Counting representations of deformed preprojective algebras

by

Hui Chen

B.S., Beijing Normal University, 2008

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
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Abstract

For any given quiver Γ, there is a preprojective algebra and deformed preprojective algebras associated to it. If the ground field is of characteristic 0, then there are no finite dimensional representations of deformed preprojective algebras with special weight 1. However, if the ground field is of characteristic p, there are many dimension vectors with nonempty representation spaces of that deformed preprojective algebras.

In this thesis, we study the representation category of deformed preprojective algebra with weight 1 over field of characteristic $p > 0$. The motivation is to count the number of rational points of the fibers $X_\lambda = \mu^{-1}(\lambda)$ of moment maps at the special weights $\lambda \in K^\times$ over finite fields, and to study the relations of the Zeta functions of these algebraic varieties X_λ which are defined over integers to Betti numbers of the manifolds $X_\lambda(\mathbb{C})$. The first step toward counting is to study the categories of representations of the deformed preprojective algebras Π^λ.

The main results of this thesis contain two types of quivers. First result shows that over finite field, the category of finite dimensional representations of deformed preprojective algebras Π^1 associated to type A quiver with weight 1 is closely related to the category of finite dimensional representations of the preprojective algebra associated to two different type A quivers. Moreover, we also give the relations between their Zeta functions. The second result shows that over algebraically closed field of characteristic $p > 0$, the category of finite dimensional representations of deformed preprojective algebras Π^1 associated to Jordan quiver with weight 1 has a close relationship with the category of finite dimensional representations of preprojective algebra associated to Jordan quiver.

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In this thesis, we study the representation category of deformed preprojective algebra with weight 1 over field of characteristic $p > 0$. The motivation is to count the number of rational points of the fibers $X_\lambda = \mu^{-1}(\lambda)$ of moment maps at the special weights $\lambda \in K^\times$ over finite fields, and to study the relations of the Zeta functions of these algebraic varieties X_λ which are defined over integers to Betti numbers of the manifolds $X_\lambda(\mathbb{C})$. The first step toward counting is to study the categories of representations of the deformed preprojective algebras Π^λ.

The main results of this thesis contain two types of quivers. First result shows that over finite field, the category of finite dimensional representations of deformed preprojective algebras Π^1 associated to type A quiver with weight 1 is closely related to the category of finite dimensional representations of the preprojective algebra associated to two different type A quivers. Moreover, we also give the relations between their Zeta functions. The second result shows that over algebraically closed field of characteristic $p > 0$, the category of finite dimensional representations of deformed preprojective algebras Π^1 associated to Jordan quiver with weight 1 has a close relationship with the category of finite dimensional representations of preprojective algebra associated to Jordan quiver.

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Chapter 1

Introduction

1.1 Background and Motivation

Given a finite type scheme X defined over $\mathbb{Z}$, there is an associated complex manifold $X(\mathbb{C})$. We are interested in the geometric and topological properties of $X(\mathbb{C})$, such as the Betti numbers $b_i = \mathrm{dim H^i(X(\mathbb{C}))}$. Also for any prime number p and any $r \geq 1$, $X(\mathbb{F}_{p^r})$ is a finite set, and we are interested in computing $|X(\mathbb{F}_{p^r})|$ for all r. Weil conjecture [40] suggests that these two are related in terms of Zeta functions of X if p is large enough relative to X, but they are quite different when p is fixed and small. The results of this thesis provide a family of examples of such X coming from quiver representations that Weil conjecture fails. The motivation is from Hausel's approach to proof of Kac's conjecture.

It is a classical question to classify the representations of any given quiver. For a given quiver Γ without loop, there is a Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(\Gamma)$[28] associated to it. If the quiver Γ is of finite type, the indecomposable representations over any algebraically closed field are in one-to-one correspondence to the positive roots of the Kac-Moody Lie algebra $\mathfrak{g}(\Gamma)$ by the dimension vectors (with coordinates being the simple roots). This was first discovered by Gabriel in 1972 [15] and proved directly by Bernstein, Gelfand and Ponomarev in [24]. For general quiver Γ, Kac proved this statement in [26]. Moreover, in [27], Kac introduced the number $A_\Gamma(\alpha, q)$ of isomorphism classes of absolutely indecomposable representations of quiver Γ over the finite field $K = \mathbb{F}_q$ of dimension vector α, and he proved [27, Proposition 1.5] that $A_\Gamma(\alpha, q)$ is a polynomial in q with integer coefficients. He went on to conjecture [27] that the constant term $A_\Gamma(\alpha, 0) = \dim \mathfrak{g}(\Gamma)_\alpha$, the root multiplicity of the root space of the Kac-Moody Lie algebra $\mathfrak{g}(\Gamma)$.

1.2 Proof of Kac's Conjecture

In 1998, Hua [23] gave a formula for the generating functions of number of isomorphism classes of a quiver over finite field $\mathbb{F}_q$. This formula encodes the coefficients of the $A-$polynomial $A_\Gamma(\alpha, q)$, which counts the number of representations of the quiver Γ over finite fields $\mathbb{F}_q$. Crawley-Boevey and Van den Bergh [9] proved that Kac's conjecture is true if α is indivisible using Weil conjecture. Finally Hausel [19] proved Kac's conjecture using the Weil conjecture and by relating the Betti numbers of the certain algebraic varieties associated to Nakajima quiver varieties and using the fact that when p is large (depending on the dimension vector α), this variety do not have rational points. However, for a fixed prime p, there are infinitely dimension vectors α of these varieties have nonempty rational points over $\mathbb{F}_{p^r}$. This makes some of the formulas in Hausel's proof not correct for any fixed prime p, however, the main result of Hausel's paper is not effected. The main goal of this paper is to find correct formulations of those formulas for every fixed prime p.

Now let us discuss the problem a little more in detail. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a finite quiver with vertices set Γ_0 and arrow set Γ_1. Given two dimension vectors $\mathbf{v} = (v_i) \in \mathbb{N}^{\Gamma_0}$ and $\mathbf{w} = (w_i) \in \mathbb{N}^{\Gamma_0}$, Nakajima [30, 31, 32] used the moment map to define a variety $\mathcal{V}_\lambda(\mathbf{v}, \mathbf{w}) = \mu^{-1}(\lambda)(\mathbb{C})$. The Nakajima quiver variety $\mathcal{M}_\lambda(\mathbf{v}, \mathbf{w})(\mathbb{C})$ is the GIT quotient of $\mathcal{V}_\lambda(\mathbf{v}, \mathbf{w})(\mathbb{C})$ with respective the group $PGL_{\mathbf{v}}(\mathbb{C})$ action. Using these varieties, Nakajima gave a geometric construction of integrable representation of highest weight $\mathbf{w}$ (the coefficients of the fundamental weights of the integral weight lattice) of the Kac-Moody Lie algebra $\mathfrak{g}(\Gamma)$. In [33], Nakajima found a combinatorial algorithm to determine the Betti numbers of Nakajima quiver varieties, then Hausel [19] proved a generating function formula for the Betti numbers of Nakajima quiver varieties in terms of counting the representations of the preprojective algebras and the deformed preprojective algebras over finite fields extending Hua's formula, which then implies Kac's conjecture on root multiplicity.

However, in Hausel's proof of Kac's conjecture [19], he used the special assumption $\Phi(\mathbf{0}) = 1$ to compute his generating function $\Phi(\mathbf{w})$. This assumption is true if the characteristic of the representation field is zero, but will never be true for any field of positive characteristics. Nevertheless, the proof of Kac's conjecture is not effected as follows. For each fixed dimension vectors $\mathbf{w}$ and $\mathbf{v}$ the Betti numbers depends on all primes sufficiently large. Thus in the computations, (for a fixed dimension vector $mathbfv$) one assumes that the characteristic is large enough. Thus, the coefficients of $X^{\mathbf{v}}$ in $\Phi(\mathbf{0})$ is zero for p large enough.

Therefore, it is an interesting question to compute the exact formula for $\Phi(\mathbf{0})$ and $\Phi(\mathbf{w})$ in any fixed prime characteristics. This is the main goal of this thesis.

1.3 Brief Introduction to This Thesis

1.3.1 Stack Function

Let $\mathcal{C}$ be an abelian category and $\mathbf{dim} : K_0(\mathcal{C}) \to \mathbb{Z}^n$ be a homomorphism of abelian groups. Assume that for any object C in $\mathcal{C}$, we have $\mathbf{dim}([C]) \in \mathbb{N}^n$ and $\mathbf{dim}([C]) = 0$ if and only if $C = 0$. Such a homomorphism $\mathbf{dim}$ is called a Chern character of the category $\mathcal{C}$. We will call it the dimension function.

Let $\mathrm{Iso}(\mathbf{v}, \mathcal{C})$ be the set of isomorphism classes of objects C in $\mathcal{C}$ with $\mathbf{dim}([C]) = \mathbf{v}$. With the following finiteness assumptions:

(1) For any object C in $\mathcal{C}$, the the automorphism group $\mathrm{Aut}(C)$ is a finite set;

(2) For each $\mathbf{v} \in \mathbb{N}^n$, the set $\mathrm{Iso}(\mathbf{v}, \mathcal{C})$ is finite.

The we can define the stack function

$$\mu_{\mathcal{C}}(X) = \sum_{\mathbf{v} \in \mathbb{N}^n} \sum_{C \in \mathrm{Iso}(\mathbf{v}, \mathcal{C})} \frac{1}{|\mathrm{Aut}(C)|} X^{\mathbf{v}} = \sum_{C \in \mathrm{Iso}(\mathcal{C})} \frac{1}{|\mathrm{Aut}(C)|} X^{\mathbf{dim}(C)}. \tag{1.1}$$

Assume that $\mathcal{D}$ is another such category with a dimension function $\mathbf{dim} : K_0(\mathcal{D}) \to \mathbb{Z}^n$. If there is a category equivalence $F : \mathcal{C} \to \mathcal{D}$ such that $\mathbf{dim}(F(C)) = \mathbf{dim}(C)$ for all C in $\mathcal{C}$. Then we have $\mu_{\mathcal{D}}(X) = \mu_{\mathcal{C}}(X)$.

Suppose $\mathcal{C}'$ and $\mathcal{C}''$ are two full subcategories of $\mathcal{C}$ closed under extensions in $\mathcal{C}$. Let $\mathcal{D}$ be the abelian full subcategory generated by $\mathcal{C}'$ and $\mathcal{C}''$ closed under extensions in $\mathcal{C}$. We will write $\mathcal{D} = \mathcal{C}' \oplus \mathcal{C}''$ if $\mathrm{Hom}_{\mathcal{C}}(C', C'') = \mathrm{Hom}_{\mathcal{C}}(C'', C') = 0$ for all C' in $\mathcal{C}'$ and C'' in $\mathcal{C}''$. Then every object in D is isomorphic to the form $C' \oplus C''$ and $\mathrm{Aut}(C' \oplus C'') = \mathrm{Aut}(C') \times \mathrm{Aut}(C'')$. With the restriction of the dimension function $\mathbf{dim}$ to each of the subcategories, we have

$$\mu_{\mathcal{D}}(X) = \mu_{\mathcal{C}'}(X)\mu_{\mathcal{C}''}(X). \tag{1.2}$$

Here the product is taken in the formal power series ring $\mathbb{Q}[[X_1, \cdots, X_n]]$.

For more general constructible abelian K-category for an algebraically closed field, one define the motive stack functions with $|\mathrm{Aut}(C)|$ being replaced by its motive measure under the assumption that $\mathrm{Aut}(C)$ is an algebraic variety of finite type and $\mathrm{Iso}(\mathbf{v}, \mathcal{C})$ is assumed to be stack of finite type. We refer to [5] for interested readers.

1.3.2 Generating Functions for $\Pi^1(\Gamma)$

For each fixed finite field $\mathbb{F}_q$, the formal power series $\Phi^1_\Gamma(\mathbf{w})$ is defined to be

$$\Phi(\mathbf{w}) = \sum_{\mathbf{v} \in \mathbb{N}^{\Gamma_0}} \frac{|\mathcal{V}_1(\mathbf{v}, \mathbf{w})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} \frac{|\mathfrak{g}_{\mathbf{v}}|}{|\mathbb{V}_{\mathbf{v},\mathbf{w}}|} X^{\mathbf{v}},$$

we have

$$\frac{|\mathfrak{gl}_{\mathbf{v}}(\mathbb{F}_q)|}{|\mathbb{V}_{\mathbf{v}}(\mathbb{F}_q)|} = q^{\langle \mathbf{v}, \mathbf{v} \rangle_\Gamma},$$

here $\langle\ ,\ \rangle_\Gamma$ is the Euler-Ringel form for the quiver Γ.

Therefore $\Phi(\mathbf{w})$ can be written as

$$\Phi(\mathbf{w}) = \sum_{\mathbf{v} \in \mathbb{N}^{\Gamma_0}} \frac{|\mathcal{V}_1(\mathbf{v}, \mathbf{w})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} q^{\langle \mathbf{v}, \mathbf{v} \rangle_\Gamma} X^{\mathbf{v}}.$$

By Jordan decomposition of the linear endomorphism $g_i \in gl(V_i)$, we have

$$\Phi(\mathbf{w}) = \Phi_{nil}(\mathbf{w})\Phi_{reg} = \Phi_{nil}(\mathbf{w})\frac{\Phi(\mathbf{0})}{\Phi_{nil}(\mathbf{0})}.$$

Hausel [19] gave the formula for $\Phi_{nil}(\mathbf{w})$, so we just need to compute $\Phi(\mathbf{0})$. It follows from the definition of $\Phi(\mathbf{0})$ in (2.10) that to compute $\Phi(\mathbf{0})$ we need to compute $|\mathcal{V}_1(\mathbf{v},\mathbf{0})(\mathbb{F}_q)|$. For a fixed dimension vector $\mathbf{v}$, the number of isomorphism classes of representations with dimension vector $\mathbf{v}$ of the deformed preprojective algebra $\Pi^1(\Gamma)$ over a finite field $\mathbb{F}_q$ is $|\mathcal{V}_1(\mathbf{v},0)(\mathbb{F}_q)/G_{\mathbf{v}}(\mathbb{F}_q)|$. Thus we will focus on classifying all representations of the deformed preprojective algebra $\Pi^1(\Gamma)$.

Remark 1.3.1. Here we did not emphasis Γ and weight λ for generating function Φ because we want to keep notations the same with Section 2.4.

Use

$$\mu^1_\Gamma(\mathbf{w}) = \sum_{\mathbf{v}\in\mathbb{N}^{\Gamma_0}} \frac{|\mathcal{V}_1(\mathbf{v},\mathbf{w})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} X^{\mathbf{v}}$$

to define the stack function for the category $\mathrm{rep}(\Pi^1(\Gamma))$.

Thus computing $\Phi(\mathbf{w})$ is equivalent to computing $\mu^1_\Gamma(\mathbf{w})$. In particular, $\Phi(\mathbf{0}) = 1$ if and only if $\mu^1_\Gamma(\mathbf{0}) = 1$.

On the other hand, we have

$$\frac{|\mathcal{V}_1(\mathbf{v},0)(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} = \sum_{M\in \mathrm{Iso}(\mathbf{v})} \frac{1}{|\mathrm{Aut}(M)|},$$

where $\mathrm{Iso}(\mathbf{v})$ is the isomorphism classes of representations of the deformed preprojective algebra $\Pi^1(\Gamma)$ of dimension vector $\mathbf{v}$ and $\mathrm{Aut}(M)$ is the automorphism group of a representation M. Thus we have

$$\mu^1_\Gamma(\mathbf{0}) = \sum_{M\in(\mathrm{Iso}\Pi^1(\Gamma)-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M)|} X^{\mathbf{dim}(M)}.$$

Therefore all what we need to do is to describe the category $\mathrm{rep}(\Pi^1(\Gamma))$ of finite dimensional representations of $\Pi^1(\Gamma)$ over the field $\mathbb{F}_q$.

Theorem 1.3.2. *[19, Lemma3] For any quiver Γ and any field K, there is a module in* $\text{rep}(\Pi^1(\Gamma))$ *of dimension vector* $\mathbf{v}$ *if and only if* $\mathbf{v} = \mathbf{0}$*, under the assumption* $\text{Char}K \nmid \sum v_i$.

Therefore, for deformed preprojective algebra with a special weight 1, if the ground field has characteristic 0, then there are no finite dimensional representations, so $\Phi(\mathbf{0}) = 1$; however, if the ground field has characteristic p, for example, a finite field, there are some dimension vectors whose representation space is nonempty, so $\Phi(\mathbf{0}) \neq 1$.

1.3.3 Main Results

In this thesis, we concentrate on classifying the category of finite dimensional representations of deformed preprojective algebra $\Pi^1(\Gamma)$ associated to two types of quiver over field of characteristic p. We use $\text{rep}(A)$ to represent the category of finite dimensional representation of an algebra A.

The first result describes the finite dimensional representations of $\Pi^1(\Gamma)$ associated to type A quiver.

Theorem 1.3.3. *Let K be any field of characteristic $p > 0$ and $N = np+s$ with $0 \leq s \leq p-1$. If the quiver is of A_N, then there is a categorical equivalence*

$$\text{rep}(\Pi^1(A_N)) \cong \text{rep}(\Pi^0(A_n))^{\oplus s+1} \oplus \text{rep}(\Pi^0(A_{n-1}))^{\oplus p-s-1}.$$

If the ground field K is a finite field $\mathbb{F}_q$, then the morphism sets are finite in these categories. For a quiver Γ, let $\Pi^0(\Gamma)$ be the preprojective algebra over the finite field $\mathbb{F}_q$. We define

$$\mu^0_\Gamma(q, X_\Gamma) = \sum_{M \in \text{rep}(\Pi^0(\Gamma))} \frac{1}{|\text{Aut}(M)|} X^{\mathbf{dim}(M)}.$$

Here the variable X_Γ is indexed by the vertex set Γ_0. If Γ is a disjoint union of two quivers Γ' and Γ', then we have

$$\mu^0_\Gamma(q, X_\Gamma) = \mu^0_{\Gamma'}(q, X_{\Gamma'})\mu^0_{\Gamma''}(q, X_{\Gamma''}). \tag{1.3}$$

Theorem 1.3.3 gives us a way to describe the representation category of deformed preprojective algebra associated to a quiver A_N of weight 1 in terms of the representation category of preprojective algebras associated to quiver A_n and A_{n-1}. Using Theorem 1.3.3 and (1.3), we can give a description of $\mu^1_{A_N}(\mathbf{0})$ in terms of $\mu^0_{A_n}(q, X_{A_n})$ and $\mu^0_{A_{n-1}}(q, X_{A_{n-1}})$.

The second result describes the finite dimensional representations of $\Pi^1(\Gamma)$ associated to the Jordan quiver over algebraic field of characteristic $p > 0$.

Theorem 1.3.4. *If the ground field K is an algebraically closed field of characteristic $p > 0$, then there is a categorical equivalence*

$$\mathrm{rep}(\Pi^1(\Gamma)) \cong \mathrm{rep}(\Pi^0(\Gamma)),$$

here Γ is Jordan quiver.

Chapter 2

Quiver and its Related Topics

The theory of representations of quivers is very rich and related to a lot of other topics. The quivers of finite representation type and tame representation type have been classified, and all their representations are known. A complete list of references can be found in [36]. In Section 2.1, we recall some basic definitions and properties which we will use. In the following sections, we introduce the related topics on Kac-Moody Lie algebra, (deformed) preprojective algebras, and Nakajima quiver varieties. We also introduce Hausel's proof of Kac's conjecture and also Weil conjecture which are the motivation of this thesis. This chapter serves as a preparation. We always fix K to be a ground field.

2.1 Basics For Quiver and its Representations

Definition 2.1.1. A quiver $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ is a quadruple consisting of two sets: Γ_0 (whose elements are called vertices) and Γ_1 (whose elements are called arrows), and two maps $s, t : \Gamma_1 \to \Gamma_0$ which associate to each arrow $e \in \Gamma_1$ its source $s(e) \in \Gamma_0$ and its target $t(e) \in \Gamma_0$, respectively.

Remark 2.1.2. If $s(e) = i$, $t(e) = j$, then e is usually denoted by $i \xrightarrow{e} j$. A quiver $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ is usually denoted briefly by $\Gamma = (\Gamma_0, \Gamma_1)$ or even simply by Γ.

Example 2.1.3.

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

is called A_N quiver, which has N vertices and $N-1$ arrows. In this case, the vertices set is denoted by $\{1, 2, \cdots, N\}$ and the arrows are $i \to i+1$, for $i = 1, 2, \cdots, N-1$.

Let $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ be a quiver, we associate with each point $i \in \Gamma_0$ a path ε_i of length $l = 0$, called the trivial path at i. Define product of 2 arrows e_1 and e_2 by

$$e_1 e_2 = \begin{cases} e_1 e_2, & \text{if } t(e_1) = s(e_2); \\ 0, & \text{otherwise.} \end{cases}$$

We also define product of the trivial path ε_i and the arrow $e \in \Gamma_1$ by

$$\varepsilon_i^2 = \varepsilon_i, \ \varepsilon_i e = \delta_{i,s(e)} e, \ e\varepsilon_i = \delta_{t(e),i} e,$$

where δ is the Kronecker delta. Then we can define the path algebra $K\Gamma$ of Γ to be an algebra generated by $\{\varepsilon_i\}_{i\in\Gamma_0}$ and $\{e\}_{e\in\Gamma_1}$ using product defined above over field K.

Example 2.1.4. If the quiver Γ is the Jordan quiver, which is 1 vertex and 1 loop, i.e.,

$$\overset{\alpha}{\circlearrowleft} \bullet \, ,$$

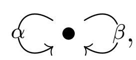
the defining basis of the path algebra $K\Gamma$ is $\{\alpha^0, \alpha, \alpha^2, \cdots, \alpha^l, \cdots\}$, where α^0 is the trivial path and the multiplication of basis vectors is given by $\alpha^l \alpha^k = \alpha^{l+k}$, for all$l, k \geq 0$. Thus $K\Gamma$ is isomorphic to the polynomial algebra $K[\alpha]$ by viewing α as one indeterminate.

Definition 2.1.5. Let K be a field, a representation (V, x) of a quiver Γ over K is a collection of K-vector spaces $\{V_i\}_{i\in\Gamma_0}$ together with a set $x = \{x(e)\}_{e\in\Gamma_1}$ of K-linear transformation $x(e) : V_{s(e)} \to V_{t(e)}$. The vector $\mathbf{dim}(V) = \mathbf{v} = (v_i)_{i\in\Gamma_0}$ is called the dimension vector of this representation, where $v_i = \dim_K(V_i)$.

Remark 2.1.6. To give a collection of vector spaces $\{V_i\}_{i\in\Gamma_0}$ is the same as to give a Γ_0-

graded vector space $V = \bigoplus_{i \in \Gamma_0} V_i$. Then each path defines an element in $\mathrm{End}(V)$. A Γ_0-graded vector space is called finite dimensional if $\dim_K V < \infty$. In this thesis, we just consider finite dimensional representations and denote the category of the finite dimensional representation category of Γ by $\mathrm{rep}(K\Gamma)$. For any finite dimensional representation (V, x) of Γ over K with $\dim(V_i) = v_i$, if we fix a basis for each K-vector space V_i, then $V_i = K^{v_i}$ for all $i \in \Gamma_0$ and every K-linear map $x(e)$ is a $v_j \times v_i$ matrix over K for arrow $i \xrightarrow{e} j$.

Example 2.1.7. Let Γ be the quiver



then $K\Gamma = K\langle \alpha, \beta \rangle$ is the free algebra. Let $I = \langle \alpha\beta - \beta\alpha - 1 \rangle \subseteq K\Gamma$, then $K\Gamma/I$ is the Weyl algebra.

Definition 2.1.8. A homomorphism f from a representation (V, x) to a representation (V', x') is a collection $\{f_i\}_{i \in \Gamma_0}$ of K-linear maps $f_i : V_i \to V'_i$ that are compatible with the structure maps $x(e)$, that is, for every arrow $i \xrightarrow{e} j$, we have $x'(e) f_i = f_j x(e)$, or equivalently, the following diagram

$$\begin{array}{ccc} V_i & \xrightarrow{x(e)} & V_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ V'_i & \xrightarrow[x'(e)]{} & V'_j \end{array}$$

is commutative. If every f_i is invertible, (V, x) and (V', x') are said to be isomorphic.

Theorem 2.1.9. *For arbitrary quiver Γ, the category rep$(K\Gamma)$ of all finite dimensional representations of Γ is equivalent to the category $K\Gamma$-mod of all finite dimensional left modules of the path algebra $K\Gamma$. Moreover, rep$(K\Gamma)$ an abelian category.*

Proof. This is a classical result. For more details, see [3]. □

Remark 2.1.10. We may also consider quiver with relations and Theorem 2.1.9 also holds for the corresponding path algebra with relations. In the future, we will abuse the notation rep($K\Gamma$) and $K\Gamma$-mod. The dimension vector actually is a Chern character, which is a group homomorphism from the Grothendieck group of the $K\Gamma$-mod to the lattice $\mathbb{Z}\Gamma_0$,

$$\mathbf{dim} : K_0(K\Gamma) \to \mathbb{Z}\Gamma_0.$$

We can use the dimension vector to classifying all representations of Γ.

For any representation (V, x) of quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with dimension vector $\mathbf{v} = (v_i)_{i\in\Gamma_0}$, we define

$$\mathbb{V}_{\mathbf{v}} = \bigoplus_{e\in\Gamma_1} \mathrm{Hom}(V_{s(e)}, V_{t(e)}).$$

To this dimension vector $\mathbf{v}$, the group

$$G_{\mathbf{v}} = \prod_{i\in\Gamma_0} GL(V_i)$$

acts on $\mathbb{V}_{\mathbf{v}}$ via the conjugate action

$$(G_i)_i \cdot (x(e))_e = (G_j x_e G_i^{-1})_{i\xrightarrow{e}j} \tag{2.1}$$

where $G_i \in GL(V_i)$.

The space $\mathbb{V}_{\mathbf{v}}$ defined above is called the space of representations of Γ with dimension vector $\mathbf{v}$, and the group $G_{\mathbf{v}}$ is called the gauge group of $\mathbb{V}_{\mathbf{v}}$. The set of $G_{\mathbf{v}}$-orbits in $\mathbb{V}_{\mathbf{v}}$ $[\mathbb{V}_{\mathbf{v}}/G_{\mathbf{v}}]$ correspond to the isomorphism classes of representations of Γ of dimension vector $\mathbf{v}$. If K is a finite field, $\mathbb{V}_{\mathbf{v}}$ and $G_{\mathbf{v}}$ are finite. If K is algebraically closed, $\mathbb{V}_{\mathbf{v}}$ are affine variety and $G_{\mathbf{v}}$ affine algebraic group acting on $\mathbb{V}_{\mathbf{v}}$. Although $\mathbb{V}_{\mathbf{v}}/G_{\mathbf{v}}$ do not have algebraic variety structure, one can use stack and we call $\mathbb{V}_{\mathbf{v}}/G_{\mathbf{v}}$ the stack of representation of Γ with dimension vector $\mathbf{v}$.

