ON GENERAL BURNSIDE PROBLEM

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CONTENTS

I. Introduction

Is every torsion group locally finite? Burnside raised the question in 1902, which became one of the most famous in group theory. This is called the general Burnside problem. There is another restricted Burnside problem which states: Let G be a torsion group in which $x^N = 1$ for all $s \in G$, N a fixed positive integer. Is G then locally finite? Until 1964 almost all work was on the restricted Burnside problem - there was no real attack on the general Burnside problem. In 1964, Golod and Shafarevitch settled the general Burnside problem in the negative.

In 1941 Kurosh asked, for algebraic algebras, the analogue of the Burnside problem. In §IV the work of Golod and Shafarevitch is used to construct a finitely generated, infinite dimensional, nil algebra, thus setteling the Kurosh problem in the negative. Using this algebra, an infinite, finitely generated, torsion group is constructed which settles the general Burnside problem in the negative.

Many types of torsion groups are locally finite, including a class of groups which may be imbedded in certain rings [1]. That work is beyond the scope of this paper, but, because of its importance, we give the special case of Matrix Groups which was settled by Burnside himself. This result is presented in §II.

Except for §III, terminology and definitions are mostly from Herstein [3]. In §III definitions are used from [2].

II. The Burnside Problem For Matrix Groups.

2.1 Definition. A group G is said to be a torsion group if every element in G is of finite order.

2.2 Definition. A group G is said to be locally finite if every finitely generated subgroup of G is finite.

2.3 Lemma. Suppose that G is a group, N a normal subgroup of G such that both N and G/N are locally finite. Then G is locally finite.

Proof. Let $g_1, \ldots, g_n$ be a finite set of elements of G: we wish to show that they generate a finite subgroup of G. If $\bar{g}_1, \ldots, \bar{g}_n$ denote their images in G/N then, by assumption, these generate a finite subgroup of G/N. Let this subgroup be $\bar{g}_1, \ldots, \bar{g}_n, \ldots, \bar{g}_t$ and let $g_{n+1}, \ldots, g_t$ be any representative inverse images of $\bar{g}_{n+1}, \ldots, \bar{g}_t$ respectively in G. For any i, j, $g_i g_j = u_{ij} g_k$ for some k and some element u_{ij} in N. Let U be the subgroup of N generated by all the u_{ij}; the local finiteness of N implies that U is a finite group. Given any three g_i, g_j, g_m then $g_i g_j g_m = u_{ij} g_k g_m = u_{ij} u_{km} g_w$, so is of the form ug_w with $u \in U$. Similarly any word in the g_i's is of the form ug_w with $u \in U$, $1 \leq w \leq t$. Hence the $g_1, \ldots, g_t$ generate a group of order at most to $t \cdot o(U)$, that is, a finite group.

2.4 Definition. Let G be a group and suppose it has a series of subgroups $1 = G_i \subset \ldots \subset G_1 \subset G_0 = G$. If each $G_r \triangleleft G_{r-1}$ for $r = 1, \ldots, i-1$, then the series is called a subnormal series for G.

2.5 Definition. A group G is said to be solvable if it has a subnormal series with G_{r-1}/G_r an abelian group $(r = 0,1,..,i-1)$.

2.6 Lemma. A solvable torsion group is locally finite.

Proof. Let G be a solvable torsion group. By the solvability of G we can find subgroups G_i where G_i is normal in G_{i-1} and G_{i-1}/G_i is abelian and where $1 = G_i \subset \ldots \subset G_1 \subset G_0 = G$. An abelian torsion group is clearly locally finite; applying Lemma 2.3 we see that we can climb up this chain to get that G is locally finite.

2.7 Lemma. A group of triangular matrices over a field is solvable.

Proof. Because a subgroup of a solvable group is obviously solvable it is enough to show that the group of invertible triangular matrices is solvable. To see this let:

$$G = G_0 = \left\{ \begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \middle| \, a_i \neq 0 \right\}, \quad G_1 = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right\},$$

$$G_2 = \left\{ \begin{bmatrix} 1 & 0 & & * \\ & \ddots & & \\ & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix} \right\}, \quad G_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 & & * \\ & \ddots & & & \\ & & \ddots & & 0 \\ & & & \ddots & 0 \\ 0 & & & & 1 \end{bmatrix} \right\}$$

and so on. Each G_i is normal in G_{i-1}, G_{i-1}/G_i is abelian and $G_n = (1)$. Thus the group of triangular matrices is solvable.

