Symmetric Surfaces in S^4

by

Malcolm H. Gabbard

B.A., Colorado College, 2019

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

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Manhattan, Kansas

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Abstract

Classical knots and their invariants provide key insights into many questions in low-dimensional topology. One invariant of knots crucial to the study of surfaces in 4-manifolds is 4-genus. We combine two variants of 4-genus (equivariant 4-genus and double-slice genus) to create a new variant of symmetric knots, which we call equivariant double-slice genus.

This dissertation works to initiate the study of this new knot invariant. Namely, we introduce the new invariant, prove elementary results about it, and prove a useful lower-bound for it. The lower bound we construct for the equivariant double-slice genus is easily computable and powerful enough to effectively distinguish the equivariant double-slice genus from existing knot invariants.

Additionally, using equivariant double-slice genus as the primary obstructive invariant, we begin to study equivariant embeddings of closed surfaces in the 4-sphere. Specifically, we prove the existence of equivariantly embedded 2-spheres in the 4-sphere which are isotopic but not equivariantly isotopic and remain equivariantly distinct after many internal stabilizations.

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Approved by:

Major Professor
Dr. David Auckly

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Classical knots and their invariants provide key insights into many questions in low-dimensional topology. One invariant of knots crucial to the study of surfaces in 4-manifolds is 4-genus. We combine two variants of 4-genus (equivariant 4-genus and double-slice genus) to create a new variant of symmetric knots, which we call equivariant double-slice genus.

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Acknowledgments

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Dedication

This work is for my friends and family, without whom it would not exist.

Chapter 1

Introduction

In recent years, properties of symmetric knots and the surfaces they bound have been well studied. Strongly invertible knots, in particular, have received considerable attention. A strongly invertible knot is a knot in S^3 invariant under an involution of S^3 with a fixed point set of S^1 intersecting the knot in two points. Recent research on strongly invertible knots has included many results about the equivariant concordance group [2, 6, 8, 9] and has included applications to non-equivariant questions [7].

One example of recent breakthroughs in studying strongly invertible knots comes from knot Floer homology. Since its introduction in the early 2000's, knot Floer homology has proven to produce powerful and computable invariants for 3-manifolds and knots. In 2022, the knot Floer homology of strongly invertible knots was studied by Dai-Mallick-Stoffregen [6], where they developed new and effective equivariant concordance invariants for strongly invertible knots. While there have been many such invariants studied in recent years, including invariants from Khovanov homology [21, 37] and g-signature [2], the work of Dai-Mallick-Stoffregen laid the groundwork for two particularly interesting results.

Of interest to people studying the equivariant concordance group, Alessio Di Prisa proved in [8], using Dai-Mallick-Stoffregen's invariants, that the equivariant concordance group is non-abelian. This is a stark contrast to the non-equivariant setting. Meanwhile, in the non-equivariant setting and using these same invariants, Dai-Kang-Mallick-Park-Stoffregen were able to close the case on a classical non-equivariant question. Namely, they proved that the $(2,1)$-cable of the figure 8 knot is not smoothly slice [7].

This second result highlights the ability to use progress in the equivariant setting to prove seemingly non-equivariant results. In the non-equivariant setting, one field of low-dimensional topology that has been flourishing is the study of closed surfaces in 4-manifolds. Particularly, there has been consistent progress in the search for topologically isotopic but smoothly distinct surfaces in various 4-manifolds. There has been a flood of new techniques coming from gauge theory, Khovanov homology, Floer theory and more which have allowed for the construction of interesting exotica.

One example of recent progress in knotted surfaces is the work of Konno-Mallick-Taniguchi [16]. In this work, they construct exotic embeddings of $\mathbb{R}P^2$ into simply connected 4-manifolds. While there are other such constructions, they are the first to construct such exotic embeddings which additionally satisfy that the complements of the embeddings are diffeomorphic. This is accomplished using Seiberg-Witten generalizations of Donaldson's diagonalization theorem. This work maps out yet another corner of the landscape of exotically knotted surfaces. Our current understanding of exotically knotted surfaces includes many interesting examples, such as knotted surfaces in simply connected 4-manifolds which remain distinct after many internal stabilizations [1]. While we have made much progress in understanding embeddings of surfaces, one major question which is still unanswered is one of the most important: are there exotically knotted spheres in S^4?

With the goal of learning more about knotted surfaces in S^4, we restrict our attention to properties of closed symmetric surfaces in S^4. To do this, we define new equivariant notions of classical knot and surface invariants. The first new concept we study is the idea of *equivariant double-slice genus*, first defined by the author in [11]. The equivariant double-slice genus of a strongly invertible knot (K,τ), which we denote $\tilde{g}_{ds}(K,\tau)$, can be thought of as the minimum genus of a symmetrically unknotted surface with cross-section K (see Chapter 3 for a precise definition). This combines the notion of equivariant 4-genus, $\tilde{g}_4(K,\tau)$, and the classical notion of double-slice genus, $g_{ds}(K)$, both of which are described in detail in Chapter 2. By combining these ideas, we provide a pathway to studying symmetric closed surfaces in S^4.

Specifically, we develop tools to study both symmetric surfaces with boundary up to equivariant isotopy rel boundary, as well as closed symmetric surfaces in (S^4, τ), for smooth involutions τ. We do this by first proving non-equivariant bounds for stabilization distance. There are many notions of stabilization distance, however we focus on internal stabilization $d_1(\Sigma_1, \Sigma_2)$ which measures the number of embedded 1-handles needed to be attached to Σ_1 and Σ_2 (surfaces in B^4 or S^4) to become isotopic. We discuss this notion of stabilization, as well as others, in Chapter 4.

For surfaces with boundary in B^4, we prove the following two results in Section 4.1 using the super-slice genus, $g_{ss}(K)$, which is a variation of the double-slice genus discussed in Chapter 2.

Theorem 1.1. *Let Σ_1 and Σ_2 be properly embedded genus h surfaces with boundary $K \subset S^3$ such that $\Sigma_1 \cup_K \Sigma_2 \subset (B^4, \Sigma_1) \cup_{(S^3,K)} (B^4, \Sigma_2)$ is unknotted. Then $d_1(\Sigma_1, \Sigma_2) \geq g_{ss}(K) - h$.*

By letting K be double-slice, this immediately yields the following corollary:

Corollary 1.2. *Let K be double-slice with $g_{ss}(K) = n$, then K admits slice disks D_1 and D_2 such that $d_1(D_1, D_2) \geq n$.*

This bound is sufficient for proving large families of disks are non-isotopic. Additionally, we use these bounds to reprove that it may take large numbers of stabilization for surfaces to become isotopic. This result is not new. However, in conjunction with our new notion of equivariant double-slice genus, it provides insight into a possible pathway to study equivariant stabilization distance. Namely, in Section 4.2 we prove the following equivariant version of Theorem 1.1:

Theorem 1.3. *Let $\Sigma_1, \Sigma_2 \subset B^4$ be properly embedded genus h surfaces with boundary K which are both $\overline{\tau}$-invariant. If $\Sigma_1 \cup_K -\Sigma_2 \subset (B^4, \Sigma_1) \cup_{(S^3,K)} (B^4, \Sigma_2)$ is equivariantly unknotted, then $\tilde{d}_1^{\tau}(\Sigma_1, \Sigma_2) \geq \frac{\tilde{g}_{ss}(K,\tau)}{2} - h$.*

In order to begin to identify knots with the desired equivariant double-slice properties, we develop an effective lower bound for equivariant double-slice genus. Using properties of involutions on handlebodies proven in [15], in Section 3.1 we discuss decompositions of involutions on embedded handlebodies in S^4. Using these results, we prove the following lower bound for equivariant double-slice genus in Section 3.2:

Theorem 1.4. *Let (K, τ) be a strongly invertible knot and let K_0 and K_1 be the knots formed from an arc of K union the half-axis h_0 and h_1 respectively. Then:*

$$\max\{g_{ds}(K_0), g_{ds}(K_1)\} \leq \tilde{g}_{ds}(K, \tau).$$

Using this lower bound, we are able to sufficiently distinguish the equivariant double-slice genus of a knot from its double-slice genus and its equivariant 4-genus. Specifically, Theorem 1.4 allows us to import the work of Orson and Powell [26] on using signature invariants to bound double-slice genus in the non-equivariant setting (discussed in Section 2.1) to answer equivariant questions. The main construction, shown in Figure 1.1, is proven in Section 3.2 to have the following properties:

Theorem 1.5. *The knot (K_n, τ) depicted in Figure 1.1 satisfies the following:*

1. *K_n is double-slice,*
2. *(K_n, τ) is equivariantly slice,*
3. *$\tilde{g}_{ds}(K_n, \tau) \geq n$.*

These results, combined with our ability to use double-slice and super-slice genus to bound stabilization distance, allow us to begin the study of equivariant stabilization distance for surfaces in S^4. Notably, we are able to prove novel results about equivariant stabilization distance for closed surfaces in S^4:

Theorem 1.6. *For every $n \in \mathbb{N}$ and every involution τ of S^4 with 2-dimensional fixed point set, there exists a τ invariant sphere S^2_n which is unknotted but has equivariant internal stabilization distance from the standard equivariant unknot at least n.*

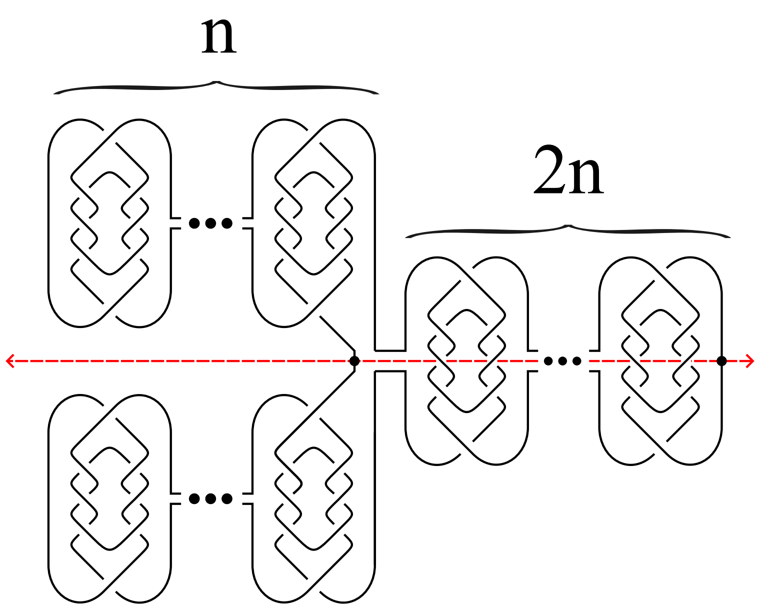

Figure 1.1: (K_n, τ)

This result is proved in Section 4.2.2 and highlights the possibility of more interesting novel behavior of equivariantly knotted surfaces in S^4. For example, this result shows that there are infinitely many strongly invertible 2-spheres in S^4 which are unknotted but not equivariantly unknotted. This fact is in stark contrast to strongly invertible classical knots, of which there is only one up to equivariant isotopy [22].

Chapter 2

Background

For the sake of clarity, we will detail here the conventions, notation, and assumptions that will be made for the remainder of this dissertation.

Unless otherwise stated, all manifolds, submanifolds, and maps are smooth. Similarly, all embeddings of manifolds with boundary are smooth and proper. The locally flat category will be briefly discussed in Section 3.3.

As for notation, we employ the following conventions:

- X is a 4-Manifold
- M is a 3-Manifold, with M_k denoting the double branched cover of S^3 along a knot K
- Σ_g is a genus g surface (possibly with boundary)
- K is a knot, i.e. the oriented image of an embedding of S^1 into S^3
- rK is the *reverse* of a knot K, i.e. K with the opposite orientation
- mK is the *mirror* of a knot K, i.e. the image of K under a reflection of S^3
- $-K$ is the *inverse* of a knot K, i.e. the reverse of the mirror
- τ is a smooth involution on either B^4 or S^4

In the following sections we will review definitions and properties related to double-slice genus and equivariant 4-genus.

2.1 Double-Slice and Super-Slice Genus

In this section, we provide a cursory introduction to both the double-slice genus and super-slice genus of a knot, highlighting key facts that will be relevant in future discussions.

First, recall that a surface $\Sigma_g \subset S^4$ is *unknotted* if it is the boundary of an embedded genus g handlebody. Since all handlebodies in S^4 are isotopic, this definition of unknotted is equivalent to being isotopic to some standard embedding of a surface. This idea of having a "standard" embedding will be less clear in the equivariant setting and possibly fail all together, making it important to distinguish the two notions for the time being. The importance of this distinction is highlighted by Theorem 1.6.