If the field K is a finite field $\mathbb{F}_q$, then for the Chern character

$$\mathbf{dim} : K_0(K\Gamma) \to \mathbb{Z}\Gamma_0,$$

we have the fibre $\mathbf{dim}^{-1}(\mathbf{v}) = \{V \in \mathrm{iso}(K\Gamma{-}\mathrm{mod}) \mid \mathbf{dim}M = \mathbf{v}\}$. Then $|\mathbf{dim}^{-1}(\mathbf{v})|$ gives the number of orbits $\mathbb{V}_{\mathbf{v}}/G_{\mathbf{v}}$, and we denote this number by $M_\Gamma(\mathbf{v}, q)$, i.e.,

$$M_\Gamma(\mathbf{v}, q) = \text{the number of isomorphism classes of representations of } \Gamma \text{ over } \mathbb{F}_q \text{ with dimension vector } \mathbf{v}.$$

We also care about the number of indecomposable representations, so let

$$I_\Gamma(\mathbf{v}, q) = \text{the number of isomorphism classes of indecomposable representations of } \Gamma \text{ over } \mathbb{F}_q \text{ with dimension vector } \mathbf{v}.$$

For finite field $K = \mathbb{F}_q$, we call module V absolutely indecomposable over K if for any finite extension field E of K, $V \bigotimes_K E$ remains indecomposable over E. Denote

$$A_\Gamma(\mathbf{v}, q) = \text{the number of isomorphism classes of absolutely indecomposable representations of } \Gamma \text{ over } \mathbb{F}_q \text{ with dimension vector } \mathbf{v}.$$

Hua [23] gave a closed formula of the generating function with coefficient $M_\Gamma(\mathbf{v}, q)$, which we will recall in Section 2.2.3 after we introduce root systems and Kac-Moody Lie algebras in Section 2.2.1 and 2.2.2.

Remark 2.1.11. If the field K is an algebraically closed field, then one define $M_\Gamma(\alpha, q) = [\mathbb{V}_{\mathbf{v}}/G_{\mathbf{v}}]$ in motivic counting. For more details, see [7].

2.2 Connection to Kac-Moody Lie Algebra

2.2.1 Root System

Definition 2.2.1. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, the Euler-Ringel form for Γ is the bilinear form on $\mathbb{Z}^{\Gamma_0}$ defined by

$$\langle \alpha, \beta \rangle_\Gamma = \sum_{i \in \Gamma_0} \alpha_i \beta_i - \sum_{i \xrightarrow{e} j \in \Gamma_1} \alpha_i \beta_j.$$

Let the corresponding symmetric bilinear form BE $(\alpha, \beta) = \langle \alpha, \beta \rangle_\Gamma + \langle \beta, \alpha \rangle_\Gamma$.

Let $\alpha_i \in \mathbb{Z}^{\Gamma_0}$ denote the coordinate vector at vertex i. If i is a loopfree vertex in Γ_0, then there is a reflection

$$\begin{aligned} r_i : \mathbb{Z}^{\Gamma_0} &\to \mathbb{Z}^{\Gamma_0}, \\ \alpha &\mapsto \alpha - (\alpha, \alpha_i)\alpha_i. \end{aligned}$$

The Weyl group is the subgroup of $\mathrm{Aut}(\mathbb{Z}^{\Gamma_0})$ generated by the r_i. The fundamental region is

$$F = \{\alpha \in \mathbb{Z}^{\Gamma_0} : \alpha \neq 0, \alpha \text{ has connected support, and } (\alpha, \alpha_i) \leq 0 \text{ for all } i\}.$$

The real roots for Γ are the orbits of coordinate vectors α_i under the Weyl group and we denote them by Δ_{re}. The imaginary roots for Γ are the orbits of $\pm\alpha$ (for $\alpha \in F$) under the Weyl group and we denote them by Δ_{im}. The α_i defined above is called simple roots.

The root system Δ is defined as

$$\Delta = \Delta_{re} \cup \Delta_{im}.$$

An element $\alpha \in \Delta \cap \mathbb{Z}_+^{\Gamma_0}$ is called positive root and we denote all the positive (resp. real or imaginary) roots by Δ^+ (resp. Δ_{re}^+ or Δ_{im}^+).

If α is a root, then so is $-\alpha$. This is true by definition for imaginary roots. It holds for real roots since $r_i(\alpha_i) = -\alpha_i$ if i is a loopfree vertex. In particular, every root α is either positive or $-\alpha$ is positive, i.e., $\Delta^- = -\Delta^+$.

Definition 2.2.2. A nonzero element $\mathbf{v} = (v_i)_{i\in\Gamma_0}$ of $\mathbb{Z}^{\Gamma_0}$ is called indivisible if $\gcd(v_i) = 1$.

Then by definition, clearly we have any real root is indivisible; if α is a real root, only $\pm\alpha$ are roots. Every imaginary root is a multiple of an indivisible root, and all other nonzero multiples are also roots.

Theorem 2.2.3. *[26] If K is an algebraically field, then we have the following result.*

(1) If there is an indecomposable representation of Γ of dimension α, then α is a root.

(2) If α is a positive real root, there is a unique indecomposable representation of Γ of dimension α up to isomorphism.

(3) If α is a positive imaginary root, then there are infinitely many indecomposable representations of Γ of dimension α up to isomorphism.

Example 2.2.4. From [3, p299], we know the positive roots of the corresponding finite dimensional simple Lie algebra, so for A_N quiver Γ, we get the indecomposable $K\Gamma$-modules have dimension vector

$$(\underbrace{0,\cdots,0}_{l_1},\underbrace{1,\cdots,1}_{l_2},\underbrace{0,\cdots,0}_{l_3})$$

here $l_1 \geq 0$, $l_2 \geq 0$, $l_3 \geq 0$ and $l_1 + l_2 + l_3 = N$. We denote this the indecomposable module by $\underline{I}(l_1 + 1, l_2)$.

2.2.2 Kac-Moody Lie Algebra

For any quiver $\Gamma = (\Gamma_0, \Gamma_1)$, under the notation in Section 2.2.1, the matrix $C = (\alpha_i, \alpha_j)_{i,j\in\Gamma_0}$ is a generalized Cartan matrix, so there is an associated Kac-Moody Lie algebra $\mathfrak{g}(\Gamma)$.

Example 2.2.5. If the quiver Γ is A_2 quiver: $\bullet \longrightarrow \bullet$, then $\alpha_1 = (1, 0)$ and $\alpha_2 = (0, 1)$. So we have,

$$\langle\alpha_1, \alpha_1\rangle_\Gamma = 1,\ \langle\alpha_2, \alpha_2\rangle_\Gamma = 1,\ \langle\alpha_1, \alpha_2\rangle_\Gamma = -1,\ \langle\alpha_2, \alpha_1\rangle_\Gamma = 0;$$

and then

$$(\alpha_1, \alpha_1) = 2,\ (\alpha_2, \alpha_2) = 2,\ (\alpha_1, \alpha_2) = -1,\ (\alpha_2, \alpha_1) = -1.$$

So the corresponding Cartan matrix

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and the associated Kac-Moody Lie algebra $\mathfrak{g}(\Gamma)$ is the semi-simple Lie algebra $\mathfrak{sl}_3$.

The Kac-Moody Lie algebra $\mathfrak{g}(\Gamma)$ is Δ-graded and

$$\mathfrak{g}(\Gamma) = (\bigoplus_{\alpha \in \Delta^- \setminus \{0\}} \mathfrak{g}_\alpha) \oplus \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta^+ \setminus \{0\}} \mathfrak{g}_\alpha).$$

Here $\mathfrak{g}_\alpha$ is the root space attached to α and note that $\mathfrak{h} = \mathfrak{g}_0$ is the Cartan subalgebra. The number $\mathrm{mult}(\alpha) := \dim_K \mathfrak{g}_\alpha$ is called the multiplicity of α.

Let V be a weight $\mathfrak{g}(\Gamma)$-module and let $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ be its weight space decomposition. We define the formal character of V by formal series

$$\mathrm{ch}V = \sum_{\lambda \in \mathfrak{h}^*} (\dim_K(V_\lambda) X^\lambda),$$

here $\mathfrak{h}^*$ is the dual K-vector space of $\mathfrak{h}$.

Let $\rho = \sum_i \frac{\alpha_i}{2}$ and use W to denote the Weyl group. Then the Weyl-Kac character formula [28] says the following:

Theorem 2.2.6. *Let $L(\Lambda)$ be an irreducible representation of $\mathfrak{g}(\Gamma)$ of highest weight Λ. Let $L(\Lambda) = \bigoplus_{\alpha \in \mathbb{N}^{\Gamma_0}} L(\Lambda)_{\Lambda - \alpha}$ denote its weight space decomposition. Then*

$$\mathrm{ch}L(\Lambda) = \frac{\sum_{w \in W} \det(w) X^{\Lambda + \rho - w(\Lambda + \rho)}}{X^\rho \prod_{\alpha \in \Delta^+} (1 - X^{-\alpha})^{\mathrm{mult}(\alpha)}}. \tag{2.2}$$

For more details of Kac-Moody Lie algebra, see [28].

2.2.3 Kac's Conjecture

Fix the field K be a finite field $\mathbb{F}_q$ in this section. Recall that $A_\Gamma(\alpha, q)$ denote the number of isomorphism classes of absolutely indecomposable representations of Γ over $\mathbb{F}_q$ of dimension vector α. The following theorem gives the relation between the representations of quivers and their root systems. It is due to Gabriel [15] in finite type, to Donovan and Freislich [12] and independently to Nazarova [34] in tame type, and to Kac [27] in general.

Theorem 2.2.7. *Let Γ be a connected quiver, then we have,*

(1) The polynomials $A_\Gamma(\alpha, q)$ are independent of the orientation of Γ.

(2) There exists an absolutely indecomposable representation of Γ over $\mathbb{F}_q$ with dimension vector α if and only if α is a positive root in Δ.

(3) $A_\Gamma(\alpha, q) = 1$ if and only if α is a positive real root.

(4) For any w in the Weyl group, $A_\Gamma(w(\alpha), q) = A_\Gamma(\alpha, q)$ provided α and $w(\alpha)$ are positive roots.

Then Kac proved in [27] that for all $\alpha \in \mathbb{N}^{\Gamma_0}$, $M_\Gamma(\alpha, q)$, $I_\Gamma(\alpha, q)$ and $A_\Gamma(\alpha, q)$ are polynomials in q with rational coefficients, which are independent of the orientation of Γ.

Example 2.2.8. [22, Appendix A] If Γ is

$$\bullet \rightrightarrows \bullet \to \bullet,$$

then

$$M_\Gamma(1,2,1,q) = 4q+7; \quad M_\Gamma(2,2,2,q) = 4q^2+13q+21;$$

$$I_\Gamma(1,2,1,q) = q+1; \quad I_\Gamma(2,2,2,q) = \frac{1}{2}q^2+\frac{1}{2}q+1;$$

$$A_\Gamma(1,2,1,q) = q+1; \quad A_\Gamma(2,2,2,q) = q+1.$$

Proposition 2.2.9. *[27, Proposition1.5] $A_\Gamma(\alpha, q)$ is a polynomial in q with integer coefficients.*

Moreover, he conjectured in [27]

Conjecture 2.2.10. *The constant term of $A_\Gamma(\alpha, q)$ equals the multiplicity of the weight α in $\mathfrak{g}(\Gamma)$:*

$$A_\Gamma(\alpha, 0) = \mathrm{mult}(\alpha).$$

Conjecture 2.2.11. *The coefficients of $A_\Gamma(\alpha, q)$ are nonnegative.*

Remark 2.2.12. Conjecture 2.2.11 was proved in [21], and we just concentrate Conjecture 2.2.10 in this thesis, so when we say Kac's conjecture in this thesis, we mean Conjecture 2.2.10.

Recall that a partition $\mu = (\mu_1, \mu_2, \cdots)$ is a finite sequence with $\mu_1 \geq \mu_2 \geq \cdots$ of non-negative integers. The integer $|\mu| = \mu_1 + \mu_2 + \cdots$ is called the weight of μ. We use $\mathcal{P}$ to denote the set of all partitions including the unique partition of 0. Note that any partition μ can be written in the exponential form $(1^{n_1}, 2^{n_2}, \cdots)$, which means that there are n_i parts equal to i in μ.

For a collection of partitions $\lambda = (\lambda^i)_{i \in \Gamma_0} \in \mathcal{P}^{\Gamma_0}$, denote $|\lambda|$ to be the vector of $(|\lambda^i|)_{i \in \Gamma_0} \in \mathbb{N}^{\Gamma_0}$. For any two partitions $\lambda^1 = (1^{m_1(\lambda^1)}, 2^{m_2(\lambda^1)}, \cdots)$ and $\lambda^2 = (1^{m_1(\lambda^2)}, 2^{m_2(\lambda^2)}, \cdots)$, we use the notation

$$\langle \lambda^1, \lambda^2 \rangle = \sum_{i,j} \min(i,j) m_i(\lambda^1) m_j(\lambda^2). \tag{2.3}$$

J. Hua [23] gives a combinatorial formula for $A_\Gamma(\alpha, q)$.

Theorem 2.2.13. *[23, Theorem 4.9] If $A_\Gamma(\alpha, q) = \sum_j t_j^\alpha q^j$, the following formal identity holds:*

$$\sum_{\lambda \in \mathcal{P}^{\Gamma_0}} \frac{\prod_{i \to j \in \Gamma_1} q^{\langle \lambda^i, \lambda^j \rangle}}{\prod_{i \in \Gamma_0} (q^{\langle \lambda^i, \lambda^j \rangle} \prod_k \prod_{j=1}^{m_k(\lambda^i)} (1 - q^{-j}))} X^{|\lambda|} = \prod_{\alpha \in \mathbb{N}^{\Gamma_0}} \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1 - q^{i+j} X^\alpha)^{t_j^\alpha}. \tag{2.4}$$

And he pointed out immediately the following:

Corollary 2.2.14. *[23, Corollary 4.10] Conjecture 2.2.10 is true if and only if the following identity holds:*

$$\lim_{q\to 0}\sum_{\lambda\in\mathcal{P}^{\Gamma_0}}\frac{\prod\limits_{i\to j\in\Gamma_1} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod\limits_{i\in\Gamma_0}(q^{\langle\lambda^i,\lambda^j\rangle}\prod\limits_{k}\prod\limits_{j=1}^{m_k(\lambda^i)}(1-q^{-j}))}X^{|\lambda|}=\sum_{w\in W}(-1)^{l(w)}X^{s(w)}. \tag{2.5}$$

Here W is the Weyl group, $l(w)$ is the length of w and $s(w)$ is the sum of positive roots mapped into negative roots by w^{-1}.

Besides $A_\Gamma(\alpha, q)$, Hua gave a closed formula for $M_\Gamma(\alpha, q)$ as follows:

Theorem 2.2.15. *[23, Theorem 4.3] The following formal identity holds:*

$$\sum_{\alpha\in\mathbb{N}^{\Gamma_0}} M_\Gamma(\alpha,q)X^\alpha=\prod_{d=1}^{\infty}(\sum_{\lambda\in\mathcal{P}^{\Gamma_0}}\frac{\prod\limits_{i\to j\in\Gamma_1} q^{d\langle\lambda^i,\lambda^j\rangle}}{\prod\limits_{i\in\Gamma_0}(q^{d\langle\lambda^i,\lambda^j\rangle}\prod\limits_{k}\prod\limits_{j=1}^{m_k(\lambda^i)}(1-q^{-dj}))}X^{d|\lambda|})^{\phi_d(q)},$$

where $\phi_1(q) = q-1$, and for $d \geq 2$, $\phi_d(q)$ is the number of monic irreducible polynomials of degree d over $\mathbb{F}_q$.

Moreover, if $A_\Gamma(\alpha, q) = \sum_{j=1}^{u_\alpha} t_j^\alpha q^j$ with $t_j^\alpha \in \mathbb{Z}$, J. Hua [23] also gave a formula for $M_\Gamma(\alpha, q)$:

Theorem 2.2.16. *[23, Theorem 4.7] The following formal identity holds:*

$$\sum_{\alpha\in\mathbb{N}^{\Gamma_0}} M_\Gamma(\alpha,q)X^\alpha=\prod_{\alpha\in\Delta^+}\prod_{j=1}^{u_\alpha}(1-q^jX^\alpha)^{-t_j^\alpha}.$$

Conjecture 2.2.10 was finally proved by Tamas Hausel in [19], and we will give a brief introduction of the proof in Section 2.4.

2.3 Preprojective Algebra Associated to a Quiver and Nakajima Quiver Variety

2.3.1 Preprojective Algebra

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver.

Definition 2.3.1. The opposite quiver Γ^{op} of Γ is the quiver obtained by reversing any arrow $i \xrightarrow{e} j$ in Γ_1. We denote arrow set of Γ^{op} by Γ_1^*.

Definition 2.3.2. The double quiver $\overline{\Gamma}$ of Γ is the quiver obtained by adjoining an arrow $j \xrightarrow{e^*} i$ for each arrow $i \xrightarrow{e} j$ in Γ_1. We denote it by $\overline{\Gamma} = (\Gamma_0, \Gamma_1 \cup \Gamma_1^*)$.

Example 2.3.3. If the quiver Γ is type A_N quiver with N vertices and $N-1$ arrows, i.e.,

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet,$$

then the corresponding opposite quiver is

$$\bullet \longleftarrow \bullet \longleftarrow \cdots \longleftarrow \bullet,$$

and the corresponding double quiver is

$$\bullet \underset{\longleftarrow}{\overset{\longrightarrow}{}} \bullet \underset{\longleftarrow}{\overset{\longrightarrow}{}} \bullet \underset{\longleftarrow}{\overset{\longrightarrow}{}} \cdots \underset{\longleftarrow}{\overset{\longrightarrow}{}} \bullet \underset{\longleftarrow}{\overset{\longrightarrow}{}} \bullet \underset{\longleftarrow}{\overset{\longrightarrow}{}} \bullet\ .$$

Definition 2.3.4. The preprojective algebra associated to quiver Γ is the associative algebra

$$\Pi(\Gamma) = K\overline{\Gamma}/(\sum_{e \in \Gamma_1} [e^*, e]).$$

More generally, the deformed preprojective algebra associated to Γ with weight $\lambda = (\lambda_i)_{i \in \Gamma_0} \in K^{\Gamma_0}$ is

$$\Pi^\lambda(\Gamma) = K\overline{\Gamma}/(\sum_{e \in \Gamma_1} [e^*, e] - \sum_{i \in \Gamma_0} \lambda_i 1_i).$$

Remark 2.3.5. For consistency, we denote $\Pi(\Gamma)$ by $\Pi^0(\Gamma)$. It is easy to know that $\Pi^\lambda(\Gamma)$ doesn't depend on the orientation of Γ. Just reverse the role of e and e^*, and change the sign of one of them.

Example 2.3.6. If the quiver Γ is Jordan quiver, i.e.

 ,

then

(1) $\Pi^0(\Gamma) = K[x, y]$ is the polynomial algebra in 2 indeterminates;

(2) $\Pi^1(\Gamma) = K\langle x, y\rangle/([x, y] = 1)$ is Weyl algebra.

We defer more details of preprojective algebra and its representations to Chapter 3.

2.3.2 Nakajima Quiver Variety

Actually the relations for the deformed preprojective algebra arise from a moment map. Let's define the Nakajima quiver variety using the moment map. Since Hausel [19] use counting over Nakajima quiver variety to prove Kac's conjecture 2.2.10, so we follow the notations in Hausel's paper [19].

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, if $\Gamma_1 = \{1, \cdots, n\}$, to each vertex i, we associate two finite dimensional K-vector spaces V_i and W_i, and denote $(\mathbf{v}, \mathbf{w}) = (v_1, \cdots, v_n, w_1, \cdots, w_n)$, where $v_i = \dim V_i$ and $w_i = \dim W_i$. To this data we have the grand vector space,

$$\mathbb{V}_{(\mathbf{v},\mathbf{w})} = \bigoplus_{e\in\Gamma_1} \mathrm{Hom}(V_{s(e)}, V_{t(e)}) \oplus \bigoplus_{i\in\Gamma_0} \mathrm{Hom}(W_i, V_i);$$

for the group

$$G_{\mathbf{v}} = \prod_{i\in\Gamma_0} GL(V_i),$$

and its Lie algebra

$$\mathfrak{g}_{\mathbf{v}} = \bigoplus_{i\in\Gamma_0} gl(V_i),$$

there is a natural representation

$$\rho_{\mathbf{v},\mathbf{w}} : G_{\mathbf{v}} \to GL(\mathbb{V}_{(\mathbf{v},\mathbf{w})}),$$

which acts on the first term by conjugation and acts from the left on the second term, and with a derivative,

$$\varrho_{\mathbf{v},\mathbf{w}} : \mathfrak{g}_{\mathbf{v}} \to gl(\mathbb{V}_{(\mathbf{v},\mathbf{w})}).$$

Let $\overline{\mathbb{V}}_{(\mathbf{v},\mathbf{w})} = \mathbb{V}_{(\mathbf{v},\mathbf{w})} \times \mathbb{V}^*_{(\mathbf{v},\mathbf{w})}$, then it has a natural symplectic bilinear form $\langle \cdot, \cdot \rangle$. The group $G_{\mathbf{v}}$ acts on $\overline{\mathbb{V}}_{(\mathbf{v},\mathbf{w})}$ preserving the symplectic form,

$$\overline{\rho}_{\mathbf{v},\mathbf{w}} := \rho_{\mathbf{v},\mathbf{w}} \times \rho^*_{\mathbf{v},\mathbf{w}} : G_{\mathbf{v}} \to GL(\overline{\mathbb{V}}_{(\mathbf{v},\mathbf{w})}),$$

with derivative,

$$\overline{\varrho}_{\mathbf{v},\mathbf{w}} : \mathfrak{g}_{\mathbf{v}} \to gl(\overline{\mathbb{V}}_{(\mathbf{v},\mathbf{w})}).$$

Concretely, if $(x, x^*) = (A(e), I_i, B(e), J_i)_{e\in\Gamma_1, i\in\Gamma_0} \in \overline{\mathbb{V}}$, $G = (G_i)_{i\in\Gamma_0} \in G_{\mathbf{v}}$ and $g = (g_i)_{i\in\Gamma_0} \in \mathfrak{g}_{\mathbf{v}}$, then

$$\overline{\rho}_{\mathbf{v},\mathbf{w}}(G)(x, x^*) = (G_{t(e)}A(e)G^{-1}_{s(e)}, G_i I_i, G_{s(e)}B(e)G^{-1}_{t(e)}, JG_i^{-1})_{e\in\Gamma_1, i\in\Gamma_0},$$

and

$$\overline{\varrho}_{\mathbf{v},\mathbf{w}}(g)(x, x^*) = (g_{t(e)}A(e) - A(e)g_{s(e)}, g_i I_i, g_{s(e)}B(e) - B(e)g_{t(e)}, -Jg_i)_{e\in\Gamma_1, i\in\Gamma_0}.$$

Definition 2.3.7. The moment map

$$\mu_{\mathbf{v},\mathbf{w}} : \overline{\mathbb{V}}_{(\mathbf{v},\mathbf{w})} \to \mathfrak{g}^*_{\mathbf{v}}$$

is defined by: if $(x, x^*) = (A(e), I_i, B(e), J_i)_{e\in\Gamma_1, i\Gamma_0}$, then

$$\mu_{\mathbf{v},\mathbf{w}}(x, x^*) = \big(I_iJ_i + \sum_{e\in s^{-1}(i)} B(e)A(e) - \sum_{e\in t^{-1}(i)} A(e)B(e)\big)^*_{i\in\Gamma_0} \in \mathfrak{g}^*_{\mathbf{v}}, \tag{2.6}$$

here we identify any elements $g = (g_i)_{i\in\Gamma_0} \in \mathfrak{g}_{\mathbf{v}}$ with the linear form $g^* : \mathfrak{g}_{\mathbf{v}} \to K$ given by $g^*((g'_i)_{i\in\Gamma_0}) = \sum_{i\in\Gamma_0} \operatorname{tr}(g_ig'_i) \in K$.

The moment map $\mu_{\mathbf{v},\mathbf{w}}$ is $G_{\mathbf{v}}$-equivariant. For $\lambda = (\lambda_i)_{i\in\Gamma_0} \in k^{\Gamma_0}$, the element $\lambda\mathbf{1}_{\mathbf{v}} = (\lambda_1 Id_{V_1}, \cdots, \lambda_n Id_{V_n}) \in \mathfrak{g}_{\mathbf{v}}$ is fixed by all element of $G_{\mathbf{v}}$. Define

$$\mathcal{V}_\lambda(\mathbf{v}, \mathbf{w}) = \mu^{-1}_{\mathbf{v},\mathbf{w}}(\lambda\mathbf{1}),$$

it is a $G_{\mathbf{v}}$-stable subset of $\overline{\mathbb{V}}_{(\mathbf{v},\mathbf{w})}$.

Remark 2.3.8. It is not hard to see if $\mathbf{w} = \mathbf{0}$, then $\mathcal{V}_\lambda(\mathbf{v}, \mathbf{w})$ is the deformed preprojective algebra $\Pi^\lambda(\Gamma)$.

For $l \in \mathbb{Z}$, we have the character

$$\begin{array}{rcl} \chi^l : G_{\mathbf{v}} & \to & K^\times \\ (g_i)_{i\in\Gamma_0} & \mapsto & \Pi_{i\in\Gamma_0}\det(g_i)^l. \end{array}$$

With these we define

$$K[\mathcal{V}_\lambda(\mathbf{v}, \mathbf{w})]^{G_{\mathbf{v}},\chi^l} := \{f \in K[\mathcal{V}_\lambda(\mathbf{v}, \mathbf{w})] \mid f(g(x)) = \chi^l(g)f(x) \text{ for all } x \in \mathcal{V}_\lambda(\mathbf{v}, \mathbf{w})\},$$

so

$$\bigoplus_{n\in\mathbb{N}} K[\mathcal{V}_\lambda(\mathbf{v}, \mathbf{w})]^{G_{\mathbf{v}},\chi^{ln}}$$

becomes an $\mathbb{N}$-graded algebra.

Definition 2.3.9. The Nakajima quiver variety [32] is the GIT quotient,

$$\mathcal{M}_{l,\lambda}(\mathbf{v},\mathbf{w}) = \mathrm{Proj}(\bigoplus_{\mathrm{n}\in\mathbb{N}} \mathrm{K}[\mathcal{V}_\lambda(\mathbf{v},\mathbf{w})]^{\mathrm{G}_\mathbf{v},\chi^{\mathrm{l}\mathrm{n}}}).$$

As an affine GIT quotient

$$\mathcal{M}_{0,\lambda}(\mathbf{v},\mathbf{w}) = \mathrm{Spec}(\mathrm{K}[\mathcal{V}_\lambda(\mathbf{v},\mathbf{w})]^{\mathrm{G}_\mathbf{v}}) = \mathrm{K}[\mathcal{V}_\lambda(\mathbf{v},\mathbf{w})]//\mathrm{G}_\mathbf{v}.$$

By the GIT construction we have the map $\mathcal{M}_{1,\lambda}(\mathbf{v},\mathbf{w}) \to \mathcal{M}_{0,\lambda}(\mathbf{v},\mathbf{w})$, which is proper and a resolution of singularities.

Next, we list some properties of $\mathcal{M}_{l,\lambda}(\mathbf{v},\mathbf{w})$ which were used by Hausel in his proof of Kac's conjecture 2.2.10.

