Descendable maps of schemes and generators of derived categories

by

Jiayu Zhao

B.A., Capital Normal University 2014

M.S., Capital Normal University 2017

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
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Approved by:

Major Professor

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Chapter 1

Introduction

The theory of triangulated categories was first developed by Jean-Louis Verdier to axiomatize what we call today the triangulated structure on the derived categories of abelian categories. Triangulated categories arise in various areas of mathematics. For example, they offer a powerful framework for studying exact sequences and spectral sequences in homological algebra[1]. They are also used to study the stable homotopy category of spectra in algebraic topology and have applications in representation theory[2].

We will focus on the triangulated categories that arise from algebraic geometry and study their generation. The concept of generation in triangulated categories was first introduced by A. Bondal and M. van den Bergh in[3]. If G is an object in a triangulated category $\mathcal{T}$, we denote by $\langle G \rangle_n$ the full subcategory of $\mathcal{T}$ whose objects can be built from G by taking finite coproducts, direct summands, shifts and at most $n-1$ cones. If $\langle G \rangle = \bigcup_{n \in \mathbb{N}} \langle G \rangle_n = \mathcal{T}$ then we say that G is a classical generator of $\mathcal{T}$, that is, every object in $\mathcal{T}$ can be built from G in finitely many steps. If there is a natural number n such that $\langle G \rangle_n = \mathcal{T}$ then we call G a strong generator (2.4).

The dimension of a triangulated category introduced by Rouquier[4] is defined to be the minimal integer n (if it exists) such that there exists a strong generator G with $\langle G\rangle_n = \mathcal{T}$. The existence of classical and strong generators, as well as the Rouquier dimension for the bounded derived categories of coherent sheaves on Noetherian schemes, have been studed in many cases[4 5 6 7 8 9] to mention a few. Orlov[7] conjectured that for any smooth quasi-projective scheme X, the Rouquier of $D^b(coh(X))$ equals the dimension of X. The conjecture has been verified in some special situations[10 11 12].

In Chapter 2, we recall some basics about triangulated categories, derived categories of schemes, derived functors, and provide the definition of generation and dimension of triangulated categories with examples. We show that for Noetherian schemes with dimension less than or equal to 1, their bounded derived categories of coherent sheaves have finite rouquier dimension.

The connection between ghost indices and generation time is discussed in Chapter 3. The ghost lemma implies the generation time of an object G has its ghost index as a lower bound, when $\langle G\rangle$ is contravariantly finite, they coinside. We give two general constructions of contravariantly finite subcategories for traingualted categories. One construction is from adjoint pairs of functors and the other is the diamond product of two such subcategories.

In Chapter 4, we study the derived categories of schemes with support on a closed subset. We first recall some basic properties of Bousfield localizations of triangulated categories. The main result is the equivalence between $D_Z(X)$ and $D_Z(U)$ for any scheme X a closed subset Z and an open subset U such that $Z \subset U \subset X$. As an application, we give a simple proof of the existence of compact generators for the derived categories of quasi-compact quasi-separated schemes.

The notation of descendable algebra objects and descendable maps was introduced by

Mathew in[13] in the content of stable ∞-categories. Descendable maps were also studied by Balmer independently under the name of nil-faithfulness[14]. We will define desccendable morphisms for schemes in Chapter 5 and prove some nice properties they have, including that they are local on the target, stable under composition, and stable under base change. Some examples will be given. Given a descendable morphism $f : X \to Y$, we will discuss how f relates the generators on $D(X)$ and on $D(Y)$.

Chapter 2

Preliminaries

2.1 Triangulated categories

Generation of triangulated categories was first introduced by A. Bondal and M. Van den Bergh[3] and further developed by R. Rouquier[4]. In this section, we recall some basic definitions and constructions of triangulated categories. For more background about category theory and homological algebra, we refer the reader to Gelfand and Manin[1] and Kashiwara[15]

Definition 2.1.1. *A triangulated category $\mathcal{T}$ is an additive category with an additive auto-equivalence* [1] *called the shift (or translation, suspension) functor equipped with a collection of triangles called distinguished triangles such that the following axioms hold:*

- *TR0: every triangle isomorphic to a distinguished triangle is a distinguished triangle.*
- *TR1: the triangle $X \xrightarrow{id_X} X \to 0 \to X[1]$ is a distinguished triangle*
- *TR2: any morphism $f : X \to Y$ can be extended to a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g}$*

$Z \xrightarrow{h} X[1]$, *this triangle is also denoted as*

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ Z \arrow[ur, dotted, "h"] & & Y \arrow[ll, "g"] \end{tikzcd}$$

Z *is called the cone of* f.

- *TR3:* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *is a distinguished triangle if and only if* $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ *is a distinguished triangle*
- *TR4: given any two distinguished triangles* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *and* $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ *and morphisms* α *and* β *making the following square commutes,*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

There exists a morphism $\gamma : Z \to Z'$ *extending the square to a morphism of triangles in that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, "\gamma"] & X[1] \arrow[d, "\alpha[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

- *TR5 (octahedral axiom): for any two composible morphisms* $X \xrightarrow{f} Y \xrightarrow{g} Z$, *there exists a distinguished triangle* $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} Z'[1]$ *as the outer triangle in the following diagram, where the objects* X', Y', Z' *are obtained from TR2.*

$$
\begin{tikzcd}
& & Z' \arrow[dl, dotted] \arrow[ddll, bend right, "u"'] & & \\
& X \arrow[rr, "f"] \arrow[dr, "g\circ f"'] & & Y \arrow[ul, "h"'] \arrow[dl, "g"] & \\
Y' \arrow[ur, dotted] \arrow[rrrr, bend right, "v"'] & & Z \arrow[ll, "l"] \arrow[rr, "k"] & & X' \arrow[ul, dotted] \arrow[uull, bend right, dotted, "w"']
\end{tikzcd}
$$

We require functors between triangulated categories respect the triangulated structures:

Definition 2.1.2. *A functor $F : \mathcal{T} \to \mathcal{T}'$ between triangulated categories is said to be exact if F is additive, with a natural isomorphism $F \circ [1] \cong [1] \circ F$ and F maps distinguished triangles to distinguished triangles.*

Definition 2.1.3. *An additive subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ is called a triangulated subcategory if $\mathcal{C}$ is closed under the shift functor $[1]$ and the construction of cones.*

Let $\mathcal{T}$ be a triangulated category. Given a triangulated subcategory $\mathcal{C}$ of $\mathcal{T}$, we can define the quotient by formally inverting the morphisms in $\mathcal{T}$ with cone in $\mathcal{C}$. The objects in $\mathcal{C}$ become zero in the quotient $\mathcal{T}/\mathcal{C}$.

Definition 2.1.4. *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{C} \subset \mathcal{T}$ be a full triangulated subcategory. The Verdier quotient $\mathcal{T}/\mathcal{C}$ is the category defined as the localization of $\mathcal{T}$ under the collection of morphisms $\mathcal{W} = \{s : X \to Y \mid X, Y \in \mathcal{T},\ cone(s) \in \mathcal{C}\}$. It consists of the following data:*

- *$Obj(\mathcal{T}/\mathcal{C}) = Obj(\mathcal{T})$*
- *$Hom_{\mathcal{T}/\mathcal{C}}(X, Y) = \{X \overset{s}{\leftarrow} Z \overset{f}{\rightarrow} Y \mid s \in \mathcal{W}\}/\sim$*
  Where f is a morphism in $\mathcal{T}$ and two roofs $X \overset{s}{\leftarrow} Z \overset{f}{\rightarrow} Y$ and $X \overset{s'}{\leftarrow} Z' \overset{f'}{\rightarrow} Y$ are equivalent if there is a roof $X \overset{t}{\leftarrow} W \overset{g}{\rightarrow} Y$ fits into the following commutative diagram:

$$\begin{tikzcd}
 & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & W \arrow[l, "t"'] \arrow[r, "g"] \arrow[u] \arrow[d] & Y \\
 & Z' \arrow[ul, "s'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

Objects in the quotient category $\mathcal{T}/\mathcal{C}$ are represented by objects in $\mathcal{T}$, but since some certain maps are changed, non-isomorphic objects in $\mathcal{T}$ can become isomorphic in $\mathcal{T}/\mathcal{C}$.

Recall a full subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is a subcategory such that for any two objects X and Y is $\mathcal{A}$ we have $Hom_{\mathcal{A}}(X,Y) = Hom_{\mathcal{B}}(X,Y)$

Definition 2.1.5. *A full subcategory $\mathcal{I}$ of a triangulated $\mathcal{T}$ is said to be*

- *thick if $\mathcal{I}$ is closed under direct summands.*
- *dense if any object in $\mathcal{T}$ is a direct summand of some object in $\mathcal{I}$*

The following statements can be found in[16] Stacks Projects 05RA.

Lemma 2.1.6. *Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{C} \subset \mathcal{T}$ be a full triangulated subcategory. Then:*

- *the Verdier quotient $\mathcal{T}/\mathcal{C}$ is a triangulated category. Its distinguished triangles are the images of the distinguished triangles in $\mathcal{T}$ under the quotient functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{C}$, $X \mapsto X$.*
- *the quotient functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{C}$ is an exact functor of triangulated categories with the following universal property:*
  If $F : \mathcal{T} \to \mathcal{T}'$ is an exact functor of triangulated categories such that objects in $\mathcal{C}$ are sent to zero objects i.e. $\mathcal{C} \subset Ker(F)$, then there exists a unique factorization:

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{T}' \\ & \mathcal{T}/\mathcal{C} \arrow[ur, "F'"'] & \end{tikzcd}$$

where F' is an exact functor.

- *The kernel of the quotient functor $Obj(Ker(Q)) = \{X \in \mathcal{T} \mid X \oplus X' \cong Y,\ Y \in \mathcal{C}\}$. In other words, it is the smallest full thick subcategory of $\mathcal{T}$ containing $\mathcal{C}$.*

Remark 2.1.7. *Localization can be defined for any category and a collection of morphisms in it. Suppose $\mathcal{C}$ is a category and let $\mathcal{W}$ be a collection of morphisms in $\mathcal{C}$, then there exists a localization $Q : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ and it is unique up to an natural equivalence of categories, the natural equivalence is unique up to a unique isomorphism. The localization $Q : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is universal among functors from $\mathcal{C}$ that send morphisms in $\mathcal{W}$ to isomorphisms.*

2.2 Derived categories of schemes

The majority of examples of triangulated categories arise from the derived categories of abelian categories and stable homotopy categories. We will mainly focus on the derived categories of schemes.

2.2.1 Derived categories of abelian categories

Definition 2.2.1. *Let $\mathcal{A}$ be an abelian category. Define*

1. *$Kom(\mathcal{A})$ the category of cochain complexes with terms in $\mathcal{A}$. Its objects are of the form:*

$$X^\bullet = \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \to \cdots$$

where $X^i \in \mathcal{A}$ with $d^{i+1} \circ d^i = 0$, a map $f : X^\bullet \to Y^\bullet$ consists of a collection of maps $f^i : X^i \to Y^i$ such that all squares commute.

2. *The ith cohomology group of $X^\bullet$ is $H^i(X^\bullet) = ker(d^i)/im(d^{i-1})$*

3. *A morphism $f : X^\bullet \to Y^\bullet$ is called a quasi-isomorphism if the induced maps $H^i(f) : H^i(X^\bullet) \to H^i(Y^\bullet)$ are isomorphisms.*

4. *The homotopic category $K(\mathcal{A})$ is $Kom(\mathcal{A})$ modulo homotopic equivalence. In other words, $Obj(K(\mathcal{A})) = Obj(Kom(\mathcal{A}))$ $Mor(K(\mathcal{A})) = Mor(Kom(\mathcal{A}))/\sim$ where two chain maps $f \sim g$ if they are homotopic equivalent.*

5. *The derived category $D(\mathcal{A})$ is the localization of $K(\mathcal{A})$ with respect to quasi-isomorphisms, see Remark 2.1.7.*

It is usually useful to restrict things to the bounded versions of the above categories. Let $\mathcal{A}$ be an abelian category. Define $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ to be the full subcategory of $D^b(\mathcal{A})$ whose objects are:

- $obj(D^+(\mathcal{A}) = \{C^\bullet \in D(\mathcal{A}) \mid H^n(C^\bullet) = 0 \ for \ all \ n \ll 0\}$
- $obj(D^-(\mathcal{A}) = \{C^\bullet \in D(\mathcal{A}) \mid H^n(C^\bullet) = 0 \ for \ all \ n \gg 0\}$
- $obj(D^b(\mathcal{A}) = \{C^\bullet \in D(\mathcal{A}) \mid H^n(C^\bullet) = 0 \ for \ all \ |n| \gg 0\}$

Similarly, one can define those subcategories for $K(\mathcal{A})$.

The categories $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ are triangulated categories, where $* \in \{\phi, b, +, -\}$.