Given an unknotted surface in S^4, we consider 1-dimensional cross-sections of this surface. We call knots which appear in such a way *double-slice*:

Definition 2.1. The *double-slice genus* of a knot K, denoted $g_{ds}(K)$, is the minimal genus of an unknotted surface $\Sigma \subset S^4$ such that Σ intersects an equatorial S^3 transversely with intersection K. If $g_{ds}(K) = 0$, we say K is *double-slice.*

The idea of a knot being double-slice has been around for a long time, first being described in the language of concordance as being *invertibly concordant* to the unknot. While we do not use the language of concordance in the primary results of this thesis, we briefly review the notion here to provide additional context for the discussions of double-slice, equivariantly slice, and equivariant double-slice knots. We begin by recalling the classical definition of concordance:

Definition 2.2. Two knots $K_0, K_1 \subset S^3$ are *concordant* if there exists a properly embedded annulus $C = S^1 \times I \subset S^3 \times I$ with $K_0 = S^1 \times \{0\}$ and $-K_1 = S^1 \times \{1\}$.

If a knot K is concordant to the unknot, then K is slice. This can be seen by taking a concordance from K to the unknot and then capping off the concordance with a disk bounded by K. We can similarly translate being double-slice into the language of concordance by considering *invertible concordance*:

Definition 2.3. We say K_0 is *invertibly concordant* to K_1 if there exists a concordance C_1 from K_1 to K_0 and a concordance C_2 from K_0 to K_1 such that the concordance $C_1 \cup_{K_0} C_2$ from K_1 to itself is isotopic rel boundary to the trivial concordance from K_1 to itself.

A knot is invertibly concordant to the unknot, then, if and only if it is double-slice. Similar to the concordance group, there is an invertible concordance group. We will not discuss the invertible concordance group here, however we do highlight one interesting open question related to it.

Question 1. *Given two knots K_1 and K_2 satisfying K_1 and $K_1 \# K_2$ are double-slice, is K_2 invertibly concordant to the unknot?*

This means that, at the moment, the invertible concordance group is actually better called the stably invertible concordance group, as it considers knots equivalent up to both invertible concordance and connect sum with a knot invertibly concordant to the unknot.

This language of invertible concordance was how double-slice knots were first discussed and lends itself well to discussions of the concordance group and invertible concordance group. In this way, concordance and invertible concordance are very useful in discussing obstructions to being double-slice. However, the language of concordance is slightly less well tailored for discussions of genus, which is why it took much longer for notions of double-slice genus to be discussed and studied. Specifically, the notion of double-slice genus was first introduced by Livingston and Meier in 2015 [19]. In this work, they discuss many obstructions to being equivariantly double-slice and give a basic coverage of the notion of genus, with truly effective lower bounds being developed only more recently in work such as Chen's paper [4] using Casson-Gordon invariants. We highlight here a particularly useful lower bound for double-slice genus from signature invariants by Orson-Powell [26] to which we will make frequent reference:

Theorem 2.4 (Orson and Powell). *Let K be a knot in S^3 and $\sigma_\omega(K)$ be its signature function, then*

$$g_{ds}(K) \geq \max|\sigma_\omega(K)|.$$

We will review here some basic facts about knot signature and the knot signature function, which is a generalization of the knot signature. The classical knot signature is a useful invariant which provides lower bounds for the classical 4-genus. The knot signature can be calculated simply from a Seifert matrix V_K for a knot K by taking the signature of the matrix $V_K + V_K^T$. The signature was first defined by Trotter in 1962 [35] and Milnor in 1968 [25], with the definitions being shown to be equivalent by Erle in 1969 [10].

Through these works, it was shown that the signature is a concordance invariant, meaning that it vanishes for slice knots. While this is a useful obstruction to knots being slice, it means that the classical signature is unhelpful for distinguishing double-slice knots from slice knots. Thankfully, the signature function is a stronger invariant. The signature function is not a concordance invariant and, therefore, we may find examples of slice knots where the signature function is not identically 0. This makes it powerful enough to identify slice knots that are not double-slice.

The signature function, sometimes called the Tristam-Levine signature function, was first discussed separately by Tristam [34] and Levine [17] in 1969. The signature function generalizes the classical signature by, instead of taking the signature of $V_K + V_K^T$, considering the signature of the following matrix

$$M_K(z) := (1-z)V_K + (1-\overline{z})V_K^T.$$

Here, z is a complex unit not equal to 1. This produces a function $\sigma_K(z)$ associated to a knot K which takes in values $z \in S^1/\{1\}$ and outputs the signature of $M_K(z)$, which is well defined as $M_K(z)$ is a hermitian matrix. The signature function of the knot will be a piecewise constant function which is only discontinuous at roots of the Alexander polynomial for the knot which fall on the unit circle.

While there are other invariants capable of identifying slice knots that are not double-slice

[19], this bound of Orson and Powell is particularly useful because the signature function is easily computable from a knot diagram. In general, given a knot diagram one can find a Seifert matrix, compute the matrix $M(z)$, find the roots of the Alexander polynomial and then compute the signature of $M_K(z)$ at, and between, the roots of the Alexander polynomial. In work done by the author for KnotInfo, all signature functions for knots up to 13 crossings were calculated, and are now available [20].

In addition to being algorithmically obtainable from a given knot diagram, the signature function is easily calculable for composite knots, as it is additive under connect sum. This is beneficial in constructions essential to Theorem 1.5, as it allows for one to grow the signature function, and therefore the double-slice genus, arbitrarily large. Namely, because the signature function is additive, Orson and Powell were able to construct a variety of families of knots with large double-slice genus. In Example 2.1 below, we review one such family that will be relevant in future discussions.

Example 2.1. As mentioned before, one important aspect of the signature function is that it is additive under connect sum. For this reason, we may pick a knot with non-zero signature function and connect sum it with itself to grow its signature function and, therefore, its double-slice genus. Consider the knot $K = 8_{20}$. K is a slice knot, with slice disk depicted in Figure 2.1.

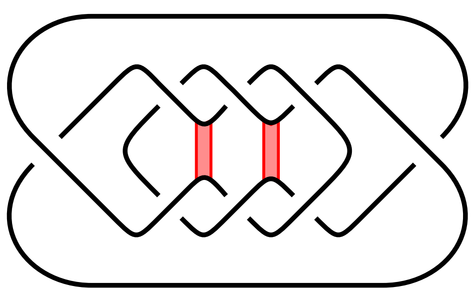

Figure 2.1: A symmetric slice disk for the knot 8_{20}.

Additionally, we can calculate that the signature function of K is not identically 0 as follows. Consider the Seifert matrix for K given here:

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Then:

$$M_K(z) = (1-z)\begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} + (1-\overline{z})\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z-1 & z-1 & z-1 & z-1 \\ 0 & 0 & z-1 & z-1 \\ 0 & z-1 & 0 & z-1 \\ 0 & 0 & z-1 & 0 \end{bmatrix} + \begin{bmatrix} \overline{z}-1 & 0 & 0 & 0 \\ \overline{z}-1 & 0 & \overline{z}-1 & 0 \\ \overline{z}-1 & \overline{z}-1 & 0 & \overline{z}-1 \\ \overline{z}-1 & \overline{z}-1 & \overline{z}-1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z+\overline{z}-2 & z-1 & z-1 & z-1 \\ \overline{z}-1 & 0 & z+\overline{z}-2 & z-1 \\ \overline{z}-1 & z+\overline{z}-2 & 0 & z-1 \\ \overline{z}-1 & \overline{z}-1 & z+\overline{z}-2 & 0 \end{bmatrix}.$$

Since the signature function will be piecewise constant with discontinuities at roots of the Alexander polynomial falling on the unit circle, we now find the roots of the Alexander polynomial. This can be calculated directly, however for simplicity we use the Alexander polynomial provided by KnotInfo, which is $\Delta_K(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$. The roots of $\Delta_K(t)$ are $e^{\pm i\frac{\pi}{3}}$. Considering $M_K(e^{i\frac{\pi}{3}})$ we get the following matrix:

$$\begin{bmatrix} e^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}-2 & e^{i\frac{\pi}{3}}-1 & e^{i\frac{\pi}{3}}-1 & e^{i\frac{\pi}{3}}-1 \\ e^{-i\frac{\pi}{3}}-1 & 0 & e^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}-2 & e^{i\frac{\pi}{3}}-1 \\ e^{-i\frac{\pi}{3}}-1 & e^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}-2 & 0 & e^{i\frac{\pi}{3}}-1 \\ e^{-i\frac{\pi}{3}}-1 & e^{-i\frac{\pi}{3}}-1 & e^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}-2 & 0 \end{bmatrix}.$$

Using Python, we calculate the signature of this matrix to be 1. We may similarly calculate the signature at $z = e^{-i\frac{\pi}{3}}$ (which is identical) and on the intervals between the roots, which have signature 0. Thus, the signature function of K is given by $\sigma_K(z) = 0$ for $z \neq e^{\pm i\frac{\pi}{3}}$ and 1 for $z = e^{\pm i\frac{\pi}{3}}$. Therefore, by Theorem 1.4, $g_{ds}(K) \geq 1$.

Now let $K_n = \#_n K$, depicted in Figure 2.2. Since K is slice, so is K_n by simply taking a boundary sum of the slice disks for K. By the additivity of the signature function, we have that $\sigma_{K_n}(e^{i\frac{\pi}{3}}) = n$. Therefore, by Theorem 2.4, $g_{ds}(K_n) \geq n$. In their paper, Orson and Powell show that the double-slice genus of K_n is exactly n by finding an explicit genus g double-slicing of it [26].

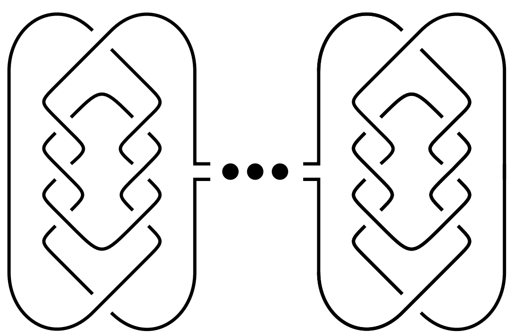

Figure 2.2: K_n, i.e. n copies of 8_{20} summed together.

One common misconception for a double-slicing surface of a knot K is that it is the relative double of a surface $\Sigma \subset B^4$ about $K \subset S^3$, i.e. that it is symmetric along K. This need not be, and oft is not, the case. Instead, the slicing surface could decompose along K into surfaces of different genus or non-isotopic surfaces of the same genus. If, however, we restrict to the case where the closed surface is the double of a surface bounded by K, we get the following definition from Chen in [4]:

Definition 2.5. The *super-slice genus*, $g_{ss}(K)$, of a knot K is the minimal genus of an unknotted surface $\Sigma \subset S^4$ such that $(S^4, \Sigma) = (B^4, \Sigma_1) \cup_{(S^3,K)} (B^4, -\Sigma_1)$. If $g_{ss}(K) = 0$, we say K is *super-slice*.

It is important to note that this definition of super-slice genus differs from Chen's original definition by a factor of two. Namely, Chen defines the super-slice genus as the genus of Σ_1 in the above construction, not Σ. Here, we make the choice of defining the super-slice genus via Σ instead, so that the following natural inequality holds: $g_{ss}(K) \geq g_{ds}(K)$.

Similar to double-slice knots, super-slice knots have been studied for a long time, with super-slice genus only being introduced recently. Of interest to us is the following lower bound for the super-slice genus coming from Chen [4]:

Theorem 2.6 (Chen)**.** *Given a knot K, let M_k be the two-fold branched cover of S^3 along K. Let n be the minimum number of generators for $H_1(M_K; \mathbb{Z})$. Then $n \leq g_{ss}(K)$.*

We now give an example using Theorem 2.6 to bound super-slice genus. We make use of this example in Section 4.2.2 in conjunction with Theorem 3.8 to obstruct the existence of an equivariant isotopy between two symmetric surfaces. While it might be possible to achieve the same result using double-slice genus and a different construction, the super-slice genus can be more useful for obstructions as super-slice knots appear to be notably rarer then the double-slice knots.

Example 2.2. Let K be the knot shown in Figure 2.3. We will find a lower bound for $g_{ss}(K)$ by calculating the homology of its double branched cover and applying Theorem 2.6. One way of calculating the homology of a double branched cover is to use a program like Snappy [5]. Here, however, we will walk through how to calculate the homology explicitly from a knot diagram.

To calculate the homology of the double branched cover, we compute the Goeritz matrix of the knot. The Goeritz matrix is useful here as it is a presentation matrix for the first homology of the double branched cover of K [18]. To calculate the Goeritz matrix, we first give a checkerboard coloring of the plane divided along K as seen on the right of Figure 2.3. We then assign to each crossing c a sign $e(c)$, with $e(c) = 1$ if there is a white region to the left before an underpass and $e(c) = -1$ otherwise (see Figure 2.4).

