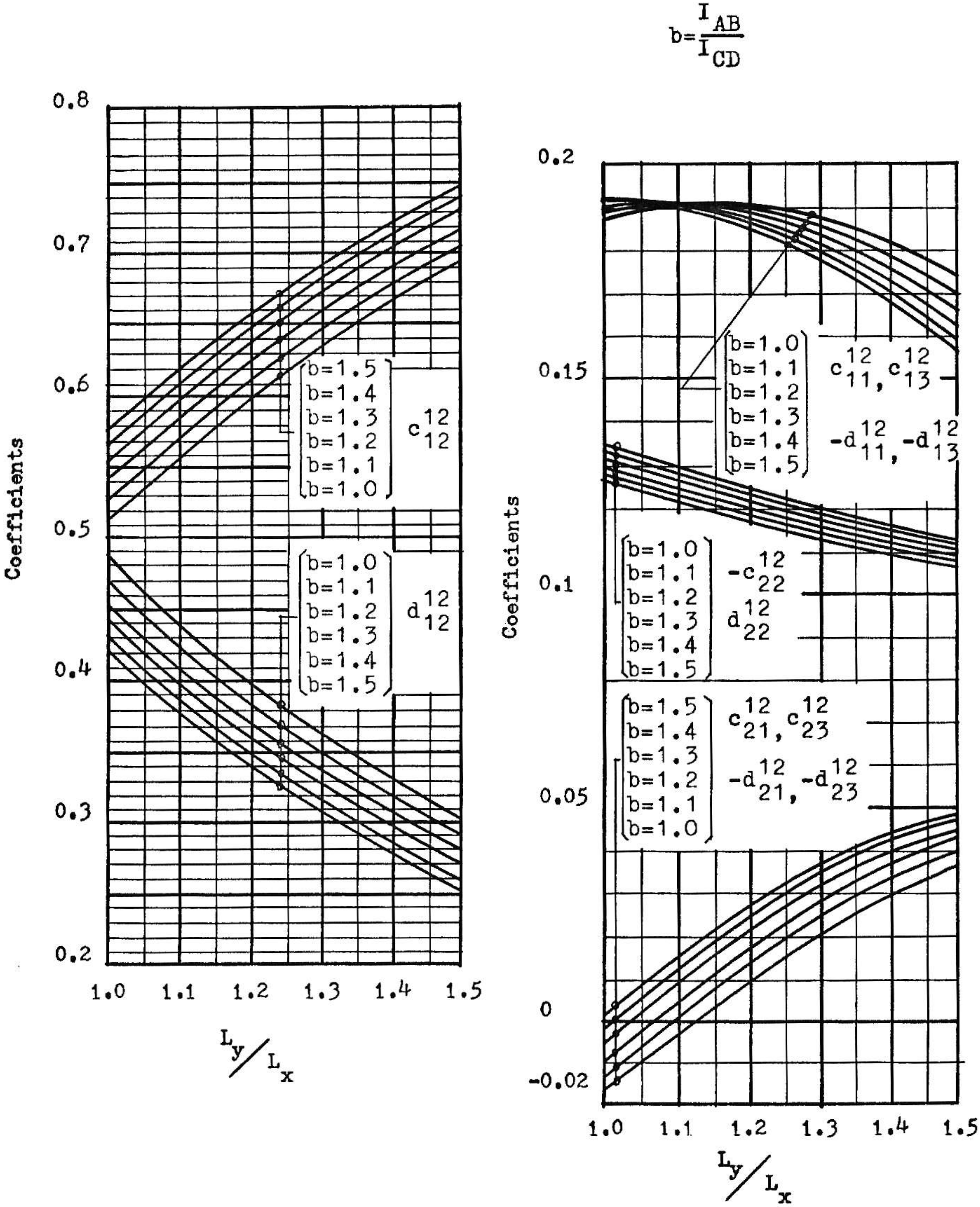


Fig. 3-11 Influence coefficients of P_x and P_y caused by P(1,2), (case 4)

$d^{13}_{11}=d^{11}_{13},\ d^{13}_{21}=d^{11}_{23},\ d^{13}_{12}=d^{11}_{12},\ d^{13}_{22}=d^{11}_{22},\ d^{13}_{13}=d^{11}_{11},\ d^{13}_{23}=d^{11}_{21},$

$c^{21}_{11}=c^{11}_{21},\ c^{21}_{21}=c^{11}_{11},\ c^{21}_{12}=c^{11}_{22},\ c^{21}_{22}=c^{11}_{12},\ c^{21}_{13}=c^{11}_{23},\ c^{21}_{23}=c^{11}_{13},$

$d^{21}_{11}=d^{11}_{21},\ d^{21}_{21}=d^{11}_{11},\ d^{21}_{12}=d^{11}_{22},\ d^{21}_{22}=d^{11}_{12},\ d^{21}_{13}=d^{11}_{23},\ d^{21}_{23}=d^{11}_{13},$

$c^{22}_{11}=c^{12}_{21},\ c^{22}_{21}=c^{12}_{11},\ c^{22}_{12}=c^{12}_{22},\ c^{22}_{22}=c^{12}_{12},\ c^{22}_{13}=c^{12}_{23},\ c^{22}_{23}=c^{12}_{13},$

$d^{22}_{11}=d^{12}_{21},\ d^{22}_{21}=d^{12}_{11},\ d^{22}_{12}=d^{12}_{22},\ d^{22}_{22}=d^{12}_{12},\ d^{22}_{13}=d^{12}_{23},\ d^{22}_{23}=d^{12}_{13},$

$c^{23}_{11}=c^{11}_{23},\ c^{23}_{21}=c^{11}_{13},\ c^{23}_{12}=c^{11}_{22},\ c^{23}_{22}=c^{11}_{12},\ c^{23}_{13}=c^{11}_{21},\ c^{23}_{23}=c^{11}_{11},$

$d^{23}_{11}=d^{11}_{23},\ d^{23}_{21}=d^{11}_{13},\ d^{23}_{12}=d^{11}_{22},\ d^{23}_{22}=d^{11}_{12},\ d^{23}_{13}=d^{11}_{21},\ d^{23}_{23}=d^{11}_{11}$

(4). By superposition, the results of the component loads $P_x(x,y)$ and $P_y(x,y)$ will be:

$$\left.\begin{aligned}
P_x(x,y) &= c^{11}_{xy}P(1,1)+c^{12}_{xy}P(1,2)+c^{13}_{xy}P(1,3)+c^{21}_{xy}P(2,1)+c^{22}_{xy}P(2,2)\\
&\quad +c^{23}_{xy}P(2,3)\\
P_y(x,y) &= d^{11}_{xy}P(1,1)+d^{12}_{xy}P(1,2)+d^{13}_{xy}P(1,3)+d^{21}_{xy}P(2,1)+d^{22}_{xy}P(2,2)\\
&\quad +d^{23}_{xy}P(2,3)
\end{aligned}\right\} \quad (3\text{-}4\text{-}A)$$

The stress Equations then can be derived in terms of those component loads $P_x(x,y)$ and $P_y(x,y)$.

From Equation (2-12), the Fixed-end Moments are:

$$M_{A_1B_1} = \frac{2}{27}(^1x) \quad |2P_x(1,1)+P_x(2,1)| \quad \ldots\ldots\ldots \quad (3\text{-}4\text{-}B_1)$$

$$M_{B_1A_1} = \frac{2}{27}(^1x) \quad |P_x(1,1)+2P_x(2,1)| \quad \ldots\ldots\ldots \quad (3\text{-}4\text{-}B_2)$$

$$M_{A_2B_2} = \frac{2}{27}(^1x) \quad |2P_x(1,2)+P_x(2,2)| \quad \ldots\ldots\ldots \quad (3\text{-}4\text{-}B_3)$$

$$M_{B_2A_2} = \frac{2}{27}(l_x) \quad [P_x(1,2)+2P_x(2,2)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}B_4)$$

$$M_{A_3B_3} = \frac{2}{27}(l_x) \quad [2P_x(1,3)+P_x(2,3)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}B_5)$$

$$M_{B_3A_3} = \frac{2}{27}(l_x) \quad [P_x(1,3)+2P_x(2,3)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}B_6)$$

$$M_{C_1D_1} = \frac{l_y}{64} \quad [9P_y(1,1)+8P_y(1,2)+3P_y(1,3)] \quad \ldots\ldots (3\text{-}4\text{-}B_7)$$

$$M_{D_1C_1} = \frac{l_y}{64} \quad [3P_y(1,1)+8P_y(1,2)+9P_y(1,3)] \quad \ldots\ldots (3\text{-}4\text{-}B_8)$$

$$M_{C_2D_2} = \frac{l_y}{64} \quad [9P_y(2,1)+8P_y(2,2)+3P_y(2,3)] \quad \ldots\ldots (3\text{-}4\text{-}B_9)$$

$$M_{D_2C_2} = \frac{l_y}{64} \quad [3P_y(2,1)+8P_y(2,2)+9P_y(2,3)] \quad \ldots\ldots (3\text{-}4\text{-}B_{10})$$

and from Equation (2-15a), the Moments at intermediate joints are:

$$M^{A_1B_1}_{(1,1)} = \frac{(l_x)}{81} \quad [8P_x(1,1)+P_x(2,1)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_1)$$

$$M^{A_1B_1}_{(2,1)} = \frac{(l_x)}{81} \quad [P_x(1,1)+8P_x(2,1)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_2)$$

$$M^{A_2B_2}_{(1,2)} = \frac{(l_x)}{81} \quad [8P_x(1,2)+P_x(2,2)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_3)$$

$$M^{A_2B_2}_{(2,2)} = \frac{(l_x)}{81} \quad [P_x(1,2)+8P_x(2,2)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_4)$$

$$M^{A_3B_3}_{(1,3)} = \frac{(l_x)}{81} \quad [8P_x(1,3)+P_x(2,3)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_5)$$

$$M^{A_3B_3}_{(2,3)} = \frac{(l_x)}{81} \quad [P_x(1,3)+8P_x(2,3)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_6)$$

$$M^{C_1D_1}_{(1,1)} = \frac{(l_y)}{128} \quad [9P_y(1,1)-P_y(1,3)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_7)$$

$$M^{C_1D_1}_{(1,2)} = \frac{(\ell_y)}{32}\,[P_y(1,1)+4P_y(1,2)+P_y(1,3)] \quad \ldots\ldots (3\text{-}4\text{-}C_8)$$

$$M^{C_1D_1}_{(1,3)} = \frac{(\ell_y)}{128}\,[9P_y(1,3)-P_y(1,1)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_9)$$

$$M^{C_2D_2}_{(2,1)} = \frac{(\ell_y)}{128}\,[9P_y(2,1)-P_y(2,3)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_{10})$$

$$M^{C_2D_2}_{(2,2)} = \frac{(\ell_y)}{32}\,[P_y(2,1)+4P_y(2,2)+P_y(2,3)] \quad \ldots\ldots (3\text{-}4\text{-}C_{11})$$

$$M^{C_2D_2}_{(2,3)} = \frac{(\ell_y)}{128}\,[9P_y(2,3)-P_y(2,1)] \quad \ldots\ldots\ldots (3\text{-}4\text{-}C_{12})$$

