THE NON-EXISTENCE OF EQUILIBRIUM IN SEQUENTIAL AUCTIONS WHEN BIDS ARE REVEALED

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ABSTRACT

Sequential auctions of homogeneous objects are common in public and private marketplaces. Weber derived equilibrium results for what is now a classic model of sequential auctions. However, Weber's results are derived in the context of two particular price quote assumptions. In this paper, we examine a model of sequential auctions based on online auctions, in which, after each auction, all bids are revealed. We show that a pure-strategic, symmetric equilibrium does not exist, regardless of whether the auctions are first- or second-price, if all bids are revealed at the end of each auction.

Keywords: sequential auctions; online auctions; e-commerce; Nash equilibrium

1. Introduction

A *sequential auction* consists of a sequence of individual auctions. Those auctions may be first-price sealed-bid (FPSB) auctions, second-price sealed-bid (Vickrey) auctions, English auctions, or reverse (Dutch) auctions. According to Krishna [2002], a single English auction is strategically equivalent to a Vickrey auction; and a single Dutch (reverse) auction is strategically equivalent to an FPSB auction.

Online auctions that encourage sniping by having fixed deadlines are considered equivalent to sealed-bid auctions. As shown by Roth and Ockenfels [2002], bidders tend to bid at the last minute, which makes them strategically distinct from traditional English auctions. According to Bajari and Hortacsu [2003], "more than 50% of final bids are submitted after 90% of the auction duration has passed." To help bidders bid at the last minute, some companies (e.g. eSnipe.com) provide software to submit bids just before the end of the auction. However, according to Lucking-Reiley [2000], sniping "destroys the English auction's attractive feature that bidders have a dominant strategy to bid up to their maximum willingness to pay." To restore this desirable feature, some online auctions, such as eBay, introduce a proxy bidding policy to auctions, in which bidders submit their maximum bids and a proxy agent will automatically outbid other competitors until reaching the maximum bid. The winner pays an amount of the second highest valuation plus a minimum increase. When the survey was done in 2000, 65 out of 142 online auction sites had adopted proxy bidding [Lucking-Reiley 2000]. With proxy bidding, online auctions, such as those on eBay, are again equivalent to the Vickrey auction [Ockenfels & Roth 2005]. Without the proxy bidding, in online auctions such as zbestoffer.com and OTWA.com, the winner pays what he/she bids. As Lucking-Reiley [2000] points out, "if all bidders were to follow a strategy of bidding only at the last minute, the game would become equivalent to a first-price sealed-bid auction."

In this paper, we study *sequences* of first-price sealed-bid auctions and second-price sealed-bid (Vickrey) auctions. It is quite common in practice to see identical or nearly-identical items sold in a sequence of auctions. Examples include auctions for satellite broadcast licenses, art, wine, fish, flowers, mineral rights, government debts, and many others [Gale & Stegeman 2001]. Among those reported in the academic literature are the sequential sale of

120 identical cases of wine in 1990 at Christie's of Chicago [McAfee & Vincent 1993] and the sale of pelts on the Seattle Fur Exchange [Lambson & Thurston 2003]. eBay, the world's largest electronic auction, can be viewed as an unending series of auctions for hundreds of thousands of nearly identical items. The practical importance of studying sequential auctions can also be partially supported by Caillau et al. [2002]. According to Caillau et al., "many goods, services and contracts are allocated in sequential auctions, sometimes with quite long time periods between two consecutive auctions, sometimes with several auctions almost in a row. As documented in the literature, estate, cattle, fish, vegetables, timber and wine are often allocated in comparable lots at sequential auctions, to a quite well-established and limited group of potential buyers." Figure 1 and Figure 2 illustrate two sequential auctions that recently occurred on eBay's and Amazon's auction sites.

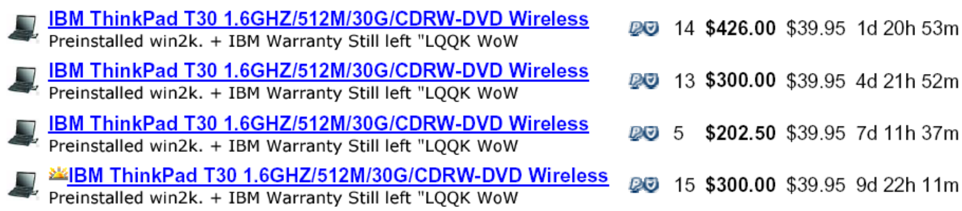

Fig 1: A Sequential Auction at eBay. Data Source: eBay 09/29/2005 12:13 AM

Fig 2: A Sequential Auction at Amazon. Data Source: Amazon Auction 12/3/2004, 8:49 PM

The model in Weber [1983] serves as a classical foundation for many of the papers that followed, and resembles our model in many ways. The most significant difference between Weber's model and our model lies in the price announcement. In Weber's model, either the winner's bid or no bid is revealed after each auction. In practice, many current online markets reveal all of the bids once the auction is over, including eBay, Yahoo! Auctions, and Amazon Auctions. This paper extends the classic model with the policy of revealing all bids, and considers the impact on the market efficiency. Our paper shows that there is no symmetric pure-strategic equilibrium in two- or more-item sequential first-price auctions, and there is no symmetric pure-strategic equilibrium in three- or more-item sequential second-price auctions. These results are significant because the literature has usually assumed the existence of equilibrium in these environments.

This paper contributes to the theory of the important sequential first-price and Vickrey auctions. Its practical importance is to show that it might not be optimal to release the bidding information if items are to be sold sequentially.