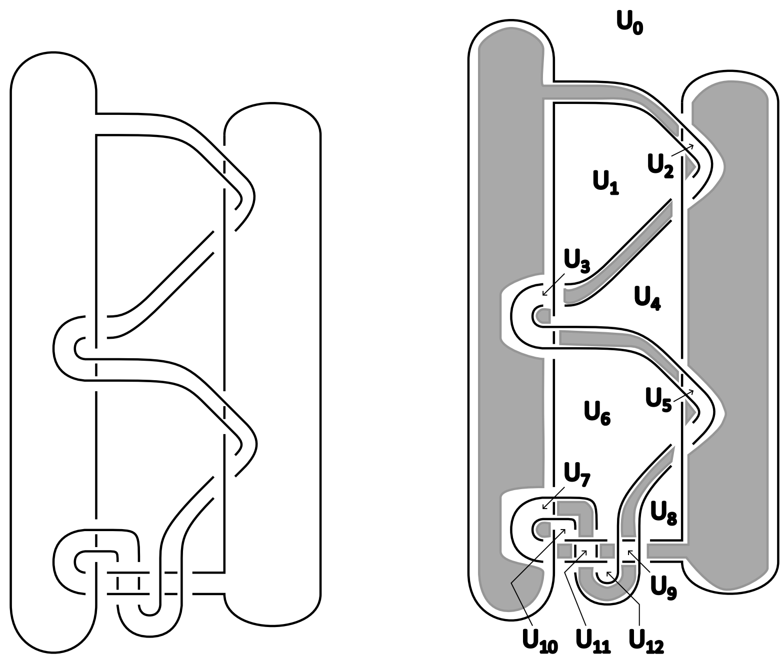

Figure 2.3: A knot K (left) with a labeled checker boarding (right).

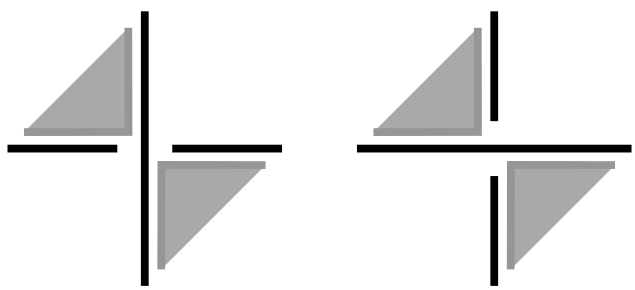

Figure 2.4: A crossing with $e(c) = -1$ (left) and a crossing with $e(c) = 1$ (right).

Let n be the number of white regions excluding the outer most region (U_0 in Figure 2.3), in this case $n = 12$. Then we can calculate the *pre-Goeritz matrix* by letting the off diagonal entries be

$$g_{ij} = \sum e(c)$$

with the sum taken over crossings where U_i and U_j meet. We then set the diagonal entries to be equal to

$$g_{ii} = -\sum_{j \neq i} e(c).$$

Lastly, to obtain the Goeritz matrix from the pre-Goeritz matrix, we delete the i-th row and column (it is standard to pick $i = 0$). With checker board and labels as in Figure 2.3, we get the following Goeritz matrix:

$$\begin{bmatrix}
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & -4 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 3 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2
\end{bmatrix}$$

This matrix is a group presentation matrix for the fundamental group, i.e. columns represent generators and rows represent relations. This presentation can be greatly reduced, for example row two says the fourth generator is just two of the first generator. After reducing the presentation additionally allowing elements to commute (as we are conserned with the homology as opposed the fundamental grou), we can identify the group as $\mathbb{Z}_5^2$. Therefore, by Theorem 2.6 we have that $g_{ss}(K) \geq 2$.

2.2 Equivariant 4-Genus

In this section, we review select introductory facts about symmetric knots and equivariant 4-genus. While knots may have many types of symmetry, in this dissertation we focus specifically on *strongly invertible knots*.

Definition 2.7. Given a knot $K \subset S^3$ and an involution τ of S^3 with fixed points set S^1, we say the pair (K, τ) is a strongly invertible knot if K is set-wise fixed by τ and $\text{fix}(\tau) \cap K$ is two points.

Given a strongly invertible knot (K, τ), it is sometimes helpful to further restrict to another object called a *directed strongly invertible knot*, denoted (K, τ, h), where h is a choice of an oriented arc of $\text{fix}(\tau)$ with endpoints on K, called a *half-axis*. For a given strongly invertible knot there are four choices of direction, and we have specific terminology relating them. Given a directed strongly invertible knot (K, τ, h), we can construct its *axis-inverse* as $i(K, \tau, h) = (K, \tau, -h)$ (where $-h$ is h with the opposite orientation) and its *antipode* $a(K, \tau, h) = (K, \tau, h')$ (where h' is the other choice of half-axis with orientation compatible with h).

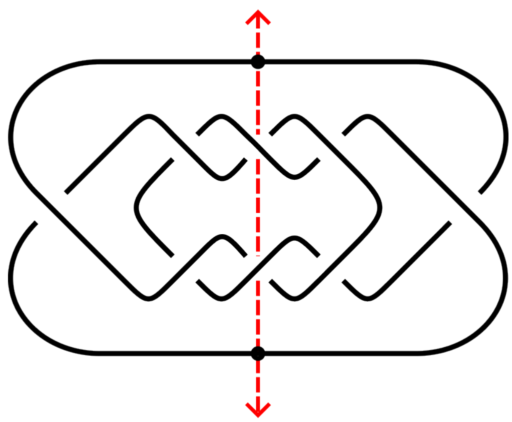

Figure 2.5: $(8_{20}, \tau)$.

Example 2.3. In Figure 2.5, the knot 8_{20} with a choice of symmetry τ can be seen, defining a strongly invertible knot $(8_{20}, \tau)$. In Figure 2.6, all four possible directed strongly invertible knots we can obtain from $(8_{20}, \tau)$ are depicted.

While there are many interesting aspects of symmetric knots worth studying, we focus primarily on how their symmetry can be extended to surfaces in 4-manifolds. The primary way this has been done classically is through the study of the *equivariant 4-genus*.

Definition 2.8. Given a strongly invertible knot (K, τ), the *equivariant 4-genus*, denoted $\tilde{g}_4(K, \tau)$, is the minimal genus of a properly embedded surface $\Sigma \subset B^4$ with $\partial\Sigma = K$ such that for some extension $\overline{\tau} : B^4 \to B^4$ of τ, $\overline{\tau}(\Sigma) = \Sigma$.

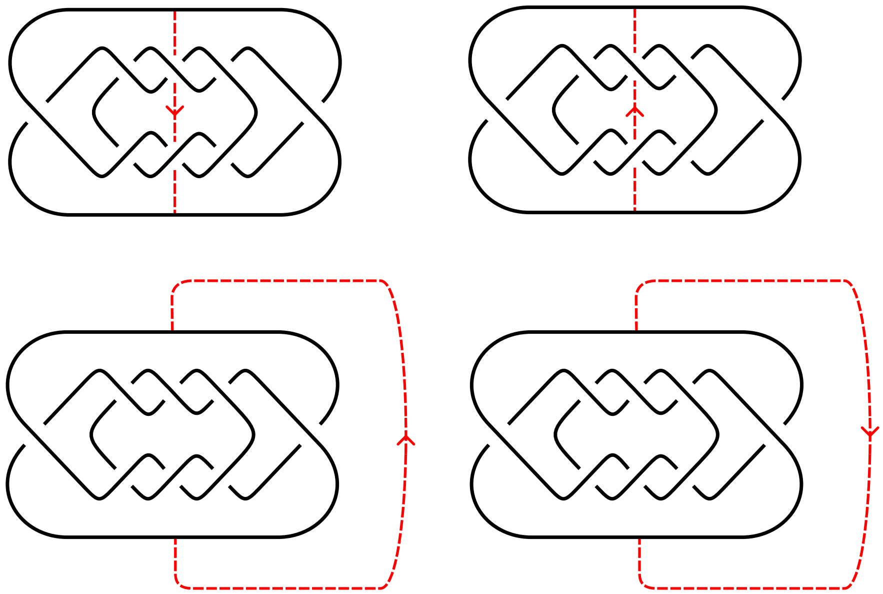

Figure 2.6: A directed strongly invertible knot $(8_{20}, \tau, h)$ (top left), the axis-inverse of (K, τ, h) (top right), the antipode of (K, τ, h) (bottom left), the antipode of the axis-inverse of (K, τ, h) (bottom right).

This notion, first defined by Boyle and Issa in [2], is a generalization of the notion of being *equivariantly slice*, i.e. having $\tilde{g}_4(K, \tau) = 0$. Being equivariantly slice has been studied much longer then the equivariant 4-genus as, similar to being slice and double-slice, it can be translated into the language of concordance, specifically equivariant concordance.

Definition 2.9. Two directed strongly invertible knots, (K_0, τ_0, h_0) and (K_1, τ_1, h_1), are *equivariantly concordant* if there exists a concordance $C \subset S^3 \times I$ from K_0 to K_1 invariant under some involution $\overline{\tau}$ of $S^3 \times I$ satisfying:

1. $\overline{\tau}|_{S^3 \times \{i\}} = \tau_i$ for $i = 0, 1$,
2. h_0 and $-h_1$ induce the same orientation on $\text{fix}(\overline{\tau})$, and
3. h_0 and $-h_1$ are contained in the same component of $\text{Fix}(\tau) \backslash C)$.

A full explanation of this is given in Sakuma's work on the equivariant concordance group of strongly invertible knots [28]. For more thorough treatments of equivariant concordance, we refer the reader to [2, 3, 8, 9, 28], where many different obstructions to being equivariantly slice are defined and studied.

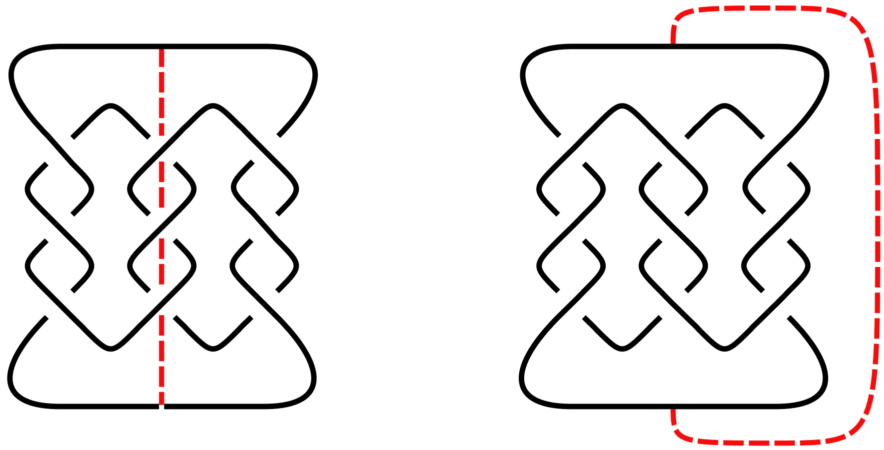

Figure 2.7: A directed strongly invertible knot $(9_{46}, \tau, h)$ and its inverse.

Similar to the case of 4-genus and double-slice genus, in the equivariant setting we have a notion of inverses. Given a directed strongly invertible knot (K, τ, h), its inverse is the knot $(-K, \tau, -h')$, which is the knot obtained by switching all crossings and picking the other half-axis with opposite orientation (see Figure 2.7). As we often do not care about a specific direction of a knot, we will regularly refer to the equivariant inverse of a strongly invertible knot (K, τ) as $(-K, \tau)$, omitting the choice of h. Note that these are the inverses in the equivariant concordance group as well.

One way to construct strongly invertible knots we will make consistent use of is doubling a classical knot. The *double* of a knot K is the strongly invertible knot $(D(K), \tau)$, where $D(K) = K\#rK$ and τ is the symmetry taking K to rK. One useful aspect of the doubling operation is that it allows us to relate the equivariant 4-genus of $(D(K), \tau)$ to the 4-genus of K via the following proposition:

Proposition 2.10. *Given a knot K with 4-genus $g_4(K) = n$, its double $(D(K), \tau)$ has equivariant 4-genus $\tilde{g}_4(D(K)) \leq 2n$.*

One application of this proposition we will use in future constructions is the following example:

Example 2.4. Let K be the knot 8_{20}. Note that 8_{20} is strongly invertible, and therefore equivalent to its reverse. Thus, its double $D(K) = K\#rK$ is exactly $K\#K$. $(D(K), \tau)$ can be seen in Figure 2.8. We can see that 8_{20} is slice by constructing an explicit slice disk for K as in Figure 2.1. Then, attaching the equivalent bands in $D(K)$ along with the image of the band under τ, we get the symmetric slice disk for $(D(K), \tau)$ seen in Figure 2.8.

