AN ANALYSIS OF SPRING-BEAMS HAVING LARGE DEFLECTIONS

by

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NOTATION

$(\zeta_n \eta_n)$	co-ordinates of pin n
ℓ_n	original undeformed length of link n
ℓ'_n	final deformed length of link n
Θ_n	tensile force of link n acting on pin n
τ_n	tensile force of link n+1 acting on pin n
T_n	moment on pin n caused by link n, and by link n+1
P_n	shearing force at right of pin n
Q_n	shearing force at left of pin n
α_n	the acute angle between x asis and link n+1, clockwise as positive
β_n	the acute angle between x axis and link n, clockwise as positive
γ_n	shear angle caused by Q_n
ϕ_n	shear angle caused by P_n
ψ	a function notation
α	pitch angle
t	a parameter in equations describing a spiral curve
a	radius of a spiral curve
i, j, k	unit vectors along x, y, z directions, respectively
T, N, B	unit vectors along the tangent, the normal and the binormal directions, respectively
$\overline{M}$	a moment vector
M_x, M_y, M_z	components of $\overline{M}$ along x, y, and z directions, respectively
M_t, M_n, M_b	components of $\overline{M}$ along T, N, and B directions, respectively

θ_y	angular displacement about y axis
$\overline{F}$	a force vector
F_t, F_n, F_b	components of $\overline{F}$ along T, N, and B directions, respectively
ρ	radius of curvature
B	equivalent bending rigidity for a spring
m_x, m_y, m_z	moments about x, y, and z directions, respectively, caused by a unit load
n	number of coils in a spring
X, Y, Z	forces along x, y, and z directions, respectively
$k_{3n+1, 0}$	spring constant of tension of link n+1 without deformation n=0, 1, 2 ---
$k_{3n+2, 0}$	spring constant of bending of link n+1 without deformation n=0, 1, 2 ---
$k_{3n+3, 0}$	spring constant of shear of link n+1 without deformation n=0, 1, 2 ---
$k_{n, 1}$	spring constant of tension of link n as deformed
$k'_{n, 2}$	spring constant of moment of pin n as deformed with same length for adjacent spring elements
$k_{n, 2}$	spring constant of moment of pin n as deformed with different length for adjacent spring elements
$k_{n, 3}$	spring constant of shear of link n as deformed
σ_t	normal stress along the T direction
τ_{tb}	shear stress perpendicular to the T direction caused by F_b
τ_{tn}	shear stress perpendicular to the T direction caused by F_n
τ_t	shear stress perpendicular to the T direction caused by M_t
τ	total shear stress

INTRODUCTION

The problem considered in this thesis is the determination of the loads carried by laterally loaded coil springs undergoing large deflections, and the stresses caused by these loads.

Three important features of the problem which are taken into account are its inherent nonlinearity, the extensibility of the spring, and its varying rigidity under load.

Since the springs considered are quite flexible, the resulting deflections are almost certain to be large, and the linear deflection theory inapplicable. These large deflections cause considerable nonuniform extensions of the springs. Since the pitch of a given spring varies from point to point under load, and the bending stiffness varies with the pitch, the deformed spring is equivalent in bending to a bar having variable rigidity.

Because, in general, the loads are distributed along the spring in an arbitrary manner, there is no simple rule in accordance with which the pitch varies. Thus, a solution based only on the differential equation for the deflected axis of the spring is not possible.

To overcome this difficulty, an approximate method has been developed. The coil spring is approximated by n link-like elastic elements pinned together with angular springs at the hinges. Concentrated loads are assumed to act on the hinges. To make the method more general, each element may be assumed to have different physical properties. With the assumptions that each element takes tension and shear only, while the connecting angular springs take moments only, the tensile bending and shear deformations of the spring are taken into account.

From the load-deflection relationships, equilibrium conditions, and geometrical relationships, a set of simultaneous algebraic equations are derived which relate external forces to the deformations they cause, or vice-versa. In the derivation, certain elastic constants occur. The relationship between these constants and the physical dimensions and material properties of given coil springs is examined. Then the solution is reduced to the solution of a set of simultaneous nonlinear algebraic equations. The stress analysis follows in a complicated but routine fashion once the unknown external forces or deflections are known.

Two basic problems are discussed in this thesis:

1. Given a set of loads, determine the deflection curve and maximum stresses which result; and

2. Given a deflection curve, determine the set of loads required to deform the spring into the given curve and the maximum stresses which result.

Although the basic sets of simultaneous nonlinear algebraic equations and methods for solution are given for both of thes problems, a numerical example is given for the second problem only. The example has been worked out with the help of a computer. Small pitch angles and symmetrical loading conditions are assumed, and three elements have been used to approximate the half coil spring. This is done only to shorten the computing time not because of any restriction on the theory. Comparison of theoretical and experimental loads calculated and found for given coil springs indicates excellent agreement between theory and experiment.

DERIVATION OF EQUATIONS

Loading Condition Given, Shear Effects Neglected (Basic Problem 1)

The equations are derived first without consideration of shear effects, and later with it.

First, the deflected spring is approximated by a finite number of link-like elements having angular springs at the hinges as shown in Fig. 1a. A typical element appears as shown in Fig. 1b. The element can stretch, as shown in the upper figure, or stretch with shear as shown in the lower figure.

Load-Deflection Relationships. From the load-deflection relationships, the following equations are obtained:

Tension: The length of link n, in terms of the coordinates of its ends, is $\sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2}$. Its original undeformed length is ℓ_n

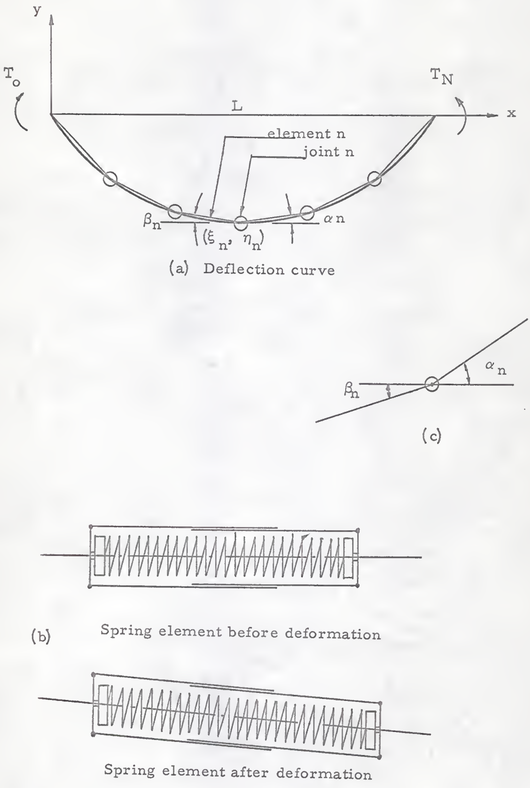


Fig. 1. Approximate deflection curve.

Assuming linearity, the tension in link n is

$$\Theta_n = k_{n,1}\left(\sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2} - \ell_n \right), \tag{1}$$

where k_{n1} is the appropriate spring constant.

Similarly, the tension in link n+1 is

$$\tau_n = k_{n+1,1}\left(\sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2} - \ell_{n+1} \right). \tag{2}$$

<u>Bending</u>: From Fig. 1c it is obvious that the angle change at joint n is

$$(\alpha_n - \beta_n) = \tan^{-1}\left(\frac{\eta_{n+1} - \eta_n}{\xi_{n+1} - \xi_n} \right) - \tan^{-1}\left(\frac{\eta_n - \eta_{n-1}}{\xi_n - \xi_{n-1}} \right). \tag{3}$$

Hence, the moment at joint n is

$$T_n = k_{n,2}(\alpha_n - \beta_n), \tag{4}$$

$$n = 1, 2, 3, \ldots . \; N-2, \; N-1.$$

Similarly,

$$T_{n+1} = k_{n+1,2}(\alpha_{n+1} - \beta_{n+1}), \tag{5}$$

and

$$T_{n-1} = k_{n-1,2}(\alpha_{n-1} - \beta_{n-1}), \tag{6}$$

where

$$\alpha_n = \tan^{-1}\left(\frac{\eta_{n+1} - \eta_{n.}}{\xi_{n+1} - \xi_n} \right), \quad n = 1, 2, \ldots . N-1, \; N, \tag{7}$$

$$\beta_n = \tan^{-1}\left(\frac{\eta_n - \eta_{n-1}}{\xi_n - \xi_{n-1}} \right), \quad \eta = 1, 2, \ldots . N-1, \; N. \tag{8}$$

Equilibrium Conditions. From the equilibrium conditions, the following equations can be obtained:

Consider pin n as a freebody, as shown in Fig. 2a. Summation of vertical forces gives

$$-F_n - \Theta_n \sin\beta_n + \tau_n \sin\alpha_n + Q_n \cos\beta_n + P_n \cos\alpha_n = 0, \tag{9}$$

and horizontal forces,

$$\tau_n \cos\alpha_n - \Theta_n \cos\beta_n - Q_n \sin\alpha_n - P_n \sin\alpha_n = 0. \tag{10}$$

Consider link n+1 as a free body, as shown in Fig. 2b. Summation of moments about the right-hand end yields

$$T_n - P_n \sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2} - T_{n+1} = 0, \tag{11}$$

or

$$P_n = \frac{T_n - T_{n+1}}{\sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2}} . \tag{11a}$$

Similarly, with link n as a free body, as shown in Fig. 2c,

$$T_{n-1} + Q_n \sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2} - T_n = 0, \tag{12}$$

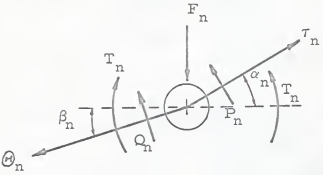
or

$$Q_n = \frac{T_n - T_{n-1}}{\sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2}} . \tag{12a}$$

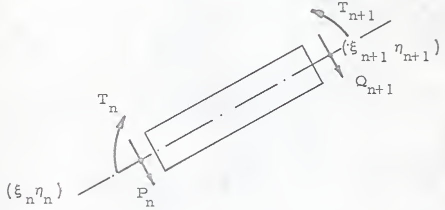
Boundary Conditions. From the boundary conditions, the following equations are known:

$$\eta_o = \xi_o = \eta_N = 0, \tag{13}$$

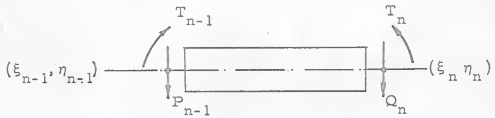
$$\xi_N = L, \tag{14}$$



(a) Joint N as a free body



(b) Link n+1 as a free body



(c) Link n as a free body

Fig. 2. Free body diagrams, shear deformation neglected (Basic Problem I).

$$T_o = C_1, \tag{15}$$

and

$$T_N = C_2, \tag{16}$$

where C_1 and C_2 are either zero or constants.

Rearrangement of these equations yields the following;

$$-F_1 - \Theta_1 \sin\beta_1 + \tau_1 \sin\alpha_1 + Q_1 \cos\beta_1 + P_1 \cos\alpha_1 = 0, \tag{17}$$

$$\tau_1 \cos\alpha_1 - \Theta_1 \cos\beta_1 - Q_1 \sin\beta_1 - P_1 \sin\alpha_1 = 0, \tag{18}$$

$$P_1 = \frac{T_1 - T_2}{\sqrt{(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2}}, \tag{19}$$

$$Q_1 = \frac{T_1 - T_o}{\sqrt{(\xi_1 - \xi_o)^2 + (\eta_1 - \eta_o)^2}}, \tag{20}$$

$$T_1 = k_{1,2}(\alpha_1 - \beta_1), \tag{21}$$

$$\vdots$$

$$-F_n - \Theta_n \sin\beta_n + \tau_n \sin\alpha_n + Q_n \cos\beta_n + P_n \cos\alpha_n = 0, \tag{22}$$

$$\tau_n \cos\alpha_n - \Theta_n \cos\beta_n - Q_n \sin\beta_n - P_n \sin\alpha_n = 0, \tag{23}$$

$$P_n = \frac{T_n - T_{n+1}}{\sqrt{\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2}}, \tag{24}$$

$$Q_n = \frac{T_n - T_{n-1}}{\sqrt{\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2}}, \tag{25}$$

$$T_n = k_{n,2}(\alpha_n - \beta_n), \tag{26}$$

$$\vdots$$

$$-F_{N-1} - \Theta_{N-1} \sin\beta_{N-1} + \tau_{N-1} \sin\alpha_{N-1} + Q_{N-1} \cos\beta_{N-1} + P_{N-1} \cos\alpha_{N-1} = 0, \tag{27}$$

$$\tau_{N-1} \cos\alpha_{N-1} - \Theta_{N-1} - Q_{N-1} \sin\beta_{N-1} - P_{N-1} \sin\alpha_{N-1} = 0, \tag{28}$$

$$P_{N-1} = \frac{T_{N-1} - T_N}{\sqrt{(\xi_N - \xi_{N-1})^2 + (\eta_N - \eta_{N-1})^2}}, \tag{29}$$

$$Q_{N-1} = \frac{T_{N-1} - T_{N-2}}{\sqrt{(\xi_{N-1} - \xi_{N-2})^2 + (\eta_{N-1} - \eta_{N-2})^2}}, \tag{30}$$

and

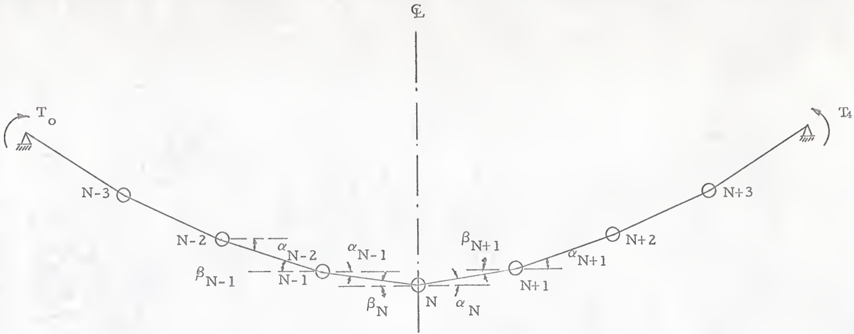
$$T_{N-1} = k_{N-1,2}(\alpha_{N-1} - \beta_{N-1}), \tag{31}$$

where T_o and T_N are defined by Eqs. (15) and (16), respectively;

and Θ_n, τ_n, α_n, and β_n are defined by Eqs. (1), (2), (7), and (8), respectively.

After substitution of α_n and β_n into Eqs. (17) through (31), there will be 2N-2 equations for 2N-2 unknowns, which are the ξ_n's and η_n's.

For symmetrical load distributions it is sufficient to consider half of the spring as shown in Fig. (3a).



(a) Deflection curve

(b) Angle positive as measured clockwise

Fig. 3. Symmetric deflection curve.

In these cases, the equilibrium equations will be identical to those for unsymmetrical loads, pin by pin, except those for pin N, which require modifications because of altered boundary conditions.

Because of symmetry, it is obvious that

$$\alpha_N = -\beta_N = -\alpha_{N-1} = \beta_{N+1}, \tag{32}$$

and

$$\alpha_{N+1} = -\beta_{N-1} = \beta_{N+2} = -\alpha_{N-2}, \tag{33}$$

where all angles are taken to be positive if measured clockwise. (See Fig. (3b).)

Thus

$$T_N = k_{N,2}(\alpha_N - \beta_N) = -2k_{N,2}\,\alpha_{N-1}, \tag{34}$$

$$T_{N+1} = k_{N+1,2}(\alpha_{N+1} - \beta_{N+1}) = -k_{N-1,2}(\alpha_{N-2} - \alpha_{N-1})$$

$$= k_{N-1,2}(\alpha_{N-1} - \beta_{N-1}) = T_{N-1}, \tag{35}$$

and

$$\xi_N = L. \tag{36}$$

Since η_N is unknown, an additional equation is required.