The rest of this paper proceeds as follows. Section 2 provides a review of the literature. In Section 3, we present a model of sequential auctions and point out the difference between Weber's model and our model. In Section 4 we discuss the symmetric equilibrium in Weber's model and show that Weber's equilibrium is not a solution to our model. In Section 4.3, we prove the non-existence of a symmetric, pure-strategic equilibrium in the model for both first-price and second-price auctions. We offer some conclusions in Section 5.

2. Literature Review

The literature on sequential auctions dates back to Vickrey [1961], in which he obtains an equilibrium solution for a sequence of first-price auctions with bidders whose single-unit-demand valuations are drawn from a uniform distribution. Since Vickrey's original work, a great deal of research has been directed towards understanding sequential auctions. Milgrom and Weber [2000][1] analyze the equilibrium solutions and price trends under more general assumptions. Hausch [1988] derives the necessary conditions for a symmetric equilibrium in Milgrom and Weber's general symmetric model by applying the signal game concept. Krishna [2002] notes that in Weber's model the price quotes of the first period have no effect on the equilibrium bids in the second period. McAfee and Vincent [1993] find a declining price pattern in symmetric sequential auctions when bidders have non-decreasing risk aversion. In another paper, the same authors [McAfee & Vincent 1997] examine the equilibrium when a seller can post a reserve price in sequential auctions. Elmaghraby [2003] studies the sequential second-price auction of heterogeneous items and concludes that the ordering of items affects the efficiency of the auction. Bernhardt and Scoones [1994] find that a more dispersed valuation distribution on one item may yield more revenue for the seller. Gale and Stegeman [2001] model two completely informed and asymmetric buyers bidding for N identical objects from N sellers sequentially under complete information by assuming that the value of one object depends on the number of objects obtained. Many other papers have addressed other variations of sequential auction models [e.g., Beggs & Graddy 1997; Branco 1997; Cai & Wurman 2005; Katzman 1999; Pitchik & Schotter 1998]. However, due to their special focuses, the above papers do not address the reveal-all-bid information policy.

Weber [1983] surveys the research on sequential auctions and concludes that, with symmetric, risk-neutral bidders and identical items, the equilibrium price in a single-unit demand, first-price, sealed-bid sequential auction is a martingale. The two different price announcement schemes in he studied have no effect on the forthcoming auction. Jeitschoko [1998] points out that it might be due to the continuous properties of valuation distributions. He also explicitly models an auction where each bidder has only two types, either a high valuation or a low valuation. In this model, the winner's price information revealed in the first auction has significant influence on the equilibrium bids for both bidders in the second auction. However, the impact of information in sequential auctions of a similar model with continuous valuation has not been explored in the literature, in part due to the computationally complexity. Our model aims to tackle this problem and shows the importance of the information revelation policy in online auction mechanism design. The critical difference between Weber's model and ours is that we look at the case in which the auctioneer reveals all of the bids -- not just the winner -- at the end of the auction, a policy that is popular in online auctions such as eBay auctions.

Another model closely related to ours is that studied by Ortega-Reichert [2000], in which two bidders bid on two items sold in a sequence of first-price, sealed-bid auctions. Ortega-Reichert derives equilibrium results for his model, and shows the signaling effects of the first bid on the second auction. However, his model differs from ours in a significant way that impacts the ability to derive a pure-strategic equilibrium. In the Ortega-Reichert model, the bidders have valuations for the two objects that are derived from a common distribution with an unknown parameter. The information revealed in the first auction affects each bidder's estimation of the value of the unknown parameter, and therefore their beliefs about their ability to win the second good. In our model, we consider a sequence of identical goods for which the bidders have a constant valuation. We show that a strategy that would reveal the bidders' valuations after the first auction would turn the remaining auctions into games of complete information.

In a sequential first-price auction model with two identical items for sale with two interested buyers who want to buy both items, Enkhbayar [2004] shows that there does not exist a weakly monotonic mixed equilibrium with a reveal-all-bid information policy. This paper is different from ours in several ways. First, the number of bidders in our model is multiple, which is a more generic assumption. Second, we study a sequence of first-price and Vickrey auctions and show the nonexistence of equilibrium for the sequential Vickrey when the number of items for sale is at least three. Finally, we assume that bidders want only one item. Caillau et al. [2002] also study the impact of information in sequential auctions, and suggest that bidders have the incentive to conceal information from the previous auction from the other bidders. Again, their model is different from ours in that they only consider a sequence of two ascending-price auctions in which the bidders' valuations are perfectly correlated across time.

On the topic of snipping in online auctions, Roth and Ockenfels [2002] and Ariely et al. [2005] use data from eBay to show that experienced bidders tend to bid at the last minute. Wang [2003] suggests that snipping is the optimal strategy for a multi-item repeated eBay auction without network traffic congestion. Caldentey and Vulcano [2006] suggest that impatient bidders might tend to participate in those auctions that are to be closed soon. Ockenfels and Roth [2005] extend their previous observations and propose a two-round auction model which includes an English

[1] This work was completed in 1982 but had not been published until 2000.

auction and a last-minute auction with network traffic congestion. They suggest that sniping can be "a best reply" in eBay auctions with private values. Cai [2006] shows that it is optimal for the bidders to bid their true valuations, as they would in a Vickrey auction, but slightly before the last moment in lieu of traffic congestion and proxy bidding. As we show previously, the existence of the snipping phenomenon supports the equivalence of online auction with proxy/non-proxy bidding to the sequential Vickrey/first-price auctions.

3. The Model

We consider the case where there are K identical items for sale in a sequence of sealed-bid auctions. Exactly one item is sold in each auction. In this paper, we discuss both first-price and second-price auctions. Due to the last-minute phenomenon, online English auctions with proxy bidding, e.g. at eBay, can be treated as Vickrey auctions [Roth & Ockenfels 2002; Ockenfels & Roth 2005] while similarly online English auctions without proxy bidding, e.g., at zbestoffer.com and otwa.com, can be approximated as first-price auctions given that bidders bid only at the last minute.