If $d_{\mathbf{v},\mathbf{w}} = (\sum_{e\in\Gamma_1} v_{s(e)}v_{t(e)} + \sum_{i\in\Gamma_0} v_i(w_i - v_i))$, then from [32, Corollary 3.12],

Lemma 2.3.10. $\mathcal{M}_{1,\lambda}(\mathbf{v},\mathbf{w})$ *is non-singular of dimension* $2d_{\mathbf{v},\mathbf{w}}$ *for all* λ.

Lemma 2.3.11. *[19, Lemma 7] For* $\lambda \neq 0$, *the variety* $\mathcal{M}_{0,\lambda}(\mathbf{v},\mathbf{w})$ *is non-singular of dimension* $2d_{\mathbf{v},\mathbf{w}}$ *and hence* $\mathcal{M}_{1,\lambda}(\mathbf{v},\mathbf{w}) \cong \mathcal{M}_{0,\lambda}(\mathbf{v},\mathbf{w})$.

Theorem 2.3.12. *[19, Theorem 8] When* $K = \mathbb{C}$ *the mixed Hodge structure on the isomorphic cohomologies* $\mathrm{H}^*(\mathcal{M}_{1,0}(\mathbf{v},\mathbf{w}) = \mathrm{H}^*(\mathcal{M}_{0,1}(\mathbf{v},\mathbf{w}))$*is pure.*

In the future, $\mathcal{M}(\mathbf{v},\mathbf{w}) = \mathcal{M}_{1,0}(\mathbf{v},\mathbf{w})(\mathbb{C})$.

2.4 Tamas Hausel'S Proof of Kac's Conjecture

Kac [27] proved the conjecture 2.2.10 for finite and tame quivers, but not for wild quivers. The next major proof was made by W. Crawley-Boevey and M. Van den Bergh [9], who proved conjecture 2.2.10 in the special case where α is indivisible by using quiver varieties [30]. They showed if α is indivisible, the polynomial $A_\Gamma(\alpha, q)$ is the Poincare polynomial of the quiver variety associated to Γ and α. In particular, its constant term equals to the dimension of the top nonvanishing cohomology group of it, which actually equals to the multiplicity of α in $\mathfrak{g}(\Gamma)$.

The main result of Tamas Hausel's [19] work is to prove conjecture 2.2.10 for arbitrary α. By counting points of the framed quiver variety $\mathcal{M}(\mathbf{v},\mathbf{w})$ over finite fields via Fourier transforms, which Hausel introduced in [18], and by Theorem 2.3.12 the mixed Hodge structure on $\mathrm{H}^*(\mathcal{M}(\mathbf{v},\mathbf{w})$ is pure, he applied Kataz's result [20, Appendix, Theorem 6.1.2] connecting the arithmetic and the cohomology of so-called polynomial-count varieties to get a generating function of Betti numbers of Nakajima quiver varieties.

Theorem 2.4.1. *[19, Theorem 1] Fix* $\mathbf{w} \in \mathbb{N}^{\Gamma_0}$*, denote*

$$b_i(\mathcal{M}(\mathbf{v},\mathbf{w})) := \dim(\mathrm{H}^i(\mathcal{M}(\mathbf{v},\mathbf{w}))).$$

Then in notation of (2.3)*, we get*

$$\sum_{\mathbf{v}\in\mathbb{N}^{\Gamma_0}}\sum_{i=0}^{d_{\mathbf{v},\mathbf{w}}} b_{2i}(\mathcal{M}(\mathbf{v},\mathbf{w}))q^{d_{\mathbf{v},\mathbf{w}}-i}X^{\mathbf{v}} = \frac{\sum\limits_{\lambda\in\mathcal{P}^{\Gamma_0}} \frac{(\prod\limits_{i\to j\in\Gamma_1} q^{\langle\lambda^i,\lambda^j\rangle})(\prod\limits_{i\in\Gamma_0} q^{\langle\lambda^i,1^{w_i}\rangle})}{\prod\limits_{i\in\Gamma_0}(q^{\langle\lambda^i,\lambda^j\rangle}\prod\limits_k\prod\limits_{j=1}^{m_k(\lambda^i)}(1-q^{-j}))} X^{|\lambda|}}{\sum\limits_{\lambda\in\mathcal{P}^{\Gamma_0}} \frac{\prod\limits_{i\to j\in\Gamma_1} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod\limits_{i\in\Gamma_0}(q^{\langle\lambda^i,\lambda^j\rangle})\prod\limits_k\prod\limits_{j=1}^{m_k(\lambda^i)}(1-q^{-j}))} X^{|\lambda|}}. \tag{2.7}$$

Then Hausel combines and compares the following results

(1) J. Hua's combinatorial formula for $A_\Gamma(\alpha,q)$ (2.4), and we rewrite it here for convenient: If $A_\Gamma(\alpha,q)=\sum\limits_j t_j^\alpha q^j$, then

$$\sum_{\lambda\in\mathcal{P}^{\Gamma_0}} \frac{\prod\limits_{i\to j\in\Gamma_1} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod\limits_{i\in\Gamma_0}(q^{\langle\lambda^i,\lambda^j\rangle}\prod\limits_k\prod\limits_{j=1}^{m_k(\lambda^i)}(1-q^{-j}))} X^{|\lambda|} = \prod_{\alpha\in\mathbb{N}^{\Gamma_0}}\prod_{i=0}^{\infty}\prod_{i=0}^{\infty}(1-q^{i+j}X^\alpha)^{t_j^\alpha}. \tag{2.8}$$

(2) The Weyl-Kac character formula (2.2) says the following:

Theorem 2.4.2. *Let* $L(\Lambda)$ *be an irreducible representation of* $\mathfrak{g}(\Gamma)$ *of highest weight* Λ*.*

Let $L(\Lambda) = \bigoplus_{\alpha \in \mathbb{N}^{\Gamma_0}} L(\Lambda)_{\Lambda-\alpha}$ *denote its weight space decomposition. Then*

$$\sum_{\alpha \in \mathbb{N}^{\Gamma_0}} \dim(L(\Lambda)_{\Lambda-\alpha}) X^{\alpha} = \frac{\sum_{w \in W} \det(w) X^{\Lambda+\rho-w(\Lambda+\rho)}}{\sum_{w \in W} \det(w) X^{\rho-w(\rho)}}.$$

(3) Nakajima in [32] gives a geometrical interpretation of the irreducible representation $L(\Lambda)$, and his description of the top cohomology of $\mathcal{M}(\mathbf{v}, \mathbf{w})$ as the weight space of the simple $\mathfrak{g}$-module with highest weight $\mathbf{w}$ which is the special case of $x = 0$ in [32, Theorem 10.2].

We use the following notations: $\alpha_i (i \in \Gamma_0)$ denote the linearly independent simple roots and $\rho = \sum_{i \in \Gamma_0} \alpha_i$, and pick the h_i satisfying $(h_i, \alpha_j) = \delta_{ij} - \frac{b_{ij}}{2}$, where $(,)$ denote the symmetric bilinear form associated to the Euler-Ringel form, and b_{ij} denote the number of edges of Γ between i and j. Finally pick Λ_i such that $(h_j, \Lambda_i) = \delta_{ij}$. For $\mathbf{w} \in \mathbb{N}^{\Gamma_0}$, let $\Lambda_{\mathbf{w}} = \sum_i w_i \Lambda_i$. Use W to denote the Weyl group.

Theorem 2.4.3. *Fix* $\mathbf{w} \in \mathbb{N}^{\Gamma_0}$, *then*

$$\sum_{\mathbf{v} \in \mathbb{N}^{\Gamma_0}} \dim(\mathrm{H}^{2d_{\mathbf{v},\mathbf{w}}}(\mathcal{M}(\mathbf{v}, \mathbf{w}))) X^{\mathbf{v}} = \sum_{\alpha \in \mathbb{N}^{\Gamma_0}} \dim(L(\Lambda_{\mathbf{w}})_{\Lambda_{\mathbf{w}-\alpha}}) X^{\mathbf{v}} \tag{2.9}$$

Finally by taking a limit as $\mathbf{w}$ goes to ∞, he manages to prove the Conjecture 2.2.10.

In Hausel's proof of (2.7), he introduced the grand generating function

$$\Phi(\mathbf{w}) = \sum_{\mathbf{v} \in \mathbb{N}^{\Gamma_0}} \frac{|\mathcal{V}_1(\mathbf{v}, \mathbf{w})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} \frac{|\mathfrak{g}_{\mathbf{v}}|}{|\mathbb{V}_{\mathbf{v},\mathbf{w}}|} X^{\mathbf{v}} \tag{2.10}$$

to compute $\sum_{\mathbf{v} \in \mathbb{N}^{\Gamma_0}} \sum_{i=0}^{d_{\mathbf{v},\mathbf{w}}} b_{2i}(\mathcal{M}(\mathbf{v}, \mathbf{w})) q^{d_{\mathbf{v},\mathbf{w}}-i} X^{\mathbf{v}}$, where $\mathcal{V}_1(\mathbf{v}, \mathbf{w}) = \mu_{\mathbf{v},\mathbf{w}}^{-1}(\mathbf{1})$ correspond to the vector $\mathbf{1} = (Id_1, \cdots, Id_n) \in \mathfrak{g}_{\mathbf{v}}$. By decomposition of the linear endomorphism $x_i \in gl(V_i)$, we have

$$\Phi(\mathbf{w}) = \Phi_{nil}(\mathbf{w}) \Phi_{reg} = \Phi_{nil}(\mathbf{w}) \frac{\Phi(\mathbf{0})}{\Phi_{nil}(\mathbf{0})}.$$

In the computation of $\Phi(\mathbf{w})$, he computed $\Phi_{nil}(\mathbf{w})$ and use

Lemma 2.4.4. *[19, Lemma 3] Let $\mathbf{w} = \mathbf{0}$ and $\mathbf{v} \in \mathbb{N}^{\Gamma_0}$ arbitrary. For $\lambda \in K^\times$, let $\lambda \mathbf{1}_\mathbf{v} = (\lambda \mathrm{Id}_{V_1}, \cdots, \lambda \mathrm{Id}_{V_n}) \in \mathfrak{g}_\mathbf{v}$. Further assume char$(K) \nmid \sum_i v_i$, then the equation $\mu_{\mathbf{v},\mathbf{0}}(v, v^*) = \lambda \mathbf{1}_\mathbf{v}{}^*$ has a solution if and only if $\mathbf{v} = \mathbf{0}$.*

to make the conclusion $\Phi(\mathbf{0}) = 1$, then get

$$\Phi(\mathbf{w}) = \frac{\Phi_{nil}(\mathbf{w})}{\Phi_{nil}(\mathbf{0})}.$$

However, Lemma 2.4.4 does not give $\Phi(\mathbf{0}) = 1$, and in some cases, $\Phi(\mathbf{0}) = 1$ will never happen.

Example 2.4.5. If $K = \mathbb{F}_q$ is a finite field of characteristic 2, the quiver is

$$\bullet \longrightarrow \bullet,$$

by computation, $\mathcal{V}_1(\mathbf{v}, \mathbf{w})$ is nonempty if and only if $\mathbf{v} = (n, n)$ with $n \in \mathbb{N}$, then we have $\Phi(\mathbf{0}) = \sum_{n \geq 0} (\prod_{i=0}^{n} \frac{q^i}{q^i - 1}) X_1^n X_2^n$.

Although Hausel's proof of Conjecture 2.2.10 is still correct, because by Lemma 2.4.4, 2.7 holds for infinitely many q for a fixed $\mathbf{v}$, it is still interesting to get the correct formula of $\Phi(\mathbf{0})$.

2.5 Weil Conjecture

Let $K = \mathbb{F}_q$ be a finite field with q elements, and X is a scheme of finite type over K. $\overline{K}$ is algebraic closure of K, and let $\overline{X} = X \times_k \overline{k}$ be the corresponding scheme over $\overline{K}$. For each integer $r \geq 1$, let N_r denote the number of points of $\overline{X}$ which are rational over the field $K_r = \mathbb{F}_{q^r}$ of q^r elements.

Definition 2.5.1. The Zeta function of X is the power series with rational coefficietns:

$$Z(t) = Z(X; t) = \exp(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}).$$

Example 2.5.2. If X is the projective line $\mathbb{P}^1$, then over any field, $\mathbb{P}^1$ has one more point than the number of elements of the field. Hence $N_r = q^r + 1$, and

$$Z(\mathbb{P}^1; t) = \exp(\sum_{r=1}^{\infty} (q^r + 1) \frac{t^r}{r}).$$

But $\sum_{r=1}^{\infty} \frac{t^r}{r} = -\ln(1-t)$ for $|t < 1|$ and $\sum_{r=1}^{\infty} \frac{q^r t^r}{r} = -\ln(1-qt)$ for $|qt < 1|$, so

$$Z(\mathbb{P}^1; t) = \frac{1}{(1-t)(1-qt)}.$$

We get this zeta function is not only a power series, but a rational function of t.

Example 2.5.3. It is not much harder to do n-dimensional projective space $\mathbb{P}^n$,

$$N_r = 1 + q^r + q^{2r} + \cdots + q^{nr},$$

so the Zeta function is

$$Z(\mathbb{P}^n; t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}.$$

If X is a scheme of finite type over $\mathbb{C}$, we can cover X with open affine subsets $Y_i = \text{Spec}(A_i)$. Each A_i is an algebra of finite type over $\mathbb{C}$, so we have

$$A_i \cong \mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_m),$$

here $f_1, \cdots, f_m$ are polynomials in $x_1, \cdots, x_n$. We can regard them as holomorphic functions on $\mathbb{C}^n$, so their set of common zeros is a complex analytic subspace $(Y_i)_h \subseteq \mathbb{C}^n$. The scheme X is obtained by glueing the Y_i, so we use the same glueing data to glue the analytic spaces $(Y_i)_h$ into an analytic space X_h.

Definition 2.5.4. The X_h defined above is called the associated complex analytic apace of the scheme X.

In 1949, Andre Weil [40] stated his famous conjectures concerning the number of solutions of polynomial equations over finite fields: let X be a smooth projective variety of dimension n defined over $\mathbb{F}_q$, and $Z(t)$ is the zeta function of X, then

(1) **Rationality:** $Z(t)$ is a rational function of t.

(2) **Functional equation:** Let E be the self-intersection number of the diagonal of $X \times X$ (which is also the top Chern class of the tangent bundle of X). Then $Z(t)$ satisfies a functional equation,

$$Z(\frac{1}{q^n t}) = \pm q^{nE/2} t^E Z(t).$$

(3) **Analogue of the Riemann hypothesis:** It is possible to write

$$Z(t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)},$$

where $P_0(t) = 1 - t$; $P_{2n}(t) = 1 - q^n t$; and for each $1 \leq i \leq 2n-1$, $P_i(t)$ is a polynomial with integer coefficients, which can be written as

$$P_i(t) = \prod(1 - \alpha_{ij} t),$$

here the α_{ij} are algebraic integers with $|\alpha_{ij}| = q^{i/2}$.

(4) **Betti numbers:** Assuming (3), we define the ith Betti number $B_i = B_i(X)$ to be the degree of the polynomial $P_i(t)$. Then we have $E = \sum(-1)^i B_i$. Furthermore, suppose that X is obtained from a variety Y defined over an algebraic number ring R, by reduction modulo a prime ideal $\mathfrak{p}$ of R. Then $B_i(X)$ is equal to the ith Betti number of the topological space $Y_h = (Y \times_R \mathbb{C})_h$, i.e., $B_i(X)$ is the rank of the ordinary cohomology group $\mathrm{H}^i(Y_h, \mathbb{Z})$.

Example 2.5.5. If $X = \mathbb{P}^1$, then its dimension is $n = 1$. Let's verify the Weil conjecture. From example 2.5.2 we have already seen $Z(t)$ is rational. The invariant E of $\mathbb{P}^1$ is 2, so we can verify the functional equation immediately,

$$Z(\frac{1}{qt}) = qt^2 Z(t).$$

The analogue of the Riemann hypothesis follows by: $P_1(t) = 1$, $P_0(t) = 1 - t$, and $P_2(t) = 1 - qt$. Hence $B_0 = B_2 = 1$ and $B_1 = 0$, and these are the usual Betti numbers of $\mathbb{P}_{\mathbb{C}}{}^1$. Moreover, $\sum(-1)^i B_i = 2 = E$.

For more details of the results of this section, we may refer to [17].

Weil proved Weil conjectures in case dim$X = 1$ earlier in [39]. Three of the Weil conjectures apart from the most difficult third conjecture (Analogue of the Riemann hypothesis) were proved by M. Artin and A. Grothendieck in the early 1960s (see [16] and SGA5); an independent proof using completely different techniques of the first one was by B.Dwork[13]. The third conjecture was proved by P. Deligne [10] in 1974.

The Weil conjecture suggested a connection between the arithmetic of algebraic varieties defined over finite fields and the topology of algebraic varieties defined over complex numbers.

In this thesis, we tried to count the number of rational points of the fibers $\mathcal{V}_1(\mathbf{v}, \mathbf{0}) = \mu_{\mathbf{v},\mathbf{0}}^{-1}(\mathbf{1})$ of moment maps at the special weights 1 over finite fields (actually it is equivalent to any nonzero weight λ) and its Zeta functions. Weil conjecture [40] suggests that over finite field and complex field $\mathbb{C}$ they are related if p is large enough, but they are quite different when p is fixed and small. From Lemma 2.4.4, we know over complex number $\mathbb{C}$, $\mathcal{V}_1(\mathbf{v}, \mathbf{0})$ is nonempty if and only if $\mathbf{v} = \mathbf{0}$; But over characteristic p field, for example finite field, things are quite different. The results of this thesis provides a family of examples of such varieties coming from quiver representations that Weil conjecture fails.

Chapter 3

Representation of Deformed Preprojective Algebra

In this chapter, we discuss representations of deformed preprojecive algebras $\Pi^\lambda(\Gamma)$ associated to a quiver Γ by representations of the path algebra $K\Gamma$. This chapter serves as a preparation of the next two chapters which contains the main results of this thesis. We separate it here because we want to generalize our results to more types of quiver. For Weyl algebra, because the results works for algebraically closed field of characteristic $p > 0$, so we also list it as a separate section.

3.1 General Results

Let K be any field.

Definition 3.1.1. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, the representations of the deformed preprojective algebra $\Pi^\lambda(\Gamma)$ associated to Γ correspond to representations of double quiver $\overline{\Gamma}$ satisfy the relation

$$\sum_{e=s^{-1}(i)} e^*e - \sum_{e=t^{-1}(i)} ee^* = \lambda_i 1_i \tag{3.1}$$

for all $i \in \Gamma_0$.

With this identification, we can speak of the dimension vector of a representation of $\Pi^\lambda(\Gamma)$.

In fact, the subalgebras $K\Gamma, K\Gamma^{op}$ of $K\bar{\Gamma}$ induces K-algebra homomorphisms,

$$\begin{array}{ccc} K\Gamma & & K\Gamma^{op} \\ & \searrow \quad \swarrow & \\ & \Pi^\lambda(\Gamma) & \end{array} .$$

These algebra homorphisms induce the natural pull back functors

$$\begin{array}{ccc} & \mathrm{rep}(\Pi^\lambda(\Gamma)) & \\ & {}^{\tau_1}\swarrow \quad \searrow^{\tau_2} & \\ \mathrm{rep}(K\Gamma) & & \mathrm{rep}(K\Gamma^{op}). \end{array}$$

These functors preserve the dimension vectors. In fact, they keep the K-vector spaces unchanged.

The representations of the deformed preprojective algebra $\Pi^\lambda(\Gamma)$ have been studied extensively in the literature. We will refer the readers to [8].

Remark 3.1.2. Since we will concentrate on τ_1 only, we denote τ_1 by τ for simplicity.

Theorem 3.1.3. *[8, Theorem 3.3] Let K be any algebraically closed field and any characteristic. Assume $\lambda \in K^{\Gamma_0}$. Then a $K\Gamma$-module M is isomorphic to $\tau(M^\lambda)$ for some $\Pi^\lambda(\Gamma)$-module M^λ if and only if for any direct summand N of M (as $K\Gamma$-module), we have*

$$\lambda \cdot \mathbf{dim}(N) := \sum_{i \in \Gamma_0} \lambda_i \dim(N_i) = 0$$

in K.

Let $\mathrm{Im}(\tau)$ denote the full subcategory of $\mathrm{rep}(K\Gamma)$ consisting of all objects of the form $\tau(M^\lambda)$ with M^λ in $\mathrm{rep}(\Pi^\lambda(\Gamma))$. $\mathrm{Im}(\tau)$ is an additive category closed under direct sums.

Corollary 3.1.4. *An indecomposable $K\Gamma$-module I is in* $\mathrm{Im}(\tau)$ *if and only if* $\lambda \cdot \mathbf{dim}(I) = 0$ *in K, here K is an algebraically closed field.*

Proof. Note that the only direct summands of I are I and $\{0\}$. Thus the corollary follows from the theorem. □

Recall that an additive category is Karoubian $\mathcal{C}$ if any idempotent endomorphism comes from a direct sum decomposition in $\mathcal{C}$. The subcategory $\mathrm{Im}(\tau)$ is closed under taking kernel and cokernels of morphism in $\mathrm{rep}(\Pi^\lambda)$ in general. However we have following

Corollary 3.1.5. $\mathrm{Im}(\tau)$ *is closed under taking direct summands in* $\mathrm{rep}(K\Gamma)$. *Thus* $\mathrm{Im}(\tau)$ *is a Karoubian additive category.*

Proof. Suppose $M = \tau(M^\lambda)$. If $M = L \oplus N$ is a direct sum decomposition in $\mathrm{rep}(K\Gamma)$. Any direct summands of L and N in $\mathrm{rep}(K\Gamma)$ are also direct summands of M. Thus both N and L satisfies the condition of the theorem. Hence both L and N are also in $\mathrm{Im}(\tau)$.

The second part follows from the fact that $\mathrm{rep}(K\Gamma)$ is Karoubian. □

We will use $\mathrm{Ind}(A)$ to denote the set of all isomorphism classes of indecomposable A-modules for a K-algebra A. Similarly, we will use $\mathrm{Ind}(\mathcal{C})$ to denote the collection of isomorphism classes of indecomposable objects in an additive category $\mathcal{C}$.

Corollary 3.1.6. *An object of* $\mathrm{Im}(\tau)$ *is indecomposable if and only if it is indecomposable in* $\mathrm{rep}(K\Gamma)$. *Thus we have*

$$\mathrm{Ind}(\mathrm{Im}(\tau)) = \mathrm{Ind}(\mathrm{rep}(K\Gamma)) \cap \mathrm{Im}(\tau) = \{I \in \mathrm{Ind}(\mathrm{rep}(K\Gamma)) \mid \lambda \cdot \mathbf{dim}(I) = 0 \in K\}$$

Remark 3.1.7. (1) Since we will just consider the case $\lambda = (\lambda_i)_{i\in\Gamma_0} \in (K^\times)^{\Gamma_0}$ with all $\{\lambda_i\}_{i\in\Gamma_0}$ are equal, and they are the same as the deformed preprojective algebra with weight $(1, \cdots, 1)$, in this dissertation, for simplicity, we always use $\Pi^1(\Gamma)$ to denote deformed preprojective algebra with weight $(1, \cdots, 1)$.

(2) From Section 2.4, we want to get the formula for $\Phi(\mathbf{0})$, what we really need to study is $\mathcal{V}_1(\mathbf{v},\mathbf{0})$, which actually is the space of representations of $\Pi^1(\Gamma)$ with dimension vector $\mathbf{v}$. For simplicity, from now on, we use $\mathcal{V}_1(\mathbf{v})$ to denote the representation space $\mathcal{V}_1(\mathbf{v},\mathbf{0})$.

For our special algebra $\Pi^1(\Gamma)$, we have

Corollary 3.1.8. *If K is characteristic 0, then $\Pi^1(\Gamma)$ has no nonzero finite dimensional module.*

So we are always interested in field K with characteristic p.

Let's recall the definition of $\mathcal{V}_1(\mathbf{v})$ in easier notation. For quiver $\Gamma = (\Gamma_0, \Gamma_1)$, if $\Gamma_1 = \{1, \cdots, n\}$, for a fixed dimension vector $\mathbf{v} = (v_1, \cdots, v_n)$, the representation space of Γ with dimension $\mathbf{v}$ is

$$\mathbb{V}_{\mathbf{v}} = \bigoplus_{e\in\Gamma_1} \mathrm{Hom}(K^{v_{s(e)}}, K^{v_{t(e)}});$$

the gauge group of $\mathbb{V}_{\mathbf{v}}$ is

$$G_{\mathbf{v}} = \prod_{i\in\Gamma_0} GL_{v_i}(K),$$

and its derivative is

$$\mathfrak{g}_{\mathbf{v}} = \bigoplus_{i\in\Gamma_0} gl(V_i).$$

Let $\overline{\mathbb{V}}_{\mathbf{v}} = \mathbb{V}_{\mathbf{v}} \times \mathbb{V}^*_{\mathbf{v}}$, then the moment map

$$\mu_{\mathbf{v}} : \overline{\mathbb{V}}_{\mathbf{v}} \to \mathfrak{g}_{\mathbf{v}}$$

is defined by: if $(x, x^*) = (x(e), x^*(e))_{e\in\Gamma_1}$,

$$\mu_{\mathbf{v}}(x, x^*)_i = \sum_{e\in s^{-1}(i)} x^*(a)x(a) - \sum_{a\in t^{-1}(i)} x(a)x^*(a) \in \mathfrak{gl}(V_i). \tag{3.2}$$

Then $\mathcal{V}_1(\mathbf{v}) = \mu_{\mathbf{v}}^{-1}(\mathbf{1})$, a $G_{\mathbf{v}}$-stable subset of $\overline{\mathbb{V}}_{\mathbf{v}}$, is the space of representations of $\Pi^1(\Gamma)$ with dimension vector $\mathbf{v}$.

When $K = \mathbb{F}_q$ is a finite field, we have

$$|\mathfrak{g}_{\mathbf{v}}(\mathbb{F}_q)| = \prod_{i\in\Gamma_0} q^{v_i^2} = q^{\mathbf{v}\cdot\mathbf{v}},$$

and

$$|\mathbb{V}_{\mathbf{v}}(\mathbb{F}_q)| = q^{\sum_{a\in\Gamma_1} v_{s(a)}v_{t(a)}}.$$

Thus

$$\frac{|\mathfrak{gl}_{\mathbf{v}}(\mathbb{F}_q)|}{|\mathbb{V}_{\mathbf{v}}(\mathbb{F}_q)|} = q^{\langle\mathbf{v},\mathbf{v}\rangle_\Gamma}.$$

Here $\langle\ ,\ \rangle_\Gamma$ is the Euler-Ringel form for the quiver Γ.