With pin N, taken as a free body, the following equations are obtained:

$$-F_N - \Theta_N \sin\beta_N + \tau_N \sin\alpha_N + Q_N \cos\beta_N + P_N \cos\alpha_N = 0, \tag{37}$$

$$P_N = \frac{T_N - T_{N+1}}{\sqrt{(\xi_{N+1} - \xi_N)^2 + (\eta_{N+1} - \eta_N)^2}}, \tag{38}$$

and

$$Q_N = \frac{T_N - T_{N-1}}{\sqrt{(\xi_N - \xi_{N-1})^2 + (\eta_N - \eta_{N-1})^2}}. \tag{39}$$

Equations (17) through (31), together with (32) through (39), form the basic nonlinear algebraic set of equations which describe the symmetric case of basic problem 1.

These equations essentially give the external vertical and horizontal loads which are required to hold the pins joining the links in given positions as functions of the coordinates of the positions. They have the form

$$
\begin{aligned}
F_1 &= F_1(\xi_i , \eta_i) \\
F_2 &= F_2(\xi_i , \eta_i) \\
&\;\vdots \\
F_N &= F_N(\xi_i , \eta_i) \\
H_1 &= H_1(\xi_i , \eta_i) \\
&\;\vdots \\
H_{N-1} &= H_{N-1}(\xi_i , \eta_i) \quad ,
\end{aligned}
\qquad
\begin{array}{c}
(2N-1) \text{ equations} \\
i = 1, 2, \dots , N
\end{array}
$$

where the F's and the H's represent the given external vertical and horizontal forces.

In the case of basic problem 1, the forces are given, and this basic set must be solved for the unknown coordinates. Although an exact solution of this set seems impossible, an approximate solution can be obtained by the summation of infinitesimal deformations found through the use of a Taylor's series expansion in which the higher order terms are neglected.

In partitioned matrix form, for an expansion about $\left\{\frac{F}{H}\right\}_o$, the truncated series is

$$\left\{\frac{F}{H}\right\} = \left\{\frac{F}{H}\right\}_o + \left(\begin{array}{c|c} \frac{\partial F_i}{\partial \xi_j} & \frac{\partial F_i}{\partial \eta_j} \\ \hline \frac{\partial H_i}{\partial \xi_j} & \frac{\partial H_i}{\partial \eta_j} \end{array}\right) \left\{\begin{array}{c} d\xi_j \\ \hline d\eta_j \end{array}\right\}$$

From this, the algorithm

$$\left\{\frac{\Delta\xi_j}{\Delta\eta_j}\right\} = \left(\begin{array}{c|c} \frac{\partial F_i}{\partial \xi_j} & \frac{\partial F_i}{\partial \eta_j} \\ \hline \frac{\partial H_i}{\partial \xi_j} & \frac{\partial H_i}{\partial \eta_j} \end{array}\right)_n^{-1} \left\{\left\{\frac{F}{H}\right\}_{n+1} - \left\{\frac{F}{H}\right\}_n\right\}$$

for computing changes in coordinates for given changes in external loads can be constructed easily.

Now, if the deflection of the laterally loaded coil spring is desired for a given loading $\left\{\frac{F}{H}\right\}$, it can be found by constructing a monotone increasing sequence of loads $\left\{\frac{F}{H}\right\}_n$ such that

$$\left\{\frac{F}{H}\right\}_o = \left\{\frac{0}{0}\right\}, \quad \left\{\frac{F}{H}\right\}_{n+1} - \left\{\frac{F}{H}\right\}_n \text{ is small, and } \lim_{n\to\infty} \left\{\frac{F}{H}\right\}_n = \left\{\frac{F}{H}\right\}$$

and computing

$$\left\{\frac{\xi}{\eta}\right\} = \sum_{n=1}^{\infty} \left\{\frac{\Delta\xi}{\Delta\eta}\right\}_n = \sum_{n=1}^{\infty} \left(\begin{array}{c|c} \frac{\partial F_i}{\partial \xi_j} & \frac{\partial F_i}{\partial \eta_j} \\ \hline \frac{\partial H_i}{\partial \xi_j} & \frac{\partial H_i}{\partial \eta_j} \end{array}\right)_n^{-1} \left\{\left\{\frac{F}{H}\right\}_{n+1} - \left\{\frac{F}{H}\right\}_n\right\}$$

Loading Condition Given, Shear Effects Considered (Basic Problem 1)

In this case, a symmetric loading is assumed.

From Fig. 4 it is obvious that

$$\alpha_n = \tan^{-1}\left(\frac{\eta_n - \eta_{n+1}}{\xi_{n+1} - \xi_n}\right), \tag{40}$$

and

$$\beta_n = \tan^{-1}\left(\frac{\eta_{n-1} - \eta_n}{\xi_n - \xi_{n-1}}\right). \tag{41}$$

In Fig. 4, $\overset{\frown}{ABC\ldots}$ represents the deflection curve due to bending only. Let the shear effect of the first element take place. Then the edge $\overline{LN}$ shifts with respect to $\overline{JP}$ to $\overline{KQ}$ and the second element translates without any rotation to the position as shown by dotted lines in Fig. 4. At this stage, the deflection curve is $\overset{\frown}{ADM\ldots}$. Now, in turn, let the shear effect of the second element, the third, and so forth take place, so as to obtain the final deflection curve $\overset{\frown}{ADE\ldots}$.

The total change in angle is $\pi - \angle ADE = \beta - \alpha$. (42)

The angle change for bending alone is $\angle ABC$.

$\overline{AB} \parallel \overline{GD}$ and $\overline{BC} \parallel \overline{DF}$ by construction.

In the actual pin connected link structure the chords $\overline{AH}$, $\overline{DI}$, and so on, are infinitesimal. Therefore, $\overline{AD} \parallel \overline{JK}$ and $\overline{DE} \parallel \overline{RS}$, from which $\angle LJK = \angle ADG = \gamma_n$, and $\angle FDE = \angle MRS = \phi_n$.

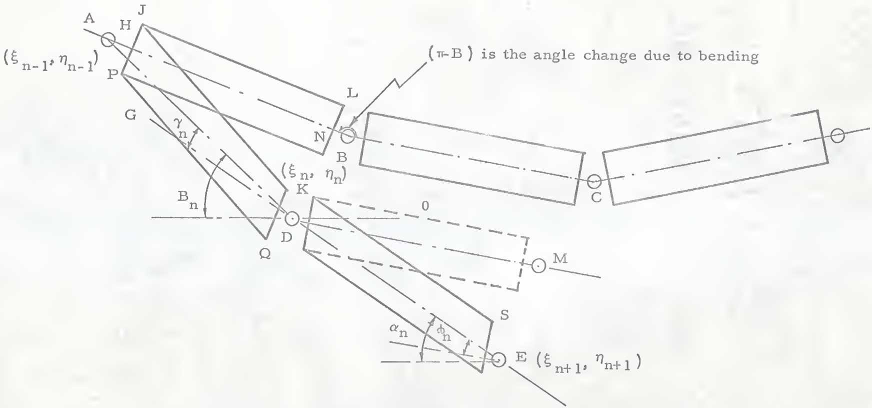


Fig. 4. Influence of shear effect in deflection curve.

The change in angle for bending is

$$\pi - \angle ABC = \pi - \angle GDF = \pi - \left[\pi - (\beta_n - \alpha_n) + \gamma_n - \phi_n\right] = \beta_n - \alpha_n + \phi_n - \gamma_n. \tag{43}$$

Load Deflection Relationships. The load-deflection relationships are:

Tension: The length of the link n after deformation is $\sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2}$, and its original length ℓ_n. The tensile forces in link n and n+1 are, respectively

$$\Theta_n = k_{n,1}\left(\sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2} - \ell_n\right) \tag{44}$$

and

$$\tau_n = k_{n+1,1}\left(\sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2} - \ell_{n+1}\right). \tag{45}$$

Shearing: The shear angles due to Q and P are γ_n and ϕ_n, respectively. Thus

$$Q_n = k_{n,3}\,\gamma_n \tag{46}$$

or

$$\gamma_n = \frac{Q_n}{k_{n,3}}\ ; \tag{46a}$$

and

$$P_n = k_{n+1,3}\,\phi_n \tag{47}$$

or

$$\phi_n = \frac{P_n}{k_{n+1,3}}\ . \tag{47a}$$

Bending: The change in angle at joint n is $(\beta_n - \alpha_n + \phi_n - \gamma_n)$.

Hence, the moment at joint n is

$$T_n = k_{n,2}\left(\beta_n - \alpha_n + \frac{P_n}{k_{n+1,3}} - \frac{Q_n}{k_{n,3}}\right). \tag{48}$$

Similarly,

$$T_{n+1} = k_{n+1,2}\left(\beta_{n+1} - \alpha_{n+1} + \frac{P_{n+1}}{k_{n+2,3}} - \frac{Q_{n+1}}{k_{n+1,3}}\right) \tag{49}$$

and

$$T_{n-1} = k_{n-1,2}\left(\beta_{n-1} - \alpha_{n-1} + \frac{P_{n-1}}{k_{n,3}} - \frac{Q_{n-1}}{k_{n-1,3}}\right). \tag{50}$$

<u>Equilibrium</u> <u>Conditions:</u> From Fig. 4

$$\angle JAH = \beta_n - \gamma_n,$$

and

$$\angle ODM = \alpha_N - \phi_n .$$

Thus, the forces Q_n and P_n are at angles

$\frac{\pi}{2} - (\beta_n - \gamma_n)$ and $\frac{\pi}{2} - (\alpha_n - \phi_n)$ with the horizontal.

Consider joint n as a free body, as shown in Fig. 5a.
Equilibrium requires that

$$-F_n + \Theta_n \sin\beta_n - \tau_n \sin\alpha_n + P_n \sin\left[\frac{\pi}{2} - (\alpha_n - \phi_n)\right] + Q_n \sin\left[\frac{\pi}{2} - (\beta_n - \gamma_n)\right] = 0,$$

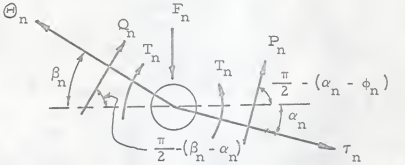
or

$$-F_n + \Theta_n \sin\beta_n - \tau_n \sin\alpha_n + P_n \cos(\alpha_n - Q_n) + Q_n \cos(\beta_n - \gamma_n) = 0, \tag{51}$$

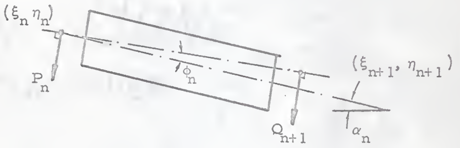
and

$$-\Theta_n \cos\beta_n + \tau_n \cos\alpha_n + P_n \sin(\alpha_n - \phi_n) + Q_n \sin(\beta_n - \gamma_n) = 0 \tag{52}$$

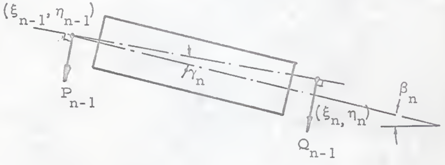
Consider link (n+1) as a free body, as shown in Fig. 5b.



(a) Joint N as a free body



(b) Link n+1 as a free body



(c) Link n as a free body

Fig. 5. Free body diagrams, shear deformation considered (Basic Problem I).

Equilibrium requires

$$T_n - P_n \sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2} \cos\phi_n - T_{n+1} = 0 \tag{53}$$

or

$$P_n = \frac{T_n - T_{n+1}}{\sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2} \quad \cos\phi_n} . \tag{53a}$$

Similarly, consider link $(n-1)$ as a free body as shown in Fig. 5c.

Then

$$Q_n = \frac{T_n - T_{n-1}}{\cos\gamma_n \sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2}} . \tag{54}$$

Boundary Conditions: From the boundary conditions, the following equations are known:

$$\xi_o = \eta_n = 0, \tag{55}$$

$$\xi_N = L, \tag{56}$$

and

$$T_o = C_1 , \tag{57}$$

where C_1 is either zero or a constant.

Because of symmetry:

$$\alpha_N = - \beta_N = - \alpha_{N-1} = \beta_{N+1} , \tag{58}$$

$$\alpha_{N+1} = - \beta_{N-1} = \beta_{N+2} = - \alpha_{N-2}, \tag{59}$$

and

$$\tau_N = \Theta_N . \tag{60}$$

(Angles are positive when measured clockwise.)

Thus,

$$T_N = k_{N,2}(\beta_N - \alpha_N + \frac{P_N}{k_{N-1,3}} - \frac{Q_N}{k_{N,3}}) = k_{N,2}(2\alpha_{N-1} + \frac{P_N}{k_{N,3}} - \frac{Q_N}{k_{N,3}}) \tag{61}$$

Substitution and rearrangement of equations yields the following set of equations:

$$\psi_{1-1} = -F_1 + \Theta_1 \sin\beta_1 - \tau_1 \sin\alpha_1 + P_1 \cos\left(\alpha_1 - \frac{P_1}{k_{2,3}}\right) + Q_1 \cos\left(\beta_1 - \frac{Q_1}{k_{1,3}}\right) \tag{62}$$

$$\psi_{2-1} = -\Theta_1 \cos\beta_1 + \tau_1 \cos\alpha_1 + P_1 \sin\left(\alpha_1 - \frac{P_1}{k_{2,3}}\right) + Q_1 \sin\left(\beta_1 - \frac{Q_1}{k_{1,3}}\right) = 0. \tag{63}$$

$$\psi_{3-1} = T_1 - k_{1,2}\left[\beta_1 - \alpha_1 + \frac{P_1}{k_{2,3}} - \frac{Q_1}{k_{1,3}}\right] = 0 \tag{64}$$

$$\psi_{4-1} = P_1 - \frac{T_1 - T_2}{\cos\left(\frac{P_1}{k_{2,3}}\right)\sqrt{(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2}} = 0 \tag{65}$$

$$\psi_{5-1} = Q_1 - \frac{T_1 - C_1}{\cos\left(\frac{Q_1}{k_{1,3}}\right)\sqrt{\xi_1^2 + \eta_1^2}} = 0 \tag{66}$$

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$$\psi_{1-n} = -F_n + \Theta_n \sin\beta_n - \tau_n \sin\alpha_n + P_n \cos\left(\alpha_n - \frac{P_n}{k_{n+1,3}}\right) + Q_n \cos\left(\beta_n - \frac{Q_n}{k_{n,3}}\right) = 0 \tag{67}$$

$$\psi_{2-n} = -\Theta_n \cos\beta_n + \tau_n \cos\alpha_n + P_n \sin\left(\alpha_n - \frac{P_n}{k_{n+1,3}}\right) + Q_n \sin\left(\beta_n - \frac{Q_n}{k_{n,3}}\right) = 0 \tag{68}$$

$$\psi_{3-n} = T_n - k_{n,2}\left[\beta_n - \alpha_n + \frac{P_n}{k_{n+1,3}} - \frac{Q_n}{k_{n,3}}\right] = 0 \tag{69}$$

$$\psi_{4-n} = P_n - \frac{T_n - T_{n+1}}{\sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2}\ \cos\left(\frac{P_n}{k_{n+1,3}}\right)} = 0 \tag{70}$$

$$\psi_{5-n} = Q_n - \frac{T_n - T_{n-1}}{\cos\left(\frac{Q_n}{k_{n,3}}\right)\ \sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2}} = 0 \tag{71}$$

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$$\psi_{4-(N-1)} = P_{N-1} - \frac{T_{N-1} - T_N}{\cos\left(\frac{P_{N-1}}{k_{N,3}}\right)\sqrt{(\xi_N - \xi_{N-1})^2 + (\eta_N - \eta_{N-1})^2}} \tag{72}$$

$$\psi_{5-(N-1)} = Q_{N-1} - \frac{T_{N-1} - T_{N-2}}{\sqrt{(\xi_{N-1} - \xi_{N-2})^2 + (\eta_{N-1} - \eta_{N-2})^2}\cos\left(\frac{Q_{N-1}}{k_{N-1,3}}\right)}$$

$$= 0 \tag{73}$$

$$\psi_{1-N} = - F_N + 2\ \Theta_N \sin\alpha_{N-1} + P_N \cos\left(\alpha_{N-1} + \frac{P_N}{k_{N,3}}\right)$$

$$+ Q_N \cos\left(\alpha_{N-1} - \frac{Q_N}{k_{N,3}}\right) = 0 \tag{74}$$

$$\psi_{2-N} = - P_N \sin\left(\alpha_{N-1} + \frac{P_N}{k_{N,3}}\right) + Q_N \sin\left(\alpha_{N-1} - \frac{Q_N}{k_{N,3}}\right) = 0 \tag{75}$$

$$\psi_{3-N} = T_N - k_{N,2}\left[2\alpha_{N-1} + \frac{P_N - Q_N}{k_{N,3}}\right] = 0 \tag{76}$$

$$\psi_{4-N} = Q_N - \frac{(T_N - T_{N-1})}{\cos\left(\frac{Q_N}{k_{N,3}}\right)\sqrt{(\xi_N - \xi_{N-1})^2 + (\eta_N - \eta_{N-1})^2}} \tag{77}$$

$$\psi_{5-N} = \xi_N - L = 0 \tag{78}$$

In these equations Θ_n and τ_n are defined by Eqs. (44) and (45), and the first subscript of the function notation ψ indicates the number of the equation, while the second subscript indicates the number of the point. This set contains 5-N equations with 5N variables. It cannot be reduced to a system containing 2N by direct substitution as was done in the case where shear effects are neglected, as shown on pages 8and 9, because the P's and Q's appear also in the arguments of trigonometric functions. The terms $\frac{P_i}{k_{i+1,3}}$ and $\frac{Q_i}{k_{i,3}}$ describe shear effects, and therefore are generally small. If they are taken to be zero, the number of equations can be reduced to 2N, with ξ_i's and η_i's as variables, and can be treated in the manner as described on pages 8 and 9. This treatment will not eliminate the shear effect completely, because the shear angles $\frac{P_i}{k_{i+1,3}}$ and $\frac{Q_i}{k_{i,3}}$ still appear in ψ_{3-i}.