We assume that there are N risk-neutral bidders, $N > K$, competing for the K items. Let A be the set of bidders. Each bidder has single-unit demand and will withdraw from the game once she wins one item. This assumption is common in theoretical models like Weber's. In actual online auctions, this assumption could be true too, although it is difficult to have the same pool of bidders in a long-period sequential auction, e.g., in Figure 1, when the sequential auction spans for about eight days; however, it becomes possible when the time span becomes shorter (i.e., in a couple of minutes) as shown in Figure 2.

The bidders' valuations are independent observations of a nonnegative random variable, V, with a commonly known continuous cumulative distribution function (CDF), F, and its associated probability density function (PDF), f. We assume that F is continuous and differentiable in the domain of the valuation variables. Each bidder knows the value of the object to herself (the private values assumption), but not that of the other bidders.

Without loss of generality, we designate Bidder 0 as the bidder whose strategy we are analyzing. Let $n = N - 1$, and let the other bidders be indexed from 1 to n. Let x be the true value of Bidder 0's valuation and let y_i be the true/concrete value of Bidder i's valuation, $i \in \{1,...,n\}$. Without loss of generality, following the notation in Krishna [2002], we let Y_j be the $(n-j+1)$-st order statistic variable of $\{y_1,...,y_n\}$. Then, we have $Y_1 \le Y_2 \le ... \le Y_n$. Let F_j be the CDF of variable Y_j and let f_j be the PDF of Y_j. We also define $f_{i,j}$ be the joint PDF of Y_i and Y_j. F^m is the multiplication of F m times. Because bidders have identical information about each other's valuations at the beginning of the sequential auction, we refer to the model as the *symmetric sequential auction model*.

The key difference between our model and Weber's model [Weber 1983] is the information revealed by the auctioneer. There are two different price quotes in Weber's model: the first announces only that an object has been sold, while the second announces also the sale price, p. Weber concludes that both price quotes yield the same equilibrium solution. We demonstrate that Weber's results do not hold if the auctioneer reveals *all bids* after each auction terminates.

Let $\beta_{k,i}(x)$ denote Bidder i's bid function, which, given her valuation, x, the bidder can use to compute her bid, b_i, in auction k, $k \in K$. In line with Krishna [2002] and Weber [1983], we assume that these strategy functions are strictly increasing and continuously differentiable in the valuation. As a result, $\beta_k(x)$ is invertible, which means that a bidder's valuation can be inferred with certainty from the bid she makes [Krishna 2002; Weber 198]. We assume $\beta(0) = 0$.

A *symmetric strategy* is a solution in which all players adopt the same strategy function $\beta_k(x)$ although the concrete strategy values vary with different x values. In our sequential auction model, a joint outcome is symmetric if $\beta_{k,i} = \beta_{k,j}$ for all bidders i and j. It is a symmetric equilibrium if no bidder can unilaterally increase her payoff by deviating from the symmetric strategy.

It has been shown that equilibrium does not exist in first-price auctions with continuous strategy space and complete information due to the discontinuity of the payoff function [Lebrun 1996]. We address this technical issue using the technique proposed by Maskin and Riley [2000]: a second round Vickrey auction is used to break the tie, if there is any. With the introduction of a second round Vickrey auction tie breaking rule, in first-price auctions with complete information there exists a pure-strategic equilibrium in which the highest type bidder bids a price equal to the second highest type bidder's valuation, and the other bidders bid their true valuations. The introduction of this tie breaking is primarily a theoretical technicality because the probability of ties is zero when the strategy space is continuous.

4. Symmetric Equilibria in Sequential Auctions

4.1. Weber's Equilibrium in First-Price Auctions

Weber [1983] derives a unique symmetric equilibrium for his model in which each bidder bids the expected value of the $(K+1)$th highest bidder assuming her own bid was the kth highest bid. That is,

$$\beta_k(x) = E[Y_{N-K} \mid Y_{N-k} < x < Y_{N-k+1}]. \tag{1}$$

It is natural to question whether the price announcement in the first auction will influence the bidders' behaviors in the second auction. However, since the winner leaves the game, the remaining bidders have the same information about the rest of the game. A proof in Krishna [2002] shows that the later period strategy is independent of the previous price announcement. As a result, for each bidder, the beliefs about the other bidders' valuation distributions remain unchanged. Weber [1983] explains that the type independence and symmetry assumptions make the equilibrium strategies independent of the two different price quotes.

However, when the auctioneer reveals all of the bids after each auction in the sequence, the above strategy is no longer an equilibrium strategy. The next section presents an example to demonstrate an individual bidder's incentive to deviate, and the following section proves the general case.

4.2. A Counter Example when Bids are Revealed in First-Price Auctions

Consider a sequence of two first-price auctions with bidders that follow the symmetric strategy in equation (1). That is, Bidder i bids $b_i = \beta_{1,i}(v_i)$. Because β is invertible, after seeing the bids, every bidder can compute $v_i = \beta_1^{-1}(b_i)$, for all bidders remaining in the auction. As a result, the second auction becomes a game of complete information.

Example 1 shows that a bidder can be better off by unilaterally deviating from equation (1) in the first auction.