Definition 2.2.2. *Let $X^\bullet \in K(\mathcal{A})$ be a chain complex, set*

- *$X^\bullet[k]$ be the complex with $X^n[k] = X^{n+k}$ and $d^n_{X^\bullet[k]} = (-1)^k d^n_{X^\bullet}$*
- *If $f : X^\bullet \to Y^\bullet$ is a morphism in $K(\mathcal{A})$, $f[k] : X^\bullet[k] \to Y^\bullet[k]$ is defined as $f[k]^n = f^{k+n}$*
- *The mapping cone $C(f)$ of a map $f : X^\bullet \to Y^\bullet$ in $K(\mathcal{A})$ is the chain complex $C(f) = X^\bullet[1] \oplus Y^\bullet$ with differential $d_{C(f)} = \begin{bmatrix} d_{X^\bullet[1]} & 0 \\ f[1] & d_{Y^\bullet} \end{bmatrix}$*
- *A triangle in $K(\mathcal{A})$ is said to be distinguished if it is isomorphic to a triangle of the*

form

$$X^\bullet \xrightarrow{f} Y^\bullet \to C(f) \to X^\bullet[1]$$

The following results can be found in S. Gelfand and Y. Manin[1]:

Theorem 2.2.3. *Let $\mathcal{A}$ be an abelian category. The Category $K(\mathcal{A})$ along with the shift functor $[1] : K(\mathcal{A}) \to K(\mathcal{A})$ and distinguished defined in 2.2.2 is a triangulated category.*

Theorem 2.2.4. *The derived category $D(\mathcal{A})$ is a triangulated category if we set its distinguished triangles as the triangles isomorphic to the image of a distinguished triangle under the quotient functor $K(\mathcal{A}) \to D(\mathcal{A})$.*

2.2.2 Derived categories of schemes

Let X be a scheme. There are several natural abelian categories defined on X. The category of $\mathcal{O}_X$-modules, the category of quasi-coherent $\mathcal{O}_X$-modules and the category of coherent $\mathcal{O}_X$-modules. We may then construct a derived category for each of them.

Definition 2.2.5. *Let X be a scheme, $Z \subset X$ a closed subset of X. Define*

1. *$D(\mathcal{O}_X)$ the derived category of $\mathcal{O}_X$-modules,*
2. *$D_{qcoh}(X)$, or $D(X)$ for short, the full subcategory of $D(\mathcal{O}_X)$ with objects whose cohomologies are quasi-coherent,*
3. *$D(qcoh(X))$ the derived category of quasi-coherent sheaves on X,*
4. *the support of $\mathcal{F} \in D(\mathcal{O}_X)$ to be $supp(\mathcal{F}) = \bigcup_{i\in\mathbb{Z}} supp(H^i(\mathcal{F}))$,*
5. *$D_Z(\mathcal{O}_X)$ (respectively $D_Z(X)$) as the full subcategory of $D(\mathcal{O}_X)$ (respectively $D_{qcoh}(X)$) with objects supported on Z.*

If X is a Noetherian scheme, denote

1. *$D_{coh}(X)$ to be the full subcategory of $D(\mathcal{O}_X)$ with objects has coherent cohomologies,*

2. *$D(coh(X))$ to be the full subcategory of $D(\mathcal{O}_X)$ consists of complexes of coherent sheaves,*

3. *a complex $E^\bullet$ of $\mathcal{O}_X$-modules is perfect if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank,*

4. $\mathrm{Perf}(X)$ *the derived category of perfect complexes on X.*

Remark 2.2.6. *The natural inclusion functor $D(qcoh(X)) \to D_{qcoh}(X)$ is an equivalence of triangulated categories when X is quasi-compact and quasi-separated by Neeman and Bökstedt's[17] Corollary 5.5.*
If X is Noetherian, then the natural functor $D(coh(X)) \to D_{coh}(X)$ is an equivalence of categories [SGA 6, II Corollaire 2.2.2.1].

Example 2.2.7. *Suppose A is a commutative Noetherian ring and $X = specA$ is the corresponding affine scheme, then there are equivalences $D(X) \cong D(Mod(A))$ and $D^b_{coh}(X) \cong D^b(mod(A))$, where $Mod(A)$ and $mod(A)$ are the category of A-modules and finitely generated A-modules. In this case, the perfect complexes on X are precisely bounded complexes of finite rank projective A-modules. An A-module M considered as a complex concentrated in degree 0 is perfect if and only of it is finitely generated and has finite projective dimension.*

Example 2.2.8. *If A is a regular ring, then it has finite global dimension and thus any finitely generated A-module M has a finite projective resolution $P^\bullet \xrightarrow{f} M$, where f is a quasi-isomorphism. So, in the derived category $D(A)$, we have $P^\bullet \cong M$. Now, it is standard to construct finite projective resolutions for bounded complexes of finitely generated A-modules (see for example Stacks Projects[16] Stacks Projects 013K). If X is a regular scheme, the above argument shows that any bounded complexes of coherent sheaves on X are locally quasi-isomorphic to a perfect complex and we deduce that* $\mathrm{Perf}(X) \cong D^b_{coh}(X)$.

Remark 2.2.9. *It is well known that all the categories defined in 2.2.5 are triangulated categories. Moreover, if one has an exact sequence of complexes $C^\bullet \xrightarrow{f} D^\bullet \xrightarrow{g} E^\bullet$, there* *exists a morphism $h : E^\bullet \to C^\bullet[1]$ in $D(\mathcal{O}_X)$ such that $C^\bullet \xrightarrow{f} D^\bullet \xrightarrow{g} E^\bullet \xrightarrow{h} C^\bullet[1]$ is a distinguished triangle.*

2.3 Derived functors

We briefly recall the theory of derived functors on schemes. We will only review some basic facts about derived direct image, derived inverse, and derived tensor product. For other functors on derived categories and more details, see Spaltenstein[18], Kashiwara[15], and[16] Stacks Projects 01DW

Derived functors are defined by replacing each complex by a K-injective or K-flat resolution. A complex is said to be acyclic if it has zero cohomology on each degree.

Definition 2.3.1. *Let $\mathcal{A}$ be an abelian category of $\mathcal{O}_X$-modules.*

1. *A complex $I^\bullet$ with terms in $\mathcal{A}$ is called K-injective if for every acyclic complex $M^\bullet$ we have $Hom_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$*
2. *A complex $K^\bullet$ is called K-flat if for every acyclic complex $M^\bullet$ the complex $Tot(M^\bullet \otimes I^\bullet)$ is acyclic.*

In the derived category of $\mathcal{O}_X$-modules or the derived categories of quasi-coherent sheaves, every complex admits a K-injective resolution and a K-flat resolution.

Theorem 2.3.2. *(Existence of K-injective complexes for Grothendieck categories[16] Stacks Projects 079P) Let $\mathcal{A}$ be a Grothendieck category. For every complex $M^\bullet$ there exists a quasi-isomorphism $M^\bullet \to I^\bullet$ such that $M^n \to I^n$ is injective and I^n is an injective object of $\mathcal{A}$ for all n and $I^\bullet$ is a K-injective complex. $M^\bullet \to I^\bullet$ is called a K-injective resolution of $M^\bullet$. Moreover, the construction is functorial in $M^\bullet$.*

Since the category $\mathcal{O}_X$-modules is a Grothendieck category, we can apply the above theorem to find K-injective resolutions for each complex.

2.3.1 Derived direct image

Let $f : X \to Y$ be a morphism of schemes.

- The direct image functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ admits a right derived functor $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, with $Rf_*(E^\bullet) = f_*(I^\bullet)$ for any complex $E^\bullet$, where $I^\bullet$ is a K-injective resolution of $E^\bullet$[16] Stacks Projects 07A5

- If f is quasi-comact and quasi-separated the functor Rf_* sends $D_{qcoh}(X)$ to $D_{qcoh}(Y)$ and is a bounded functor.[16] Stacks Projects 08D5

- If Y is quasi-compact then for any $E^\bullet \in D_{qcoh}(X)$, $Rf_*(E^\bullet)$ is bounded above.[16] Stacks Projects 08D5

- If X, Y are Noetherian schemes and f is proper, then Rf_* sends $D^b_{coh}(X)$ to $D^b_{coh}(Y)$ Hartshorne[19] Theorem 8.8. Conversely, if Rf_* sends $\mathrm{Perf}(X)$ to $D^b_{coh}(Y)$ then f is proper by Neeman[5]Lemma 0.20

- If f is a proper morphisms of finite Tor-dimension between Noetherian schemes, in other words f is perfect, then Rf_* sends $\mathrm{perf}(X)$ to $\mathrm{perf}(Y)$. Flat proper morphisms are perfect.[5][16] Stacks Projects 0B6F

Summarizing the above statements, we get the following diagram:

$$
\begin{tikzcd}
D(\mathcal{O}_X) \arrow[r, "Rf_*"] & D(\mathcal{O}_Y) \\
D_{qcoh}(X) \arrow[u, hook] \arrow[r, "f\ qcqs"] & D_{qcoh}(Y) \arrow[u, hook] \\
D^b_{coh}(X) \arrow[u, hook] \arrow[r, "f\ proper", "X\ Y\ noetherian"'] & D^b_{coh}(Y) \arrow[u, hook] \\
\mathrm{perf}(X) \arrow[u, hook] \arrow[r, "f\ perfect"] & \mathrm{perf}(Y) \arrow[u, hook]
\end{tikzcd}
$$

2.3.2 Derived inverse image

Let X be a scheme and $C^\bullet$ be any complex of sheaves of $\mathcal{O}_X$-modules, then there is a K-flat complex $K^\bullet$ and a quasi-isomorphism $K^\bullet \to C^\bullet$. In other words $C^\bullet$ has a K-flat resolution[16] Stacks Projects 06YF.

Let $f : X \to Y$ be a morphism of schemes.

- The inverse image functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ admits a left derived functor $Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$, with $Lf^*(E^\bullet) = f^*(K^\bullet)$ for any complex $E^\bullet$, where $K^\bullet$ is a K-flat resolution of $E^\bullet$. Spaltenstein[18]Proposition 5.6
- Lf^* sends $D_{qcoh}(Y)$ to $D_{qcoh}(X)$ and ends $D^-_{qcoh}(Y)$ to $D^-_{qcoh}(X)$, Lipman[20] Proposition (3.9.1)
- Lf^* sends $D^-_{coh}(Y)$ to $D^-_{coh}(X)$, if f has finite Tor-dimension, then Lf^* sends $D^b_{coh}(Y)$ to $D^b_{coh}(X)$
- Lf^* sends $\text{Perf}(Y)$ to $\text{Perf}(X)$ for any f[16] Stacks Projects 09UA.

We have the following diagram:

$$\begin{tikzcd}
D(\mathcal{O}_Y) \arrow[r, "Lf^*"] & D(\mathcal{O}_X) \\
D_{qcoh}(Y) \arrow[u, hook] \arrow[r] & D_{qcoh}(X) \arrow[u, hook] \\
D^b_{coh}(Y) \arrow[u, hook] \arrow[r, "finite", "Tor-dimension"'] & D^b_{coh}(X) \arrow[u, hook] \\
\text{Perf}(Y) \arrow[u, hook] \arrow[r, "any\ f"] & \text{Perf}(X) \arrow[u, hook]
\end{tikzcd}$$

2.3.3 Derived tensor product

Let $F^\bullet$ be an object in $D(\mathcal{O}_X)$, and $K^\bullet \to F^\bullet$ be a K-flat resolution of $F^\bullet$, Then the functor

$$K(\mathcal{O}_X) \to K(\mathcal{O}_X), \quad G^\bullet \mapsto Tot(G^\bullet \otimes_{\mathcal{O}_X} K^\bullet)$$

has a left derived functor

$$- \otimes^L_{\mathcal{O}_X} F^\bullet : D(\mathcal{O}_X) \to D(\mathcal{O}_X)$$

Moreover, if both $F^\bullet$ and $G^\bullet$ are in $D_{qcoh}(X)$ then so is $F^\bullet \otimes^L_{\mathcal{O}_X} G^\bullet$[20](2.5.8). If K, L are perfect complexes, then so is $K \otimes^L_{\mathcal{O}_X} L$[16] Stacks Projects 09J5

We have the following diagram:

$$\begin{tikzcd}
D(\mathcal{O}_X) \times D(\mathcal{O}_X) \arrow[r, "- \otimes^L -"] & D(\mathcal{O}_X) \\
D_{qcoh}(X) \times D_{qcoh}(X) \arrow[r] \arrow[u, hook] & D_{qcoh}(X) \arrow[u, hook] \\
D^-_{coh}(X) \times D^-_{coh}(X) \arrow[r] \arrow[u, hook] & D^-_{coh}(X) \arrow[u, hook] \\
\mathrm{Perf}(X) \times \mathrm{Perf}(X) \arrow[r] \arrow[u, hook] & \mathrm{Perf}(X) \arrow[u, hook]
\end{tikzcd}$$

Finally, let us recall a nice formula that relates the three derived functors above:

Theorem 2.3.3. *(Lipman[20] Proposition 3.9.4) Let $f : X \to Y$ be a quasi-compact, quasi-separated morphism of schemes, and let $F \in D_{qcoh}(X)$, $G \in D_{qcoh}(Y)$ be two complexes. Then the natural map $Rf_*F \otimes_L G \to Rf_*(F \otimes_L Lf^*G)$ is an isomorphism*

2.4 Generation of triangulated categories

2.4.1 Generators and dimensions of triangulated categories

Let $\mathcal{T}$ be a triangulated category. For a collection of objects $\mathcal{S}$ in $\mathcal{T}$, denote by $add(\mathcal{S})$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{S}$ and closed under finite direct sums, direct summands and shifts. And denote by $Add(\mathcal{S})$ the smallest full subcategory closed under set indexed coproducts, direct summands and shifts.