Deflection Curve Given, Shear Effect Considered
(Basic Problem II)

In this case, a symmetric deflection curve is assumed, and N points (not including the origin) are chosen arbitrarily along the first half of the curve (See Fig. 3a). These points should correspond to end points of the elements. In general, both horizontal and vertical external forces H_n and F_n will be required at the pins joining the links, since the coordinates of these pins will not be known, even if a deflection curve is specified. The equations are derived in a manner similar to that used for basic problem I.

Load-Deflection Relationships. The equations are the same as those given on pages 16 and 17 for basic problem I.

Equilibrium Conditions. Consider pin n as a free body, as shown in Fig. 6a. Then summation of forces in the vertical direction gives

$$-F_n + \Theta_n \sin\beta_n - \tau_n \sin\alpha_n + Q_n \sin\left[\frac{\pi}{2} - (\beta_n - \gamma_n)\right] - P_n \sin\left[\frac{\pi}{2} - (\alpha_n - \phi_n)\right] = 0$$

or

$$-F_n + \Theta_n \sin\beta_n - \tau_n \sin\alpha_n + Q_n \cos(\beta_n - \gamma_n) - P_n \cos(\alpha_n - \phi_n) = 0 \tag{79}$$

Summation in the horizontal direction gives

$$-H_n - \Theta_n \cos\beta_n + \tau_n \cos\alpha_n + Q_n \cos\left[\frac{\pi}{2} - (\beta_n - \gamma_n)\right] - P_n \cos\left[\left(\frac{\pi}{2} - (\alpha_n - \phi_n)\right)\right] = 0,$$

or

$$-H_n - \Theta_n \cos\beta_n + \tau_n \cos\alpha_n - P_n \sin(\alpha_n - Q_n) + Q_n \sin(\beta_n - \gamma_n) = 0. \tag{80}$$

Consider pin (n+1) as a free body, as shown in Fig. 6b.

Then

$$T_n + P_n \sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2} \cos\phi_n - T_{n+1} = 0 \tag{81}$$

or

$$P_n = \frac{T_{n+1} - T_n}{\cos\left(\frac{P_n}{k_{n+1,3}}\right) \sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2}}. \tag{81b}$$

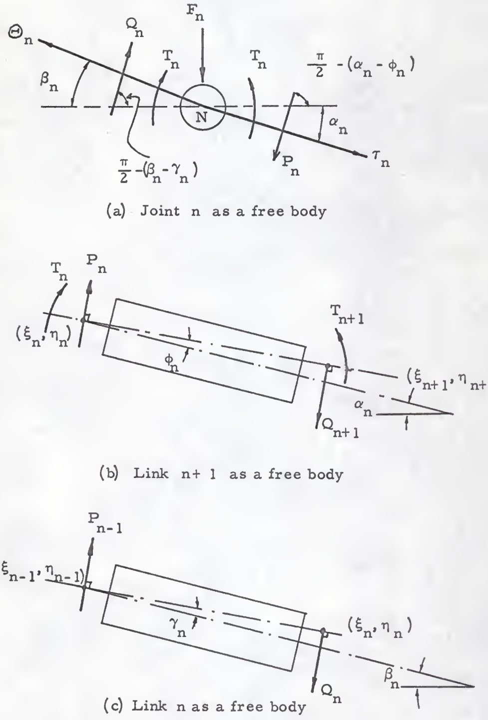

Fig. 6. Free body diagrams, shear deformation considered (Basic Problem II).

Consider pin n as a free body, as shown in Fig. 6c.

Then

$$Q_n = \frac{T_n - T_{n-1}}{\cos\left(\frac{Q_n}{k_{n,3}}\right)\sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2}} \,. \tag{82}$$

Boundary Condition. These are

$$\xi_o = \eta_o = 0, \tag{83}$$

$$\xi_N = L, \tag{84}$$

$$\alpha_N = -\beta_N = -\alpha_{N-1} = \beta_{N+1}\,, \tag{85}$$

$$\alpha_{N+1} = -\beta_{N-1} = \beta_{N+2} = -\alpha_{N-2}\,, \tag{86}$$

$$\tau_N = \Theta_N\,, \tag{87}$$

and

$$T_o = k_{o,2}(\beta_o - \alpha_o), \tag{88}$$

where β_o is either equal to α_o or is a constant.

Substitution and rearrangement of these equations yields the following set:

$$\psi_{1-1} = -F_1 + \Theta_1 \sin\beta_1 - \tau_1 \sin\alpha_1 + Q_1\cos\left(\beta_1 - \frac{Q_1}{k_{1,3}}\right) - P_1\cos\left(\alpha_1 - \frac{P_1}{k_{2,3}}\right) = 0 \tag{89}$$

$$\psi_{2-1} = -H_1 - \Theta_1 \cos\beta_1 + \tau_1 \cos\alpha_1 + Q_1 \sin\left(\beta_1 - \frac{Q_1}{k_{1,3}}\right) - P_1 \sin\left(\alpha_1 - \frac{P_1}{k_{2,3}}\right) = 0 \tag{90}$$

$$\psi_{3-1} = T_1 - k_{1,2}\left(\beta_1 - \alpha_1 + \frac{P_1}{k_{2,3}} - \frac{Q_1}{k_{1,3}}\right) = 0 \tag{91}$$

$$\psi_{4-1} = P_1 - \frac{T_2 - T_1}{\cos\left(\frac{P_1}{k_{2,3}}\right)\sqrt{(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2}} = 0 \tag{92}$$

$$\psi_{5-1} = Q_1 - \frac{T_2 - T_1}{\cos \frac{Q_1}{k_{1,3}} \sqrt{\xi_1^2 + \eta_1^2}} = 0 \tag{93}$$

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$$\psi_{1-n} = -F_n + \Theta_n \sin\beta_n - \tau_n \sin\alpha_n + Q_n \cos(\beta_n - \frac{Q_n}{k_{n,3}}) - P_n \cos(\alpha_n - \frac{P_n}{k_{n+1,3}}) = 0 \tag{94}$$

$$\psi_{2-n} = -H_n - \Theta_n \cos\beta_n + \tau_n \cos\alpha_n + Q_n \sin(\beta_n - \frac{Q_n}{k_{n,3}}) - P_n \sin(\alpha_n - \frac{P_n}{k_{n+1,3}}) = 0 \tag{95}$$

$$\psi_{3-n} = T_n - k_{n,2}(\beta_n - \alpha_n + \frac{P_n}{k_{n+1,3}} - \frac{Q_n}{k_{n,3}}) \tag{96}$$

$$\psi_{4-n} = P_n - \frac{T_{n+1} - T_n}{\cos(\frac{P_n}{k_{n+1,3}}) \sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2}} \tag{97}$$

$$\psi_{5-n} = Q_n - \frac{T_n - T_{n-1}}{\cos(\frac{Q_n}{k_{n,3}}) \sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2}} \tag{98}$$

.
.
.

$$\psi_{4-(N-1)} = P_{N-1} - \frac{T_N - T_{N-1}}{\cos(\frac{P_{n-1}}{k_{n,3}}) \sqrt{(\xi_N - \xi_{N-1})^2 + (\eta_N - \eta_{N-1})^2}} = 0 \tag{99}$$

$$\psi_{5-(N-1)} = Q_{N-1} - \frac{T_{N-1} - T_{N-2}}{\cos(\frac{Q_{N-1}}{k_{N-1,3}}) \sqrt{(\xi_{N-1} - \xi_{N-2})^2 + (\eta_{N-1} - \eta_{N-2})^2}} = 0 \tag{100}$$

$$\psi_{1-N} = -F_N + 2\Theta_N \sin\alpha_{N-1} + Q_N \cos(\alpha_{N-1} - \frac{Q_N}{k_{N,3}}) - P_N \cos(\alpha_{N-1} + \frac{P_N}{k_{N,3}}) = 0 \tag{101}$$

$$\psi_{2-N} = - H_N + Q_N \sin(\alpha_{N-1} - \frac{Q_N}{k_{N,3}}) + P_N \sin(\alpha_{N-1} + \frac{P_N}{k_{N,3}}) = 0 \quad (102)$$

$$\psi_{3-N} = T_N - k_{N,2} (2\alpha_{N-1} + \frac{P_N - Q_N}{k_{N,3}}) = 0 \quad (103)$$

$$\psi_{4-N} = Q_N - \frac{T_N - T_{N-1}}{\cos(\frac{Q_N}{k_{N-3}}) \sqrt{(\xi_N - \xi_{N-1})^2 + (\eta_N - \eta_{N-1})^2}} \quad (104)$$

$$\psi_{5-N} = \xi_N - L = 0. \quad (105)$$

In the preceding equations,

$$\alpha_n = \tan^{-1} (\frac{\eta_n - \eta_{n+1}}{\xi_{n+1} - \xi_n})$$

$$\beta_n = \tan^{-1} (\frac{\eta_{n-1} - \eta_n}{\xi_n - \xi_{n-1}}),$$

$$\Theta_n = k_{n,1} (\sqrt{(\xi_n - \xi_{n-1})^2 + (\eta_n - \eta_{n-1})^2} - \ell_n),$$

and $$\tau_n = k_{n+1,1} (\sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2} - \ell_{n+1}).$$

PROPERTIES OF THE ELEMENTAL LINKS IN TERMS OF THE PROPERTIES OF GIVEN COIL SPRINGS

In the equations which were derived for Basic Problems I and II in the previous sections, certain elastic constants occur.

The relationships between these constants and the physical dimensions and material properties of given coil springs is examined. The helical spring as shown in Fig. 7a, can be described by the space curve

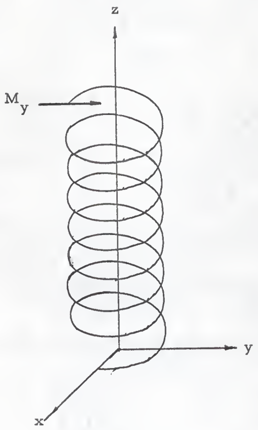


Fig. 7 (a). A helical spring subjected to pure bending moment.

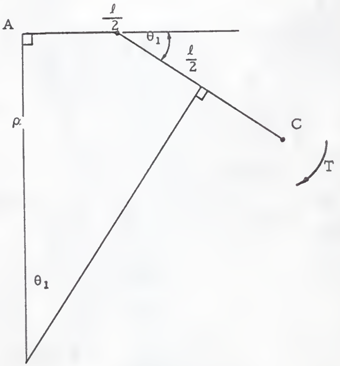


Fig. 7(b) Same rigidity, same length in adjacent elements

$$x = a \cos t$$
$$y = a \sin t \qquad (106)$$
$$z = a t \tan \alpha,$$

where a is the radius of the spring, t is a parameter, and α is the pitch angle.

The unit vectors of the curve along the tangent, the normal, and the binomal directions are, respectively.

$$\hat{T} = \frac{-\sin t \ \hat{i} + \cos t \ \hat{j} + \tan \alpha \ \hat{k}}{\sec \alpha}, \qquad (107)$$

$$\hat{N} = -(\cos t \ \hat{i} + \sin t \ \hat{j}), \qquad (108)$$

and

$$\hat{B} = \hat{i} \ (\sin \alpha \ \sin t) - \hat{j} \ \sin \alpha \ \cos t + \hat{k} \ \cos \alpha, \qquad (109)$$

respectively. (See Appendix 1)

<u>Constants</u> <u>for</u> <u>Pure</u> <u>Bending</u>. The angular displacement about the y axis caused by the bending moment My, as shown in Fig. 7a, is θy.

Since $M_x = M_z = 0$, and $\overline{M} = M_y \ \hat{j}$,

$$M_t = \overline{M} \cdot \hat{T} = M_y \cos t \cos \alpha,$$

$$M_n = \overline{M} \cdot \hat{N} = -M_y \sin t,$$

and

$$M_b = -M_y \sin \alpha \cos t.$$

Now, from Langhaar [3], and Hodgman [2], it follows that

$$\theta_y = \int_0^{2n_1\pi} \left(\frac{M_n \frac{\partial M_n}{\partial M_y}}{EI} + \frac{M_b \frac{\partial M_b}{\partial M_y}}{EI} + \frac{M_t \frac{\partial M_t}{\partial M_n}}{GI_p} \right) a \sec \alpha \; dt$$

$$= \int^{2n_1\pi} \left(\frac{M_y \sin^2 t}{EI} + \frac{M_y \sin^2 \alpha \cos^2 t}{EI} + \frac{M_y \cos^2 t \cos^2 \alpha}{GI_p} \right) a \sec \alpha \; dt$$

$$= M_y \, a \sec \alpha \left(\frac{n_1 \pi}{EI} + \frac{\sin^2 \alpha \; n_1 \pi}{EI} + \frac{\cos^2 \alpha}{GI_p} n_1 \pi \right). \tag{110}$$

(note that $a \sec \alpha \; dt = ds$)

By definition,

$$k_{2,0} = \frac{M_y}{\theta_y} . \tag{111}$$

Substitution of Eq. (110) into (111) yields

$$k_{2,0} = \frac{\cos \alpha}{a n_1 \pi \left(\frac{1 + \sin^2 \alpha}{EI} + \frac{\cos^2 \alpha}{GI_p} \right)} . \tag{112}$$

This constant, $k_{2,0}$, may also be obtained in another manner. In Fig. 7b, $\widehat{ABC}$ ... represents the approximate deflection curve. Approximately,

$$\theta_1 = \frac{\ell}{\rho} . \tag{113}$$

For elastic deflections,

$$\frac{1}{\rho} = \frac{M}{B} \tag{114}$$

where $B = \dfrac{\sin \alpha}{\dfrac{1 + \sin^2 \alpha}{2EI} + \dfrac{\cos^2 \alpha}{2GI_p}}$

according to Timoshenko [6].

Substitution of Eq. (114) into (113) yields

$$\theta_1 = \frac{\frac{\ell M}{\sin\alpha}}{\frac{1+\sin^2\alpha}{2EI} + \frac{\cos^2\alpha}{2GI_p}} , \qquad (115)$$

from which

$$k_{2,0} = \frac{M}{\theta_1} = \frac{\sin\alpha}{\left(\frac{1+\sin^2\alpha}{2EI} + \frac{\cos^2\alpha}{2GI_p}\right)\ell} ; \qquad (116)$$

and since $\ell = 2an_1\pi\tan\alpha$,

$$k_{2,0} = \frac{\cos\alpha}{a\ n_1\ \pi\left(\frac{1+\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p}\right)} . \qquad (117)$$

According to Frisch-Fay [1], when the spring is stretched, $k_{2,0}$ will be increased in the amount

$$k'_{n,2} = \frac{\ell'_n}{\ell_n}\ k_{2,0} , \qquad (118)$$

Where ℓ'_n is the final length of the spring element, and ℓ_n is the original undeformed length.