Example 1. Suppose there are $N=10$ bidders in a sequence of two, first-price auctions. If Bidder 0 uses the strategy suggested by Weber, she will bid $\beta_1(x)$ in the first auction. Similarly, Bidder i will bid $\beta_1(y_i)$. After the first auction, the second auction becomes a complete information auction. The second highest bidder wins the second item at the price of the third highest valuation. We have the following cases:

1. If $x > Y_9$, Bidder 0 wins the first item and pays $E[Y_8 \mid x > Y_9]$.
2. If $Y_9 > x > Y_8$, Bidder 0 loses the first item, but will win the second auction with an expected payment of $E[Y_8 \mid Y_9 > x > Y_8]$. The payoff is an expectation because Bidder 0 will not obtain the true information about y_8, the concrete valuations of Bidder 8, until the first auction completes.
3. If $Y_8 > x$, Bidder 0 loses both the first and the second auctions.

Now, suppose Bidder 0 deviates from $\beta_1(x)$ to $\beta_1'(x) = 0$, while the other bidders stick to Weber's strategy. After the first auction, the other bidders infer that Bidder 0's valuation is 0, and is therefore not a factor in their decisions. Although Bidder 0 will always lose the first auction, she can benefit from this deception, as evidenced by the following four exhaustive cases.

1. If $x > Y_9$, Bidder 9 wins the first item. In the second auction, all of the other bidders will believe that Bidder 8 has the highest valuation and Bidder 7 is the second highest. Thus, Bidder 8 will bid y_7. On average, Bidder 0 will be able to win the second item at price $E[Y_7 \mid x > Y_9]$. Since $E[Y_8 \mid x > Y_9] \ge E[Y_7 \mid x > Y_9]$, $x - E[Y_7 \mid x > Y_9] \ge x - E[Y_8 \mid x > Y_9]$. Thus, Bidder 0 will be better off by deviating in this case.
2. If $Y_9 > x > Y_8$, Bidder 0 will again win the second auction at $E[Y_7 \mid Y_9 > x > Y_8]$.
3. If $Y_8 > x > Y_7$, Bidder 8 will believe that the second highest valuation in the second auction is Y_7, and will

bid y_7. Again, Bidder 0 can bid $y_7+\varepsilon$ and win the second item, where the ε term is included to avoid the tie with Y_8. Thus, Bidder 0 will expect to pay $E[Y_7 \mid Y_8 > x > Y_7] - \varepsilon$.

4. If $Y_7 > x$, Bidder 0 will lose both items.

Thus, Bidder 0 will have a greater expected payoff by unilaterally deviating in the first auction. A comparison of the above cases is shown in Table 1.

Table 1: The expected utility of Bidder 0 in the sequential first-price auctions.

	Utility When Using Weber's Strategy	Utility When Deviating to Strategy $z=0$	Increase in Utility When Deviating
$x > Y_9$	$x - E[Y_8 \mid x > Y_9]$	$x - E[Y_7 \mid x > Y_9]$	>0
$Y_9 > x > Y_8$	$x - E[Y_8 \mid Y_9 > x > Y_8]$	$x - E[Y_7 \mid Y_9 > x > Y_8]$	>0
$Y_8 > x > Y_7$	0	$x - E[Y_7 \mid Y_8 > x > Y_7] - \varepsilon$	>0
$Y_7 > x$	0	0	$=0$

Example 1 demonstrates that a bidder would be better off by unilaterally deviating from Weber's strategy when the other bidders use Weber's strategies in the first auction. Thus, Weber's equilibrium strategies cannot be equilibrium in this new model of sequential auctions. We now address the question of whether any pure strategic, symmetric equilibrium exists in this game.

4.3. Non-Existence of Symmetric Equilibrium in Sequential First-Price Auctions

We now address the general question of the existence of symmetric equilibrium in the sequential auction with bid revelation. We continue to consider a sequence of two auctions. Assume that there exists a symmetric, pure-strategic equilibrium, such that every bidder uses the same strategic function, β, in the first auction. With this assumption, and the previous assumption that β is strictly increasing and invertible, every bidder can infer every other bidder's true valuation after the first auction. As a result, the second and future auctions become complete information games.

Thus, we restrict our analysis to the first auction, in which the bidders have incomplete information. The definition of a symmetric equilibrium requires that a bidder cannot be better off by unilaterally deviating from β when all other bidders are playing β. Let $u(x)$ denote the payoff to a bidder if she bids $\beta(x)$. Let $u(x,z \mid z \ge x)$ denote the payoff of this bidder if she deviates from $\beta(x)$ to a higher bid $\beta(z)$. Similarly, we let $u(x,z \mid z \le x)$ denotes the payoff of this bidder if she deviates from $\beta(x)$ to a lower bid $\beta(z)$.

When $z \ge x$, Bidder 0 will win the first item if z is larger than the highest valuation of the other bidders. Otherwise, she will win the second item if x is larger than the second highest valuation of the other bidders. Bidder 0's payoff function can be written as

$$\begin{aligned} u(x,z \mid z \ge x) &= \Pr(Y_n < z)[x - \beta(z)] \\ &\quad + \Pr(Y_{n-1} < x \le z < Y_n) \times [x - E[Y_{n-1} \mid Y_{n-1} < x \le z < Y_n]]. \end{aligned} \tag{2}$$

The first term results from the event $Y_n < z$ in which Bidder 0 wins the first auction. The second term captures the case that Bidder 0 loses the first auction and wins the second by bidding the revealed value of the third highest bidder.

We now consider the case where $z \le x$. If z is larger than the highest valuation of the other bidders, Bidder 0 will win the first item. Otherwise, Bidder 0 may still be able to win the second item, depending upon the revealed valuations of the other bidders. In the following analysis, Bidder 0 may bid against a bidder who has a type greater than Bidder 0's. In such a case, Bidder 0 would lose the tie-breaker unless she bids slightly above the expected bid of the bidder with the higher type.