2.8 <u>Lemma.</u> Let G be a finitely generated torsion group of matrices over a field F. Then there exists a positive integer N such that $a^N = 1$ for any a which is a characteristic root of any element of G.

<u>Proof.</u> Let $G \subset F_k$ be generated by $a_1, \ldots, a_r$. If P is the prime field of F let F_1 be the field obtained by adjoining all the entries of $a_1, \ldots, a_r$ to P. Clearly every element in G has entries in F_1. Since F_1 is finitely generated over P we can find a subfield K of F_1 which is purely transcendental over P and such that $[F_1 : K] = m$ is finite. Using the regular representation of F over K [5] we can write F_1 as a set of $m \times m$ matrices over K. Substituting these matrices for the entries of F_1 in the elements of G, we realize G as a group of $mk \times mk$ matrices over the field K. In other words we may consider that $G \subset K_t$ for some t where K is a finitely generated purely transcendental extension of P.

Let a be a characteristic root of any element of G; since G is a torsion group, each $g \in G$ satisfies an equation $x^n - 1 = 0$. Thus a is a root of unity so is algebraic over the prime field P. Since $G \subset K_t$, by the Cayley-Hamilton Theorem any element of G satisfies a polynomial over K of degree t therefore the characteristic roots of the elements of F satisfy polynomials over K of degree t. Since a is such and since K is purely transcendental over P we deduce that a is algebraic over P of degree at most t.

The argument now divides according to the characteristic of P.

1. If P is the field of p elements, p a prime, then $[P(a) : P] \leq t$ as we have just seen, hence $P(a)$ is a finite field having p^k elements with $k \leq t$. Thus $a^{p^k-1} = 1$; since $k \leq t$ for all the characteristic roots a of G the result follows.

2. If the characteristic of P is 0, P is the rational field and all the characteristic roots of the elements of G lie in extension fields of degree at most t over P. Since a primitive m-th root of unity has as its minimal polynomial the cyclotomic polynomial which is irreducible and of degree $\phi(m)$, the Euler ϕ-function, and since $\phi(m)$ goes to infinity with m we conclude that there are only a finite number of roots of unity present. Hence there is a positive integer N such that $a^N = 1$ for all such a.

2.9 <u>Lemma.</u> Let S be an irreducible semigroup of $n \times n$ matrices over a field F. Suppose that tr a, the trace of a, takes on k distinct values as a ranges over S. Then S has at most k^{n^2} elements.

<u>Proof.</u> Let $\overline{F}$ be the algebraic closure of F; then clearly $S \subset F_n \subset \overline{F}_n$. In other words, we may assume that F is algebraically closed.

Let $A = \{\sum c_i a_i \mid c_i \in F,\ a_i \in S\}$ be the linear span of S. Since S is a semigroup, A is a subalgebra of F. Moreover A acts irreducibly and faithfully on V, the set of n-tuples over F, since S itself is already irreducible. The commuting ring of A on V being a finite dimensional division algebra over F must be F itself. By Wedderburn's theorem $A = F_n$ follows.

Since S spans $A = F_n$ over F there must be matrices $a_1, \ldots,$ a_{n^2} in S which form a basis of F_n over F. Let $d_1, \ldots, d_k$ be the k values assumed by the traces of the elements of S. If $x \in S$ then $\operatorname{tr} a_1 x, \ldots, \operatorname{tr} a_{n^2} x$ is an n^2-tuple of elements each of which is one of $d_1, \ldots, d_k$. Since there are k^{n^2} such n^2-tuples we will be done if we can show that the system of equations $\operatorname{tr} a_1 x = b_1, \ldots, \operatorname{tr} a_{n^2} x = b_{n^2}$, $b_i \in F$ has at most one solution in F_n. However, these equations are linear, so it is enough to show that $\operatorname{tr} a_i x = 0$, $i = 1, \ldots, n^2$ has only $x = 0$ as solution.