In case the adjacent elements are different in length, as shown in Fig. 7c and 7d, the equivalent bending constant for the spring can be found as follows:

If the stiffness of both elements are the same, from Fig. 7c,

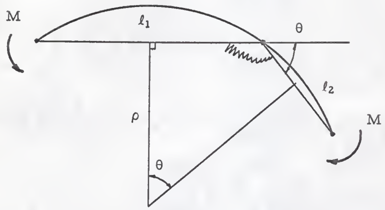
$$\frac{B}{\rho} = M = k_{n,2}\,\theta = k_{n,2}\,\frac{\ell_1 + \ell_2}{\rho\, 2} ,$$

from which

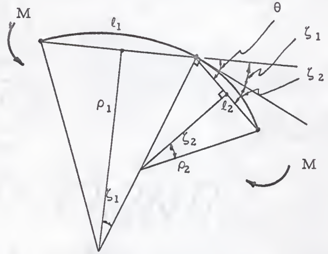
$$k_{n,2} = \frac{2\,B}{\ell_1 + \ell_2} , \qquad (119)$$

or

$$k_{n,2} = \frac{2\,\ell'_n\ k'_{n,2}}{\ell'_n + \ell'_{n+1}} . \qquad (120)$$



(c) Same rigidity, different length in adjacent elements.



(d) Different rigidity, different length in adjacent elements.

Fig. 7. Derivation of the spring constant for bending.

If the stiffnesses of the two links are different, then from Fig. 7d,

$$\frac{B_1}{\rho_1} = M = k_1 \zeta_1 ,$$

$$\frac{B_2}{\rho_2} = M = k_2 \zeta_2$$

and $$M = k\theta = k(\zeta_1 + \zeta_2),$$

from which

$$k_2 \zeta_2 = k_1 \zeta_1 = k(\zeta_1 + \zeta_2),$$

or $$k_1 = k\left(1 + \frac{\zeta_2}{\zeta_1}\right) \text{ and } k_2 = k\left(\frac{\zeta_1}{\zeta_2} + 1\right).$$

Since

$$\frac{\zeta_1}{\zeta_2} = \frac{k_2}{k_1} .$$

Substitution yields

$$k_1 = k\left(1 + \frac{k_1}{k_2}\right)$$

or

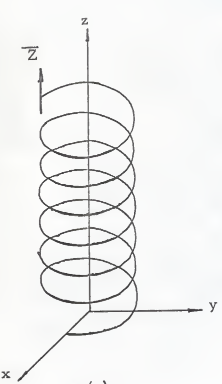
$$k = \frac{k_1 k_2}{k_1 + k_2} . \tag{121}$$

If $k_2 \to \infty$,

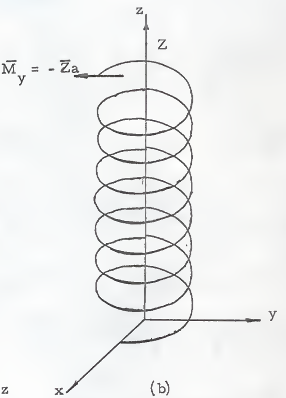
$$\lim_{k_2 \to \infty} k = k_1 . \tag{122}$$

<u>Constants for Tension.</u> The loading condition as shown in Fig. 8a can be replaced by that as shown in Fig. 8b. Let δ_z'' represent the displacement of the free end along the z direction caused by the eccentric load Z; and δ_z' that caused by M_y. Then the displacement due to the center load Z is

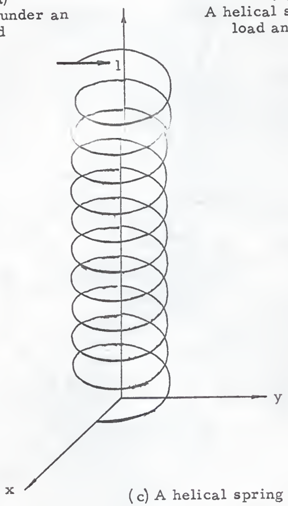
$$\delta_z = \delta_z'' - \delta_z' .$$



(a)
A helical spring under an eccentric load



(b)
A helical spring under a central load and a bending moment



(c) A helical spring under a unit bending shear

Fig. 8. Derivation of the spring constant for tension.

For an eccentric load, as shown in Fig 8,

$$\overline{M}_x = - Z_y \hat{i}, \quad \overline{M}_y = - Z(a-x)\hat{j}, \quad \overline{M}_z = 0.$$

Since $y = a \sin t$, $x = a \cos t$, and $z = a\, t \tan\alpha$,

$$M_t = \overline{M}\cdot\hat{T} = \left[- Z_y \hat{i} - Z(a-x)\hat{j}\right]\cdot\hat{T} = (Z a \cos\alpha - Za \cos t \cos\alpha), \tag{123}$$

$$M_n = \overline{M}\cdot\hat{N} = \left[-Z a \sin t\,\hat{i} - Z(a - a\cos t)\hat{j}\right]\cdot\hat{N} = Z a \sin t, \tag{124}$$

and
$$M_b = \overline{M}\cdot\hat{B} = \left[-Z a \sin t\,\hat{i} - Z a(1-\cos t)\hat{j}\right]\cdot\overline{B}$$
$$= (-Z a \sin\alpha + Z a \sin\alpha \cos t). \tag{125}$$

It follows that

$$\delta^{*}_z = \int_0^{2\pi n}\left(\frac{M_n \frac{\partial M_n}{\partial Z}}{EI} + \frac{M_b \frac{\partial M_b}{\partial Z}}{EI} + \frac{M_t \frac{\partial M_t}{\partial Z}}{GI_p}\right) a \sec\alpha\, dt$$

$$= \int_0^{2\pi n}\left[\frac{(Z a \sin t)(a \sin t)}{EI} + \frac{(-Z a \sin\alpha + Z a \sin\alpha \cos t)(-a \sin\alpha + a \sin\alpha \cos t)}{EI} + \frac{Z a \cos\alpha\,(1-\cos t)\, a\cos\alpha\,(1-\cos t)}{GI_p}\right] a \sec\alpha\, dt$$

$$= Z a^3 \sec\alpha \int_0^{2n\pi}\left(\frac{\sin^2 t}{EI} + \frac{\sin^2\alpha\,(\cos t - 1)^2}{EI} + \frac{\cos^2\alpha\,(1-\cos t)^2}{GI_p}\right) dt$$

$$= Za^3 \sec\alpha\left[\frac{n\pi}{EI} + \left(\frac{\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p}\right)\int_0^{2\pi n}(\cos^2 t - 2\cos t + 1)\, dt\right]$$

$$= Z a^3 \sec\alpha\left[\frac{n\pi}{EI} + \left(\frac{\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p}\right)(n\pi + 2n\pi)\right],$$

from which

$$\delta^{*}_z = Z a^3 \sec\alpha\left[\frac{n\pi(1+3\sin^2\alpha)}{EI} + \frac{3n\pi\cos^2\alpha}{GI_p}\right]. \tag{126}$$

The angular displacement of the free end about the y axis caused by Z is θ_y, which can be determined by the unit dummy load method [3] (See Fig. 8c).

The moments about the x, y and z axes caused by the unit load as shown in Fig. 8c, are represented by m_x, m_y, and m_z.

Now,

$$m_x = 0, \qquad m_y = 1, \quad \text{and} \quad m_z = 0.$$

Thus $\hat{m} = \hat{j}$.

It follows that

$$m_n = \hat{j} \cdot \hat{N} = -\sin t,$$

$$m_t = \hat{j} \cdot \hat{T} = \cos t \cos \alpha,$$

and

$$m_b = \hat{j} \cdot \hat{B} = -\sin \alpha \cos t.$$

Then

$$\theta_y = \int_0^{2n\pi} \left(\frac{M_n m_n}{GI} + \frac{M_b m_b}{GI} + \frac{M_t m_t}{GI_p} \right) a \sec \alpha \, dt$$

$$= \int_0^{2n\pi} \left[\frac{Z a \sin t}{EI} (-\sin t) + \frac{1}{EI}(-Z a \sin\alpha + Z a \sin\alpha \cos t)(-\sin\alpha \cos t) \right.$$

$$\left. + \frac{1}{GI_p}(Z a \cos\alpha - Z a \cos t \cos\alpha) \cos t \cos\alpha \right] a \sec \alpha \, dt$$

$$= Za^2 \sec \alpha \int_0^{2n\pi} \left[\frac{-\sin^2 t}{EI} + \frac{\sin^2\alpha}{EI}(\cos t - \cos^2 t) + \frac{\cos^2\alpha}{GI_p}(\cos t - \cos^2 t) \right] dt$$

$$= Z a^2 \sec \alpha \left[\left(-\frac{n\pi}{EI} \right) + \left(\frac{\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p} \right) (-n\pi) \right],$$

and finally

$$\theta_y = -n\pi Z a^2 \sec\alpha \left[\frac{1}{EI} + \left(\frac{\sin^2 \alpha}{EI} + \frac{\cos^2 \alpha}{GI_p} \right) \right]. \tag{127}$$

From Maxwell's reciprocal theorem, the displacement δ_z' can be obtained by replacing Z in Eq. (127) by M_y which equals $-Za$. Thus,

$$\delta'_z = -n\pi M_y a^2 \sec\alpha \left(\frac{1}{EI} + \frac{\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p} \right). \tag{128}$$

From Eqs. (126) and (128), the displacement caused by the central load Z is

$$\delta_z = \delta''_z - \delta'_z = 2\, Z a^3 \sec\alpha \; n\pi \left(\frac{\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p} \right).$$

Now,

$$k_{1,0} = \frac{Z}{\delta_z} = \frac{\cos\alpha}{2\, a^3 n\, \pi \left(\frac{\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p} \right)} . \tag{129}$$

This constant changes with length, and is given by

$$k_{n,1} = \frac{\ell_n}{\ell'_n}\, k_{1,0} \tag{130}$$

for stretched springs.

Constants for Shear. The displacement in the x direction of the free end caused by the load X, as shown in Fig. 9, contains two parts, the displacement caused by bending, and that caused by shear. Let δ, δ_1, and δ_2 represent the displacements caused by shear, moment, and the total displacement, respectively.

Then, $\delta = \delta_2 - \delta_1$.

The displacement caused by bending is (See Timoshenko [6])

$$\delta_1 = \frac{X\ell^3}{3B} , \tag{131}$$

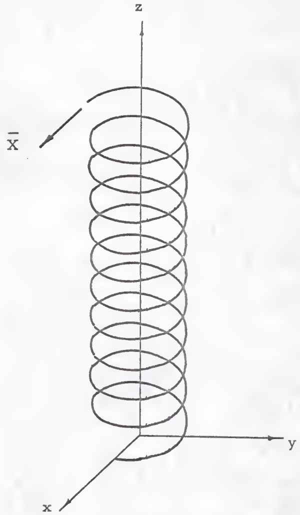


Fig. 9. Derivation of the spring constant for shear.

$$\text{where } B = \frac{\sin\alpha}{\left(\dfrac{1+\sin^2\alpha}{2EI} + \dfrac{\cos^2\alpha}{2GI_p}\right)} \tag{132}$$

and

$$\ell = 2n\pi a \tan\alpha . \tag{133}$$

Substitution of Eqs. (132) and (133) into Eq. (131) yields

$$\delta_1 = \frac{8 \ X a^3 n^3 \pi^3 \tan^3\alpha}{3 \sin\alpha}\left(\frac{1+\sin^2\alpha}{2EI} + \frac{\cos^2\alpha}{2GI_p}\right),$$

or

$$\delta_1 = \frac{4}{3} Z n^3 \pi^3 a^3 \tan^2\alpha \ \sec\alpha \left(\frac{1+\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p}.\right) . \tag{134}$$

The total displacement caused by the load X, as shown in Fig. 9, is found by the energy method as before.

Since

$$M_x = 0,$$

$$M_y = X(2an\pi\tan\alpha - at\tan\alpha),$$

and

$$M_z = Z a \sin t,$$

$$\overline{M} = X(2an\pi\tan\alpha - a t \tan\alpha)\hat{j} + X a \sin t \hat{k}, \tag{135}$$

from which

$$M_t = \overline{M} \cdot \hat{T} = \frac{X a \tan\alpha}{\sec\alpha}(2n\pi\cos t - t\cos t + \sin t)$$

or

$$M_t = X a \sin\alpha \ (2n\pi\cos t - t\cos t + \sin t); \tag{136}$$

$$M_n = \overline{M} \cdot \hat{N} = - X a \tan\alpha \ (2n\pi - t)\sin t; \tag{137}$$

$$M_b = \overline{M} \cdot \hat{B} = - X a \tan\alpha \sin\alpha \cos t (2n\pi - t) + X a \sin t \cos\alpha; \tag{138}$$

and

$$\delta_2 = \int_0^{2n\pi} \left\{ \frac{Xa^2\tan^2\alpha\ (2n\pi - t)^2\sin^2 t}{EI} + \frac{a^2\left[-\tan\alpha\sin\alpha\cos t(2n\pi - t) + \sin t\cos\alpha\right]^2}{EI} + \frac{Xa^2\sin^2\alpha\ (2n\pi\cos t - t\cos t + \sin t)^2}{GI_p} \right\} a\sec\alpha\ dt$$

$$= Xa^3\sec\alpha\ (I_1 + I_2 + I_3).$$

This can be written as

$$\delta_2 = Xa^3\sec\alpha \left\{ \frac{\tan^2\alpha}{EI}\left(\frac{4}{3}n^3\pi^3 - \frac{n\pi}{2}\right) + \frac{1}{EI}\left[\left(\frac{4}{3}n^3\pi^3 + \frac{n\pi}{2}\right) + \tan^2\alpha\sin^2\alpha - n\pi\sin^2\alpha + n\pi\cos^2\alpha\right] + \frac{\sin^2\alpha}{GI_p}\left(\frac{4}{3}n^3\pi^3 + \frac{5n\pi}{2}\right)\right\}. \qquad (139)$$

(See Appendix II.)

Finally, the displacement due to shear can be given as

$$\delta = \delta_2 - \delta_1 = Xa^3\sec\alpha\left[\frac{\tan^2\alpha}{EI}\left(-\frac{n\pi}{2}\right) + \frac{1}{EI}\left(\frac{n\pi}{2}\tan^2\alpha\sin^2\alpha - n\pi\sin^2\alpha + n\pi\cos^2\alpha\right) + \frac{\sin^2\alpha}{GI_p}\ \frac{5n\pi}{2}\right] \qquad (140)$$

or

$$\delta = Xa^3\sec\alpha\left[\frac{n\pi}{2}\ \frac{\tan^2\alpha}{EI}\ (-\cos^2\alpha) + \frac{n\pi}{EI}(\cos^2\alpha - \sin^2\alpha) + \frac{\sin^2\alpha}{GI_p}\ \frac{5n\pi}{2}\right],$$

which reduces to

$$\delta = Xa^3\sec\alpha\left(\frac{-3n\pi\sin^2\alpha}{2EI} + \frac{n\pi}{EI}\cos^2\alpha + \frac{\sin^2\alpha}{GI_p}\ \frac{5n\pi}{2}\right). \qquad (140a)$$

The shear angle γ is

$$\gamma = \frac{\delta}{\ell} = \frac{Xa^3\sec\alpha}{2n\pi a\tan\alpha}\left(\frac{-3n\pi}{2GI}\sin^2\alpha + \frac{n\pi}{EI}\cos^2\alpha + \frac{\sin^2\alpha}{GI_p}\ \frac{5n\pi}{2}\right),$$

or

$$\gamma = \frac{Xa^3}{2\sin\alpha}\left[\frac{1}{EI}\left(\cos^2\alpha - \frac{3}{2}\sin^2\alpha\right) + \frac{\sin^2\alpha}{GI_p}\ \frac{5}{2}\right].$$

Hence,

$$k_3 = \frac{X}{\gamma} = \frac{2\sin\alpha}{a^2\left[\frac{1}{EI}\left(\cos^2\alpha - \frac{3}{2}\sin^2\alpha\right) + \frac{\sin^2\alpha}{GI_p}\ \frac{5}{2}\right]}. \qquad (141)$$

If α is small, (say $\alpha < 10^{\circ}$), approximately $\cos^2\alpha = 1$ and $\sin^2\alpha = 0$.