There are four variations of non-zero outcomes:

- Case 1: $Y_{n-1} < z < Y_n$. Bidder 0 loses the first item; however, both z and x are larger than the second highest valuation of the other bidders such that Bidder 0 will win the second item and expect to pay $E[Y_{n-1} \mid Y_{n-1} < z < Y_n]$.
- Case 2: $Y_{n-2} < z < Y_{n-1}$. In this case, Bidder 0 still loses the first item. However, the bidder with the second highest valuation infers from the first auction that the third highest valuation is z. As a result, he will bid z in the second auction. Bidder 0 can bid $z+\varepsilon$ to outbid the bidder with rank Y_{n-1}.
- Case 3: $z < Y_{n-2} < x < Y_{n-1}$. In this case, the bidder with the second highest valuation will bid what appears to be the third highest value: $E[Y_{n-2} \mid z < Y_{n-2} < x < Y_{n-1}]$. Bidder 0 needs to bid $E[Y_{n-2} \mid z < Y_{n-2} < x < Y_{n-1}] + \varepsilon$ to outbid the bidder with the second highest type.
- Case 4: $z < Y_{n-2} < Y_{n-1} < x$. This case is the same as the above case with the exception that Bidder 0 will win the tie breaker and so does not need to add ε to her bid to win the second item.

Thus, when $z \le x$, the payoff function can be written as

$$\begin{aligned} u(x,z,\varepsilon \mid z \le x) = \Pr(Y_n < z)[x-\beta(z)] \\ &+ \Pr(Y_{n-1} < z < Y_n)[x - E[Y_{n-1} \mid Y_{n-1} < z < Y_n]] \\ &+ \Pr(Y_{n-2} < z < Y_{n-1})[x-(z+\varepsilon)] \\ &+ \Pr(z < Y_{n-2} < x < Y_{n-1}) \times \\ &[x - (E[Y_{n-2} \mid z < Y_{n-2} < x < Y_{n-1}] + \varepsilon)] \\ &+ \Pr(z < Y_{n-2} < Y_{n-1} < x) \times \\ &[x - E[Y_{n-2} \mid z < Y_{n-2} < Y_{n-1} < x]]. \end{aligned}$$

In the above equation, the first term represents the case when Bidder 0 wins the first auction. The next four terms represent the above cases 1–4.

As ε goes to zero, $u(x,z,\varepsilon \mid z \le x)$ asymptotically goes to

$$\begin{aligned} u(x,z \mid z \le x) = \Pr(Y_n < z)[x-\beta(z)] \\ &+ \Pr(Y_{n-1} < z < Y_n)[x - E[Y_{n-1} \mid Y_{n-1} < z < Y_n]] \\ &+ \Pr(Y_{n-2} < z < Y_{n-1})[x-z] \\ &+ \Pr(z < Y_{n-2} < x)[x - E[Y_{n-2} \mid z < Y_{n-2} < x]]. \end{aligned} \tag{3}$$

If $\beta(x)$ is Bidder 0's best response to the other bidders playing β, it must be true that $u(x,z \mid z \ge x) \le u(x)$, and $u(x,z,\varepsilon \mid z \le x) \le u(x)$. As ε goes to zero, the symmetric equilibrium requires

$$\begin{aligned} &u(x,z \mid z \ge x) \le u(x), \text{and} \\ &u(x,z \mid z \le x) \le u(x). \end{aligned} \tag{4}$$

It follows from equations (2) and (3) that $u(x,z \mid z \le x)$ and $u(x,z \mid z \ge x)$ are continuous and differentiable because the probability functions are continuous and differentiable. Also, we know $u(x,z \mid z \le x) = u(x,z \mid z \ge x) = u(x)$ when $z = x$. We let $\beta_{LHS}(x)$ denote the solution for $u(x,z \mid z \le x)$ and let $\beta_{RHS}(x)$ denote the solution for $u(x,z \mid z \ge x)$. If there exists a symmetric, pure-strategic equilibrium, we should have $\beta_{LHS}(x) = \beta_{RHS}(x)$. We now present our main result.

Theorem 1 *In the sequential first-price auctions with full bid revelation, there does not exist a symmetric, pure-strategic equilibrium.*

We prove the result by contradiction in detail in the Appendix. We prove that for every x, $\beta_{LHS}(x) < \beta_{RHS}(x)$, which contradicts the assumption that β is strictly increasing and differential and thus we should have

$\beta_{LHS}(x) = \beta_{RHS}(x)$ if there exists such a symmetric, pure-strategic equilibrium. It is worth noting that $\beta_{LHS}(x) < \beta_{RHS}(x)$ does not imply that there exist two different equilibrium strategy functions. The whole proof process shows that there does not exist a definition for β at any specified point x because the asymptotic limits from either side are not equal.

4.4. Non-Existence of Symmetric Equilibrium in Sequential Vickrey Auctions

Weber [1983] characterized the equilibrium in the sequential Vickrey auction scenario with only the winner's bid revealed as

$$\beta_k(x) = E[Y_{N-K} \mid Y_{N-k-1} < x < Y_{N-k}], \forall k < K. \tag{5}$$

In the last auction, where $k = K$, each bidder bids her true valuation. In the auctions prior to the last, each bidder bids her expectation of the ($K+1$)-th highest bidder assuming that she is at or above K.

To analyze the sequential Vickrey model with all bids revealed, we again assume that there exists a symmetric, pure-strategic equilibrium such that every bidder uses the same strictly increasing and invertible bidding function, β, in the first auction. Thus, every bidder can infer every other bidder's true valuation after the first auction, and the second and future auctions become complete information games.