From Equation (2-17) or (2-17a),

$$\left.\begin{aligned} Y^{C_1D_1}_{(1,1+\Delta y)} &= P_y(1,1)\cdot C_y+P_y(1,2)\cdot D_y+P_y(1,3)\cdot E_y \\ Y^{C_1D_1}_{(1,1-\Delta y)} &= P_y(1,1)\cdot F_y+P_y(1,2)\cdot G_y+P_y(1,3)\cdot H_y \end{aligned}\right\} \quad \ldots (3\text{-}20a)$$

$$\left.\begin{aligned} Y^{C_2D_2}_{(2,1+\Delta y)} &= P_y(2,1)\cdot C_y+P_y(2,2)\cdot D_y+P_y(2,3)\cdot E_y \\ Y^{C_2D_2}_{(2,1-\Delta y)} &= P_y(2,1)\cdot F_y+P_y(2,2)\cdot G_y+P_y(2,3)\cdot H_y \end{aligned}\right\} \quad \ldots (3\text{-}20b)$$

$$\left.\begin{aligned} Y^{C_1D_1}_{(1,3+\Delta y)} &= P_y(1,1)\cdot E_y+P_y(1,2)\cdot D_y+P_y(1,3)\cdot C_y \\ Y^{C_1D_1}_{(1,3-\Delta y)} &= P_y(1,1)\cdot H_y+P_y(1,2)\cdot G_y+P_y(1,3)\cdot F_y \end{aligned}\right\} \quad \ldots (3\text{-}20c)$$

$$\left.\begin{aligned} Y^{C_2D_2}_{(2,3+\Delta y)} &= P_y(2,1)\cdot E_y+P_y(2,2)\cdot D_y+P_y(2,3)\cdot C_y \\ Y^{C_2D_2}_{(2,3-\Delta y)} &= P_y(2,1)\cdot H_y+P_y(2,2)\cdot G_y+P_y(2,3)\cdot F_y \end{aligned}\right\} \quad \ldots (3\text{-}20d)$$

where $C_y = \frac{({}^1y-y_2)^2}{6EI_{CD}\cdot 64} \quad [9\,{}^1y-10({}^1y-y_2)]$

$$D_y = \frac{4y_2^2}{6EI_{CD}\cdot 64} \quad [6\,{}^1y-8y_2]$$

$$E_y = \frac{y_2^2}{6EI_{CD}\cdot 64} \quad [9\,{}^1y-10y_2]$$

$$F_y = \frac{9y_1^2}{6EI_{CD}\cdot 64} \quad [3\,{}^1y-6y_1]$$

$$G_y = \frac{4y_1^2}{6EI_{CD}\cdot 64} \quad [6\,{}^1y-8y_1]$$

$$H_y = \frac{y_1^2}{6EI_{CD}\cdot 64} \quad [9\,{}^1y-10y_1]$$

in which $y_1 = \frac{{}^1y}{4} - \Delta y$, $y_2 = \frac{{}^1y}{4} + \Delta y$, where Δy is any arbitrary value.

and
$$\left.\begin{aligned} Y^{A_1B_1}_{(1+\Delta x,1)} &= P_x(1,1)\cdot C_x + P_x(2,1)\cdot D_x \\ Y^{A_1B_1}_{(1-\Delta x,1)} &= P_x(1,1)\cdot E_x + P_x(2,1)\cdot F_x \end{aligned}\right\} \quad \cdots\cdots\cdots (3\text{-}21a)$$

$$\left.\begin{aligned} Y^{A_2B_2}_{(1+\Delta x,2)} &= P_x(1,2)\cdot C_x + P_x(2,2)\cdot D_x \\ Y^{A_2B_2}_{(1-\Delta x,2)} &= P_x(1,2)\cdot E_x + P_x(2,2)\cdot F_x \end{aligned}\right\} \quad \cdots\cdots\cdots (3\text{-}21b)$$

$$\left.\begin{aligned} Y^{A_3B_3}_{(1+\Delta x,3)} &= P_x(1,3)\cdot C_x + P_x(2,3)\cdot D_x \\ Y^{A_3B_3}_{(1-\Delta x,3)} &= P_x(1,3)\cdot E_x + P_x(2,3)\cdot F_x \end{aligned}\right\} \quad \cdots\cdots\cdots (3\text{-}21c)$$

$$\left.\begin{aligned} Y^{A_1B_1}_{(2+\Delta x,1)} &= P_x(1,1)\cdot D_x + P_x(2,1)\cdot C_x \\ Y^{A_1B_1}_{(2-\Delta x,1)} &= P_x(1,1)\cdot F_x + P_x(2,1)\cdot E_x \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3\text{-}21d)$$

$$\left.\begin{aligned} Y^{A_2B_2}_{(2+\Delta x,2)} &= P_x(1,2)\cdot D_x + P_x(2,2)\cdot C_x \\ Y^{A_2B_2}_{(2-\Delta x,2)} &= P_x(1,2)\cdot F_x + P_x(2,2)\cdot E_x \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3\text{-}21e)$$

$$\left.\begin{aligned} Y^{A_3B_3}_{(2+\Delta x,3)} &= P_x(1,3)\cdot D_x + P_x(2,3)\cdot C_x \\ Y^{A_3B_3}_{(2-\Delta x,3)} &= P_x(1,3)\cdot F_x + P_x(2,3)\cdot E_x \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3\text{-}21f)$$

where

$$C_x = \frac{6l_x(l_x - x_2)^2 - 7(l_x - x_2)^3}{6EI_{AB}\cdot 27}$$

$$D_x = \frac{6l_x x_2^2 - 7x_2^3}{6EI_{AB}\cdot 27}$$

$$E_x = \frac{12l_x x_1^2 - 20x_1^3}{6EI_{AB}\cdot 27}$$

$$F_x = \frac{6l_x x_1^2 - 7x_1^3}{6EI_{AB}\cdot 27}$$

in which $x_1 = \frac{l_x}{3} - \Delta x$, $\Delta x_2 = \frac{l_x}{3} + \Delta x$, where Δx is any arbitrary value.

Then

$$\theta^x_{(1,1)} = \frac{Y^{C_1D_1}_{(1,1+\Delta y)} - Y^{C_1D_1}_{(1,1-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots \quad (3\text{-}22a)$$

$$\theta^x_{(2,1)} = \frac{Y^{C_2D_2}_{(2,1+\Delta y)} - Y^{C_2D_2}_{(2,1-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots \quad (3\text{-}22b)$$

$$\theta^{x}_{(1,3)} = \frac{Y^{C_1D_1}_{(1,3+\Delta y)} - Y^{C_1D_1}_{(1,3-\Delta y)}}{2\Delta y} \qquad \text{. . . . . (3-22c)}$$

$$\theta^{x}_{(2,3)} = \frac{Y^{C_2D_2}_{(2,3+\Delta y)} - Y^{C_2D_2}_{(2,3-\Delta y)}}{2\Delta y} \qquad \text{. . . . . (3-22d)}$$

and

$$\theta^{y}_{(1,1)} = \frac{Y^{A_1B_1}_{(1+\Delta x,1)} - Y^{A_1B_1}_{(1-\Delta x,1)}}{2\Delta x} \qquad \text{. . . . . (3-23a)}$$

$$\theta^{y}_{(1,2)} = \frac{Y^{A_2B_2}_{(1+\Delta x,2)} - Y^{A_2B_2}_{(1-\Delta x,2)}}{2\Delta x} \qquad \text{. . . . . (3-23b)}$$

$$\theta^{y}_{(1,3)} = \frac{Y^{A_3B_3}_{(1+\Delta x,3)} - Y^{A_3B_3}_{(1-\Delta x,3)}}{2\Delta x} \qquad \text{. . . . . (3-23c)}$$

$$\theta^{y}_{(2,1)} = \frac{Y^{A_1B_1}_{(2+\Delta x,1)} - Y^{A_1B_1}_{(2-\Delta x,1)}}{2\Delta x} \qquad \text{. . . . . (3-23d)}$$

$$\theta^{y}_{(2,2)} = \frac{Y^{A_2B_2}_{(2+\Delta x,2)} - Y^{A_2B_2}_{(2-\Delta x,2)}}{2\Delta x} \qquad \text{. . . . . (3-23e)}$$

$$\theta^{y}_{(2,3)} = \frac{Y^{A_3B_3}_{(2+\Delta x,3)} - Y^{A_3B_3}_{(2-\Delta x,3)}}{2\Delta x} \qquad \text{. . . . . (3-23f)}$$

Substituting Equations (3-20a) to (3-20d) and (3-21a) to (3-21f) into Equations (3-22a) to (3-22d) and (3-23a) to (3-23f), respectively. $\theta^{x}_{(x,y)}$ and $\theta^{y}_{(x,y)}$ can be found. Then the torsional moments can be derived as follows:

$$T^{x}_{(1,1)} = \frac{3\cdot G\cdot J_{AB}\cdot \theta^{x}_{(1,1)}}{l_x} \qquad \text{. . . . . (3-4-}D_1\text{)}$$

$$T^{x}_{(2,1)} = \frac{3\cdot G\cdot J_{AB}\cdot [\theta^{x}_{(2,1)} - \theta^{x}_{(1,1)}]}{l_x} \qquad \text{. . . . . (3-4-}D_2\text{)}$$

$$T^{x}_{(3,1)} = \frac{3\cdot G\cdot J_{AB}\cdot \theta^{x}_{(2,1)}}{l_x} \qquad \text{. . . . . (3-4-}D_3\text{)}$$

$$T^{x}_{(1,3)} = \frac{3\cdot G\cdot J_{AB}\cdot \theta^{x}_{(1,3)}}{l_x} \qquad \text{. . . . . (3-4-}D_4\text{)}$$

$$T^{x}_{(2,3)} = \frac{3\cdot G\cdot J_{AB}\cdot[\theta^{x}_{(2,3)}-\theta^{x}_{(1,3)}]}{l_x} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_5)$$

$$T^{x}_{(3,3)} = \frac{3\cdot G\cdot J_{AB}\cdot \theta^{x}_{(2,3)}}{l_x} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_6)$$

$$T^{y}_{(1,1)} = \frac{4\cdot G\cdot J_{CD}\cdot \theta^{y}_{(1,1)}}{l_y} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_7)$$

$$T^{y}_{(1,2)} = \frac{4\cdot G\cdot J_{CD}\cdot[\theta^{y}_{(1,2)}-\theta^{y}_{(1,1)}]}{l_y} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_8)$$

$$T^{y}_{(1,3)} = \frac{4\cdot G\cdot J_{CD}\cdot[\theta^{y}_{(1,2)}-\theta^{y}_{(1,3)}]}{l_y} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_9)$$

$$T^{y}_{(1,4)} = \frac{4\cdot G\cdot J_{CD}\cdot \theta^{y}_{(1,3)}}{l_y} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_{10})$$

$$T^{y}_{(2,1)} = \frac{4\cdot G\cdot J_{CD}\cdot \theta^{y}_{(2,1)}}{l_y} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_{11})$$

$$T^{y}_{(2,2)} = \frac{4\cdot G\cdot J_{CD}\cdot[\theta^{y}_{(2,2)}-\theta^{y}_{(2,1)}]}{l_y} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_{12})$$

$$T^{y}_{(2,3)} = \frac{4\cdot G\cdot J_{CD}\cdot[\theta^{y}_{(2,2)}-\theta^{y}_{(2,3)}]}{l_y} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_{13})$$

$$T^{y}_{(2,4)} = \frac{4\cdot G\cdot J_{CD}\cdot \theta^{y}_{(2,3)}}{l_y} \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}D_{14})$$

From Equations (2-20) and (2-20a),

$$R_{A_1} = \frac{20}{27}P_x(1,1) + \frac{7}{27}P_x(2,1) \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}E_1)$$

$$R_{B_1} = \frac{7}{27}P_x(1,1) + \frac{7}{27}P_x(2,1) \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}E_2)$$

$$R_{A_2} = \frac{20}{27}P_x(1,2) + \frac{7}{27}P_x(2,1) \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}E_3)$$

$$R_{B_2} = \frac{7}{27}P_x(1,2) + \frac{20}{27}P_x(2,2) \qquad \ldots\ldots\ldots\ldots (3\text{-}4\text{-}E_4)$$

$$R_{A_3} = \frac{20}{27} P_x(1,3) + \frac{7}{27} P_x(2,3) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}4\text{-}E_5)$$

$$R_{B_3} = \frac{7}{27} P_x(1,3) + \frac{20}{27} P_x(2,3) \quad \ldots\ldots\ldots\ldots \quad (3\text{-}4\text{-}E_6)$$

$$R_{D_1} = \frac{27}{32} P_y(1,1) + \frac{1}{2} P_y(1,2) + \frac{5}{32} P_y(1,3) \quad \ldots\ldots \quad (3\text{-}4\text{-}E_7)$$

$$R_{C_1} = \frac{5}{32} P_y(1,1) + \frac{1}{2} P_y(1,2) + \frac{27}{32} P_y(1,3) \quad \ldots\ldots \quad (3\text{-}4\text{-}E_8)$$

$$R_{D_2} = \frac{27}{32} P_y(2,1) + \frac{1}{2} P_y(2,2) + \frac{5}{32} P_y(2,3) \quad \ldots\ldots \quad (3\text{-}4\text{-}E_9)$$

$$R_{C_2} = \frac{5}{32} P_y(2,1) + \frac{1}{2} P_y(2,2) + \frac{27}{32} P_y(2,3) \quad \ldots\ldots \quad (3\text{-}4\text{-}E_{10})$$

(b). Numerical Example

The gridwork loaded with 36 kip concentrated loads on each joint as shown in Figure 3-12 will be considered in this example. The factors I, E, G, and J are assumed to be the same for all beams.