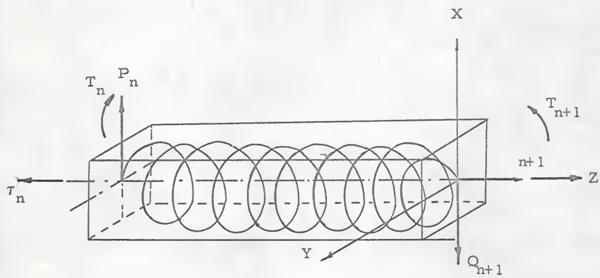
In this case,

$$k_{3,0} = \frac{2 \sin\alpha \ EI}{a^2 \cos \alpha} = \frac{2 \tan \alpha \ EI}{a^2} = \frac{\ell \ EI}{a^3 n \pi}. \qquad (141a)$$

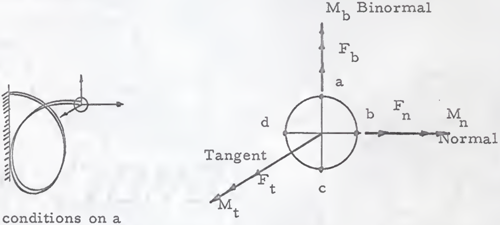
Under stretching, $$k_{n,3} = \frac{\ell'_n}{\ell_n} k_{3,0} \cdot \qquad (142)$$

STRESS ANALYSIS

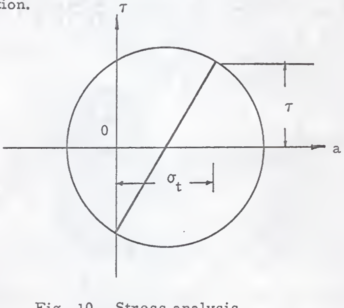
It is very difficult to determine exactly where and how large the maximum stresses are in an element with the loading condition as shown in Fig. 10a. However, they can be approximated without difficulty under the assumption that T_{n+1} is greater than T_n (this assumption causes no loss of generality), that shear forces are perpendicular to the axis of the element, and that a right-hand coordinate system is attached to the right end of the element, as shown in Fig. 10a. It is reasonable to expect that the maximum stresses happen at a section between $t = 0$ and $t = 2\pi$ because M_y is larger at this section than at any other section, while other moments and forces may be the same. One of the sections at $t = 0$, $\frac{-\pi}{2}$, $-\pi$ and $\frac{-3\pi}{2}$ is probably the critical section because at least one of the moments M_x, M_y, and M_z is maximum or minimum (which implies a greatest negative value). For a critical section, one of the points a, b, c, and d, as shown in Fig. 10b, is probably the critical point because the section is circular and at



(a) Loading conditions on a spring element.



(b) Loading conditions on a cross section.



(c) Mohr's circle

Fig. 10. Stress analysis.

least the stress component due to one of the forces and moments (See Fig. 10b) is maximum. (See Timoshenko [6].)

Critical Section

Case I. $t = 0$

$$M_z = M_x = 0$$

$$M_y = -\tau_n a - T_{n+1}$$

Hence,

$$\overline{M} = (-\tau_n a - T_{n+1})\,\hat{j}, \tag{143}$$

and

$$\overline{F} = P_n\,\hat{i} - \tau_n\,\hat{k}. \tag{144}$$

From Eqs. (107), (108), and (109), (143), (144), the following equations are obtained.

$$M_t = \overline{M}\cdot\hat{T} = (-\tau_n a - T_{n+1})\,\hat{j}.\left(\frac{-\sin t\,\hat{i} + \cos t\,\hat{j} + \tan\alpha\,\hat{k}}{\sec\alpha}\right)$$

or

$$M_t = -(\tau_n a + T_{n+1})\frac{\cos t}{\sec\alpha} = -(\tau_n a + T_{n+1})\cos\alpha; \tag{145}$$

$$M_n = \overline{M}\cdot\hat{N} = -(\tau_n a + T_{n+1})\,\hat{j}\cdot\left[-(\cos t\,\hat{i} + \sin t\,\hat{j})\right] = 0; \tag{146}$$

$$M_b = \overline{M}.\hat{B} = -(\tau_n a + T_{n+1})\,\hat{j}\cdot(\sin\alpha\,\sin t\,\hat{i} - \sin\alpha\cos t\,\hat{j} + \cos\alpha\,\hat{k})$$

$$= +(\tau_n a + T_{n+1})\sin\alpha\,\cos t,$$

or

$$M_b = (\tau_n a + T_{n+1})\sin\alpha; \tag{147}$$

$$F_t = \overline{F}\cdot\hat{T} = (P_n\,\hat{i} - \tau_n\,\hat{k}).\left(\frac{-\sin t\,\hat{i} + \cos t\,\hat{j} + \tan\alpha\,\hat{k}}{\sec\alpha}\right)$$

$$= -\frac{P_n\sin t}{\sec\alpha} - \tau_n\sin\alpha,$$

or

$$F_t = -\tau_n \sin\alpha \; ; \tag{148}$$

$$F_n = \overline{F}\cdot \hat{N} = (P_n \hat{i} - \tau_n \hat{k}) \cdot \left[-(\cos t\, \hat{i} + \sin t\, \hat{j})\right] = -P_n \cos t$$

or

$$F_n = -P_n \; ; \tag{149}$$

$$F_b = \overline{F}\cdot \hat{B} = (P_n \hat{i} - \tau_n \hat{k}) \cdot (\sin\alpha\, t\, \hat{i} - \sin\alpha\, \cos t\, \hat{j} + \hat{k}\cos\alpha)$$

$$= P_n \sin\alpha\, \sin t - \tau_n \cos\alpha$$

or

$$F_b = -\tau_n \cos\alpha \; . \tag{150}$$

Case II: $t = \frac{-\pi}{2}$,

$M_z = -P_n a$,

$M_x = -\tau_n a$,

and $M_y \cong -T_{n+1}$.

Hence,

$$\overline{M} = -\tau_n a\, \hat{i} - T_{n+1}\, \hat{j} - P_n a\, \hat{k} \tag{151}$$

and

$$\overline{F} = P_n \hat{i} - \tau_n \hat{k} . \tag{152}$$

Following the procedure in Case I,

$$M_t = \overline{M}\cdot \hat{T} = + \frac{\tau_n a \sin t}{\sec\alpha} - T_{n+1} \cos t - P_n a \sin\alpha$$

or

$$M_t = -\frac{\tau_n a}{\sec\alpha} - P_n a \sin\alpha \; ; \tag{153}$$

$$M_n = \overline{M}\cdot \overline{N} = + \tau_n a \cos t + T_{n+1} \sin t$$

or

$$M_n = -T_{n+1} \; ; \tag{154}$$

$$M_b = \bar{M} \cdot \bar{B} = -\tau_n\, a \sin\alpha \sin\tau + T_{n+1} \sin\alpha \cos t - P_n a \cos\alpha$$

or

$$M_b = \tau_n\, a \sin\alpha - P_n\, a \cos\alpha \; ; \qquad (155)$$

$$F_t = -\frac{P_n \sin t}{\sec\alpha} - \tau_n \sin\alpha = P_n \cos\alpha - \tau_n \sin\alpha ; \qquad (156)$$

$$F_n = -P_n \cos t = 0 ; \qquad (157)$$

and

$$F_n = P_n \sin\alpha \sin t - \tau_n \cos\alpha = -P_n \sin\alpha - \tau_n \cos\alpha . \qquad (158)$$

<u>Case III:</u> at $t = -\pi$

$$M_z = M_x = 0,$$

and

$$M_y \cong -\tau_n\, a + T_{n+1} .$$

Since the magnitude of M_y in this case is obviously smaller than that in Case I while other quantities are almost the same, this section is not likely the critical one.

<u>Case IV:</u> at $t = -\frac{3\pi}{2}$

$$M_z = P_n\, a ,$$

$$M_x = \tau_n\, a ,$$

and approximately

$$M_y \cong -T_{n+1} .$$

Hence

$$\bar{M} = \tau_n\, a\, \hat{i} - T_{n+1}\, \hat{j} + P_n a\, \hat{k} \qquad (159)$$

and

$$\bar{F} = P_n\, \hat{i} - \tau_n\, \hat{k} ; \qquad (160)$$

$$\bar{M}_t = -\frac{\tau_n a}{\sec\alpha} \sin t - T_{n+1} \cos t + P_n\, a \sin\alpha = -\frac{\tau_n a}{\sec\alpha} + P_n\, a \sin\alpha ; \qquad (161)$$

$$M_n = -\tau_n a \cos t + T_{n+1} \sin \tau = T_{n+1} ; \tag{162}$$

$$M_b = \tau_n a \sin\alpha \sin t + T_{n+1} \sin\alpha \cos t + P_n a \cos\alpha$$

$$= \tau_n a \sin\alpha + P_n a \cos\alpha ; \tag{163}$$

$$F_t = \frac{-P_n \sin t}{\sec\alpha} - \tau_n \sin\alpha = -P_n \cos\alpha - \tau_n \sin\alpha ; \tag{164}$$

$$F_n = - P_n \cos t = 0 ; \tag{165}$$

and

$$F_b = P_n \sin\alpha \sin t - \tau_n \cos\alpha = P_n \sin\alpha - \tau_n \cos\alpha . \tag{166}$$

Note that in all the equations of this section P_n, τ_n, and T_{n+1} are always positive.

Critical Point

In any section, if the six components of forces and moments, as shown in Fig. 10b, are known, the stress at any point can be computed easily. The maximum stresses are determined by a Mohr's circle construction [7]. (See Fig. 10c.) The shearing forces are assumed to be uniformly distributed over the cross section. The normal stress and shear stress are, following Langhaar [3]

$$\sigma_t = \frac{F_t}{A} + \frac{M_n B}{I_n} - \frac{M_b N}{I_b} , \tag{167}$$

$$\tau_{tb} = \frac{F_b}{A} , \tag{168}$$

$$\tau_{tn} = \frac{F_n}{A} , \tag{169}$$

and

$$\tau_t = \pm \frac{M_t r}{I_p} , \tag{170}$$

where r is the radius of the cross section of the spring. At b and c positive signs are used for τ_t, while at a and d the negative sign applies.

Case I: At point a the normal and shear stress are

$$\sigma_t = \frac{F_t}{A} + \frac{M_n r}{I_n} \tag{171}$$

and

$$\tau = -\frac{M_t r}{I_p} + \frac{F_n}{A}, \tag{172}$$

from which

$$\sigma_{\substack{max\\min}} = \frac{\frac{F_t}{A} + \frac{M_n r}{I_n}}{2} \pm \frac{1}{2}\sqrt{\left(\frac{F_t}{A} + \frac{M_n r}{I_n}\right)^2 + 4\left(\frac{M_t r}{I_p} - \frac{F_n}{A}\right)^2} \tag{173}$$

and

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\frac{F_t}{A} + \frac{M_n r}{I_n}\right)^2 + 4\left(\frac{M_t r}{I_p} - \frac{F_n}{A}\right)^2}. \tag{174}$$

Case II: At point b the stresses are

$$\sigma_t = \frac{F_t}{A} - \frac{M_b r}{I_b}, \tag{175}$$

$$\tau = \frac{M_t r}{I_p} + \frac{F_b}{A}, \tag{176}$$

$$\sigma_{\substack{max\\min}} = \frac{\frac{F_t}{A} - \frac{M_b r}{I_b}}{2} \pm \frac{1}{2}\sqrt{\left(\frac{F_t}{A} - \frac{M_b r}{I_b}\right)^2 + 4\left(\frac{M_t r}{I_p} + \frac{F_b}{A}\right)^2}, \tag{177}$$

and

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\frac{F_t}{A} - \frac{M_n r}{I_b}\right)^2 + 4\left(\frac{M_t r}{I_p} + \frac{F_b}{A}\right)^2}. \tag{178}$$

Case III: At point c the stresses are

$$\sigma_t = \frac{F_t}{A} - \frac{M_n r}{I_n}, \tag{179}$$

$$\tau = \frac{M_t r}{I_p} + \frac{F_n}{A}, \tag{180}$$

$$\sigma_{\substack{max\\min}} = \frac{\frac{F_t}{A} - \frac{M_n r}{I_n}}{2} \pm \frac{1}{2}\sqrt{\left(\frac{F_t}{A} - \frac{M_n r}{I_n}\right)^2 + 4\left(\frac{M_t r}{I_p} + \frac{F_n}{A}\right)^2}, \tag{181}$$

and

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\frac{F_t}{A} - \frac{M_n r}{I_n}\right)^2 + 4\left(\frac{M_t r}{I_p} + \frac{F_n}{A}\right)^2}. \tag{182}$$

Case IV: At point d the stresses are

$$\sigma_t = \frac{F_t}{A} + \frac{M_b r}{I_b}, \tag{183}$$

$$\tau = -\frac{M_t r}{I_p} + \frac{F_b}{A}, \tag{184}$$

$$\sigma_{\substack{max\\min}} = \frac{\frac{F_t}{A} + \frac{M_b r}{I_b}}{2} \pm \frac{1}{2}\sqrt{\left(\frac{F_t}{A} + \frac{M_b r}{I_b}\right)^2 + 4\left(\frac{M_t r}{I_p} - \frac{F_b}{A}\right)^2}, \tag{185}$$

and

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\frac{F_t}{A} + \frac{M_b r}{I_b}\right)^2 + 4\left(\frac{M_t r}{I_p} - \frac{F_b}{A}\right)^2}. \tag{186}$$

In general, the maximum stresses occur at point b of the section $t = 0$.

NUMERICAL EXAMPLE

A detailed example is shown in this section to illustrate solution of Basic Problem II for a given coil spring. This coil spring is used as an idler roller for conveyer belts, as manufactured by the J. B. Ehrsam Company of Enterprise, Kansas. A parabolic deflection curve is assumed. It is further assumed that only vertical external loads are applied at the pins.

Design Data:

Spring used (See Fig. 11) : 48 H-D

Span: 33"

Free length of idler: 54.25"

Unstretched length of spring: 36"

Pitch: $\frac{5}{8} + \frac{1}{8} = \frac{6}{8} = 0.75''$

Diameter of wire: $d = \frac{5}{8}'' = 0.625''$

Diameter of the spring coil:

(center to center) $(3\frac{7}{8} + \frac{5}{8}) = 4.5''$

(radius) $a = 2.25''$

Acting spring length* $36'' - 2 \times 2\frac{1}{2} \times \frac{6}{8} = 32.25''$

Acting number of coils $= 32.25 \times \frac{4}{3} = 43$

Rigid part of the idler at each end: $\frac{1}{2}(54.25 - 32.25) = 11''$

---

* Two and a half coils are taken out from each end of the spring according to the specification of the Ehrsam Company.

Trough angle β_o: 36^o

Maximum deflection: $(16.75 - 6 + 2.5) = 13.25''$

Moment of inertia $I = \frac{\pi d^4}{64}$

$I_p = \frac{\pi d^4}{32}$

Young's modulus: $E = 30 \times 10^6$ psi

Modulus of rigidity: $G = 11.5 \times 10^6$ psi

Deflection curve assumed: $(y + m\ell^2) = m(x - \ell)^2$
(See Fig. 11)

Total number of links used: $2N = 6.$

<u>Calculation of Spring Constants:</u> The pitch angle α can be found from

$$\alpha = \tan^{-1} \frac{p}{2\pi a} = \tan^{-1} \frac{0.75}{2\pi \times 2.25} = \tan^{-1} 0.053 = 3.04^o.$$

Since α is small, approximately $\sin^2 \alpha = 0$

$\cos^2 \alpha \cong \cos \alpha \cong 1.$

With the assumption of small pitch angle and the design data listed in Paragraph 1, Design Data, the three spring constants for tension, bending and shear of each element can be obtained from Eqs. (117), (129), and (141a).