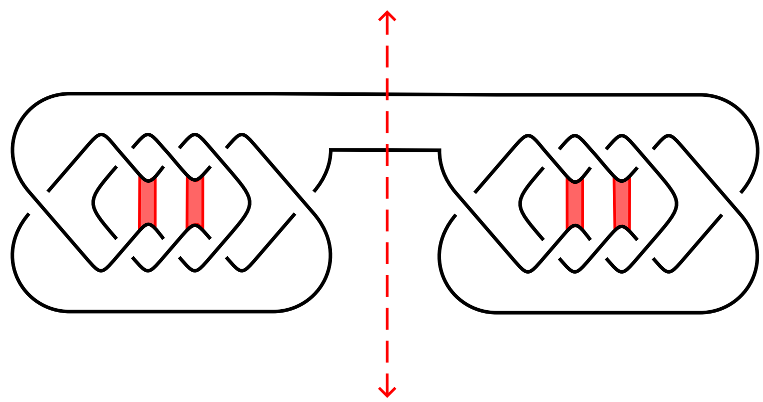

Figure 2.8: A band diagram of a symmetric slice disk for the strongly invertible knot $(D(8_{20}), \tau)$.

Chapter 3

Equivariant Double-Slice and Super-Slice Genus

As in the case of equivariant 4-genus, equivariant double-slice and super-slice genus are 4D invariants of symmetric knots which study extensions of the symmetry of S^3 to a 4-manifold. For equivariant 4-genus, this was an arbitrary extension to B^4. Now, however, we consider extensions to S^4. Letting τ be the standard rotation of S^3 around S^1, and viewing S^3 as an equator of S^4, we will refer to an orientation preserving extension of τ to S^4 as $\overline{\tau}$. Notably, one can check that this extension is a $\mathbb{Z}_2$ action on S^4 with fixed point set a, possibly knotted, S^2 intersecting S^3 transversely. While one could also consider orientation reversing extensions of τ, we will not study such extensions here.

We now briefly define a few objects which we will refer back to in future discussions. The following definition first appeared in the author's work [11].

Definition 3.1. Given a directed strongly invertible knot $(K, \tau, h) \subset S^3$ and a smooth extension $\overline{\tau}$ of τ to S^4, an *equivariant slicing handlebody* of (K, τ, h) is a handlebody $H \subset S^4$ such that $\overline{\tau}(H) = H$ and H intersects the standard S^3 containing K transversely with $h \subset H$ and $\partial H \cap S^3 = K$. We call ∂H an *equivariant slicing surface* and say that (K, τ, h) *divides* ∂H.

The existence of an equivariant slicing handlebody for a strongly invertible knot is guaranteed by taking an equivariant Seifert surface for the knot, which is guaranteed to exist by [2], and thickening it into S^4. Additionally, it is easy to check that $\overline{\tau}|_H$ is orientation reversing. With this, we are able to now define an equivariant notion of the double-slice genus. We begin by defining the equivariant double-slice genus of a directed strongly invertible knot, from which we can define the equivariant double-slice genus of a non-directed strongly invertible knot.

Definition 3.2. The *equivariant double-slice genus* of a directed strongly invertible knot (K, τ, h), which we denote $\tilde{g}_{ds}(K, \tau, h)$, is the minimal genus of an equivariant slicing handlebody for (K, τ, h). If $\tilde{g}_{ds}(K, \tau, h) = 0$, we say (K, τ, h) is *equivariantly double-slice.*

We can then define the *equivariant double-slice genus* of a strongly invertible knot (K, τ) to be the minimum of the equivariant double-slice genus of (K, τ, h_1) and (K, τ, h_2), where h_1 and h_2 are the two choices of half-axis for (K, τ). We can also then define the *equivariant super-slice genus* of a knot to be the minimal genus of an equivariant slicing handle-body for (K, τ) whose boundary is symmetric about K. In other words, the equivariant super-slice genus is the same as the super-slice genus simply requiring all surfaces and handlebodies to be $\overline{\tau}$-invariant for some orientation preserving extension of τ to S^4.

The existence of equivariant slicing handlebodies for a strongly invertible knot guarantees that the equivariant double-slice genus is finite. We immediately get the following bounds for equivariant double-slice genus:

$$g_{ds}(K) \leq \tilde{g}_{ds}(K, \tau) \quad \text{and} \quad 2\tilde{g}_4(K, \tau) \leq \tilde{g}_{ds}(K, \tau).$$

These lower bounds make it easy to distinguish the equivariant double-slice genus from the double-slice genus and from the equivariant 4-genus. However, they are not sufficient to simultaneously distinguish it from both.

Example 3.1. Consider the strongly invertible knot $(9_{46}, \tau)$ depicted in Figure 2.7. It is easy to check, either explicitly or by checking KnotInfo [20], that 9_{46} is double-slice. However, using g-signatures, Boyle and Issa show in [2] that $\tilde{g}_4(K, \tau) \neq 0$. Therefore, by the second of the two inequalities above, we have that $\tilde{g}_{ds}(K,\tau) \geq 2\tilde{g}_4(K,\tau) \geq 2 \gneq g_{ds}(K)$. If instead we considered the strongly invertible knot $(8_{20},\tau)$ in Figure 2.5, we could similarly find that while it is equivariantly slice, with an explicit equivariant slice disk shown in Figure 2.1, a quick search on KnotInfo verifies that its double-slice genus is non-zero. From the first of the two inequalities above, this gives that $\tilde{g}_{ds}(8_{20},\tau) \geq 1 > \tilde{g}_4(8_{20})$.

In order to simultaneously distinguish the equivariant double-slice genus of a knot from its double-slice genus and equivariant 4-genus, we will prove Theorem 1.4, which provides a lower bound relying on the symmetry of the equivariant slicing handlebody.

As with double-slice genus and equivariant 4-genus, one can similarly consider equivariant double-slice knots in terms of concordance. Namely, to be equivariant double-slice is to be *equivariantly invertibly concordant* to the unknot:

Definition 3.3. We say (K_0,τ_0,h_0) is *equivariantly invertibly concordant* to (K_1,τ_1,h_1) if there exists an equivariant concordance C_1 from (K_1,τ_1,h_1) to (K_0,τ_0,h_0) and a concordance C_2 from (K_0,τ_0,h_0) to (K_1,τ_1,h_1) such that the concordance $C_1 \cup_{K_0} C_2$ from K_1 to itself is equivariantly isotopic rel boundary to the trivial concordance from K_1 to itself.

Unlike equivariant concordance and invertible concordance, it is not currently known if there is a well-defined equivariant invertible concordance group. This is unknown even considered stably, as in the case of invertible concordance. While it seems entirely likely that there would be inverses for knots up to equivariant invertible concordance, it is not immediately obvious to the author how to verify their existence. The problem here is that the slice disks guaranteed to exist in the equivariant concordance setting are constructed fundamentally differently from the pairs of slice disks used to construct the unknotted 2-spheres in the invertible concordance setting.

3.1 Decompositions of Involutions of Handlebodies

In order to prove Theorem 1.4 we must first review some technical results about involutions on handlebodies. Namely, we make use of a decomposition of actions on handlebodies called the vertical-horizontal decompostion, defined and studied by Kalliongis and McCullough in [15].

An involution on a bundle $\Sigma \times I$ is said to be *vertical* if it can be written as $1_\Sigma \times r$ and *horizontal* if it can be written as $\sigma \times id$ for σ some involution of Σ, see Figure 3.1.

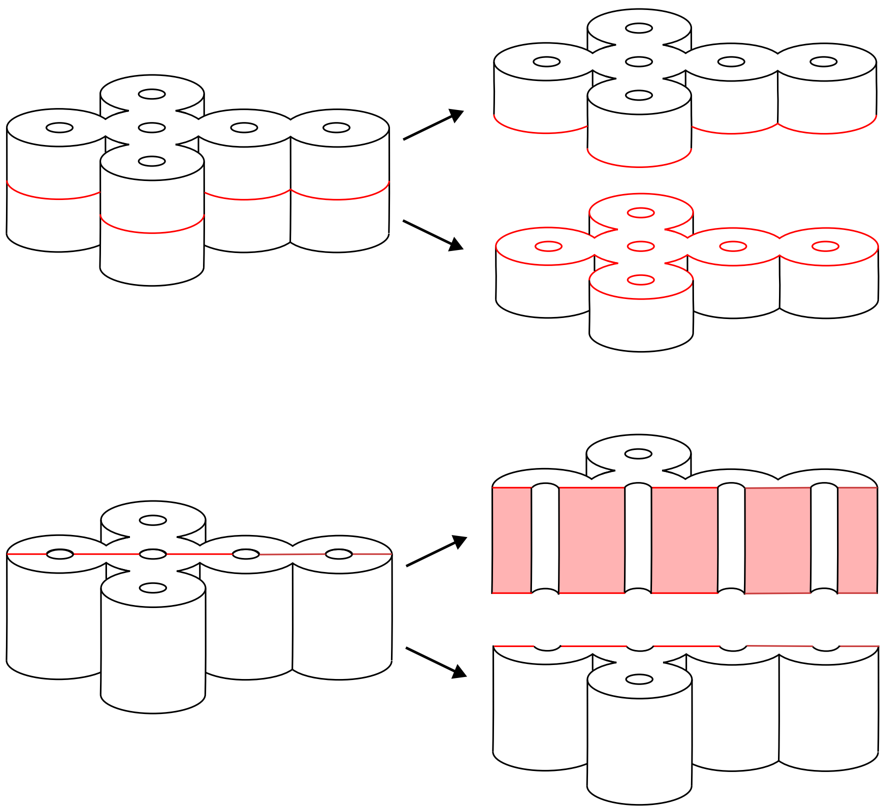

Figure 3.1: Example of vertical H_i components (above) and horizontal H_0 components (below) with fixed point set in red.

Theorem 3.4 (Theorem 5.1 in [15]). *Let τ be an orientation reversing involution of a handlebody H, and suppose that some component of fix(h) is 2-dimensional. Then H has a decomposition $H = H_0 \cup (\bigcup_{j=1}^{r} H_j)$, where each piece is h-invariant, such that*

1. *H_0 is an I-bundle over a connected 2-manifold, and the restriction of h to H_0 is horizontal. This action may be chosen to be a product action if and only if no component of fix(h) is a point or a Möbius band.*

2. *Each H_j is an I-bundle over a surface of negative Euler characteristic, which is a deformation retract of a component of fix(h), and the restriction of h to H_j is vertical.*

3. *$\{H_1, \ldots, H_r\}$ are pairwise disjoint, and for $1 \leq i \leq r$ each $H_0 \cap H_i$ is a single 2-disk.*

If we restrict our attention to equivariant slicing handlebodies we may further restrict this decomposition. In the case of an equivariant slicing handlebody, we get added restrictions on the vertical-horizontal decomposition coming from our knowledge of $\overline{\tau}$. We first notice the following property of the fixed point set of an equivariant slicing handlebody.

Lemma 3.5 ([11])**.** *The fixed point set of an equivariant slicing handlebody for a directed strongly invertible knot (K, τ, h) is the disjoint union of some number of planar surfaces.*

With this in mind, we are able to get the following stronger version of Theorem 3.4 in the case of equivariant slicing handlebodies. The following proposition and proof are from the author's work in [11].

Proposition 3.6. *Let H be an equivariant slicing handlebody for a directed strongly invertible knot (K, τ, h). Then H has a decomposition $H = H_0 \cup (\bigcup_{j=1}^{r} H_j)$, where each piece is τ-invariant, such that*

1. *H_0 is an I-bundle over a connected planar surface, and the restriction of τ to H_0 is horizontal and a product action.*

2. *Each H_j is an I-bundle over a planar surface which is not a disk or annulus, which is a deformation retract of a component of fix(τ), and the restriction of τ to H_j is vertical.*

3. *$\{H_1, \ldots, H_r\}$ are pairwise disjoint, and for $1 \leq i \leq r$ each $H_0 \cap H_i$ is a single 2-disk.*

Proof. We begin by verifying that the assumptions of Theorem 3.4 hold for an arbitrary equivariant slicing handlebody H, i.e. that $\tau|_H$ is orientation reversing and that $\text{fix}(\tau|_H)$ contains a 2-dimensional component.

Since $H \cap S^3$ contains an arc h of the fixpoint set, $\overline{\tau}$ preserves the tangent vector to h at a point of h. Since it setwise fixes the hemispheres of S^4, it also fixes the normal vector to

S^3 at that point. Furthermore, the fixed point set is two-dimensional, so we conclude that $\overline{\tau}|_H$ is orientation reversing.

The fact that $\text{fix}(\tau|_H)$ contains a 2-dimensional component comes from the fact that it contains a half-axis of $\text{fix}(\tau) \subset S^3$. Since this subset is 1-dimensional, and since $\tau|_H$ is orientation reversing, it must then be a subset of a 2-dimensional subset of $\text{fix}(\tau|_H)$. Thus, we know that H has a vertical horizontal decomposition as in Theorem 3.4.

We now look at the changes to (1), namely that H_0 is an I-bundle over a planar surface and that the restriction of h to H_0 is horizontal and a product action. The fact that H_0 is an I-bundle over a planar surface, as opposed to an arbitrary 2-manifold as in Theorem 3.4, is a direct result of Lemma 3.5. Similarly, by Lemma 3.5 no component of $\text{fix}(\tau)$ is a point or Mobius band. So by (1) of Theorem 3.4, we have that the restriction of τ to H_0 is horizontal and is a product action.