Therefore the definition of (2.10) can be written as

$$\Phi(\mathbf{w}) = \sum_{\mathbf{v}\in\mathbb{N}^{\Gamma_0}} \frac{|\mathcal{V}_1(\mathbf{v},\mathbf{w})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} q^{\langle\mathbf{v},\mathbf{v}\rangle_\Gamma} X^{\mathbf{v}}.$$

Thus $\Phi(\mathbf{0})$ can be written as

$$\Phi(\mathbf{0}) = \sum_{\mathbf{v}_1\in\mathbb{N}^{\Gamma_0}} \frac{|\mathcal{V}(\mathbf{v})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} q^{\langle\mathbf{v},\mathbf{v}\rangle_\Gamma} X^{\mathbf{v}}. \tag{3.3}$$

As what we said in section , we define a function

$$\mu_\Gamma^1(\mathbf{w}) = \sum_{\mathbf{v}\in\mathbb{N}^{\Gamma_0}} \frac{|\mathcal{V}_1(\mathbf{v},\mathbf{w})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} X^{\mathbf{v}},$$

then

$$\mu_\Gamma^1(\mathbf{0}) = \sum_{\mathbf{v}\in\mathbb{N}^{\Gamma_0}} \frac{|\mathcal{V}_1(\mathbf{v})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} X^{\mathbf{v}}.$$

Thus computing $\Phi(\mathbf{w})$ is equivalent to computing $\mu_\Gamma^1(\mathbf{w})$. In particular, $\Phi(\mathbf{0}) = 1$ if and only if $\mu_\Gamma^1(\mathbf{0}) = 1$.

On the other hand, we have

$$\frac{|\mathcal{V}_1(\mathbf{v})(\mathbb{F}_q)|}{|G_{\mathbf{v}}(\mathbb{F}_q)|} = \sum_{M \in \mathrm{Iso}(\mathbf{v})} \frac{1}{|\mathrm{Aut}(M)|},$$

where $\mathrm{Iso}(\mathbf{v})$ denotes the isomorphism classes of representations of the deformed preprojective algebra $\Pi^1(\Gamma)$ of dimension vector $\mathbf{v}$ and $\mathrm{Aut}(M)$ is the automorphism group of a representation M. Thus we have

$$\mu^1_\Gamma(\mathbf{0}) = \sum_{M \in (\mathrm{Iso}\Pi^1(\Gamma)-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M)|} X^{\mathbf{dim}(M)}. \tag{3.4}$$

Thus all what we need to do is to describe the category $\Pi^1(\Gamma)$-Mod of all finite dimensional representations of $\Pi^1(\Gamma)$ over the field $\mathbb{F}_q$.

Example 3.1.9. We now assume that K has characteristic $p > 0$ and consider a Dynkin quiver A_p with the vertex set $\{1, 2, \cdots, p\}$ and the arrow set $\{i \to i+1 \mid i = 1, \ldots, p-1\}$. The only (non-zero) indecomposable KA_p-module N satisfying the condition $\lambda \cdot \mathbf{dim}(N) = 0$ has dimension vector $\dim N_i = 1$ for all $i \in \Gamma_0$. This is the unique indecomposable module I_p corresponding to the highest positive root α_0. Thus any $\Pi^1(A_p)$-module restricting to Γ is isomorphic to $I_p^{\oplus n}$.

Thus we have $\mathcal{V}_1(\mathbf{v}) \neq \emptyset$ if and only if $\mathbf{v} = m\alpha_0$. Any $\Pi^1(A_p)$-module has the following form

$$K^m \underset{x_1}{\overset{\mathrm{Id}}{\rightleftarrows}} K^m \underset{x_2}{\overset{\mathrm{Id}}{\rightleftarrows}} K^m \underset{x_3}{\overset{\mathrm{Id}}{\rightleftarrows}} \cdots \underset{x_{p-3}}{\overset{\mathrm{Id}}{\rightleftarrows}} K^m \underset{x_{p-2}}{\overset{\mathrm{Id}}{\rightleftarrows}} K^m \underset{x_{p-1}}{\overset{\mathrm{Id}}{\rightleftarrows}} K^m$$

with x_i being a $m \times m$-matrix for $i = 1, \ldots, p-1$. Using the relation in (3.1), we have $x_i = iI$ for $i = 1, \ldots, p-1$.

Thus for $\mathbf{v} = m\alpha_0$, there is exactly one isomorphism class of $\Pi^1(A_p)$-modules with dimension vector $\mathbf{v}$. In particular, the module category $\mathrm{rep}\Pi^1(A_p)$ has exactly one irreducible representation with dimension vector α_0 and all other representations are direct sums of the irreducible representation. If M is a $\Pi^1(A_p)$-module with dimension vector $m\alpha_0$, then $\mathrm{Aut}(M) = GL_m(k)$.

If $K = \mathbb{F}_q$ is a finite field, then

$$|GL_m(\mathbb{F}_q)| = q^{m^2}(1-q^{-1})(1-q^{-2})\cdots(1-q^{-m}),$$

which implies that

$$\mu^1_{A_p}(\mathbf{0}) = \sum_{m=0}^{\infty} \frac{q^{-m^2}}{\prod_{i=1}^{m}(1-q^{-i})}(X^{\alpha_0})^m. \tag{3.5}$$

Knowing that for the quiver A_p, we have $\langle \alpha_0, \alpha_0 \rangle = 1$, then $\langle m\alpha_0, m\alpha_0 \rangle = m^2$ and

$$\Phi(\mathbf{0}) = \sum_{m=0}^{\infty} \frac{1}{\prod_{i=1}^{m}(1-q^{-i})}(X^{\alpha_0})^m. \tag{3.6}$$

Remark 3.1.10. Although Theorem 3.1.3 and Corollary 3.1.4 work for algebraically closed field K, the conclusions still hold for type A quiver for any characteristic $p > 0$ field which will be proved in Section 4.3.

We know that the function $\mu^1_\Gamma(\mathbf{0})$ is invariant under category equivalence of the $\Pi^1(\Gamma)$ provided the Chern character $K_0(\Pi^1(\Gamma)\text{Mod}) \to \mathbb{Z}^{\Gamma_0}$ is invariant under the equivalence. Thus we will concentrate on computing the function $\mu^1_\Gamma(\mathbf{0})$ (with respect to the fixed Chern character). The main goal of this thesis is to use this property to compute the stack function μ.

3.2 Full Subquivers and Their Deformed Preprojective Algebras

Let Γ be a quiver, and $\Gamma'_0 \subseteq \Gamma_0$ be a subset of vertices. We denote by Γ' the induced subquiver by Γ'_0, here Γ'_1 consists all arrows a in Γ_1 such that both $s(a)$ and $t(a)$ are in Γ'_0. We will call such subquiver full subquiver. Then Γ'^{op} is the induced subquiver of Γ^{op} by Γ'_0 and $\bar{\Gamma}'$ is the induced subquiver of $\bar{\Gamma}$ by Γ'_0 as well.

For any $\mathbf{v} \in \mathbb{N}^{\Gamma_0}$, the support $\text{supp}(\mathbf{v}) = \{i \in \Gamma_0 \mid v_i \neq 0\}$. Then $\mathbb{N}^{\Gamma'_0}$ can be thought as the subset of $\mathbb{N}^{\Gamma_0}$ consisting of all $\mathbf{u}$ with $\mathrm{supp}(\mathbf{u}) \subseteq \Gamma_0'$. In particular for any such $\mathbf{v}$ we have

$$\mathcal{V}_\lambda(\mathbf{v}, \Gamma) = \mathcal{V}_{\lambda'}(\mathbf{v}, \Gamma') \text{ and } G_{\mathbf{v}}(\Gamma) = G_{\mathbf{v}}(\Gamma'),$$

with $\lambda' \in K^{\Gamma_0'}$, $\lambda' = \lambda|_{\Gamma_0'}$. Since we have different quivers here, we add the notation of quivers here to distinguish. Then the algebra homomorphism $k\bar{\Gamma}' \to k\bar{\Gamma}$ induces an algebra homomorphism $\Pi^{\lambda'}(\Gamma') \to \Pi^\lambda(\Gamma)$. In fact this map identifies $\Pi^{\lambda'}(\Gamma')$ as the subalgebra of $\Pi^\lambda(\Gamma)$. On the other hand, $\Pi^{\lambda'}(\Gamma')$ is the quotient algebra of $\Pi^\lambda(\Gamma)$ modulo the ideal generated by the idempotents 1_i with $i \in \Gamma_0 \setminus \Gamma_0'$.

Thus we have two functors between these two categories. The functor

$$\pi_* : \mathrm{rep}(\Pi^{\lambda'}(\Gamma')) \to \mathrm{rep}(\Pi^\lambda(\Gamma))$$

identifies each $\Pi^{\lambda'}(\Gamma')$-module as $\Pi^\lambda(\Gamma)$-module with the same dimension vector. This corresponds to extension by zero of sheaves over a constructible subset. This functor is also the pullback functor of algebra homomorphism

$$\Pi^\lambda(\Gamma) \to \Pi^{\lambda'}(\Gamma').$$

The other functor is

$$\pi^* : \mathrm{rep}(\Pi^\lambda(\Gamma)) \to \mathrm{rep}(\Pi^{\lambda'}(\Gamma'))$$

by assigning each $\Pi^\lambda(\Gamma)$-module M to $\sum_{i \in \Gamma_0'} M_i$.

These two functors are adjoint pairs.

Theorem 3.2.1. *If Γ' is a full subquiver of Γ, for any M in $\Pi^1(\Gamma)$ with $\mathrm{supp}(\mathbf{dim}(M)) \subseteq \Gamma_0'$, M is irreducible (or indecomposable) in $\mathrm{rep}(\Pi^1(\Gamma'))$ if and only if M is irreducible (or indecomposable) in $\mathrm{rep}(\Pi^1(\Gamma))$.*

3.3 Some Results for Weyl Algebra

For Weyl algebra, it is the deformed preprojective algebra of the Jordan quiver, and

Lemma 3.3.1. *For Weyl algebra* $W = K\langle x, y\rangle/([x, y] = 1)$, *we have*

(1) $[x^n, y] = nx^{n-1}$, *and* $[x, y^n] = ny^{n-1}$;

(2) A basis for W *is* $\{x^i y^j, i, j \geq 0\}$;

(3) W *can be viewed as the algebra of polynomial differential operators in one variable.*

Proof. (1) use the relation $[x, y] = 1$ and induction;

(2) see Proposition 2.7.1 in [14];

(3) see Remark 2.7.2 in [14].

□

3.3.1 Characteristic 0

If the ground field is of characteristic 0, then by Corollary 3.1.8, we have

Theorem 3.3.2. *There are no finite dimensional representations of Weyl algebra.*

But for our special case Weyl algebra, we can prove it simply by the fact: for any two square matrices X, Y, $\mathrm{tr}[X, Y] = 0$.

Proposition 3.3.3. *[37] Let* $K = \mathbb{C}$, W *be the Weyl algebra,* G *be a finite subgroup of* $\mathrm{Aut}_{\mathbb{C}}(W)$, *then* G *is conjugate to one of the following special subgroups of the canonical image of* $SL_2(\mathbb{C})$:

1. $\mathbb{C}_n$: *a cyclic group of order* n.

2. $\mathbb{D}_n$: *a binary dihedral group of order* $4n$.

3. $\mathbb{T}$: *a binary tetrahedral group of order 24.*

4. $\mathbb{O}$: a binary octahedral group of order 48

5. $\mathbb{I}$: a binary icosahedral group of order 120.

Corollary 3.3.4. *[1] Let $K = \mathbb{C}$, W be the Weyl algebra, G and T be two finite subgroups of $\mathrm{Aut}_{\mathbb{C}}(W)$. Then $W^G \cong W^T$ if and only if $G \cong T$.*

Actually, the automorphism group of Weyl algebra of index n is given by [6, Conjecture 1]. The automorphism group of our Weyl algebra W is a positive example of the Conjecture. We summarize it to the following Theorem: by the work of [25], [11] and [29],

Theorem 3.3.5. *The automorphism group of the Weyl algebra W over $\mathbb{C}$ is isomorphic to the group of the polynomial symplectomorphisms of a 2-dimensional affine space*

$$\mathrm{Aut}_{\mathbb{C}}(W) \cong \mathrm{Aut}_{\mathbb{C}}(P),$$

here P is the Poisson algebra over $\mathbb{C}$ which is the usual polynomial algebra $\mathbb{C}[x,y] \cong \mathcal{O}({\mathbb{A}_{\mathbb{C}}}^2)$ endowed with the Poisson bracket:

$$\begin{pmatrix} \{x,x\} & \{x,y\} \\ \{y,x\} & \{y,y\} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

.

For more details of the automorphism of W and general Weyl algebra of index n over $\mathbb{C}$, see [1], [6]. A lot of the work of the automorphism of W and general Weyl algebra of index n comes from reduce to the Weyl algebra of characteristic $p > 0$.

3.3.2 Characteristic p

Definition 3.3.6. An Azumaya algebra over a commutative ring R is an R-algebra A which is finitely generated and projective over R, and is such that the action map $A \otimes_R A^{op} \to \mathrm{End}_R(A)$ is an isomorphism of R-algebras.

Lemma 3.3.7. *[38] Let K be a field of characteristic p, for Weyl algebra*

$$W = K\langle x, y\rangle/([x, y] = 1),$$

we have the following facts,

(1) x^p, y^p belongs to the center of W.

(2) More precisely, the center of W is $Z = K[x^p, y^p]$.

(3) W is a free Z module of rank p^2.

In particular, W is an Azumaya algebra over Z of rank p^2.

For more details of Azumaya algebra, see [2], [35], [4].

Lemma 3.3.8. *(1) Any finite dimensional representation of W has dimension divisible by p.*

(2) Over algebraically closed field of characteristic $p > 0$, the finite irreducible representations of W are p dimensional.

Proof. (1) This is just from Theorem 3.1.3.

(2) This is also from Theorem 3.1.3 and the definition of irreducible representation. We may also prove it directly: let V be an irreducible finite dimensional representation of W, and let v be an eigenvector of y in V. Then show that the collection of vectors $\{v, xv, x^2v, \cdots, x^{p-1}v\}$ is a basis of V.

□

Chapter 4

Type A Quiver Case

In this chapter, we always assume K is a field of characteristic p.

4.1 A_p Full Subquivers

In this subsection we assume that the field K has positive characteristic p and $\lambda = 1$. Let Γ be a quiver and we consider the category $\text{rep}(\Pi^1(\Gamma))$.

Let $\iota : A_p \to \Gamma$ be an embedding of quivers such that the image $\Gamma' = \iota(A_p)$ is a full subquiver. We will use the $\text{rep}(\iota)$ to denote the full subcategory of $\Pi^1(\Gamma)$ consisting of modules with dimension vectors $\mathbf{v}$ having support in Γ'_0. We know from Example 3.1.9 that $\text{rep}(\iota)$ is a semisimple category with exactly one irreducible object and thus is equivalent to the category of finite dimensional K-vector spaces. Let $L(\iota)$ be the image of the unique irreducible module of $\Pi^1(A_p)$ in $\text{rep}(\Pi^1(\Gamma))$. Then $L(\iota)$ is uniquely determined by its image in Γ, and is an irreducible representation of $\Pi^1(\Gamma)$ since $\mathbf{dim}(L(\iota))$ is minimal among all $\mathbf{v} \in \mathbb{N}^{\Gamma_0}$ under the partial order relation $\mathbf{v} \geq \mathbf{u}$ if and only if $v_i \geq u_i$ for all $i \in \Gamma_0$.

By Remark 3.1.10, the set of dimension vectors $\mathbf{v}$ with an irreducible $\Pi^\lambda(\Gamma)$-module of dimension vector $\mathbf{v}$ is described as those satisfying the following two conditions,

(1) There is an indecomposable $K\Gamma$-module of dimension $\mathbf{v}$ and $\lambda \cdot \mathbf{v} = 0$ in K;

(2) For any decomposition $\mathbf{v} = \sum_{j=1}^{r} \mathbf{u}^j$ with $r \geq 2$ and $\mathbf{u}^j$ satisfying (1), one has

$$1 - \langle \mathbf{v}, \mathbf{v} \rangle > \sum_{j=1}^{r} (1 - \langle \mathbf{u}^j, \mathbf{u}^j \rangle)$$

.

In particular $\mathbf{dim}(L(\iota))$ satisfies these two conditions.

We call a quiver Γ is A_p-coverable if every vertex of Γ is in the image of an embedding $\iota : A_p \to \Gamma$.

Examples of A_p-coverable quivers include Dynkin quivers of type A_N with $N \geq p$, type D_N with $N \geq p+1$ and type E_6 if $p = 2, 3$, E_7 and E_8 with $p = 2, 3, 5$.

We now define a new quiver with vertices be all A_p full subquivers of Γ and the number of arrows from ι_1 to ι_2 is $\dim \operatorname{Ext}(L(\iota_1), L(\iota_2))$. We are interested in describing this quiver. The representation category of $\Pi^1(\Gamma)$ is closely related to representation category of the preprojective algebras of this new quiver.

Example 4.1.1. Let $\Gamma = A_N$ with vertices being $\{1, \cdots, N\}$ Γ is A_p-coverable. There are exactly $N - p + 1$ such A_p subquivers with ι_r generated by the vertex set $\{r, \cdots, r + p - 1\}$. Checking the above conditions for $\mathbf{v}$ with an irreducible $\Pi^1(A_N)$-module of dimension $\mathbf{v}$, one sees that $\mathbf{dim}(L(\iota_j))$ are the only dimension vectors with an irreducible $\Pi^1(A_N)$-module. By Theorem 3.2.1, $\{L(\iota_1), \cdots, L(\iota_{N-p+1})\}$ is the full set of all irreducible modules in $\operatorname{rep}(\Pi^1(A_N))$.

4.2 Duality

Let $\sigma : \Gamma \to \Gamma^{op}$ be the standard anti-isomorphism of quivers which is identity on the set of vertices and $\sigma(a) = a^*$ such that $s(a) = t(a^*)$ and $t(a) = s(a^*)$. For any $\lambda \in K^{\Gamma_0}$, there is an induced K-algebra anti-automorphism $\sigma : \Pi^\lambda(\Gamma) \to \Pi^\lambda(\Gamma)$ defined by $\sigma(a) = a^*$ and $\sigma(a^*) = -a$ and $\sigma(1_i) = 1_i$. Then $\sigma^2(a) = -a$ and $\sigma^2(a^*) = -a^*$ defines an order 4 automorphism of the K-algebra $\Pi^\lambda(\Gamma)$. An interesting question is to determine the Automorphism group

$\mathrm{Aut}(\Pi^\lambda(\Gamma))$, and Weyl algebra we mentioned in Section 3.3 is just a special case.

For any $\Pi^\lambda(\Gamma)$-module M, let M^* be the Γ_0-graded dual vector space defined by

$$(M^*)_i = \mathrm{Hom}_K(M_i, K).$$

Then M^* has a natural $\Pi^\lambda(\Gamma)$-module structure defined by $(xf)(v) = f(\sigma(a)v)$ for all $x \in \Pi^\lambda(\Gamma)$, $f \in M^*$ and $v \in M$. The functor $(\cdot)^* : \mathrm{rep}(\Pi^\lambda(\Gamma)) \to \mathrm{rep}(\Pi^\lambda(\Gamma))$ is contravariant and exact.

If $\underline{M}$ is a $K\Gamma$-module, then $(\underline{M})^*$ is a $K\Gamma^{op}$-module with $(a^*f)(x) = -f(a(x))$ for any arrow $a \in \Gamma_1$. Then the functor $(\cdot)^* : \mathrm{rep}(K\Gamma) \to \mathrm{rep}(K\Gamma^{op})$ is exact and defines a category equivalence $\mathrm{rep}(K\Gamma)^{op} \cong \mathrm{rep}(K\Gamma^{op})$

Lemma 4.2.1. *The following diagram of functors commmute.*

$$\begin{array}{ccc} \mathrm{rep}(\Pi^\lambda(\Gamma)) & \xrightarrow{(\cdot)^*} & \mathrm{rep}(\Pi^\lambda(\Gamma)) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau^{op}} \\ \mathrm{rep}(K\Gamma) & \xrightarrow{(\cdot)^*} & \mathrm{rep}(K\Gamma^{op}). \end{array}$$

It follows from the definition that for any $\Pi^\lambda(\Gamma)$-module M, one always have $\mathbf{dim}(M^*) = \mathbf{dim}(M)$.

Proposition 4.2.2. $(\cdot)^*$ *send irreducibles to irreducibles, and indecomposables to indecomposables. In particular,* $\mathrm{Ext}^1_{\Pi^\lambda(\Gamma)}(M, N) = \mathrm{Ext}^1_{\Pi^\lambda(\Gamma)}((N)^*, (M)^*)$.

4.3 Block Decomposition $\mathrm{rep}(\Pi^1(A_N))$

Recall from Section 3.1 the algebra homomorphism

$$K\Gamma \to \Pi^\lambda(\Gamma),$$

induces an exact functor on their representation categories,

$$\tau_1 : \mathrm{rep}(\Pi^\lambda(\Gamma)) \to \mathrm{rep}(K\Gamma).$$

For future use, we define some notations as follows.

(1) We will just use τ_1 in this chapter, so for simplicity, we denote it by $\tau : \mathrm{rep}(\Pi^\lambda(\Gamma)) \to \mathrm{rep}(K\Gamma)$.

(2) To distinguish with $\Pi^\lambda(\Gamma)$-modules, we will use the underlined $\underline{M}$ to denote $K\Gamma$-module. In particular, when M is a $\Pi^\lambda(\Gamma)$-module, then $\underline{M} = \tau(M)$.

For any $K\Gamma$ module $\underline{M}$, to each vertex $i \in \Gamma_0$, there is a finite dimensional K vector space $\underline{M}_i$, and we define its support to be $\mathrm{supp}(\underline{M}) = \{i \in \Gamma_0 \mid \underline{M}_i \neq 0\}$, i.e., $\mathrm{supp}(\underline{M}) = \mathrm{supp}(\mathbf{dim}(\underline{M}))$. In the future, we may abuse the notation $\mathrm{supp}(\underline{M})$ and $\mathrm{supp}(\mathbf{dim}(\underline{M}))$.

For any $\Pi^1(\Gamma)$ module M, we have the decomposition in $\mathrm{rep}(K\Gamma)$

$$\underline{M} \cong \bigoplus_{\underline{I} \in \mathrm{Ind}(K\Gamma)} \underline{I}^{n_{\underline{I}}} \tag{4.1}$$

as $K\Gamma$-modules. By Corollary 3.1.5, any $\underline{I}$ with $n_{\underline{I}} \neq 0$ has to be in $\mathrm{Im}(\tau)$. Thus $\underline{I} \in \mathrm{Ind}(\mathrm{Im}(\tau))$.

For any $i \in \Gamma_0$, we have

$$M_i = \bigoplus_{\underline{I} \in \mathrm{Ind}(K\Gamma)} \underline{I}_i^{n_{\underline{I}}} \tag{4.2}$$

as a vector space. The $\Pi^1(\Gamma)$-module structure on M is determined by the collection of pairs $(x(e), x^*(e))$ for each $e \in \Gamma_1$. For each $i \in \Gamma_0$, and the arrow $e : i \to i'$, the linear maps

$$x(e) : \bigoplus_{\underline{I}} \underline{I}_i^{n_{\underline{I}}} \to \bigoplus_{\underline{I}} \underline{I}_{i'}^{n_{\underline{I}}} \quad \text{and} \quad x^*(e) : \bigoplus_{\underline{I}} \underline{I}_{i'}^{n_{\underline{I}}} \to \bigoplus_{\underline{I}} \underline{I}_i^{n_{\underline{I}}} \tag{4.3}$$

can be decomposed into blocks $x(e) = (x(e)_{I,I'})$ and $x^*(e) = (x^*(e)_{I,I'})$ with

$$x(e)_{I,I'} : \underline{I}_i^{n_{\underline{I}}} \to (\underline{I'}_{i'})^{n_{\underline{I'}}} \quad \text{and} \quad x^*(a)_{I,I'} : \underline{I}_{i'}^{n_{\underline{I}}} \to (\underline{I}'_i)^{n_{\underline{I'}}} \tag{4.4}$$

such that

$$x(e)_{I,I'} = \begin{cases} 0 & \text{if } \underline{I} \neq \underline{I'}, \\ x_{\underline{I}}(e)^{\oplus n_{\underline{I}}} & \text{if } \underline{I} = \underline{I'}. \end{cases} \tag{4.5}$$

Here $x_{\underline{I}}(e)$ is the corresponding linear map of the $K\Gamma$-module $\underline{I}$ at the arrow $e \in \Gamma_1$.

Thus to determine a $\Pi^\lambda(\Gamma)$-module structure, all we need to determine are the linear maps $x^*(e)_{\underline{I},\underline{I'}}$ satisfying the following conditions. For any $i \in \Gamma_0$ and any two $\underline{I}$ and $\underline{I'}$, we have

$$\sum_{e \in s^{-1}(i)} x^*(e)_{\underline{I},\underline{I'}} x_{\underline{I}}(e)^{\oplus n_{\underline{I}}} - \sum_{a \in t^{-1}(i)} x_{\underline{I'}}(e)^{\oplus n_{\underline{I'}}} x^*(e)_{\underline{I},\underline{I'}} = \begin{cases} \lambda_i \mathrm{Id}_{\underline{I}_i} & \text{if } \underline{I} = \underline{I'}, \\ 0 & \text{if } \underline{I} \neq \underline{I'}. \end{cases} \tag{4.6}$$

In the rest of this section, we assume K is a field of characteristic p and the quiver Γ is always type A_N, vertices set is $\Gamma_0 = \{1, 2, \cdots, N\}$, arrows are $i \to i+1$.

We now focus on the quiver $\Gamma = A_N$ and $\lambda = (1, \cdots, 1)$. Recall the notation in Example 2.2.4, all indecomposable $K\Gamma$-modules are of the form $\underline{I}(i,l)$ such that $\mathrm{supp}(\mathbf{dim}(\underline{I})) = [i, i+l-1] = \{i, i+1, \ldots, i+l-1\}$. Here l is called the length of $\underline{I}(i,l)$. In fact $\underline{I}(i,l)$ is uniquely determined by the segment $[i, i+l-1]$, and up to isomorphism, we have

$$\begin{array}{ccccccc} & & i & & i+l-1 & & \\ & & \downarrow & & \downarrow & & \\ 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \to & & K & \xrightarrow{\mathrm{Id}} \cdots \xrightarrow{\mathrm{Id}} K \xrightarrow{\mathrm{Id}} & K & & \xrightarrow{0} 0 \xrightarrow{0} \cdots \xrightarrow{0} 0. \end{array}$$

Lemma 4.3.1. *An indecomposable $K\Gamma$-module $\underline{I}(i,l)$ is a direct summand of $\tau(M) = \underline{M}$ for some M in $\mathrm{rep}(\Pi^1(A_N))$ if and only if $p \mid l$. In other words, the indecomposable objects in $\mathrm{Im}(\tau)$ are exactly $\underline{I}(i,kp)$, for $i \in \Gamma_0$ and $k \in K$ with $i + kp - 1 \leq N$.*

Proof. Suppose $\underline{M} = \underline{N} \oplus \underline{I}(i,l)$. The vertices involving $\underline{I}(i,l)$ are $i, i+1, \cdots, i+l-1$. For simplicity, we denote $\underline{I}(i,l)$ by $\underline{I}$, then $\underline{M} = \underline{N} \oplus \underline{I}$. We just need to consider the vertices

$i-1, i, \cdots, i+l$, and the representation of $\underline{M}$ is isomorphic to

$$\underline{N}_{i-1} \xrightarrow{\begin{pmatrix} A_0 \\ 0 \end{pmatrix}} \underline{N}_i \oplus K \xrightarrow{\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} A_{l-1} & 0 \\ 0 & 1 \end{pmatrix}} \underline{N}_{i+l-1} \oplus K \xrightarrow{\begin{pmatrix} A_l & 0 \end{pmatrix}} \underline{N}_{i+l},$$

here the A_j are K-linear map from $\underline{N}_{i+j-1}$ to $\underline{N}_{i+j}$ in the representation of $\underline{N}$.