In a sequence of $K-1$ Vickrey auctions with complete information, it is Nash equilibrium for the top $K-1$ players to bid the K-th highest remaining valuation, while all others bid their true valuations. This conclusion, however, leads to the observation that Weber's strategy is not equilibrium in the first auction when the bids of all bidders are revealed. Similar to Example 1, an example of 3-item, 4-bidder sequence of Vickrey auctions illustrates how a bidder can improve her expected utility by misrepresenting her valuation in the first auction. The four conditions and their expected payoffs are shown in Table 2.

Table 2: The expected utility of Bidder 0 in the sequential Vickrey auctions.

	Utility when Using Weber's Strategy	Utility Deviating to $z=0$	Increase in Utility when Deviating
$x > Y_3$	$x - E[Y_2 \mid Y_3 > x > Y_2]$	x	>0
$Y_3 > x > Y_2$	$x - E[Y_2 \mid Y_2 > x > Y_1]$	x	>0
$Y_2 > x > Y_1$	$x - E[Y_1 \mid Y_2 > x > Y_1]$	x	>0
$Y_1 > x$	0	x	>0

We now examine a sequence of three Vickrey auctions with an arbitrary number of bidders. The extension to an arbitrary number of auctions follows easily from the three auction case. We concentrate our analysis on the first auction, in which the bidders have incomplete information, and show that a symmetric, pure-strategic equilibrium does not exist.

When all bidders play β and Bidder 0 selects $z \geq x$, she will win the first item if z is larger than the highest valuation of the other bidders, and will pay the second highest bid, $\beta(Y_n)$. Otherwise, she will win the second item if x is larger than the second highest valuation of the other bidders. Bidder 0's payoff function can be written as

$$\begin{aligned} u(x,z \mid z \geq x) = {} & \Pr(Y_n < z)[x - E[\beta(Y_n) \mid Y_n < z]] \\ & + \Pr(Y_{n-2} < x \leq z < Y_n)[x - E[Y_{n-2} \mid Y_{n-2} < x \leq z < Y_n]]. \end{aligned} \tag{6}$$

The first term results from the event $Y_n < z$ in which Bidder 0 wins the first auction. The second term captures the case in which Bidder 0 loses the first auction and wins the second by bidding the revealed valuation of the fourth highest bidder.

We now consider the case where Bidder 0 chooses $z \leq x$. If z is larger than the highest valuation of the other bidders, Bidder 0 will win the first item. Otherwise, Bidder 0 may still be able to win the second or third item, depending upon the revealed valuations of the other bidders.

There are four non-zero outcomes:

- Case 1: $Y_{n-2} < z < Y_n$. Bidder 0 loses the first item; however, both z and x are larger than the third highest valuation among the other bidders and Bidder 0 will be able to win one of the next two auctions and expect to pay $E[Y_{n-2} \mid Y_{n-2} < z < Y_n]$.

- Case 2: $Y_{n-3} < z < Y_{n-2}$. In this case, Bidder 0 still loses the first item. However, the bidders with the second and third highest valuations infer from the first auction that the fourth highest valuation is z and, as a result, will bid z in the second auction. This allows Bidder 0 to outbid bidder Y_{n-2} by bidding $z+\varepsilon$, thus stealing an item when she is not one of the three highest valuing bidders.
- Case 3: $z < Y_{n-3} < x$. In this case, the bidder with the third highest valuation will bid what appears to be the fourth highest value. There are two sub-cases. The first sub-case is $x < Y_{n-2}$, so Bidder 0 needs to bid $E[Y_{n-2} \mid z < Y_{n-3} < x < Y_{n-2}] + \varepsilon$ to outbid the bidder with the third highest type. The second sub-case is $z < Y_{n-3} < Y_{n-2} < x$, so Bidder 0 does not need to add ε to her bid to win the second/third item.

As ε goes to zero, asymptotically, the payoff function can be written as

$$\begin{aligned} u(x,z \mid z \le x) = {} & \Pr(Y_n < z)[x - E[\beta(Y_n) \mid Y_n < z]] \\ & + \Pr(Y_{n-2} < z < Y_n)[x - E[Y_{n-2} \mid Y_{n-2} < z < Y_n]] \\ & + \Pr(Y_{n-3} < z < Y_{n-2})[x - z] \\ & + \Pr(z < Y_{n-3} < x)[x - E[Y_{n-3} \mid z < Y_{n-3} < x]]. \end{aligned} \tag{7}$$

As a condition of symmetric equilibrium, equation 4 is also true for sequential Vickrey auctions. We now present our second main result.

Theorem 2 *In the sequential Vickrey auctions with full bid revelation, there does not exist a symmetric, pure-strategic equilibrium.*

We prove the result by contradiction again, as detailed in the Appendix. Similar to the previous discussion, the above theorem is true when the number of items for sale is at least three. However, in a two-stage sequential Vickrey auction (only two items for sale), it is easy to prove that there exists a symmetric equilibrium despite the reveal-all-bid price quote. This result holds because, in the second auction, bidders will bid their true valuations whether they have complete or incomplete information about the other bidders' valuations. Similar solutions can be found in much of the literature on second-price sequential auctions, see Elmaghraby [2003].

5. Discussion and Conclusion

This paper extends the classic Weber [1983]'s model to consider an auction policy in which all bids are revealed after the auction is over, which is a common rule both online and in traditional auctions like English auctions. We show that the existence of a pure-strategic equilibrium is not guaranteed in some important classes of sequential auctions. In particular, we prove that there does not exist a symmetric, pure-strategic equilibrium in sequences of either first- or second-price auctions. The nonexistence of equilibrium holds in first-price auctions when the number of item for sale is at least two, and in Vickrey auctions when the number of items for sale is at least three.