For (2) the only change made from Theorem 3.4 is noting that the only surfaces that can appear are, by Lemma 3.5, planar surfaces. By (2) of Theorem 3.4, each must have a negative Euler characteristic, hence is not a disk or annulus.

As there were no changes to (3), this completes the proof.

□

We are now ready to prove Theorems 1.4 and 1.5.

3.2 Proofs of Theorems 1.4 and 1.5

To prove Theorem 1.4, we first prove the following equivalent statement for directed strongly invertible knots. The following Theorem and proof are excerpted from the author's work in [11].

Theorem 3.7. *Let (K, τ, h) be a directed strongly invertible knot and K_0 be the union of h with an arc of K ending on the two fixed points. Then $g_{ds}(K_0) \leq \tilde{g}_{ds}(K, \tau, h)$.*

Proof. Let H be a minimal equivariant slicing handlebody for (K, τ, h). We will start by creating a decomposition of H which allows us to construct a useful slicing handlebody for

K_0.

We start by decomposing H into $H = H_0 \cup (\bigcup_{j=1}^{r} H_j)$ as described in Proposition 3.6. Using this decomposition, we will show that the fixed point set $\overline{\tau}|_H$, which we will abuse notation and refer to as simply τ, separates H into two identical components, H^1 and $H^2 = \tau(H^1)$. Smith proved that the fixed point set of an involution on a sphere is a $\mathbb{Z}_2 - \check{C}ech$ homology sphere [29–32]. This means an involution on a circle has zero or two fixed points. It follows by extending the involution on a planar surface Σ to an involution on S^2 that a 1-dimensional fixed point set of an involution on Σ is either S^1 or a collection of properly embedded arcs, either of which separates the surface into identical components.

Thus, since τ acts trivially on fibers of H_0, the fixed point set of τ separates H_0 into two identical components, H_0^1 and H_0^2, satisfying $g(H_0) \geq g(H_0^i)$ as shown in Figure 3.1. For the H_j, the fixed point set is a planar surface separating the H_j into identical components, H_j^1 and H_j^2, with $H_j^1 = \tau(H_j^2)$ and $g(H_j) = g(H_j^i)$ for $i = 1, 2$ as shown in Figure 3.1. Note that all the attaching regions in both H_0 and the H_j are disks containing an arc of the fixed point set which are identified. Since both fixed point sets separate, the fixed point set of $H_0 \cup H_j$ separates. Iterating, we get that the fixed point set of H separates it into two handlebodies, H^1 and H^2, with decompositions given by connect sums of the H_j^1 and H_j^2, respectively. Thus, we have:

$$g(H) = g(H_0) + \sum_{j=1}^{r} g(H_j) \geq g(H_0^i) + \sum_{j=1}^{r} g(H_j^i) = g(H^i).$$

By construction, the H^i are splitting handlebodies for the two "halves" of K corresponding to the two different arcs. □

As an immediate result of Theorem 3.7 and the definition of equivariant double-slice genus for non-directed strongly invertible knots, we get the following:

Theorem 1.4. *Let (K, τ) be a strongly invertible knot and let K_0 and K_1 be the knots formed from an arc of K union the half-axis h_0 and h_1 respectively. Then:*

$$\min\{g_{ds}(K_0), g_{ds}(K_1)\} \leq \tilde{g}_{ds}(K, \tau).$$

Proof. By the definition of equivariant double-slice genus for non-directed strongly invertible knots, $\tilde{g}_{ds}(K,\tau)$ is the minimum of $\tilde{g}_{ds}(K,\tau,h_1)$ and $\tilde{g}_{ds}(K,\tau,h_2)$. Therefore, letting K_0 and K_1 be as in the statement of the theorem, by Theorem 3.7 we get that $\tilde{g}_{ds}(K,\tau) \geq \min\{g_{ds}(K_0), g_{ds}(K_1)\}$. □

Additionally, we may consider equivariant super-slice genus bounds in much the same way, bounding it below by the non-equivariant superslice genus of the knot K_0.

Theorem 3.8. *Let (K,τ) be a strongly invertible knot and let K_0 and K_1 be the knots formed from an arc of K union the half-axis h_0 and h_1 respectively. Then:*

$$\min\{g_{ss}(K_0), g_{ss}(K_1)\} \leq \tilde{g}_{ss}(K,\tau).$$

The proof of Theorem 3.8 follows exactly the proof of Theorem 1.4 by taking all slicing handlebodies to be super-slicing handlebodies.

Using Theorem 1.4, we are now able to simultaneously distinguish equivariant double-slice genus from double-slice genus and equivariant 4-genus.

Theorem 1.5. *The knot (K_n,τ) depicted in Figure 1.1 satisfies the following:*

1. *K_n is double-slice,*

2. *(K_n,τ) is equivariantly slice,*

3. *$\tilde{g}_{ds}(K_n,\tau) \geq n$.*

The following proof is excerpted from the author's work in [11].

Proof. We will show the knot K_n pictured in Figure 1.1 satisfies the desired properties. To see that K_n is double-slice, we note that $K_n = (8_{20}\#-8_{20})^{2n}$. Since $8_{20}\#-8_{20}$ is double-slice, so is the $2n$ connect sum of it with itself, i.e. K_n. Now we show that it is equivariantly slice. First, note that 8_{20} is the pretzel knot $(3,-3,2)$ and so, by Sakuma [28], is equivariantly slice. Additionally note that since 8_{20} is slice, $D(8_{20})$ is equivariantly slice. Since K_n can be seen as an equivariant connect sum of $2n$ copies of 8_{20} and n copies of $D(8_{20})$, which we just

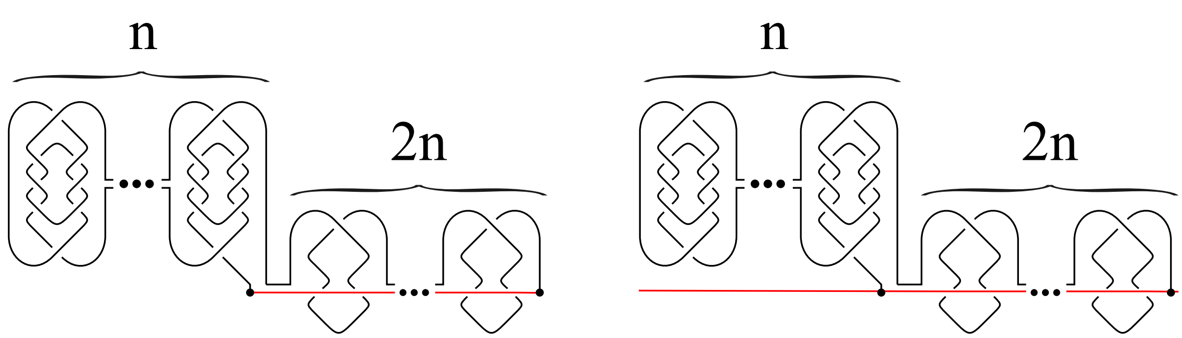

Figure 3.2: The decomposition of (K_n, τ) with both half-axis.

said are equivariantly slice, we know that K_n is also equivariantly slice. Lastly, we show that $\tilde{g}_{ds}(K_n, \tau, h_n) \geq n$. To do this, we first see in Figure 3.2 that, regardless of which half-axis we pick, the knot we get from taking an arc of K_n union the half-axis is $\#^n 8_{20}$. Thus, by Theorem 1.4, we have that, for either choice of half-axis, $\tilde{g}_{ds}(K_n, \tau, h_n) \geq g_{ds}(\#^n 8_{20})$. As discussed in Example 2.1, using signature invariants and the work of Orson-Powell [26], we can see $g_{ds}(\#^n 8_{20}) = n$. Thus, we get our last property, that $\tilde{g}_{ds}(K_n, \tau) \geq n$.

□

In Section 4.2 we will use these results to obstruct specific symmetric 2-spheres from being equivariantly isotopic. We now highlight an additional interesting example of a non-equivariantly double-slice knot.

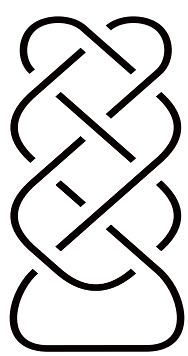

Figure 3.3: 8_9

Example 3.2. Here we use a fundamentally different construction from Theorem 1.5 to construct a double-slice and equivariantly slice knot which is not equivariantly double-slice.

Consider the knot 8_9 depicted in Figure 3.3. 8_9 is one of three knots under twelve crossings with an interesting combination of invariants (accompanied by $12a_{435}$ and $12a_{477}$). Specifically, 8_9 is slice, has double-slice genus 1, and is fully amphichiral (meaning it is isotopic to its reverse and mirror).

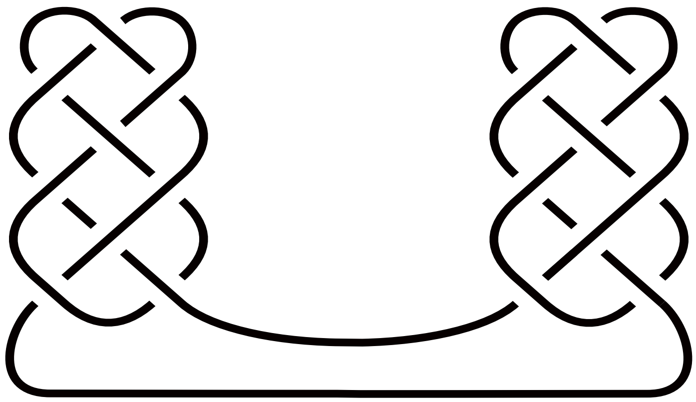

Figure 3.4: $D(8_9)$

Because 8_9 is isotopic to its reverse, we may consider the strongly invertible knot $(D(8_9), \tau)$ depicted in Figure 3.4. From Proposition 2.10, it is immediate that $\tilde{g}_4(D(8_9), \tau) = 0$. Similarly, because $g_{ds}(8_9) = 1$, by Theorem 1.4 we immediately get that $\tilde{g}_4(D(8_9), \tau) > 1$. Lastly, we can also see that the double-slice genus of $D(8_9)$ is zero, because 8_9 is fully amphichiral and therefore $D(8_9) = 8_9 \# -8_9$. Thus, we get another construction providing a simultaneous gap between the equivariant double-slice genus and the double-slice and equivariant genus.

Moreover, we may additionally take an equivariant invertible concordance of this knot $D(8_9)$ to get the strongly invertible knot shown in Figure 3.5. This satisfies all the conditions above, i.e. is double-slice and equivariantly slice but not equivariantly double-slice, and has the additional property of being hyperbolic. The fact that it is hyperbolic is confirmed computationally using SnapPy's verified algorithm to confirm hyperbolicity of a knot [5].

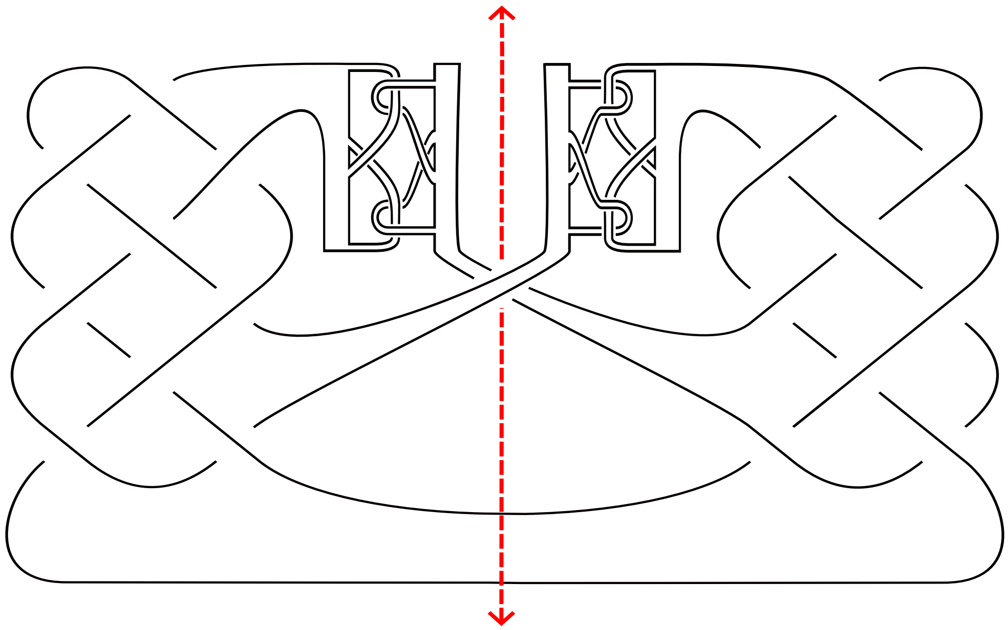

Figure 3.5: A hyperbolic knot equivariantly invertibly concordant to $(D(8_9), \tau)$.

When considering equivariantly super-slice knots, there are fewer options for construction. This is because, unlike double-slice knots, we cannot simply sum a knot with its inverse to get a super-slice knot. This, combined with the fact that they are rare among prime knots, makes constructions much more limited. In the following example, we consider a super-slice knot which will be central to constructions of symmetric surfaces in Chapter 4.