Since M is the pull back of $\underline{M}$ in $\text{rep}(\Pi^1(A_N))$, we get for the vertices $i-1, i, \cdots, i+l$, and the representation of M is isomorphic to

$$M_{i-1} \underset{\begin{pmatrix} A_0^* & B_0^* \end{pmatrix}}{\overset{\begin{pmatrix} A_0 \\ 0 \end{pmatrix}}{\rightleftarrows}} M_i \underset{\begin{pmatrix} A_1^* & B_1^* \\ C_1^* & D_1^* \end{pmatrix}}{\overset{\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}}{\rightleftarrows}} \cdots\cdots\cdots \underset{\begin{pmatrix} A_{l-1}^* & B_{l-1}^* \\ C_{l-1}^* & D_{l-1}^* \end{pmatrix}}{\overset{\begin{pmatrix} A_{l-1} & 0 \\ 0 & 1 \end{pmatrix}}{\rightleftarrows}} M_{i+l-1} \underset{\begin{pmatrix} A_l^* \\ B_l^* \end{pmatrix}}{\overset{\begin{pmatrix} A_l & 0 \end{pmatrix}}{\rightleftarrows}} M_{i+l} \ ,$$

with the relations:

$$\begin{cases} \begin{pmatrix} A_1^* & B_1^* \\ C_1^* & D_1^* \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} A_0 \\ 0 \end{pmatrix} \begin{pmatrix} A_0^* & B_0^* \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} A_k^* & B_k^* \\ C_k^* & D_k^* \end{pmatrix} \begin{pmatrix} A_k & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} A_{k-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{k-1}^* & B_{k-1}^* \\ C_{k-1}^* & D_{k-1}^* \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ 0 & 1 \end{pmatrix}, \ 2 \leq k \leq l-1 \\ \begin{pmatrix} A_l^* \\ B_1^* \end{pmatrix} \begin{pmatrix} A_l & 0 \end{pmatrix} - \begin{pmatrix} A_{l-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{l-1}^* & B_{l-1}^* \\ C_{l-1}^* & D_{l-1}^* \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

By matrix computation, the above relations give

$$\begin{cases} \begin{pmatrix} A_1^*A_1 - A_0A_0^* & B_1^* - A_0B_0^* \\ C_1^*A_1 & D_1^* \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} A_k^*A_k - A_{k-1}A_{k-1}^* & B_k^* - A_{k-1}B_{k-1}^* \\ C_k^*A_k - C_{k-1}^* & D_k^* - D_{k-1} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 1 \end{pmatrix}, \ 2 \le k \le l-1 \\ \begin{pmatrix} A_l^*A_l - A_{l-1}A_{l-1}^* & -A_{l-1}B_{l-1}^* \\ B_l^*A_l - C_{l-1}^* & -D_{l-1}^* \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

In particular, we have

$$\begin{cases} D_1^* = 1, \\ D_k^* - D_{k-1}^* = 1, \ 2 \le k \le l-1 \\ -D_{l-1}^* = 1. \end{cases}$$

The solutions are

$$\begin{cases} D_k^* = k, \ 1 \le k \le l-1 \\ -D_{l-1}^* = 1. \end{cases}$$

so $l-1 = -1$ then $p \mid l$.

Then we get if $\underline{M} = \underline{N} \oplus \underline{I}(i,l)$, then $p \mid l$. The rest of the theorem is easy to prove. $\square$

Since τ is an additive functor with the property that $\tau(M) \cong 0$ if and only if $M \cong 0$, any M in $\mathrm{rep}(\Pi^1(\Gamma))$ such that $\tau(M)$ is indecomposable must be indecomposable in $\mathrm{rep}(\Pi^1(\Gamma))$. Also $\underline{I}(i,p)$ in $\mathrm{Im}(\tau)$ are of minimal length among all non-zero indecomposable objects, and recall that an object in $\mathrm{Im}(\tau)$ is called irreducible if it has no proper sub-object in $\mathrm{Im}(\tau)$, so we get,

Corollary 4.3.2. *The irreducible objects in* $\mathrm{Im}(\tau)$ *are* $\underline{I}(i,p)$*, for any* $i \in \Gamma_0$ *with* $i+p-1 \le N$.

Lemma 4.3.3. *Suppose an indecomposable* $K\Gamma$*-module* $\underline{I}$ *is a direct summand of* $\underline{M} \cong \bigoplus_{\underline{I} \in \mathrm{Ind}(K\Gamma)} \underline{I}^{n_{\underline{I}}}$ *for some* M *in* $\mathrm{rep}(\Pi^1(\Gamma))$*, and* $i \xrightarrow{e} i' \xrightarrow{e'} i''$ *is a path of length 2 in* $\Gamma = A_N$ *such that all three vertices are in* $\mathrm{supp}(\underline{I})$*. Then* $x^*(e')_{\underline{I},\underline{I}} = x^*(e)_{\underline{I},\underline{I}} + \mathrm{Id}$.

Proof. Using the fact that $x(e)_{\underline{I},\underline{I'}} = 0$ if $\underline{I} \neq \underline{I'}$,

$$x^*(e')x(e') - x(e)x^*(e) = \mathrm{Id}_{\bigoplus_{\underline{I}} \underline{I}_{i'}^{n_{\underline{I}}}}$$

and $\underline{I}'_{i'} \neq 0$ we can easily get

$$\mathrm{Id}_{\underline{I}_{i'}^{n_{\underline{I}}}} = (x^*(e')x(e'))_{\underline{I}\underline{I}} - (x(e)x^*(e))_{\underline{I},\underline{I}} = x^*(e')_{\underline{I},\underline{I}}x(e')_{\underline{I},\underline{I}} - x(e)_{\underline{I},\underline{I}}x^*(e)_{\underline{I},\underline{I}}.$$

Since i, i', and i'' are in the support of $\mathbf{dim}(\underline{I})$ we have $x(e')_{\underline{I}\underline{I}} = \mathrm{Id}$ and $x(e)_{\underline{I},\underline{I}} = \mathrm{Id}$. Hence the lemma follows. □

Remark 4.3.4. (1) In case i' and i'' are in $\mathrm{supp}(\mathbf{dim}(\underline{I}))$, but i is not in $\mathrm{supp}(\mathbf{dim}(\underline{I}))$ we have $x(e)_{I,I} = 0$ and $x(e')_{I,I} = \mathrm{Id}$. Hence we have $x^*(e') = \mathrm{Id}_{\underline{I}_{i'}^{n_I}}$.

(2) For any vertex $i' \in \Gamma_0$, and $I \neq I'$, the two step path $i \xrightarrow{a} i' \xrightarrow{a'} i''$ defines a map

$$\phi_{i'} = x^*(e')_{\underline{I},\underline{I'}}x(e')_{\underline{I},\underline{I}} = x(e)_{\underline{I'},\underline{I'}}x^*(e)_{\underline{I},\underline{I'}} : \underline{I}_{i'} \to \underline{I'}_{i'}$$

such that $\phi = (\phi_{i'})_{i'\in\Gamma_0} : \underline{I}^{\oplus n_{\underline{I}}} \to \underline{I'}^{\oplus n_{\underline{I'}}}$ is a $K\Gamma$-module homomorphism.

Lemma 4.3.5. *If I is any $\Pi^1(\Gamma)$-module such that $\tau(I) = \underline{I} = \underline{I}(i, kp)$ is indecomposable $K\Gamma$, then for any arrow $i' \xrightarrow{e'} i''$, we have $x^*(e') = (i'' - i)\mathrm{Id}$ if $x(e') = \mathrm{Id}$.*

Proof. Apply Lemma 4.3.3 to $M = I$ and induction for $i' \geq i$. When $i' = i$, we have $i'' = i + 1$ and the remark following the proof of Lemma 4.3.3 shows that $x^*(e') = \mathrm{Id}$. Assume the formula holds for $i' > i$, we apply Lemma 4.3.3 to get $x^*(e') = (i' - i + 1)\mathrm{Id}$. □

This lemma implies that $x^*(e')$ is uniquely determined. Thus we have

Corollary 4.3.6. *For any $\underline{I} \in \mathrm{Ind}(\mathrm{Im}(\tau))$, there is a unique I in $\mathrm{Ind}(\Pi^1(\Gamma))$ such that $\tau(I) = \underline{I}$. We will simply denote this I by $\tau^{-1}(\underline{I})$. In this case, $\tau^{-1}(\underline{I}(i, kp)$ is uniserial with simple quotient $\tau^{-1}(\underline{I}(i, p))$ and simple submodule $\tau^{-1}(\underline{I}(i + (k-1)p, p))$. In particular $\tau^{-1}(\underline{I}(i, p))$ are all possible irreducible objects in $\mathrm{rep}(\Pi^1(\Gamma))$.*

We note that since the duality functor $(\cdot)^*$ does not change the dimension vector. In this case the dimension vectors uniquely determine the simple $\Pi^1(\Gamma)$-modules. We have for any irreducible object $\tau^{-1}(\underline{I}(i,p))$, $(\tau^{-1}(\underline{I}(i,p)))^* \cong \tau^{-1}(\underline{I}(i,p))$. However, it is not true that $(\tau^{-1}(\underline{I}(i,kp)))^* \cong \tau^{-1}(\underline{I}(i,kp))$. In fact, $\mathrm{rep}(\Pi^1(\Gamma))$ has more indecomposable objects than the set $\{(\tau^{-1}(\underline{I}(i,kp)))^*\} \cup \{\tau^{-1}(\underline{I}(i,kp))\}$. See Example 4.3.14.

Definition 4.3.7. If $\underline{I} = \underline{I}(i,kp)$, an arrow $i' \xrightarrow{e'} i'' \in \Gamma$ a called a p-arrow for $\underline{I}$ if $i'' - i = 0$ in K. In this case, the map $x^*(e') = 0$ for $I = \tau^{-1}(\underline{I})$.

Lemma 4.3.8. *Let $\underline{I} = \underline{I}(i,kp)$ be a direct summand appearing the decomposition of $\underline{M}$ above. If $i' \xrightarrow{e'} i''$ is an arrow in Γ such that $\underline{I}_{i'} \neq 0$ and $\underline{I}_{i''} \neq 0$, then*

$$x^*(e')_{\underline{I}I} = \begin{cases} 0 & \text{if } e' \text{ is a } p\text{-arrow for } \underline{I}, \\ (i''-i)\mathrm{Id}_{\underline{I}^n_{i'} I} & \text{if } e' \text{ is not a } p\text{-arrow for } \underline{I}. \end{cases} \tag{4.7}$$

Proof. By direct computation applying Lemma 4.3.3, we get $x(a')^*_{\underline{I}I} = (i''-i)\mathrm{Id}$. Then the lemma follows. □

Theorem 4.3.9. *Suppose M is a $\Pi^1(\Gamma)$-module such that $\underline{M} = \underline{I}(i_1,k_1p) \oplus \underline{I}(i_2,k_2p)$. If $p \nmid (i_1 - i_2)$ for $i_1, i_2 \in \Gamma_0$, then $M \cong \tau^{-1}(\underline{I}(i_1,k_1p)) \oplus \tau^{-1}(\underline{I}(i_2,k_2p))$.*

Proof. Define

$$A = \mathrm{supp}(\underline{I}(i_1,k_1p)) \cap \mathrm{supp}(\underline{I}(i_2,k_2p)).$$

For notational simplicity, we denote $\underline{I} = \underline{I}(i_1,k_1p)$ and $\underline{I'} = \underline{I}(i_2,k_2p)$. Since $\mathrm{supp}(\underline{I}(i_1,k_1p)) = [i_1, i_1+k_1p-1]$ and $\mathrm{supp}(\underline{I}(i_2,k_2p)) = [i_2, i_2+k_2p-1]$ are intervals, we can assume $i_1 < i_2$.

We first assume $A \neq \emptyset$, then we have $\sharp(A) = l > 0$. Thus $i_2 < i_1 + k_1p$ and $A = [i_2, i_1+k_1p-1]$ or $A = [i_2, i_2+k_2p-1]$. Thus $l = i_1+k_1p-i_2$ or k_2p depending on whether $i_1+k_1p < i_2+k_2p$ or $i_1+k_1p \geq i_2+k_2p$.

Case 1: $i_1+k_1p < i_2+k_2p$. Under this assumption, we have $\mathrm{Hom}_{K\Gamma}(\underline{I}, \underline{I'}) = 0$ and $\mathrm{Hom}_{K\Gamma}(\underline{I'}, \underline{I}) = K$.

We have $A = \{i'_1, \cdots, i'_l\}$ with $i'_1 = i_2$ and $i'_l = i_1 + k_1 p - 1$. There are $i'_0, i'_{l+1} \in \Gamma_0$, such that the following path

$$i'_0 \stackrel{e_0}{\to} i'_1 \stackrel{e_1}{\to} i'_2 \stackrel{e_2}{\to} \cdots \stackrel{e_{l-1}}{\to} i'_l \stackrel{e_l}{\to} i'_{l+1}$$

satisfies to conditions $\underline{I}_{i'_k} \neq 0$ for $k = 0, 1, \cdots, l$ and, $\underline{I}_{i'_{l+1}} = 0$ while $\underline{I}'_{i'_k} \neq 0$ for $k = 1, 2, \cdots, l+1$ and $\underline{I}'_{i'_0} = 0$.

For the module $\underline{M}$, the corresponding arrows defines the sequence of linear maps

$$\underline{I}_{i'_0} \oplus \underline{I}'_{i'_0} \stackrel{x(e_0)}{\to} \underline{I}_{i'_1} \oplus \underline{I}'_{i'_1} \stackrel{x(e_1)}{\to} \underline{I}_{i'_2} \oplus \underline{I}'_{i'_2} \stackrel{x(e_2)}{\to} \cdots \stackrel{x(e_{l-1})}{\to} \underline{I}_{i'_l} \oplus \underline{I}'_{i'_l} \stackrel{x(e_l)}{\to} \underline{I}_{i'_{l+1}} \oplus \underline{I}'_{i'_{l+1}}.$$

By Lemma 4.3.3 we have $x(e_k)_{\underline{I},\underline{I}} = \mathrm{Id}$ for $k = 0, \cdots, l$, $x(e_k)_{\underline{I}',\underline{I}'} = \mathrm{Id}$ for $k = 1, \cdots, l+1$, and $x(e_k)_{\underline{I},\underline{I}'} = 0 = x(e_k)_{\underline{I}'\underline{I}}$ for $k = 0, \cdots, l+1$. Using the matrix notation, we have

$$x(e_0) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}; \quad x(e_l) = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix};$$

$$x(e_k) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \quad \text{for} \quad k = 1, 2, \cdots, l-1.$$

To get the $\Pi^1(\Gamma)$-module structure on M determine the maps

$$x^*(e_k) = \begin{pmatrix} x^*(e_k)_{\underline{I},\underline{I}} & x^*(e_k)_{\underline{I}',\underline{I}} \\ x^*(e_k)_{\underline{I},\underline{I}'} & x^*(e_k)_{\underline{I}',\underline{I}'} \end{pmatrix}.$$

By Lemma 4.3.8 we have $x^*(e_k)_{\underline{I},\underline{I}} = (i'_{k+1} - i_1)\mathrm{Id}$ and $x^*(e_k)_{\underline{I}',\underline{I}'} = (i'_{k+1} - i_2)\mathrm{Id}$ for $k = 1, \cdots, l$. By Remark 4.3.4(b), $x^*(e_k)_{\underline{I},\underline{I}'} : \underline{I}_{i'_k} \to \underline{I}'_{i'_{k+1}} = \underline{I}'_{i'_k}$ defines a homomorphism of $K\Gamma$-modules $\underline{I} \to \underline{I}'$. Similarly $x^*(e_k)_{\underline{I}',\underline{I}} : \underline{I}'_{i'_k} \to \underline{I}_{i'_{k+1}} = \underline{I}'_{i'_k}$ defines a homomorphism of

$K\Gamma$-modules $\underline{I'} \to \underline{I}$. Using matrix form we have

$$x^*(e_0) = \begin{pmatrix} (i'_1 - i_1)\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}; \quad x^*(e_l) = \begin{pmatrix} 0 & \phi_{i'_l} \\ 0 & (i'_{l+1} - i_2)\mathrm{Id} \end{pmatrix};$$

$$x^*(e_k) = \begin{pmatrix} (i'_{k+1} - i_1)\mathrm{Id} & \phi_{i'_k} \\ 0 & (i'_{k+1} - i_2)\mathrm{Id} \end{pmatrix}, \text{ for } k = 1, 2, \cdots, l-1.$$

Here $\phi : \underline{I'} \to \underline{I}$ is a homomorphism of $K\Gamma$-module. Thus there is $x \in K$ such that $\phi_{i'_k} = x$ for $k = 1, \cdots, l$. By assumption $c = i'_1 - i_1 = i_2 - i_1 \neq 0$ in K and $c + l = 0$.

We note that the automorphism group $\mathrm{Aut}_{K\Gamma}(\underline{I} \oplus \underline{I'})$ has the matrix form

$$\begin{pmatrix} G_{\underline{I},\underline{I}} & G_{\underline{I'},\underline{I}} \\ 0 & G_{\underline{I'},\underline{I'}} \end{pmatrix}.$$

Now now use the group $G = \prod_{i \in \Gamma_0} G_i \subseteq \mathrm{Aut}_{K\Gamma}(\underline{M})$ action on M to find simpler matrix presentation of the morphisms $x^*(e)$. We now take the $G = (G_i)_{i \in \Gamma_0} \in \mathrm{Aut}_{K\Gamma}(\underline{I} \oplus \underline{I'})$

$$G_i = \begin{cases} \mathrm{Id} & \text{if } i \notin A, \\ \begin{pmatrix} \mathrm{Id} & c^{-1}x \\ 0 & \mathrm{Id} \end{pmatrix} & \text{if } i \in A. \end{cases}$$

and define $(y(e), y^*(e)) = G \cdot (x(e), x^*(e))$, which defines a $\Pi^1(\Gamma)$-module M' which is isomorphic to M. Here $y(e) = x(e)$ for all $e \in \Gamma_1$ and

$$y^*(e_0) = \begin{pmatrix} c \cdot \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}; \quad y^*(e_l) = \begin{pmatrix} 0 & 0 \\ 0 & l \cdot \mathrm{Id} \end{pmatrix};$$

$$y^*(e_k) = \begin{pmatrix} (c+k)\mathrm{Id} & 0 \\ 0 & k \cdot \mathrm{Id} \end{pmatrix}, \text{ for } k = 1, 2, \cdots, l-1.$$

Hence the $M \cong \tau^{-1}(\underline{I}(i_1, k_1p)) \oplus \tau^{-1}(\underline{I}(i_2, k_2p))$ in this case.

Case 2: $i_1 + k_1p \geq i_2 + k_2p$. Under this assumption, we have $\mathrm{Hom}_{K\Gamma}(\underline{I}, \underline{I'}) = 0$ and $\mathrm{Hom}_{K\Gamma}(\underline{I'}, \underline{I}) = 0$.

Using the same notation as in *Case 1* for the arrows e_k with $k = 0, 1, \ldots, l+1$. The only difference is that $\underline{I'}_{i'_{l+1}} = 0$ and $\underline{I}_{i'_{l+1}} \neq 0$. Similar matrix computation shows that $x^*(e_k)$ are diagonal matrix already and the theorem follows without using the automorphism group action.

If $A = \emptyset$, then we have $i_1 + k_1p < i_2$. Under this assumption, $\mathrm{Hom}_{K\Gamma}(\underline{I}, \underline{I'}) = 0$ and $\mathrm{Hom}_{K\Gamma}(\underline{I'}, \underline{I}) = 0$. In this case, all components of $x(e)$ are either Id or zero. Noting that $\underline{M}_{i_1+k_1p} = \underline{M}_{i_1+k_1p+1} = 0$. Hence the full subquiver generated by $\mathrm{supp}(M)$ has two connected components. Thus the theorem follows directly. □

Remark 4.3.10. Under the notation in Theorem 4.3.9 and from its proof, we also have if $\mathrm{supp}(\underline{I}(i_1, k_1p)) \cap \mathrm{supp}(\underline{I}(i_2, k_2p)) = \emptyset$, then $M \cong \tau^{-1}(\underline{I}(i_1, k_1p)) \oplus \tau^{-1}(\underline{I}(i_2, k_2p))$ unless $i_2 = i_1 + k_1p$ or $i_1 = i_2 + k_2p$.

Theorem 4.3.11. *For any $i_1, i_2 \in \Gamma_0$ such that $i_1 + p - 1 \leq N$ and $i_2 + p - 1 \leq N$, we have*

$$\mathrm{Ext}^1_{\Pi^1(\Gamma)}(\tau^{-1}(\underline{I}(i_1, p)), \tau^{-1}(\underline{I}(i_2, p))) = \begin{cases} K & \text{if } i_2 = i_1 + p \text{ or } i_1 = i_2 + p; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let M be a $\Pi^1(\Gamma)$-module with the short exact sequence

$$0 \to \tau^{-1}(\underline{I}(i_2, p)) \to M \to \tau^{-1}(\underline{I}(i_1, p)) \to 0$$

of $\Pi^1(\Gamma)$-modules.

(1) If $p \nmid (i_1 - i_2)$, Theorem 4.3.9 implies that $M = \tau^{-1}(\underline{I}(i_1, p)) \oplus \tau^{-1}(\underline{I}(i_2, p))$. Thus the sequence splits as $\Pi^1(\Gamma)$-module, and $\mathrm{Ext}^1_{\Pi^1(\Gamma)}(\tau^{-1}(\underline{I}(i_1, p)), \tau^{-1}(\underline{I}(i_2, p))) = 0$.

Moreover, by Remark 4.3.10, we know $\mathrm{Ext}^1_{\Pi^1(\Gamma)}(\tau^{-1}(\underline{I}(i_1, p)), \tau^{-1}(\underline{I}(i_2, p))) = 0$ unless $i_2 = i_1 + p$, $i_1 = i_2 + p$, or $i_1 = i_2$.

(2) we consider the case $i_1 = i_2$. In this case, $\tau(M) = \underline{I}(i_1, p)^2$. By direct computation using (4.6), M is a direct sum of two copies of the irreducible module $\tau^{-1}(\underline{I}(i_1, p))$, and $\mathrm{Ext}^1_{\Pi^1(\Gamma)}(\tau^{-1}(\underline{I}(i_1, p)), \tau^{-1}(\underline{I}(i_2, p))) = 0$.

(3) We now discuss the case $i_2 + p = i_1$. Since $\mathrm{Hom}_{K\Gamma}(\underline{I}(i_2, p), \underline{I}(i_2, 2p)) = 0$ and $\underline{M}$ has a submodule $\underline{I}(i_2, p)$, then $\underline{M} \neq \underline{I}(i_2, 2p)$, and $\underline{M} = \underline{I}(i_1, p) \oplus \underline{I}(i_2, p)$. Let $i_2 + p - 1 \xrightarrow{e} i_1$ be the arrow of interest, we have the $\Pi^1(\Gamma)$-module structure of M as follows

$$\cdots \underset{x^*(e')}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{x^*(e)}{\overset{x(e)=0}{\rightleftarrows}} K \underset{x^*(e'')}{\overset{\mathrm{Id}}{\rightleftarrows}} \cdots .$$

Here $x^*(e') = (p-1) \cdot \mathrm{Id}$ and $x^*(e'') = \mathrm{Id}$, and $x^*(e) = k \cdot \mathrm{Id}$, where k is any element of K. Hence $\mathrm{Ext}^1_{\Pi^1(\Gamma)}(\tau^{-1}(\underline{I}(i_1, p)), \tau^{-1}(\underline{I}(i_2, p))) = K$.

(4) Finally, we consider the case $i_2 = i_1 + p$. Apply the duality functor $(\cdot)^*$ and by Proposition 4.2.2, the proof is similar to the case $i_1 = i_2 + p$.

$\square$

Corollary 4.3.12. *If $p \nmid (i_1 - i_2)$ for $i_1, i_2 \in \Gamma_0$, then*

$$\mathrm{Ext}^1_{\Pi^1(\Gamma)}(\tau^{-1}(\underline{I}(i_1, k_1 p)), \tau^{-1}(\underline{I}(i_2, k_2 p))) = 0.$$

Proof. This follows from the fact that all composition factors of $\tau(\underline{I}(i_1, k_1 p))$ are $\tau(\underline{I}(i_1 + jp, p))$ with $j = 0, \ldots, k_1 - 1$ and all composition factors of $\tau(\underline{I}(i_2, k_1 p))$ are $\tau(\underline{I}(i_2 + jp, p))$ with $j = 0, \ldots, k_2 - 1$. Now the corollary follows Theorem 4.3.11. $\square$

Remark 4.3.13. The results of this will work for functor τ^{op} with indices reversed. In particular, for indecomposable $K\Gamma^{op}$-module $\underline{I}$ of length kp, there is a unique indecomposable $\Pi^1(\Gamma)$-module N such that $\tau^{op}(N) = \underline{I}$. One might ask whether every indecomposable $\Pi^1(\Gamma)$-module M has the property that either $\tau(M)$ is indecomposable or $\tau^{op}(M)$ is indecomposable. The answer is negative in the following example.

Example 4.3.14. Consider the $\Pi^1(\Gamma)$-module M for $\Gamma = A_{3p}$ as follows:

$$K \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \cdots \underset{(p-1)\cdot\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{0}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{-\mathrm{Id}}{\overset{(p-1)\cdot\mathrm{Id}}{\rightleftarrows}} K \underset{-\mathrm{Id}}{\overset{(p-2)\cdot\mathrm{Id}}{\rightleftarrows}} \cdots \underset{-\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{\mathrm{Id}}{\overset{0}{\rightleftarrows}} K \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{2\cdot\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \cdots \underset{(p-1)\cdot\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K\ .$$

Then M has three composition factors $\tau^{-1}(\underline{I}(1,p))$, $\tau^{-1}(\underline{I}(p+1,p))$, and $\tau^{-1}(\underline{I}(2p+1,p))$ with $\tau^{-1}(\underline{I}(p,p))$ being the unique simple submodule, which is also the radical. M has Loewy length 2. But neither $\tau(M)$ nor $\tau^{op}(M)$ is indecomposable.

For the quiver A_N, let $L(i) = \tau^{-1}(\underline{I}(i,p))$ be the irreducible $\Pi^1(\Gamma)$-module. Then $\mathrm{rep}(\Pi^1(\Gamma))$ has exactly blocks $\mathcal{C}_i$ with $i = 1, \cdots, p$, where $\mathcal{C}_i$ is full subcategory of $\mathrm{rep}(\Pi^1(\Gamma))$ generated by the irreducible modules $L(i+kp)$ closed under extensions in $\mathrm{rep}(\Pi^1(\Gamma))$. Then we have a category decomposition

$$\mathrm{rep}(\Pi^1(\Gamma)) = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots \oplus \mathcal{C}_p.$$

More precisely, any $\Pi^1(\Gamma)$-module M in $\mathcal{C}_i$,

$$\tau(M) = \bigoplus_{i' \equiv i \mathrm{Mod} p, i'+kp-1 \leq N} \underline{I}(i', kp)^{n_{i',k}}.$$

Corollary 4.3.15. *If M is a $\Pi^1(\Gamma)$-module such that there are i_1 and i_2, with $p \nmid (i_1 - i_2)$ and both $\underline{I}(i_1, k_1 p)$ and $\underline{I}(i_1, k_1 p)$ are direct summands of $\tau(M)$, then M is not indecomposable.*

4.4 Describing a block of $\mathrm{rep}(\Pi^1(A_N))$

In this section we describe a block $\mathcal{C}_i$ of the category $\mathrm{rep}(\Pi^1(\Gamma))$ for the quiver $\Gamma = A_N$. Here we fix $1 \leq i \leq p$.

