ESTIMATING THE PARAMETERS OF THE WEIBULL DISTRIBUTION

by

JOHN LAWRENCE STEDL

B. S., Wisconsin State University, 1965

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

Approved by:

A. M. Feyerherm
Major Professor

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I INTRODUCTION

The Weibull distribution was originally derived in a paper by Fisher and Tippett (2). However, as the name implies, it is referred to as the Weibull distribution because of a paper by Weibull (8) some 10 years later. It is a three parameter distribution but without undue loss of generality, the location parameter c can be set equal to zero. The two-parameter case will be considered in this report.

The Weibull distribution plays an important role in describing many natural phenomenon. Some examples given in Weibull (9) are: (1) Yield strength of steel, (2) Fiber strength of Indian cotton, (3) Statures of adult males, born in the British Isles. Dubay (1) describes how the Weibull can be used to describe the deterioration pattern of certain inexpensive industrial products. In his examples, time to deterioration is the random variable which obeys a Weibull probability law.

One of the difficulties encountered in working with the Weibull distribution involves estimating its parameters. The purpose of this report is to show four methods of estimating the shape and scale parameters of the Weibull distribution; namely, the method of moments, maximum likelihood method, variance method, and Rand Corporation's method. These will be derived in some detail and properties of these estimators will be included, when possible.

II CHARACTERISTICS OF THE WEIBULL DISTRIBUTION

A random variable X is said to distributed as the Weibull distribution if its density is:

$$f(x;a,b) = abx^{a-1} \exp\{-b(x)^a\} \; ; \; x \geq 0 \qquad (1)$$
$$= 0 \qquad x < 0 \; .$$

This is a two parameter family of distributions, the parameters being a and b. Both the shape parameter a and the scale parameter b must be greater than zero. Figures 1, 2, and 3 show graphs of the Weibull distribution for various values of a and b. Figure 1 shows the distribution to be exponential when both a and b are set equal to 1. Figure 3 shows the distribution to be "almost" bell-shaped when $a = 3.26$ and $b = 0.70$.

The mean, variance and median will be derived for the density given in (1). To derive these quantities, one needs to find $\int_0^\infty xf(x)\,dx$, $\int_0^\infty x^2 f(x)\,dx$ and the value m such that $\int_0^m f(x)\,dx = 1/2$ respectively. By using the transformation $y = bx^a$,

$$\int_0^\infty xf(x)\,dx = \int_0^\infty (y/b)^{1/a} \cdot \exp(-y)\,dy$$

$$= \Gamma(1+1/a) \cdot b^{-1/a} \cdot \int_0^\infty \frac{(y)^{1/a}}{\Gamma(1+1/a)} \cdot \exp(-y)\,dy$$

$$= \Gamma(1+1/a) \cdot b^{-1/a} \cdot 1,$$

because the above integrand is just a gamma density function. Hence the mean of the Weibull distribution is given by:

$$\mu = b^{-1/a} \cdot \Gamma(1+1/a) \; . \qquad (2)$$

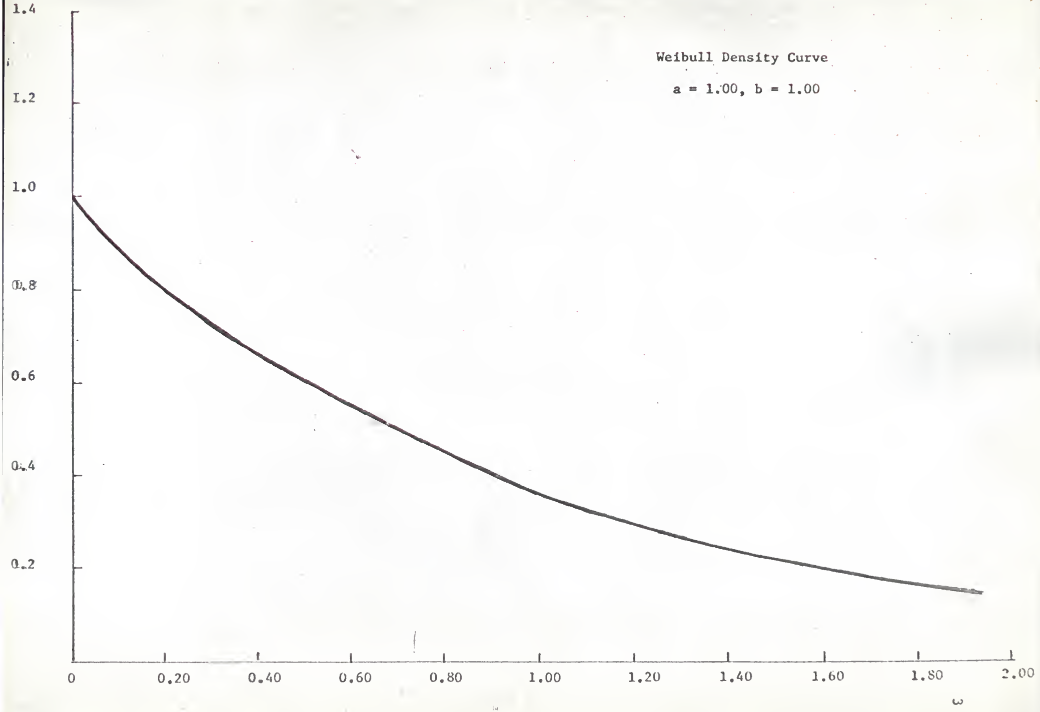
Weibull Density Curve
a = 1.00, b = 1.00
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
1.80
2.00
ω

