SINGULAR INTEGRATION WITH APPLICATIONS TO BOUNDARY VALUE PROBLEMS

by

ADELINA E KAYE

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B.A., California State University Fullerton, 2013

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Approved by:

Co-Major Professor
Nathan Albin

Approved by:

Co-Major Professor
Pietro Poggi-Corradini

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Abstract

This report explores singular integration, both real and complex, focusing on the the Cauchy type integral, culminating in the proof of generalized Sokhotski-Plemelj formulae and the applications of such to a Riemann-Hilbert problem.

Table of Contents

List of Figures

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Dedication

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Chapter 1

Real Singular Integration

Boundary value problems are well studied in mathematics, partially due to their abundance in every day life. For example, a pan on a hot stove. The temperature of the stove below the pan is different from the room temperature above the pan. In such a situation, the pan would be the boundary, and it may be of interest to understand how the temperature below relates to the temperature above. The first tool needed to study boundary value problems is integration, and in particular, singular integration, which will be defined later in the paper. Almost as soon as the concept of integration is introduced in calculus, students are informed that there are certain functions for which the integral is undefined on specific intervals. A classic example would be integration of the function $f(x) = \frac{1}{x}$ on the interval $(-1, 1)$ (as in Figure 1.1).

In terms of Riemann integration, the integral

$$\int_a^b \frac{dx}{x - c}$$

such that $a < c < b$ is undefined. Fixing $a = -1$, $b = 1$, and $c = 0$, we can consider the integral as indefinite, and recall why this it is undefined:

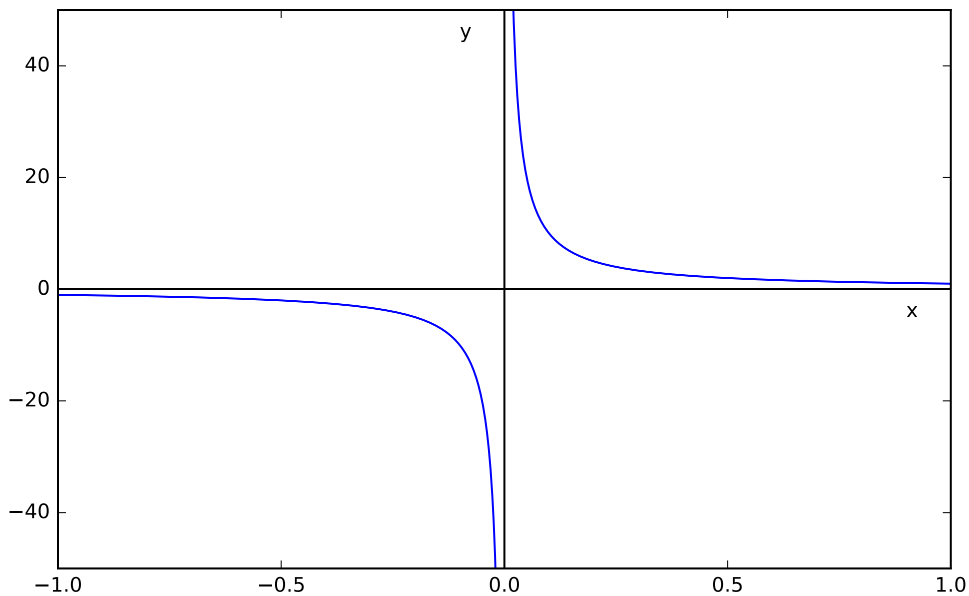

Figure 1.1: $f(x) = \frac{1}{x}$ *on the interval* -1 *to* 1*.*

$$\begin{aligned}\int_{-1}^{1} \frac{dx}{x} &= \lim_{\varepsilon_1 \to 0^+} \left(\int_{-1}^{-\varepsilon_1} \frac{dx}{x} \right) + \lim_{\varepsilon_2 \to 0^+} \left(\int_{\varepsilon_2}^{1} \frac{dx}{x} \right) \\ &= \lim_{\varepsilon_1 \to 0^+} \left(\ln(|x|) \Big|_{-1}^{-\varepsilon_1} \right) + \lim_{\varepsilon_2 \to 0^+} \left(\ln(|x|) \Big|_{\varepsilon_2}^{1} \right) \\ &= \lim_{\varepsilon_1 \to 0^+} (\ln(\varepsilon_1)) - \lim_{\varepsilon_2 \to 0^+} (\ln(\varepsilon_2))\end{aligned}$$

Each limit is undefined, and we can proceed no further without stepping outside of the confines of Riemann integration. One way to attempt to make sense of the integral would be to connect the rates at which ε_1 and ε_2 approach zero such that $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = k\varepsilon$ (see