Now the a_i are a basis of F_n so if $\operatorname{tr} a_i x = 0$ for all i then $\operatorname{tr} yx = 0$ for all matrices y in F_n. Since the trace is a non-degenerate bilinear form this indeed forces $x = 0$.

2.10 Theorem (Burnside). A torsion group of matrices over a field is locally finite.

Proof. Let $G' \subset F_n$ be a torsion group of matrices. We go by induction on n.

If $n = 1$ then $G' \subset F$ and so the result is trivial. Suppose the result true for matrices of order less than n. If $G' \subset F_n$ is a torsion group of matrices let G be a finitely generated subgroup of G'. We would like to prove that G is finite.

By Lemma 2.8 we have that there exists an integer $N > 0$ such that $a^N = 1$ for any a which is a characteristic root of an element of G. In consequence, tr g, as g runs over G, takes on only a finite number of values. If G should be an irreducible group of matrices it would be finite by applying Lemma

2.9. So suppose that G is reducible. By a change of basis we can assume that $g \in G$ is of the form

$$\begin{pmatrix} g_1 & 0 \\ b_1 & g_2 \end{pmatrix}$$

where $g_1 \in F_m$, $g_2 \in F_{n-m}$ for $0 < m < n$. The set G_1 of g_1 arising this way is a torsion group of m x m matrices over F so by induction it is locally finite. (In fact it is finitely generated so is actually even finite.) Similarly G_2, the set of g_2 arising, is locally finite group. Given

$$\begin{pmatrix} g_1 & 0 \\ b_1 & g_2 \end{pmatrix} \quad \text{in G map it onto} \quad \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

this map ϕ is clearly a homomorphism of G into a locally finite group. Moreover, Ker ϕ is a subgroup of the group of triangular matrices and, as a subgroup of a torsion group is itself a torsion group. Invoking Lemma 2.6 we deduce that Ker ϕ is locally finite. Knowing that Ker ϕ and G/Ker ϕ are locally finite we have by Lemma 2.3 that G itself is locally finite. Since G is finitely generated it must therefore be finite. We have proved that G' is locally finite. Thus the theorem is established.

III Graded Algebra

3.1 <u>Definition.</u> A grading in a module A is defined by a family of submodules A^n (n running through all integers) such that A is the direct sum ΣA^n. Each $a \in A$ has then a unique representation $a = \Sigma a^n$; $a^n \in A^n$ where only a finite number of a^n's are different from zero. We call a^n the homogeneous component of degree n. A module with a grading defined in it is called a graded module.

3.2 <u>Definition.</u> A submodule B of a graded module A is called homogeneous if $B = \Sigma B^n$ where $B^n = B \cap A^n$. The quotient A/B may then be regarded as a graded module by setting

$$(A/B)^n = (A^n + B)/B \cong A^n/B^n .$$

It will be convenient to identigy A/B with $\Sigma A^n/B^n$.

3.3 <u>Definition.</u> An R-algebra A is a right R-module which is a ring in such a way that addition in the ring is the addition in the module and $(ab)r = (ar)b = a(br)$ for all a and b in A and r in R. A homomorphism $A \dashrightarrow A'$ of R-algebras A, A' is a function on the set A to the set A' which is a homomorphism both of R-modules and of rings.

Hence an Algebra is defined to be a module A which is also a ring in such a way that the product $(a,b) \dashrightarrow ab$ in the ring satisfies $(ab)r = a(br) = (ar)b$ for all $r \in R$. This condition, with the distributive law for multiplication, states that the product $(a,b) \dashrightarrow ab$ is a bilinear function $A \otimes A \dashrightarrow A$. By the universality of the tensor product, there is a unique homomorphism $f: A \otimes A \dashrightarrow A$ of modules with $f(a \otimes b) = ab$ for all a,b in A. The unit element 1 of the ring also determines a homomorphism $g: R \dashrightarrow A$, defined by $g(r) = 1 \cdot r \in A$.