For any two subcategories $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{T}$, define

- $\mathcal{I} * \mathcal{J} = \{cone(\phi) : \phi \in Hom_{\mathcal{T}}(\mathcal{J}, \mathcal{I})\})$
- $\mathcal{I} \diamond \mathcal{J} = add(\mathcal{I} * \mathcal{J})$

Definition 2.4.1. *Let $\mathcal{S}$ be a collection of objects in $\mathcal{T}$.*

1. *$\langle \mathcal{S} \rangle_0$ is the full subcategory of objects isomorphic to the zero object.*
2. *$\langle \mathcal{S} \rangle_1 = add(\mathcal{S})$*
3. *$\langle \mathcal{S} \rangle_n = \langle \mathcal{S} \rangle_1 \diamond \langle \mathcal{S} \rangle_{n-1}$*
4. *$\langle \mathcal{S} \rangle = \bigcup\limits_{i \in \mathbb{N}} \langle \mathcal{S} \rangle_i$*
5. *Define $\overline{\langle \mathcal{S} \rangle}_i$ the same way as above with add replaced by Add. In other words, all small coproducts are allowed.*

Example 2.4.2. *Let $\mathcal{A}$ be an abelian category, $D^b(\mathcal{A})$ its derived category. Then $D^b(\mathcal{A}) \subset \langle \mathcal{A} \rangle_2$. This is because for any complex $C^\bullet \in D^b(\mathcal{A})$, the short exact sequence $0 \to Z(C^\bullet) \to C^\bullet \to B(C^\bullet) \to 0$ induces a distinguished triangle $Z(C^\bullet) \to C^\bullet \to B(C^\bullet) \to Z(C^\bullet[1])$ and since $Z(C^\bullet)$ and $B(C^\bullet)$ have zero differentials, they are objects in $add(\mathcal{A})$.*

Definition 2.4.3. *Let $\mathcal{T}$ be a triangulated category and G an object of $\mathcal{T}$.*

1. *G is a generator if for any object $X \in \mathcal{T}$ and any $i \in \mathbb{Z}$, $Hom_{\mathcal{T}}(G, X[i]) = 0$ implies*

$X = 0$,

2. *G is a classical generator if $\langle G \rangle = \mathcal{T}$,*

3. *G is a strong generator if $\langle G \rangle_n = \mathcal{T}$ for some natural number n,*

4. *G is a classical $\oplus$-generator if $\overline{\langle \mathcal{G} \rangle} = \mathcal{T}$,*

5. *G is a strong $\oplus$-generator if $\overline{\langle \mathcal{G} \rangle}_n = \mathcal{T}$ for some n.*

Definition 2.4.4. *Let $\mathcal{T}$ be a triangulated category, X be an object in $\mathcal{T}$, $\mathcal{S}$ be a collection of objects in $\mathcal{T}$. Define the $\mathcal{S}$-level of X in $\mathcal{T}$ as*

$$Level_{\mathcal{T}}^{\mathcal{S}}(X) := \inf\{n \in \mathbb{N} | X \in \langle \mathcal{S} \rangle_n\}.$$

When $\mathcal{S}$ consists of a single object G, we write $Level_{\mathcal{T}}^{G}(X)$ for $Level_{\mathcal{T}}^{\{G\}}(X)$.

Clearly, if an object $G \in \mathcal{T}$ such that for any object $X \in \mathcal{T}$ the G-level of X is finite, i.e. $Level_{\mathcal{T}}^{G}(X) < \infty$, then G is a classical generator of $\mathcal{T}$. If moreover $Level_{\mathcal{T}}^{G}(X) < n$, then G is a strong generator of $\mathcal{T}$.

Remark 2.4.5. *Let $\{X_i\}$ be a finite collection of objects in a triangulated category $\mathcal{T}$, and let $\mathcal{S}$ be any collection of objects in $\mathcal{T}$. By taking direct sums of distinguished triangles when building the objects $\{X_i\}$, one easily sees that $X_i \in \mathcal{T}$, $Level_{\mathcal{T}}^{\mathcal{S}}(\bigoplus X_i) = max_i\{Level_{\mathcal{T}}^{\mathcal{S}}(X_i)\}$*

Definition 2.4.6. *Let $\mathcal{T}$ be a triangulated category. The Rouquier dimension of $\mathcal{T}$, denoted by $Rdim(\mathcal{T})$, is the minimal integer n such that there exists an object $G \in \mathcal{T}$ with $\mathcal{T} \subseteq \langle G \rangle_{n+1}$. The Rouquier is defined to be ∞ if there is no such G.*

Remark 2.4.7. *If $\mathcal{T}$ admits a strong generator G, then all classical generators are strong generators as the level of G under any classical generator must be finite.*

2.4.2 Some examples

For affine regular schemes, their Rouquier dimension coinsides with Krull dimension.

Example 2.4.8. *Let A be a ring with finite global dimension i.e. A is regular. Then $D(mod(A)) = \langle A \rangle_{glodim(A)+1}$, see Christensen*[21] *Corollary 8.4.*

In general, Neeman showed that all Noetherian regular schemes have finite Rouquier dimension.

Theorem 2.4.9. *(Neeman*[5] *Theorem 0.15) Let X be a Noetherian regular scheme. Then $D^b_{coh}(X)$ has a strong generator.*

Aoki was able to enlarge the collection of schemes that have finite Rouquier dimension to all quasi-excellent schemes.

Theorem 2.4.10. *(Aoki*[8] *Main theorem) If X is a quasi-compact separated quasi-excellent scheme of finite dimension, $D^b_{coh}(X)$ has a strong generator.*

Example 2.4.11. *The following example can be found in*[6]*. Let E be an elliptic curve with identity e. Then the bounded derived category of coherent sheaves $D^b_{coh}(E)$ of E is strongly generated. Specifically, we have*

1. $D^b(Coh(E)) = \langle \mathcal{O}(-3e) \oplus \mathcal{O} \oplus \mathcal{O}(3e) \rangle_2$
2. $D^b_{coh}(E) = \langle \mathcal{O} \oplus \mathcal{O}(3e) \rangle_3$
3. $D^b_{coh}(E) = \langle \mathcal{O} \oplus \mathcal{O}_{2e} \rangle_4$
4. $D^b_{coh}(E) = \langle \mathcal{O} \oplus \mathcal{O}_e \rangle_5$

Example 2.4.12. *If X is a zero-dimensional Noetherianscheme, then $D^b(coh(X))$ admits a strong generator. We may assume $X = SpecA$ is affine by 5.2.10. Then A is an Artinian ring and we have $A \cong \bigoplus_{p \in SpecA} A_p$ (recall that $SpecA$ is a finite set). For any finitely generated*

A-module M, there is an isomorphism $M \cong \bigoplus M_p^{n_p}$ see Theorem 2.13[22]*. By Example 2.4.5 we may then assume $A = (A, p)$ is a local Artinian ring. Since $p^n = 0$ for some n, in the filtration $0 = p^n M \subset p^{n-1} M \subset \cdots \subset M$ each sunquotient is a finite dimensional A/p-vector space and thus in $\langle A/p \rangle_1$, this implies that $M \in \langle A/p \rangle_n$. By example 2.4.2, $D^b(mod(A)) \subset \langle A/p \rangle_{2n}$.*

Example 2.4.13. *Let C be a one-dimensional Noetherian scheme with finitely many singular points, then $D^b(coh((X))$ admits a strong generator. Again, we may assume $X = SpecA$ is an affine scheme. By Letz*[23] *Theorem 3.1, $G \in D^b(mod(A))$ is a strong generator if and only if G_p is a strong generator in $D^b(mod(A_p))$ for every p. If $p \in SpecA$ is a regular point, then A_p is a strong generator of $D^b(mod(A_p))$ by 2.4.8. If $p \in SpecA$ is a singular point, consider the completion $\hat{A}$ of A_p at p. $\hat{A}$ is a one-dimensional excellent local ring. By Takahashi*[24] *Corolary 4.3, $D^b(mod(\hat{A})) = \langle \hat{A} \oplus \hat{k} \rangle$, and by Aoki*[8] *Main theorem, $\hat{A} \oplus \hat{k}$ is a strong generator. Then by Letz*[23] *corollary 2.16, $A_p \oplus A_p/p$ is a strong generator of $D^b(mod(A_p))$. Since the canonical functor $D^b(mod(A)) \to D^b(mod(A_p))$ is essentially surjective accroding to Lemma 3.9*[23]*, there is an object $G(p) \in D^b(mod(A))$ such that $G(p)_p \cong A_p \oplus A_p/p$ for singular p, let $G = \bigoplus G(p)$ then $A \oplus G$ strongly generates $D^b(mod(A))$.*

Chapter 3

Ghost lemma and Contravariantly finite subcategories

The level of an object $Level^{\mathcal{S}}_{\mathcal{T}}X$ is generally difficult to compute directly. However, we can relate it to the (co)ghost index, which provides a lower bound for the level. When $\mathcal{S}$ is contravariantly finite, the level coincides with the ghost index. We will recall the ghost lemma and provide some constructions of contravariantly finite subcategories in this chapter.

3.1 Ghost lemma and ghost index

Recall for a collection of objects $\mathcal{S} \subset \mathcal{T}$, $\mathcal{S}$-ghost maps are maps in $\mathcal{T}$ that are invisible by $\mathcal{S}$. The following is the precise definition.

Definition 3.1.1. *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S}$ be a collection of objects in $\mathcal{T}$.*

- *A map $f : A \to B$ is called a $\mathcal{S}$-ghost if the induced map*

$$Hom(S, f[n]) : Hom(S, A[n]) \xrightarrow{f[n]\circ -} Hom(S, B[n])$$

vanishes for any $S \in \mathcal{S}$ and any $n \in \mathbb{Z}$.

- *An n-fold $\mathcal{S}$-ghost map is a composition of n $\mathcal{S}$-ghost maps.*
- *The ($\mathcal{S}$-)ghost index of an object $A \in \mathcal{T}$ denoted by $gin^{\mathcal{S}}_{\mathcal{T}}(A)$ is the smallest natural number n such that all n-fold ghost maps of the form $A \to B$ vanishes.*
- *Dually, one can define coghost maps and coghost index via the functor $Hom(-, \mathcal{S})$.*

Recall the ghost lemma[25]

Lemma 3.1.2. *(Ghost Lemma) Let $\mathcal{T}$ be a triangulated category and let A, G be objects in $\mathcal{T}$. If $A \in \langle G \rangle_n$, then $gin^{G}_{\mathcal{T}}(A) \leq n$, in other words, $gin^{G}_{\mathcal{T}}(A) \leq Level^{G}_{\mathcal{T}}(A)$*

So, the least number of cones needed to build A from G has a lower bound given by the (co)ghost index.

3.2 Contravariantly finite subcategories of triangulated categories

The converse of the ghost lemma 3.1.2 holds when $\langle G \rangle$ is a contravariantly finite subcategory. We now recall the definition:

Definition 3.2.1. *Let $\mathcal{C}$ be an additive category and $\mathcal{S}$ be a subcategory of $\mathcal{C}$.*

1. *A morphism $\phi : S \to X$ from an object in $\mathcal{S}$ to an object in $\mathcal{C}$ is said to be a right $\mathcal{S}$-approximation of X if for any morphism $\phi' : S' \to X$ with $S' \in \mathcal{S}$, there is a factorization:*

$$\begin{tikzcd} S \arrow[r, "\phi"] & X \\ S' \arrow[u, "g"] \arrow[ur, "\phi'"'] & \end{tikzcd}$$

2. *The subcategory $\mathcal{S}$ is called a contravariantly finite subcategory if every object in $\mathcal{C}$ has*

a right $\mathcal{S}$-approximation.

We provide some constructions of contravariantly finite subcategories of triangulated categories. The following lemma says they can arise from adjoint functors, so in particular, admissible subcategories are contravariantly finite.

Lemma 3.2.2. *Consider a pair of adjoint functors* $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ *with* $(L \dashv R)$. *Then the full subcategory* $L(\mathcal{C}) \subset \mathcal{D}$ *is a contravariantly finite subcategory.*

Proof. For any object $X \in \mathcal{D}$, there is the unit map $\eta_X : LRX \to X$, and if there is a morphism $f : LY \to X$ for some $Y \in \mathcal{C}$, then f corresponds to a map $g : Y \to RX$ via $Hom_{\mathcal{D}}(LY, X) \cong Hom_{\mathcal{C}}(Y, RX)$, and we have

$$\begin{tikzcd} LRX \arrow[r, "\eta_X"] & X \\ LY \arrow[u, "Lg"] \arrow[ur, "f"'] & \end{tikzcd}$$

□

Given two contravariantly finite subcategories, one can take their $*$ product to build a new one.