Figure 1

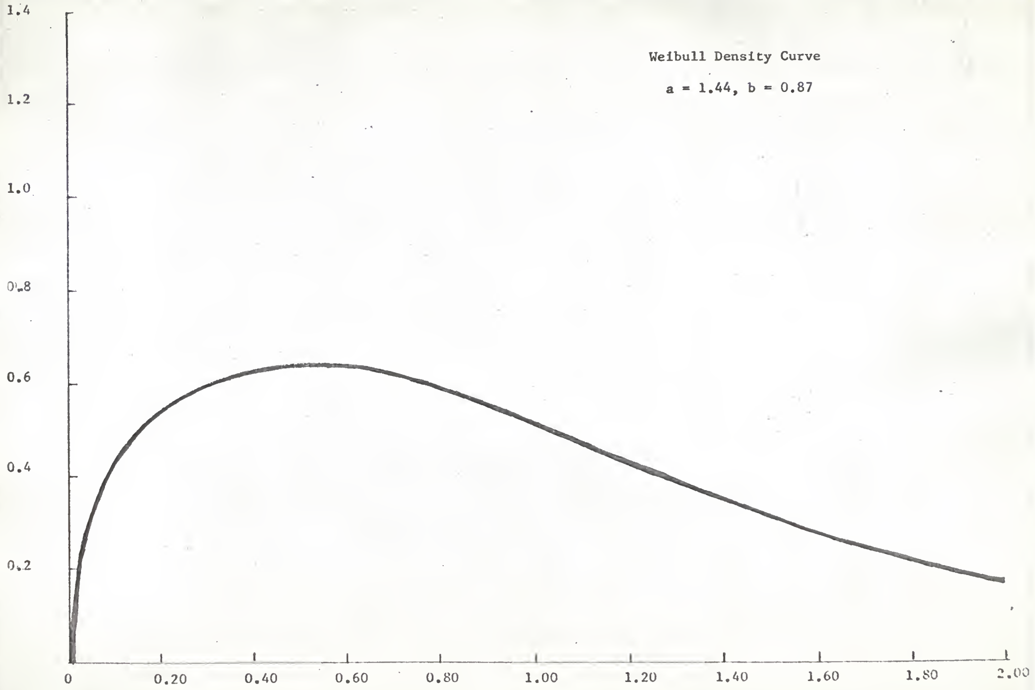
Weibull Density Curve
a = 1.44, b = 0.87
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
1.80
2.00