3.4 <u>Definition.</u> A graded algebra A is a graded module with two homomorphisms $f: A \otimes A \dashrightarrow A$ and $g: R \dashrightarrow A$ of graded modules such that the diagrams below are commutative:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1 \otimes f"] \arrow[d, "f \otimes 1"'] & A \otimes A \arrow[d, "f"] \\ A \otimes A \arrow[r, "f"'] & A \end{tikzcd} \quad , \quad \begin{tikzcd} R \otimes A \arrow[d, "g \otimes 1"'] & = & A \arrow[d, "1"'] & = & A \otimes R \arrow[d, "1 \otimes g"] \\ A \otimes A \arrow[rr, "f"'] & & A & & A \otimes A \arrow[ll, "f"] \end{tikzcd}$$

3.5 Example of a graded Algebra

Let G be any field and let $T = F[x_1, x_2, \ldots, x_d]$ be the polynomial ring over F in the noncommuting variables $x_1, \ldots, x_d$. This is also called the free associative algebra on $x_1, x_2, \ldots, x_d$. over F. We can write T as

$$T = T_0 \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots$$

where $T_0 = F$ and T_n has a basis the d^n elements of $x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_n}$, where the x_{i_j} are chosen from $x_1, \ldots, x_d$. The elements of T_n are called homogeneous elements of degree n. Clearly the T_n are submodules. If we define $f: T \otimes T \dashrightarrow T$ by ordinary multiplication in polynomial ring, and define $g: F \dashrightarrow T$ by $g(r) = 1 \cdot r \in T$ where 1 is the identity of F, then the following two diagrams are commutative:

$$\begin{tikzcd} T\otimes T\otimes T \arrow[r, dashed, "1\otimes f"] \arrow[d, "f\otimes 1"'] & T\otimes T \arrow[d, "f"] \\ T\otimes T \arrow[r, dashed, "f"'] & T \end{tikzcd}\quad,\quad \begin{tikzcd} F\otimes T \arrow[r, equal] \arrow[d, "g\otimes 1"'] & T \arrow[r, equal] \arrow[d, "1"] & T\otimes F \arrow[d, "1\otimes g"] \\ T\otimes T \arrow[r, dashed, "f"'] & T & T\otimes T \arrow[l, dashed, "f"] \end{tikzcd}$$

Hence T is a graded algebra.

IV Golod-Shafarevitch Theorem

4.1 <u>Lemma.</u> Let T be the graded algebra defined in 3.5. Let $\mathfrak{a} = (f_1, f_2, \ldots)$ be the two sided ideal of T generated by the homogeneous elements $f_1, f_2, \ldots$ of degrees $2 \leq n_1 \leq n_2 \ldots$ respectively. Then $A = T/\mathfrak{a}$ inherits the grading of T, in fact

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_n \oplus \ldots \quad \text{where } A_i \cong T_i/\mathfrak{a} \cap T_i.$$

<u>Proof.</u> Let $a \in \mathfrak{a}$. Then $a = \sum c_{iq} f_q b_{qj}$ where c_{iq}, b_{qj} are homogeneous. Therefore $c_{iq} f_q b_{qj} \in T_n$ for some n. Hence $a \in \sum_{n=0}^{\infty} \mathfrak{a} \cap T_n$. Clearly $\mathfrak{a} \subset \sum \mathfrak{a} \cap T_n \subset \mathfrak{a}$. Hence $\mathfrak{a}$ is a homogeneous submodule. By definition 3.2,

$$A_i = (T_i + \mathfrak{a})/\mathfrak{a} \cong T_i/T_i \cap \mathfrak{a})$$

and $$A = A_0 \oplus A_1 \oplus \ldots$$

4.2 <u>Theorem (Golod-Shafarevitch).</u> For A as defined above, furthermore let r_i be the number of n_j which are equal to i. Let $b_n = \dim_F (A_n)$.

1. $b_n \geq db_{n-1} - \sum_{n_i \leq n} b_{n-n_i}$ for $n \geq 1$.