From Figure 3-10 and 3-11,

$c^{11}_{11}=0.5=c^{13}_{13}=c^{21}_{21}=c^{23}_{23}$, $\quad c^{11}_{21}=-0.167=c^{13}_{23}=c^{21}_{11}=c^{23}_{13}$,

$c^{11}_{12}=0.186=c^{13}_{12}=c^{21}_{22}=c^{23}_{22}$, $\quad c^{11}_{22}=0.023=c^{13}_{22}=c^{21}_{12}=c^{23}_{12}$,

$c^{11}_{13}=0.003=c^{13}_{11}=c^{21}_{23}=c^{23}_{21}$, $\quad c^{11}_{23}=0.03=c^{13}_{21}=c^{21}_{13}=c^{23}_{11}$,

$c^{12}_{11}=0.186=c^{22}_{21}$, $\quad c^{12}_{21}=0.023=c^{22}_{11}$, $\quad c^{12}_{12}=0.645=c^{22}_{22}$,

$c^{12}_{22}=-0.12=c^{22}_{12}$, $\quad c^{12}_{13}=0.186=c^{22}_{23}$, $\quad c^{12}_{23}=0.023=c^{22}_{13}$,

$d^{11}_{11}=0.5=d^{13}_{13}=d^{21}_{21}=d^{23}_{23}$, $\quad d^{11}_{21}=0.167=d^{13}_{23}=d^{21}_{11}=d^{23}_{13}$,

$d^{11}_{12}=-0.186=d^{13}_{12}=d^{21}_{22}=d^{23}_{22}$, $\quad d^{11}_{22}=-0.023=d^{13}_{22}=d^{21}_{12}=d^{23}_{12}$,

$d^{11}_{13}=-0.003=d^{13}_{11}=d^{21}_{23}=d^{23}_{21}$, $\quad d^{11}_{23}=-0.03=d^{13}_{21}=d^{21}_{13}=d^{23}_{11}$,

$d_{11}^{12}=-0.186=d_{21}^{22}$, $d_{21}^{12}=-0.023=d_{11}^{22}$, $d_{12}^{12}=0.355=d_{22}^{22}$,

$d_{22}^{12}=0.12=d_{12}^{22}$, $d_{13}^{12}=-0.186=d_{23}^{22}$, $d_{23}^{12}=-0.023=d_{13}^{22}$,

Substituting into Equation (3-4-A),

$P_x(1,1)=(0.5+0.186+0.003-0.167+0.023+0.03)\times 36 = 20.7$ kips

$P_x(2,1)=(-0.167+0.023+0.03+0.5+0.186+0.003)\times 36 = 34.0$ kips

$P_x(1,2)=(0.186+0.645+0.186+0.023-0.12+0.023)36 = 34.0$ kips

$P_x(2,2)=(0.023-0.12+0.023+0.186+0.645+0.186)36 = 34.0$ kips

$P_x(1,3)=(0.003+0.186+0.5+0.03+0.023-0.167)\times 36 = 20.7$ kips

$P_x(2,3)=(0.03+0.023-0.167+0.003+0.186+0.5)\times 36 = 20.7$ kips

$P_y(1,1)=(0.5-0.003+0.167-0.03-0.186-0.023)\times 36 = 15.3$ kips

$P_y(2,1)=(0.167-0.03+0.5-0.003-0.023-0.186)\times 36 = 15.3$ kips

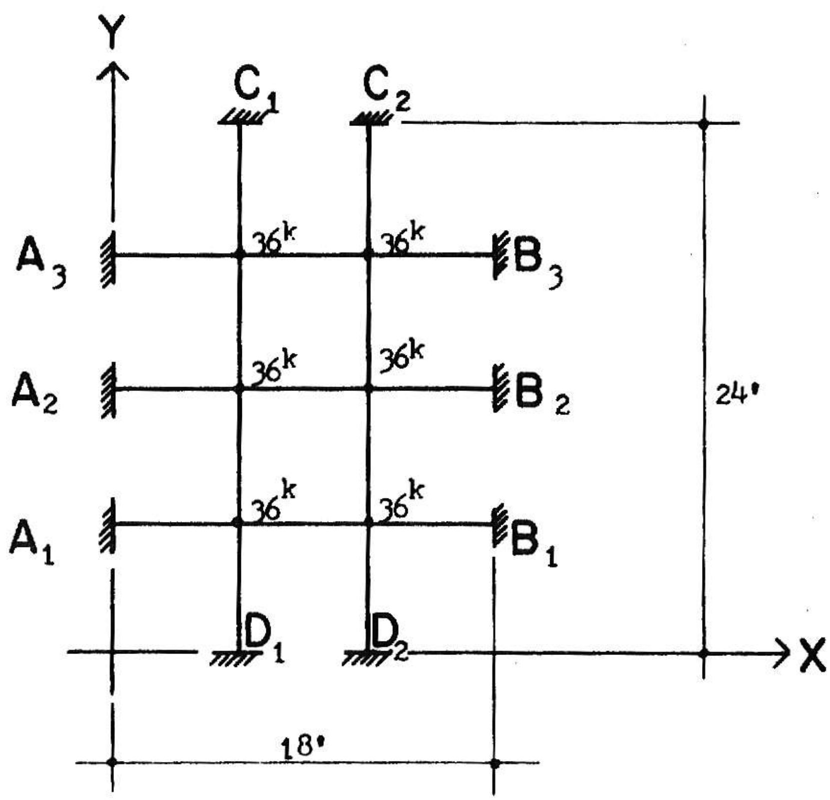


Fig. 3-12 Calculation example for case four

$P_y(1,2)=(-0.186-0.186-0.023-0.023+0.355+0.12)36 = 2.0$ kips

$P_y(2,2)=(-0.023-0.023-0.186-0.186+0.12+0.355)36 = 2.0$ kips

$P_y(1,3)=(-0.003+0.5-0.03+0.167-0.186-0.023) \times 36 = 15.3$ kips

$P_y(2,3)=(-0.03+0.167-0.003+0.5-0.023-0.186) \times 36 = 15.3$ kips

Substituting into Equations (3-4-B_1) to (3-4-C_{12}),

$$M_{A_1B_1} = \frac{2}{27}(18)(3\times20.7) = 83^{\text{k-ft}} = M_{B_1A_1} = M_{A_3B_3} = M_{B_3A_3}$$

$$M_{A_2B_2} = \frac{2}{27}(18)(3\times34) = 136^{\text{k-ft}} = M_{B_2A_2}$$

$$M_{C_1D_1} = \frac{24}{64}(9\times15.3+8\times2+3\times15.3) = 74.8^{\text{k-ft}} = M_{D_1C_1} = M_{C_2D_2} = M_{D_2C_2}$$

$$M^{A_1B_1}_{(1,1)} = \frac{18}{81}(9\times20.7) = 41.4^{\text{k-ft}} = M^{A_1B_1}_{(2,1)} = M^{A_3B_3}_{(1,3)} = M^{A_3B_3}_{(2,3)}$$

$$M^{A_2B_2}_{(1,2)} = \frac{18}{81}(9\times34) = 68^{\text{k-ft}} = M^{A_2B_2}_{(2,2)}$$

$$M^{C_1D_1}_{(1,1)} = \frac{24}{128}(8\times15.3) = 22.9^{\text{k-ft}} = M^{C_1D_1}_{(1,3)} = M^{C_2D_2}_{(2,1)} = M^{C_2D_2}_{(2,3)}$$

$$M^{C_1D_1}_{(1,2)} = \frac{24}{32}(2\times15.3+4\ 2) = 28.9^{\text{k-ft}} = M^{C_2D_2}_{(2,2)}$$

By substituting $P_x(x,y)$ and $P_y(x,y)$ into Equations (3-20a) to (3-21f) and assuming $\Delta x=\Delta y=1^{\text{ft}}$,

$$x_1=5^{\text{ft}}=y_1, \qquad x_2=7^{\text{ft}}=y_2$$

$$C_x = \frac{6\times18(18-7)^2-7(18-7)^3}{6\times27} = 23.2$$

$$D_x = \frac{6\times18\times7^2-7\times7^3}{6\times27} = 17.8$$

$$E_x = \frac{12\times18\times5^2-20\times5^3}{6\times27} = 17.9$$

$$F_x = \frac{6\times18\times5^2-7\times5^3}{6\times27} = 11.3$$

$$Y^{A_1B_1}_{(1+\Delta x,1)} = 20.7\ (23.2+17.8) = 848$$

$$Y^{A_1B_1}_{(1-\Delta x,1)} = 20.7\ (17.9+11.3) = 605$$

$$Y^{A_2B_2}_{(1+\Delta x,2)} = 34\ (23.2+17.8) = 1390$$

$$Y^{A_2B_2}_{(1-\Delta x,2)} = 34\ (17.9+11.3) = 992$$

$$C_y = \frac{(24-7)^2}{6\times64}\ [9\times24-10(24-7)] = 34.6$$

$$D_y = \frac{4\times7^2}{6\times64}\ (6\times24-8\ 7) = 44.8$$

$$E_y = \frac{7^2}{6\times64}\ (9\times24-10\ 7) = 18.6$$

$$F_y = \frac{9\times5^2}{6\times64}\ (3\times24-6\ 5) = 24.6$$

$$G_y = \frac{4\times5^2}{6\times64}\ (6\times24-8\ 5) = 27.1$$

$$H_y = \frac{5^2}{6\times64}\ (9\times24-10\ 5) = 10.8$$

$$Y^{C_1D_1}_{(1,1+\Delta y)} = 15.3\times34.6+2\times44.8+15.3\times18.6 = 904$$

$$Y^{C_1D_1}_{(1,1-\Delta y)} = 15.3\times24.6+2\times27.1+15.3\times10.8 = 597$$

$$\theta^x_{(1,1)} = \frac{904-597}{2\times1} = 153.5 = \theta^x_{(2,1)} = \theta^x_{(1,3)} = \theta^x_{(2,3)}$$

$$\theta^x_{(1,2)} = \theta^x_{(2,2)} = 0$$

$$\theta^y_{(1,1)} = \frac{848-605}{2x1} = 121.5 = \theta^y_{(1,3)} = \theta^y_{(2,1)} = \theta^y_{(2,3)}$$

$$\theta^y_{(1,2)} = \frac{1390-992}{2x1} = 199 = \theta^y_{(2,2)}$$

$$T^x_{(1,1)} = \frac{3x1x1x153.5}{18} = 25.6^{k-ft} = T^x_{(2,1)} = T^x_{(1,3)} = T^x_{(2,3)}$$

$$T^y_{(1,1)} = \frac{4x1x1x121.5}{24} = 20.2^{k-ft} = T^y_{(1,4)} = T^y_{(2,1)} = T^y_{(2,4)}$$

$$T^y_{(1,2)} = \frac{4x1x1x(199-121.5)}{24} = 12.9^{k-ft} = T^y_{(2,2)} = T^y_{(1,3)} = T^y_{(2,3)}$$

$$R_{A_1} = R_{B_1} = R_{A_3} = R_{B_3} = \frac{(20+7)}{27}(20.7) = 20.7 \text{ kips}$$

$$R_{A_2} = R_{B_2} = \frac{(20+7)}{27}(34) = 34.0^{\text{kips}}$$

$$R_{C_1} = R_{C_2} = R_{D_1} = R_{D_2} = (\frac{27}{32}+\frac{5}{32})(15.3) + \frac{1}{2}(2) = 16.3^{\text{kips}}$$

End

CHAPTER FOUR

COMPUTER ANALYSIS FOR COMPLICATED CASES

(1). Case five

Consider a gridwork which consists of n transverse beams and m

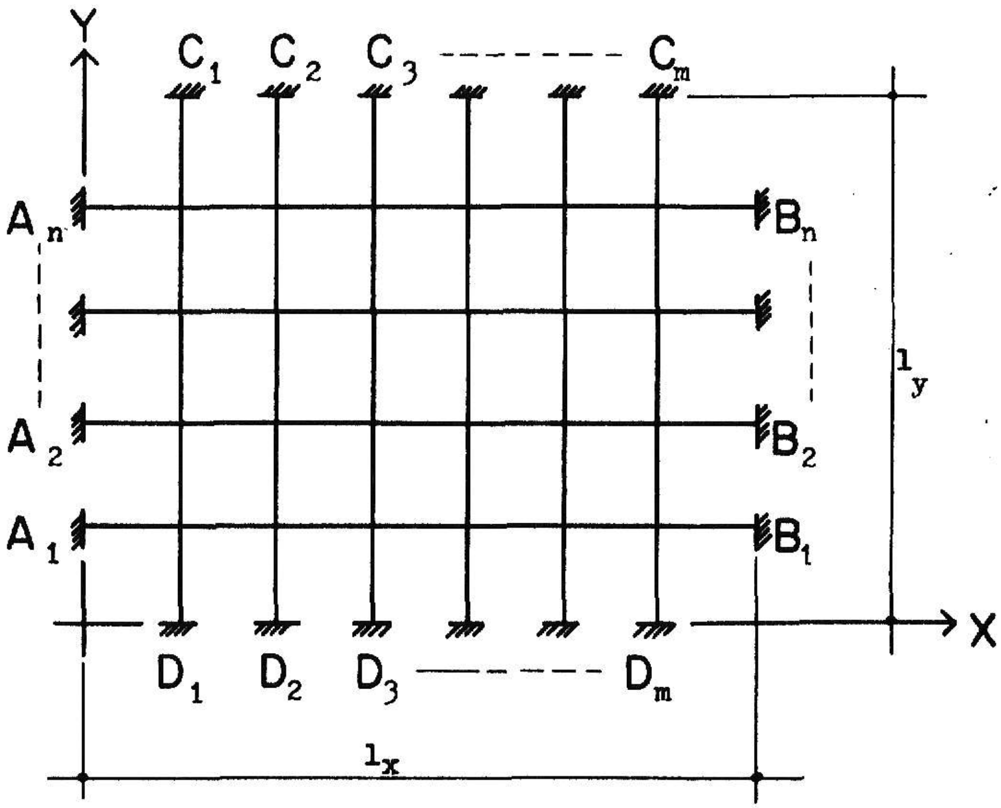


Fig. 4-1 Gridwork with n transverse beams and m Longitudinal beams

longitudinal beams as shown in Figure 4-1.