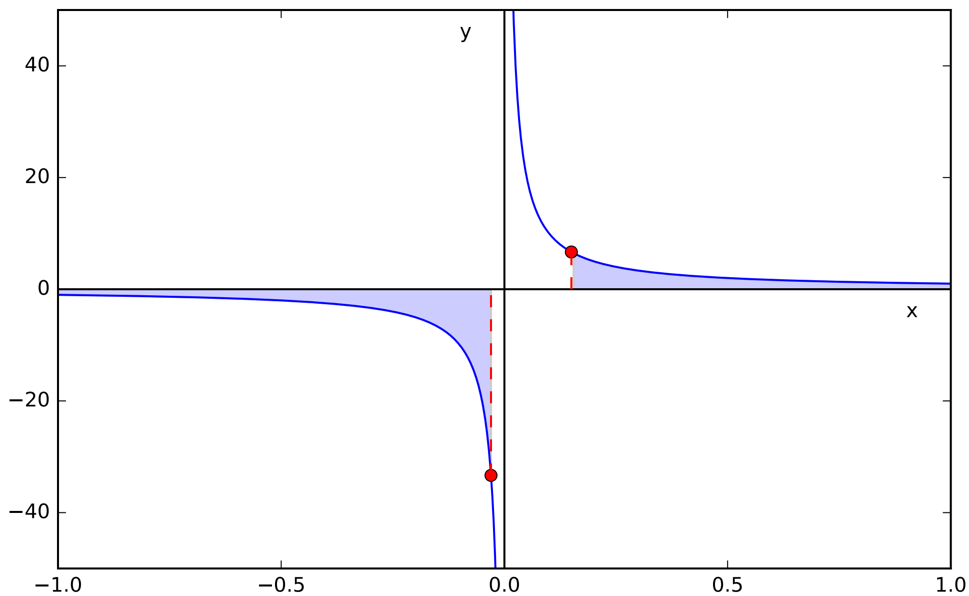

Figure 1.2: *Generalized principal value integral for* $f(x) = \frac{1}{x}$.

Figure 1.2). Consider the following, with $k > 0$:

$$\begin{aligned}
\int_a^b \frac{dx}{x-c} &:= \lim_{\varepsilon \to 0^+} \left(\int_a^{c-\varepsilon} \frac{dx}{x-c} + \int_{c+\varepsilon k}^b \frac{dx}{x-c} \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\ln(|x-c|)\Big|_a^{c-\varepsilon} + \ln(|x-c|)\Big|_{c+k\varepsilon}^b \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\ln(|c-\varepsilon-c|) - \ln(|a-c|) + \ln(|b-c|) - \ln(|c+k\varepsilon-c|) \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\ln(\varepsilon) - \ln(c-a) + \ln(b-c) - \ln(k\varepsilon) \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\ln\left(\frac{\varepsilon}{k\varepsilon}\right) + \ln\left(\frac{b-c}{c-a}\right) \right) \\
&= \ln\left(\frac{1}{k}\right) + \ln\left(\frac{b-c}{c-a}\right) \\
&= \ln\left(\frac{b-c}{k(c-a)}\right)
\end{aligned}$$

To see how intuitively pleasing this is, we can return to the case $a = -1$, $b = 1$, and

$c = 0$, fixing $k = 1$:

$$\int_{-1}^{1} \frac{dx}{x} = \ln\left(\frac{1}{1}\right) = 0.$$

When k is fixed at 1, this is understood as singular integration. However, when we do not fix $k = 1$, an issue of well-defined-ness arises. Now,

$$\int_{-1}^{1} \frac{dx}{x} = \ln\left(\frac{1}{k}\right) = -\ln(k)$$

which will vary depending on k. This occurs as a result of the "speed" at which ε and $k\varepsilon$ approach zero.

Recall from calculus that not all indefinite integrals are undefined in the Riemann sense, for example $\int_{-1}^{1} x^{-\frac{1}{3}} dx$ (see Figure 1.3).

$$\begin{aligned}
\int_{-1}^{1} x^{-\frac{1}{3}} dx &= \lim_{\varepsilon_1 \to 0^+} \int_{-1}^{-\varepsilon_1} x^{-\frac{1}{3}} dx + \lim_{\varepsilon_2 \to 0^+} \int_{\varepsilon_2}^{1} x^{-\frac{1}{3}} dx \\
&= \lim_{\varepsilon_1 \to 0^+} \left(\frac{3}{2} x^{\frac{2}{3}} \Big|_{-1}^{-\varepsilon_1} \right) + \lim_{\varepsilon_2 \to 0^+} \left(\frac{3}{2} x^{\frac{2}{3}} \Big|_{\varepsilon_2}^{1} \right) \\
&= \lim_{\varepsilon_1 \to 0^+} \left(\frac{3}{2}(-\varepsilon_1)^{\frac{2}{3}} - \frac{3}{2}(-1)^{\frac{2}{3}} \right) + \lim_{\varepsilon_2 \to 0^+} \left(\frac{3}{2}(1)^{\frac{2}{3}} - \frac{3}{2}(\varepsilon_2)^{\frac{2}{3}} \right) \\
&= -\frac{3}{2} + \frac{3}{2} = 0
\end{aligned}$$