Lemma 3.2.3. *Let $\mathcal{T}$ be a triangulated category. If there are two contravariantly finite subcategories $\mathcal{I}$ and $\mathcal{J}$, then $\mathcal{I} * \mathcal{J}$ is also contravariantly finite.*

Proof. Let X be an arbitrary object in $\mathcal{T}$. Let $\phi : I_X \to X$ be an $\mathcal{I}$-approximation of X and make it into a distinguished triangle: $I_X \xrightarrow{\phi} X \xrightarrow{f} Y \to I_X[1]$. Then let $\psi : J_Y \to Y$ be an $\mathcal{J}$-approximation of Y and make it into a distinguished triangle: $J_Y \xrightarrow{\psi} Y \xrightarrow{g} Z \to J_Y[1]$. By octahedral axiom, we get a distinguished triangle: $I_X[1] \to W \to J_Y[1]$ as in the following diagram:

$$
\begin{tikzcd}
 & & I_X[1] \arrow[dl, dotted] \arrow[dll, bend right, "u"'] & & \\
 & X \arrow[rr, "f"] \arrow[dr, "g\circ f"'] & & Y \arrow[ul] \arrow[dl, "g"] & \\
W \arrow[ur, dotted] \arrow[rrrr, bend right, "v"'] & & Z \arrow[ll] \arrow[rr] & & J_Y[1] \arrow[ul, dotted] \arrow[uull, dotted, bend right, "w"']
\end{tikzcd}
$$

So $W \in \mathcal{I} \diamond \mathcal{J}$. We claim that the map $W[-1] \xrightarrow{l} X$ is an $\mathcal{I} \diamond \mathcal{J}$-approximation of X. To see this, let $E \xrightarrow{h} X$ be any map with $E \in \mathcal{I} \diamond \mathcal{J}$. Then E fits into a distinguished triangle $E_I \to E \to E_J \rightsquigarrow$ with $E_I \in \mathcal{I}$, $E_J \in \mathcal{J}$. Consider the following diagram:

$$
\begin{tikzcd}
E_I \arrow[rr] \arrow[dr, "\alpha"] \arrow[dd] & & E \arrow[rr] \arrow[dr, "h"] \arrow[dd, "\delta"' near end] & & E_J \arrow[dr, "\beta"] \arrow[dd, "\gamma"' near end] & \\
 & I_X \arrow[rr, "\phi"] & & X \arrow[rr, "f"] & & Y \\
I_X \arrow[rr] \arrow[ur, "id"'] & & W[-1] \arrow[rr] \arrow[ur, "l"'] & & J_Y \arrow[ur, "\psi"'] &
\end{tikzcd}
$$

We have the map

- α because $\phi : I_X \to X$ is an $\mathcal{I}$-approximation of X
- β because of TR4
- γ because $\psi : J_Y \to Y$ is an $\mathcal{J}$-approximation of Y
- δ because of TR4

We need to check the diagram commutes, specifically, the composition $E \xrightarrow{\delta} W[-1] \xrightarrow{l} X$ is $h : E \to X$. This is not guaranteed from the construction, but we can modify the map δ as follows: We have $f \circ l \circ \delta = f \circ h$ because the above diagram commutes except for the middle triangle, in other words, the map $l \circ \delta - h \in Hom(E, X)$ is sent to zero under f_* in the commutative diagram below

$$
\begin{tikzcd}
Hom(E, I_X) \arrow[r, "\phi_*"] & Hom(E, X) \arrow[r, "f_*"] & Hom(E, Y) \\
Hom(E, I_X) \arrow[r] \arrow[u, "="] & Hom(E, W[-1]) \arrow[r] \arrow[u, "l_*"] & Hom(E, J_Y) \arrow[u, "\psi_*"]
\end{tikzcd}
$$

And thus $l \circ \delta - h \in Hom(E, X)$ is the image of a map $a \in Hom(E, I_X)$, suppose $\tilde{a}$ is the image of a in $Hom(E, W[-1])$, then $l \circ (\delta - \tilde{a}) = h$ as desired. And this shows that $W[-1] \to X$ is an $\mathcal{I} \diamond \mathcal{J}$-approximation of X. $\square$

Corollary 3.2.4. *Let $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$ be as in Lemma 3.2.3. Assume further that $\mathcal{I} = \mathcal{I}[1]$ and $\mathcal{J} = \mathcal{J}[1]$, in other words they are closed under shift. Then $\mathcal{I} \diamond \mathcal{J}$ is a contravariantly finite subcategory of $\mathcal{T}$*

Proof. It is clear that if $\mathcal{I}$ and $\mathcal{J}$ are closed under shift, then so is $\mathcal{I} * \mathcal{J}$. In this case, $\mathcal{I} \diamond \mathcal{J}$ consists of objects that are direct summands of objects in $\mathcal{I} * \mathcal{J}$. For any object $X \in \mathcal{T}$, let $W \xrightarrow{\phi} X$ be its $\mathcal{I} * \mathcal{J}$-approximation. Let $Y \xrightarrow{f} X$ be a morphisms with $Y \in \mathcal{I} \diamond \mathcal{J}$. Then there is an object Y' such that $Y \oplus Y'$ is an object in $\mathcal{I} * \mathcal{J}$. Since the map $Y \oplus Y' \xrightarrow{(f,0)} X$ factor through $W \xrightarrow{\phi} X$, $Y \xrightarrow{f} X$ also fator through $W \xrightarrow{\phi} X$. $\square$

Chapter 4

Derived subcategories with support

4.1 Exact sequences of triangulated categories

Derived subcategories with support are usually not strongly generated, but since they fit into exact sequence of triangulated categories, we can use them to construct generators and classical generators.

The following definition is from[4], see also[17].

Definition 4.1.1. *Let $\mathcal{T}$ be a triangulated category, and $i_* : \mathcal{I} \to \mathcal{T}$ be a thick subcategory. Denote by $j^* : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ the quotient. The sequence is called an exact sequence of triangulated categories:*

$$0 \to \mathcal{I} \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}/\mathcal{I} \to 0$$

An object $C \in \mathcal{T}$ is called $\mathcal{I}$-local if it is in the right orthogonal of $\mathcal{I}$ i.e. $Hom_{\mathcal{T}}(\mathcal{I}, C) = 0$. $\mathcal{I}$ is called a Bousfield subcategory if the functor j^* admits a right adjoint j_*.

Example 4.1.2. *If X is a quasi-compact, quasi-separated scheme and $Z \subset X$ is a closed subscheme, denote by $U = X - Z$. The $i_* : D_Z(X) \to D(X)$ is a Bousfield subcategoy, and* *we have an exact sequence of triangulated categories*

$$0 \to D_Z(X) \xrightarrow{i_*} D(X) \xrightarrow{j^*} D(U) \to 0$$

where i_* is the inclusion functor and j^* is the pullback functor.

The following fact is well known, for completeness we provide a proof.

Lemma 4.1.3. *(Neeman[17] lemma 2.9) For any two objects $C, D \in \mathcal{T}$ with D is $\mathcal{I}$-local, we have $Hom_{\mathcal{T}}(C, D) \cong Hom_{\mathcal{T}/\mathcal{I}}(j^*C, j^*D)$.*

Proof. A morphism in $Hom_{\mathcal{T}/\mathcal{I}}(j^*C, j^*D)$ is of the form $C \xleftarrow{s} Z \xrightarrow{f} D$ with $X = cone(s) \in \mathcal{I}$ that is, in the triangle $Z \to C \to X \to Z[1]$, $X \in \mathcal{I}$. Then from the exact sequence

$$0 = Hom(X, D) \to Hom(C, D) \to Hom(Z, D) \to Hom(X[-1], D) = 0$$

we deduce that $Hom(C, D) \cong Hom(Z, D)$ and thus, there is a unique factorization:

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ C \arrow[rr, "g"] & & D \end{tikzcd}$$

which shows that the natural map $Hom_{\mathcal{T}}(C, D) \to Hom_{\mathcal{T}/\mathcal{I}}(j^*C, j^*D)$ induced by the quotient functor is surjective.

On the other hand, if a morphism $g : C \to D$ is sent to zero in $\mathcal{T}/\mathcal{I}$, i.e. $C \xleftarrow{id_C} C \xrightarrow{g} D$ is equivalent to a roof $C \xleftarrow{s} Z \xrightarrow{0} D$, thus $g \circ s = 0$. Let $I = cone(s) \in \mathcal{I}$, apply $Hom(-, D)$ to the distinguished triangle $X \xrightarrow{s} C \to I$ one gets an exact sequence

$$0 = Hom(I, D) \to Hom(C, D) \xrightarrow{s_*} Hom(X, D)$$

g is mapped to $g \circ s = 0$ under an injective map so $g = 0$. $\square$

It will be useful to know some equivalent conditions for an object being $\mathcal{I}$-local.

Lemma 4.1.4. *Let $i_* : \mathcal{I} \to \mathcal{T}$ be a Bousfield subcategory and $C \in \mathcal{T}$ be an object. Then the following are equivalent:*

1. *C is $\mathcal{I} - local$*
2. *The unit map $\eta_C : C \to j_* j^* C$ is an isomorphism*
3. *There is an object $C' \in \mathcal{T}/\mathcal{I}$ such that $C \cong j_* C'$*

Proof. For any object B in $\mathcal{T}$, we have $Hom_{\mathcal{T}}(B, C) \cong Hom_{\mathcal{T}/\mathcal{I}}(j^* B, j^* C) \cong Hom_{\mathcal{T}}(B, j_* j^* C)$ by 4.1.3. Thus 1 implies 2 by Yoneda Lemma. Clearly, 2 implies 3. Suppose $C \cong j_* C'$ for an object C' in $\mathcal{T}/\mathcal{I}$, then for any $I \in \mathcal{I}$, $Hom_{\mathcal{T}}(I, C) \cong Hom_{\mathcal{T}}(I, j_* C') \cong Hom_{\mathcal{T}/\mathcal{I}}(j^* I, C') = 0$ since $j^* I \cong 0$. $\square$

Bousfield subcategories are particularly useful because there is a distinguished triangle for each object arising from the corresponding exact sequence of triangulated categories. The statement of the following lemma can be found in Rouquier[4] 5.2.2.

Lemma 4.1.5. *Suppose $i_* : \mathcal{I} \to \mathcal{T}$ is a Bousfield subcategory. Then,*

1. *The functor $i_* : \mathcal{I} \to \mathcal{T}$ has a right adjoint $i^!$*
2. *The unit map $id_{\mathcal{I}} \to i^! i_*$ is an isomorphism*
3. *The counit map $j^* j_* \to id_{\mathcal{T}/\mathcal{I}}$ is an isomorphism*
4. *For any object $C \in \mathcal{T}$, there is a distinguished triangle*

$$i_* i^! C \to C \to j_* j^* C \to i_* i^! C[1]$$

where the first map is the counit map, the second map is the unit map.

Proof. Consider the counit map $C \to j_* j^* C$ and let C' be the cocone, in other words, there is a distinguished triangle

$$C' \to C \to j_*j^*C \to C'[1] \tag{$*$}$$

Apply the functor j^* to the triangle $*$, we get the distinguished triangle

$$j^*C' \to j^*C \to j^*j_*j^*C \to j^*C'[1]$$

By Miyachi[26] Proposition 2.3, the functor j_* is fully faithful, and thus the map $j^*C \to j^*j_*j^*C$ is an isomorphism. So $j^*C' = 0$ and hence $C' \in \mathcal{I}$.

We claim that the cocone C' is defined up to a unique isomorphism: Suppose C'' is also a cocone, Apply $Hom(C'', -)$ to the distinguished triangle, since j_*j^*C is $\mathcal{I}$-local, we get an exact sequence:

$$0 = Hom(C'', j_*j^*C[-1]) \to Hom(C'', C') \to Hom(C'', C) \to Hom(C'', j_*j^*C) = 0$$

So, we have $Hom(C'', C') \cong Hom(C'', C)$ which implies there is a unique map $\alpha : C'' \to C'$ that get mapped to the cocone map $C'' \to C$, similarly there is a unique map $\beta : C' \to C''$ with $\alpha \circ \beta = id_{C'}$ and $\beta \circ \alpha = id_{C''}$

Thus the cocone construction forms a functor $i^! : \mathcal{T} \to \mathcal{I}$ and the map $C' = i_*i^!C \to C$ defines a natural transformation $i_*i^! \to id_{\mathcal{T}}$ which shows that $(i_* \dashv i^!)$. This shows 1, 4.

2, 3 follows from Miyachi[26] Proposition 2.3. □

Lemma 4.1.6. *In the Bousfield localization*

$$\mathcal{I} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{T} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{T}/\mathcal{I}$$

as in 4.1.5, if $\mathcal{I}$ and $\mathcal{T}$ are cocomplete i.e. they admits coproducts, and if j_* preserves coproducts, then so is $i^!$

Proof. Since i_* and j^* have right adjoint functors, they preserve colimits and coproducts.