(a). Equations

Assume the applied load P at joint (x,y) to be replaced by unknown components $P_x(x,y)$ and $P_y(x,y)$, where P_x and P_y are the loads carried by the beams parallel to the X-axis and Y-axis at the (x,y) intersection, respectively.

From equilibrium:

$$P_x(x,y) + P_y(x,y) + P(x,y) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4\text{-}1)$$

The deflections at joint (x,y) caused by the component loads P_x and P_y will be derived as follows:

Using Equation (2-8),

$$\left\{ Y^{P_x}_{(x,y)} \middle| \begin{array}{l} x = 1 \to m \\ y = 1 \to n \end{array} \right\} = \sum_{i=1}^{m} P_x(i,y) \cdot [AP_x(x,i)] \quad \ldots\ldots\ldots \quad (4\text{-}2)$$

$$\left\{ Y^{P_x}_{(x,y)} \middle| \begin{array}{l} x = 1 \to m \\ y = 1 \to m \end{array} \right\} = \sum_{i=1}^{n} P_y(x,i) \cdot [AP_y(i,y)] \quad \ldots\ldots\ldots \quad (4\text{-}3)$$

where

$$AP_x(x,i) = \frac{(NX-i)^2(x)^2(1_x)^3}{6EI(NX)^6} \left[3(NX)(i) - 3(i)(x) - (NX-i)(x)\right]$$

$$\ldots\ldots\ldots\ldots\ldots \text{ when } x \leq i$$

$$AP_x(x,i) = \frac{(i)^2(NX-x)^2(1_x)^3}{6EI(NX)^6} \left[3(NX)(NX-i) - 3(NX-i)(NX-x) - (NX-x)i\right]$$

$$\ldots\ldots\ldots\ldots\ldots \text{ when } x > i$$

$$AP_y(i,y) = \frac{(NY-i)^2(y)^2(1_y)^3}{6EI(NY)^6} \left[3(NY)(i) - 3(i)(y) - (NY-i)(y)\right]$$

$$\ldots\ldots\ldots\ldots\ldots \text{ when } y \leq i$$

$$AP_y(i,y) = \frac{(i)^2(NY-y)^2(1_y)^3}{6EI(NY)^6} \left[3(NY)(NY-i) - 3(NY-i)(NY-y) - (NY-y)i\right]$$

$$\ldots\ldots\ldots\ldots\ldots \text{ when } y > i$$

in which

$$NX = m + 1, \quad NY = n + 1$$

From Equation (2-10), the deflections at joints (x,y), caused by the Torsional Moments induced in the orthogonal beams, are as follows:

$$\left\{ Y^{T_x}_{(x,y)} \middle| \begin{array}{l} x = 1 \to m \\ y = 1 \to n \end{array} \right\} = \sum_{i=1}^{m} T^x_{(i,y)} \cdot [AT_x(x,i)] \quad \ldots\ldots \quad (4\text{-}4)$$

$$\left\{ Y^{T_y}_{(x,y)} \middle| \begin{array}{l} x = 1 \to m \\ y = 1 \to n \end{array} \right\} = \sum_{i=1}^{n} T^y_{(x,i)} \cdot [AT_y(i,y)] \quad \ldots\ldots \quad (4\text{-}5)$$

$$AT_x(x,i) = \frac{(i)^2(l_x)^2}{6EI(NX)^2} \; [2C_{xi} + C_{xi}\frac{(NX-i)}{NX} + d_{xi}\frac{i}{NX} - \frac{i}{NX}]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{ when } x \leq i$$

$$AT_x(x,i) = \frac{(NX)(l_x) - (i)(l_x)^{\;2}}{6EI(NX)^2} \; [2d_{xi} + d_{xi}\frac{i}{NX} + C_{xi}\frac{(NX-i)}{NX} + \frac{(NX-i)}{NX}] \ldots\ldots\ldots\ldots \text{ when } x > i$$

$$AT_y(i,y) = \frac{(i)^2(l_y)^2}{6EI(NY)^2} \; [2C_{yi} + C_{yi}\frac{(NY-i)}{NY} + d_{yi}\frac{i}{NY} - \frac{i}{NY}]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{ when } y \leq i$$

$$AT_y(i,y) = \frac{(NY)(l_y)-(i)(l_y)^{\;2}}{6EI(NY)^2} \; [2d_{yi} + d_{yi}\frac{i}{NY} + C_{yi}\frac{(NY-i)}{NY}] + \frac{(NY-i)}{NY}] \ldots\ldots\ldots\ldots \text{ when } y > i$$

in which

$$NX = m + 1, \quad NY = n + 1$$

and

$$\left.\begin{aligned}
c_{xi} &= \frac{i}{(NX)^3}\,[-(NX)^3 + 4(NX)(i) - 3(NX)(i)^2] \\
d_{xi} &= \frac{i}{(NX)^3}\,[3(NX)(i)^2 - 2(NX)^2(i)] \\
c_{xi} &= \frac{i}{(NY)^3}\,[-(NY)^3 + 4(NY)(i) - 3(NY)(i)^2] \\
d_{yi} &= \frac{i}{(NY)^3}\,[3(NY)(i)^2 - 2(NY)^2(i)]
\end{aligned}\right\} \quad \ldots\ldots (4\text{-}6)$$

The final deflection at each joint will be:

$$\left\{ Y^{x}_{(x,y)} \middle| \begin{matrix} x = 1 \cdot m \\ y = 1 \cdot n \end{matrix} \right\} = Y^{P_x}_{(x,y)} + Y^{T_x}_{(x,y)} \quad \ldots\ldots\ldots\ldots (4\text{-}7)$$

$$\left\{ Y^{y}_{(x,y)} \middle| \begin{matrix} x = 1 \cdot m \\ y = 1 \cdot n \end{matrix} \right\} = Y^{P_y}_{(x,y)} + Y^{T_y}_{(x,y)} \quad \ldots\ldots\ldots\ldots (4\text{-}8)$$

Deflection compatibility, yields

$$Y^{x}_{(x,y)} = Y^{y}_{(x,y)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4\text{-}9)$$

At the beginning, $T^{x}_{(x,y)}$ and $T^{y}_{(x,y)}$ are assumed to be zero. Then from Equation (4-1) and (4-9), where $Y^{T_x}_{(x,y)}$ and $Y^{T_y}_{(x,y)}$ are zero, the unknown component loads $P_x(x,y)$ and $P_y(x,y)$ can be found.

The Torsional Moments $T^{x}_{(x,y)}$ and $T^{y}_{(x,y)}$ caused by these component loads $P_x(x,y)$ and $P_y(x,y)$ can then be found by using Equations (4-C).

Equations (4-7) and (4-8) can then be modified to reflect the values of $Y^{T_x}_{(x,y)}$ and $Y^{T_y}_{(x,y)}$, and they yield a new set of values of $Y^{x}_{(x,y)}$ and $Y^{y}_{(x,y)}$. From the new set Equations (4-9) and the original Equations (4-1), a new set of $P_x(x,y)$ and $P_y(x,y)$, and a new set of $T^{x}_{(x,y)}$ and $T^{y}_{(x,y)}$, respectively, are calculated.

This procedure is then continued until the $T^{x}_{(x,y)}$ and $T^{y}_{(x,y)}$ from the last two cycles approach each other to the desired accuracy.

The stress equation can then be derived in terms of component loads $P_x(x,y)$ and $P_y(x,y)$.

From Equation (2-13), we find Fixed-end Moments as:

$$|M^{x}_{(o,y)}|\ y = 1 \rightarrow n| = \sum_{i=1}^{m} P_x(i,y)(i)(NX-i)^2(\ell_x) \Big/ (NX)^3 +$$

$$\sum_{i=1}^{m} T^{x}_{(i,y)} \ [-(NX)^3 + 4(NX)^2(i) - 3(NX)(i)^2] \Big/ (NX)^3$$

$$\left\{ M^{x}_{(NX,y)} \,\middle|\, y = 1 \rightarrow n \right\} = \sum_{i=1}^{m} P_x(i,y)(i)^2(NX-i)(l_x) \Big/ (NX)^3 +$$

$$\sum_{i=1}^{m} T^{x}_{(i,y)} \left[3(NX)(i)^2 - 2(NX)^2(i) \right] \Big/ (NX)^3$$

$$\ldots\ldots\ldots \quad (4\text{-}A)$$

$$\left\{ M^{y}_{(x,o)} \,\middle|\, x = 1 \rightarrow m \right\} = \sum_{i+1}^{n} P_x(x,i)(i)(NY-i)^2(l_y) \Big/ (NY)^3 +$$

$$\sum_{i=1}^{n} T^{y}_{(x,i)} \left[-(NY)^3 + y(NY)^2(i) - 3(NY)(i)^2 \right] \Big/ (NY)^3$$

$$\left\{ M^{y}_{(x,NY)} \,\middle|\, x = 1 \rightarrow m \right\} = \sum_{i=1}^{n} P_y(x,i)(i)^2(NY-i)(l_y) \Big/ (NY)^3 +$$

$$\sum_{i=1}^{n} T^{y}_{(x,i)} \left[3(NY)(i)^2 - 2(NY)^2(i) \right] \Big/ (NY)^3$$

From Equation (2-15a) and from equilibrium, i.e., the algebraic sum of moments at each joint should be zero, the moments at intermediate joints are:

$$\left.\begin{aligned} \left\{ M^{x}_{(x,y)} \,\middle|\, \begin{matrix} x = 1 \rightarrow m \\ y = 1 \rightarrow n \end{matrix} \right\} &= \left(\sum_{i=1}^{m} M_{(x,i)} \right) \pm \frac{1}{2} \left(T^{y}_{(x,y)} + T^{y}_{(x,y+1)} \right) \\ \left\{ M^{y}_{(x,y)} \,\middle|\, \begin{matrix} x = 1 \rightarrow m \\ y = 1 \rightarrow n \end{matrix} \right\} &= \left(\sum_{i=1}^{n} M_{(i,y)} \right) \pm \frac{1}{2} \left(T^{x}_{(x,y)} + T^{x}_{(x+1,y)} \right) \end{aligned}\right\} \ldots (4\text{-}B)$$

where

$$M_{(x,i)} = P_x(i,y)(NX-i)^2(l_x) \left\{ \frac{[3(i)+(NX-i)]\,x}{(NX)^4} - \frac{i}{(NX)^3} \right\} \quad \ldots \text{ when } x \leq i$$

$$M_{(x,i)} = P_x(i,y)(i)^2(l_x) \left\{ \frac{[3(NX-i)+i]\,(NX-x)}{(NX)^4} - \frac{(NX-i)}{(NX)^3} \right\} \quad \ldots \text{when } x > i$$

$$M_{(i,y)} = P_y(x,i)(NY-i)^2(l_y) \left\{ \frac{[3(i)+(NY-i)]\,y}{(NY)^4} - \frac{i}{(NY)^3} \right\} \quad \ldots \text{ when } y \leq i$$

$$M_{(i,y)} = P_y(x,i)(i)^2(l_y) \left\{ \frac{[3(NY-i)+i]\,(NY-y)}{(NY)^4} - \frac{(NY-i)}{(NY)^3} \right\} \quad \ldots \text{ when } y > i$$