Such functions have integrable singularities–the integral does not depend of values of k.

$$\begin{aligned}
\int_{-1}^{1} x^{-\frac{1}{3}} dx &= \lim_{\varepsilon \to 0^+} \int_{-1}^{-\varepsilon} x^{-\frac{1}{3}} dx + \lim_{\varepsilon \to 0^+} \int_{k\varepsilon}^{1} x^{-\frac{1}{3}} dx \\
&= \lim_{\varepsilon \to 0^+} \left(\frac{3}{2} x^{\frac{2}{3}} \Big|_{-1}^{-\varepsilon} \right) + \lim_{\varepsilon \to 0^+} \left(\frac{3}{2} x^{\frac{2}{3}} \Big|_{k\varepsilon}^{1} \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\frac{3}{2}(-\varepsilon)^{\frac{2}{3}} - \frac{3}{2}(-1)^{\frac{2}{3}} \right) + \lim_{\varepsilon \to 0^+} \left(\frac{3}{2}(1)^{\frac{2}{3}} - \frac{3}{2}(k\varepsilon)^{\frac{2}{3}} \right) \\
&= -\frac{3}{2} + \frac{3}{2} = 0
\end{aligned}$$

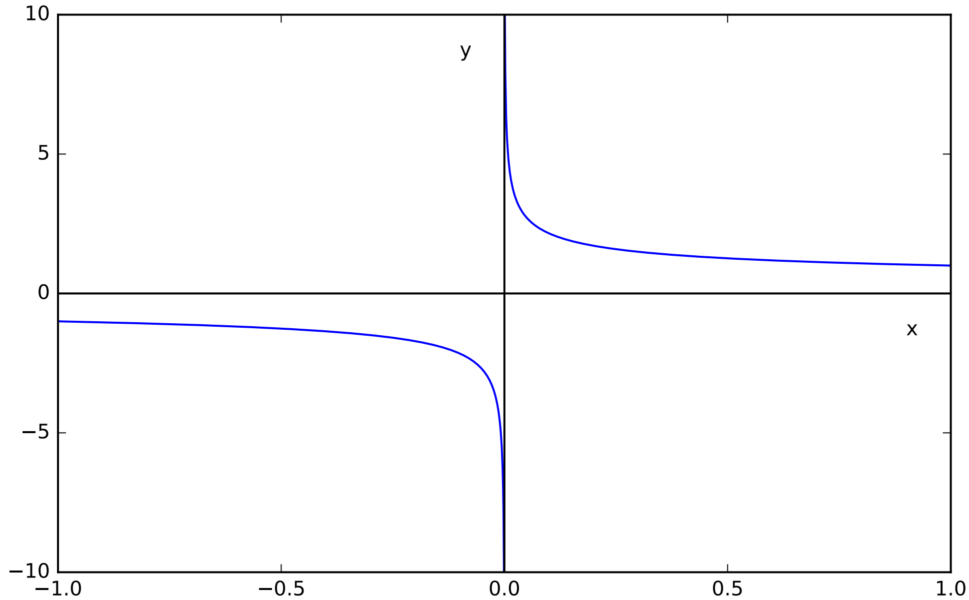

Figure 1.3: *Graph of* $f(x) = x^{-\frac{1}{3}}$.

Chapter 2

Complex Singular Integration

Integrals of functions with zeros in the denominator are so prevalent in the study of complex analysis that mathematicians sought for cases in which the integrals might be able to be defined. One result of this study was the Cauchy Residue Theorem:

Theorem 1 (Cauchy Residue Theorem)**.** *Let L be a simple, closed, positively oriented contour. If f is analytic both on L and in the interior of L except for a finite number of singular points z_k, where $k \in \{1, 2, ..., n\}$ in the interior of L, then*

$$\int_L f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k).$$

Definition 2.1 (Residue)**.** Recall the Laurent expansion of a complex-valued function, f, in a disk centered at a point z_0:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} b_k (z - z_0)^{-k}.$$

The Residue of f at z_0, $\text{Res}(f(z), z_0) = b_1$.

Because it applies to a contour, L, Cauchy's Residue Theorem can become extremely useful as we seek to solve boundary value problems. Such problems, however, would address not just L and its interior, referred to as D^+, but also its exterior, D^-.