Let the first element have 7.5 coils, and the second and the third element have 7 coils each. For the latter case

$$k_{10} = \frac{\cos\alpha}{2a^3 n_1 \pi\left(\frac{\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p}\right)} \cong \frac{G\,Ip}{2a^3 n_1 \pi},$$

$$k_{10} = \frac{11.5 \times 10^6 \times \frac{0.625^4}{32}\pi}{2 \times (2.25)^3 \times 7 \times \pi} = \frac{11.5 \times 10^6 \times 0.153}{2 \times 11.4 \times 7 \times 32} = 345.0;$$

$$k_{20} = \frac{\cos\alpha}{a\, n_1 \pi\left(\frac{1+\sin^2\alpha}{EI} + \frac{\cos^2\alpha}{GI_p}\right)} \cong \frac{1}{a\, n_1 \pi\left(\frac{1}{EI} + \frac{1}{GI_p}\right)}$$

$$= \frac{1}{2.25 \times 7 \times \pi \left(\frac{1}{30 \times 10^6 \times \frac{0.625^4}{64}} + \frac{1}{11.5 \times 10^6 \times \frac{0.625^4}{32} \times \pi} \right)} = 1980.0;$$

and

$$k_{30} = \frac{\ell_1 EI}{a^3 n \pi} = \frac{32.25 \times 30 \times 0.625^4 \times 10^6}{(2.25)^3 \times 43 \times 64} = \frac{32.25 \times 30 \times 0.153 \times 10^6}{11.4 \times 43 \times 64}$$

$$= 4680.0.$$

For the former case (7.5 coils)

$$k_{40} = \frac{7}{7.5} \times k_1 = 322.0$$

and

$$k_{50} = \frac{7}{7.5} \times k_2 = 1850.0$$

Let $R_1 = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$,

$R_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,

and

$R_3 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$,

and h_1, h_2, h_3, be the original length of the first, second and third elements, respectively. Then

$$k_{21} = k_1 = \frac{R_2}{h_2} k_{10} ,$$

$$k'_{22} = k_2 = \frac{R_2}{h_2} k_{20} ,$$

$$k_{23} = k_3 = \frac{R_2}{h_2} k_{30} ,$$

$$k_{11} = k_4 = \frac{R_1}{h_1} k_{40} ,$$

$$k'_{12} = k_5 = \frac{R_1}{h_1} k_{50} ,$$

$$k_{12} = k_6 = 2k_5 \frac{R_1}{R_1 + R_2} ,$$

$$k_{13} = k_7 = \frac{R_1}{h_1} k_{30} ,$$

$$k_{31} = k_8 = \frac{R_3}{h_3} k_{10} ,$$

$$k'_{32} = k_9 = \frac{R_3}{h_3} k_{20}$$

$$k_{33} = k_{10} = \frac{R_3}{h_3} k_{30} ,$$

and

$$k_{32} = k_{11} = 2 k_2 \frac{R_2}{R_2 + R_3} .$$

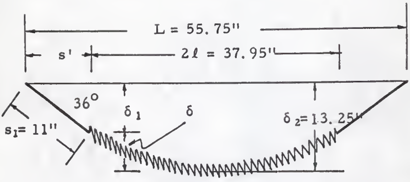
Calculation of Coordinates: The assumed deflection curve of the idler is shown in Fig. 11a. A rigid "can" is attached to each end of the spring to connect the spring to a bearing. The deflection curve of the spring is shown in Fig. 11b.

Because of this "can", and the assumed deflection curves,

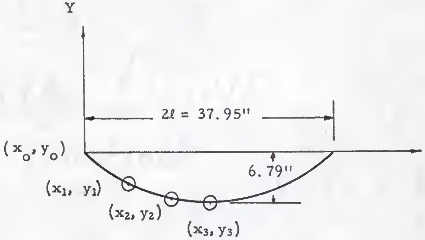
$$2\ell = L - 2s' = 55.75 - 22 \cos 36^\circ = 55.75 - 17.8 = 37.95''$$

and

$$\delta = \delta_2 - \delta_1 = 13.25 - 11 \sin 36^\circ = 13.25 - 6.46 = 6.79''$$



(a) The whole idler



(b) The spring part of the idler

Fig. 11. A deflection curve of 48HD spring idler.

The deflection curve is assumed to be parabolic, of the

form $\quad y + m\ell^2 = m(x - \ell)^2$, (187)

with the deflection $y = -6.79''$ at $x = \ = \ell = \frac{37.95}{2} = 18.96''$.

Thus,

$$m = \frac{6.79}{18.96^2} = \frac{6.79}{36\ 0} = 0.0188, \tag{188}$$

With x_1, x_2 arbitrary,

$$y_1 = m(x_1 - \ell)^2 - m\ell^2 \tag{189}$$

and

$$y_2 = m(x_2 - \ell)^2 - m\ell^2. \tag{190}$$

Note that $x_3 = \ell = 18.96''$ (191)

and $\quad y_3 = -6.79''$. (192)

<u>Calculation of Angles and Tensile Forces:</u>

$$\alpha_1 = \tan^{-1}\frac{y_1 - y_2}{x_2 - x_1} \tag{193}$$

$$\beta_1 = \tan^{-1}\frac{y_0 - y_1}{x_1 - x_0} \tag{194}$$

$$\alpha_2 = \tan^{-1}\frac{y_2 - y_3}{x_3 - x_2} \tag{195}$$

$$\beta_2 = \alpha_1 \tag{196}$$

$$\Theta_1 = k_4(R_1 - h_1) \tag{197}$$

$$\Theta_2 = \tau_1 = k_1(R_2 - h_2) \tag{198}$$

$$\tau_2 = k_8(R_3 - h_3). \tag{199}$$

Simultaneous Equations:

$$-F_1 + \Theta_1 \sin\beta_1 - \tau_1 \sin\alpha_1 + Q_1 \cos\left(\beta_1 - \frac{Q_1}{k_7}\right) - P_1 \cos\left(\alpha_1 - \frac{P_1}{k_3}\right) = 0$$

$$-H_1 - \Theta_1 \cos\beta_1 + \tau_1 \cos\alpha_1 + Q_1 \sin\left(\beta_1 - \frac{Q_1}{k_7}\right) - P_1 \sin\left(\alpha_1 - \frac{P_1}{k_3}\right) = 0$$

$$T_1 - k_6\left(\beta_1 - \alpha_1 + \frac{P_1}{k_3} - \frac{Q_1}{k_7}\right) = 0$$

$$p_1 - \frac{T_2 - T_1}{\cos\left(\frac{P_1}{k_3}\right)\sqrt{x_2 - x_1)^2 + (y_2 - y_1)^2}} = 0 \tag{200}$$

$$Q_1 - \frac{T_1 - T_0}{\cos\left(\frac{Q_1}{k_7}\right)\sqrt{x_1^2 + y_1^2}} = Q_1 - \frac{T_1 - k_5\left(\beta_0 - \alpha_0 + \frac{Q_1}{k_7}\right)}{\cos\left(\frac{Q_1}{k_7}\right)\sqrt{x_1^2 + y_1^2}} = 0$$

$$-F_2 + \Theta_2 \sin\beta_2 - \tau_2 \sin\alpha_2 + Q_2 \cos\left(\beta_2 - \frac{Q_2}{k_3}\right) - P_2 \cos\left(\alpha_2 - \frac{P_2}{k_{10}}\right) = 0$$

$$-H_2 - \Theta_2 \cos\beta_2 + \tau_2 \cos\alpha_2 + Q_2 \sin\left(\beta_2 - \frac{Q_2}{k_3}\right) - P_2 \sin\left(\alpha_2 - \frac{P_2}{k_{10}}\right) = 0$$

$$T_2 - k_{11}\left(\beta_2 - \alpha_2 + \frac{P_2}{k_{10}} - \frac{Q_2}{k_3}\right) = 0$$

$$P_2 + \frac{1}{\cos\left(\frac{P_2}{k_{10}}\right) R_3}\left[k_{11}\left(\beta_2 - \alpha_2 + \frac{P_2}{k_{10}} - \frac{Q_2}{k_3}\right) - k_9\left(2\alpha_2 + \frac{P_3}{k_{10}} - \frac{Q_3}{k_{10}}\right)\right] = 0$$

$$Q_2 - \frac{T_2 - T_1}{\cos\left(\frac{P_1}{k_3}\right)\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = 0$$

$$-F_3 + 2\tau_2 \sin\alpha_2 + Q_3 \cos\left(\alpha_2 - \frac{Q_3}{k_{10}}\right) - P_3 \cos\left(\alpha_2 + \frac{P_3}{k_{10}}\right) = 0$$

$$Q_3 \sin\left(\alpha_2 - \frac{Q_3}{k_{10}}\right) + P_3 \sin\left(\alpha_2 + \frac{P_3}{k_{10}}\right) = 0$$

$$T_3 - k_9\left(2\alpha_2 + \frac{P_3 - Q_3}{k_{10}}\right) = 0$$

$$Q_3 + \frac{1}{\cos\left(\frac{Q_3}{k_{10}}\right) R_3}\left[k_{11}(\beta_2 - \alpha_2) + \left(\frac{P_2}{k_{10}} - \frac{Q_2}{k_3}\right) - k_9\left(2\alpha_2 + \frac{P_3}{k_{10}} - \frac{Q_3}{k_{10}}\right)\right] = 0$$

$$F_0 - k_5\left(\beta_0 - \beta_1 + \frac{Q_1}{k_9}\right) / S_1 + Q_1 \cos(\beta_0 - \beta_1) - \Theta_1 \sin(\beta_0 - \beta_1) = 0$$

<u>Programming:</u> In order to adapt this set of equations for solution by computer, they are rewritten in the matrix form AX = B where X is a column matrix with the independent variables as elements, A is a square coefficient matrix and P is another column matrix with constants as elements. The following notation is used in the computer program which was constructed to solve the given set. The first subscript in Aij indicates the equation number, and the second subscript indicates the variable number.

$x_1 = P_1$ $\quad x_2 = P_2$ $\quad x_3 = P_3$ $\quad x_4 = Q_1$ $\quad x_5 = Q_2$ $\quad x_6 = Q_3$

$x_7 = T_1$ $\quad x_8 = T_2$ $\quad x_9 = T_3$ $\quad x_{10} = F_1$ $\quad x_{11} = F_2$ $\quad x_{12} = F_3$

$x_{13} = H_1$ $\quad x_{14} = H_2$ $\quad FSO = F_0 \cos\beta_0$

$A_{1,1}* = -\cos\left(\alpha_1 - \frac{P_1}{k_3}\right)$ $\qquad A_{1,4} = \cos\left(\beta_1 - \frac{Q_1}{k_7}\right)$

$A_{1,10} = -1$ $\qquad A_{2,1} = -\sin\left(\alpha_1 - \frac{P_1}{k_3}\right)$

$A_{2,4} = \sin\left(\beta_1 - \frac{Q_1}{k_7}\right)$ $\qquad A_{2,13} = -1$

$A_{3,1} = -\frac{k_6}{k_3}$ $\qquad A_{3,4} = \frac{k_6}{k_7}$ $\qquad A_{3,7} = 1$

$A_{4,1} = 1$ $\qquad A_{4,7} = \dfrac{1}{\cos\left(\frac{P_1}{k_3}\right)\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$

---

* only non-zero elements are listed here.

$A_{4,8} = -A_{4,7}$ $\quad$ $A_{5,4} = 1$ $\quad$ $A_{5,7} = -\dfrac{1}{\sqrt{x_1^2 + y_1^2}\ \cos(\frac{Q_1}{k_7})}$

$A_{6,2} = -\cos\left(\alpha_2 - \dfrac{P_2}{k_{10}}\right)$ $\quad$ $A_{6,5} = \cos\left(\beta_2 - \dfrac{Q_2}{k_3}\right)$

$A_{6,11} = -1$ $\quad$ $A_{7,2} = -\sin\left(\alpha_2 - \dfrac{P_2}{k_{10}}\right)$

$A_{7,5} = \sin\left(\beta_2 - \dfrac{Q_2}{k_3}\right)$ $\quad$ $A_{7,14} = -1$ $\quad$ $A_{8,2} = -\dfrac{k_{11}}{k_{10}}$

$A_{8,5} = \dfrac{k_{11}}{k_3}$ $\quad$ $A_{8,8} = 1$ $\quad$ $A_{9,2} = 1 + \dfrac{k_{11}}{k_{10}\ R_3\ \cos(\frac{P_2}{k_{10}})}$

$A_{9,3} = -\dfrac{k_{11}}{k_{10}\ R_3\ \cos(\frac{P_2}{k_{10}})}$ $\quad$ $A_{9,5} = -\dfrac{k_{11}}{k_3\ R_3\ \cos(\frac{P_2}{k_{10}})}$

$A_{9,6} = -A_{9,3}$ $\quad$ $A_{10,5} = 1$ $\quad$ $A_{10,7} = A_{4,7}$

$A_{10,8} = -A_{4,7}$ $\quad$ $A_{11,3} = -\cos\left(\alpha_2 + \dfrac{P_3}{k_{10}}\right)$

$A_{11,6} = \cos\left(\alpha_2 - \dfrac{Q_3}{k_{10}}\right)$ $\quad$ $A_{11,12} = -1$

$A_{12,13} = \sin\left(\alpha_2 + \dfrac{P_3}{k_{10}}\right)$ $\quad$ $A_{12,6} = \sin\left(\alpha_2 - \dfrac{Q_3}{k_{10}}\right)$

$A_{13,3} = -\dfrac{k_9}{k_{10}}$ $\quad$ $A_{13,6} = -A_{13,3}$ $\quad$ $A_{13,9} = 1$

$A_{14,2} = \dfrac{k_{11}}{k_{10}\ R_3\ \cos\left(\frac{Q_3}{k_{10}}\right)}$ $\quad$ $A_{14,3} = -A_{14,2}$

$A_{14,5} = - A_{14,2}$ $\qquad$ $A_{14,6} = 1 + A_{14,2}$

$A_{1,15}^{*} = \tau_1 \sin \alpha_1 - \quad \sin \beta_1$ $\qquad$ $A_{2,15} = \quad \cos \beta_1 - \tau_1 \cos \alpha_1$

$A_{3,15} = k_6 (\beta_1 - \alpha_1)$ $\qquad$ $A_{5,15} = - \dfrac{k_5 (\beta_0 - \beta_1 + \frac{Q_1}{k_7})}{\cos (\frac{Q_1}{k_7}) \sqrt{x_1^2 + y_1^2}}$

$A_{6,15} = \tau_2 \sin \alpha_2 - \quad \sin \beta_2$ $\qquad$ $A_{7,15} = \quad {}_2 \cos \beta_2 - \tau_2 \cos \alpha_2$

$A_{8,15} = k_{11} (\beta_2 - \alpha_2)$ $\qquad$ $A_{9,15} = \dfrac{- k_{11}(\beta_2 - \alpha_2) + 2k_9 \alpha_2}{R_3 \cos (\frac{P_2}{k_{10}})}$

$A_{11,15} = - 2 \tau_2 \sin \alpha_2$ $\qquad$ $A_{13,15} = 2 k_9 \alpha_2$

$A_{14,15} = A_{9,15}$

SL = ℓ $\qquad$ SM = m $\qquad$ CHI = h_1 $\qquad$ CH2 = $h_2 = h_3$ $\qquad$ CH3 = s_1

GKJ = K_j 0 $\qquad$ J = 1, 2, 3, 4, 5 $\qquad$ SN = n * *

If the spring is loaded laterally only, as is the case for idler rollers, such as considered in this example, the horizontal forces applied to the pins connecting the links should be zero.

---

* $A_{1,15}$ to $A_{14,15}$ are the elements in the P matrix.

** For a short span idler, a second degree parabolic deflection curve may turn out to be unrealistic (F_0 is upward). Therefore a fourth degree parabolic curve, $(y + m\ell^2 + n\ell^4) = m (x - \ell)^2 + n (x - \ell)^4$ should be used.

For arbitrarily selected positions of these pins along a given deflection curve, the horizontal forces at the pins will not be zero. They are given by the solutions of the fundamental set of equations, however, and once known indicate how the assumed positions should be changed to reduce the horizontal forces to zero.

The program developed for computation determines the correct positions of the pins by successive approximations, using the magnitudes of the horizontal forces as criteria for determining the changes in positions required to reduce these forces to zero.

In order to implement this, the program includes the following terms and their meanings for specification of errors and corrections to positions:

VADI: Error allowed for H_1 and H_2 to be different from zero.