Using Equation (2-17) or (2-17a),

$$\left\{ Y^{P_x}_{(x\pm x,y)} \,\middle|\, \begin{array}{l} x = 1\rightarrow m \\ y = 1\rightarrow n \end{array} \right\} = \sum_{i=1}^{m} P_x(i,y)\cdot\, [AP'_x(x,i)] \quad \ldots\ldots (4\text{-}10)$$

$$\left\{ Y^{P_y}_{(x,y\pm y)} \,\middle|\, \begin{array}{l} x = 1\rightarrow m \\ y = 1\rightarrow n \end{array} \right\} = \sum_{i=1}^{n} P_y(x,i)\cdot\, [AP'_y(i,y)] \quad \ldots\ldots (4\text{-}11)$$

where

$$AP'_x(x,i) = \frac{(NX-i)^2\left(\frac{x}{NX}\,{}^1x\pm\Delta x\right)^2}{6EI(NX)^3}\,[\,3(i)({}^1x)-3(i)\left(\frac{x\cdot{}^1x}{NX}\pm\Delta x\right)$$

$$-(NX-i)\left(\frac{x}{NX}\,{}^1x\pm\Delta x\right)] \quad \ldots\ldots \text{when } \left(\frac{x}{NX}\,{}^1x\pm\Delta x\right)\leq\frac{i}{NX}\,{}^1x$$

$$AP'_x(x,i) = \frac{(i)^2\left[{}^1x-\left(\frac{x}{NX}\,{}^1x\pm\Delta x\right)\right]^2}{6EI(NX)^3}\left\{3(NX-i)\,{}^1x-3(NX-i)\left[{}^1x-\left(\frac{x}{NX}\,{}^1x\pm\Delta x\right)\right]\right.$$

$$\left.-i\left[{}^1x-\left(\frac{x}{NX}\,{}^1x\pm\Delta x\right)\right]\right\} \quad \ldots\ldots \text{when } \left(\frac{x}{NX}\,{}^1x\pm\Delta x\right)>\frac{i}{NX}\,{}^1x$$

$$AP'_y(i,y) = \frac{(NY-i)^2\left(\frac{y}{NY}\,{}^1y\pm\Delta y\right)^2}{6EI(NY)^3}\,[3(i)({}^1y)-3(i)\left(\frac{y\cdot{}^1y}{NY}\pm\Delta y\right)$$

$$-(NY-i)\left(\frac{y}{NY}\,{}^1y\pm\Delta y\right)] \quad \ldots\ldots \text{when } \left(\frac{y}{NY}\,{}^1y\pm\Delta y\right)\leq\frac{i}{NY}\,{}^1y$$

$$AP'_y(i,y) = \frac{(i)^2\left[{}^1y-\left(\frac{y}{NY}\,{}^1y\pm\Delta y\right)\right]^2}{6EI(NY)^3}\left\{3(NY-i)\,{}^1y-3(NY-i)\left[{}^1y-\left(\frac{y}{NY}\,{}^1y\pm\Delta y\right)\right]\right.$$

$$\left.-i\left[{}^1y-\left(\frac{y}{NY}\,{}^1y\pm\Delta y\right)\right]\right\} \quad \ldots\ldots \text{when } \left(\frac{y}{NY}\,{}^1y\pm\Delta y\right)>\frac{i}{NY}\,{}^1y$$

And from Equation (2-17b) or (2-17c),

$$\left\{ Y^{T_x}_{(x\pm x,y)} \,\middle|\, \begin{array}{l} x = 1\rightarrow m \\ y = 1\rightarrow n \end{array} \right\} = \sum_{i=1}^{m} T^{x}_{(i,y)}\cdot [AT'_x(x,i)] \quad \ldots\ldots\ldots (4\text{-}11)$$

$$\left\{ Y^{T_y}_{(x,y\pm y)} \,\middle|\, \begin{array}{l} x = 1\rightarrow m \\ y = 1\rightarrow n \end{array} \right\} = \sum_{i=1}^{n} T^{y}_{(x,i)}\cdot [AT'_y(i,y)] \quad \ldots\ldots\ldots (4\text{-}12)$$

where

$$AT'_x(x,i) = \frac{\left(\frac{x}{NX} l_{x\pm\Delta x}\right)^2}{6EI} \left\{ 2C_{xi} + C_{xi} \frac{\left[l_x - \left(\frac{x}{NX} l_{x\pm\Delta x}\right)\right]}{l_x} + d_{xi} \frac{\left(\frac{x}{NX} l_{x\pm\Delta x}\right)}{l_x} - \frac{\left(\frac{x}{NX} l_{x\pm\Delta x}\right)}{l_x} \right\} \quad \ldots\ldots \text{when } \left(\frac{x}{NX} l_{x\pm\Delta x}\right) \le \frac{i}{NX} l_x$$

$$AT'_x(x,i) = \frac{\left[l_x - \left(\frac{x}{NX} l_{x\pm\Delta x}\right)\right]^2}{6EI} \left\{ 2d_{xi} + d_{xi} \frac{\left(\frac{x}{NX} l_{x\pm\Delta x}\right)}{l_x} + C_{xi} \frac{\left[l_x - \left(\frac{x}{NX} l_{x\pm\Delta x}\right)\right]}{l_x} + \frac{\left[l_x - \left(\frac{x}{NX} l_{x\pm\Delta x}\right)\right]}{l_x} \right\} \quad \ldots\ldots \text{when } \left(\frac{x}{NX} l_{x\pm\Delta x}\right) > \frac{i}{NX} l_x$$

$$AT'_y(i,y) = \frac{\left(\frac{y}{NY} l_{y\pm\Delta y}\right)^2}{6EI} \left\{ 2C_{yi} + C_{yi} \frac{\left[l_y - \left(\frac{y}{NY} l_{y\pm\Delta y}\right)\right]}{l_y} + d_{yi} \frac{\left(\frac{y}{NY} l_{y\pm\Delta y}\right)}{l_y} - \frac{\left(\frac{y}{NY} l_{y\pm\Delta y}\right)}{l_y} \right\} \quad \ldots\ldots \text{when } \left(\frac{y}{NY} l_{y\pm\Delta y}\right) \le \frac{i}{NY} l_y$$

$$AT'_y(i,y) = \frac{\left[l_y - \left(\frac{y}{NY} l_{y\pm\Delta y}\right)\right]^2}{6EI} \left\{ 2d_{yi} + d_{yi} \frac{\left(\frac{y}{NY} l_{y\pm\Delta y}\right)}{l_y} + C_{yi} \frac{\left[l_y - \left(\frac{y}{NY} l_{y\pm\Delta y}\right)\right]}{l_y} + \frac{\left[l_y - \left(\frac{y}{NY} l_{y\pm\Delta y}\right)\right]}{l_y} \right\} \quad \ldots\ldots \text{when } \left(\frac{y}{NY} l_{y\pm\Delta y}\right) > \frac{i}{NY} l_y$$

in which C_{xi}, d_{xi}, C_{yi}, d_{yi} are as shown in Equation (4-6)

Then

$$\left\{ Y^x_{(x\pm\Delta x,y)} \;\middle|\; \begin{array}{l} x = 1 \to m \\ y = 1 \to n \end{array} \right\} = Y^{P_x}_{(x\pm\Delta x,y)} + Y^{T_x}_{(x\pm\Delta x,y)} \quad \ldots\ldots (4\text{-}13)$$

$$\left\{ Y^y_{(x,y\pm\Delta y)} \;\middle|\; \begin{array}{l} x = 1 \to m \\ y = 1 \to n \end{array} \right. = Y^{P_y}_{(x,y\pm\Delta y)} + Y^{T_y}_{(x,y\pm\Delta y)} \quad \ldots\ldots (4\text{-}14)$$

Substituting into Equation (2-18), yields

$$\theta^x_{(x,y)} = \frac{Y^y_{(x,y+\Delta y)} - Y^y_{(x,y-\Delta y)}}{2\Delta y} \quad \ldots\ldots\ldots\ldots (4\text{-}15)$$

$$\theta^y_{(x,y)} = \frac{Y^x_{(x+\Delta x,y)} - Y^x_{(x-\Delta x,y)}}{2\Delta x} \quad \ldots\ldots\ldots\ldots (4\text{-}16)$$

The Torsional Moments $T^x_{(x,y)}$ and $T^y_{(x,y)}$ on beams parallel to the x-axis and y-axis in the (x,y) intersection are determined as follows:

$$\left\{ T^x_{(x,y)} \middle| \begin{matrix} x = 1 \to m \\ y = 1 \to n \end{matrix} \right\} = G \cdot J_x \cdot \frac{\theta^x_{(x,y)} - \theta^x_{((x-1),y)}}{l_x}$$

where $\theta^x_{((x-1),y)} = 0$, when $x = 1$.

and

$$\left\{ T^x_{(NX,y)} \middle| y = 1 \to n \right\} = -G \cdot J_x \frac{\theta^x_{(m,y)}}{l_x}$$

$$\left\{ T^y_{(x,y)} \middle| \begin{matrix} x = 1 \to m \\ y = 1 \to n \end{matrix} \right\} = G \cdot J_y \cdot \frac{\theta^y_{(x,y)} - \theta^y_{(x,(y-1))}}{l_y} \quad \ldots (4\text{-C})$$

where $\theta^y_{(x,(y-1))} = 0$, when $y = 1$

and

$$\left\{ T^y_{(x,NY)} \middle| x = 1 \to m \right\} = -G \cdot J_y \cdot \frac{\theta^y_{(x,n)}}{l_y}$$

From Equations (2-20) and (2-20a),

$$\left\{ R^x_{(o,y)} \middle| y = 1 \to n \right\} = \sum_{i=1}^{m} P_x(i,y)(NX-i)^2[3(i)+(NX-i)] \Big/ (NX)^3$$

$$\left\{ R^x_{(NX,y)} \middle| y = 1 \to n \right\} = \sum_{i=1}^{m} P_x(i,y)(i)^2[i+3(NX-i)] \Big/ (NX)^3$$

$$\left\{ R^y_{(x,o)} \middle| x = 1 \to m \right. = \sum_{i=1}^{n} P_y(x,i)(NY-i)^2[3(i)+(NY-i)] \Big/ (NY)^3 \quad \ldots (4\text{-D})$$

$$\left\{ R^y_{(x,NY)} \middle| x = 1 \to m \right. = \sum_{i=1}^{n} P_y(x,i)(i)^2[i+3(NY-i)] \Big/ (NY)^3$$

(b) Numerical Example

Consider a gridwork as shown in Figure 4-2 which is loaded with a 36 kip concentrated load at each intersection. The factors I, E, G, and J are assumed to be the same for all beams.