2. If for each i the $r_i \leq [(d-1)/2]^2$ then A is infinite dimensional over F.

Proof. We will exhibit linear mappings ϕ, ψ so that the sequence

$$(1) \quad A_{n-n_1} \oplus \cdots \oplus A_{n-n_k} \oplus \cdots \xrightarrow{\phi} A_{n-1} \oplus \cdots \oplus A_{n-1} \xrightarrow{\psi} A_n \to 0$$

is exact, where the first sum runs over all $n_i \leq n$ and where the second sum is that of d copies of A_{n-1}.

Note that if we are able to do this then the equality expressed in the theorem would be proved for then

$$db_{n-1} = b_n + \dim (\mathrm{Ker}\, \psi)$$

and since $\mathrm{Ker}\, \psi$ is a homomorphic image of $\oplus_{n_i \leq n} A_{n-n_i}$ we would have $\dim (\mathrm{Ker}\, \psi) \leq \sum_{n_i \leq n} b_{n-n_i}$, the net result of which would be $db_{n-1} \leq b_n + \sum_{n_i \leq n} b_{n-n_i}$, the desired conclusion.

Our objective then becomes that of defining the ϕ and ψ. To this end we shall first define mappings Φ and Ψ for the sequence

$$(2) \quad T_{n-n_1} \oplus \cdots \oplus T_{n-n_k} \oplus \cdots \xrightarrow{\Phi} \overbrace{T_{n-1} \oplus \cdots \oplus T_{n-1}}^{d\text{-times}} \xrightarrow{\Psi} T_n \to 0$$

where Φ and Ψ are linear. Although the sequence will not be exact at the T-level it will turn out to be so at the A-level; that is, we shall induce the proper ϕ and ψ from these Φ and Ψ.

The mapping Ψ is defined simply by:

$$\Psi: \ t_1 \oplus \cdots \oplus t_d \to \sum_{i=1}^{d} t_i x_i \quad \text{for } t_i \in T_{n-1}.$$

To get Φ we proceed as follows: if

$$s_{n-n_1} \oplus \cdots \oplus s_{n-n_k} \oplus \cdots \in T_{n-n_1} \oplus \cdots \oplus T_{n-n_k} \oplus \cdots$$

then $\sum s_{n-n_i} f_i \in T_n$. As an element in T_n we can write

$$\sum s_{n-n_i} f_i = \sum_{i=1}^{d} u_i x_i$$

where the u_i are uniquely determined elements in T_{n-1}. Define Φ by

$$\Phi: \ s_{n-n_1} \oplus \cdots \oplus s_{n-n_k} \oplus \cdots \to u_1 \oplus \cdots \oplus u_d$$

It is trivial that the Φ and Ψ defined are linear and have the proper ranges and domains. It is equally clear that Ψ is onto T_n so that the sequence (2) is exact at T_n.

Let $\mathfrak{A}_i = \mathfrak{A} \cap T_i$; our aim is to induce mappings ϕ and ψ from our Φ and Ψ for the sequence (1).

If $t_1, \ldots, t_d \in \mathfrak{A}_{n-1}$ then since $\mathfrak{A}$ is an ideal of T, $\sum t_i x_i \in \mathfrak{A}$; by the properties of the grading $\sum t_i x_i \in T_n$. In short, it is in $\mathfrak{A}_n$. Thus the mapping Ψ induces down to ψ:

$$\psi: A_{n-1} \oplus \cdots \oplus A_{n-1} \to A_n \to 0$$

We now consider Φ. Suppose that $s_{n-n_1}, s_{n-n_2}, \ldots, s_{n-n_k}, \ldots$ are in $\mathfrak{A}_{n-n_1}, \mathfrak{A}_{n-n_2}, \ldots, \mathfrak{A}_{n-n_k}, \ldots$ respectively. We must show that $u_1, \ldots, u_d$ defined by $\sum s_{n-n_i} f_i = \sum u_i x_i$ are in $\mathfrak{A}_{n-1}$. Since Φ is linear it suffices to do so for each s_{n-n_i} in $\mathfrak{A}_{n-n_i}$.