Figure 2

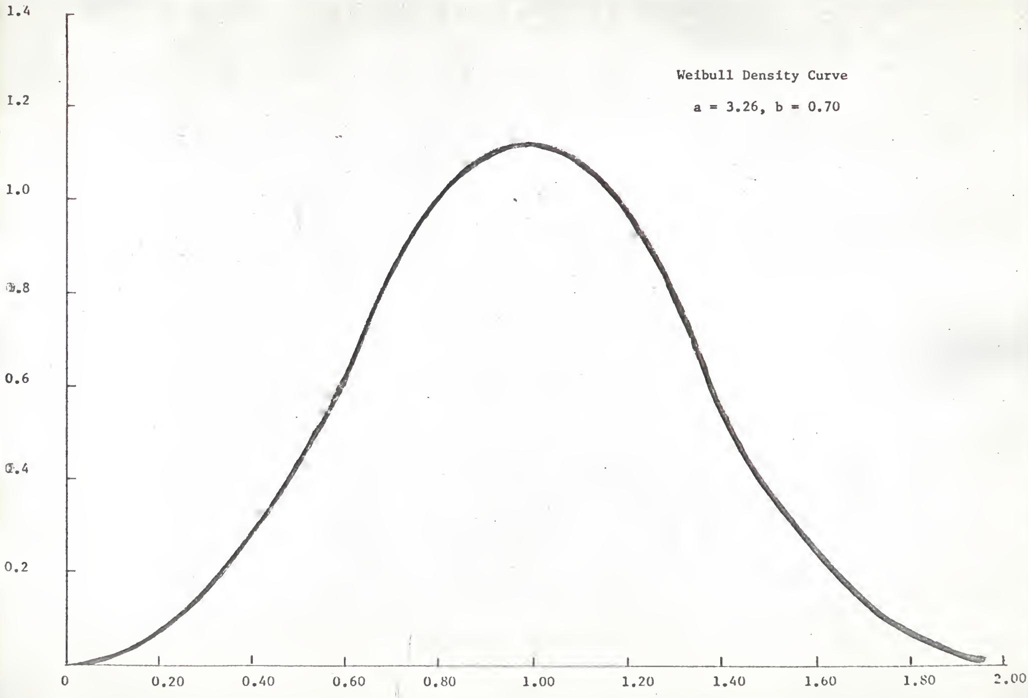
Weibull Density Curve
a = 3.26, b = 0.70
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
1.80
2.00
s

Figure 3

By using the transformation $y = bx^a$ again

$$\int_0^\infty x^2 f(x)\, dx = \int_0^\infty (y/b)^{2/a} \cdot \exp(-y)\, dy$$

$$= \Gamma(1+2/a) \cdot b^{-2/a} \cdot \int_0^\infty \frac{y^{2/a}}{\Gamma(1+2/a)} \cdot \exp(-y)\, dy$$

$$= b^{-2/a} \cdot \Gamma(1+2/a) \cdot 1 \qquad (3)$$

because the integrand is just a gamma density function again. The variance of a random variable is $E(X^2) - E^2(X)$. Thus, the variance of the Weibull distribution is given by:

$$\sigma^2 = b^{-2/a} \{\Gamma(1+2/a) - \Gamma^2(1+1/a)\} \quad . \qquad (4)$$

Finally, it can be shown that

$$\int_0^{b^{-1/a}(\ln 2)^{1/a}} f(x)\, dx = 1/2$$

(by using the transformation $y = bx^a$ again). Hence the median, m, is given by:

$$m = b^{-1/a}(\ln 2)^{1/a} \quad .$$

In a search for "good" estimators for a and b in (1), one should consider the property of sufficiency. Sufficiency is an especially important property because if sufficient estimators exist then one can proceed to find minimum varianced unbiased estimators. In general, an unbiased estimator based on a sufficient statistic will have smaller variance than one which is not based on a sufficient statistic (see Mood-Graybill (7), p 176).

Unfortunately, a set of joint-sufficient statistics does not exist. To show this, consider that in a random sample of size n ($n \geq 2$), the likelihood is given by:

$$L = b^n a^n \prod_{i=1}^{n} x_i^{a-1} \exp(-b \sum_{i=1}^{n} x_i^a) \qquad (5)$$

In order to obtain a set of joint-sufficient statistics for a and b, say $\hat{\theta}_1$ and $\hat{\theta}_2$, the likelihood must be expressible as:

$$L = k(x_1, \ldots, x_n) \; g(a;b,\hat{\theta}_1,\hat{\theta}_2)$$

where k does not contain a or b. Consider the term in (5),

$$\exp(-bx_1^a) \exp(-bx_2^a) \; . \; . \; . \; \exp(-bx_n^a) \; .$$

Clearly, it is impossible to separate the x_i's from a. Thus joint-sufficient statistics for a and b do not exist.