VAD2: Shift in position of hinges to reduce H_1 and H_2 to be zero.

VAD3: Error allowed for P's and Q's between first iteration and second iteration.

VAD4 = β_o

Within this process of successive approximations, a minor iterative procedure is included to make use of the almost linear nature of the basic set of equations. Let $BJ = P_i$ and $CJ = Q_j$, where $J = 1, 2, 3$, in the arguments of the trigonometric functions which appear in the basic equations. This set is first solved with

BJ = CJ = 0, and the P's and Q's so obtained substituted into BJ and CJ, and the basic set solved again to obtain improved values.

The program for solving the set of equations and the results obtained can be found in Appendix 3.

The forces found for the Basic Problem II, for the data presented on page 49 are

$$F_0 = 60\ \# \qquad F_1 = 155.37\# \qquad F_2 = 138.53\# \qquad F_3 = 156.9\#.$$

The total load, including the weight of the spring itself, is

$$2 \times (60 + 155.37 + 138.53) + 156.9 = 864.7\ \#.$$

Other cases for the idler rollers designated 48HD with maximum displacement 9.75" and 4"; and the case for 24D with maximum displacement 8.57" (using a fourth parabolic curve for the deflection curve for this case) have also been worked out. The computed results * are shown in the following table.

Table 1. Comparison of theoretical and experimental loads.

Idler No.	Max. Def. δ_2	Calculated Load, lb.	Experimental Load - lbs.		
			Load Added	Weight	Total
48 HD	4.00" **	73.5	0	100	100
	9.75"	394.0	300	100	400
	13.25"	864.7	750	100	850***
24D	8.57"	232	200	30	230***

* The experimental results were furnished by the Ehrsam Company.

** The theory is very sensitive to the maximum deflection. The measurement of this quantity should be very accurate.

*** These are the maximum loads allowed to be taken by the respective idlers.

Stress Analysis: The stresses at the right end, i.e. $t = 0$, of each element are determined in this section to illustrate the stress analysis procedure. A computer program for this stress analysis is given in Appendix IV.

The forces and moments acting on each element, determined from results of the previous section, are shown in Fig. 12 a, b, and c.

The vector moment and vector force for the first element are given in the equations

$$\overline{M} = -(671 \times 2.5 + 466.4)\,\hat{j} = -(1675 + 466)\,\hat{j} = -2141\,\hat{j}$$

and

$$\overline{F} = 32.8\,\hat{i} - 671\,\hat{k}.$$

For the second element, the corresponding equations are

$$\overline{M} = -(599.3 \times 2.5 + 556.8)\,\hat{j} = -(1480 + 557)\,\hat{j} = -2037\,\hat{j}$$

and

$$\overline{F} = 13.6\,\hat{i} - 599.3\,\hat{k}.$$

For the third element they are

$$\overline{M} = -(562.7 \times 2.5 + 611.4)\,\hat{j} = -(1400 + 611)\,\hat{j} = -2011\,\hat{j}$$

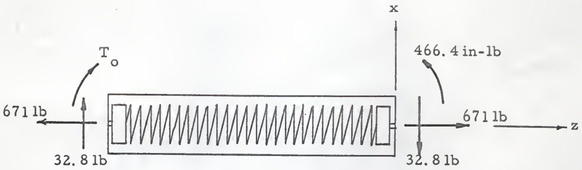
and

$$\overline{F} = 8.3\,\hat{i} - 562.7\,\hat{k}.$$

The formulas for the maximum stresses are obtained by substituting the expressions for I_p, I and A into Eqs. (173) through (186).

At point a (See Fig. 10b)

$$\sigma_{\substack{max\\min}} = \frac{\dfrac{F_t}{A} + \dfrac{M_n r}{I_n}}{2} \pm \frac{1}{2}\sqrt{\left(\frac{F_t}{A} + \frac{M_n r}{I_n}\right)^2 + 4\left(\frac{M_t r}{I_p} - \frac{F_n}{A}\right)^2}$$



(a) First element

(b) Second element

(c) Third element

Fig. 12. Loads on various elements.

$$= \frac{1}{2}\left[\left(\frac{4F_t}{\pi d^2} + \frac{32M_n}{\pi d^3} + \sqrt{\left(\frac{4F_t}{\pi d^2} + \frac{32M_n}{\pi d^3}\right)^2 + \left(\frac{32M_t}{\pi d^3} - \frac{4F_n}{\pi d^2}\right)^2}\right]\right.$$

or

$$\sigma_{\substack{max\\min}} = \frac{16}{\pi d^3}\left[\left(\frac{F_t d}{8} + M_n\right) + \sqrt{\left(\frac{F_t d}{8} + M_n\right)^2 + \left(M_t - \frac{F_n d}{8}\right)^2}\right]$$

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{\left(\frac{F_t d}{8} + M_n\right)^2 + \left(M_t - \frac{F_n d}{8}\right)^2}\,. \tag{201}$$

At point b,

$$\sigma_{\substack{max\\min}} = \frac{16}{\pi d^3}\left[\left(\frac{F_t d}{8} - M_b\right) \pm \sqrt{\left(\frac{F_t d}{8} - M_b\right)^2 + \left(M_t + \frac{F_b d}{8}\right)^2}\right]$$

and

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{\left(\frac{F_t d}{8} - M_b\right)^2 + \left(M_t + \frac{F_b d}{8}\right)^2}. \tag{202}$$

At point c

$$\sigma_{\substack{max\\min}} = \frac{16}{\pi d^3}\left[\left(\frac{F_t d}{8} - M_n\right) \pm \sqrt{\left(\frac{F_t d}{8} - M_n\right)^2 + \left(M_t + \frac{F_n d}{8}\right)^2}\right]$$

and

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{\left(\frac{F_t d}{8} - M_n\right)^2 + \left(M_t + \frac{F_n d}{8}\right)^2}. \tag{203}$$

At point d

$$\sigma_{\substack{max\\min}} = \frac{16}{\pi d^3}\left[\left(\frac{F_t d}{8} + M_b\right) \pm \sqrt{\left(\frac{F_t d}{8} + M_b\right)^2 + \left(M_t - \frac{F_b d}{8}\right)^2}\right]$$

and

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{\left(\frac{F_t d}{8} + M_b\right)^2 + \left(M_t - \frac{F_b d}{8}\right)^2}\,. \tag{204}$$

In the computer program given in Appendix IV the following notations are used:

$A_1 = M_i$	$A_2 = M_j$	$A_3 = M_k$
$B_1 = F_i$	$B_2 = F_j$	$B_3 = F_k$

D = d (diameter of wire)

c = α (pitch angle)	$T_1 = t$
FMOT = M_t	FMON = M_n
FMOB = M_b	FFOT = F_t
FFON = F_n	FFUB = F_b

The stresses found for idler roller 48HD under 850 lb load are given in Table 2.

Table 2. Stress analysis for 48HD idler roller under 850 pounds vertical load.

Critical Points	First Element			Second Element			Third Element		
	Tensile Stress*	Compres-sive Stress	Shearing Stress	Tensile Stress*	Compres-sive Stress	Shearing Stress	Tensile Stress*	Compres-sive Stress	Shearing Stress
a	44467.3	-44583.5	44525.4	42359.1	-42463.0	42411.0	41829.3	-41926.8	41878.0
b	43304.4	-48166.5	45735.5	41159.3	-45780.6	43469.9	40589.3	-45146.6	42867.9
c	44574.2	-44690.4	44632.3	42403.5	-42507.3	42455.4	41856.4	-41953.9	41905.1
d	45863.2	-41233.5	43548.4	43723.4	-39309.7	41516.5	43215.0	-38852.6	41033.8

* All stresses are given in psi.

The maximum stresses, therefore, are

$$\sigma = -48166.5 \text{ psi}$$

$$\tau = 45735.5 \text{ psi}$$

They occur at point b of the first element.

Table 3. Stress analysis for 24D idler roller under 230 pounds vertical load.

Critical Points	First Element			Second Element			Third Element		
	Tensile Stress*	Compres-sive Stress	Shearing Stress	Tensile Stress*	Compres-sive Stress	Shearing Stress	Tensile Stress*	Compres-sive Stress	Shearing Stress
a	27653.3	-27704.6	27678.9	31611.0	-31657.7	31634.4	32186.6	-32226.9	32206.8
b	27071.1	-29609.7	28340.4	30811.2	-33700.5	32255.8	31267.3	-34199.5	32733.4
c	27774.6	-27825.9	27800.3	31747.0	-31793.8	31770.4	32275.3	-32315.6	32295.4
d	28412.5	-25976.5	27194.5	32610.6	-29814.7	31212.6	33259.3	-30407.8	31833.6

* All stresses are given in psi.

The maximum stresses are

$\sigma = - 34199.5$ psi

$\tau = 32733.4$ psi

They occur at point b of the third element.

CONCLUSIONS

From Table 1, it is seen that the theoretical values check very well with the experimental ones, especially when the laterally-loaded springs are under heavy loads. The theory is very sensitive to maximum deflections. Under a heavy load, half an inch error in measurement may have only a little effect on the final results, while under a light load it may cause a great error, for in the latter case half an inch is probably 20 per cent of the maximum deflection.

In the numerical example given in the preceding section, the deflection curve of the spring idler has been replaced by six elements and symmetry been assumed. This has been done only to shorten the computing time. The theory can be used equally well for a more general case. The program given in Appendix I, however, can be used only for symmetrical deflections.

The time required for solution of the basic set of simultaneous equations depends greatly on the accuracy of the first guess of the pin positions along the assumed deflection curve. By printing out outputs of the first round and noting values of H_1 and H_2, the first guess can be adjusted to make the positions of the pins closer to the correct values.

ACKNOWLEDGMENT

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REFERENCES

[1] Frisch-Fay, R. Flexible Bars. Washington: Butterworth, 1962.

[2] Hodgman, C. D. Handbook of Mathematical Tables. Cleveland: Chemical Rubber Pub. Co., 1962.

[3] Langhaar, H. L. Energy Method in Applied Mechanics. New York: John Wiley and Sons, 1962.

[4] Lass, H. Vector and Tensor Analysis. New York: McGraw-Hill Book Co., 1950.

[5] Taylor, A. E. Advanced Calculus. Boston: Ginn and Co., 1955.

[6] Timoshenko, S. Strength of Materials - Parts I and II, 3d ed., New York: Van Nostrand Co., 1956.

[7] Timoshenko, S. and J. N. Goodier. Theory of Elasticity, 2d ed., New York: McGraw-Hill Book Co., 1951.

APPENDICES

APPENDIX I

Deviation of Unit Vectors along Tangent, Normal and Binormal Directions at Any Point on the Spiral Curve

The spiral curve, as shown in Fig. 12a, is described by the following parametric equations:

$$x = a \cos t$$
$$y = a \sin t \tag{A1}$$
$$z = a\, t \tan \alpha$$

where α is the pitch angle.

Then, the position vector $\overline{R}$ is

$$\overline{R} = a \cos t\, \hat{i} + a \sin t\, \hat{j} + a\, t \tan \alpha\, \hat{k} \tag{A2}$$

and the arc length

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = a \sec \alpha \; dt, \tag{A3}$$

where i, j, k are unit vectors along the x, y, and z axis, respectively.

From (As) and (A3) (See Lass |4|)

$$\overline{T} = \frac{d\overline{R}}{ds} = \frac{\frac{d\overline{R}}{dt}}{\frac{ds}{dt}} = \frac{-a \sin t\, \hat{i} + a \cos t\, \hat{j} + a \tan \alpha\, \hat{k}}{a \sec \alpha} \tag{A4}$$

From (A4) the curvature κ is

$$\kappa = \left| \frac{d\overline{T}}{ds} \right| = \left| \frac{\frac{d\overline{T}}{dt}}{\frac{ds}{dt}} \right| = \left| \frac{-a \cos t\, \hat{i} - a \sin t\, \hat{j}}{a^2 \sec^2 \alpha} \right| = \frac{1}{a \sec^2 \alpha}. \tag{A5}$$

The following relationship can be used to find $\overline{N}$.

$$\frac{d\overline{T}}{ds} = \kappa \overline{N} = \frac{1}{a \sec^2 \alpha} \overline{N} = \frac{(\cos t \hat{i} + \sin t \hat{j})}{a \sec^2 \alpha}, \tag{A6}$$

whence,
$$\overline{N} = -(\cos t \hat{i} + \sin t \hat{j}). \tag{A7}$$

$\overline{B}$ is obtained from the definition (See Lass [7]),

$$\overline{B} = \overline{T} \times \overline{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-\sin t}{\sec \alpha} & \frac{\cos t}{\sec \alpha} & \frac{\tan \alpha}{\sec \alpha} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \hat{i}\left(\frac{\tan \alpha \ \sin t}{\sec \alpha}\right)$$

$$-\hat{j}\left(\frac{\tan \alpha \ \cos t}{\sec \alpha}\right) + \hat{k} \cos \alpha = \hat{i}(\sin \alpha \ \sin t) - \hat{j} \sin \alpha \ \cos t + \hat{k} \cos \alpha \tag{A8}$$

Collecting Eqs. (A4), (A7), and (A8) gives

$$\overline{T} = \frac{-\sin t \hat{i} + \cos t \hat{j} + \tan \alpha \hat{k}}{\sec \alpha},$$

$$\overline{N} = -(\cos t \hat{i} + \sin t \hat{j}), \tag{A9}$$

and

$$\overline{B} = \hat{i}(\sin \alpha \ \sin t) - \hat{j} \sin \alpha \ \cos t + \hat{k} \cos \alpha .$$

APPENDIX II

Evaluation of Some Definite Integrals

$$I_1 = \int_0^{2n\pi} \frac{\tan^2\alpha}{EI} \; (2n\pi - t)^2 \sin^2 t \, dt$$

$$= \int_0^{2n\pi} \frac{\tan^2\alpha}{EI} \; \sin^2 t(4n^2\pi^2 - 4n\pi t + t^2)\, dt$$

$$= \frac{\tan^2\alpha}{EI}\left[4n^2 \pi^2 n\pi - 4n\pi \int_0^{2n\pi} \frac{t(1-\cos 2t)}{2} dt + \int_0^{2n\pi} t^2 \frac{1 - \cos 2t}{2} dt\right]$$

$$= \frac{\tan^2\alpha}{EI} \quad 4n^3\pi^3 - 4n\pi\left[\frac{t^2}{4} - \frac{1}{8}(\cos 2t + 2t \sin 2t)\right] + \frac{t^3}{6}$$

$$- \frac{1}{16}\left[4t\cos 2t + (4t^2 - 2)\sin 2t\right] \Bigg|_{t=0}^{t=2n\pi}$$

$$= \frac{\tan^2\alpha}{EI}\left[4n^3\pi^3 - 4n^3\pi^3 + \frac{8}{6}n^3\pi^3 - \frac{8n\pi}{16}\right] = \frac{\tan^2\alpha}{EI}\left(\frac{4}{3} n^3\pi^3 - \frac{n\pi}{2}\right)$$

$$I_2 = \int_0^{2n\pi} \frac{1}{EI}\left[- \tan\alpha \; \sin\alpha \; \cos t\,(2n\pi - t) + \sin t \cos\alpha\right]^2 dt$$

$$= \int_0^{2n\pi} \frac{1}{EI}\left[\tan^2\alpha \; \sin^2\alpha \; \cos^2 t\,(4n^2\pi^2 - 4n\pi t + t^2) - 2\tan\alpha \; \sin\alpha \; \cos\alpha\right.$$

$$\left. \times \sin t \cos t\,(2n\pi - t) + \sin^2 t \cos^2\alpha\right] dt$$

$$= \int_0^{2n\pi} \frac{1}{EI}\left[\tan^2\alpha \; \sin^2\alpha \; \frac{1+\cos 2t}{2}(4n^2\pi^2 - 4n\pi t + t^2) - \sin^2\alpha \; \sin^2 t(2n\pi - t)\right.$$

$$\left. + \sin^2 t \cos^2\alpha\right] dt$$

$$= \frac{1}{EI}\left\{\frac{\tan^2\alpha\ \sin^2\alpha}{2}\left(4n^2\pi^2 t - 2n\pi t^2 + \frac{t^3}{3} + \frac{4n^2\pi^2 \sin 2t}{2} - n\pi(\cos 2t + 2t\sin 2t)\right]\right.$$

$$+ \frac{1}{8}\left[4t\cos 2t + (4t^2-2)\sin 2t\right] - \sin^2\alpha\left[-\frac{2n\pi\cos 2t}{2} - \frac{1}{4}(\sin 2t - 2t\cos 2t)\right]$$

$$\left. + \cos^2\alpha\ \frac{t - \frac{1}{2}\sin 2t}{2}\right\}\Bigg|_{t=0}^{t=2n\pi}$$

$$I_2 = \frac{1}{EI}\left\{\frac{\tan^2\alpha\ \sin^2\alpha}{2}\left[8n^3\pi^3 - 8n^3\pi^3 + \frac{8n^3\pi^3}{3} + \frac{2n\pi}{2}\right] + \frac{\sin^2\alpha}{4}(-4n\pi) + \cos^2\alpha\, n\pi\right\}$$

$$= \frac{1}{EI}\left[\left(\frac{4n^3\pi^3}{3} + \frac{n\pi}{2}\right)\tan^2\alpha\ \sin^2\alpha + n\pi\cos^2\alpha - n\pi\sin^2\alpha\right]$$

$$= \frac{1}{EI}\left[\left(\frac{4n^3\pi^3}{3} + \frac{n\pi}{2}\right)\tan^2\alpha\ \sin^2\alpha + n\pi\cos 2\alpha\right]$$

$$I_3 = \int_0^{2n\pi}\frac{\sin^2\alpha}{GI_p}\left[\cos^2 t\,(2n\pi - t)^2 + 2\cos t\sin t\,(2n\pi - t) + \sin^2 t\right] dt$$

$$= \frac{\sin^2\alpha}{GI_p}\left[\frac{4}{3}n^3\pi^3 + \frac{n\pi}{2} - \frac{1}{4}(\sin 2t - 2t\cos 2t) + n\pi\right]_0^{2n\pi}$$

$$= \frac{\sin^2\alpha}{GI_p}\left[\frac{4}{3}n^3\pi^3 + \frac{n\pi}{2} + n\pi + n\pi\right]$$

$$= \frac{\sin^2\alpha}{GI_p}\left[\frac{4}{3}n^3\pi^3 + \frac{5}{2}n\pi\right].$$