Let $\mathrm{Irr}(\mathcal{C}_i)$ be the set of isomorphism classes of irreducible modules in $\mathcal{C}_i$. Then

$$\mathrm{Irr}(\mathcal{C}_i) = \{L(i+kp) \mid k = 0, 1, \ldots, N_i\}.$$

Here $N_i = [\frac{N-i}{p}]$ is the maximal integer not to exceed $\frac{N-i}{p}$.

Let $\Gamma(i)$ be the ext-quiver of the block $\mathcal{C}_i$ with the vertex set being $\mathrm{Irr}(\mathcal{C}_i)$, the number of arrows from L to L' is $\dim \mathrm{Ext}^1_{\mathcal{C}_i}(L, L')$ for two irreducible objects L L' in $\mathcal{C}_i$. By using Theorem 4.3.11, we see that $\Gamma(i) = \overline{A_{N_i+1}}$, the double quiver of the quiver A_{N_i+1}. Our goal is to show that $\mathcal{C}_i$ is related to the preprojective algebra $\Pi^0(A_{N_i+1})$.

4.4.1 Block Decomposition of $\mathrm{Im}(\tau)$

To describe the objects in $\mathcal{C}_i$, we first construct the block decompositions of $\mathrm{Im}(\tau)$. We note that the full subcategory $\mathrm{Im}(\tau)$ of $\mathrm{rep}(K\Gamma)$ is a not an abelian category.

Definition 4.4.1. We say that two indecomposable objects $\underline{I}$ and $\underline{I}'$ in $\mathrm{Im}(\tau)$ are Π^1-related if there is an indecomposable object M in $\mathrm{rep}(\Pi^1(\Gamma))$ such that both $\underline{I}$ and $\underline{I}'$ are direct summands of $\tau(M)$. Extend the Π^1-related relation to an equivalence relation on the set of isomorphism classes of indecomposable objects in $\mathrm{Im}(\tau)$.

We decompose the indecomposable objects in $\mathrm{Im}(\tau)$ into p sets, $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_p$ with $\mathcal{I}_i = \{\underline{I}(i + mp, kp)\}$ with $i + mp + kp - 1 \leq N$. By Corollary 4.3.12, each $\mathcal{I}_i$ are in the same equivalence class. Corollary 4.3.15 implies that two indecomposable modules in two different $\mathcal{I}_i$'s are not Π^1 related. Thus $\{\mathcal{I}_1, \cdots, \mathcal{I}_p\}$ are the equivalence classes of indecomposable modules in $\mathrm{Im}(\tau)$.

Let $\mathcal{D}_i$ be the full subcategory of $\mathrm{Im}(\tau)$ consisting of all modules that are direct sums of objects in $\mathcal{I}_i$. Note that we do not have a category decomposition $\mathrm{Im}(\tau) = \bigoplus_{i=1}^{p} \mathcal{D}_i$ since objects in different $\mathcal{D}_i$ can have non-zero morphism in $\mathrm{rep}(K\Gamma)$.

Lemma 4.4.2. *Every object in $\mathcal{D}_i$ is of the form $\tau(M)$ for some $M \in \mathcal{C}_i$ (although such M is not unique).*

Proof. We use the fact that $\mathrm{Im}(\tau)$ is Karoubian (see Corollary 3.1.5). Then the lemma follows from Corollary 4.3.6 and Corollary 4.3.15. $\square$

To study the $\mathcal{D}_i$, we need to study $\mathcal{I}_i$ first. If $N = np + s$, then

$$|\mathcal{I}_i| = \begin{cases} \frac{n(n+1)}{2} & \text{for } i = 1, 2, \cdots, s+1; \\ \frac{n(n-1)}{2} & \text{for } i = s+1, s+2, \cdots, p. \end{cases}$$

Proposition 4.4.3. *The category $\mathcal{D}_i$ are isomorphic to*

$$\begin{cases} \mathrm{rep}(KA_n), & \text{if } i = 1, 2, \cdots, s+1; \\ \mathrm{rep}(KA_{n-1}), & \text{if } i = s+1, s+2, \cdots, p. \end{cases}$$

Proof. We just prove $i = 1$ case, and the other cases are similar.

We know $\mathcal{I}_1 = \{\underline{I}(1 + mp, lp)\}$, with $0 \leq m \leq n-1$, and $(m+l) \leq n$, and these are the indecomposable modules in $\mathcal{D}_1$. Under similar notation, we denote the indecomposable modules in $\mathrm{rep}(KA_n)$ to be $\{\underline{I}^{(n)}(k, l)\}$, with $1 \leq k \leq n$ and $(k + l - 1) \leq n$. Then we define a functor

$$\mathcal{F} : \mathcal{D}_1 \to \mathrm{rep}(KA_n)$$

to be,

$$\mathcal{F}(\underline{I}(1 + mp, lp)) = \underline{I}^{(n)}(m + 1, l).$$

It is easy to define on the morphisms and check that $\mathcal{F}$ gives the category equivalence. $\square$

Remark 4.4.4. It is easy to see that all the indecomposable objects in $\mathcal{D}_i$ have the same p-arrows; indecomposable objects in different $\mathcal{D}_i$ have no common p-arrows. Then we may define p-arrows in $\mathcal{D}_i$ to be p-arrows for indecomposable objects in $\mathcal{D}_i$.

Lemma 4.4.5. *Any module in $\mathcal{D}_i$ is isomorphic to a module $\underline{M} \in \mathcal{D}_i$, such that if the arrow $i' \to i''$ is not a p-arrow, then $\underline{M}_{i'} \to \underline{M}_{i''}$ is identity.*

Proof. If $i' \to i''$ is not a $p-$arrow in $\mathcal{D}_i$, we know the indecomposable modules are of the form $\underline{I}(mp+i, lp)$, so we may assume $\underline{M} \cong \bigoplus_k \underline{I}(m_k p + i, l_k p)$. The conclusion is just an easy computation. $\square$

4.4.2 Pull Back of $\mathrm{Im}(\tau)$

To pull back $\mathrm{Im}(p)$, we need to pull back each $\mathcal{D}_i$, which are just $\mathcal{C}_i$, for $i = 1, 2, \cdots, p$.

Lemma 4.4.6. *For $M \in \mathrm{rep}(\Pi^1(\Gamma))$, if the arrows $i_1 \xrightarrow{e_1} i_2 \xrightarrow{e_2} \cdots i_{l-1} \xrightarrow{e_{l-1}} i_l$ satisfy $M_{i_j} \xrightarrow{x(e_j)} M_{i_{j+1}}$ is Id, for any $j = 1, 2, \cdots, l-1$, then $M_{i_j} \xleftarrow{x^*(e_j)} M_{i_{j+1}}$ is $\alpha + j \cdot id = \beta - (l-j) \cdot \mathrm{Id}$, where the α and β are the following morphisms,*

$$\overset{\alpha}{\circlearrowleft} M_{i_1} \underset{x^*(e_1)}{\overset{x(e_1)}{\leftrightarrows}} M_{i_2} \underset{x^*(e_2)}{\overset{x(e_2)}{\leftrightarrows}} \cdots M_{i_{l-1}} \underset{x^*(e_{l-1})}{\overset{x(e_{l-1})}{\leftrightarrows}} M_{i_l} \overset{\beta}{\circlearrowright}. \tag{4.8}$$

here, if i_1 is the initial vertex of Γ, then $\alpha = 0$, and if i_l is the ending vertex of Γ, then $\beta = 0$.

Proof. First, for $j = 1$ we have $x^*(e_1)x(e_1) - \alpha = \mathrm{Id}$, so $x^*(e_1) = \alpha + \mathrm{Id}$; then use induction we can get $x^*(e_j) = \alpha + j \cdot \mathrm{Id}$.

To prove $x^*(e_j) = \beta - (l-j) \cdot \mathrm{Id}$, we also use $\beta - x(e_l)x^*(e_l) = \mathrm{Id}$ and induction from $j = l$. □

Then from Lemma 4.4.5 and 4.4.6, we easily get

Theorem 4.4.7. *Any module in $\mathcal{C}_i$ is isomorphic to a module $M \in \mathcal{C}_i$, such that M is determined by all $\{ M_{i'} \underset{x^*(e)}{\overset{x(e)}{\rightleftarrows}} M_{i''} \}$ with $\{i' \xrightarrow{e} i''\}$ p-arrows in $\mathcal{D}_i$.*

From Example 3.1.9, we know that $\mathrm{rep}(\Pi^1(A_p))$ is a semisimple category and there is only one irreducible object L in it. Recall that L is

$$K \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{2\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{3\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \cdots \underset{(p-3)\cdot\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{(p-2)\cdot\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K \underset{(p-1)\cdot\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} K .$$

Using L and Theorem 4.4.7, let's define a functor $L \overset{A_p}{\underset{m}{\bigotimes}}$ from $\mathrm{rep}(\Pi^0(A_m))$ to $\mathrm{rep}(\Pi^1(A_{mp}))$,

$$L \overset{A_p}{\underset{m}{\bigotimes}} : \mathrm{rep}(\Pi^0(A_m)) \to \mathrm{rep}(\Pi^1(A_{mp})).$$

For any $(M,x) \in \text{rep}(\Pi^0(A_m))$, M is represented by

$$M_{i-1} \underset{x^*(e)}{\overset{x(e)}{\rightleftarrows}} M_i \underset{x^*(e')}{\overset{x(e')}{\rightleftarrows}} M_{i+1}$$

with $x^*(e')x(e') - x(e)x^*(e) = 0$ for any consecutive arrows $i-1 \xrightarrow{e} i \xrightarrow{e'} i+1$. Here, if $i-1=0$, then $i=1$ is the initial vertex for A_m, then there is no M_{i-1} and no morphisms $x(e)$ and $x^*(e)$, for definition and computation consistency, we keep it here and treat $x(e)=0$ and $x^*(e)=0$. We make similar convention for $i+1=m+1$, i.e., i is the ending vertex of A_m.

Define $L \overset{A_p}{\underset{m}{\bigotimes}} M \in \text{rep}(\Pi^1(A_{mp}))$ to be

$$L \overset{A_p}{\underset{m}{\bigotimes}} M_{i-1} \underset{x^*(e)}{\overset{x(e)}{\rightleftarrows}} L \overset{A_p}{\underset{m}{\bigotimes}} M_i \underset{x^*(e')}{\overset{x(e')}{\rightleftarrows}} L \overset{A_p}{\underset{m}{\bigotimes}} M_{i+1}\ ,$$

where $L \overset{A_p}{\underset{m}{\bigotimes}} M_i$ is

$$M_i \underset{\alpha_i(1)}{\overset{\text{Id}}{\rightleftarrows}} M_i \underset{\alpha_i(2)}{\overset{\text{Id}}{\rightleftarrows}} M_i \underset{\alpha_i(3)}{\overset{\text{Id}}{\rightleftarrows}} \cdots \underset{\alpha_i(p-3)}{\overset{\text{Id}}{\rightleftarrows}} M_i \underset{\alpha_i(p-2)}{\overset{\text{Id}}{\rightleftarrows}} M_i \underset{\alpha_i(p-1)}{\overset{\text{Id}}{\rightleftarrows}} M_i$$

with $\alpha_i(k) = x(e)x^*(e) + k \cdot \text{Id} = x^*(e')x(e') + k \cdot \text{Id}$. It is easy to check $L \overset{A_p}{\underset{m}{\bigotimes}} M$ is a $\Pi^1(A_{mp})$-module.

For any $\Pi^0(A_m)$-modules (M,x) and (M',x'), if there is a morphism f from M to M', with $f = \{f_i\}_{i=1}^m$ which makes the diagram

$$\begin{array}{ccc} M_{i-1} & \underset{x^*(e)}{\overset{x(e)}{\rightleftarrows}} & M_i \\ \downarrow f_{i-1} & & \downarrow f_i \\ M'_{i-1} & \underset{x'^*(e)}{\overset{x'(e)}{\rightleftarrows}} & M'_i \end{array}$$

commutative, then we have $x'(e)f_{i-1} = f_i x(e)$ and $x'^*(e)f_i = f_{i-1}x^*(e)$.

We define $L\overset{A_p}{\underset{m}{\bigotimes}}(f) : L\overset{A_p}{\underset{m}{\bigotimes}} M \to L\overset{A_p}{\underset{m}{\bigotimes}} M'$ to be

$$\begin{array}{ccc} L\overset{A_p}{\underset{m}{\bigotimes}} M_{i-1} & \underset{x^*(e)}{\overset{x(e)}{\rightleftarrows}} & L\overset{A_p}{\underset{m}{\bigotimes}} M_i \\ {\scriptstyle L\overset{A_p}{\underset{m}{\bigotimes}}(f_{i-1})}\downarrow & & \downarrow{\scriptstyle L\overset{A_p}{\underset{m}{\bigotimes}}(f_i)} \\ L\overset{A_p}{\underset{m}{\bigotimes}} M'_{i-1} & \underset{x'^*(e)}{\overset{x'(e)}{\rightleftarrows}} & L\overset{A_p}{\underset{m}{\bigotimes}} M'_i \end{array} ,$$

here $L\overset{A_p}{\underset{m}{\bigotimes}} M'_i$ is

$$M'_i \underset{\alpha'_i(1)}{\overset{\mathrm{Id}}{\rightleftarrows}} M'_i \underset{\alpha'_i(2)}{\overset{\mathrm{Id}}{\rightleftarrows}} M'_i \underset{\alpha'_i(3)}{\overset{\mathrm{Id}}{\rightleftarrows}} \cdots \underset{\alpha'_i(p-3)}{\overset{\mathrm{Id}}{\rightleftarrows}} M'_i \underset{\alpha'_i(p-2)}{\overset{\mathrm{Id}}{\rightleftarrows}} M'_i \underset{\alpha'_i(p-1)}{\overset{\mathrm{Id}}{\rightleftarrows}} M'_i$$

with $\alpha'_i(k) = x'(e)x'^*(e) + k \cdot \mathrm{Id} = x'^*(e')x'(e') + k \cdot \mathrm{Id}$ and $L\overset{A_p}{\underset{m}{\bigotimes}}(f_i) = (f_i, \cdots, f_i)$.

To check that $L\overset{A_p}{\underset{m}{\bigotimes}}(f)$ is a morphism from $L\overset{A_p}{\underset{m}{\bigotimes}} M$ to $L\overset{A_p}{\underset{m}{\bigotimes}} M'$, we just need to check that the diagram above is commutative. There are two cases:

(1) For any $k = 1, 2, \cdots, p-1$, we need to prove the diagrams

$$\begin{array}{ccc} M_{i-1} & \underset{\alpha_{i-1}(k)}{\overset{\mathrm{Id}}{\rightleftarrows}} & M_{i-1} \\ {\scriptstyle f_{i-1}}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ M'_{i-1} & \underset{\alpha'_{i-1}(k)}{\overset{\mathrm{Id}}{\rightleftarrows}} & M'_{i-1} \end{array}$$

and

$$\begin{array}{ccc} M_i & \underset{\alpha_i(k)}{\overset{\mathrm{Id}}{\rightleftarrows}} & M_i \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ M'_i & \underset{\alpha'_i(k)}{\overset{\mathrm{Id}}{\rightleftarrows}} & M'_i \end{array}$$

are commutative. The proof of the two types are similar, so for simplicity, we just prove the later cases. Note that $\alpha_i(k) = x(e)x^*(e) + k \cdot \mathrm{Id}$, $\alpha'_i(k) = x'(e)x'^*(e) + k \cdot \mathrm{Id}$,

$x'(e)f_{i-1} = f_i x(e)$ and $x'^*(e)f_i = f_{i-1}x^*(e)$, so we have

$$\begin{aligned}
\alpha_i'(k)f_i &= (x'(e)x'^*(e) + k \cdot \mathrm{Id})f_i \\
&= x'(e)(x'^*(e)f_i) + k \cdot f_i \\
&= x'(e)(f_{i-1}x^*(e)) + k \cdot f_i \\
&= (x'(e)f_{i-1})x^*(e) + k \cdot f_i \\
&= (f_i x(e))x^*(e) + k \cdot f_i \\
&= f_i(x(e))x^*(e) + k \cdot \mathrm{Id}) \\
&= f_i \alpha_i(k).
\end{aligned}$$

(2) We need to prove the middle diagram

$$\begin{array}{ccc}
M_{i-1} & \underset{x^*(e)}{\overset{x(e)}{\rightleftarrows}} & M_i \\
{\scriptstyle f_{i-1}}\downarrow & & \downarrow{\scriptstyle f_i} \\
M'_{i-1} & \underset{x'^*(e)}{\overset{x'(e)}{\rightleftarrows}} & M'_i
\end{array}$$

is commutative, and it is just by definition of f.

For any $m \le m'$, A_m is a full subquiver of A'_m. There are $m'-m+1$ natural embedding of quiver A_m into $A_{m'}$, and we denote them by $\mathcal{F}_{m,m'}$, $\tau \circ \mathcal{F}_{m,m'}$, $\tau^2 \circ \mathcal{F}_{m,m'}$, $\cdots$, $\tau^{m'-m} \circ \mathcal{F}_{m,m'}$. More precisely, $\tau^k \circ \mathcal{F}_{m,m'}$ maps the ith vertex of A_m to the $i+k$th vertex of A'_m.

Recall from Section 3.2, these embedding induce natural embedding of $\mathrm{rep}(\Pi^1(A_m))$ into $\mathrm{rep}(\Pi^1(A'_m))$, and we denote them by $\pi_{0*}(m,m')$, $\pi_{1*}(m,m')$, $\pi_{2*}(m,m')$, $\cdots$, $\pi_{(m'-m)*}(m,m')$, respectively.

Since $\mathrm{rep}(\Pi^1(A_N)) = \bigoplus_{i=1}^{p} \mathcal{C}_i$, there is a canonical projection of $\mathrm{rep}(\Pi^1(A_N))$ to $\mathcal{C}_i$, and we denote it by p_i for any $i = 1, 2, \cdots, p$. Finally combine the $L\bigotimes_{m}^{A_p}$ functor, the embedding functor π_* and projection functor p_i, we get the following p functors:

(1) for $i = 1, 2, \cdots, s+1$, there are $s+1$ functors,

$$\mathcal{F}_i : \mathrm{rep}(\Pi^0(A_n)) \xrightarrow{L \overset{A_p}{\underset{n}{\otimes}}} \mathrm{rep}(\Pi^1(A_{np})) \overset{\pi_{(i-1)*}(np,N)}{\longrightarrow} \mathrm{rep}(\Pi^1(A_N)) \xrightarrow{p_i} \mathcal{C}_i;$$

(2) for $i = s+2, s+3, \cdots, p$, there are $p-s-1$ functors,

$$\mathcal{G}_i : \mathrm{rep}(\Pi^0(A_{n-1})) \xrightarrow{L \overset{A_p}{\underset{n-1}{\otimes}}} \mathrm{rep}(\Pi^1(A_{(n-1)p})) \overset{\pi_{(i-1)*}((n-1)p,N)}{\longrightarrow} \mathrm{rep}(\Pi^1(A_N)) \xrightarrow{p_i} \mathcal{C}_i.$$

Then we can start to prove our main theorem in this section.

Theorem 4.4.8. *The category $\mathcal{C}_i$ are isomorphic to*

$$\begin{cases} \mathrm{rep}(\Pi^0(A_n)), & \text{if } i = 1, 2, \cdots, s+1; \\ \mathrm{rep}(\Pi^0(A_{n-1})), & \text{if } i = s+1, s+2, \cdots, p. \end{cases}$$

Proof. Now from Lemma 4.4.5, Theorem 4.4.7, and combinatorics we get the functors $\mathcal{F}_i$ $(i = 1, 2, \cdots, s+1)$ and $\mathcal{G}_i$ $(i = s+2, s+3, \cdots, p)$ induce category equivalence.

□

Therefore, we get Theorem 4.4.8, and combine the decomposition $\mathrm{rep}(\Pi^1(A_N)) = \bigoplus_{i=1}^{p} \mathcal{C}_i$, we get Theorem 1.3.3: if $N = np + s$ with $0 \le s \le p-1$, then for type A_N quiver, there is a categorical equivalence

$$\mathrm{rep}(\Pi^1(A_N)) \cong \mathrm{rep}(\Pi^0(A_n))^{\oplus s+1} \oplus \mathrm{rep}(\Pi^0(A_{n-1}))^{\oplus p-s-1}.$$

4.5 Computation of Stack Functions

In this section , we always fix the ground field be a finite field $\mathbb{F}_q$ with characteristic p and $N = np + s$ with $0 \le s \le p-1$. We study the stack functions $\Phi(\mathbf{0})$ and $\mu(\mathbf{0})$ in this section and we get some interesting identities using Theorem 1.3.3.

From Section 3.1, we have

$$\mu_\Gamma^1(\mathbf{0}) = \sum_{M \in (\mathrm{Iso}\Pi^1(\Gamma)-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M)|} X^{\mathbf{dim}(M)}.$$

and

$$\Phi(\mathbf{0}) = \sum_{M \in (\mathrm{Iso}\Pi^1(\Gamma)-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M)|} q^{\langle \mathbf{dim}(M), \mathbf{dim}(M) \rangle_\Gamma} X^{\mathbf{dim}(M)}.$$

Since we will have different quivers and different deformed preprojective algebras, and the $\mathbf{0}$ is always fixed, so we change the notation a little bit here to make things clearly.

$$\mu_\Gamma^\lambda(q, X) = \sum_{M \in (\mathrm{Iso}\Pi^\lambda(\Gamma)-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M)|} X^{\mathbf{dim}(M)}.$$

and

$$\Phi_\Gamma^\lambda(q, X) = \sum_{M \in (\mathrm{Iso}\Pi^\lambda(\Gamma)-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M)|} q^{\langle \mathbf{dim}(M), \mathbf{dim}(M) \rangle_\Gamma} X^{\mathbf{dim}(M)}.$$

We denote by $K[[\Gamma_0]]$ the ring of formal power series of with commuting variables $\{X^{\mathbf{d}} \mid \mathbf{d} \in \mathbb{Z}^{\Gamma_0}\}$. Elements in $K[[\Gamma_0]]$ can be uniquely written as

$$\sum_{\mathbf{d} \in \mathbb{Z}^{\Gamma_0}} f_{\mathbf{d}} X^{\mathbf{d}} \text{ with } f_{\mathbf{d}} \in K.$$

We note that the K-algebra structure on $K[[\Gamma_0]]$ depends on the vertex set Γ_0 only.

Recall the Euler-Ringel form on $\mathbb{Z}^{\Gamma_0}$ is defined by

$$\langle \mathbf{d}, \mathbf{d}' \rangle = \sum_{i \in \Gamma_0} d_i d_i' - \sum_{a \in \Gamma_1} d_{s(a)} d_{t(a)}'$$

for any $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^{\Gamma_0}$. The corresponding symmetric bilinear form is $(\mathbf{d}, \mathbf{d}') = \langle \mathbf{d}, \mathbf{d}' \rangle + \langle \mathbf{d}', \mathbf{d} \rangle$. We define a new algebra structure on $K[[\Gamma_0]]$ with multiplication deformed by

$$X^{\mathbf{d}} \cdot X^{\mathbf{d}'} = q^{(\mathbf{d}, \mathbf{d}')} X^{\mathbf{d}+\mathbf{d}'}.$$

We use $K[[\Gamma]]$ to denote this algebra structure (still commutative). Let $F : K[[\Gamma_0]] \to K[[\Gamma]]$ defined by

$$F(X^{\mathbf{d}}) = q^{\langle \mathbf{d}, \mathbf{d} \rangle} X^{\mathbf{d}}.$$

Then F is a K-algebra isomorphism. In particular, we have

Lemma 4.5.1. $\Phi_\Gamma^\lambda(X) = F(\mu_\Gamma^\lambda(q, X))$.

Therefore, we just consider the $\mu_\Gamma^\lambda(X)$.

We know that $\mathrm{rep}(\Pi^1(A_N)) = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots \oplus \mathcal{C}_p$, so from the discussion for 1.2 in Section 3.1, we have

$$\begin{aligned} \mu_{A_N}^1(q, X) &= \prod_{i=1}^{p} \sum_{M(i) \in (\mathrm{Iso}\mathcal{C}_i)} \frac{1}{|\mathrm{Aut}(M(i))|} X^{\mathbf{dim}(M(i))} \\ &=: \prod_{i=1}^{p} \mu_{\mathcal{C}_i}(q, X). \end{aligned}$$

Remark 4.5.2. By Theorem 4.4.8, we have $\mathrm{rep}(\Pi^0(A_n)) \cong \mathcal{C}_i$ by functors $\mathcal{F}_i$ for $i = 1, 2, \cdots, s+1$; and $\mathrm{rep}(\Pi^0(A_{n-1})) \cong \mathcal{C}_i$ by functors $\mathcal{G}_i$ for $i = s+2, s+3, \cdots, p$. For simplicity, we denote these functor by Θ_i in general, i.e., $\Theta_i = \mathcal{F}_i$ for $i = 1, 2, \cdots, s+1$, and for $i = s+2, s+3, \cdots, p$, $\Theta_i = \mathcal{G}_i$. Moreover, we also use n_i to denote n and $n-1$ in general, i.e., $n_i = n$ for $i = 1, 2, \cdots, s+1$, and for $i = s+2, s+3, \cdots, p$, $n_i = n-1$.

Under the notation in Remark 4.5.2, we have $\mathrm{rep}(\Pi^0(A_{n_i})) \cong \mathcal{C}_i$ by Θ_i. However, for any $M \in \mathrm{rep}(\Pi^0(A_{n_i}))$, $\mathbf{dim}(\Theta_i(M)) \neq \mathbf{dim}(M)$, so $\mu_{A_{n_i}}^0(q, X) \neq \mu_{\mathcal{C}_i}(q, X)$. But $\mu_{A_{n_i}}^0(q, X)$ and $\mu_{\mathcal{C}_i}(q, X)$ still have close relationship.

For quiver $\Gamma = A_N$, we know it has N vertices; for quiver $\Gamma(i) = A_{n_i}$, it has n_i vertices. Note that $\mathcal{C}_i$ are subcategory of $\mathrm{rep}(\Pi^0(A_N))$. We use $\{\epsilon_i \mid i = 1, \cdots, N\} \subseteq \mathbb{Z}^{\Gamma_0}$ to denote the standard basis of the abelian group $\mathbb{Z}^{\Gamma_0}$. We will use the same notation ϵ_k to denote the basis of $\mathbb{Z}^{\Gamma(i)_0}$. It should not cause any confusion. We define a homomorphism of abelian groups

$$\theta_i : \mathbb{Z}^{\Gamma(i)_0} \to \mathbb{Z}^{\Gamma_0} \text{ by } \theta_i(\epsilon_k) = \sum_{r=0}^{p-1} \epsilon_{i+(k-1)p+r}.$$

Note that $\theta_i(\mathbb{Z}^{\Gamma(i)_0} \subseteq \mathbb{Z}^{\Gamma_0}$. In fact for the functor $\Theta_i : \mathrm{rep}(\Pi^0(A_{n_i})) \to \mathcal{C}_i$, we have

$$\mathbf{dim}(\Theta_i(M)) = \theta_i(\mathbf{dim}(M))$$

for all $M \in \mathrm{rep}(\Pi^0(A_{n_i}))$.