Recall that for the Residue Theorem, we required f to be analytic both on L and in D^+. From this, we obtained

$$\frac{1}{2\pi i}\int_L \frac{f(\tau)d\tau}{\tau - z} = \begin{cases} f(z), & z \in \mathrm{D}^+; \\ 0, & z \in \mathrm{D}^- . \end{cases}$$

(It is worth observing here that

$$\frac{1}{2\pi i}\int_L \frac{f(\tau)d\tau}{\tau - z}$$

is referred to as a Cauchy Type Integral, and that for such integrals, $f(\tau)$ is referred to as the density, and $\frac{1}{\tau - z}$ the Cauchy kernel.)

However, this does not address our primary interest, which is what happens when z is on the boundary.

Since $\int_a^b \frac{dx}{x-c}$ is such an iconic example of singular integration on the real line, let's explore its complex counterpart, the Cauchy integral in the special case of the density function, $f(\tau) = 1$:

$$\int_L \frac{d\tau}{\tau - t},$$

where L is a contour running from a to b, and $t \in L$. Again, this cannot be computed in the Riemann sense, so we integrate from a to t_1 and t_2 to b, where t_1 and t_2 are points on the curve with t in between them. Thus we integrate over L, but avoiding a small region containing t, referred to as ℓ (as seen in Figure 2.1). We allow $t_1, t_2 \in L$ to approach t such that $\lim_{t_1,t_2 \to t}\left(\frac{|t_1 - t|}{|t_2 - t|}\right) = k$, and then take that limit.

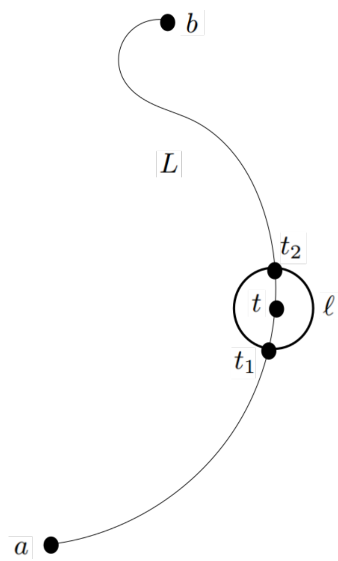

Figure 2.1: *The curve, L, and region, ℓ.*

$$
\begin{aligned}
\int_L \frac{d\tau}{\tau - t} &= \lim_{t_1, t_2 \to t} \left(\int_{L-\ell} \frac{d\tau}{\tau - t} \right) \\
&= \lim_{t_1, t_2 \to t} \left(\int_a^{t_1} \frac{d\tau}{\tau - t} + \int_{t_2}^{b} \frac{d\tau}{\tau - t} \right) \\
&= \lim_{t_1, t_2 \to t} \left(\log(\tau - t)\Big|_a^{t_1} + \log(\tau - t)\Big|_{t_2}^{b} \right) \\
&= \lim_{t_1, t_2 \to t} \left(\log(t_1 - t) - \log(a - t) + \log(b - t) - \log(t_2 - t) \right) \\
&= \log\left(\frac{b-t}{a-t}\right) + \lim_{t_1, t_2 \to t} \left(\log\left(\frac{t_1 - t}{t_2 - t}\right) \right) \\
&= \log\left(\frac{b-t}{a-t}\right) + \lim_{t_1, t_2 \to t} \left(\log\left(\left| \frac{t_1 - t}{t_2 - t} \right| \right) \right) + \lim_{t_1, t_2 \to t} \left(i \left(\arg(t_1 - t) - \arg(t_2 - t) \right) \right) \\
&= \log\left(\frac{b-t}{a-t}\right) + \log(k) + i\pi
\end{aligned}
$$

Figure 2.2 demonstrates $\lim_{t_1, t_2 \to t} (\arg(t_1 - t) - \arg(t_2 - t)) = \pi$. We observe that this only holds when L is a curve without corners or cusps. In such situations, the angle can still be computed, but it must be done in a different way. Notice that θ is measured in such a way that is does not intersect the branch cut for the logarithm.

It is also worth observing here that if L is simple, closed curve, that is $a = b$, then under

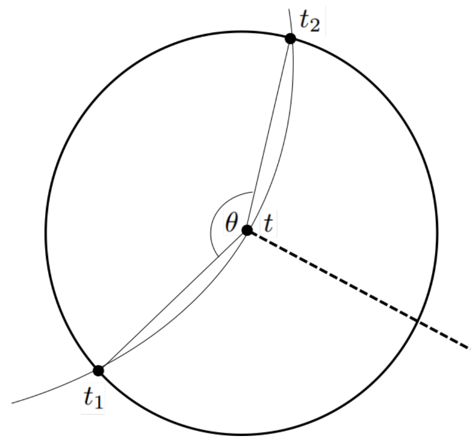

Figure 2.2: *The region, ℓ, and branch cut of the logarithm.*

our assumption that $\lim_{t_1,t_2\to t}\left(\frac{|t_1-t|}{|t_2-t|}\right)=k$, we have

$$\int_L \frac{d\tau}{\tau-t}=\log(k)+i\pi.$$

This method of assigning values to previously non-integrable integrals is standard in complex analysis and its applications. The case of $k=1$ is most commonly used, and is referred to as the principal value of the singular integral. However, in the interest of generality, this paper will continue to leave $k\in\mathbb{R}$ such that $k>0$. Additionally, we will assume L is closed unless otherwise stated.