Now $\quad s_{n-n_i}f_i = \sum_{j=1}^{d} s_{n-n_i}g_{ij}x_j$

where $f_i = \sum_{j=1}^{d} g_{ij}x_j$. Thus $u_j = s_{n-n_i}g_{ij}$ and so is in $\mathfrak{A}$ as s_{n-n_i} is in the ideal $\mathfrak{A}$. Being of the correct grade it is in $\mathfrak{A}_{n-1}$. Therefore Φ too induces down to a map ϕ from

$$A_{n-n_1} \oplus \cdots \oplus A_{n-n_k} \oplus \cdots \quad \text{to} \quad A_{n-1} \oplus \cdots \oplus A_{n-1}.$$

We must still show exactness at $A_{n-1} \oplus \cdots \oplus A_{n-1}$. We first establish that $\phi\psi = 0$. If $s_{n-n_1}, \ldots, s_{n-n_k}$ are in $T_{n-n_1}, \ldots, T_{n-n_k}$ respectively then

$$(s_{n-n_1} \oplus \cdots \oplus s_{n-n_k})\Psi\Phi = \sum_{i=1}^{d} u_i x_i$$

where $\sum u_i x_i = \sum s_{n-n_i} f_i$; since the f_i are in $\mathfrak{A}$ the sum, $\sum s_{n-n_i} f_i$, is in $\mathfrak{A}$ hence $\sum u_i x_i \in \mathfrak{A}$. In other words, $\Psi\Phi$ maps $T_{n-n_1} \oplus \cdots \oplus T_{n-n_k} \oplus \cdots$ into $\mathfrak{A}$ hence $A_{n-n_1} \oplus \cdots \oplus A_{n-n_k} \oplus \cdots$ is mapped into 0 by $\phi\psi$.

We must still show that if $(t_1 \oplus \cdots \oplus t_d)\Phi \in \mathfrak{A}$ then we can find elements $u_1, \ldots, u_d$ in T_{n-1} such that $t_i - u_i \in \mathfrak{A}$ for $i = 1, 2, \ldots, d$ and such that $\sum u_i x_i = \sum s_{n-n_i} f_i$ for some s_{n-n_i} in the appropriate T_{n-n_i}. Suppose then that $(t_1 \oplus \cdots \oplus t_d)\Phi = \sum t_i x_i \in \mathfrak{A}$; being in $\mathfrak{A}$, which is generated as a two sided ideal by the f_j, we have that

$$\sum t_i x_i = \sum a_{kq} f_q b_{kq} + \sum c_q f_q$$

where the a_{kq}, b_{kq}, c_q are homogeneous and where the degree of b_{kq} is at least 1. On comparing degree on both sides we may even assume that the $a_{kq} f_q b_{kq}$, $c_q f_q$ are all in T_n. Since the b_{kq} are of degree at least, $b_{kq} = \sum_{m=1}^{d} d_{kqm} x_m$ hence

$$\Sigma\, a_{kq} f_q b_{kq} = \Sigma\, a_{kq} f_q d_{kqm} x_m = \Sigma\, d_m x_m$$

where $d_m = \sum_{k,q} a_{kq} f_q d_{kqm}$. But since $f_q \in \mathfrak{A}$ we have that $d_m \in \mathfrak{A}$. If we write

$$\Sigma\, c_q f_q = \sum_{j=1}^{d} u_i x_i$$

we have that

$$\sum_{i=1}^{d} t_i x_i = \sum_{i=1}^{d} d_i x_i + \sum_{i=1}^{d} u_i x_i$$

hence $t_i - u_i = d_i \in \mathfrak{A}$. But $(c_1 \oplus \ldots \oplus c_i \oplus \ldots)\Phi = u_1 \oplus \ldots \oplus u_d$ by the definition of Φ ; hence we have proved the exactness of (1) at $A_{n-1} \oplus \ldots \oplus A_{n-1}$. This proves part (1) of the theorem.

We now consider part (2). For formal power series in t with integer coefficients we declare

$$\sum_{n=0}^{\infty} c_n t^n \geq \sum_{n=0}^{\infty} d_n t^n$$

if $c_i \geq d_i$ for all i.