The homomorphism $theta_i$ also induces K-algebra homomorphism $\theta_{i*} : K[[\Gamma(i)_0]] \to K[[\Gamma_0]]$, which is defined by, for $g(Y) = \sum\limits_{\mathbf{d}\in\mathbb{Z}^{\Gamma(i)_0}} g_{\mathbf{d}} Y^{\mathbf{d}}$,

$$\theta_{i*}(\sum_{\mathbf{d}\in\mathbb{Z}^{\Gamma(i)_0}} g_{\mathbf{d}} Y^{\mathbf{d}}) = \sum_{\mathbf{d}\in\mathbb{Z}^{\Gamma(i)_0}} g_{\mathbf{d}} X^{\theta_i(\mathbf{d})} = \sum_{\mathbf{v}\in\mathbb{Z}^{\Gamma_0}} (\sum_{\mathbf{d}\in\theta_i^{-1}(\mathbf{v})} g_{\mathbf{d}}) X^{\mathbf{v}} \in K[[\Gamma_0]].$$

Note that for $\mathbf{d} = (d_i)_{i\in\Gamma_0} \in \mathbb{Z}^{\Gamma_0}$, then $X^{\mathbf{v}} = \prod\limits_{i\in\Gamma_0} (X^{\epsilon_i})^{d_i}$. If we denote by $\theta_{i*}(X^{\epsilon_k}) = X^{\theta_i(\epsilon_k)}$ for all $k \in \Gamma(i)_0$, then the map θ_{i*} is the variable subsititution $Y^{\epsilon_k} \mapsto X^{\theta_i(\epsilon_k)}$ of formal power series. We simply write this change of variable by $Y \mapsto \theta_i(Y)$, then $\theta_{i*}(g(X)) = g(\theta_i(Y))$. Noted that $g(\theta_i(Y))$ is a formal power series of X.

We can now take K to be any commutative ring containing $\mathbb{Z}$ and all |Aut(M)| for all M. (For finite field $K = \mathbb{Q}$ will be good enough. But for general case, q becomes a variable and K should contain necessary inverses of automorphism groups.)

Note that

$$\mu^0_{A_{n_i}}(q, X) = \sum_{M(i)\in \mathrm{Iso}\Pi^0(A_{n_i})-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M(i))|} X^{\mathbf{dim}(M(i))},$$

and

$$\mu_{\mathcal{C}_i}(q, X) = \sum_{M(i)\in(\mathrm{Iso}\mathcal{C}_i)} \frac{1}{|\mathrm{Aut}(M(i))|} X^{\mathbf{dim}(M(i))};$$

then under the equivalence Θ_i we get

$$\begin{aligned}
\mu_{\mathcal{C}_i}(q,X) &= \sum_{M(i)\in(\mathrm{Iso}\Pi^0(A_{n_i})-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M(i))|} X^{\mathbf{dim}(\Theta_i(M(i)))} \\
&= \sum_{M(i)\in(\mathrm{Iso}\Pi^0(A_{n_i})-\mathrm{Mod})} \frac{1}{|\mathrm{Aut}(M(i))|} X^{\theta_i(\mathbf{dim}(M(i)))} \\
&= \theta_{i*}(\mu^0_{A_{n_i}}(q, X_{A_{n_i}})) \\
&= \mu^0_{A_{n_i}}(q, \theta_i(X_{A_{n_i}})) \in K[[A_N]].
\end{aligned}$$

Here $X = (X_1, \cdots, X_N)$, and we are using $X_{A_{n_i}}$ to indicate the variables used the quiver A_{n_i}. Remember that for $i = 1, 2, \cdots, s+1$, $n_i = n$, and for $i = s+2, s+3, \cdots, p$, $n_i = n-1$.

If $n_i = 0$, then the quiver A_{n_i} has no vertex and thus has no representations of any kind (with an understanding of having a zero representation). We will use the convention $\mu^0_{A_{n_i}}(X_{A_{n_i}}) = 1$.

Theorem 4.5.3. $\mu^1_{A_N}(q, X) = \prod_{i=1}^{p} \mu^0_{A_{n_i}}(q, \theta_i(X_{A_{n_i}}))$.

Corollary 4.5.4. *For the quiver A_N we have $\Phi(\mathbf{0}) = \prod_{i=1}^{p} F(\mu^0_{A_{n_i}}(q, \theta_i(X_{A_{n_i}})))$.*

Chapter 5

Jordan Quiver Case

In this chapter, we assume K is an algebraically closed field of characteristic p, and $W = K\langle x, y\rangle/[x, y] = 1$ is Weyl algebra. To classify all finite dimensional representations of Weyl algebra, we just need to classify its representation space of a fixed dimension.

5.1 Decomposition

Now for a fixed dimension N, in this section, we use the eigenvalues of y acting on W-module to decompose the W-module.

Let's list some notations we will use first.

(1) I_N denotes the $N \times N$ identity matrix, and sometimes for simplicity, we ignore the size and just write I.

(2) We have the variety $\mathcal{V}_N = \{(X, Y) \in \mathfrak{gl}_N(K) \times \mathfrak{gl}_N(K) \mid [X, Y] = I_N\}$ is the N dimensional representation space for Weyl algebra W, so we use a pair of $N \times N$ matrices (X, Y) to represent an N dimensional representation of W.

(3) For any matrix M, if we decompose it into $s \times s$ blocks, then we denote the blocks to be $\{M_{i,j}\}_{i,j=1}^{s}$ and write $M = (M_{i,j})_{i,j=1}^{s}$.

(4) $J_N(\alpha)$ denotes the $N \times N$ Jordan block with eigenvalue α. If the eigenvalue is 0, then we denote $J_N(0)$ to be J_N for simplicity, i.e.

$$J_N(0) = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{N\times N}.$$

(5) Q_N denotes the following matrix:

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & N-1 \\ & & & & 0 \end{pmatrix}_{N\times N}.$$

There are two natural maps

$$\begin{array}{ccc} & \mathcal{V}_N = \{(X,Y) \in \mathfrak{gl}_N(K) \times \mathfrak{gl}_N(K) \mid [X,Y] = I_N\} & \\ \swarrow^{\tau_1} & & \searrow^{\tau_2} \\ \mathfrak{gl}_N(K) & & \mathfrak{gl}_N(K) \end{array}.$$

To classify the N dimensional representation space, we fix a class in $\mathrm{Im}(\tau_2)$ first, pull it back to $\mathcal{V}_N$, then push forward to $\mathrm{Im}(\tau_1)$. Therefore we just need to classify the push forward.

Fixing a class in $\mathrm{Im}(\tau_2)$ is equivalent to fixing the eigenvalues of Y.

Theorem 5.1.1. *For any finite dimensional representation (X,Y) of Weyl algebra, if under* *a fixed basis,*

$$Y = \begin{pmatrix} J_m(\alpha_1) & 0 \\ 0 & J_n(\alpha_2) \end{pmatrix}$$

with $\alpha_1 \neq \alpha_2$, then the same size block decomposition of X is

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix}.$$

Then we will have $[X_{11}, J_m(\alpha_1)] = I_m$, $X_{1,2} = 0$, $X_{2,1} = 0$ and $[X_{22}, J_n(\alpha_2)] = I_n$, i.e., this representation is decomposable.

Proof. Use the relation $[X, Y] = I$ we can get

$$\begin{cases} X_{1,1}J_m(\alpha_1) - J_m(\alpha_1)X_{1,1} = I_m, \\ X_{1,2}J_n(\alpha_2) - J_m(\alpha_1)X_{1,2} = 0, \\ X_{2,1}J_m(\alpha_1) - J_n(\alpha_2)X_{2,1} = 0, \\ X_{2,2}J_n(\alpha_2) - J_n(\alpha_2)X_{2,2} = I_n. \end{cases}$$

The first equation and the fourth equation tell $[X_{11}, J_m(\alpha_1)] = I_m$ and $[X_{22}, J_n(\alpha_2)] = I_n$ directly.

The second equation gives $X_{1,2}J_n(\alpha_2) = J_m(\alpha_1)X_{1,2}$, and $\alpha_1 \neq \alpha_2$, so $X_{1,2}$ is a morphism between different eigenspaces , which tells $X_{1,2} = 0$.

For the proof $X_{2,1} = 0$, it's similar.

□

Then we know different eigenvalues of Y give a decomposition of the module (X, Y) into indecomposable modules, so we just need to consider the case $Y \cong J_N(\alpha)$ with only one eigenvalue. Denote $G_{Y,N} = \{G \in GL_N(K) \mid GY = YG\}$, since $J_N(\alpha) = \alpha I_N + J_N$, $[X, J_N(\alpha)] = [X, \alpha I_N] + [X, J_N] = [X, J_N]$, we have $G_{J_N(\alpha),N} = G_{J_N,N}$. Therefore we can reduce to the nilpotent elements in $\mathrm{Im}(\tau_2)$.

Remark 5.1.2. (1) We have similar decomposition results for the 2 variable polynomial algebra $K[x, y]$.

(2) Set

$$\mathcal{X}_{Y,N} = \{X \in \mathfrak{gl}_N(K) \mid [X, Y] = I_N \text{ with } Y \text{ fixed nilpotent}\},$$

then $\mathcal{X}_{Y,N}$ is the push forward of Y, and the conjugation action of $GL_N(K)$ on $\mathfrak{gl}_N(K)$ induces the conjugation action of $G_{Y,N}$ on $\mathcal{X}_{Y,N}$. To classify the isomorphic class, we just need to get the orbits $\mathcal{X}_{Y,N}/G_{Y,N}$.

(3) Fix a nilpotent isomorphic class $[Y]$ in $\mathrm{Im}(\tau_2) \subseteq \mathfrak{gl}_N(K)$ is equivalent to fix a nilpotent Jordan matrix, which is also equivalent to fix a partition of N.

In the rest of this chapter, we always fix a basis, such that under this basis, the W-module (X, Y) is a pair of matrices with Y a nilpotent Jordan matrix.

5.2 Count the Orbits

In this section, we want to get the orbits $\mathcal{X}_{Y,N}/G_{Y,N}$. We first found a subset $\widetilde{\mathcal{X}}_{Y,N}$ of $\mathcal{X}_{Y,N}$ which intersects each orbit of $\mathcal{X}_{Y,N}/G_{Y,N}$ nonempty, and then use this subset to count the orbits $\mathcal{X}_{Y,N}/G_{Y,N}$.

5.2.1 Nilpotent Elements in Im(τ_2)

By Lemma 3.3.8, we know for algebraically closed field of characteristic p, W-modules are np dimensional, here n is some integer number; and the irreducible W-modules are p dimensional. For any W-module (X, Y), we want to know more information of Y in this section, and then we can give a precise description of the irreducible W-module.

From now on, we fix

$$Y = \begin{pmatrix} J_{N_1} & & & \\ & J_{N_2} & & \\ & & \ddots & \\ & & & J_{N_s} \end{pmatrix}_{N\times N},$$

then $\sum_{i=1}^{s} N_i = N = np$.

Lemma 5.2.1. *$\mathcal{X}_{Y,N}$ is not empty if and only if the size of every Jordan block of Y is divisible by p, i.e., $p \mid N_i$ for each $i = 1, \cdots, s$.*

Proof. Without loss of generality, assume $N_1 \geq N_2 \geq \cdots \geq N_s$, and $X = (X_{i,j})_{i,j=1}^{s}$ is the same size block decomposition of X, so $X_{i,j}$ is an $N_i \times N_j$ matrix. Then we have

$$X_{i,j}J_{N_j} - J_{N_i}X_{i,j} = \begin{cases} 0, & \text{if } i \neq j; \\ I_{N_i}, & \text{if } i = j. \end{cases} \tag{5.1}$$

When $i = j$, (5.1) gives $[X_{i,i}, J_{N_i}] = I_{N_i}$. Since $\mathrm{tr}[X_{i,i}, J_{N_i}] = 0 = \mathrm{tr}I_{N_i}$, we need $N_i = 0$, which means $p \mid N_i$. This proves $\mathcal{X}_{Y,N}$ is not empty only if the size of every Jordan block of Y is divisible by p.

For the other side, if $p \mid N_i$ for all $i = 1, \cdots, s$, by computing directly and notice that the characteristic of K is p, we have $[Q_N, J_N] = I_N$. □

From now on, we always assume $N_1 \geq N_2 \geq \cdots \geq N_s$. Let's define some special matrices for future use.

(1) If $i \leq j$, then $N_i \geq N_j$, then define $J^{\alpha}_{i,j} = \begin{pmatrix} 0 \\ J^{\alpha}_{N_j} \end{pmatrix}_{N_i \times N_j}$.

(2) If $i \geq j$, then $N_i \leq N_j$, then define $J^{\alpha}_{i,j} = \begin{pmatrix} J^{\alpha}_{N_i} & 0 \end{pmatrix}_{N_i \times N_j}$.

(3) If $i = j$, then $J^{\alpha}_{i,i} = J^{\alpha}_{N_i}$.

Using (5.1) to compute $X = (X_{i,j})_{i,j=1}^{s}$ directly, we have

1. If $i < j$, then $N_i \geq N_j$, so

$$
\begin{aligned}
X_{i,j} &= \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
x_0^{(ij)} & & & & \\
x_1^{(ij)} & x_0^{(ij)} & & & \\
\vdots & x_1^{(ij)} & \ddots & & \\
\vdots & & \ddots & x_0^{(ij)} & \\
x_{N_j-1}^{(ij)} & \cdots & \cdots & x_1^{(ij)} & x_0^{(ij)}
\end{pmatrix}_{N_i \times N_j} \\
&= \sum_{\alpha=0}^{N_j-1} x_\alpha^{(ij)} J_{i,j}^{\alpha}.
\end{aligned}
$$

2. If $i > j$, then $N_i \leq N_j$, so

$$
\begin{aligned}
X_{i,j} &= \begin{pmatrix}
x_0^{(ij)} & & & & & 0 & \cdots & 0 \\
x_1^{(ij)} & x_0^{(ij)} & & & & 0 & \cdots & 0 \\
\vdots & x_1^{(ij)} & \ddots & & & \vdots & \ddots & \vdots \\
\vdots & & \ddots & x_0^{(ij)} & & 0 & \cdots & 0 \\
x_{N_i-1}^{(ij)} & \cdots & \cdots & x_1^{(ij)} & x_0^{(ij)} & 0 & \cdots & 0
\end{pmatrix}_{N_i \times N_j} \\
&= \sum_{\alpha=0}^{N_i-1} x_\alpha^{(ij)} J_{i,j}^{\alpha}.
\end{aligned}
$$

3. If $i = j$, then

$$X_{i,i} = \begin{pmatrix} x_0^{(ii)} & 1 & & & \\ x_1^{(ii)} & x_0^{(ii)} & 2 & & \\ \vdots & x_1^{(ii)} & \ddots & \ddots & \\ \vdots & & \ddots & x_0^{(ii)} & N_i - 1 \\ x_{N_i-1}^{(ii)} & \cdots & \cdots & x_1^{(ii)} & x_0^{(ii)} \end{pmatrix}_{N_i \times N_i}$$

$$= \sum_{\alpha=0}^{N_i-1} x_\alpha^{(ij)} J_{i,i}^\alpha + Q_{N_i}.$$

Since the irreducible W-modules are p dimensional, so we can easily get

Corollary 5.2.2. *Any irreducible W-module (X, Y) with Y nilpotent is isomorphic to $\big(\sum_{\alpha=0}^{p-1} x_\alpha J_p^\alpha + Q_p, J_p\big)$, with $x_\alpha \in K$.*

5.2.2 Push Forward to Im(τ_1)

We will find the subset $\widetilde{\mathcal{X}}_{Y,N}$ of $\mathcal{X}_{Y,N}$ which intersects each orbit of $\mathcal{X}_{Y,N}/G_{Y,N}$ nonempty by studying the push forward a fixed Y in Im(τ_2) to Im(τ_1) in this section.

Let's fix N satisfying $p \mid N$ and fix

$$Y = \begin{pmatrix} J_{N_1} & & & \\ & J_{N_2} & & \\ & & \ddots & \\ & & & J_{N_s} \end{pmatrix}_{N \times N},$$

here $N_1 \geq N_2 \geq \cdots \geq N_s$, $N_i = k_i p$ for some k_i and $N = \sum_{i=1}^{s} N_i$.

Fix

$$Q = \begin{pmatrix} Q_{N_1} & & & \\ & Q_{N_2} & & \\ & & \ddots & \\ & & & Q_{N_s} \end{pmatrix}_{N\times N},$$

it is easy to get $Q = Q_N$ because $p \mid N_i$ for each $i = 1, \cdots, s$ and characteristic of K is p.

Since $[Q, Y] = I_N$, then we can write $\mathcal{X}_{Y,N} = \mathrm{End}_Y(K^N) + Q$, where $\mathrm{End}_Y(K^N) = \{X \in \mathfrak{gl}_N(K) \mid [X, Y] = 0\}$. Therefore for all $X \in \mathcal{X}_{Y,N}$, we can write $X = \widetilde{X} + Q = (\widetilde{X}_{i,j})_{i,j=1}^s + Q$ with $\widetilde{X} \in \mathrm{End}_Y(K^N)$. In the future, we just write $X = (X_{i,j})_{i,j=1}^s + Q$ for simplicity. Pay attention the $X_{i,j}$ a little bit different from the above one, and here

(1) If $i < j$, $X_{i,j} = \sum_{\alpha=0}^{N_j-1} x_\alpha^{(ij)} J_{i,j}^\alpha$.

(2) If $i > j$, $X_{i,j} = \sum_{\alpha=0}^{N_i-1} x_\alpha^{(ij)} J_{i,j}^\alpha$.

(3) If $i = j$, $X_{i,i} = \sum_{\alpha=0}^{N_i-1} x_\alpha^{(ij)} J_{i,i}^\alpha$.

Use $N_i = k_i p$, and set $k_{(ij)} = \min\{k_i, k_j\}$, we can write $X_{i,j} = \sum_{\alpha=0}^{k_{(ij)}p-1} x_\alpha^{(ij)} J_{i,j}^\alpha$ in general, where $x_\alpha^{(ij)} \in K$. Also, under this notation, we have

$$\mathcal{X}_{Y,N} = \{X = (X_{i,j})_{i,j=1}^s + Q \mid X_{i,j} = \sum_{\alpha=0}^{k_{(ij)}p-1} x_\alpha^{(ij)} J_{i,j}^\alpha, x_\alpha^{(ij)} \in K\}$$

$$G_{Y,N} = \{G = (G_{i,j})_{i,j=1}^s \text{ is invertible} \mid G_{i,j} = \sum_{\alpha=0}^{k_{(ij)}p-1} g_\alpha^{(ij)} J_{i,j}^\alpha, g_\alpha^{(ij)} \in K\}.$$

By matrix computation, we give some formulas for future use under this notation.

Lemma 5.2.3. *We have the following computation formulas relating to X, Q and J.*

(1)

$$J_{i,j}^\alpha J_{j,l}^\beta = J_{i,l}^{\alpha+\beta+\delta p}. \tag{5.2}$$

(2)

$$X_{i,j}X_{j,l} = \sum_{\alpha=0}^{k_{(ij)}p-1} \sum_{\beta=0}^{k_{(jl)}p-1} x_{\alpha}^{(ij)} x_{\beta}^{(jl)} J_{i,l}^{\alpha+\beta+\delta p}, \tag{5.3}$$

here $\delta = k_{(il)} - min\{k_i, k_j, k_l\} \geq 0$.

(3)

$$Q_{N_i} J_{i,j}^{n} - J_{i,j}^{n} Q_{N_j} = n J_{i,j}^{n-1}. \tag{5.4}$$

Now let's prove our key theorem in this section.

Theorem 5.2.4. *For any* $X = (X_{i,j})_{i,j=1}^{s} + Q \in \mathcal{X}_{Y,N}$*, there exists* $G \in G_{Y,N}$*, such that the following holds*

$$GXG^{-1} \in \{\Phi = (\Phi_{i,j})_{i,j=1}^{s} + Q \mid \Phi_{i,j} = \sum_{\alpha=0}^{k_{(ij)}} \phi_{\alpha p-1}^{(ij)} J_{i,j}^{\alpha p-1}, \phi_{\alpha p-1}^{(ij)} \in K\} =: \widetilde{\mathcal{X}}_{Y,N}.$$

Here we always make a convention that $J_{i,j}^{-1} = 0$.

Actually Theorem 5.2.4 tells us each element in $\mathcal{X}_{Y,N}$ is isomorphic to an element in $\mathcal{X}_{Y,N}$ with the coefficients of $J_{i,j}^{n}$ is 0 if $n \neq \alpha p - 1$ for some $\alpha = 1, \cdots, k_{(ij)}$. Then the subset $\widetilde{\mathcal{X}}_{Y,N}$ of $\mathcal{X}_{Y,N}$ intersects each orbit of $\mathcal{X}_{Y,N}/G_{Y,N}$ nonempty.

To prove Theorem 5.2.4, set

$$\mathcal{X}_{Y,N}(\theta) = \{X = (X_{i,j})_{i,j=1}^{s} + Q \mid X_{i,j} = \sum_{\alpha=0}^{[\frac{\theta}{p}]} x_{\alpha p-1}^{(ij)} J_{i,j}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_{(ij)}p-1} x_{\alpha}^{(ij)} J_{i,j}^{\alpha}\},$$

here $[\frac{\theta}{p}]$ is the maximal integer not to exceed $\frac{\theta}{p}$.

It is easy to get the following properties of $\mathcal{X}_{Y,N}(\theta)$:

(1) $\mathcal{X}_{Y,N}(kp-1) = \mathcal{X}_{Y,N}(kp)$ for any integer k.

(2) There is a filtration

$$\mathcal{X}_{Y,N} = \mathcal{X}_{Y,N}(0) \supset \mathcal{X}_{Y,N}(1) \supset \cdots \supset \mathcal{X}_{Y,N}(\infty) = \widetilde{\mathcal{X}}_{Y,N}.$$

Remark 5.2.5. Actually when θ is big enough, we have

$$\mathcal{X}_{Y,N}(\theta) = \mathcal{X}_{Y,N}(\theta+1) = \cdots = \mathcal{X}_{Y,N}(\infty) = \widetilde{\mathcal{X}}_{Y,N}.$$

To prove Theorem 5.2.4, we prove the following Lemma first.

Lemma 5.2.6. *If $\theta + 1 \neq 0$ in K, then for any $X = (X_{i,j})_{i,j=1}^s + Q \in \mathcal{X}_{Y,N}(\theta)$, there exists $G \in G_{Y,N}$, such that $GXG^{-1} = \Phi$, where $\Phi = (\Phi_{i,j})_{i,j=1}^s + Q \in \mathcal{X}_{Y,N}(\theta+1)$.*

Proof. X has $s \times s$ blocks, so we set any $G \in G_{Y,N}$ the same size blocks. For simplicity, we denote $[\frac{\theta}{p}]$ by γ, then

$$X_{i,j} = \sum_{\alpha=0}^{\theta} x_{\alpha p-1}^{(ij)} J_{i,j}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_{(ij)}p-1} x_{\alpha}^{(ij)} J_{i,j}^{\alpha}. \tag{5.5}$$

What we need to prove is for every (i,j),

$$\Phi_{i,j} = \sum_{\alpha=0}^{\theta+1} \phi_{\alpha p-1}^{(ij)} J_{i,j}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{k_{(ij)}p-1} \phi_{\alpha}^{(ij)} J_{i,j}^{\alpha}. \tag{5.6}$$

Step 1: For $i = j$, let's prove there exists G, such that all the $\Phi_{i,i}$ are of the form 5.6. Take

$$G = \begin{pmatrix} G_1 & & \\ & \ddots & \\ & & G_s \end{pmatrix}_{N\times N},$$

with $G_i = I + g_i J_{i,i}^{\theta+1}$ for some $g_i \in K$, then

$$G^{-1} = \begin{pmatrix} G_1^{-1} & & \\ & \ddots & \\ & & G_s^{-1} \end{pmatrix}_{N\times N},$$

with $G_i^{-1} = \sum_{\alpha=0}^{k_ip-1}(-1)^\alpha g_i^\alpha J_{i,i}^{\alpha(\theta+1)}$. Denote

$$G((X_{i,j})_{i,j=1}^s + Q)G^{-1} = (\Phi_{i,j})_{i,j=1}^s + Q,$$

so

$$\begin{cases} \Phi_{i,j} & = G_i X_{i,j} G_j^{-1}, & \text{if } i \neq j; \\ \Phi_{i,i} & = G_i(X_{i,i} + Q_{N_i})G_i^{-1} - Q_{N_i} & . \end{cases}$$

Note $k_{(ii)} = k_i$, use (5.3) and (5.4), we get

$$\begin{aligned}
\Phi_{i,i} &= G_i(X_{i,i} + Q_{N_i})G_i^{-1} - Q_{N_i} \\
&= (I + g_i J_{i,i}^{\theta+1})(\sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(ii)} J_{i,i}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_ip-1} x_\alpha^{(ii)} J_{i,i}^\alpha + Q_{N_i}) \\
&\quad (\sum_{\beta=0}^{k_ip-1} (-1)^\beta g_i^\beta J_{i,i}^{\beta(\theta+1)}) - Q_{N_i} \\
&= \sum_{\alpha=0}^{\gamma} \sum_{\beta=0}^{k_ip-1} (-1)^\beta (x_{\alpha p-1}^{(ii)} g_i^\beta J_{i,i}^{\beta(\theta+1)} + x_{\alpha p-1}^{(ii)} g_i^{\beta+1} J_{i,i}^{(\beta+1)(\theta+1)}) J_{i,i}^{\alpha p-1} \\
&\quad + \sum_{\alpha=\theta}^{k_ip-1} \sum_{\beta=0}^{k_ip-1} (-1)^\beta (x_\alpha^{(ii)} g_i^\beta J_{i,i}^{\beta(\theta+1)} + x_\alpha^{(ii)} g_i^{\beta+1} J_{i,i}^{(\beta+1)(\theta+1)}) J_{i,i}^\alpha \\
&\quad + \sum_{\beta=0}^{k_ip-1} (-1)^\beta (g_i^\beta Q_{N_i} J_{i,i}^{\beta(\theta+1)} + g_i^{\beta+1} J_{i,i}^{\theta+1} Q_{N_i} J_{i,i}^{\beta(\theta+1)}) - Q_{N_i} \\
&= \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(ii)} J_{i,i}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_ip-1} x_\alpha^{(ii)} J_{i,i}^\alpha - Q_{N_i} \\
&\quad + \sum_{\beta=0}^{k_ip-1} (-1)^\beta \{g_i^\beta Q_{N_i} J_{i,i}^{\beta(\theta+1)} + g_i^{\beta+1}[Q_{N_i} J_{i,i}^{\theta+1} - (\theta+1) J_{i,i}^\theta] J_{i,i}^{\beta(\theta+1)}\} \\
&= \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(ii)} J_{i,i}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_ip-1} x_\alpha^{(ii)} J_{i,i}^\alpha - Q_{N_i} - \sum_{\beta=0}^{k_ip-1} (-1)^\beta (\theta+1) g_i^{\beta+1} J_{i,i}^{\theta+\beta(\theta+1)} \\
&\quad + \sum_{\beta=0}^{k_ip-1} (-1)^\beta (g_i^\beta Q_{N_i} J_{i,i}^{\beta(\theta+1)} + g_i^{\beta+1} Q_{N_i} J_{i,i}^{(\beta+1)(\theta+1)})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha=0}^{\gamma} x^{(ii)}_{\alpha p-1} J^{\alpha p-1}_{i,i} + \sum_{\alpha=\theta}^{k_i p-1} x^{(ii)}_{\alpha} J^{\alpha}_{i,i} - Q_{N_i} \\
&\quad - \sum_{\beta=0}^{k_i p-1} (-1)^{\beta}(\theta+1) g_i^{\beta+1} J^{\theta+\beta(\theta+1)}_{i,i} + Q_{N_i} \\
&= \sum_{\alpha=0}^{\gamma} x^{(ii)}_{\alpha p-1} J^{\alpha p-1}_{i,i} + x^{(ii)}_{\theta} J^{\gamma_{(ii)p}+\theta}_{i,i} - (\theta+1) g_i J^{\theta}_{i,i} + \sum_{\alpha=\theta+1}^{k_i p-1} \widetilde{x}^{(ii)}_{\alpha} J^{\alpha}_{i,i}
\end{aligned}
$$

$\theta+1 \neq 0$, so $\theta+1$ is invertible, take $g_i = \frac{1}{\theta+1} x^{(ii)}_{\theta}$, then

$$
\begin{aligned}
\Phi_{i,i} &= \sum_{\alpha=0}^{\gamma} x^{(ii)}_{\alpha p-1} J^{\alpha p-1}_{i,i} + x^{(ii)}_{\theta} J^{\theta}_{i,i} - (\theta+1) g_i J^{\theta}_{i,i} \\
&= \sum_{\alpha=0}^{\gamma} x^{(ii)}_{\alpha p-1} J^{\alpha p-1}_{i,i} + x^{(ii)}_{\theta} J^{\theta}_{i,i} - (\theta+1) g_i J^{\theta}_{i,i} + \sum_{\alpha=\theta+1}^{k_i p-1} \widetilde{x}^{(ii)}_{\alpha} J^{\alpha}_{i,i} \\
&= \sum_{\alpha=0}^{\gamma} x^{(ii)}_{\alpha p-1} J^{\alpha p-1}_{i,i} + x^{(ii)}_{\theta} J^{\theta}_{i,i} - (\theta+1) \frac{1}{\theta+1} x^{(ii)}_{\theta} J^{\theta}_{i,i} + \sum_{\alpha=\theta+1}^{k_i p-1} \widetilde{x}^{(ii)}_{\alpha} J^{\alpha}_{i,i} \\
&= \sum_{\alpha=0}^{\gamma} x^{(ii)}_{\alpha p-1} J^{\alpha p-1}_{i,i} + \sum_{\alpha=\theta+1}^{k_i p-1} \widetilde{x}^{(ii)}_{\alpha} J^{\alpha}_{i,i},
\end{aligned}
$$

and $\Phi_{i,i}$ is of the form (5.6).