Consider the cases presented in Figure 2.3. We claim that

$$\int_L \frac{d\tau}{\tau-z}=\begin{cases}2\pi i, & z\in \mathrm{D}^+;\\ 0, & z\in \mathrm{D}^-;\\ \pi i+\log(k), & z\in L.\end{cases}$$

For $z\in \mathrm{D}^+$ and $z\in \mathrm{D}^-$ we obtain these values via the Cauchy Residue Theorem. For $z\in \mathrm{D}^+$,

$$\int_L \frac{d\tau}{\tau-z}=2\pi i\,\mathrm{Res}\left(\frac{1}{\tau-z},z\right)=2\pi i.$$

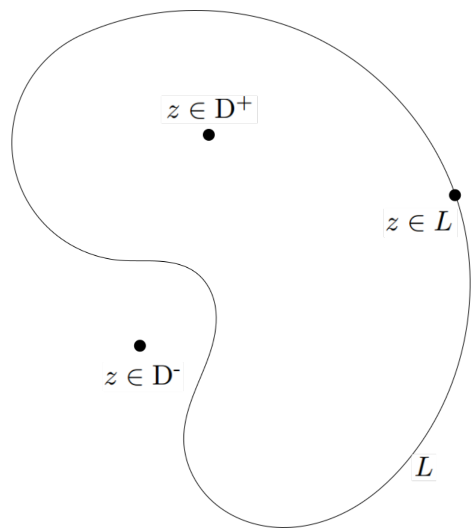

Figure 2.3: *Location of* z.

Notice that z is the only possible singularity, so if $z \in D^-$, there are no singularities in D^+, meaning

$$\int_L \frac{d\tau}{\tau - z} = 2\pi i(0) = 0.$$

This special case of the Residue Theorem is often stressed in elementary complex analysis by stating that integration of an analytic function over a smooth, closed curve with no singularities on the interior is equal to zero.

For $z \in L$, the value of the integral was already obtained above for a general simple, closed curve.

With the aide of these ideas, we can now address integrals that would have been previously impossible to integrate, such as $\int_{|\tau|=1} \frac{3\tau^2-(7+2i)\tau+\frac{1}{2}(3+7i)}{(\tau-3)(\tau-\frac{1}{2})(\tau-i)} d\tau$.

Applying the method of partial fraction decomposition,

$$\int_{|\tau|=1} \frac{3\tau^2 - (7+2i)\tau + \frac{1}{2}(3+7i)}{(\tau-3)(\tau-\frac{1}{2})(\tau-i)} d\tau = \int_{|\tau|=1} \left(\frac{1}{\tau-3} + \frac{1}{\tau-\frac{1}{2}} + \frac{1}{\tau-i} \right) d\tau.$$

This splits the integrand into three expressions, one with a singularity inside the unit circle, one with a singularity outside the unit circle, and one on the unit circle itself, see 2.4. In light of this, we split this into three separate integrals, obtaining

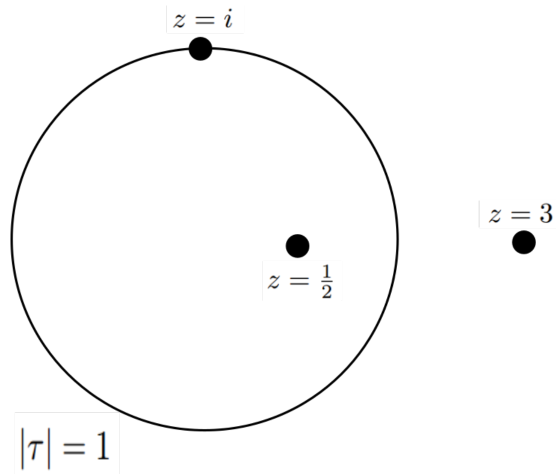

Figure 2.4: *Partial Fractions Illustration*

$$\int_{|\tau|=1} \frac{d\tau}{\tau - 3} + \int_{|\tau|=1} \frac{d\tau}{\tau - \frac{1}{2}} + \int_{|\tau|=1} \frac{d\tau}{\tau - i}.$$

Each of these integrals can now be evaluated based on where its respective singularity lies.

$$\int_{|\tau|=1} \frac{d\tau}{\tau - 3} + \int_{|\tau|=1} \frac{d\tau}{\tau - \frac{1}{2}} + \int_{|\tau|=1} \frac{d\tau}{\tau - i} = 0 + 2\pi i + (\log(k) + \pi i).$$

Thus,

$$\int_{|\tau|=1} \frac{3\tau^2 - (7+2i)\tau + \frac{1}{2}(3+7i)}{(\tau - 3)(\tau - \frac{1}{2})(\tau - i)} d\tau = 3\pi i + \log(k).$$

Chapter 3

Limiting Value Functions

Now, let us consider a function, Φ, whose value is a Cauchy Type integral with density φ, a Hölder continuous function (as defined on the following page) defined on L, an arbitrary closed curve in the complex plane.