Suppose $\{X_r\}$ is a collection of objects in $\mathcal{T}$ Consider the following two distinguished triangles

$$\begin{tikzcd}
\bigoplus i_*i^!X_r \arrow[r] \arrow[d] & \bigoplus X_r \arrow[r] \arrow[d, "id"] & \bigoplus j_*j^*X_r \arrow[d, "\cong"] \\
i_*i^! \bigoplus X_r \arrow[r] & \bigoplus X_r \arrow[r] & j_*j^* \bigoplus X_r
\end{tikzcd}$$

By assumption j_* preserves coproduct, the natural map $\bigoplus j_*j^*X_r \to j_*j^* \bigoplus X_r$ is an isomorphism. So, the map $\bigoplus i_*i^!X_r \to i_*i^! \bigoplus X_r$ is also an isomorphism. □

It turns out that a generator in the subcategory $\mathcal{I}$ can be used to test whether an object in $\mathcal{T}$ is $\mathcal{I}$-local.

Corollary 4.1.7. *Suppose G is a generator of $\mathcal{I}$, then an object $C \in \mathcal{T}$ is $\mathcal{I}$-local if and only if $Hom_{\mathcal{T}}(G, C[n]) = 0$ for any $n \in \mathbb{Z}$. In other words, C is right orthogonal to $\mathcal{I}$ iff it is right orthogonal to G.*

Proof. The only if part is trivial. Suppose $C \in \mathcal{T}$ is right orthogonal to G. for any $i \in \mathbb{Z}$. By adjunction $Hom_{\mathcal{I}}(G, i^!C[n]) \cong Hom_{\mathcal{T}}(G, C[n]) = 0$. Since G is a generator in $\mathcal{I}$, $i^!C \cong 0$. The distinguished triangle in 4.1.5 (4):

$$i_*i^!C \to C \to j_*j^*C$$

shows that $C \cong j_*j^*C$ and so C is $\mathcal{I}$-local by 4.1.4. □

4.2 Derived subcategories with support

4.2.1 An excision lemma

The derived subcategory of $D(X)$ with support in a closed subset Z, $D_Z(X)$, has objects consists of chain complexes with cohomology supported in Z, the sheaves on each degree are not required to be supported in Z, but it is harmless to restrict such complexes to an open subset containing Z.

Lemma 4.2.1. *Let X be a quasi-compact, quasi-separated scheme and $Z \subset X$ a closed subset. Suppose $j : U \to X$ is an open subscheme of X that contains Z. Then the functor $j_* : D_Z(U) \to D_Z(X)$ is an equivalence of categories.*

Proof. Let $V = X - Z$ be the complement of Z. Then $\{U, V\}$ forms an open covering of X. Denote by $k : V \to X$ and $l : U \cap V \to X$ the open immersions as in the following diagram:

$$\begin{tikzcd}
 & X & \\
U \arrow[ur, "j"] & & V \arrow[ul, "k"'] \\
 & U \cap V \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, "l"] &
\end{tikzcd}$$

Complete the unit maps $\mathcal{O}_X \to k_* k^* \mathcal{O}_X$ and $\mathcal{O}_U \to \alpha_* \alpha^* \mathcal{O}_U$ into a distinguished triangles, one gets

$$Q \to \mathcal{O}_X \to k_* k^* \mathcal{O}_X \rightsquigarrow \tag{1}$$

$$Q' \to \mathcal{O}_U \to \alpha_* \alpha^* \mathcal{O}_U \rightsquigarrow \tag{2}$$

Apply the functor j^* on triangle (1),

$$j^* Q \to \mathcal{O}_U \to j^* k_* k^* \mathcal{O}_X \rightsquigarrow$$

By flat base change, one gets $j^* k_* k^* \mathcal{O}_X \cong \alpha_* \beta^* k^* \mathcal{O}_X \cong \alpha_* \mathcal{O}_{U \cap V} \cong \alpha_* \alpha^* \mathcal{O}_U$.
It follows that $j^* Q \cong Q'$

Apply j_*j^* on triangle (1), one gets

$$j_*j^*Q \to j_*j^*\mathcal{O}_X \to j_*j^*k_*k^*\mathcal{O}_X \rightsquigarrow$$

which can be written as

$$j_*Q' \to j_*j^*\mathcal{O}_X \to l_*l^*\mathcal{O}_X \rightsquigarrow \tag{3}$$

By triangulated 9-lemma one gets the following diagram where all columns and all rows are distinguished triangles

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & j_*j^*\mathcal{O}_{\mathcal{X}} \arrow[r] \arrow[d] & j_*j^*\mathcal{O}_{\mathcal{X}} \arrow[d] \\
\mathcal{O}_X \arrow[r] \arrow[d] & j_*j^*\mathcal{O}_X \oplus k_*k^*\mathcal{O}_X \arrow[r] \arrow[d] & l_*l^*\mathcal{O}_X \arrow[d] \\
\mathcal{O}_X \arrow[r] & k_*k^*\mathcal{O}_X \arrow[r] & Q[1]
\end{tikzcd}$$

The first two columns and the first row are trivial triangles, the second row is the Mayer-Vietoris triangle, the bottom row is the triangle (1). Compare the third column and triangle (3) one found $j_*Q' \cong Q$

For an arbitrary object $C \in D(X)$, apply the functor $-\otimes C$ on triangle (1), one gets

$$Q \otimes C \to C \to j_*j^*\mathcal{O}_X \otimes C \rightsquigarrow$$

By projection formula, $j_*j^*\mathcal{O}_X \otimes C \cong j_*j_*C$. So one has

$$Q \otimes C \to C \to j_*j^*C \rightsquigarrow \tag{4}$$

Now compare this triangle (4) with the triangle $i_*i^!C \to C \to j_*j^*C \rightsquigarrow$ in lemma 4.1.5 we found that $i^!C \cong Q \otimes C$. Since $i^!i_* = id_{D_Z(X)}$ the functor $i^!$ is essentially surjective and thus every object in $D_Z(X)$ is of the form $Q \otimes C$ for some object $C \in D(X)$.

Consider the functor j_* induced by the open immersion j:

$$D_Z(U) \xrightarrow{j_*} D_Z(X)$$

For any object $Q \otimes C \in D_Z(X)$, one has $Q \otimes C \cong j_*Q' \otimes C \cong j_*(Q' \otimes j^*C)$, so j_* is essentially surjective. j_* is fully faithful follows from formal nonsense. □

4.2.2 Compact generators

As an application, we provide a simple proof for the existence of compact generators in the derived category of quasi-coherent sheaves for quasi-compact, quasi-separated schemes. For the original statement of the theorem see[3] Theorem 3.1.1

Definition 4.2.2. *Let $\mathcal{C}$ be an additive category. An object $c \in \mathcal{C}$ is compact if for any collection of objects $\{X_i\}_{i \in I}$, where I is a set, the canonical map*

$$\bigoplus_{i \in I} Hom(C, X_i) \to Hom(C, \bigoplus_{i \in I} X_i)$$

is an isomorphism.

Example 4.2.3. *For a quasi-compact, quasi-separated scheme X, the compact objects in $D(X)$ are precisely the perfect complexes by Theorem 3.1.1[3]. By Corollary 6.17 in[4], if X is a separated Noetherian scheme, then the compact objects in the bounded derived category of quasi-coherent sheaves consist of bounded complexes with coherent cohomology, in other words $D^b(X)^c = D_{coh}(X)$.*

Lemma 4.2.4. *Let $\mathcal{I}$ be a Bousfield subcategory of $\mathcal{T}$, if both $\mathcal{I}$ and $\mathcal{T}/\mathcal{I}$ have a generator, then so does $\mathcal{T}$. Moreover, if $\mathcal{I}$ and $\mathcal{T}$ admit arbitrary coproducts, and compact generators,* *and the right adjoint $j_* : \mathcal{T}/\mathcal{I} \to \mathcal{T}$ of the quotient functor j^* commutes with coproducts, then $\mathcal{T}$ has a compact generator.*

Proof. We only prove the second statement, the generator in the first statement follows from the same construction. In the following diagram, all four functors commute with coproducts by 4.1.6

$$\mathcal{I} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{T} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{T}/\mathcal{I}$$

Suppose C' and C'' are compact generators of $\mathcal{I}$ and $\mathcal{T}/\mathcal{I}$ respectively. We first show that the object i_*C' is a compact object in $\mathcal{T}$. For any collection of object $\{X_r\}_{r\in I}$ in $\mathcal{T}$,

$$\bigoplus_{r\in I} Hom_{\mathcal{T}}(i_*C', X_r) \cong \bigoplus_{r\in I} Hom_{\mathcal{I}}(C', i^! X_r) \tag{4.1}$$

$$\cong Hom_{\mathcal{I}}(C', \bigoplus_{r\in I} i^! X_r) \tag{4.2}$$

$$\cong Hom_{\mathcal{I}}(C', i^! \bigoplus_{r\in I} X_r) \tag{4.3}$$

$$\cong Hom_{\mathcal{I}}(i_*C', \bigoplus_{r\in I} X_r) \tag{4.4}$$

We have (3.1) and (3.4) is because of the adjunction $i_* \dashv i^!$, (3.2) is due to C' is compact in $\mathcal{I}$ and (3.3) follows from the fact that $i^!$ preserves coproducts.

Now consider the object $C'' \oplus C''[1]$ in $\mathcal{T}/\mathcal{I}$, it is compact and by Thomason-Trobaugh-Neeman localization theorem, see[27] Theorem 2.1, there is a compact object $C \in \mathcal{T}$ such that $j^*C \cong C'' \oplus C''[1]$. We claim that the object $D = C \oplus i_*C'$ is the desired compact generator of $\mathcal{T}$. Let X be an object in $\mathcal{T}$ such that for any $n \in \mathbb{Z}$

$$Hom_{\mathcal{T}}(D, X[n]) \cong Hom_{\mathcal{T}}(i_*C', X[n]) \oplus Hom_{\mathcal{T}}(C, X[n]) = 0$$

So $Hom_{\mathcal{T}}(i_*C', X[n]) \cong Hom_{\mathcal{I}}(C', i^! X[n]) = 0$ for any $n \in \mathbb{Z}$ and since C' is a compact generator in $\mathcal{I}$, $i^!X = 0$. It follows from the distinguished triangle $i^!X \to X \to j^*j_*X \rightsquigarrow$

that $X \cong j^* j_* X$ and thus X is $\mathcal{I}$-local. Finally we have

$$0 = Hom_{\mathcal{T}}(C, X[n]) \cong Hom_{\mathcal{T}}(C, j_* j^* X[n]) \cong Hom_{\mathcal{T}/\mathcal{I}}(j^* C, j^* X[n])$$

Since $j^* C \cong C'' \oplus C''[1]$ is a compact generator of $\mathcal{T}/\mathcal{I}$, $j^* X = 0$ and $X \cong j_* j^* X = 0$ $\square$

Theorem 4.2.5. *Let X be a quasi-compact, quasi-separated scheme. Then the category $D_{qcoh}(X)$ admits a compact generator.*

Proof. Since X is quasi-compact, there exists a finite affine open cover $\{U_i\}_{i=1,\ldots,n}$ of X. Apply induction on n. If $n = 1$, X is an affine scheme and $\mathcal{O}_X$ is a compact generator for $D(X)$. Let $Y = \bigcup_{i=2,\ldots,n} U_i$, then $D(Y)$ has a compact generator by induction hypothesis. Let $Z = X - Y = U_1 - U_1 \cap Y$, since X is quasi-separated, $U_1 \cap Y$ is quasi-compact, then $Z = V(f_1, f_2, \ldots f_r)$ is defined by finitely many elements in $\Gamma(U_1, \mathcal{O}_{U_1})$. By[17] Proposition 6.1, $D_Z(U_1)$ has $Q = \bigotimes_{i=1,\ldots,r}(\mathcal{O}_{U_1} \xrightarrow{f_i} \mathcal{O}_{U_1})$ as a compact generator. So, by Lemma 4.2.1 $D_Z(X) \cong D_Z(U_1)$ has a compact generator. Consider the exact sequence of triangulated categories,

$$D_Z(X) \underset{i^!}{\overset{i_*}{\rightleftarrows}} D(X) \underset{j_*}{\overset{j^*}{\rightleftarrows}} D(Y)$$

The functor j_* preserves coproducts since it has a right adjoint functor, so by Lemma 4.2.4 $D(X)$ has compact generator.

$\square$

Remark 4.2.6. *According to Rouquier[4] Theorem 4.22, if a cocomplete triangulated category $\mathcal{T}$ has a compact generator G, then G is a classical generator of the full subcategory $\mathcal{T}^c$ consisting of compact objects of $\mathcal{T}$. It follows immediately from the example 4.2.3 that for any quasicompact, quasi-separated scheme X, the category $Perf(X)$ has a classical generator. By Pat Lank and Noah Olander[9] Theorem 1.1, this classical generator is a strong generator if and only if X is a regular scheme.*

Chapter 5

Descendable morphisms of schemes

The notation of descendable algebra objects and descendable maps was introduced by Mathew in[13] in the content of stable ∞-categories. Descendable maps were also studied by Balmer independently under the name of nil-faithfulness[14]. Balmer showed that descent-up-to-nilpotence happens for nil-faithful tensor triangulated rings of finite degree (Theorem 3.19[14]). Aoki noticed the relation between descendable maps and dimensions of tensor triangulated categories[8] which motivated us to define descendable morphisms for schemes.