Step 2: Let's prove that there exists G, such that the non-diagonal block matrices $\Phi_{i,j}$ are of the form (5.6). For simplicity, we still use $X = (X_{i,j})_{i,j=1}^{s} + Q$ to denote the matrix after step 1, i.e.,

$$
\begin{cases}
X_{i,j} &= \sum\limits_{\alpha=0}^{\gamma} x^{(ij)}_{\alpha p-1} J^{\alpha p-1}_{i,j} + \sum\limits_{\alpha=\theta}^{k_{(ij)}p-1} x^{(ij)}_{\alpha} J^{\alpha}_{i,j}, \text{ if } i \neq j; \\
X_{i,i} &= \sum\limits_{\alpha=0}^{\gamma} x^{(ii)}_{\alpha p-1} J^{\alpha p-1}_{i,i} + \sum\limits_{\alpha=\theta+1}^{k_{(ii)}p-1} x^{(ii)}_{\alpha} J^{\alpha}_{i,i}.
\end{cases}
$$

For notation easier, in some computation we will use $X_{i,j} = \sum\limits_{\alpha=0}^{k_{(ij)}p-1} x^{(ij)}_{\alpha} J^{\alpha}_{i,j}$ for any (i,j).

Pick

$$G = \begin{pmatrix} I & & \\ & \ddots & \\ & & I \end{pmatrix}_{N\times N} + (G_{mn})_{N\times N}.$$

Here $m \neq n$, and (G_{mn}) is a matrix which satisfies for $(i,j) \neq (m,n)$, the matrix in (i,j) block is 0, and the (m,n) block $G_{m,n} = g_{\theta+1}^{(mn)} J_{m,n}^{\theta+1}$ with $g_{\theta+1}^{(mn)} \in K$.

Then

$$G^{-1} = \begin{pmatrix} I & & \\ & \ddots & \\ & & I \end{pmatrix}_{N\times N} - (G_{mn})_{N\times N},$$

so

$$\begin{aligned} G((X_{i,j})_{i,j=1}^s + Q)G^{-1} &= (I + (G_{mn}))((X_{i,j})_{i,j=1}^s + Q)(I - (G_{mn})) \\ &= (G_{mn})(X_{i,j})_{i,j=1}^s - (X_{i,j})_{i,j=1}^s(G_{mn}) - (G_{mn})(X_{i,j})_{i,j=1}^s(G_{mn}) \\ &\quad +(X_{i,j})_{i,j=1}^s + (G_{mn})Q - Q(G_{mn}) - (G_{mn})Q(G_{mn}) + Q \\ &=: (\Phi_{i,j})_{i,j=1}^s + Q. \end{aligned}$$

Since $m \neq n$, (G_{mn}) is a matrix with only (m,n) block non-zero, and Q is a matrix with only diagonal blocks nonzero, we have $(G_{mn})Q(G_{mn}) = 0$, and then

$$\begin{aligned} (\Phi_{i,j})_{i,j=1}^s &= (X_{i,j})_{i,j=1}^s + (G_{mn})(X_{i,j})_{i,j=1}^s - (X_{i,j})_{i,j=1}^s(G_{mn}) \\ &\quad -(G_{mn})(X_{i,j})_{i,j=1}^s(G_{mn}) + (G_{mn})Q - Q(G_{mn}). \end{aligned}$$

Therefore we have

$$\begin{cases} \Phi_{i,j} &= X_{i,j}, & \text{if } i \neq m \text{ and } j \neq n; \\ \Phi_{m,j} &= X_{m,j} + G_{m,n}X_{n,j}, & \text{if } j \neq n; \\ \Phi_{i,n} &= X_{i,n} - X_{i,m}G_{m,n}, & \text{if } i \neq m; \\ \Phi_{m,n} &= X_{m,n} + G_{m,n}X_{n,n} - X_{m,m}G_{m,n} & \\ & \quad -G_{m,n}X_{n,m}G_{m,n} + G_{m,n}Q_n - Q_mG_{mn}, & \end{cases}$$

$\theta+1 \neq 0$ tells us $\theta+1$ is invertible, take $g^{(mn)}_{\theta+1} = \frac{1}{\theta+1}x^{(mn)}_{\theta}$. Use $G_{m,n} = g^{(mn)}_{\theta+1}J^{\theta+1}_{m,n}$ and $Q_{N_i}J^n_{i,j} - J^n_{i,j}Q_j = nJ^{n-1}_{i,j}$, we have

$$\begin{aligned} G_{m,n}Q_{N_n} - Q_{N_m}G_{m,n} &= -(\theta+1)g^{(mn)}_{\theta+1}J^{\theta}_{m,n} \\ &= -(\theta+1)g^{(mn)}_{\theta+1}J^{\theta}_{m,n} \\ &= -(\theta+1)\frac{1}{\theta+1}x^{(mn)}_{\theta}J^{\theta}_{m,n} \\ &= -x^{(mn)}_{\theta}J^{\theta}_{m,n}. \end{aligned}$$

Pick $\delta \geq 0$, and use $X_{i,j} = \sum\limits_{\alpha=0}^{k_{(ij)}p-1} x^{(ij)}_{\alpha}J^{\alpha}_{i,j}$ for $X_{n,n}$, $X_{m,m}$, and $X_{n,m}$, then

(1) For the (m,n) block,

$$\begin{aligned} \Phi_{m,n} &= X_{m,n} + G_{m,n}X_{n,n} - X_{m,m}G_{m,n} + G_{m,n}Q_{N_n} - Q_{N_m}G_{m,n} - G_{m,n}X_{n,m}G_{m,n} \\ &= X_{m,n} + \sum_{\alpha=0}^{k_np-1} g^{(mn)}_{\theta+1}x^{(nn)}_{\alpha}J^{\theta+1+\alpha}_{m,n} - \sum_{\alpha=0}^{k_mp-1} x^{(mm)}_{\alpha}g^{(mn)}_{\theta+1}J^{\theta+1+\alpha}_{m,n} - x^{(mn)}_{\theta}J^{\theta}_{m,n} \\ &\quad - \sum_{\alpha=0}^{k_{(nm)}p-1} g^{(mn)}_{\theta+1}x^{(nm)}_{\alpha}g^{(mn)}_{\theta+1}J^{2\gamma p+2\theta+2+\alpha+\delta p}_{m,n} \\ &= (\sum_{\alpha=0}^{\gamma} x^{(mn)}_{\alpha p-1}J^{\alpha p-1}_{m,n} + \sum_{\alpha=\theta}^{k_{(mn)}p-1} x^{(mn)}_{\alpha}J^{\alpha}_{m,n}) + \sum_{\alpha=\theta+1}^{k_{(mn)}p-1} \widetilde{x}^{(mn)}_{\alpha}J^{\alpha}_{m,n} - x^{(mn)}_{\theta}J^{\theta}_{m,n} \\ &= (\sum_{\alpha=0}^{\gamma} x^{(mn)}_{\alpha p-1}J^{\alpha p-1}_{m,n} + \sum_{\alpha=\theta+1}^{k_{(mn)}p-1} x^{(mn)}_{\alpha}J^{\alpha}_{m,n}) + \sum_{\alpha=\theta+1}^{k_{(mn)}p-1} \widetilde{x}^{(mn)}_{\alpha}J^{\alpha}_{m,n} \end{aligned}$$

$$= \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(mn)} J_{m,n}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{k_{(mn)}p-1} \phi_{\alpha}^{(mn)} J_{m,n}^{\alpha}.$$

(2) For $j \neq n$, there are two cases:

if $j \neq m$, use $X_{n,j} = \sum_{\alpha=0}^{k_{(nj)}p-1} x_{\alpha}^{(nj)} J_{n,j}^{\alpha}$ and $X_{m,j} = \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(mj)} J_{m,j}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_{(mj)}p-1} x_{\alpha}^{(mj)} J_{m,j}^{\alpha}$, then

$$\begin{aligned}
\Phi_{m,j} &= X_{m,j} + G_{m,n} X_{n,j} \\
&= \left(\sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(mj)} J_{m,j}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_{(mj)}p-1} x_{\alpha}^{(mj)} J_{m,j}^{\alpha}\right) + \sum_{\alpha=0}^{k_{(nj)}p-1} g_{\theta+1}^{(mn)} x_{\alpha}^{(nj)} J_{m,j}^{\theta+1+\alpha+\delta p} \\
&= \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(mj)} J_{m,j}^{\alpha p-1} + x_{\theta}^{(mj)} J_{m,j}^{\theta} + \sum_{\alpha=\theta+1}^{k_{(mj)}p-1} \widetilde{x}_{\alpha}^{(mj)} J_{m,j}^{\alpha};
\end{aligned}$$

if $j = m$, use $X_{n,m} = \sum_{\alpha=0}^{k_{(nm)}p-1} x_{\alpha}^{(nm)} J_{n,m}^{\alpha}$ and $X_{m,m} = \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(mm)} J_{m,m}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{k_m p-1} x_{\alpha}^{(mm)} J_{m,m}^{\alpha}$, then

$$\begin{aligned}
\Phi_{m,m} &= X_{m,m} + G_{m,n} X_{n,j} \\
&= \left(\sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(mm)} J_{m,m}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{k_m p-1} x_{\alpha}^{(mm)} J_{m,m}^{\alpha}\right) + \sum_{\alpha=0}^{k_{(nm)}p-1} g_{\theta+1}^{(mn)} x_{\alpha}^{(nm)} J_{m,m}^{\theta+1+\alpha+\delta p} \\
&= \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(mj)} J_{m,m}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{k_m p-1} \widetilde{x}_{\alpha}^{(mj)} J_{m,m}^{\alpha}.
\end{aligned}$$

(3) For $i \neq m$, there are also two cases:

if $i \neq n$, use $X_{i,m} = \sum_{\alpha=0}^{k_{(im)}p-1} x_{\alpha}^{(im)} J_{i,m}^{\alpha}$ and $X_{i,n} = \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(in)} J_{i,n}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_{(in)}p-1} x_{\alpha}^{(in)} J_{i,n}^{\alpha}$, then

$$\begin{aligned}
\Phi_{i,n} &= X_{i,n} - X_{i,m} G_{m,n} \\
&= \left(\sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(in)} J_{i,n}^{\alpha p-1} + \sum_{\alpha=\theta}^{k_{(in)}p-1} x_{\alpha}^{(in)} J_{i,n}^{\alpha}\right) - \sum_{\alpha=0}^{k_{(im)}p-1} x_{\alpha}^{(im)} g_{\theta+1}^{(mn)} J_{i,n}^{\theta+1+\alpha+\delta p} \\
&= \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(in)} J_{i,n}^{\alpha p-1} + x_{\theta}^{(in)} J_{i,n}^{\theta} + \sum_{\alpha=\theta+1}^{k_{(in)}p-1} \widetilde{x}_{\alpha}^{(in)} J_{i,n}^{\alpha};
\end{aligned}$$

if $i = n$, use $X_{n,m} = \sum_{\alpha=0}^{k_{(nm)}p-1} x_\alpha^{(nm)} J_{n,m}^\alpha$ and $X_{n,n} = \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(nn)} J_{n,n}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{k_n p-1} x_\alpha^{(nn)} J_{n,n}^\alpha$, then

$$\begin{aligned}\Phi_{n,n} &= X_{n,n} - X_{n,m}G_{m,n} \\ &= (\sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(nn)} J_{n,n}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{k_n p-1} x_\alpha^{(nn)} J_{n,n}^\alpha) - \sum_{\alpha=0}^{k_{(nm)}p-1} x_\alpha^{(nm)} g_{\theta+1}^{(mn)} J_{n,n}^{\theta+1+\alpha+\delta p} \\ &= \sum_{\alpha=0}^{\gamma} x_{\alpha p-1}^{(nn)} J_{n,n}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{k_n p-1} \widetilde{x}_\alpha^{(nn)} J_{n,n}^\alpha\end{aligned}$$

Therefore, we get for $m \neq n$, $\exists G \in G_{Y,N}$, such that

$$G((X_{ij})_{i,j=1}^s + Q)G^{-1} = (\Phi_{ij})_{i,j=1}^s + Q$$

with $\Phi_{m,n}$ and all Φ_{ii} are of the form (5.6). More precisely, G will change X_{mn} into the form (5.6) we want, keep all the other blocks to be the original form (either (5.5) or (5.6)) but with maybe different coefficients.

Using this method, and going over all (m,n) with $m \neq n$, we get there exists $G \in G_{Y,N}$, such that $GXG^{-1} = (\Phi_{i,j})_{i,j=1}^s + Q$, with

$$\Phi_{i,j} = \sum_{\alpha=0}^{\gamma} \phi_{\alpha p-1}^{(ij)} J_{i,j}^{\alpha p-1} + \sum_{\alpha=\theta+1}^{kp-1} \phi_\alpha^{(ij)} J_{i,j}^\alpha$$

then we complete the proof.

□

Since we have

(1) $\mathcal{X}_{Y,N}(kp-1) = \mathcal{X}_{Y,N}(kp)$ for any integer k,

(2) When θ is big enough, $\mathcal{X}_{Y,N}(\theta) = \mathcal{X}_{Y,N}(\theta+1) = \cdots = \mathcal{X}_{Y,N}(\infty) = \widetilde{\mathcal{X}}_{Y,N}$,

using Lemma 5.2.6 and induction on θ, we can complete the proof of Theorem 5.2.4.

Then combine Lemma 5.2.2 and Theorem 5.2.4, we have

Corollary 5.2.7. *Any irreducible W-module (X, Y) with Y nilpotent is isomorphic to $\big(xJ_p^{p-1}+Q_p, J_p\big)$, with $x \in K$.*

Moreover, we have

Corollary 5.2.8. *Up to isomorphism, all the irreducible W-module (X, Y) are bijective to (x, y) with $x, y \in K$, i.e., bijective to the closed points of Spec$K[x, y]$.*

There is a natural map

$$\begin{aligned} \pi : \widetilde{\mathcal{X}}_{Y,N} &\to \mathcal{X}_{Y,N}/G_{Y,N} \\ X &\mapsto [X] \end{aligned}$$

Then by Theorem 5.2.4, we get

Corollary 5.2.9. *π is a surjective map.*

5.2.3 Orbits Counting

Now we have found $\widetilde{\mathcal{X}}_{Y,N}$ which intersects each orbit in $\mathcal{X}_{Y,N}/G_{Y,N}$ nonempty, then let's count the orbits $\mathcal{X}_{Y,N}/G_{Y,N}$ by $\widetilde{\mathcal{X}}_{Y,N}$ in this section.

Lemma 5.2.10. *For any $X \in \widetilde{\mathcal{X}}_{Y,N}$, set*

$$A_X := \{G \in G_{Y,N} \mid GXG^{-1} \in \widetilde{\mathcal{X}}_{Y,N}\},$$

$$B := \{G \in G_{Y,N} \mid G = (G_{i,j})_{i,j=1}^{s}, G_{i,j} = \sum_{\alpha=0}^{k_{(ij)}-1} g_{\alpha p}^{(ij)} J_{ij}^{\alpha p}, g_{\alpha p}^{(ij)} \in K\},$$

then $A_X = B$.

Proof. For one side, it is easy to check $B \subseteq A_X$;

Now , let's prove $A_X \subseteq B$.

For all $X \in \widetilde{\mathcal{X}}_{Y,N}$, $G \in G_{Y,N}$, if there exists $\Phi \in \widetilde{\mathcal{X}}_{Y,N}$, such that $GXG^{-1} = \Phi$, then we have $GX = \Phi G$, we use the notation that $X = (X_{i,j})_{i,j=1}^{s} + Q$, $G = (G_{i,j})_{i,j=1}^{s}$ and

$\Phi = (\Phi_{i,j})_{i,j=1}^{s} + Q$. $\forall (m,n)$, then:

$$\sum_{l=1}^{s}(G_{m,l}X_{l,n} - \Phi_{m,l}G_{l,n}) = Q_{N_m}G_{m,n} - G_{m,n}Q_{N_n};$$

since $X_{i,j} = \sum_{\alpha=0}^{k_{(ij)}-1} x_{\alpha p-1}^{(ij)} J_{i,j}^{\alpha p-1}$, $\Phi_{i,j} = \sum_{\alpha=0}^{k_{(ij)}-1} \phi_{\alpha p-1}^{(ij)} J_{i,j}^{\alpha p-1}$ and $G_{i,j} = \sum_{\alpha=0}^{k_{(ij)}p-1} g_{\alpha}^{(ij)} J_{i,j}^{\alpha}$, by direct computation we have

$$\begin{aligned}\sum_{l=1}^{s}(G_{m,l}X_{l,n} - \Phi_{m,l}G_{l,n}) &= \sum_{l=1}^{s}\Big(\sum_{\alpha=0}^{k_{(ml)}-1}\sum_{\beta=0}^{k_{(ln)}-1} g_{\alpha p}^{(ml)} x_{\beta p-1}^{(ln)} J_{m,n}^{(\alpha+\beta+\delta)p-1} \\ &\quad - \sum_{\alpha=0}^{k_{(ml)}-1}\sum_{\beta=0}^{k_{(ln)}-1} \phi_{\alpha p-1}^{(ml)} g_{\beta p}^{(ln)} J_{m,n}^{(\alpha+\beta+\delta')p-1}\Big),\end{aligned}$$

and

$$Q_{N_m}G_{m,n} - G_{m,n}Q_{N_n} = \sum_{\alpha=0}^{k_{(mn)}p-1} \alpha g_{\alpha}^{(mn)} J_{m,n}^{\alpha-1}. \tag{5.7}$$

Compare the coefficients of $J_{m,n}$, we get the α in (5.7) is a multiple of p, so $G \in B$. □

Define

$$\widetilde{G}_{Y,N} := \{G \in G_{Y,N} \mid G\widetilde{\mathcal{X}}_{Y,N}G^{-1} \subseteq \widetilde{\mathcal{X}}_{Y,N}\},$$

then the action $G_{Y,N} \curvearrowright \mathcal{X}_{Y,N}$ induces an action $\widetilde{G}_{Y,N} \curvearrowright \widetilde{\mathcal{X}}_{Y,N}$, where the action is still conjugation.

From Lemma 5.2.10, we know $\widetilde{G}_{Y,N} = B = A_X$, and also

$$\pi : \widetilde{\mathcal{X}}_{Y,N} \to \mathcal{X}_{Y,N}/G_{Y,N}$$

induces a map

$$\widetilde{\pi} : \widetilde{\mathcal{X}}_{Y,N}/\widetilde{G}_{Y,N} \to \mathcal{X}_{Y,N}/G_{Y,N}.$$

From Corollary 5.2.9 it is easy to get

Theorem 5.2.11. *$\widetilde{\pi}$ is bijective.*

5.3 Categorical Equivalence

We got an easier subset $\widetilde{\mathcal{X}}_{Y,N}$ to compute the orbits $\mathcal{X}_{Y,N}/G_{Y,N}$. Now for $W = K\langle x, y\rangle/[x, y] = 1$, and $A = K[x, y]$, let

$$\mathcal{C}_n^o = \text{the representation space of all } n\text{-dim } A\text{-modules with } y \text{ acts nilpotently},$$

$$\mathcal{D}_{np}^o = \text{the representation space of all } np\text{-dim } W\text{-modules with } y \text{ acts nilpotently}.$$

We still use a pair of matrices $(X(0), Y(0))$ and (X, Y) to denote modules in $\mathcal{C}_n^o$ and $\mathcal{D}_{np}^o$ respectively. Let's construct a functor

$$\begin{aligned} \mathcal{F}_n : \mathcal{C}_n^o &\rightarrow \mathcal{D}_{np}^o \\ (X(0), Y(0)) &\mapsto (X, Y) \end{aligned}$$

as follows, (here we denote $N = np$ for consistent notation in the above sections)

(1) For objects, if under fixed basis,

$$Y(0) = \begin{pmatrix} J_{k_1} & & & \\ & J_{k_2} & & \\ & & \ddots & \\ & & & J_{k_s} \end{pmatrix}_{n\times n},$$

with $k_1 \geq k_2 \geq \cdots \geq k_s$, then $X(0) = (X(0)_{i,j})_{i,j=1}^s$ with $X(0)_{i,j} = \sum_{\alpha=0}^{k_{(ij)}-1} x_\alpha^{(ij)} \widetilde{J}_{i,j}^\alpha$ by direct computation. Here we use the notation $\widetilde{J}_{i,j}^\alpha$ which is similar as $J_{i,j}^\alpha$:

(i) If $i \leq j$, then $k_i \geq k_j$, $\widetilde{J}_{i,j}^\alpha = \begin{pmatrix} 0 \\ J_{k_j}^\alpha \end{pmatrix}_{k_i\times k_j}$ with $J_{i,j}^\alpha = \begin{pmatrix} 0 \\ J_{N_j}^\alpha \end{pmatrix}_{N_i\times N_j}$.

(ii) If $i \geq j$, then $k_i \leq k_j$, $\widetilde{J}_{i,j}^\alpha = \begin{pmatrix} J_{k_i}^\alpha & 0 \end{pmatrix}_{k_i\times k_j}$ with $J_{i,j}^\alpha = \begin{pmatrix} J_{N_i}^\alpha & 0 \end{pmatrix}_{N_i\times N_j}$.

(iii) If $i = j$, then $\widetilde{J}_{i,i}^\alpha = J_{k_i}^\alpha$ with $J_{i,i}^\alpha = J_{N_i}^\alpha$.

Define

$$\mathcal{F}_n(Y(0)) = Y = \begin{pmatrix} J_{k_1 p} & & & \\ & J_{k_2 p} & & \\ & & \ddots & \\ & & & J_{k_s p} \end{pmatrix}_{N\times N},$$

and $\mathcal{F}_n(X(0)) = X = (X_{i,j})_{i,j=1}^s + Q_N$ with $X_{i,j} = \sum_{\alpha=0}^{k_{(ij)}-1} x_\alpha^{(ij)} J_{i,j}^{\alpha p+p-1}$.

(2) For morphisms, if $H(0) = (H(0)_{i,j})_{i,j=1}^s \in \mathrm{Hom}((X(0)_1, Y(0)), (X(0)_2, Y(0)))$, then by direct computation we have $H(0)_{i,j} = \sum_{\alpha=0}^{k_{(ij)}-1} h_\alpha^{(ij)} \widetilde{J}_{i,j}^{\alpha}$.

Define $\mathcal{F}_n(H(0)) = F_n(H(0)) = (H_{i,j})_{i,j=1}^s$ with $H_{i,j} = \sum_{\alpha=0}^{k_{(ij)}-1} h_\alpha^{(ij)} J_{i,j}^{\alpha p}$.

Then from what we get in Section 5.2.3, the functor $\mathcal{F}_n$ gives a categorical equivalence between $\mathcal{C}_n^o$ and $\mathcal{D}_{np}^o$.

Note if

$$\mathcal{C}_n = \text{the representation space of all } n\text{-dim } A\text{-modules},$$

$$\mathcal{D}_{np} = \text{the representation space of all } np\text{-dim } W\text{-modules},$$

then by the decomposition discussion in Section 5.1, we have

$$\mathcal{C}_n \cong \prod_{k\in K} \mathcal{C}_n^o; \quad \mathcal{D}_{np} \cong \prod_{k\in K} \mathcal{D}_{np}^o,$$

so we get the following Corollary:

Corollary 5.3.1. *There is a categorical equivalence between $\mathcal{C}_n$ and $\mathcal{D}_N$.*

Since

$$\mathrm{rep}(\Pi^0(\Gamma)) = \prod_{n\in\mathbb{Z}} \mathcal{C}_n$$

$$\mathrm{rep}(\Pi^1(\Gamma)) = \prod_{n\in\mathbb{Z}} \mathcal{D}_{np},$$

then we get Theorem 1.3.4: for Jordan quiver Γ, there is a categorical equivalence

$$\mathrm{rep}(\Pi^1(\Gamma)) \cong \mathrm{rep}(\Pi^0(\Gamma)).$$

Remark 5.3.2. Since we are not using finite field, we can not get the stack function μ as what we did in 4.5, but we can denote $\mathbb{L} =\mid K \mid$, and using motivic counting to get a similar function μ.

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