Definition 3.1 (Hölder Continuity)**.** A function, φ is Hölder continuous if, for all τ and t

$$|\varphi(\tau) - \varphi(t)| < A|\tau - t|^{\lambda},$$

where $A, \lambda \in \mathbb{R}$ such that $A > 0$, and $0 < \lambda \leq 1$.

Fixing $t \in L$,

$$\begin{aligned}
\Phi(z) &= \int_L \frac{\varphi(\tau)d\tau}{\tau - z} \\
&= \int_L \frac{\varphi(\tau) - \varphi(t) + \varphi(t)}{\tau - z} d\tau \\
&= \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - z} d\tau + \varphi(t) \int_L \frac{d\tau}{\tau - z}.
\end{aligned}$$

If we let

$$\psi(z) = \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - z} d\tau,$$

then

$$\Phi(z) = \psi(z) + \varphi(t) \int_L \frac{d\tau}{\tau - z}.$$

As previously shown, $\varphi(t) \int_L \frac{d\tau}{\tau - z}$ is defined for all $z \in \mathbb{C}$, using Cauchy Type integration when necessary. $\psi(z)$ is also defined for $z \in \mathrm{D}^+$ and $z \in \mathrm{D}^-$, so to fully understand $\Phi(z)$, we must examine $\psi(t)$ for $t \in L$.

Recall, φ is a Hölder continuous function. From the definition of Hölder continuity, we obtain

$$\left| \frac{\varphi(\tau) - \varphi(t)}{\tau - t} \right| < \frac{A}{|\tau - t|^{1-\lambda}},$$

where $0 \leq 1 - \lambda < 1$.

Therefore

$$\int_L \left| \frac{\varphi(\tau) - \varphi(t)}{\tau - t} \right| < \int_L \frac{A}{|\tau - t|^{1-\lambda}}$$

is Riemann integrable, in much the same way as the real example $f(x) = \frac{1}{x^{\frac{1}{3}}}$, which we explored in Chapter 1. Thus Φ is defined on all of $\mathbb{C}$. In fact, the following lemma [1] provides an even better result.

Lemma 1 (Gakhov's Basic Lemma)**.** *Let φ be a Hölder continuous function. If*

$$\psi(t) := \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau,$$

then not only does $\lim_{z \to t}(\psi(z))$ exist for both $z \in \mathrm{D}^+$ and $z \in \mathrm{D}^-$ approaching t along any path which does not intersect L (referred to as $\lim_{z \to t^+}(\psi(z))$ and $\lim_{z \to t^-}(\psi(z))$ repectively), but also

$$\psi(t) = \lim_{z \to t}(\psi(z)),$$

yielding continuity of $\psi(z)$ on $\mathbb{C}$.

Thus, $\Phi^+(t) = \lim_{z \to t}$and $\Phi^-(t) = \lim_{z \to t}$ are defined for $z \in \mathrm{D}^+$ and $z \in \mathrm{D}^-$, respectively.

Since we are motivated by boundary value problems, a natural question to address is finding the limiting values of the Cauchy-type integral for a fixed $t \in L$,

$$\begin{aligned}\Phi(t) &= \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)d\tau}{\tau - t}\\ &= \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)-\varphi(t)}{\tau - t}d\tau + \varphi(t)\frac{1}{2\pi i}\int_L \frac{d\tau}{\tau - t}.\end{aligned}$$

We will use our understanding of ψ to find these limiting values.

$$\begin{aligned}\psi(t) &= \lim_{z\to t^+}(\psi(z))\\ &= \lim_{z\to t^+}\left(\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)-\varphi(t)}{\tau - z}d\tau\right)\\ &= \lim_{z\to t^+}\left(\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}d\tau - \frac{1}{2\pi i}\varphi(t)\int_L \frac{d\tau}{\tau - z}\right)\\ &= \lim_{z\to t^+}\left(\Phi(z) + \frac{1}{2\pi i}\varphi(t)\int_L \frac{d\tau}{\tau - z}\right)\\ &= \Phi^+(t) - \frac{1}{2\pi i}\varphi(t)(2\pi i)\\ &= \Phi^+(t) - \varphi(t).\end{aligned}$$