By writing Equations (4-1), (4-9), (4-A), (4-B), (4-C) and (4-D) into a computer program, the stresses in the gridwork are found as follows:

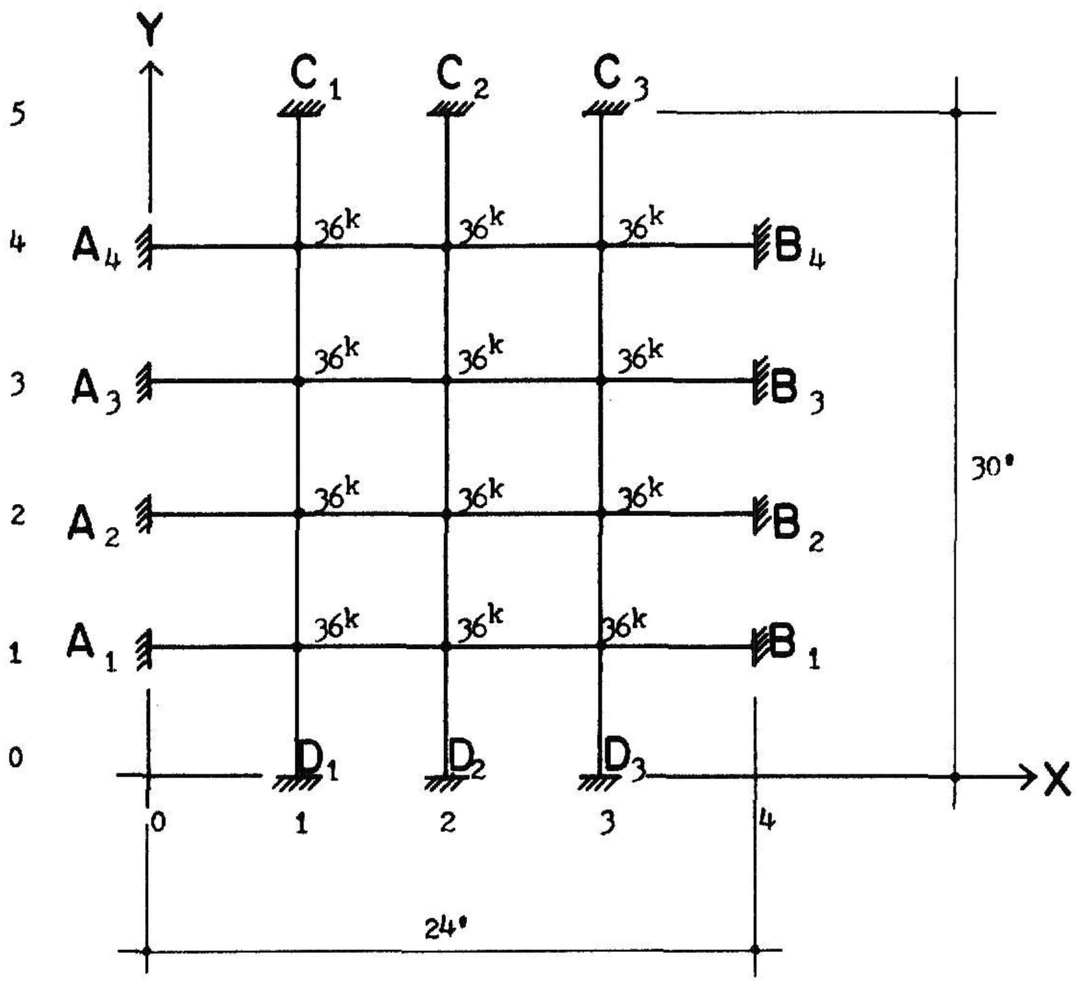


Fig. 4-2 Numerical example for case five

$$M_{A_1B_1} = 100.29\ ^{k\text{-}ft} = M_{B_1A_1} = M_{A_4B_4} = M_{B_4A_4}$$

$$M_{A_2B_2} = 206.95\ ^{k\text{-}ft} = M_{B_2A_2} = M_{A_3B_3} = M_{B_3A_3}$$

$$M_{C_1D_1} = 100.69 \text{ k-ft} = M_{D_1C_1} = M_{C_3D_3} = M_{D_3C_3}$$

$$M_{C_2D_2} = 182.20 \text{ k-ft} = M_{D_2C_2}$$

$$M^{A_1B_1}_{(1,1)} = 28.89 \text{ k-ft} = M^{A_1B_1}_{(3,1)} = M^{A_4B_4}_{(1,4)} = M^{A_4B_4}_{(3,4)}$$

$$M^{A_1B_1}_{(2,1)} = 49.65 \text{ k-ft} = M^{A_4B_4}_{(2,4)}$$

$$M^{A_2B_2}_{(1,2)} = 65.69 \text{ k-ft} = M^{A_2B_2}_{(3,2)} = M^{A_3B_3}_{(1,3)} = M^{A_3B_3}_{(3,3)}$$

$$M^{A_2B_2}_{(2,2)} = 124.22 \text{ k-ft} = M^{A_3B_3}_{(2,3)}$$

$$M^{C_1D_1}_{(1,1)} = 24.98 \text{ k-ft} = M^{C_1D_1}_{(1,4)} = M^{C_3D_3}_{(3,1)} = M^{C_3D_3}_{(3,4)}$$

$$M^{C_1D_1}_{(1,2)} = 39.10 \text{ k-ft} = M^{C_1D_1}_{(1,3)} = M^{C_3D_3}_{(3,2)} = M^{C_3D_3}_{(3,3)}$$

$$M^{C_2D_2}_{(2,1)} = 45.30 \text{ k-ft} = M^{C_2D_2}_{(2,4)}$$

$$M^{C_2D_2}_{(2,2)} = 96.57 \text{ k-ft} = M^{C_2D_2}_{(2,3)}$$

$$T^x_{(1,1)} = 37.87 \text{ k-ft} = T^x_{(4,1)} = T^x_{(1,4)} = T^x_{(4,4)}$$

$$T^x_{(2,1)} = 24.65 \text{ k-ft} = T^x_{(3,1)} = T^x_{(2,4)} = T^x_{(3,4)}$$

$$T^x_{(1,2)} = 14.72 \text{ k-ft} = T^x_{(4,2)} = T^x_{(1,3)} = T^x_{(4,3)}$$

$$T^x_{(2,2)} = 9.44 \text{ k-ft} = T^x_{(2,3)} = T^x_{(3,2)} = T^x_{(3,3)}$$

$$T^y_{(1,1)} = 37.20 \text{ k-ft} = T^y_{(3,1)} = T^y_{(1,5)} = T^y_{(3,5)}$$

$$T^y_{(1,2)} = 32.44^{\text{ k-ft}} = T^y_{(1,4)} = T^y_{(3,2)} = T^y_{(3,4)}$$

$$T^y_{(2,1)} = 0 = T^y_{(2,5)} = T^y_{(2,2)} = T^y_{(2,4)} = T^y_{(2,3)} = T^y_{(1,3)} = T^y_{(3,3)}$$

$$R_{A_1} = 21.25^{\text{ k}} = R_{B_1} = R_{A_4} = R_{B_4}$$

$$R_{A_2} = 45.44^{\text{ k}} = R_{B_2} = R_{A_3} = R_{B_3}$$

$$R_{C_1} = 21.34^{\text{ k}} = R_{D_1} = R_{C_3} = R_{D_3}$$

$$R_{C_2} = 39.36^{\text{ k}} = R_{D_2}$$

(c) Guide for use of the computer program

The program is written for the analysis of gridworks with fixed-edges. It can be used to analyze gridworks with all joints loaded or with part of the joints loaded. There are only two data cards needed to use this program.

The first data card expresses L_x, L_y, N_x, N_y, K_{xy}, E, XI, YI, G, XJ, YJ, and must be punched in a form to be compatible with the following FORTRAN format statement,

```
FORMAT (5I5,/6E12.4)
```

where

L_x = length of beams in the x-direction;

L_y = length of beams in the y-direction;

N_x = number of grid spacings in the x-direction, the maximum value is 10;

N_y = number of grid spacings in the y-direction, the maximum value is 10;

K_{xy} = unit length in the same factor as L_x;

E = modulus of elasticity;

XI = I for the beams in the x-direction;

YI = I for the beams in the y-direction;

G = modulus of elasticity in shear;

XJ = J for the beams in the x-direction;

YJ = J for the beams in the y-direction;

I = moment of inertia of the beam cross section with respect to the neutral axis;

J = For a circular cross section is the polar moment of inertia. For a rectangular cross section it is equal to βbd^3, where b is the width, d is the depth of the section, and β is a coefficient depending on the cross-section properties.

The second card expresses the applied load at each joint. The loads must be punched in sequence as $P_{(1,1)}$, $P_{(2,1)}$, $P_{(3,1)}$, . . . $P_{(1,2)}$, $P_{(2,2)}$, $P_{(3,2)}$, ... and in a form compatible with the following FORTRAN format statement

```
FORMAT (6E10.2)
```

Zero must be punched when $P_{(x,y)}$ does not exist.

The joint and member coordinates used in the output are shown in Figure 4-3.

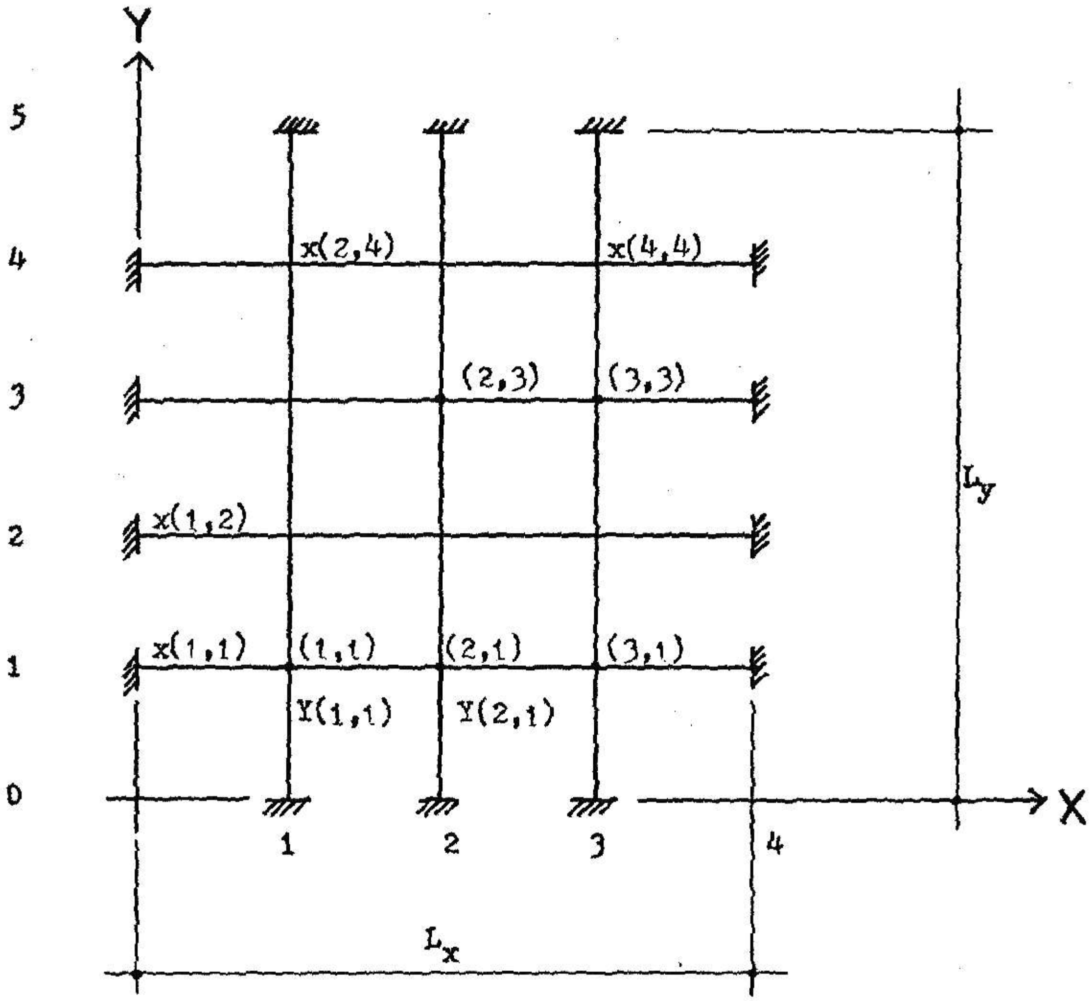


Fig. 4-3 Joint and member coordinates

CHAPTER FIVE

COMPARISON OF THE RESULTS FROM THE APPROXIMATE SOLUTION AND THE STRUDL PROGRAM

(1). Case 1

Fixed-end Moment

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
A, B	33.2	32.24	+2.97
C, D	24.3	25.37	-4.22

Intermediate Moment

Point		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
Beam parallel to x-axis	0	33.2	32.24	+2.97
Beam parallel to y-axis	0	24.3	25.37	-4.22

Reaction

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
A, B	11.0	10.75	+2,32
C, D	7.0	7.25	-3.45

(2). Case 2

Fixed-end Moment

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
A_1, B_1, A_2, B_2	43.2	40.15	+7.6
C, D	28.8	33.23	-13.3

Intermediate Moment

Joint Coordinate		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
Beam parallel to x-axis	(1,1),(1,2)	43.2	40.15	+7.6
Beam parallel to y-axis	(1,1),(1,2)	14.4	22.15	-34.7

Torsional Moment

Member Coordinate		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
Beam parallel to x-axis	(1,1),(2,1)	6.25	5.53	+12.8

Reaction

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
A_1, B_1, A_2, B_2	14.4	13.38	+7.6
C, D	7.2	9.23	-22