From part (1) we have that

$$\sum_{n=1}^{\infty} b_n t^n \geq \sum_{n=1}^{\infty} d b_{n-1} t^n - \sum_{n=1}^{\infty} \sum_{n_i \leq n} b_{n-n_i} t^n.$$

From the definition of r_i we can write

$$\sum_{n_i, m} b_m t^{n_i + m} = \Sigma\, t^{n_i} \sum_{m=0} b_m t^m = \Big(\sum_{i=2} r_i t^i\Big)\Big(\sum_{m=0} b_m t^m\Big).$$

Let $P_A(t) = \sum_{m=0}^{\infty} b_m t^m$. The above relations become:

$$P_A(t) - 1 \geq dtP_A(t) - (\sum_{i=2}^{\infty} r_i t^i)\, P_A(t);$$

therefore

$$P_A(t)\,(1 - dt + \sum_{i=2} r_i t^i) \geq 1.$$

Now if the coefficients in the formal power series expansion of

$$(1 - dt + \sum_{i=2}^{\infty} r_i t^i)^{-1}$$

are nonnegative we get that

$$P_A(t) \quad (1 - dt + \sum_{i=2}^{\infty} r_i t^i)^{-1}$$

and so, an infinite number of the b_n must be different from 0. Hence A is infinite dimensional.

This itself is of great interest and we single it out as a theorem before completing the proof of part (2) of the Theorem 4.2.

4.3 Theorem. A is infinite dimensional over F if the coefficients in the power series expansion of

$$(1 - dt + \sum_{i=2}^{\infty} r_i t^i)^{-1}$$

are nonnegative.

To finish the proof of Theorem 4.2 we need to show that if each $r_i \leq [(d-1)/2]^2$ then the criterion of Theorem 4.3 is satisfied and so the theorem is established.

Proof.

$$1 - dt + \sum_{i=2} r_i t^i \leq 1 - dt\, (\frac{d-1}{2})^2 \sum_{i=2} t^i$$

$$(1 - dt + \sum_{i=2}^{\infty} r_i t^i)^{-1} \geq \left[1 - dt + \frac{(\frac{d-1}{2})^2 \cdot t^2}{1 - t}\right]^{-1}$$

$$= \frac{1 - t}{1-(d+1)t + dt^2 + (\frac{d-1}{2})^2 \ t^2} \qquad \text{with } u = \frac{d+1}{2} \quad \text{this becomes}$$

$$\frac{1 - t}{1 - 2ut + u^2t^2} = \frac{1 - t}{(1 - ut)^2}$$

$$= (1-t)\ (1 + ut + u^2t^2 + \ldots)^2 = (1-t)\ (1 + 2ut + 3u^2t^2 + \ldots (u+1)u^nt^n + \ldots)$$

$$= 1 + (2u-1)t + u(3u-2)t^2 + \ldots + u^{n-1}\left[(n+1)u-n\right] t^n + \ldots$$

As $u > 1$ each coefficient is positive.

From Theorem 4.2 we can construct a finitely generated nil algebra which is not nilpotent, hence settling the Kurosh Problem (If A is an algebraic algebra over F, does a finite number of elements of A always generate a finite dimensional subalgebra of A?) in the negative.

4.4 <u>Theorem.</u> If F is any countable field there exists an infinite dimensional nil algebra over F generated by three elements.

<u>Proof.</u> Let $T = F[x_1, x_2, x_3]$ be the free algebra over F in the three noncommuting variables x_1, x_2, x_3. Since T is graded we can write $T = F \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots$ where T_i is homogeneous of degree i. The ideal $T' = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots$ of T is countable and we enumerate its elements as $s_1, s_2, \ldots$ Pick $m_1 \geq 2$ and let $s_1^{m_1} = s_{12} + s_{13} + \ldots + s_{1k_1}$ where $s_{1j} \in T_j$. Now pick an integer

$m_2 > 0$ so that $s_2^{m_2} \in T_{k_1+1} \oplus T_{k_1+2} \oplus \ldots$, hence

$$s_2^{m_2} = s_{2,k_1+1} + \ldots + s_{2,k} \quad \text{where } s_{2_j} \in T_j.$$