III METHOD OF MOMENTS

This method of estimation simply equates the first two sample moments with the first two population moments. The first two population moments as derived in II are:

$$\mu_1 = E(x) = b^{-1/a}\,\Gamma(1+1/a),$$

$$\mu_2 = E(x^2) = b^{-2/a}\,\Gamma(1+2/a)\ .$$

The sample moments are given by:

$$m_1 = \sum_{i=1}^{n} x_i/n = \bar{x}, \tag{6}$$

$$m_2 = \sum x_i^2/n\ . \tag{7}$$

Now proceeding to derive the moment estimators by the usual method, let

$$\mu_1 = m_1,$$

$$\mu_2 = m_2.$$

Then

$$\bar{x} = \hat{b}^{-1/a}\,\Gamma(1+1/\hat{a}), \tag{8}$$

$$\sum_{i=1}^{n} x_i^2/n = b^{-2/\hat{a}}\,\Gamma(1+2/\hat{a})\ . \tag{9}$$

The quantity $\hat{b}$ can be eliminated by dividing equation (9) by equation (8) squared which yields:

$$\frac{\sum_{i=1}^{n} x_i^2/n}{\bar{x}^2} = \frac{\Gamma(1+2/\hat{a})}{\Gamma^2(1+1/\hat{a})} . \tag{10}$$

Clearly, (10) cannot be solved explicitly for $\hat{a}$, but with the aid of tables of the Gamma function one can determine the value of $\hat{a}$ which satisfies (10). This is the method of moments estimator for a.

After $\hat{a}$ has been determined, equation (8) can be solved for $\hat{b}$. The solution is the method of moments estimator for b and is given by:

$$\hat{b} = \{(1/\bar{x}) \cdot \Gamma(1+1/\hat{a})\}^{\hat{a}} . \tag{11}$$

IV METHOD OF MAXIMUM LIKELIHOOD

The likelihood function for a random sample $x_1, x_2, \ldots, x_n$ from a Weibull distribution can be written as

$$L = a^n b^n \prod_{i=1}^{n} x_i^{a-1} \cdot \exp\{-b(\sum_{i=1}^{n} x_i^a)\}$$

or upon taking the naperian logarithm of L, one obtains,

$$L^* = \ln L = n \ln(b) + n \ln(a) + (a-1) \sum_{i=1}^{n} \ln(x_i) - b \sum_{i=1}^{n} x_i^a .$$

To get the maximum likelihood estimators, L^* is differentiated with respect to a and then with respect to b. The resulting equations are set equal to zero and solved for $\hat{a}$ and $\hat{b}$. Thus

$$\frac{dL^*}{da} = \frac{n}{a} + \sum_{i=1}^{n} \ln(x_i) - b \sum_{i=1}^{n} x_i^a \ln(x_i) ,$$

$$\frac{dL^*}{db} = \frac{n}{b} - \sum_{i=1}^{n} x_i^a ,$$

which yield the following equations in $\hat{a}$ and $\hat{b}$,

$$\frac{n}{\hat{a}} + \sum_{i=1}^{n} \ln(x_i) - \hat{b} \sum_{i=1}^{n} x_i^{\hat{a}} \ln(x_i) = 0, \qquad (12)$$

$$\frac{n}{\hat{b}} + \sum_{i=1}^{n} x_i^{\hat{a}} = 0 . \qquad (13)$$

However, these equations cannot be solved explicitly for $\hat{a}$ and $\hat{b}$, (see Menon (6)). In fact (12) cannot be solved for $\hat{a}$ even if b were known. Intuitively, the reason is that equations of the form $d^x + e^x = k$ cannot be solved explicitly for x where d, e, and k are constants. Thus an iterative scheme is sought which will yield the values of $\hat{a}$ and $\hat{b}$ in equations (12) and (13).

The iterative technique developed by Hardy (3) is to write equations (12) and (13) as:

$$a_{j+1} = \frac{n}{b_j \sum_{i=1}^{n} x_i^{a_j} \ln(x_i) - \sum_{i=1}^{n} \ln(x_i)} , \qquad (14)$$

$$b_j = \frac{n}{\sum_{i=1}^{n} x_i^{a_j}} . \qquad (15)$$

Now guess an initial value of a, call it a_o. This is put in equation (15) to yield a value of b_o. These two values of a_o and b_o are substituted in (14) to yield a value of a_1. This process is repeated until a_j and b_j converge. These convergent values, a_j and b_j are the maximum likelihood estimators for a and b respectively. Lloyd and Lipow (4) claim that this method will converge if a reasonable estimate of a_o is made. The moment estimator for a given in (10) can be used as the initial value a_o.