5.1 Tensor triangulated categories

5.1.1 Thick tensor ideals

We first recall the definition of tensor triangulated categories and tensor ideals.

Definition 5.1.1. *A tensor triangulated category $(\mathcal{T}, \otimes, 1_{\mathcal{T}})$ is a triangulated category $\mathcal{T}$ with a symmetric monoidal structure compatible with the triangulated structure in the following sense:*

- *For all objects A, B in $\mathcal{T}$, a natural isomorphism*

$$e_{A,B} : A[1] \otimes B \cong A \otimes B[1]$$

- *For any object E, the functor $- \otimes E$ is additive*
- *For any object E, and any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$, $A \otimes E \xrightarrow{f \otimes id_E} B \otimes E \xrightarrow{g \otimes id_E} C \otimes E \xrightarrow{h \otimes id_E} A \times E[1]$ is a distinguished triangle.*
- *The natural isomorphisms $e_{A,B}$ are compatible with the associator in a natural way.*

A tensor triangulated functor $F : \mathcal{T} \to \mathcal{K}$ is a functor which is both exact and lax monoidal such that the two natural maps from $F(A) \otimes F(B)[1]$ to $F(A \otimes B[1])$ in the following diagram agree:

$$\begin{tikzcd}
& F(A) \otimes F(B[1]) \arrow[dr] & \\
F(A) \otimes F(B)[1] \arrow[ur] \arrow[dr] & & F(A \otimes B[1]) \\
& F(A \otimes B)[1] \arrow[ur] &
\end{tikzcd}$$

Example 5.1.2. *The derived categories $D_{qcoh}(X)$ and $\mathrm{Perf}(X)$ together with the derived tensor product are tensor triangulated categories, if $f : X \to Y$ is a morphism of schemes, then Rf_* and Lf^* are tensor triangulated functors.*

Definition 5.1.3. *Let $\mathcal{T}$ be a tensor triangulated category. A full subcategory $\mathcal{I} \subset \mathcal{T}$ is called a $\otimes$-ideal if, whenever $I \in \mathcal{I}$ and $T \in \mathcal{T}$, the tensor product $I \otimes T$ belongs to $\mathcal{I}$.*

Example 5.1.4. *If $H \in \mathcal{T}$ is an object, the full subcategory $H \otimes \mathcal{T} = \{H \otimes T \mid Y \in \mathcal{T}\}$ is a $\otimes$-ideal.*

In fact, for any object $H \in \mathcal{T}$, we have a $\otimes$-ideals $\langle H \otimes \mathcal{T} \rangle_n$ for each $n \in \mathbb{N}$. First, notice that exact functors between triangulated categories commute with generation:

Lemma 5.1.5. *Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories, and let $\mathcal{S} \subset \mathcal{T}$ be a collection of objects. Then $F(\langle \mathcal{S} \rangle_n) \subset \langle F(\mathcal{S}) \rangle_n$ for any $n \in \mathbb{N}$.*

Proof. Since F is an exact functor, it preserves distinguished triangles and in particular it preserves finite coproducts and direct summands. We will prove the statement by induction on n.

When $n = 1$, let $X \in \langle \mathcal{S} \rangle_1$ be an arbitrary object. Then there exists an object X' such that $X \oplus X' \cong \bigoplus_{i=1,\dots,n} S_i[n_i]$, where $S_i \in \mathcal{S}$. Then $F(X) \oplus F(X') \cong \bigoplus_{i=1,\dots,n} F(S_i)[n_i] \in \langle F(\mathcal{S}) \rangle_1$. Assuming the statement holds for all j with $1 \leq j \leq n-1$. Let $X_n \in \langle \mathcal{S} \rangle_n$. Then there is an object X_n' such that $X = X_n \oplus X_n'$ fits into a distinguished triangle: $X_1 \to X_{n-1} \to X$ with $X_1 \in \langle \mathcal{S} \rangle_1$ and $x_{n-1} \in \langle \mathcal{S} \rangle_{n-1}$. Apply the functor F on this triangle, we get $F(X_1) \to F(X_{n-1}) \to F(X)$. By induction hypothesis, $F(X_1) \in \langle F(\mathcal{S}) \rangle_1$ and $F(x_{n-1}) \in \langle F(\mathcal{S}) \rangle_{n-1}$. So, $F(X) \in \langle F(\mathcal{S}) \rangle_n$ and it follows that $F(X_n) \in \langle F(\mathcal{S}) \rangle_n$ as well. □

Corollary 5.1.6. *Let $\mathcal{T}$ be a tensor triangulated category, H be an object in $\mathcal{T}$. For each natural number n, $\langle H \otimes \mathcal{T} \rangle_n = \langle \{H \otimes E : E \in \mathcal{T}\} \rangle_n$ is a $\otimes$-ideal.*

Proof. We need to show that for each $T \in \mathcal{T}$ there is the following inclusion:

$$T \otimes \langle \{H \otimes E : E \in \mathcal{T}\} \rangle_n \subset \langle \{H \otimes E : E \in \mathcal{T}\} \rangle_n$$

But this follows directly from Lemma 5.1.5 since $T \otimes - : \mathcal{T} \to \mathcal{T}$ is an exact functor, take $\mathcal{S} = \{H \otimes E : E \in \mathcal{T}\}$. One gets

$$T \otimes \langle \{H \otimes E : E \in \mathcal{T}\} \rangle_n \subset \langle \{T \otimes H \otimes E : E \in \mathcal{T}\} \rangle_n \subset \langle \{H \otimes E : E \in \mathcal{T}\} \rangle_n$$

□

It is useful for us to consider those $\otimes$-ideals which are also triangulated subcategories and closed under direct summands, these are called thick $\otimes$-ideals.

Lemma 5.1.7. *Let $\mathcal{T}$ be a tensor triangulated category, and let H be an object in $\mathcal{T}$, then the smallest thick $\otimes$-ideal containing H is $\langle\{H \otimes E : E \in \mathcal{T}\}\rangle = \bigcup_{n\in\mathbb{N}} \langle\{H \otimes E : E \in \mathcal{T}\}\rangle_n$. We will call this subcategory the thick $\otimes$-ideal generated by H.*

Proof. By corollary 5.1.6 $\langle\{H \otimes E : E \in \mathcal{T}\}\rangle$ is a $\otimes$-ideal. It is clear that this is a thick triangulated subcategory and any thick $\otimes$-ideal containing H also contains this subcategory. □

Suppose an object H generates the category $\mathcal{T}$ if tensor products are also allowed when generating (we can take finite coproducts, direct summands, shifts, tensor products and cones) then there is an upper bound for the number of cones needed to generate each object.

Theorem 5.1.8. *Let $E \in \mathcal{T}$ be an object in a tensor triangulated category $(\mathcal{T}, \otimes, 1_{\mathcal{T}})$. If $\bigcup_{n\in\mathbb{N}} \langle\{H \otimes E : E \in \mathcal{T}\}\rangle_n = \mathcal{T}$ then $\langle\{H \otimes E : E \in \mathcal{T}\}\rangle_n = \mathcal{T}$ for some n.*

Proof. The unit $1_{\mathcal{T}} \in \mathcal{T} = \bigcup_{n\in\mathbb{N}} \langle\{H \otimes E : E \in \mathcal{T}\}\rangle_n$ implies that $1_{\mathcal{T}} \in \langle\{H \otimes E : E \in \mathcal{T}\}\rangle_n$ for some n. By corollary 5.1.6, $\langle\{H \otimes E : E \in \mathcal{T}\}\rangle_n = \mathcal{T}$ □

Remark 5.1.9. *In fact, with the above assumption, there exists an object $E \in \mathcal{T}$ such that $1_{\mathcal{T}} \in \langle H \otimes E\rangle_n$. This is because to build $1_{\mathcal{T}}$ one only need finitely many objects to tensor with throughout the process, one can take E to be the direct sum of these objects.*

5.1.2 Tensor ghosts

Similar to the definition of E-ghost maps which are defined by the functor $Hom(E, -)$, we can use the functor $E\otimes$- to define $\otimes$-ghost maps.

Definition 5.1.10. *Let $\mathcal{T}$ be a tensor triangulated category, let $\mathcal{S}$ be a collection of objects in $\mathcal{T}$. A morphism $f : A \to B$ in $\mathcal{T}$ is called a $\mathcal{S}$-$\otimes$-ghost if the morphism $f \otimes id_E : A \otimes E \to$*

$B \otimes E$ is zero for any $E \in \mathcal{S}$.

If a map is a E-$\otimes$-ghost, it is immediate that the map is killed by more objects.

Lemma 5.1.11. *If a morphism $f : A \to B$ in $\mathcal{T}$ is a E-$\otimes$-ghost then it is also a $\langle \{E \otimes F : F \in \mathcal{T}\} \rangle_1$-$\otimes$-ghost.*

Proof. It is clear that if $f \otimes id_E = 0$ then $f \otimes id_{E \otimes F} \cong f \otimes id_E \otimes id_F = 0$ for any $F \in \mathcal{T}$. Also $f \otimes id_{\bigoplus_{i=1,\ldots r} E \otimes F_i} \cong \bigoplus_{i=1,\ldots r} f \otimes id_{E \otimes F_i} = 0$.
For the shift functor [1], we have the following commutative diagram by the definition of tensor triangulated categories:

$$\begin{tikzcd} A \otimes E[1] \arrow[r, "f \otimes id_{E[1]}"] \arrow[d, "\cong"] & B \otimes E[1] \arrow[d, "\cong"] \\ (A \otimes E)[1] \arrow[r, "(f \otimes id_E)[1]"] & (B \otimes E)[1] \end{tikzcd}$$

Thus $f \otimes id_{E[1]} = 0$.
It remains to check for direct summands. Let F' be a direct summand of an object of the form $\bigoplus_{i=1,\ldots r} E \otimes F_i$, say $F' \otimes F'' \cong \bigoplus_{i=1,\ldots r} E \otimes F_i$, then $f \otimes id_{F'} \oplus f \otimes id_{F''} \cong f \otimes id_{F' \oplus F''} = 0$ and thus $f \otimes id_{F'} = 0$. $\square$

The following lemma is parallel to the ghost lemma.

Lemma 5.1.12. *If $f : A \to B$ is a $\mathcal{I}$-$\otimes$-ghost, $g : B \to C$ is a $\mathcal{J}$-$\otimes$-ghost, then $g \circ f$ is a $\mathcal{I} \diamond \mathcal{J}$-$\otimes$-ghost. In particular, the composition of n E-$\otimes$-ghost is a $\langle \{E \otimes X : X \in \mathcal{T}\} \rangle_n$-$\otimes$-ghost*

Proof. Take an arbitrary object $\tilde{M} \in \mathcal{I} \diamond \mathcal{J}$, then $\tilde{M}$ is a direct summand of an object M which fits into a distinguished triangle $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with $M' \in \mathcal{J}$ and $M'' \in \mathcal{I}$. By 5.1.11, we may assume $\tilde{M} = M$. Tensor $A \xrightarrow{f} B \xrightarrow{g} C$ with this triangle, we get the following diagram:

$$\begin{tikzcd}
A \otimes M' \arrow[r, "f\otimes id_{M'}"] \arrow[d, "id_A\otimes\alpha"] & B \otimes M' \arrow[r, "g\otimes id_{M'}"] \arrow[d, "id_B\otimes\alpha"] & C \otimes M' \arrow[d, "id_C\otimes\alpha"] \\
A \otimes M \arrow[r, "f\otimes id_M"] \arrow[d, "id_A\otimes\beta"] & B \otimes M \arrow[r, "g\otimes id_M"] \arrow[d, "id_B\otimes\beta"] & C \otimes M \arrow[d, "id_C\otimes\beta"] \\
A \otimes M'' \arrow[r, "f\otimes id_{M''}"] & B \otimes M'' \arrow[r, "f\otimes id_{M''}"] & C \otimes M''
\end{tikzcd}$$

The columns are distinguished triangles and the maps $g \otimes id_{M'}$ and $f \otimes id_{M''}$ are zero maps by assumption. Apply the functor $Hom(A \otimes M, -)$ to this diagram, and diagram chasing shows that $(g \circ f) \otimes id_M = 0$. □

Corollary 5.1.13. *The tensor product of an $\mathcal{I}$-$\otimes$-ghost and a $\mathcal{J}$-$\otimes$-ghost is a $\mathcal{J} \diamond \mathcal{I}$-$\otimes$-ghost and an $\mathcal{I} \diamond \mathcal{J}$-$\otimes$-ghost.*

Proof. Suppose $f : A \to B$ is a $\mathcal{I}$-$\otimes$-ghost, $g : C \to D$ is a $\mathcal{J}$-$\otimes$-ghost. Then $f \otimes g$ is an $\mathcal{I} \diamond \mathcal{J}$-$\otimes$-ghost and a $\mathcal{J} \diamond \mathcal{I}$-$\otimes$-ghost because of the following decomposition

$$\begin{tikzcd}
 & B \otimes C \arrow[dr, "id_B\otimes g"] & \\
A \otimes C \arrow[ur, "f\otimes id_C"] \arrow[rr, "f\otimes g"] \arrow[dr, "id_A\otimes g"'] & & B \otimes D \\
 & A \otimes D \arrow[ur, "f\otimes id_D"'] &
\end{tikzcd}$$