Similarly, by Gakhov's Basic Lemma,

$$\begin{aligned}\psi(t) &= \lim_{z\to t^-}(\psi(z))\\ &= \Phi^-(t) - \frac{1}{2\pi i}\varphi(t)(0)\\ &= \Phi^-(t)\end{aligned}$$

and

$$\begin{aligned}\psi(t) &= \Phi(t) - \frac{1}{2\pi i}\varphi(t)(\pi i + \log(k))\\ &= \Phi(t) + \frac{1}{2}\varphi(t)\left(\frac{\log(k)}{\pi i} - 1\right).\end{aligned}$$

Thus,

$$\psi(t) = \Phi^+(t) - \varphi(t) = \Phi^-(t) = \Phi(t) + \frac{1}{2}\varphi(t)\left(\frac{\log(k)}{\pi i} - 1\right),$$

ultimately yielding the important result,

$$\Phi^{\pm}(t) = \Phi(t) + \frac{1}{2}\varphi(t)\left(\frac{\log(k)}{\pi i} \pm 1\right).$$

For the case $k = 1$, the above formulae are known as the Sokhotski-Plemelj formulae, and are vital for the solutions of boundary value problems.

Chapter 4

A Riemann-Hilbert Problem

We now finally have all the tools needed to solve a particular case of boundary value problems, known as the Riemann-Hilbert problem, and will find Φ such that $\Phi^{-} = \Phi^{+}$ for $|x| > 1$ and $\Phi^{-} = (1 + k^2)\,\Phi^{+}$ for $|x| < 1$ if $\Phi \to 1$ as $z \to \infty$ and $k \in \mathbb{R}$.

First, we notice this is a homogeneous Riemann-Hilbert problem, taking on the form

$$\Phi^{+}(t) = g(t)\,\Phi^{-}(t),$$

where

$$g(t) = \frac{1}{(1 + k^2)}.$$

Observe that g is Hölder continuous, so Sokhotski-Plemelj will apply. A restatement of the standard Sokhotski-Plemelj formulae yields:

$$\Phi^{+}(t) - \Phi^{-}(t) = \varphi(t).$$

Thus, working with sums is preferred over working with products. This can be easily dealt with by using the properties of the logarithm.

First, define

$$\Gamma(z) := \frac{1}{2\pi i} \int_L \frac{\log(g(\tau))d\tau}{\tau - z}$$

and set

$$\Phi(z) = e^{\Gamma(z)}.$$

Next, we observe $|x| > 1$ gives us an open contour. According to [2], the fundamental solution, $X(t)$, of a Riemann-Hilbert problem on an open, non-self-intersecting contour with endpoints a and b takes on the form

$$X(t) = (z-a)^{\lambda}(z-b)^{\mu}e^{\Gamma(z)}.$$

Here, and λ and μ are integers determined by the index. For a closed contour, the index of $\varphi(t)$ is the difference of the zeros and poles inside the contour. Here, our contour is open, so by [2] λ and μ must obey

$$-1 < \lambda + \alpha < 1$$

$$-1 < \mu + \beta < 1,$$

where

$$\alpha + iA = -\frac{1}{2\pi i}\log(g(a))$$

$$\beta + iB = \frac{1}{2\pi i}\log(g(a)).$$

The specific solution to the Riemann-Hilbert problem is

$$\Phi(z) = X(t)p(z),$$

where $p(z)$ is an polynomial determined by the behavior of Φ at infinity. Thus, since our particular problem has $a = -1$ and $b = 1$,our solution will take on the form

$$\Phi(z) = (z+1)^{\lambda}(z-1)^{\mu}e^{\Gamma(z)}p(z).$$

By definition,

$$
\begin{aligned}
\Gamma(z) &:= \frac{1}{2\pi i}\int_L \frac{\log(g(\tau))d\tau}{\tau - z} \\
&= \frac{1}{2\pi i}\int_{-1}^{1} \frac{-\log(1+k^2))d\tau}{\tau - z} \\
&= -\frac{\log(1+k^2)}{2\pi i}\int_{-1}^{1} \frac{d\tau}{\tau - z} \\
&= -\frac{\log(1+k^2)}{2\pi i}(\log(\tau - z))|_{-1}^{1} \\
&= -\frac{\log(1+k^2)}{2\pi i}(\log(1-z) - \log(-1-z)) \\
&= -\frac{\log(1+k^2)}{2\pi i}\log\left(\frac{1-z}{-1-z}\right) \\
&= -\frac{\log(1+k^2)}{2\pi i}\log\left(\frac{z-1}{z+1}\right).
\end{aligned}
$$

Setting

$$\nu = -\frac{\log(1+k^2)}{2\pi i},$$

we obtain

$$\Gamma(z) = \log\left(\left(\frac{z-1}{z+1}\right)^{\nu}\right).$$

Thus,

$$
\begin{aligned}
\Phi(z) &= (z+1)^{\lambda}(z-1)^{\mu} e^{\log\left(\left(\frac{z-1}{z+1}\right)^{\nu}\right)} p(z) \\
&= (z+1)^{\lambda}(z-1)^{\mu}\left(\frac{z-1}{z+1}\right)^{\nu} p(z).
\end{aligned}
$$

Since $1 + k^2$ is positive and real,

$$\alpha + iA = -\frac{1}{2\pi i}\log(1+k^2)$$

is a strictly complex number, meaning $\alpha = 0$. A similar argument yields $\beta = 0$ as well.