V RAND CORPORATION'S METHOD

This method of estimation is due to Wilson (10). A bayesian approach is used; namely, it is assumed that the scale parameter b has a prior gamma distribution. Hence one can obtain the joint density of $x_1, \ldots, x_n$ and b and then integrate this joint density with respect to b to obtain a marginal likelihood function which contains only the shape parameter a. Finally asymptotic properties of this likelihood function yield an estimator for a.

In a random sample of size n, the joint density of the x's given b is:

$$h(x_1, \ldots, x_n | b) = a^n b^n g^{n(a-1)} \exp(-bnr(a)^a)$$

where

$$g = \prod_{i=1}^{n} x_i^{1/n} ,$$

and

$$r(a) = \left[\sum_{i=1}^{n} x_i^a / n \right]^{1/a} .$$

Now assume the density for b to be given by:

$$g(b) = \begin{cases} \frac{(T')^{N'}}{\Gamma(N')} b^{N'-1} \exp(-bT') & ; b \geq 0 \\ 0 & ; b < 0 . \end{cases}$$

Since

$$f(b,x_1, \ldots, x_n) = h(x_1, \ldots, x_n|b)\, g(b),$$

$\int_0^\infty f(b,x_1, \ldots, x_n)db = \int_0^\infty f\, db$ will contain only a, T', and N' .

Now

$$\int_0^\infty f db = \int_0^\infty a^n b^n g^{n(a-1)} \exp(-bnr(a)^a) \frac{(T')^{N'}}{\Gamma(N')} b^{N'-1} \exp(-bT')db,$$

$$= a^n g^{n(a-1)} \frac{(T')^{N'}}{\Gamma(N')} \int_0^\infty b^{n+N'-1} \exp\{-b(T'+nr(a)^a)\} .$$

Therefore

$$\int_0^\infty f db = a^n g^{n(a-1)} (T')^{N'} \frac{\Gamma(N'+n)}{\Gamma(N')} \cdot \frac{1}{\left(T'+nr(a)^a\right)^{N'+n}} \tag{16}$$

since the integrand can be put in the form

$$\frac{\left(T'+nr(a)^a\right)^{N'+n}}{\Gamma(N'+n)}\; b^{n+N'-1} \cdot \exp\{-b(T'+nr(a)^a)\}$$

which is a gamma density function. Hence (16) is the marginal likelihood function for a, call it $L(a)$. From (16), $L(a)$ is proportional to

$$a^n (g)^{na} \left(1/(T'+nr(a)^a)\right)^{N'+n} . \tag{17}$$

From (17), one can obtain the maximum likelihood estimator directly; however, it is a burdensome computation. Therefore asymptotic properties of $L(a)$ are exploited to arrive at an estimator for a.

A Theorem from Mann and Wald (5) states that if a sequence of random numbers converge in the probability sense to c, then $g(x_n)$ converges to g(c) for any continuous function g. It is used to prove the following:

Lemma 1:

$$p \lim_{n\to\infty} r(a) = \frac{r(1)}{\Gamma(1+1/a)}$$

where $p \lim_{n\to\infty}$ defines a limit in the probability sense.

Proof:

From (2), $E(X) = b^{-1/a}\,\Gamma(1+1/a)$. By applying the Weak Law of Large Numbers to E(X) the following is obtained:

$$p \lim_{n\to\infty} \bar{X} = b^{-1/a}\,\Gamma(1+1/a) \ , \tag{18}$$

or since $\bar{X} = r(1)$,

$$p \lim_{n\to\infty} r(1) = b^{-1/a}\,\Gamma(1+1/a) \ . \tag{19}$$

Clearly, the distribution of $W = X^a$ is exponential with mean given by 1/b. Again by applying the Weak Law of Large Numbers to E(W) the following is obtained:

$$p \lim_{n\to\infty} \bar{W} = 1/b \ .$$

Now consider the function $\bar{W}^{1/a}$ (which is just r(a))

Hence, by the Mann Wald Theorem the above equation becomes:

$$p \lim_{n\to\infty} r(a) = (1/b)^{1/a} \ . \tag{20}$$

Combining (19) and (20), it follows that

$$p \lim_{n\to\infty} r(a) = \frac{r(1)}{\Gamma(1+1/a)} \quad .$$

This completes the proof of Lemma 1.