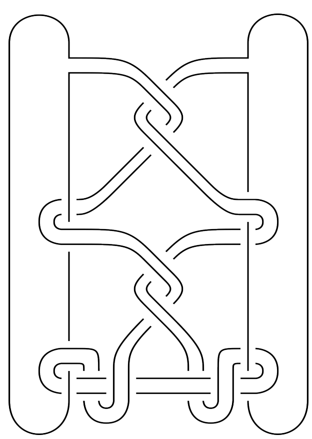

Figure 3.6: A strongly invertible knot (K, τ) which is super slice but not equivariantly super-slice.

Example 3.3. Consider the strongly invertible knot (K, τ) in Figure 3.6, with the obvious strong inversion. K is constructed by taking a symmetric band connecting two disjoint unknots. This produces a symmetric ribbon disk for K, meaning it is equivariantly slice. Moreover, because the core of the band is homotopically trivial in the complement of the two unknots, this means that K is super-slice [19].

To see that K is not equivariantly super-slice, we apply Theorem 3.8. When we construct the knot K_0 by taking half of K union one of its half-axis, we get the knot discussed in Example 2.2. As shown in the example, K_0 has super-slice genus at least two. Therefore, by Theorem 3.8, we have that K has equivariant super-slice genus at least two.

3.3 Similar Results in the Locally Flat Category

Considering the differences between locally flat and smooth 4-genus is an active and interesting field of study. In relation to the double-slice genus, one can define the *topological double-slice genus*, denoted $g_{ds}^{top}(K)$, exactly the same as the double-slice genus, simply requiring that the unknotted 2-sphere be locally flat instead of smooth.

In [23], Meier constructs an infinite family of topologically double-slice knots which are not smoothly double-slice. This was shown by analyzing correction terms from Heegaard Floer homology. This discrepancy between the locally flat and smooth genus is more easily detectable for the super-slice genus, where every knot with Alexander polynomial equal to 1 is topologically super-slice. In fact, Ruberman was able to construct smoothly double-slice, topologically super-slice knots which were not smoothly super-slice [27].

So far, we have been restricting our analysis of equivariant genus to the smooth setting. One reason for this is because smooth actions on S^4 are much more tractable then topological actions. That said, we may still analyze differences between the locally flat and smooth setting while keeping the actions themselves smooth. If we keep all actions smooth but allow the surfaces to be locally flat in the definition of equivariant double-slice and equivariant super-slice genus, we get what we call *equivariant topological double-slice genus* and *equivariant topological super-slice genus* which we denote $\tilde{g}_{ds}^{top}(K,\tau)$ and $\tilde{g}_{ss}^{top}(K,\tau)$, respectively.

Considering the equivariant topological double-slice and super-slice genus, we are able to get the following topological versions of Theorem 1.4 and 3.8:

Theorem 3.9. *Let (K, τ) be a strongly invertible knot and let K_0 and K_1 be the knots formed from an arc of K union the half-axis h_0 and h_1 respectively. Then:*

$$\min\{g_{ds}^{top}(K_0), g_{ds}^{top}(K_1)\} \leq \tilde{g}_{ds}^{top}(K, \tau).$$

and:

Theorem 3.10. *Let (K, τ) be a strongly invertible knot and let K_0 and K_1 be the knots formed from an arc of K union the half-axis h_0 and h_1 respectively. Then:*

$$\min\{g_{ss}^{top}(K_0), g_{ss}^{top}(K_1)\} \leq \tilde{g}_{ss}^{top}(K, \tau).$$

The proofs of both of these theorems follows exactly as in Theorem 1.4 and 3.8, as the construction only utilizes smoothness in the action. Notably, this makes Theorems 1.4 and 3.8 capable of detecting equivariant exotica.

One first goal in differentiating between topological and smooth equivariant double/super-slice genus is the construction of knots which are equivariantly topologically double/super-slice but not smoothly equivariantly double/super-slice. The easiest way to construct such a knot is by simply taking the double of a topologically double/super-slice knot. The resulting knot will then be equivariantly topologically double/super-slice, via a connect sum of the two double-slicing spheres, but cannot be smoothly double-slice by Theorems 1.4 and 3.8. We highlight one such construction here.

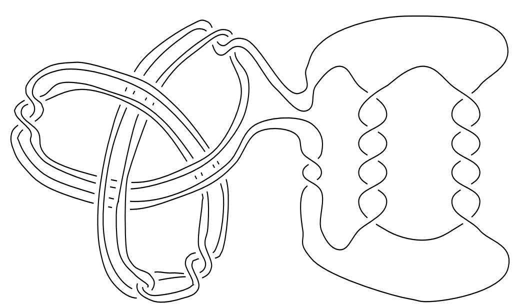

Figure 3.7: A topologically but not smoothly double-slice knot.

Example 3.4. Let K be the knot in Figure 3.7. It was shown by Meier in [23] that K is smoothly slice and topologically double-slice but not smoothly double-slice. When we take the double $(D(K), \tau)$ described in Figure 3.8, we will get interesting equivariant properties which we now explore.

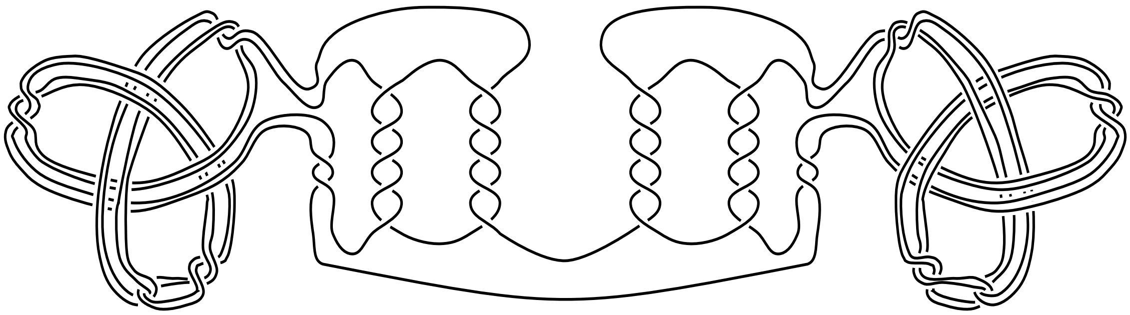

Figure 3.8: An equivariantly topologically double-slice but not equivariantly smoothly double-slice knot.

Since K was smoothly slice, by Proposition 2.10 we have that $(D(K), \tau)$ is equivariantly smoothly slice. Additionally, since K is topologically double-slice, we may construct an equivariant topological double-slicing sphere for $(D(K), \tau)$ by taking a connect sum of the slicing sphere for K and the equivalent slicing sphere for rK along the band used for the connect sum of the knots. Therefore, $\tilde{g}_{ds}^{top}(D(K), \tau) = 0$.

Lastly, we may apply Theorem 1.4 to show that $(D(K), \tau)$ is not equivariantly double-slice. This is immediate as the two knots K_0 and K_1 constructed from $(D(K), \tau)$ as described in Theorem 1.4 are exactly K which, by Meier [23], is not smoothly double-slice. Thus, $(D(K), \tau)$ is an example of a smoothly equivariantly slice, topologically equivariantly double-slice, but not smoothly equivariantly double-slice knot.

What would make the above example even more interesting would be if the knot was, in the non-equivariant setting, smoothly double-slice. It is unknown if such a knot could exist, but it would be interesting if it did.

Question 2. *Does there exist a strongly invertible knot* (K, τ) *with* $g_{ds}(K) = \tilde{g}_4(K, \tau) = \tilde{g}_{ds}^{top}(K, \tau) = 0$ *but* $\tilde{g}_{ds}(K, \tau) > 0$*?*

One possible construction of a knot answering Question 2 in the affirmative would follow as in Example 3.4, but starting with a fully amphichiral knot. Then, the double of the knot would be double-slice and all of the other properties would follow as in Example 3.4. Unfortunately, to the best of the author's knowledge, it is not currently known if such a fully amphichiral knot exists.

As mentioned above, this brief analysis is only a half-step into the topological setting, as the action is still smooth. It might be possible to extend the analysis of this dissertation to more general topological actions. At this time, however, this is the extent of the work done in the topological setting by the author and further analysis of the topological setting will be reserved for future works.

Chapter 4

Stabilization and Equivariant Isotopy

One well-studied property of exotic (homeomorphic but not diffeomorphic) 4-manifolds is stabilization distance. That is, for homeomorphic orientable 4-manifolds X_1 and X_2, the smallest n such that $X_1 \# n(S^2 \times S^2)$ is diffeomorphic to $X_2 \# n(S^2 \times S^2)$. By [36], n is finite. Of relevance to this dissertation is various notions of stabilization distances for embedded surfaces. In Section 4.1 we review various notions of stabilization distance for embedded surfaces, prove Theorem 1.1, and look at a variety of interesting examples. In Section 4.2 we introduce equivariant isotopy of surfaces in S^4, define new notions of equivariant stabilization distance, and use our previous results about equivariant double-slice genus to prove Theorem 1.3 and construct other interesting symmetric surfaces in S^4.

4.1 Non-Equivariant Stabilization Bounds

Given two surfaces Σ_1 and Σ_2 in S^4, we have various notions of equivalence, such as homotopy, relative diffeomorphism, topological ambient isotopy, smooth ambient isotopy, etc. Given two surfaces which are not equivalent in your equivalence of choice, it is often useful to ask just how in-equivalent they are. One way that we do this is by defining some notion of stabilization after which our surfaces become equivalent. While there are many different versions of stabilization for surfaces, we are interested specifically in stabilizations which, eventually, make non-isotopic surfaces become isotopic. The two specific versions of stabilization we care about are the following:

Definition 4.1. Let $\Sigma_1, \Sigma_2 \subset S^4$ be genus g surfaces, then the *1-handle stabilization distance* $d_1(\Sigma_1, \Sigma_2)$ is the minimal number of smoothly embedded orientation preserving 1-handles $\{h_{1,i}\}$ and $\{h_{2,i}\}$ attached to Σ_1, Σ_2 respectively such that $\Sigma_1 \cup \{h_{1,i}\}$ and $\Sigma_2 \cup \{h_{2,i}\}$ are isotopic.

Slightly more restrictive, we also consider the following version of stabilization:

Definition 4.2. Let $\Sigma_1, \Sigma_2 \subset S^4$ be genus g surfaces, then the *internal stabilization distance* $d(\Sigma_1, \Sigma_2)$ is the minimal number of smoothly embedded orientation preserving 1-handles $\{h_{1,i}\}$ and $\{h_{2,i}\}$ attached in a local chart to Σ_1, Σ_2 respectively such that $\Sigma_1 \cup \{h_{1,i}\}$ and $\Sigma_2 \cup \{h_{2,i}\}$ are isotopic.

We will revisit these exact notions of stabilization in the following section, where we define equivariant versions of them. For the moment, though, we modify these notions slightly to work for surfaces with boundary in B^4. For properly embedded surfaces with boundary in B^4, we understand *1-handle stabilization* and *internal stabilization* to mean the same as above, with all attaching regions taken to be away from the boundary and all isotopies to be isotopies rel boundary. These notions of stable equivalence of surfaces rel boundary have been studied by various others, including [12, 13, 24, 33]. We highlight some of the techniques previously used in the following example.

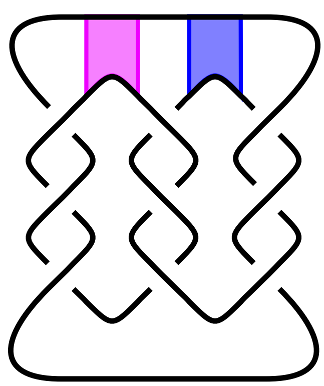

Figure 4.1: Two slice disks for 9_{46} which, together, form an unknotted 2-sphere in S^4.

Example 4.1. Consider the two slice disks for 9_{46} in Figure 4.1, each depicted by a different colored band. These slice disks can be shown to be non-isotopic in a variety of ways. In [12], their Khovanov homology is used. Specifically, Hayden and Sundberg analyze the maps that different slice disks induce on the Khovanov homology of 9_{46}, showing that each disk induces a distinct map on Khovanov homology and are therefore non-isotopic. More classically, using the language of [14], these disks can also be distinguished by their *peripheral map*. That is, they can be distinguished by their maps

$$h_D : \pi_1(S^3 \backslash K) \to \pi_1(D^4 \backslash D)$$

induced by the embedding of the slice disk D.

Miller and Powell [24] also reprove that these disks are distinct by similarly using Alexander modules, analyzing the kernels of the map

$$H_1(S^3 \backslash K; \mathbb{Q}[t^{\pm 1}]) \to H_1(D^4 \backslash D; \mathbb{Q}[t^{\pm 1}]).$$

Their result has the added benefit of being able to not just show that the disks are distinct, but bound their stabilization distance. This allows them to construct, for their Theorem B, pairs of disks with arbitrarily large stabilization distance by taking boundary sum of these disks.