This paper demonstrates the importance of the information revelation policy in an auction mechanism design. Normally, the selection of the auction type is usually determined by the sellers, who want to optimize the revenue. The lack of an equilibrium for the buyers means that the outcome of the auction is less predictable, which may negatively impact the revenue achieved by the seller. Moreover, the market is more likely to be inefficient.

When the majority of existing literature is focused on sequential auction models having equilibria, the discovery that the majority of sequential online auctions are of a form that have no symmetric equilibrium helps us better understand online behavior. The insights provided by this analysis could guide sellers towards methods of timing the listing of auctions such that they avoid listing substitutable items on the same website within a short time period, thereby attracting the same subset of bidders.

These results do not rule out symmetric or asymmetric mixed-strategy equilibria in the models although we believe that it will be analytically challenging. In a mixed-strategic equilibrium, bidders cannot accurately infer others' valuations after the auction. As a result, information in the next auction remains incomplete. Suppose in a two-item sequential first-price auction, it is optimal for bidders to use a Weber's equilibrium strategy in the second auction. However, in the first auction in which bidders use a mixed-strategy, it is likely that bidders will not bid higher than their second-auction strategies; as a result, the expected revenue from the first auction might be less than the revenue from the second auction. Hence, if there were a mixed strategy equilibrium, the seller might be worse off by offering two identical items for sale to the same pool of bidders.

We recognize that the assumption that strategy functions be strictly increasing and differentiable, though commonly used in sequential auction models, is restrictive. Changing this, or other assumptions, might affect the results. For example, if bidders become asymmetric (e.g. they do not have the same payoff functions) or bidders have multiple-unit demand or the number of bidders is not fixed, the final results will most likely be different. In actual online auctions, situations can be even more complicated by the details of real markets. For instance, few real products have strictly independent values, and bidders often adjust their estimates of their valuations as they see what others are willing to pay. Exploring these potential extensions provides new venues for future research.

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APPENDIX

Proof of Theorem 1: We prove the result by contradiction. We first assume that in the sequential first-price auction model in which all bids are revealed, there exists a symmetric, pure-strategic equilibrium, β.

When $z \ge x$, we refer to it as a right hand side (RHS) deviation. Similarly, $z \le x$ is a left hand side (LHS) deviation. In the following discussion, we replace β with β_{RHS} in $u(x,z \mid z \ge x)$ and replace β with β_{LHS} in $u(x,z \mid z \le x)$. Our target is to solve β_{RHS} and β_{LHS} respectively from equations (2) and (3). From the assumption that β is a symmetric pure strategy that is continuous, strictly increasing, and invertible, it follows that $\beta_{\text{RHS}}(x) = \beta_{\text{LHS}}(x)$.

We can rewrite equation (4) as

$$\frac{u(x,z \mid z \ge x) - u(x)}{z-x} \le 0, \forall z \ge x, \text{and}$$
$$\frac{u(x,z \mid z \le x) - u(x)}{z-x} \ge 0, \forall z \le x.$$

As a result, there must exist a small enough deviation $|z-x|$ such that taking the first order condition on both sides of x yields

$$\frac{\partial u(x,z \mid z \ge x)}{\partial z} \le 0, \forall z \ge x, and \qquad \text{(A−1)}$$
$$\frac{\partial u(x,z \mid z \le x)}{\partial z} \ge 0, \forall z \le x. \qquad \text{(A−2)}$$

Equation (A-1) can be derived from equation (2). When $z \ge x$, we have

$$\begin{aligned} & u(x,z \mid z \ge x) \\ &= F_n(z)[x - \beta_{\text{RHS}}(z)] \\ &\quad + n(1-F(z))F^{n-1}(x) \times \\ &\quad [x - \frac{\int_{-\infty}^{x}\int_{z}^{+\infty} y_{n-1} f_{n-1,n}(y_{n-1},y_n)dy_n dy_{n-1}}{n(1-F(z))F^{n-1}(x)}] \\ &= F_n(z)[x-\beta_{\text{RHS}}(z)] + n(1-F(z))F^{n-1}(x)x \\ &\quad - \int_{-\infty}^{x}\int_{z}^{+\infty} y_{n-1} n(n-1) f(y_{n-1}) f(y_n) F^{n-2}(y_{n-1}) dy_n dy_{n-1} \\ &= F_n(z)[x-\beta_{\text{RHS}}(z)] + n(1-F(z))F^{n-1}(x)x \\ &\quad - \int_{-\infty}^{x} y_{n-1} n(n-1) f(y_{n-1}) F^{n-2}(y_{n-1}) dy_{n-1}[1-F(z)]. \end{aligned}$$

Solving the first order condition, we obtain

$$\begin{aligned} \frac{\partial u(x,z \mid z \ge x)}{\partial z} &= f_n(z)[x-\beta_{\text{RHS}}(z)] - F_n(z)\beta_{\text{RHS}}'(z) - nf(z)F^{n-1}(x)x \\ &\quad + \int_{-\infty}^{x} y_{n-1} n(n-1) f(y_{n-1}) F^{n-2}(y_{n-1}) dy_{n-1} f(z) \\ &\le 0. \end{aligned}$$

From the definition of equilibrium, we know that $u(x,z \mid z \ge x)$ is maximized at $z = x$. Because $F_n(x) = F^n(x)$ and $f_n(x) = nf(x)F^{n-1}(x)$, setting $z = x$ allows us to reduce the above equation to

$$\frac{\partial u(x,z\mid z\ge x)}{\partial z}\Big|_{z=x} = -[F_n(x)\beta_{\text{RHS}}(x)]' + \int_{-\infty}^{x} y_{n-1}n(n-1)f(y_{n-1})F^{n-2}(y_{n-1})dy_{n-1}f(x)$$
$$= 0.$$