□

Lemma 5.1.14. *Suppose the composition of any n E-$\otimes$-ghost maps vanish or the tensor product of any n E-$\otimes$-ghost maps vanish, and suppose there is a E-$\otimes$-ghost map $f : F \to 1_{\mathcal{T}}$ with $cone(f) \in \langle\{E \otimes X : X \in \mathcal{T}\}\rangle_m$, then $\langle\{E \otimes X : X \in \mathcal{T}\}\rangle_{mn} = \mathcal{T}$*

Proof. Denote by $C = cone(f)$. We have a distinguished triangle $F \to 1_{\mathcal{T}} \to C$. Tensor

$F^{\otimes m}$ with this triangle for $m = 1, 2, ..., n-1$ we got n distinguished triangles

$$F^{\otimes(m+1)} \xrightarrow{f \otimes id_{F^{\otimes m}}} F^{\otimes m} \to C \otimes F^{\otimes m}$$

The maps $f \otimes id_{F^{\otimes m}}$ are all E-tensor ghost and thus their composition $F^{\otimes n} \xrightarrow{f^{\otimes n}} 1_{\mathcal{T}}$ is the zero by assumption. So, $1_{\mathcal{T}}$ is a direct summand of $cone(f^{\otimes n})$. But by octahedral axiom, $cone(f^{\otimes n})$ is in $\langle \{E \otimes X : X \in \mathcal{T}\} \rangle_{mn}$ $\square$

Example 5.1.15. *Let A be a commutative algebra object in $\mathcal{T}$, and let $\eta : 1_{\mathcal{T}} \to A$ be the unit map. Then if one completes η into a distinguished triangle $C \to 1_{\mathcal{T}} \to A$, the map $C \to 1_{\mathcal{T}}$ is a A-$\otimes$-ghost. This is because the map $A \xrightarrow{\eta \otimes id_A} A^{\otimes 2}$ has the multiplication $A \otimes A \to A$ as its left inverse and thus it is a split monomorphism.*

Example 5.1.16. *Let $P \in D_{qcoh}(X)$ be any perfect complex, and let $C \to \mathcal{O}_X \to P \otimes P^*$ is a distinguished triangle, where the second map is the coevaluation map. Then the map $C \to \mathcal{O}_X$ is a P-$\otimes$-ghost. Because the map $P \to P \otimes P \otimes P^*$ obtained by tensoring P on the coevaluation map is a split monomorphism*[28] *Theorem 3.8.*

5.2 Descendable morphisms of schemes

In this section we define descendable morphisms of schemes and prove some basic properties and provide some examples.

The following Proposition in[8] is the motivation of the definition of descendable morphisms of schemes:

Proposition 5.2.1. *Let $f : X \to Y$ be a morphism of quasi-compact, quasi-separated schemes. If there is a complex $E \in D_{qcoh}(X)$ such that the thick $\otimes$-ideal generated by Rf_*E is $D_{qcoh}(Y)$, then Rf_* preserves strong $\oplus$-generators.*

Proof. Suppose $D_{qcoh}(X) = \overline{\langle G \rangle}_m$, Then

$$\begin{aligned} D_{qcoh}(Y) &= \langle Rf_*(E) \otimes F : F \in D_{qcoh}(Y) \rangle_n \\ &= \langle Rf_*(E \otimes Lf^* F) : F \in D_{qcoh}(Y) \rangle_n \\ &\subset \langle Rf_*(D_{qcoh}(X)) \rangle_n \\ &\subset \langle Rf_* \overline{\langle G \rangle}_m \rangle_n \\ &\subset \overline{\langle Rf_* G \rangle}_{mn} \end{aligned}$$

□

Remark 5.2.2. *Since we will also be interested in the bounded derived categories of coherent sheaves later, we want the objects E and Rf_*E in 5.2.1 to be in $D^b_{coh}(X)$ and $D^b_{coh}(Y)$ respectively. But there is no guarantee that if $E \in D^b_{coh}(X)$ then $Rf_*E \in D^b_{coh}(Y)$ for non-proper f. The way to solve this issue is to find an object $F \in D^b_{coh}(Y)$ such that $Rf_*E \in \overline{\langle F \rangle}_n$ then use F as the strong $\oplus$-generator. It is due to Aoki*[8] (*Theorem 3.5*) *that we are able to find such a F when f is a finite type morphism between Noetherian schemes.*

Definition 5.2.3. *A morphism $f : X \to Y$ is said to be descendable if there is an object $E \in D_{qcoh}(X)$ such that the thick tensor ideal generated by Rf_*E in $D_{qcoh}(Y)$ is $D_{qcoh}(Y)$ in other words $\langle Rf_*E \otimes D_{qcoh}(Y) \rangle = D_{qcoh}(Y)$.*

Example 5.2.4. *Let X be a variety over a field of characteristic zero that has rational singularity. Then by definition, X has a resolution of singularity $\tilde{X} \xrightarrow{f} X$ such that $\mathcal{O}_X \cong Rf_*\mathcal{O}_{\tilde{X}}$, and since the thick tensor ideal generated by $\mathcal{O}_X$ is $D_{qcoh}(X)$, we have $\langle Rf_*\mathcal{O}_{\tilde{X}} \otimes D_{qcoh}(X) \rangle_1 = D_{qcoh}(X)$ and the morphism f is descendable. We will see later that any resolution of singularity is descendable.*

Example 5.2.5. *Let $X \to S$, $Y \to S$ be morphisms of schemes. The projection $X \times_S Y \xrightarrow{pr_X}$*

X is descendable, because we have

$$id_X = pr_X \circ (id_X, f) : X \xrightarrow{(id_X, f)} X \times_S Y \xrightarrow{pr_X} X$$

for any morphism $f : X \to Y$.

Lemma 5.2.6. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f : X \to Z$ is descendable, then g is descendable.*

Proof. If the thick tensor ideal generated by $Rg_* Rf_* E$ is $D(Z)$ for some $E \in D_{qcoh}(X)$, then the object $Rf_* E \in D_{qcoh}(Y)$ has the desired property. □

We shall show that descendable morphisms satisfy some nice properties. We begin by showing that they are closed under composition.

Proposition 5.2.7. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If f, g are descendable, then $g \circ f : X \to Z$ is descendable.*

Proof. By assumption, there are objects $E \in D_{qcoh}(X)$ and $F \in D_{qcoh}(Y)$ such that

$$\langle Rf_* E \otimes D_{qcoh}(Y) \rangle = D_{qcoh}(Y)$$

$$\langle Rg_* F \otimes D(Z) \rangle = D(Z)$$

By Remark 5.1.9 we have

$$F \in \langle Rf_* E \otimes H \rangle_n = \langle Rf_* (E \otimes Lf^* H) \rangle_n$$

$$\mathcal{O}_Z \in \langle Rg_* F \otimes L \rangle_m$$

for some object $H \in D_{qcoh}(Y)$, $L \in D(Z)$ and some natural numbers n, m. And it follows that

$$\begin{aligned}\mathcal{O}_Z &\in \langle Rg_*F \otimes L\rangle_m \\ &\subset \langle Rg_*\langle Rf_*(E \otimes Lf^*H)\rangle_n \otimes L\rangle_m \\ &\subset \langle Rg_*Rf_*(E \otimes Lf^*H) \otimes L\rangle_{mn}\end{aligned}$$

Where the second inclusion is due to Lemma 5.1.5.

Thus we have

$$D(Z) = \langle Rg_*Rf_*(E \otimes Lf^*H) \otimes D(Z)\rangle$$

and the object $E \otimes Lf^*H \in D_{qcoh}(X)$ satisfies the requirement. □

The derived direct image of the structure sheaf $\mathcal{O}_X$ of X can actually generates the derived direct image of any other complexes in $D_{qcoh}(X)$ which simplifies the definition of descendable morphisms: we can just test whether the thick tensor ideal generated by $Rf_*\mathcal{O}_X$ is $D_{qcoh}(Y)$.

Lemma 5.2.8. *Let $f : X \to Y$ be a morphism of schemes. For any object $E \in D_{qcoh}(X)$, $Rf_*E \in \langle Rf_*\mathcal{O}_X \otimes D_{qcoh}(Y)\rangle_1$.*

Proof. Denote by $\epsilon : id \to Rf_*Lf^*$ the unit map and $\eta : Lf^*Rf_* \to id$ the counit map. We have the following commutative diagram

$$\begin{tikzcd} Hom_{D_{qcoh}(X)}(Lf^*Rf_*E, Lf^*Rf_*E) \arrow[r, "\cong"] \arrow[d, "\eta_{E*}"] & Hom_{D_{qcoh}(Y)}(Rf_*E, Rf_*Lf^*Rf_*E) \arrow[d, "Rf_*\eta_{E*}"] \\ Hom_{D_{qcoh}(X)}(Lf^*Rf_*E, E) \arrow[r, "\cong"] & Hom_{D_{qcoh}(Y)}(Rf_*E, Rf_*E) \end{tikzcd}$$

Keep track on the identity map in $Hom_{D_{qcoh}(X)}(Lf^*Rf_*E, Lf^*Rf_*E)$ one found $Rf_*\eta_E \circ \epsilon_{Rf_*E} = id_{Rf_*E}$. Thus Rf_*E is a direct summand of $Rf_*Lf^*Rf_*E$.

But then $Rf_*Lf^*Rf_*E \cong Rf_*E \otimes Rf_*\mathcal{O}_X \in \langle Rf_*\mathcal{O}_X \otimes D_{qcoh}(Y)\rangle$ as desired $\square$

Lemma 5.2.9. *Let $f : X \to Y$ be a morphism of schemes, f is descendable if and only if $\langle Rf_*\mathcal{O}_X \otimes D_{qcoh}(Y)\rangle = D_{qcoh}(Y)$.*

Proof. If the thick tensor ideal generated by $Rf_*\mathcal{O}_X$ is $D_{qcoh}(Y)$ then f is descendable by definition.

Conversely, suppose there is some $E \in D_{qcoh}(X)$ such that $\langle Rf_*E \otimes D_{qcoh}(Y)\rangle = D_{qcoh}(Y)$, then by Lemma 5.2.8,

$$\begin{aligned} D_{qcoh}(Y) &= \langle Rf_*E \otimes D_{qcoh}(Y)\rangle \\ &\subset \langle \langle Rf_*\mathcal{O}_X \otimes D_{qcoh}(Y)\rangle_1 \otimes D_{qcoh}(Y)\rangle \\ &\subset \langle Rf_*\mathcal{O}_X \otimes D_{qcoh}(Y)\rangle \end{aligned}$$

$\square$

Next, we show that the morphisms induced by any Zariski open cover is descendable. Assume first that the covering is finite.

Proposition 5.2.10. *Let $\{U_i\}_{i=1,2\ldots n}$ be an open covering of X. Denote by j_i the inclusion maps. Then the morphism $\coprod U_i \to X$ is descendable, i.e. the thick tensor ideal generated by $\bigoplus_{i=1,2\ldots n} Rj_{i*}\mathcal{O}_{U_i}$ is $D_{qcoh}(X)$*

Proof. When $n = 1$ the statement is trivial. Suppose the open cover consists of two open subsets $\{U, V\}$ and denote the inclusion maps as follows

$$\begin{tikzcd} & X & \\ U \arrow[ur, "i"] & & V \arrow[ul, "j"'] \\ & U \cap V \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, "l"] & \end{tikzcd}$$

We have the Mayer-Vietoris triangle:

$$\mathcal{O}_X \to Ri_*\mathcal{O}_U \oplus Rj_*\mathcal{O}_V \to Rl_*\mathcal{O}_{U\cap V} \rightsquigarrow$$

Notice that $Rl_*\mathcal{O}_{U\cap V} \cong Ri_*R\alpha_*\mathcal{O}_{U\cap V} \cong Ri_*(R\alpha_*\mathcal{O}_{U\cap V} \otimes \mathcal{O}_U) \cong Ri_*R\alpha_*\mathcal{O}_{U\cap V} \otimes Ri_*\mathcal{O}_U$ Where the last isomorphism follows from the fact that derived pushforward of an open immersion commutes with derived tensor product. To be more specific, one has $Li^*Ri_* \cong id_{D(U)}$ by Lemma 4.1.5 (3) and thus a trivial triangle

$$Li^*Ri_*\mathcal{O}_U \to \mathcal{O}_U \to 0 \rightsquigarrow$$

Apply the functor $Ri_*(- \otimes R\alpha_*\mathcal{O}_{U\cap V})$ one gets

$$Ri_*\mathcal{O}_U \otimes Ri_*R\alpha_*\mathcal{O}_{U\cap V} \to Ri_*(\mathcal{O}_U \otimes R\alpha_*\mathcal{O}_{U\cap V}) \to 0 \rightsquigarrow$$

So, we have $Rl_*\mathcal{O}_{U\cap V} \in \langle (Ri_*\mathcal{O}_U \oplus Rj_*\mathcal{O}_V) \otimes D_{qcoh}(X)\rangle_1$ and thus

$$\mathcal{O}_X \in \langle (Ri_*\mathcal{O}_U \oplus Rj_*\mathcal{O}_V) \otimes D_{qcoh}(X)\rangle_2$$

The general case follows from induction on the number of open subsets in the cover. Denote $U = U_1$ and $V = \bigcup\limits_{i=2\ldots n} U_i$. Then we have the following factorization

$$\coprod_{i=1,2\ldots n} U_i \to U \coprod V \to X$$

Where the first map is identity on U_1 and send the rest to V in the obvious way, both maps are descendable by the induction hypothesis and the above argument. Thus, $\coprod U_i \to X$ is descendable by Prop 5.2.7 and $\mathcal{O}_X \in \langle \bigoplus\limits_{i=1\ldots n} Rj_{i*}\mathcal{O}_{Ui} \otimes D_{qcoh}(X)\rangle_n$.