We introduce here new bounds for stabilization distance for certain families of slice disks and surfaces, using the previously discussed double-slice and super-slice genus. The following theorem and proof are excerpted from the author's work in [11].

Theorem 1.1. *Let Σ_1 and Σ_2 be properly embedded genus h surfaces with boundary $K \subset S^3$ such that $\Sigma_1 \cup_K -\Sigma_2 \subset (B^4, \Sigma_1) \cup_{(S^3,K)} (B^4, -\Sigma_2)$ is unknotted. Then $d_1(\Sigma_1, \Sigma_2) \geq \frac{g_{ss}(K)}{2} - h$.*

Proof. Let $d = d_1(\Sigma_1, \Sigma_2)$ and let $\Sigma_1' = \Sigma_1 \cup \{h_i\}_{i=1}^d$ and $\Sigma_2' = \Sigma_2 \cup \{h_i'\}_{i=1}^d$ be stabilized surfaces that are isotopic rel boundary. Since $\Sigma_1 \cup_K -\Sigma_2 \subset (B^4, \Sigma_1) \cup_{(S^3,K)} (B^4, \Sigma_2)$ is unknotted, the stabilized surface $\Sigma' = \Sigma_1' \cup_K -\Sigma_2' \subset (B^4, \Sigma_1') \cup_{(S^3,K)} (B^4, \Sigma_2')$ is also unknotted, as it is an unknotted handlebody $\Sigma_1 \cup_K -\Sigma_2$ union handles. Since Σ_1' is isotopic rel boundary to Σ_2', we can isotope Σ' in S^4 rel B^4 (the hemisphere containing Σ_1') to get the double of (B^4, Σ_1'). This means that the double of (B^4, Σ_1') is a superslicing surface for K and therefore $h + d \geq \frac{g_{ss}(K)}{2}$, i.e. $d \geq \frac{g_{ss}(K)}{2} - h$. Note we have $\frac{g_{ss}(K)}{2}$, not $g_{ss}(K)$, because we want the genus of the surface bounded by the knot, not its double. Thus, $d_1(\Sigma_1, \Sigma_2) \geq \frac{g_{ss}(K)}{2} - h$. □

Corollary 1.2. *Let K be double-slice with $g_{ss}(K) = n$, then K admits slice disks D_1 and D_2 such that $d_1(D_1, D_2) \geq \frac{n}{2}$.*

Proof. If K is double-slice, it appears as the cross-section of an unknotted 2-sphere $S^2 \subset S^4$. We can then decompose (S^4, S^2) as $(B^4, D_1) \cup_{(S^3,K)} (B^4, -D_2)$. The disks D_1 and D_2 then satisfy the conditions of Theorem 1.1. Since they are genus 0 we then have $d_1(D_1, D_2) \geq \frac{g_{ss}(K)}{2}$. □

This corollary provides yet another obstruction to the two slice disks for 9_{46} in Example 4.1 being isotopic.

Example 4.2. Consider again the two slice-disks in Figure 4.1. Since, together, they form an unknotted 2-sphere, they satisfy the conditions of Theorem 1.1. Moreover, $H_1(M_{9_{46}}; \mathbb{Z}) = \mathbb{Z}_3^2$. This means, by Theorem 2.6, $g_{ss}(9_{46}) \geq 2$. Thus, by Theorem 1.1, we have that $d_1(D_1, D_2) \geq \frac{g_{ss}(9_{46})}{2} = 1$. Therefore, they are not isotopic.

In fact, looking to [19], we see a tabulation of double-slice knots with fewer then 13 crossings. From this tabulation, they identify 17 knots which are double-slice with non-zero Alexander polynomial. Moreover, for these 17 knots, they provide explicit band descriptions for the unknotted spheres of which the knots appear as cross-sections. Applying Corollary 1.2, this immediately yields the following corollary:

Corollary 4.3. *The band diagrams in [19] depict two non-isotopic disks for the knots:*

$$9_{46}, 10_{99}, 10_{123}, 10_{155}, 11n_{49}, 11n_{74}, 12a_{427}, 12a_{1105}, 12n_{268},$$

$$12n_{309}, 12n_{397}, 12n_{414}, 12n_{605}, 12n_{636}, 12n_{706}, 12n_{817}, 12n_{838}.$$

A quick search on KnotInfo will verify that for the following 15 of these knots $H_1(M_K, \mathbb{Z}) = \mathbb{Z}_p^2$ for some p (where M_K is the two-fold branched cover of S^3 along K).

$$9_{46}, 10_{99}, 10_{123}, 10_{155}, 11n_{74}, 12a_{427}, 12a_{1105}, 12n_{268},$$

$$12n_{397}, 12n_{414}, 12n_{605}, 12n_{636}, 12n_{706}, 12n_{817}, 12n_{838}.$$

This allows us to take boundary sums of the slice disks in [19] to get surfaces with large stabilization distance rel boundary.

Example 4.3. Let K be one of the 15 knots listed above. Then $H_1(M_K, \mathbb{Z}) = \mathbb{Z}_p^2$ for some p. Therefore, letting $K_n = \#^n K$, we have that $H_1(M_{K_n}, \mathbb{Z}) = \mathbb{Z}_p^{2n}$. By Chen [4], this implies that $g_{ss}(K_n) \geq n$. Meanwhile, letting D_1 be the disk described by the blue bands in [19] and D_2 be described by the green bands, we get that the band system describes a connect sum of unknotted spheres, and therefore a double-slicing of K. By Corollary 1.2, this means that their stabilization distance is at least n. Moreover, it is easy to check that taking any subset u of n bands containing a single band from each K and letting v be the complementary set of n bands, u and v describe non-isotopic slice disks with stabilization distance n as well.

While this method is computationally simple, generally requiring less work than examining the Khovanov homology or peripheral maps of the knot, it does have some drawbacks. Namely, it is restricted to surfaces which together form unknotted spheres, and it is a topological obstruction. The fact that the isotopy between the surfaces could be smooth or topological means that disks proven to be non-isotopic this way will be topologically non-isotopic as well. This means that Theorem 1.1 is not helpful in the pursuit of disks which are topologically but not smoothly isotopic. That said, as demonstrated above, this method is useful for generating large families of non-isotopic disks quickly, and is additionally helpful as it can be extended quickly to the equivariant setting as we see in the following section.

4.2 Equivariant Isotopy and Stabilization

In this section, we define equivariant notions of stabilization distance and use the newly defined equivariant double-slice and super-slice genus to extend our non-equivariant techniques from Section 4.1 to this setting. Additionally, we identify and study interesting equivariant embeddings of closed surfaces, using equivariant double-slice genus as an obstruction to equivariant isotopy.

While there are many 4-manifolds with interesting symmetry, here we concern ourselves solely with S^4. While in some ways the simplest closed 4-manifold, many questions about surfaces in S^4 remain unanswered, making it a good starting point for examining equivariant isotopy of surfaces. Additionally, S^4 has rich symmetric structures in the sense that it admits $\mathbb{Z}_p$ actions with knotted 2-spheres as the fixed point set. In order to begin looking at these symmetric closed surfaces in S^4, we start by discussing symmetric surfaces with boundary in B^4.

4.2.1 Equivariant Isotopy and Equivariant Internal Stabilization Distances for Symmetric Surfaces rel Boundary

When looking at symmetric surfaces in (B^4, τ), we have new notions of equivalence we can analyze. Namely, instead of simply considering the surfaces up to isotopy, with the added information about the symmetry τ we can consider the surfaces up to equivariant isotopy. This allows us to begin to develop a notion of equivariantly exotic surfaces. That is to say, symmetric surfaces which are isotopic but not equivariantly isotopic. Given two such surfaces with common boundary in B^4, we extend the definition of 1-handle stabilization distance and internal stabilization distance to the equivariant setting.

Definition 4.4. Let τ be a smooth $\mathbb{Z}_p$ action on B^4, $K \subset S^3$ be a τ-invariant knot, and $\Sigma_1, \Sigma_2 \subset B^4$ be τ-invariant properly embedded surfaces with common boundary K. The *equivariant 1-handle stabilization distance*, denoted $\tilde{d}_1^{\tau}(\Sigma_1, \Sigma_2)$, is the minimal number of orientation preserving ambient 1-handles $\{h_i\}$ and $\{h_i'\}$ needed so that $\tau(\Sigma_i \cup \{h_i\}) = \Sigma_i \cup \{h_i\}$ for $i \in \{1, 2\}$ and $\Sigma_1 \cup \{h_i\}$ is smoothly equivariantly isotopic rel boundary to $\Sigma_2 \cup \{h_i'\}$.

If we additionally require that the attached 1-handles are attached in local charts, as in the case of internal stabilization distance, we get the *equivariant internal stabilization distance*, which we denote $\tilde{d}^{\tau}(\Sigma_1, \Sigma_2)$.

While the work of Dai-Mallick-Stoffregen highlights ways to bound classical stabilization distance using equivariant techniques, they do not provide any immediate insight into equivariant stabilization distance. As of writing, to the authors knowledge, discussion of equivariant stabilization is non-existent in the current literature and, therefore, lacking in obstructions. While it is reasonable to assume, and would be interesting to see, that some of the techniques detailed in Example 4.1 could be given equivariant analogous, here we use our newly defined equivariant double-slice genus and equivariant super-slice genus to extend Theorem 1.2 to the equivariant setting.

If we replace double-slice and super-slice with equivariantly double-slice and equivariantly super-slice in the proof of Theorem 1.1, we immediately get a proof of the following result:

Theorem 1.3. *Let $\Sigma_1, \Sigma_2 \subset B^4$ be properly embedded genus h surfaces with boundary K which are both $\overline{\tau}$-invariant. If $\Sigma_1 \cup_K -\Sigma_2 \subset (B^4, \Sigma_1) \cup_{(S^3,K)} (B^4, \Sigma_2)$ is equivariantly unknotted, then $\tilde{d}_1^{\tau}(\Sigma_1, \Sigma_2) \geq \frac{\tilde{g}_{ss}(K,\tau)}{2} - h$.*

One possible application of Theorem 1.3 is the construction of isotopic but not equivariantly isotopic disks bounded by a strongly invertible knot. As of the writing of this dissertation, there are no published examples of such disks, however it seems highly likely that they exist.

Question 3. *Can Theorem 1.3 be used to detect isotopic but not equivariantly isotopic disks with common boundary?*

4.2.2 Equivariant Isotopy and Equivariant Internal Stabilization Distances for Symmetric Surfaces in S^4

So far, our discussions have been restricted to surfaces with boundary in B^4. Here we broaden our discussion to closed symmetric surfaces in S^4. When considering symmetric surfaces in S^4, we start by considering them up to equivariant isotopy. Specifically, we consider surfaces in (S^4, τ), where τ is an involution with a 2-dimensional fixed point set. Given a surface

$\Sigma \subset S^4$ invariant under such a τ, we call (Σ, τ) a *strongly invertible surface.* Using equivariant double-slice genus, we prove some fundamental results about strongly invertible surfaces.

In the case of classical knots, Marumoto showed in 1977 that if U is the classical unknot, any two strongly invertible knots (U, τ_1) and (U, τ_2) are equivariantly isotopic [22]. This is a crucial fact, as it allows for a clear identity element in the equivariant concordance group. The most obvious equivalent version of this statement for strongly invertable surfaces would be that letting τ be an orientation preserving involution with 2D fixed point set and S_1 and S_2 be unknotted 2-spheres which are also τ-invariant, then (S_1, τ) is equivariantly isotopic to (S_2, τ). That is to say, up to equivariant isotopy there is a unique strongly invertible 2-sphere which bounds a 3-ball. Using equivariant double-slice genus, we prove that not only is this false, but there are isotopic but non-equivariantly isotopic unknotted spheres, and that the equivariant internal stabilization distance between them can be arbitrarily large.

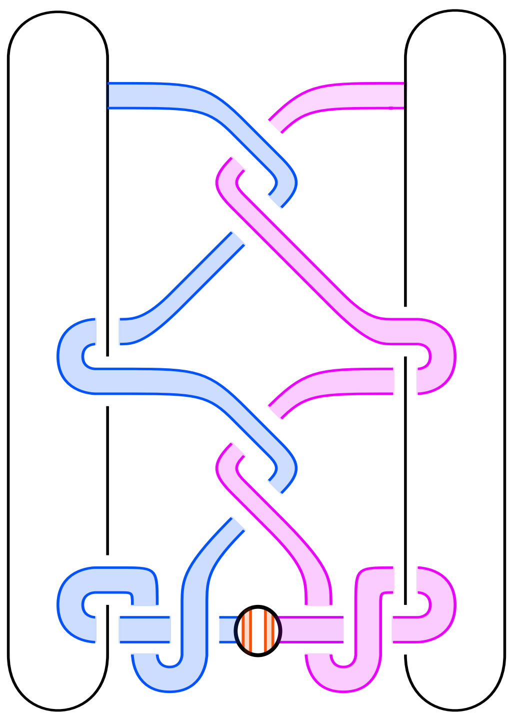

Figure 4.2: Band diagram for S, a symmetric 2-sphere.