APPENDIX III

Program for Solving Simultaneous Equations in Basic Problem II

```
C      NUMERICAL EXAMPLE FOR SPRING BEAM
    1 FORMAT (I3)
    3 FORMAT (I3,F14.6)
      DIMENSION A(14,15)
    4 FORMAT (4F14.6)
    5 FORMAT (5F14.6)
    6 FORMAT (5F14.6)
   41 FORMAT (5F14.6)
   42 FORMAT (3F14.6)
   43 FORMAT (4F14.6)
C*****ENTER DATA HERE
      READ 4, X0,X1,X2,X3
      READ 5, SL,SM,CH1,CH2,CH3
      READ 41, GK1,GK2,GK3, Y0,Y3
      READ 42, GK4,GK5,SN
      READ 43, VAD1,VAD2,VA3,VAD4
      READ 1,N
      N1=N+1
C*****EQUQTIONS BECOME LINEAR HERE
      B1=0.0
      C1=0.0
      B2=0.0
      C2=0.0
      B3=0.0
      C3=0.0
C*****LINE 9 TO 110 SUBPROGRAM FOR SOLVING SIMULTANEOUS LINEAR
    9 DO 10 I=1,N                                        EQUATIONS
      DO 10 J=1,N1
   10 A(I,J)=0.0
C*****ENTER EXPRESSIONS FOR GEOMETRIC RELATIONS HERE
      Y1=SM*(X1-SL)**2+SN*(X1-SL)**4-SM*SL**2-SN*SL**4
      Y2=SM*(X2-SL)**2+SN*(X2-SL)**4-SM*SL**2-SN*SL**4
      AL1=ATANF((Y1-Y2)/(X2-X1))
      BE1=ATANF((Y0-Y1)/(X1-X0))
      AL2=ATANF((Y2-Y3)/(X3-X2))
      BE2=AL1
      R1=SQRTF((X1-X0)**2+(Y1-Y0)**2)
      R2=SQRTF((X2-X1)**2+(Y2-Y1)**2)
      R3=SQRTF((X3-X2)**2+(Y3-Y2)**2)
C*****ENTER EXPRESSIONS FOR SPRING CONSTANTS HERE
```

```
      SK1=R2*GK1/CH2
      SK2=R2*GK2/CH2
      SK3=R2*GK3/CH2
      SK4=R1*GK4/CH1
      SK5=R1*GK5/CH1
      SK6=SK5*2.0*R1/(R1+R2)
      SK7=R1*GK3/CH1
      SK8=R3*GK1/CH2
      SK9=R3*GK2/CH2
      SK10=R3*GK3/CH2
      SK11=SK2*2.0*R2/(R2+R3)
C*****ENTER EXPRESSIONS FOR TENSILE FORCES HERE
      HT1=SK4*(SQRTF((X1-X0)**2+(Y1-Y0)**2)-CH1)
      TA1=SK1*(SQRTF((X2-X1)**2+(Y2-Y1)**2)-CH2)
      HT2=TA1
      TA2=SK8*(SQRTF((X3-X2)**2+(Y3-Y2)**2)-CH2)
C*****ENTER  EXPRESSIONS FOR COEFFICIENTS A IN THE MATRICE EQUA-
      A(1,1)=-COSF(AL1-B1/SK3)                             TIONS AX ≐ p
      A(1,4)=COSF(BE1-C1/SK7)
      A(1,10)=-1.0
      A(2,1)=-SINF(AL1-B1/SK3)
      A(2,4)=SINF(BE1-C1/SK7)
      A(2,13)=-1.0
      A(3,1)=-SK6/SK3
      A(3,4)=SK6/SK7
      A(3,7)=1.0
      A(4,1)=1.0
      A(4,7)=+1.0/(SQRTF((X2-X1)**2+(Y2-Y1)**2)*COSF(B1/SK3))
      A(4,8)=-A(4,7)
      A(5,4)=1.0
      A(5,7)=-1.0/(SQRTF(X1**2+Y1**2)*COSF(C1/SK7))
      A(6,2)=-COSF(AL2-B2/SK10)
      A(6,5)=COSF(BE2-C2/SK3)
      A(6,11)=-1.0
      A(7,2)=-SINF(AL2-B2/SK10)
      A(7,5)=SINF(BE2-C2/SK3)
      A(7,14)=-1.0
      A(8,2)=-SK11/SK10
      A(8,5)=SK11/SK3
      A(8,8)=1.0
      A(9,2)=1.0+SK11/(SK10*R3*COSF(B2/SK10))
      A(9,3)=-SK11/(SK10*R3*COSF(B2/SK10))
      A(9,5)=-SK11/(SK3*R3*COSF(B2/SK10))
      A(9,6)=-A(9,3)
      A(10,5)=1.0
```

```
      A(10,7)=A(4,7)
      A(10,8)=-A(4,7)
      A(11,3)=-COSF(AL2+B3/SK10)
      A(11,6)=COSF(AL2-C3/SK10)
      A(11,12)=-1.0
      A(12,3)=SINF(AL2+B3/SK10)
      A(12,6)=SINF(AL2-C3/SK10)
      A(13,3)=-SK9/SK10
      A(13,6)=-A(13,3)
      A(13,9)=1.0
      A(14,2)=SK11/(SK10*R3*COSF(C3/SK10))
      A(14,3)=-A(14,2)
      A(14,5)=-A(14,2)
      A(14,6)=1.0+A(14,2)
ENTER EXPRESSIONS FOR P IN THE MATRICE EQUATION AX=P
      A(1,15)=TA1*SINF(AL1)-HT1*SINF(BE1)
      A(2,15)=HT1*COSF(BE1)-TA1*COSF(AL1)
      A(3,15)=SK6*(BE1-AL1)
      A(5,15)=-SK5*(VAD4*3.14159/180.0-ATANF((Y0-Y1)/(X1-X0))+C1/SK7)
     C/(COSF(C1/SK7)*SQRTF(X1**2+Y1**2))
      A(6,15)=TA2*SINF(AL2)-HT2*SINF(BE2)
      A(7,15)=HT2*COSF(BE2)-TA2*COSF(AL2)
      A(8,15)=SK11*(BE2-AL2)
      A(9,15)=(-SK11*(BE2-AL2)+SK9*(2.0*AL2))/(R3*COSF(B2/SK10))
      A(11,15)=-2.0*TA2*SINF(AL2)
      A(13,15)=SK9*2.0*AL2
      A(14,15)=A(9,15)
   23 DO 100 J=1,N
      IF ( SENSE SWITCH 1 ) 25,26
   25 PRINT 1, J
   26 IF (SENSE SWITCH 2) 28,59
   28 IF (J-N) 29,59,59
   29 X = 0.0
      DO 40 I=J,N
      IF ( ABSF(X) - ABSF(A(I,J)) ) 32,40,40
   32 X = A(I,J)
      IC = I
   40 CONTINUE
      DO 50 K=J,N1
      X = A(IC,K)
      A(IC,K) = A(J,K)
   50 A(J,K) = X
   59 X = 1.0/A(J,J)
      DO 60 K=J,N1
```

```
   60 A(J,K) = A(J,K)*X
      DO 80 I=1,N
      IF (I-J) 65,80,65
   65 AIJ = - A(I,J)
      DO 70 K=J,N1
   70 A(I,K) = A(I,K) + AIJ*A(J,K)
   80 CONTINUE
  100 CONTINUE
      DO 110 I=1,N
  110 PUNCH 3, I, A(I,N1)
      IF (A(13,15)**2-VAD1) 140,140,149
  140 IF(A(14,15)**2-VAD1) 200,200,149
  149 IF (A(13,15)**2-A(14,15)**2) 150,150,151
  151 IF (A(13,15)) 152,150,153
C*****ADJUEST POSITIONS OF POINTS HERE
  152 X1=X1-VAD2
      GO TO 9
  153 X1=X1+VAD2
      GO TO 9
  150 IF (A(14,15)) 154,200,155
  154 X2=X2-VAD2
      GO TO 9
  155 X2=X2+VAD2
      GO TO 9
  200 IF (A(1,15)-B1-VA3) 210,210,220
  210 IF (A(4,15)-C1-VA3) 211,211,220
  211 IF (A(2,15)-B2-VA3) 212,212,220
  212 IF (A(5,15)-C2-VA3) 213,213,220
  213 IF (A(3,15)-B3-VA3) 214,214,220
  214 IF (A(6,15)-C3-VA3) 230,230,220
C*****ITERATIONS BEGIN HERE
  220 B1=A(1,15)
      C1=A(4,15)
      B2=A(2,15)
      C2=A(5,15)
      B3=A(3,15)
      C3=A(6,15)
      GO TO 9
  230 FSO=SK5*(VAD4*3.1416/180.0-BE1+C1/SK3)/CH3-A(4,15)*COS
     C(VAD4*3.1416/180.0-BE1)+HT1*SIN(VAD4*3.1416/180.0-BE1)
      PUNCH 6, HT1,TA1,HT2,TA2,FSO
      STOP
      END
```

APPENDIX IV

Program for Stress Analysis

```
STRESS ANALYSIS OF SPRINGS
    1 FORMAT (3F14.6)
    2 FORMAT (3F14.6)
    3 FORMAT (3F14.6)
    4 FORMAT (3F14.6)
    5 FORMAT (3F14.6)
    6 FORMAT (3F14.6)
    7 FORMAT (3F14.6)
      READ1, A1,A2,A3
      READ2,B1,B2,B3
      READ3,D,T1,C
      T2=C*3.1416/180.0
      C1=-SINF(T1)*COSF(T2)
      C2=COSF(T1)*COSF(T2)
      C3=SINF(T2)
      D1=-COSF(T1)
      D2=-SINF(T1)
      D3=0.0
      E1=SINF(T1)*SINF(T2)
      E2=-SINF(T2)*COSF(T1)
      E3=+COSF(T2)
      FMOT=A1*C1+A2*C2+A3*C3
      FMON=A1*D1+A2*D2+A3*D3
      FMOB=A1*E1+A2*E2+A3*E3
      FFOT=B1*C1+B2*C2+B3*C3
      FFON=B1*D1+B2*D2+B3*D3
      FFOB=B1*E1+B2*E2+B3*E3
      TCA11=16.0/(3.1416*(D**3))*((FFOT*D/8.0+FMON)+SQRTF((FFOT*
     CD/8.0+FMON)**2+(FMOT-FFON*D/8.0)**2))
      TCA12=16.0/(3.1416*(D**3))*((FFOT*D/8.0+FMON)-SQRTF((FFOT*
     CD/8.0+FMON)**2+(FMOT-FFON*D/8.0)**2))
      SCA1=+16.0/(3.1416*(D**3))*(SQRTF((FFOT*
     CD/8.0+FMON)**2+(FMOT-FFON*D/8.0)**2))
      TCA21=16.0/(3.1416*(D**3))*((FFOT*D/8.0-FMOB)+SQRTF((FFOT*
     CD/8.0-FMOB)**2+(FMOT+FFOB*D/8.0)**2))
      TCA22=16.0/(3.1416*(D**3))*((FFOT*D/8.0-FMOB)-SQRTF((FFOT*
     CD/8.0-FMOB)**2+(FMOT+FFOB*D/8.0)**2))
      SCA2=+16.0/(3.1416*(D**3))*(SQRTF((FFOT*
     CD/8.0-FMOB)**2+(FMOT+FFOB*D/8.0)**2))
```

```
 TCA31=16.0/(3.1416*(D**3))*((FFOT*D/8.0-FMON)+SQRTF((FFOT*
CD/8.0-FMON)**2+(FMOT+FFON*D/8.0)**2))
 TCA32=16.0/(3.1416*(D**3))*((FFOT*D/8.0-FMON)-SQRTF((FFOT*
CD/8.0-FMON)**2+(FMOT+FFON*D/8.0)**2))
 SCA3=+16.0/(3.1416*(D**3))*(SQRTF((FFOT*
CD/8.0-FMON)**2+(FMOT+FFON*D/8.0)**2))
 TCA41=16.0/(3.1416*(D**3))*((FFOT*D/8.0+FMOB)+SQRTF((FFOT*
CD/8.0+FMOB)**2+(FMOT-FFOB*D/8.0)**2))
 TCA42=16.0/(3.1416*(D**3))*((FFOT*D/8.0+FMOB)-SQRTF((FFOT*
CD/8.0+FMOB)**2+(FMOT-FFOB*D/8.0)**2))
 SCA4=+16.0/(3.1416*(D**3))*(SQRTF((FFOT*
CD/8.0+FMOB)**2+(FMOT-FFOB*D/8.0)**2))
 PUNCH4,TCA11,TCA12,SCA1
 PUNCH5,TCA21,TCA22,SCA2
 PUNCH 6,TCA31,TCA32,SCA3
 PUNCH7,TCA41,TCA42,SCA4
 STOP
 END
```

AN ANALYSIS OF SPRING-BEAMS HAVING
LARGE DEFLECTIONS

by

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B. S., National Taiwan University, China, 1962

AN ABSTRACT OF A MASTER'S THESIS

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ABSTRACT

This thesis presents an approximate solution for the stress analysis of laterally loaded coil springs undergoing large deflections.

The coil spring is approximated by n link-like elastic elements, pinned together, with elastic restraints (or angular springs) at the hinges. Each element may have different physical properties. With the assumptions that each link-like element takes tension and shear only, while the connecting angular springs take moments only, the analysis is reduced to the solution of a set of simultaneous nonlinear algebraic equations for the determination of the forces on the elements, from which the stress analysis follows in a complicated, but routine manner.

Two basic problems are discussed:

1. Given a set of loads, determine the deflection curve and maximum stresses which result; and

2. Given a deflection curve, determine the set of loads required to deform the spring into the given curve, and the maximum stresses which result.

Although the basic sets of simultaneous nonlinear algebraic equations and methods for solution are discussed for both of these problems, a complete numerical solution is given for the second problem only.

Comparison of theoretical and experimental loads calculated and found for given coil springs indicates excellent agreement between theory and experiment.