To obtain an asymptotic expression of L(a) in (17), one can use Lemma 1 and the Mann Wald Theorem to write (17) as:

$$L(a) = a^n g^{na} \left[\frac{1}{T'+n \left[\frac{r(1)}{\Gamma(1+1/a)} \right]^a} \right]^{N'+n} + \text{Error term.} \qquad (21)$$

Now for practical reasons the major result is the following.

Lemma 2:

$$p \lim_{n\to\infty} \left(\frac{1}{n} \ln (L(a)) + \ln n \right) = \ln(a) + a \ln \left[\frac{g \cdot \Gamma(1+1/a)}{r(1)} \right] \qquad (22)$$

Proof:

The proof follows immediately if one takes the logarithm of both sides of (22). This operation yields the following:

$$\frac{1}{n} \ln (L(a)) + \ln(n) = \ln(a) + a \ln g$$

$$- \left(1 + \frac{N'}{n}\right) \ln \left[T'/n + \frac{r(1)^a}{\Gamma(1+1/a)^a} \right]$$

$$- \frac{N'}{n} \ln(n) + \text{Error term} \; .$$

Now take $p \lim_{n\to\infty}$ of both sides to get

$$p \lim_{n \to \infty} \left(\frac{1}{n} \ln(L(a)) + \ln(n)\right) = \ln(a)$$

$$+ a \ln\left[\frac{g \cdot \Gamma(1+1/a)}{r(1)}\right] .$$

The maximum asymptotic-marginal-likelihood estimate of the shape parameter is defined as the value of $\hat{a}$ maximizing the likelihood function (22). Note that this estimator must be distinguished from the asymptotic form of the maximum marginal likelihood estimator that can be derived explicitly from (16).

If the derivative of the right hand member of (22) with respect to a is set equal to zero the following implicit relation for $\hat{a}$ results:

$$\left(\psi(1/\hat{a}) - 1\right) \quad \hat{a} - \ln \left(\Gamma(1+1/\hat{a})\right) = \ln\left(g/r(1)\right) ,$$

where

$$\psi(x) = \Gamma'(x+1)/\Gamma(x+1) .$$

Values of $g/r(1)$ are tabulated in Table 1 for selected values of $\hat{a}$ $\left(\text{recall } r(1) = \sum_{i=1}^{n} x_i/n\right)$. This table is a reprint from Wilson (10) .

Hence to obtain $\hat{a}$ one must calculate $g/\bar{x}$ and then go to the table to find the value of $\hat{a}$.

Table 1

MAXIMUM ASYMPTOTIC-MARGINAL-LIKELIHOOD ESTIMATES FOR THE SHAPE PARAMETER OF A WEIBULL DISTRIBUTION

$g/\bar{x}$	$\hat{a}$	$g/\bar{x}$	$\hat{a}$
.1459	.0500	.4285	.5000
.1496	.0526	.5615	1.0000
.1536	.0556	.5825	1.1111
.1580	.0588	.6060	1.2500
.1627	.0625	.6324	1.4286
.1680	.0667	.6625	1.6667
.1737	.0714	.6970	2.0000
.1801	.0769	.7372	2.5000
.1873	.0833	.7846	3.3333
.1953	.0909	.8117	4.0000
.2046	.1000	.8416	5.0000
.2152	.1111	.8964	8.3333
.2278	.1250	.9116	10.0000
.2428	.1429	.9276	12.5000
.2612	.1667	.9531	20.0000
.2845	.2000	.9620	25.0000
.3155	.2500	.9805	50.0000
.3594	.3333	.9901	100.0000

VI VARIANCE METHOD

In considering this method due to Menon (6), the Weibull density is written in a slightly different form, namely:

$$f(x) = \begin{cases} (a/b')(x/b')^{a-1} \exp\{-(x/b')^{a}\}; & x \geq 0 \\ 0 & ; x < 0 . \end{cases} \tag{23}$$

In this form of the Weibull distribution a and b' are the shape and scale parameter respectively. The relation of (1) to (24) is that $b = (1/b')^{a}$. In this case a different shape parameter is being estimated.