Theorem 1.6. *For every $n \in \mathbb{N}$ and every involution τ of S^4 with 2-dimensional fixed point set, there exists a τ invariant sphere S^2_n which is unknotted but has equivariant internal stabilization distance from the standard equivariant unknot at least n.*

Proof. Let $S \subset S^4$ be the 2-sphere described by the band diagram in Figure 4.2. Given any involution τ of S^4 with 2-dimensional fixed point set, we can find some neighborhood of a point on the fixed point set where the action is linear. We may assume that S is contained in this neighborhood and that the action restricted to S is the symmetry visible in the band diagram taking the blue band to the pink and the red to itself. We now construct a new symmetric sphere S_1 by taking an equivariant connect sum of S with itself, as shown in Figure 4.3.

We can see that S_1 is the unknotted 2-sphere by performing the series of band moves and isotopies shown in Figure 4.4. While these band moves appear to be done symmetrically it would be reasonable to assume they represent an equivariant isotopy, however this is not the case. In Figure 4.4, going from the first to second diagram and the fourth to fifth diagram, we see that the bands pass through the fixed point axis of the fixed point set. Absent symmetry, this is not a problem. However, in the presence of symmetry, this is not allowed, as it would require the cores of bands to intersect at the fixed point meaning it is no-longer an isotopy. We now obstruct such an equivariant isotopy from existing via the equivariant super-slice genus.

If we resolve the blue and pink bands we get a strongly invertible knot which we call (K_1, τ). (K_1, τ) is exactly the knot from Example 3.3. As shown in the example, (K_1, τ) is not equivariantly super-slice. Therefore, we have that S_1 is not equivariantly unknotted.

Letting S_n be the n-fold equivariant connect sum of S_1 with itself we then may get a cross section of S_n to be given by (K_n, τ), the n-fold equivariant connect sum of K_1 with itself. Following the same argument as above we get that $\tilde{g}_{ss}(K_n, \tau) \geq 2n$. Thus, if we were to take $2n$ equivariant internal stabilizations S_n, without loss of generality taking the attaching regions for the 1-handles to be symmetric about K, the resulting surface would be a genus $2n$ τ-invariant surface symmetric about (K_n, τ). Because $\tilde{g}_{ss}(K_n, \tau) \geq 2n$, this surface can not be equivariantly unknotted, completing the proof. □

This result highlights just how different equivariant isotopy classes of surfaces can be from smooth isotopy, even in simple 4-manifolds like S^4. Two obvious generalization of this work include considering equivariant embedding of surfaces in S^4 with other group actions as well, as considering equivariant embeddings of surfaces in other 4-manifolds.

Question 4. *Given a 4-manifold X with symmetry ρ, do there exist equivariantly embedded surfaces which are isotopic but not equivariantly isotopic that remain equivariantly non-isotopic after arbitrarily many stabilizations?*

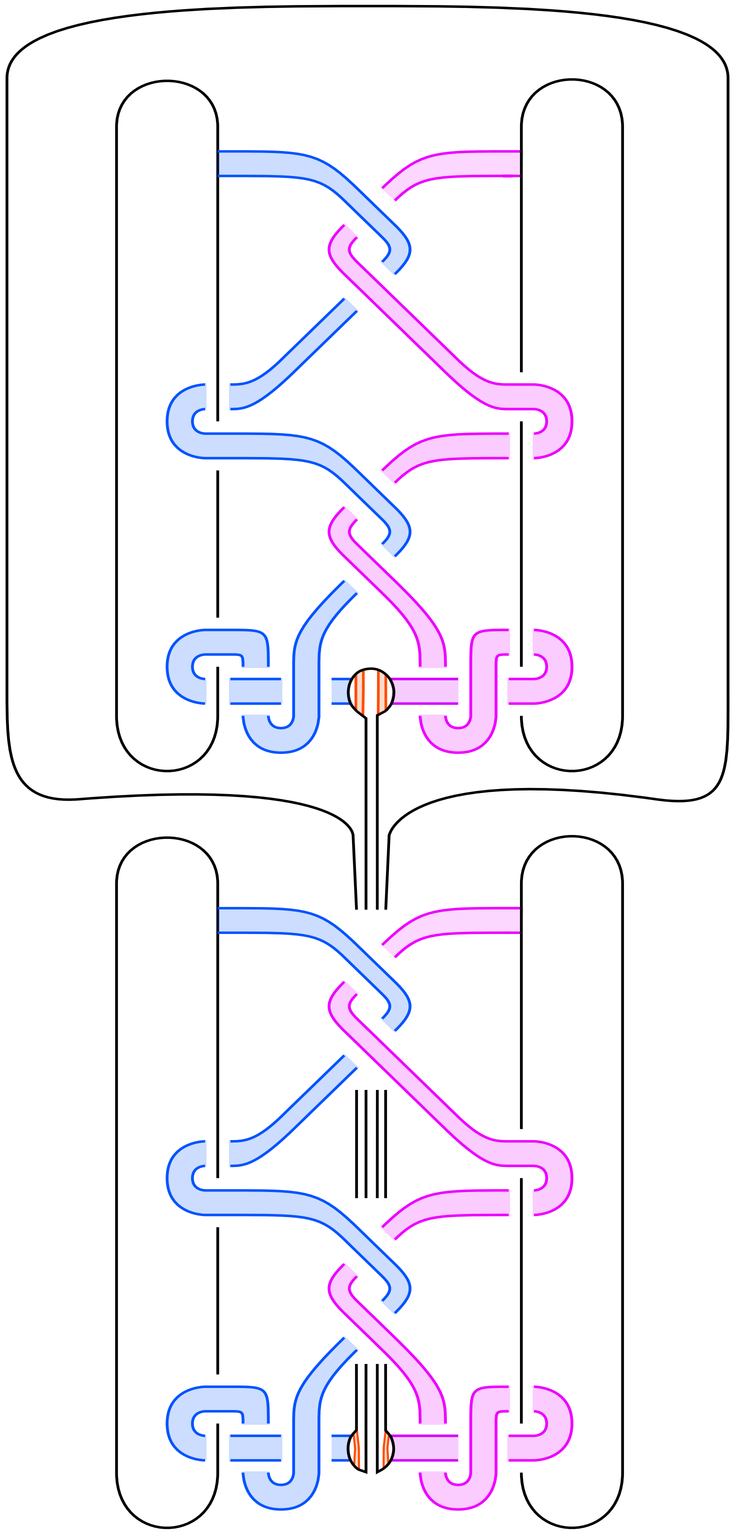

Figure 4.3: Band diagram for S_1, an equivariant connect sum of S with itself.

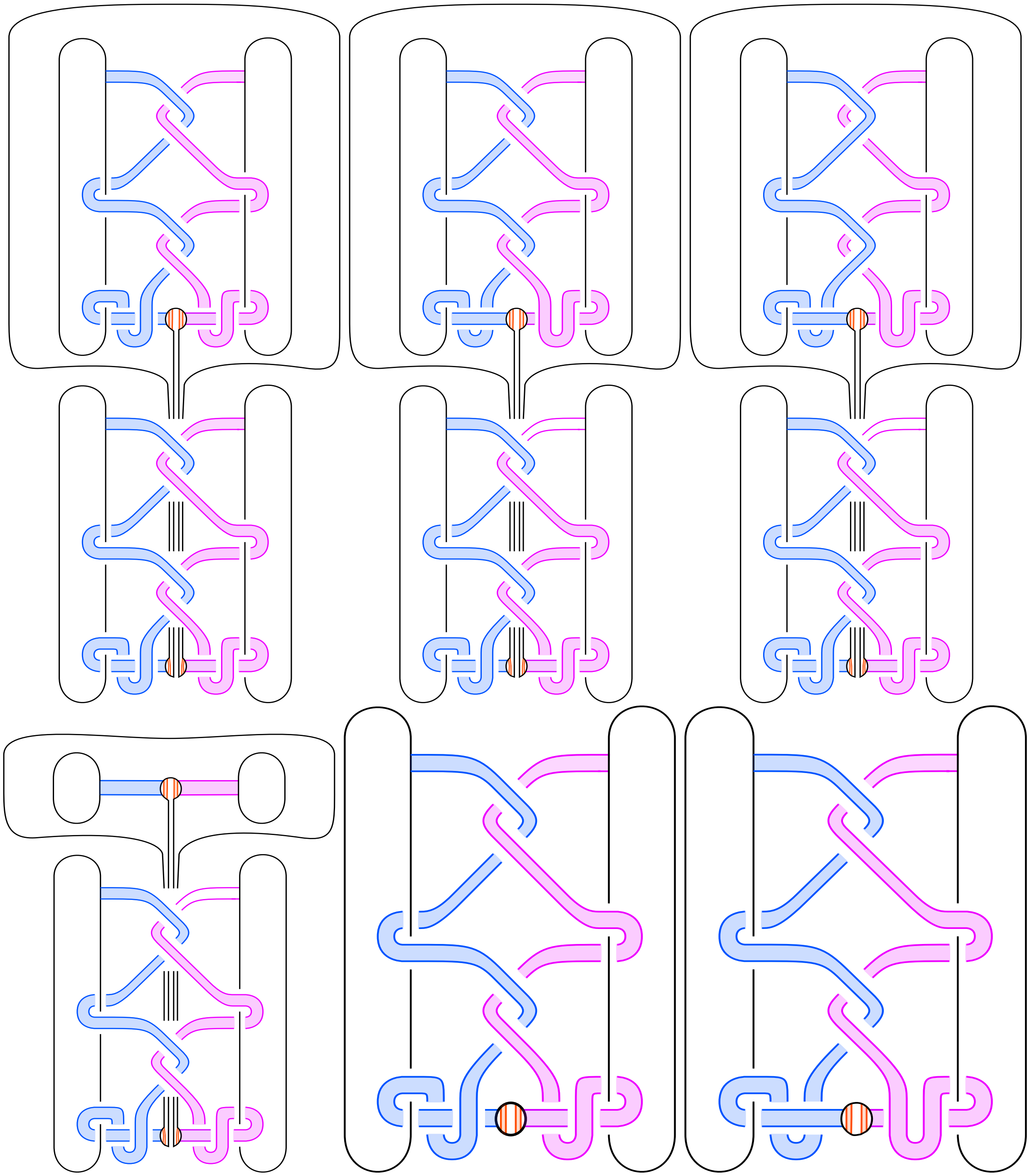

Figure 4.4: A sequence of band moves taking the 2-sphere from Figure 4.3 to an unknot, continued on the next page.

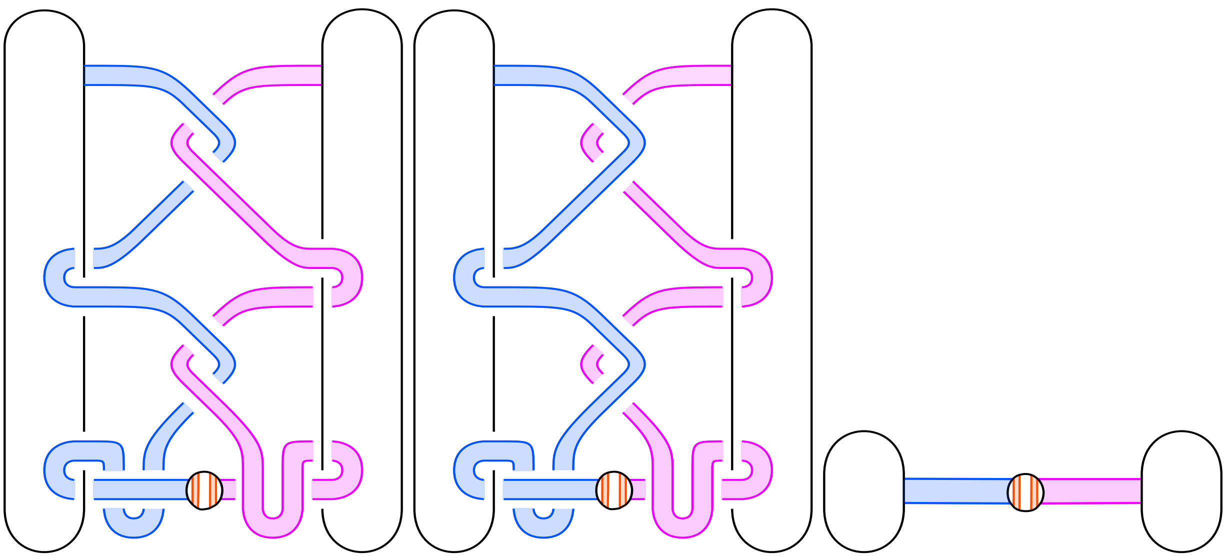

Figure 4.5: A continuation of Figure 4.4, with the first diagram (left) the same as the last diagram of Figure 4.4.

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