□

Corollary 5.2.11. *Assume X is a quasicompact, quasiseparated scheme. Let $\{U_i\}_{i\in I}$ be an open cover of X. Denote by j_i the inclusion maps. Then the morphism $\coprod U_i \to X$ is descendable, i.e. the thick tensor ideal generated by $\bigoplus_{i\in I} Rj_{i*}\mathcal{O}_{U_i}$ is $D_{qcoh}(X)$*

Proof. Since X is quasi-compact, there is a finite sub-cover of $\{U_i\}_{i\in I}$, say $\{U_i\}_{i=1,2\ldots n}$. Then the statement follows from the fact that $\bigoplus_{i=1,2\ldots n} Rj_{i*}\mathcal{O}_{U_i}$ is a direct summand of $\bigoplus_{i\in I} Rj_{i*}\mathcal{O}_{U_i}$ $\square$

The following Theorem is due to Hopkins see Neeman[29] Lemma 1.2 or Thomason[28] Lemma 3.14.

Theorem 5.2.12. *Let X be a quasicompact, quasiseparated scheme. Let $\mathcal{E}$, $\mathcal{F}$ be two perfect complexes. Then $\mathcal{F} \in \langle \mathcal{E} \otimes \mathrm{Perf}(X)\rangle$ in $\mathrm{Perf}(X)$ if and only if $supp(\mathcal{F}) \subset supp(\mathcal{E})$.*

Corollary 5.2.13. *Let X be a quasicompact, quasiseparated scheme and let $E \in \mathrm{Perf}(X)$ that has full support. Then $\langle \mathcal{E} \otimes D_{qcoh}(X)\rangle = D_{qcoh}(X)$*

Proof. This is because $\mathcal{O}_X \in \langle \mathcal{E} \otimes \mathrm{Perf}(X)\rangle \subset \langle \mathcal{E} \otimes D_{qcoh}(X)\rangle$. $\square$

Corollary 5.2.14. *Proper flat surjective morphisms are descendable.*

Proof. Suppose $f : X \to Y$ is a proper flat surjective morphism of schemes, then Rf_* sends perfect complexes to perfect complexes, more over, since f is surjective, $Rf_*(\mathcal{O}_X)$ has full support. $\square$

Descendable morphisms are local on the target, and can be checked on an open covering.

Proposition 5.2.15. *Let $f : X \to Y$ be a morphism of quasicompact, quasiseparated schemes, and let $\{U_r\}$ be a finite open cover of Y. Denote by $i_r : U_r \to Y$ the open immersions and let $V_r = f^{-1}(U_r)$. Then $\langle f_*\mathcal{O}_X \otimes D_{qcoh}(Y)\rangle = D_{qcoh}(Y)$ if and only if $\langle Rf_*\mathcal{O}_{V_r} \otimes D(U_r)\rangle = D(U_r)$ for each r.*

Proof. Consider the commutative diagram of morphism of schemes:

$$\begin{tikzcd} \coprod_{r=1,2\ldots n} V_r \arrow[r, "(f|_{V_r})"] \arrow[d] & \coprod_{r=1,2\ldots n} U_r \arrow[d, "(i_r)"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose $\langle Rf_*\mathcal{O}_{V_r} \otimes D(U_r)\rangle = D(U_r)$ for each r, then the morphism $(f|_{V_r})$ is descendable. By Prop 5.2.10, (i_r) is descendable. So their composition is descendable by 5.2.7. It follows from Lemma 5.2.8 that $\langle f_*\mathcal{O}_X \otimes D_{qcoh}(Y)\rangle = D_{qcoh}(Y)$.

Conversely, assume $f : X \to Y$ is descendable. Then $\langle Rf_*\mathcal{O}_X \otimes D_{qcoh}(Y)\rangle = D_{qcoh}(Y)$ and thus

$$\begin{aligned}
\langle Rf|_{V_r*}\mathcal{O}_{V_r} \otimes D(U_r)\rangle &\cong \langle Rf|_{Vr*}(\mathcal{O}_X|_{V_r}) \otimes D(U_r)\rangle \\
&\cong \langle Li_r^* Rf_*\mathcal{O}_X \otimes D(U_r)\rangle \\
&\cong \langle Li_r^* Rf_*\mathcal{O}_X \otimes Li_r^* D_{qcoh}(Y)\rangle \\
&\cong \langle Li_r^*(Rf_*\mathcal{O}_X \otimes D_{qcoh}(Y))\rangle \\
&\cong Li_r^*(D_{qcoh}(Y)) \\
&\cong D(U_r)
\end{aligned}$$

Where the second isomorphisms is due to flat base change, the third isomorphism is because the map Li_r^* is essentially surjective. □

Corollary 5.2.16. *Let $f : E \to X$ be any fiber bundle, then f is descendable.*

Proof. There is an open covering $\{U_i\}$ of X such that f can be written as the projection $U_i \times E_i \to U_i$, thus f is descendable. □

Apply 5.2.15 to the identity morphism $id_X : X \to X$ one can show that the thick $\otimes$-ideal generated by $E \in D_{qcoh}(X)$ is $D_{qcoh}(X)$ if and only if E generates locally.

Corollary 5.2.17. *Let X be a quasicompact, quasiseparated scheme and $\{U_i\}$ a finite open cover of X. An object $E \in D_{qcoh}(X)$ such that $\langle E \otimes D_{qcoh}(X) \rangle = D_{qcoh}(X)$ if and only if $\langle E|_{U_I} \otimes D(U_i) \rangle = D(U_i)$ for each i.*

The following statement is a direct corollary of the fact that descendable morphisms preserves strong $\oplus$-generators. See Lank[30] Theorem D for a version for $D^b_{coh}(X)$.

Corollary 5.2.18. *Let X be a Noetherian scheme and let $\{U_i\}_{i=1,2 \dots n}$ be an open covering of X. Then $D_{qcoh}(X)$ admits a strong $\oplus$-generator if and only if for each i, $D_{qcoh}(U_i)$ has a strong $\oplus$-generator.*

Next, we show that descendable morphisms are stable under base change.

Proposition 5.2.19. *Let $f : X \to Y$ be a descendable morphism of quasicompact, quasiseparated schemes. Then any base change of f is descendable.*

Proof. Let $Y' \to Y$ be any morphism of schemes. Denote by X' the fiber product.

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $\{U_i\}$ be an affine open cover of Y and suppose $g^{-1}(U_i) = \bigcup V_{ij}$ and $f^{-1}(U_i) = \bigcup W_{ik}$ with V_{ij} and W_{ik} affine open subsets of Y' and X respectively. Then $X' = X \times_Y Y'$ is covered by $\{V_{ij} \times_{U_i} W_{ik}\}$. Consider the following commutative diagram, in which all squares are cartisian:

$$\begin{tikzcd}
\coprod_{i,j,k} W_{ik} \times_{U_i} V_{ij} \arrow[r] \arrow[d] & \coprod_{i,j} X \times_Y V_{ij} \arrow[r] \arrow[d] & \coprod_{i,j} V_{ij} \arrow[d] \\
\coprod_{i,k} W_{ik} \times_Y Y' \arrow[r] \arrow[d] & X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
\coprod_{i,k} W_{ik} \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

By Corollary 5.2.11, Lemma 5.2.15, and Lemma 5.2.6, to show f' is descendable, we only need to show for each pair of i, j the morphism $\coprod_k V_{ij} \times_{U_i} W_{ik} \to V_{ij}$ is descendable. Thus, we may assume X, Y, Y' are affine schemes. Assume $X = SpecA$, $Y = SpecB$, $Y' = SpecB'$ and denote by $A' = A \otimes_B B'$ the tensor product. We have the following commutative diagram:

$$\begin{tikzcd}
A \otimes_B B' & B' \arrow[l, "f'"'] \\
A \arrow[u, "g'"] & B \arrow[l, "f"'] \arrow[u, "g"]
\end{tikzcd}$$

By abuse notations, we use the same names for the corresponding ring maps. Now being descendable means there is a chain complex of A-modules $E^\bullet$ when considered as a chain complex of B-modules generates all chain complexes of B-modules in $D(Mod(B))$ in other words, $\langle E^\bullet \otimes^L D(Mod(B))\rangle = D(Mod(B))$. Then one can check that the chain complex $E^\bullet \otimes^L B' \in D(Mod(A \otimes_B B'))$ makes f' descendable. □

If a morphisms $f : X \to Y$ preserve classical generators of the bounded derived category of coherent sheaves, then f is descendable.

Lemma 5.2.20. *Let $f : X \to Y$ be a morphism of schemes and $E \in D^b_{coh}(X)$ be a classical generator of $D^b_{coh}(X)$. If Rf_*E is a classical generator of $D^b_{coh}(Y)$ then f is descendable.*

Proof. By assumption, we have $\mathcal{O}_Y \in \langle Rf_*E\rangle = D^b_{coh}(Y)$ and thus $\mathcal{O}_Y \in \langle Rf_*E \otimes D_{qcoh}(Y)\rangle = D_{qcoh}(Y)$. □

Remark 5.2.21. *The assumption of the existence of a classical generator in Lemma 5.2.20 can be removed for most schemes. For example, if the regular locus of a scheme X contains a non-empty open subset, then $D^b_{qcoh}(X)$ has classical generators*[31] *Proposition 4.3 and Theorem 4.15.*

Surjective proper morphisms are descendable.

Corollary 5.2.22. *Let $f : X \to Y$ be a surjective proper morphisms of schemes, if Y is a J-2 Noetherian scheme of finite Kurll dimension and X has a classical generator, then f is descendable.*

Proof. By Pat Lank[30] Theorem 4.1, such a morphism preserves classical generators, and thus descendable by 5.2.20. □

The following observation is due to Neeman[5] see Lemma 2.6, Lemma 2.7: If $D_{qcoh}(X)$ has a strong $\oplus$-generator, then $D^b_{coh}(X)$ has a strong generator. It allows us to relate descendable morphisms and strong generators for the bounded derived categories of coherent sheaves.

Lemma 5.2.23. *Let X be a Noetherian scheme. If $D_{qcoh}(X)$ admits a strong (resp. classical) generator G with $G \in D^b_{coh}(X)$, then G is a strong (resp. classical) generator of $D^b_{coh}(X)$.*

Proof. By assumption we have $D_{qcoh}(X) = \overline{\langle G \rangle}_n$ for some n, then

$$D^b_{coh}(X) = D^b_{coh}(X) \cap \overline{\langle G \rangle}_n \subset \langle G \rangle_{2n}$$

Where the inclusion is Neeman[5] Lemma 2.6. □

Remark 5.2.24. *We do not know whether the converse of Lemma 5.2.23 is true in general. But in*[4] *Theorem 7.38, Rouquier showed that when X is a separated scheme of finite type over a perfect field, then there is a $E \in D^b_{coh}(X)$ such that E is a strong generator for $D^b_{coh}(X)$ and a strong $\oplus$-generator of $D_{qcoh}(X)$.*

And in[5] *Theorem 2.3 Neeman showed that if X is a Noetherian scheme such that every closed closed subscheme admits a regular alteration (*[5] *Reminder 0.13, as examples, any regular schemes or schemes admits a resolution of singularities has regular alternations, de Jong showed in*[32] *that any schemes separated and finite type over an excellent scheme of dimension ≤ 2 has regular alternations), then there is a $E \in D^b_{coh}(X)$ such that E is a strong generator for $D^b_{coh}(X)$ and a strong $\oplus$-generator of $D_{qcoh}(X)$.*

Combining Lemma 5.2.1 Remark 5.2.2 Lemma 5.2.23 and Remark 5.2.24 we get the following:

Corollary 5.2.25. *Let X, Y be Noetherian schemes and let $f : X \to Y$ be a descendable morphism of finite type. If $D_{qcoh}(X)$ admits a strong $\oplus$-generator in $D^b_{coh}(X)$, then $D_{qcoh}(Y)$ admits a strong $\oplus$-generator F in $D^b_{coh}(Y)$, which implies that F is a strong generator of $D^b_{coh}(Y)$.*

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