(3) Case 3

Fixed-end Moment

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)}$ x 100%
A_1, B_1, A_2, B_2 C_1, D_1, C_1, D_1	72	67.5	+6.67

Intermediate Moment

Joint Coordinate		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)}$ x 100%
Beams parallel to x-axis	(1,1),(1,2) (2,1),(2,2)	36	40.5	-11.0
Beams parallel to y-axis	(1,1),(1,2) (2,1),(2,2)	36	40.5	-11.0

Torsional Moment

Member Coordinate		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)}$ x 100%
Beams parallel to x-axis	(1,1),(3,1) (1,2),(3,2)	17.7	13.5	+31
Beams parallel to y-axis	(1,1),(1,3) (2,1),(2,3)	17.7	13.5	+31

Reaction

Joint	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)}$ x 100%
A_1, B_1, A_2, B_2 C_1, D_1, C_2, D_2	18	18	0

(4) Case 4

Fixed-end Moment

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
A_1, B_1, A_3, B_3	83	78.1	+6.37
A_2, B_2	136	120.1	+13.3
C_1, D_1, C_2, D_2	74.8	74.37	+0.6

Intermediate Moment

Joint Coordinate		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
Beams parallel to x-axis	(1,1),(2,1) (1,3),(2,3)	41.4	43.47	-4.75
	(1,2),(2,2)	68	68.5	-0.73
Beams parallel to y-axis	(1,1),(1,3) (2,1),(2,3)	22.9	33.7	-32
	(1,2),(2,2)	28.9	27.16	+6.4

Torsional Moment

Member Coordinate		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
Beams parallel to x-axis	(1,1),(1,3) (3,1),(3,3)	25.6	20.3	+26
Beams parallel to y-axis	(1,1),(1,4) (2,1),(2,4)	20.2	17.31	+16.6
	(1,2),(1,3) (1,2),(2,3)	12.9	8.47	+51.8

Reaction

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
A_1, B_1, A_3, B_3	20.7	20.26	+2.47
A_2, B_2	34	31.42	+8.3
C_1, D_1, C_2, D_2	16.3	18.02	-9.55

(5) Case 5

Fixed-end Moment

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
A_1, B_1, A_4, B_4	100.29	106.51	-5.85
A_2, B_2, A_3, B_3	206.95	202.39	+2.26
C_1, D_1, C_3, D_3	100.69	105.11	-4.2
C_2, D_2	182.2	175.41	+3.87

Intermediate Moment

Joint Coordinate		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
Beams parallel to x-axis	(1,1),(3,1) (1,4),(3,4)	28.89	30.55	-5.45
	(2,1),(2,4)	49.65	51.87	-4.28
	(1,2),(1,3) (3,2),(3,3)	65.69	63.42	+3.58
	(2,2),(2,3)	124.22	107.05	+16.0
Beams parallel to y-axis	(1,1),(1,4) (3,1),(3,4)	24.98	26.9	-7.15
	(1,2),(1,3) (3,2),(3,3)	39.1	36.35	+7.55
	(2,1),(2,4)	45.3	50.7	-10.6
	(2,2),(2,3)	96.57	70	+37.8

Torsional Moment

Member Coordinate		Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
Beams parallel to x-axis	(1,1),(4,1) (1,4),(4,4)	37.87	39.09	-3.13
	(2,1),(3,1) (2,4),(3,4)	24.65	23.23	+6.11
	(1,2),(1,3) (4,2),(4,3)	14.72	15.39	-4.35
	(2,2),(2,3) (3,2),(3,3)	9.44	9.8	-3.66
Beams parallel to y-axis	(1,1),(3,1) (1,5),(3,5)	37.2	37.97	-2.03
	(1,2),(1,4) (3,2),(3,4)	32.44	31.5	+2.99
	(2,1),(2,5)	$\doteq 0$	$\doteq 0$	0
	(2,2),(2,4) (1,3),(3,3)	$\doteq 0$	$\doteq 0$	0

Reaction

Point	Approximate Solution (1)	STRUDL Solution (2)	$\frac{(1)-(2)}{(2)} \times 100\%$
A_1, B_1, A_4, B_4	21,53	22.84	-5.75
A_2, B_2, A_3, B_3	45.44	44.3	+2.57
C_1, D_1, C_3, D_3	21.34	22.0	-3.0
C_2, D_2	39.36	37.69	+4.44

DISCUSSION

The method presented in this report is based on the simplest mathematical processes and the most fundamental structural concepts for solving for the stresses in gridworks. It is similar to the "Beam method" in the analysis of Folded Plate Structures. The equations developed by the writer can be easily followed and readily applied.

In simple cases the writer neglects the beam deflections caused by the effect of torsional moments, since these deflections are small in comparison with the total deflections. It then becomes a simple procedure to find the approximate stresses in the gridworks. By using the same process, we can develop solutions for many other cases.

In a comparison of the results from this method and ICES STRUDL results, it can be seen that the stresses, in the structure with biaxial symmetry in both geometry and loading, from the former method are close to those from latter method. Simultaneously, we find the number of unknowns in this type of gridwork will reduce to one quarter of the number for an unsymmetric situation, hence it will be the easiest type of gridwork to solve using this approximate method.

CONCLUSIONS

The analysis of gridworks subjected to normal loads is normally a tedious process, the writer has attempted to circumvent this difficulty. This is only the beginning of doing this kind of work. Even though computers are presently popular and widely available, a simple method of analysis for plane gridworks is worth while for the practicing engineer. We can use this method for preliminary design or for estimating the required sections or even for the final analysis of simple gridwork structures.

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APPENDIX

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$JOB
C      ANALYSIS OF GRIDWORK STRUCTURES WITH FIXED EDGES
       DIMENSION A(200,201),PX(10,10),PY(10,10),FEMXA(10),FEMXB(10),
      CFEMYC(10),FEMYD(10),CMX(10,10),CMY(10,10),DEFX1(10,10),DEFX2(10,1
      C),DEFY1(10,10),DEFY2(10,10),TORY(11,11),TORX(11,11),REAXA(10),
      CREAXB(10),REAYC(10),REAYD(10),THETAY(10,10),THETAX(10,10),TX(10,
      C10),TY(10,10),DEFT(100),YY(10,10),YX(10,10),AB(100)
   50 FORMAT(5I5,/6E12.4)
      READ(5,50) LX,LY,NX,NY,KXY,E,XI,YI,G,XJ,YJ
      MX=NX-1
      MY=NY-1
      N=MX*MY
      N1=2*N
      N2=N1+1
  100 FORMAT(6E10.2)
      READ(5,100) (AB(K),K=1,N)
      DO 80 K=1,MX
      DO 80 J=1,MY
      YY(K,J)=0.0
   80 YX(K,J)=0.0
      AMX=0.
      BMX=0.
      AMY=0.0
      BMY=0.0
      DO 82 K=1,MY
      DO 82 J=1,MX
      TX(J,K)=0.0
      TORX(J,K)=0.
      TY(J,K)=0.
   82 TORY(J,K)=0.
      JJJ=0
  500 DO 90 K=1,N1
      DO 90 J=1,N2
   90 A(K,J)=0.0
      JJJ=JJJ+1
      DO 101 K=1,N
      L=(K-1)/MX
      IS=L*MX
      IF(K-MX) 111,111,112
  111 J1=K
      GO TO 113
  112 J1=K-IS
  113 DO 101 M=1,MX
      I=K+N
      IF(J1-M) 103,103,162
  103 A(I,M+IS)=((NX-M)**2*LX**3*J1**2*(3.*NX*M-3.*M*J1-(NX-M)*J1))/
     C(6.*E*XI*NX**6)
      GO TO 101
  162 A(I,M+IS)=(M**2*LX**3*(NX-J1)**2*(3.*NX*(NX-M)-3.*(NX-M)*(NX-J1)
     C-(NX-J1)*M))/(6.*E*XI*NX**6)
  101 CONTINUE
      DO 105 K=1,N
      L1=(K-1)/MX
      J3=K-L1*MX
      J2=L1+1
      DO 105 M1=1,MY
      MM=M1+((J3-1)*MY+N)
      I=K+N
      IF(J2-M1) 107,107,166
  107 A(I,MM)=-((NY-M1)**2*LY**3*J2**2*(3.*NY*M1-3.*M1*J2-(NY-M1)*J2)
```

```
      C)/(6.*E*YI*NY**6)
       GO TO 105
  166 A(I,MM)=-(M1**2*LY**3*(NY-J2)**2*(3.*NY*(NY-M1)-3.*(NY-M1)*(NY-
     CJ2)-(NY-J2)*M1))/(6.*E*YI*NY**6)
  105 CONTINUE
      DO 140 K=1,MY
      DO 140 J=1,MX
      KKJ=J+(K-1)*MX
  140 DEFT(KKJ)=YY(J,K)-YX(J,K)
      DO 142 K=1,N
  142 A(K+N,N2)=DEFT(K)
      DO 109 K=1,N
      A(K,N2)=AB(K)
  109 A(K,K)=1.
      DO 110 K=1,N
      K1=(K-1)/MX
      K2=K-MX*K1
      I1=((K2-1)*MY)+(N+1+K1)
  110 A(K,I1)=1.
      DO 123 K=1,N1
      K11=0
      IF(A(K,K).NE.0.0) GO TO 123
  124 K11=K11+1
      KL=K11+K
      IF(A(KL,K).EQ.0.0) GO TO 124
      DO 122 KM=1,N2
      TEMP=A(K,KM)
      A(K,KM)=A(KL,KM)
  122 A(KL,KM)=TEMP
  123 CONTINUE
      DO 121 K=1,N1
      DIV=A(K,K)
      S=1.0/DIV
      DO 128 KJ=K,N2
  128 A(K,KJ)=A(K,KJ)*S
      A(K,K)=1.
      DO 129 KI=1,N1
      IF(KI-K) 130,129,130
  130 AIJ=-A(KI,K)
      DO 132 KJ=K,N2
  132 A(KI,KJ)=A(KI,KJ)+AIJ*A(K,KJ)
  129 CONTINUE
  121 CONTINUE
      DO 133 K=1,MY
      DO 133 II=1,MX
      IKJ=(K-1)*MX+II
  133 PX(II,K)=A(IKJ,N2)
      DO 135 II=1,MX
      DO 135 K=1,MY
      IJ=((II-1)*MY+K+N)
  135 PY(II,K)=A(IJ,N2)
      DO 300 KX=1,MY
      DO 300 JX=1,MX
      KX1=(JX*LX)/NX-KXY
      KX2=(JX*LX)/NX+KXY
      KXX1=KX1*NX/LX
      KXX2=KX2*NX/LX
      DEFX1(JX,KX)=0.
      DEFX2(JX,KX)=0.
      X1=KX1*1.
```