The method employed is to find an estimate of (1/a), say $\hat{d} = (\widehat{1/a})$ and then set $\hat{a} = (1/\hat{d})$. Similarly, an estimate of $\ln \hat{b}'$, ln b' is obtained and then $\hat{b}'$ is set equal to $\exp(\ln \hat{b}')$. Of course, it is known that $\hat{a}$ and $\hat{b}'$ do not necessarily possess the same statistical properties that $\hat{d}$ and $\ln \hat{b}'$ possess. Nevertheless, the justification of $\hat{b}'$ and $\hat{a}$ as estimators will be in the "good qualities" of $\ln \hat{b}'$ and $\hat{d}$.

In order to derive the estimator $\hat{a}$, consider the random variable $W = (X/b')^{a}$. Since $dw = (a/b')(x/b')^{a-1}\,dx$, clearly the distribution of w is:

$$g(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & w < 0 . \end{cases}$$

Hence W is distributed independently of a and b'. In particular ln W is independent of a and b'. By definition, the variance of ln W is given by $E(\ln^2 W) - E^2(\ln W)$. Menon (6) has calculated the value of Var(ln W) to be $\Pi^2/6$. Also since $W = (X/b')^{a}$, the following is obtained:

$$\text{Var}(\ln W) = \text{Var}\{\ln(X/b')^a\}$$

$$= \text{Var}\{a(\ln X - \ln b')\}$$

$$= a^2 \text{Var}(\ln X) .$$

Thus by combining the two expressions for Var (ln W), the following results:

$$1/a = \{\text{Var}(\ln X) \cdot 6/\Pi^2\}^{1/2} \qquad (24)$$

Now set $d = 1/a$ and the estimator $\hat{d}$, of d (also of 1/a) is obtained by putting the sample variance of ln X in (24), or:

$$\hat{d} = \left[6/\Pi^2 \left(\sum_{i=1}^{n} (\ln X_i)^2 - \frac{\left(\sum_{i=1}^{n} \ln X_i\right)^2}{n} \right) \Big/ n-1 \right]^{1/2}$$

As stated previously, the estimator of a denoted $\hat{a}$ is:

$$\hat{a} = 1/\hat{d} .$$

Menon (6), reports the following statistical properties of $\hat{d}$. The quantity $\hat{d}$ is not an unbiased estimator, however, its expected value and variance is given by:

$$E(\hat{d}) = d + O(1/n)$$

$$\text{Var}(\hat{d}) = 1.1 \frac{d^2}{n} + d^2 O(1/n^2) .$$

For large n, $\hat{d}$ is asymptotically normal with mean and variance given above. Its asymptotic efficiency is 55%.

One is really interested in the properties of $\hat{a}$. Not much is known about the statistical properties of $\hat{a}$. By Menon (6) , $\hat{a}$ is asymptotically normal with mean and variance given by a and $1.1 \frac{a^2}{n}$ respectively.

The estimator $\hat{b}'$ can be obtained after the value of $\hat{d}$ from above is known by using the following argument. Consider:

$$E(\ln(X)) = \int_0^\infty \ln x \left(\frac{a}{b'}\right)\left(\frac{x}{b'}\right)^{a-1} \exp\{-\left(\frac{x}{b'}\right)^a\} \, dx$$

Let $y = \left(\frac{x}{b'}\right)^a$, then $dy = \left(\frac{a}{b'}\right)\left(\frac{1}{b'}\right)^{a-1}$ and $E(\ln(X))$ becomes:

$$E(\ln(X)) = \int_0^\infty \ln(b' \cdot y^{1/a}) \, e^{-y} \, dy$$

$$= \int_0^\infty (\ln b') \, e^{-y} + \int_0^\infty \ln(y^{1/a}) \cdot e^{-y} dy$$

$$= \ln b' + \frac{1}{a} \int_0^\infty \ln y \cdot e^{-y} \, dy \, .$$

Recall that $d = 1/a$ and since $\int_0^\infty (\ln y) \, e^{-y} \, dy = -.5772$ the above equation becomes:

$$E(\ln(X)) = \ln b' + d(-.5772) \, .$$

Now, in the above equation one can estimate $E(\ln(X))$ by $\sum_{i=1}^{n} \ln x_i / n$, $\ln b'$ by $\ln \hat{b}'$ and d by $\hat{d}$ to get the following estimator for $\ln b'$:

$$\ln \hat{b}' = \sum_{i=1}^{n} \ln x_i + \hat{d} \, (.5772),$$

or upon substituting the value of $\hat{d}$ obtained previously the following estimator for ln b is obtained:

$$\ln \hat{b}' = \sum_{i=1}^{n} \ln x_i/n + .5772 \left[(6/\pi^2) \{ (\sum_{i=1}^{n} (\ln x_i)^2 - \frac{(\sum_{i=1}^{n} \ln x_i)^2}{n})/n-1 \} \right]^{1/2} .$$

Now it seems reasonable to define the estimator for b' to be

$$\hat{b}' = \exp(\ln \hat{b}') .$$

Menon (6) reports the following statistical properties of $\ln \hat{b}'$:

$$E(\ln \hat{b}') = \ln b + O(1/n)$$

$$\mathrm{Var}\ (\ln b') = 1.2\ d^2/n + d^2\ O(n^{-3/2}) .$$

The quantity $\ln \hat{b}'$ is asymptotical normal with mean and variance given above. Again, little is known about the statistical properties of the estimator for b', $\exp(\ln \hat{b}')$. But by Menon (6)-- $\hat{b}'$ is asymptotical normal with mean and variance given by b' and $1.2\ d^2b^2/n$ respectively.

VII CONCLUSION

Four methods of estimating the shape and scale parameters of the Weibull distribution were presented. It was shown at the onset that sufficient estimators do not exist. The estimators derived by the variance method were shown to be asymptotically normal. With this exception, properties of the proposed estimators are a subject of further investigation.

For further study it is recommended that the properties of these estimators such as bias, variance, consistency, and efficiency be studied, perhaps by Monte Carlo methods.

VIII ACKNOWLEDGEMENT

This writer is indebted to Dr. A.M. Feyerherm for his help in the preparation of this report.

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ESTIMATING THE PARAMETERS OF THE WEIBULL DISTRIBUTION

by

JOHN LAWRENCE STEDL

B. S., Wisconsin State University, 1965

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

A random variable X is said to be distributed as the Weibull distribution if its density is given by:

$$f(x;a,b) = abx^{a-1}\exp\left(-b(x)^a\right) \quad ; \quad x \geq 0$$

$$= 0 \quad ; \quad x < 0 .$$

This is a two parameter family of distributions in which the shape parameter a and scale parameter b must both be greater than zero. In its most general form the Weibull distribution contains a location parameter, but without loss of generality the location parameter is assumed to be zero.

Four methods of estimating the parameters of the Weibull distribution are presented: the method of moments, maximum likelihood method, Rand Corporation method, and the variance method.

The method of moments estimators are derived by equating the first two population moments about the origin with the first two sample moments about the origin. This procedure yields an implicit equation for $\hat{a}$ which may be solved when the sample values are known. When $\hat{a}$ is determined the explicit form of $\hat{b}$ is given.

The maximum likelihood estimators are derived by differentiating the likelihood function with respect to a and b. However when the resulting equations are set equal to zero, explicit expressions for $\hat{a}$ and $\hat{b}$ do not exist. But by using iterative techniques, solutions for $\hat{a}$ and $\hat{b}$ can be found.

Rand Corporation develops an estimator for the shape parameter $\hat{a}$ by using a bayesian technique. It is assumed that the scale parameter b has a prior gamma density. Then the joint density of $x_1,\ldots.,x_n$ and b is obtained and integrated with respect to b to obtain a marginal likelihood function which contains only the shape parameter a. Asymptotic properties of this likelihood function yield an implicit equation in $\hat{a}$. However with the aid of Table 1, $\hat{a}$ can be determined after the sample data is known.

The variance method first obtains an estimate $\hat{d}$ of $1/\hat{a}$ and then the estimate of a is given by $\hat{a} = 1/\hat{d}$. Similarly an estimate $\widehat{\ln b}$ is derived and then the estimator of b is given by $\hat{b} = \exp(\widehat{\ln b})$. Both $\hat{a}$ and $\hat{d}$ are asymptotically normal.