Continue in this pattern. Hence we have chosen integers $m_i > 0$ and $k_1 < k_2 < \ldots$ such that

$$s_i^{m_i} = s_{i,k_{i-1}+1} + \ldots + s_{i,k_i} \quad \text{with } s_{ij} \in T_j.$$

Let $\mathfrak{A}$ be the ideal of T generated by all the s_{ij}. By our choice of the s_{ij} the integers r_k in Theorem 4.2 are all at most 1. Since $d = 3$, $r_i \leq 1 \leq [(d-1)/2]^2$ holds true, therefore applying Theorem 4.2 we get that $T/\mathfrak{A}$ is infinite dimensional. Since $\mathfrak{A} \subset T'$ we have that $T'/\mathfrak{A}$ is infinite dimensional over F. By construction $T'/\mathfrak{A}$ is a nil algebra. Since it is generated by three elements $T'/\mathfrak{A}$ is the required example.

We close the paper by settling the general Burnside Problem in the negative.

4.5 <u>Theorem.</u> If p is any prime number there exists an infinite group G generated by three elements in which every element has finite order a power of p.

<u>Proof.</u> Let F be the prime field with p elements and let $\mathfrak{A}$ be the ideal in $T = F[x_1, x_2, x_3]$ constructed in the course of proving Theorem 4.4. Let $A = T/\mathfrak{A}$ and let a_1, a_2, a_3 be the elements $x_1 + \mathfrak{A}$, $x_2 + \mathfrak{A}$, $x_3 + \mathfrak{A}$ respectively. Let G be the multiplicative semigroup in A generated by the elements $1 + a_1$, $1 + a_2$,

$1+a_3$. Any element in G is of the form $1+a$ where $a \in T'/\mathfrak{A}$ (so is nilpotent). For large enough n, $a^{p^n} = 0$ hence $(1+a)^{p^n} = 1 + a^{p^n} = 1$ since we are in characteristic p. Hence G is a group - in fact a torsion group - and every element of G has order a power of p. We claim that G is infinite. For if G is finite the linear combinations of its elements would form a finite dimensional algebra B over F; since 1, $1+a_i$ are in G the element $a_i = (1+a_i) - 1 \in B$. Since 1, a_1, a_2, a_3 generate A we get $A = B$ contradicting that A is infinite dimensional over F. This finishes the proof.

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ON GENERAL BURNSIDE PROBLEM

by

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ABSTRACT

The general Burnside problem asked, is every torsion group locally finite? Many types of torsion groups are locally finite. However, the general Burnside problem is answered in the negative

We begin by showing that a torsion group of matrices is locally finite. This is done by Burnside himself.

Next we define Graded Algebras and give an example of a graded algebra, $T = F[x_1, \dots, x_d]$ be the polynomial ring over F in the d noncommuting variables.

Next we want to prove the Golod-Shafarevitch Theorem. Let $\mathfrak{A} = (f_1, f_2, \dots)$ be the two-sided ideal of T generated by the homogeneous elements $f_1, f_2, \dots$ of degrees $2 \le n_1 \le n_2 \le \dots$ respectively. Furthermore let r_i be the number of n_j which are equal to i. Since $\mathfrak{A}$ is homogeneously generated the algebra $A = T/\mathfrak{A}$ inherits the grading of T. Let $A = A_0 \oplus A_1 \oplus \dots \oplus A_n \oplus \dots$ and let $b_n = \dim_F(A_n)$, the Golod-Shafarevitch Theorem furnishes a sufficient condition that A be infinite dimensional over F.

Theorem (Golod-Shafarevitch). For A as described above

1. $b_n \ge d b_{n-1} - \sum_{n_i \le n} b_{n-n_i}$ for $n \ge 1$.
2. If for each i the $r_i \le [(d-1)/2]^2$ then A is infinite dimensional over F.

Then we construct a counterexample which settles the Kurosh problem in the negative, namely, if F is any countable field there exists an infinite dimensional nil algebra over F generated by three elements. Using this we then settle the general Burnside problem in the negative.