Thus, λ and μ, both integers, must satisfy

$$-1 < \lambda < 1$$

and

$$-1 < \mu < 1,$$

meaning $\lambda = \mu = 0$. This yields

$$\begin{aligned} \Phi(z) &= (z+1)^0 (z-1)^0 e^{\log\left(\left(\frac{z-1}{z+1}\right)^\nu\right)} p(z) \\ &= \left(\frac{z-1}{z+1}\right)^\nu p(z). \end{aligned}$$

Since

$$\lim_{z\to\infty} \Phi(z) = 1 = \lim_{z\to\infty} \left(\frac{z-1}{z+1}\right)^\nu,$$

it must follow that $p(z) = 1$, finally yielding

$$\Phi(z) = \left(\frac{z-1}{z+1}\right)^\nu.$$

See figures 4.1 and 4.2 for visual representations of Φ for the value $k = 1$.

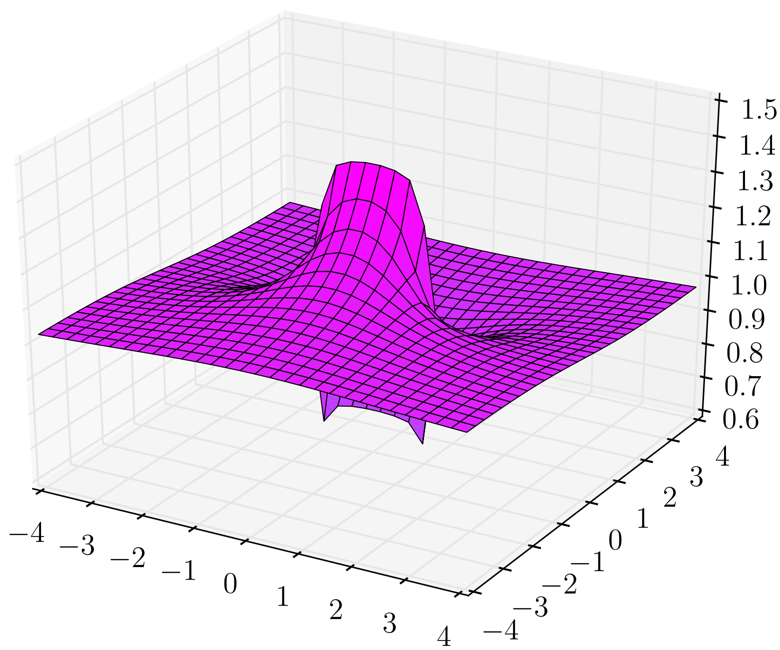

Figure 4.1: *Graph of the real part of* Φ*.*

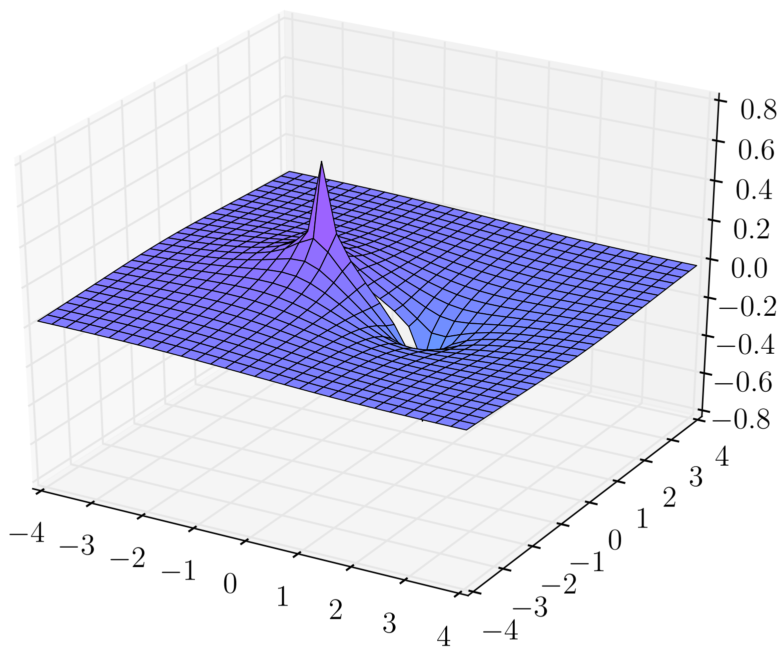

Figure 4.2: *Graph of the imaginary part of* Φ.

Bibliography

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