```
      X2=KX2*1.
      DO 302 MI=1,MX
      AMX=(TX(MI,KX)/NX**3)*(-NX**3+4.*NX**2*MI-3.*NX*MI**2)
      BMX=(TX(MI,KX)/NX**3)*(3.*NX*MI**2-2.*NX**2*MI)
      IF(KXX1-MI) 303,304,304
  303 AA1=PX(MI,KX)*(NX-MI)**2*X1**2*(3.*MI*LX-3.*MI*X1-(NX-MI)*X1)/
     C(6.*E*XI*NX**3)
      AAT1=X1**2/6*E*XI*(2*AMX+AMX*(LX-X1)/LX+BMX*X1/LX-TX(MI,KX)*X1/LX
      DEFX1(JX,KX)=DEFX1(JX,KX)+AA1+AAT1
      GO TO 330
  304 AA1=PX(MI,KX)*MI**2*(LX-X1)**2*(3.*(NX-MI)*LX-3.*(NX-MI)*(LX-X1)
     C-MI*(LX-X1))/(6.*E*XI*NX**3)
      AAT1=(LX-X1)**2/6.*E*XI*(2*BMX+BMX*X1/LX+AMX*(LX-X1)/LX+TX(MI,KX)
     C(LX-X1)/LX)
      DEFX1(JX,KX)=DEFX1(JX,KX)+AA1+AAT1
  330 IF(KXX2-MI) 333,334,334
  333 AA2=PX(MI,KX)*(NX-MI)**2*X2**2*(3.*MI*LX-3.*MI*X2-(NX-MI)*X2)/
     C(6.*E*XI*NX**3)
      AAT2=X2**2/6*E*XI*(2*AMX+AMX*(LX-X2)/LX+BMX*X2/LX-TX(MI,KX)*X2/LX
      DEFX2(JX,KX)=DEFX2(JX,KX)+AA2+AAT2
      GO TO 302
  334 AA2=PX(MI,KX)*MI**2*(LX-X2)**2*(3.*(NX-MI)*LX-3.*(NX-MI)*(LX-X2)
     C-MI*(LX-X2))/(6.*E*XI*NX**3)
      AAT2=(LX-X2)**2/6.*E*XI*(2*BMX+BMX*X2/LX+AMX*(LX-X2)/LX+TX(MI,KX)
     C(LX-X2)/LX)
      DEFX2(JX,KX)=DEFX2(JX,KX)+AA2+AAT2
  302 CONTINUE
      THETAY(JX,KX)=(DEFX2(JX,KX)-DEFX1(JX,KX))/2.*KXY
      IF(KX-1) 305,305,306
  305 TORY(JX,KX)=THETAY(JX,KX)*G*YJ*NY/LY
      GO TO 300
  306 TORY(JX,KX)=(THETAY(JX,KX)-THETAY(JX,KX-1))*G*YJ*NY/LY
  300 CONTINUE
      DO 307 JX=1,MX
  307 TORY(JX,NY)=-THETAY(JX,MY)*G*YJ*NY/LY
      DO 310 KY=1,MX
      DO 310 JY=1,MY
      KY1=(JY*LY)/NY-KXY
      KY2=(JY*LY)/NY+KXY
      KYY1=KY1*NY/LY
      KYY2=KY2*NY/LY
      DEFY1(KY,JY)=0.
      DEFY2(KY,JY)=0.
      Y1=KY1*1.
      Y2=KY2*1.
      DO 312 MI=1,MY
      AMY=(TY(KY,MI)/NY**3)*(-NY**3+4.*NY**2*MI-3.*NY*MI**2)
      BMY=(TY(KY,MI)/NY**3)*(3.*NY*MI**2-2.*NY**2*MI)
      IF(KYY1-MI) 313,314,314
  313 B1=PY(KY,MI)*(NY-MI)**2*Y1**2*(3.*MI*LY-3.*MI*Y1-(NY-MI)*Y1)/
     C(6.*E*YI*NY**3)
      BT1=Y1**2/6.*E*YI*(2*AMY+AMY*(LY-Y1)/LY+BMY*Y1/LY-TY(KY,MI)*Y1/LY
      DEFY1(KY,JY)=DEFY1(KY,JY)+B1+BT1
      GO TO 340
  314 B1=PY(KY,MI)*MI**2*(LY-Y1)**2*(3.*(NY-MI)*LY-3.*(NY-MI)*(LY-Y1)
     C-MI*(LY-Y1))/(6.*E*YI*NY**3)
      BT1=(LY-Y1)**2/6.*E*YI*(2*BMY+BMY*Y1/LY+AMY*(LY-Y1)/LY+TY(KY,MI)*
     C(LY-Y1)/LY)
      DEFY1(KY,JY)=DEFY1(KY,JY)+B1+BT1
  340 IF(KYY2-MI) 343,344,344
```

```
343 B2=PY(KY,MI)*(NY-MI)**2*Y2**2*(3.*MI*LY-3.*MI*Y2-(NY-MI)*Y2)/
   C(6.*E*YI*NY**3)
    BT2=Y2**2/6.*E*YI*(2*AMY+AMY*(LY-Y2)/LY+BMY*Y2/LY-TY(KY,MI)*Y2/LY)
    DEFY2(KY,JY)=DEFY2(KY,JY)+B2+BT2
    GO TO 312
344 B2=PY(KY,MI)*MI**2*(LY-Y2)**2*(3.*(NY-MI)*LY-3.*(NY-MI)*(LY-Y2)
   C-MI*(LY-Y2))/(6.*E*YI*NY**3)
    BT2=(LY-Y2)**2/6.*E*YI*(2*BMY+BMY*Y2/LY+AMY*(LY-Y2)/LY+TY(KY,MI)*
   C(LY-Y2)/LY)
    DEFY2(KY,JY)=DEFY2(KY,JY)+B2+BT2
312 CONTINUE
    THETAX(KY,JY)=(DEFY2(KY,JY)-DEFY1(KY,JY))/2.*KXY
    IF(KY-1) 315,315,316
315 TORX(KY,JY)=THETAX(KY,JY)*G*XJ*NX/LX
    GO TO 310
316 TORX(KY,JY)=(THETAX(KY,JY)-THETAX(KY-1,JY))*G*XJ*NX/LX
310 CONTINUE
    DO 317 JY=1,MY
317 TORX(NX,JY)=-THETAX(MX,JY)*G*XJ*NX/LX
    DO 350 K=1,MX
    DO 350 J=1,MY
    TX(K,J)=TORY(K,J)-TORY(K,(J+1))
350 TY(K,J)=TORX(K,J)-TORX((K+1),J)
    DO 355 K=1,MY
    DO 355 J=1,MX
    YX(J,K)=0.
    DO 355 MI=1,MX
    AMX=(TX(MI,K)/NX**3)*(-NX**3+4.*NX**2*MI-3.*NX*MI**2)
    BMX=(TX(MI,K)/NX**3)*(3.*NX*MI**2-2.*NX**2*MI)
    IF(J-MI) 361,361,362
361 C=(J**2*LX**2/(6.*E*XI*NX**2))*(2*AMX+AMX*(NX-J)/NX+BMX*J/NX-TX(MI
   C,K)*J/NX)
    GO TO 355
362 C=((NX*LX-J*LX)**2/(6.*E*XI*NX**2))*(2*BMX+BMX*J/NX+AMX*(NX-J)/NX
   C+TX(MI,K)*(NX-J)/NX)
355 YX(J,K)=YX(J,K)+C
    DO 365 K=1,MX
    DO 365 J=1,MY
    YY(K,J)=0.
    DO 365 MI=1,MY
    AMY=(TY(K,MI)/NY**3)*(-NY**3+4.*NY**2*MI-3.*NY*MI**2)
    BMY=(TY(K,MI)/NY**3)*(3.*NY*MI**2-2.*NY**2*MI)
    IF(J-MI) 371,371,372
371 C=(J**2*LY**2/(6.*E*YI*NY**2))*(2*AMY+AMY*(NY-J)/NY+BMY*J/NY-TY(K,
   CMI)*J/NY)
    GO TO 365
372 C=((NY*LY-J*LY)**2/(6.*E*YI*NY**2))*(2*BMY+BMY*J/NY+AMY*(NY-J)/NY+
   CTY(K,MI)*(NY-J)/NY)
365 YY(K,J)=YY(K,J)+C
    IF(JJJ.GT.3) GO TO 502
    GO TO 500
502 DO 200 K=1,MY
    FEMXA(K)=0.
    FEMXB(K)=0.
    DO 200 II=1,MX
    A1=PX(II,K)*II*(NX-II)**2*LX/NX**3
    AM=(TX(II,K)/NX**3)*(-NX**3+4.*NX**2*II-3.*NX*II**2)
    A2=PX(II,K)*II**2*(NX-II)*LX/NX**3
    BM=(TX(II,K)/NX**3)*(3.*NX*II**2-2.*NX**2*II)
    FEMXA(K)=FEMXA(K)+A1+AM
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200 FEMXB(K)=FEMXB(K)+A2+BM
    DO 202 K=1,MX
    FEMYD(K)=0.
    FEMYC(K)=0.
    DO 202 II=1,MY
    A3=PY(K,II)*II*(NY-II)**2*LY/NY**3
    AM=(TY(K,II)/NY**3)*(-NY**3+4.*NY**2*II-3.*NY*II**2)
    A4=PY(K,II)*II**2*(NY-II)*LY/NY**3
    BM=(TY(K,II)/NY**3)*(3.*NY*II**2-2.*NY**2*II)
    FEMYD(K)=FEMYD(K)+A3+AM
202 FEMYC(K)=FEMYC(K)+A4+BM
    DO 210 K=1,MY
    DO 210 J=1,MX
    CMX(J,K)=+(ABS(TX(J,K)))/2.
    DO 210 II=1,MX
    IF(J-II) 213,213,214
213 B=PX(II,K)*(NX-II)**2*LX*((3.*II+(NX-II))*J/NX**4-II*1./NX**3)
    GO TO 210
214 B=PX(II,K)*II**2*LX*(((3.*(NX-II)+II)*(NX-J))/NX**4-(NX-II)*1./
   CNX**3)
210 CMX(J,K)=CMX(J,K)+B
    DO 220 K=1,MX
    DO 220 J=1,MY
    CMY(K,J)=+(ABS(TY(K,J)))/2.
    DO 220 II=1,MY
    IF(J-II) 223,223,224
223 B=PY(K,II)*(NY-II)**2*LY*((3.*II+(NY-II))*J/NY**4-II*1./NY**3)
    GO TO 220
224 B=PY(K,II)*II**2*LY*(((3.*(NY-II)+II)*(NY-J))/NY**4-(NY-II)*1./
   CNY**3)
220 CMY(K,J)=CMY(K,J)+B
    DO 400 K=1,MY
    REAXA(K)=0.
    REAXB(K)=0.
    DO 400 J=1,MX
    R1=PX(J,K)*(NX-J)**2*(3*J+(NX-J))/NX**3
    R2=PX(J,K)*J**2*(J+3*(NX-J))/NX**3
    REAXA(K)=REAXA(K)+R1
400 REAXB(K)=REAXB(K)+R2
    DO 402 K=1,MX
    REAYD(K)=0.
    REAYC(K)=0.
    DO 402 J=1,MY
    R3=PY(K,J)*(NY-J)**2*(3*J+(NY-J))/NY**3
    R4=PY(K,J)*J**2*(J+3*(NY-J))/NY**3
    REAYD(K)=REAYD(K)+R3
402 REAYC(K)=REAYC(K)+R4
    KB=0
150 FORMAT(1H0,12X,'FIXED-END MOMENTS AND REACTIONS ON BEAMS')
151 FORMAT(/8X,'X',5X,'Y',5X,'FIXED-END MOMENT',23X,'REACTION'/)
152 FORMAT(/7X,I2,4X,I2,5X,E15.7,20X,E15.7)
    WRITE(6,150)
    WRITE(6,151)
    WRITE(6,152) (K,KB,FEMYD(K),REAYD(K),K=1,MX)
    WRITE(6,152) (K,NY,FEMYC(K),REAYC(K),K=1,MX)
    WRITE(6,152) (KB,K,FEMXA(K),REAXA(K),K=1,MY)
    WRITE(6,152) (NX,K,FEMXB(K),REAXB(K),K=1,MY)
153 FORMAT(1H0,12X,'INTERMEDIATE MOMENTS AND TORSIONAL MOMENTS ON BEA
   CS-'/13X,'PARALLEL TO X-AXIS')
154 FORMAT(/8X,'X',5X,'Y',5X,'INTERMEDITATE MOMENT',15X,'TORSIONAL MO
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    CENT')
     WRITE(6,153)
     WRITE(6,154)
     WRITE(6,152) ((K,J,CMX(K,J),TORX(K,J),K=1,MX),J=1,MY)
 155 FORMAT(/7X,I2,4X,I2,40X,E15.7)
     WRITE(6,155) (NX,J,TORX(NX,J),J=1,MY)
 156 FORMAT(1H0,12X,'INTERMEDIATE MOMENTS AND TORSIONAL MOMENTS ON BEA
    CS-'/13X,'PARALLEL TO Y-AXIS')
     WRITE(6,156)
     WRITE(6,154)
     WRITE(6,152) ((K,J,CMY(K,J),TORY(K,J),J=1,MY),K=1,MX)
     WRITE(6,155) (K,NY,TORY(K,NY),K=1,MX)
     STOP
     END

$ENTRY
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SIMPLIFIED AND APPROXIMATE SOLUTION

FOR GRIDWORKS WITH FIXED EDGES

by

Chung-Yih Lee

Diploma, Taipei Institute of Technology, 1965

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1973

ABSTRACT

This report presents a simplified method for solving for the approximate stresses in gridworks which are fixed at their edges. The method assumes that the applied load at each node can be replaced by two components acting separately upon the beams which are intersect at that node. The deflections of the longitudinal and transverse beams caused by these component loads should be compatible at each intersection, where the deflections caused by the torsional moments are neglected, since these deflections are small in comparison with the total deflections. From this relationship the component loads can be found. Then all of the stresses in the gridwork can be derived in terms of these components.

Using this process the writer derived simplified equations for four simple special cases, and developed a computer program for complicated cases. A numerical example is given for each case and the results of the numerical examples are compared with the results from the ICES STRUDL program.