As a result,

$$[F_n(x)\beta_{\text{RHS}}(x)]' = \int_{-\infty}^{x} n(n-1)y_{n-1}f(y_{n-1})F^{n-2}(y_{n-1})dy_{n-1}f(x). \qquad \text{(A−3)}$$

Now, let us consider the case where $z \le x$. Equation (3) can be rewritten as:

$$\begin{aligned}
&u(x,z\mid z\le x)\\
&= \Pr(Y_n < z)[x-\beta_{\text{LHS}}(z)]\\
&\quad + \Pr(Y_{n-1} < z < Y_n)[x - E[Y_{n-1}\mid Y_{n-1} < z < Y_n]]\\
&\quad + \Pr(Y_{n-2} < z < Y_{n-1})[x-z]\\
&\quad + \Pr(z < Y_{n-2} < x)[x - E[Y_{n-2}\mid z < Y_{n-2} < x]]\\
&= F_n(z)[x-\beta_{\text{LHS}}(z)] + n(1-F(z))F^{n-1}(z)x\\
&\quad - \int_{-\infty}^{z}\int_{z}^{+\infty} y_{n-1}n(n-1)f(y_n)f(y_{n-1})F^{n-2}(y_{n-1})dy_n dy_{n-1}\\
&\quad + \int_{-\infty}^{z}\int_{z}^{+\infty} n(n-1)(n-2)F(y_{n-2})^{n-3}f(y_{n-2})f(y_{n-1})\times\\
&\quad [1-F(y_{n-1})]dy_{n-1}dy_{n-2}[x-z]\\
&\quad + x\int_{z}^{x} f_{n-2}(y_{n-2})dy_{n-2} - \int_{z}^{x} y_{n-2}f_{n-2}(y_{n-2})dy_{n-2}\\
&= F_n(z)[x-\beta_{\text{LHS}}(z)] + n(1-F(z))F^{n-1}(z)x\\
&\quad - \int_{-\infty}^{z} n(n-1)y_{n-1}f(y_{n-1})F^{n-2}(y_{n-1})dy_{n-1}[1-F(z)]\\
&\quad + n(n-1)(n-2)\int_{-\infty}^{z} F(y_{n-2})^{n-3}f(y_{n-2})dy_{n-2}\times\\
&\quad \int_{z}^{+\infty} f(y_{n-1})[1-F(y_{n-1})]dy_{n-1}[x-z]\\
&\quad + x\int_{z}^{x} f_{n-2}(y_{n-2})dy_{n-2} - \int_{z}^{x} y_{n-2}f_{n-2}(y_{n-2})dy_{n-2}\\
&= F_n(z)[x-\beta_{\text{LHS}}(z)] + n(1-F(z))F^{n-1}(z)x\\
&\quad - \int_{-\infty}^{z} n(n-1)y_{n-1}f(y_{n-1})F^{n-2}(y_{n-1})dy_{n-1}[1-F(z)]\\
&\quad + \frac{n(n-1)}{2}F^{n-2}(z)[1-F(z)]^2[x-z]\\
&\quad + x\int_{z}^{x} f_{n-2}(y_{n-2})dy_{n-2} - \int_{z}^{x} y_{n-2}f_{n-2}(y_{n-2})dy_{n-2}.
\end{aligned}$$

The first order condition is

$$\begin{aligned}
\frac{\partial u(x,z\mid z\le x)}{\partial z} &= f_n(z)[x-\beta_{\text{LHS}}(z)] - F_n(z)\beta_{LHS}'(z)\\
&\quad + n(n-1)F^{n-2}(z)f(z)[1-F(z)]x - nF^{n-1}(z)f(z)x\\
&\quad - n(n-1)zf(z)F^{n-2}(z)[1-F(z)]\\
&\quad + \int_{-\infty}^{z} n(n-1)y_{n-1}f(y_{n-1})F^{n-2}(y_{n-1})dy_{n-1}f(z)\\
&\quad + \frac{n(n-1)(n-2)}{2}F^{n-3}(z)f(z)[1-F(z)]^2[x-z]\\
&\quad - n(n-1)F^{n-2}(z)[1-F(z)]f(z)[x-z]\\
&\quad - \frac{n(n-1)}{2}F^{n-2}(z)[1-F(z)]^2\\
&\quad - f_{n-2}(z)x + zf_{n-2}(z)\\
&\ge 0.
\end{aligned}$$

At $z = x$,

$$\begin{aligned}
&\frac{\partial u(x,z\mid z\le x)}{\partial z}\Big|_{z=x}\\
&= f_n(x)[x-\beta_{\text{LHS}}(x)] - F_n(x)\beta_{\text{LHS}}'(x) - nF^{n-1}(x)f(x)x\\
&\quad + \int_{-\infty}^{x} n(n-1)y_{n-1}f(y_{n-1})F^{n-2}(y_{n-1})dy_{n-1}f(x)\\
&\quad - \frac{n(n-1)}{2}F^{n-2}(x)[1-F(x)]^2
\end{aligned}$$

$$
\begin{aligned}
&= -[F_n(x)\beta_{\text{LHS}}(x)]' \\
&\quad + \int_{-\infty}^{x} n(n-1) y_{n-1} f(y_{n-1}) F^{n-2}(y_{n-1}) dy_{n-1} f(x) \\
&\quad - \frac{n(n-1)}{2} F^{n-2}(x)[1-F(x)]^2 \\
&\geq 0.
\end{aligned}
$$

Thus, we have

$$[F_n(x)\beta_{\text{LHS}}(x)]' \leq \int_{-\infty}^{x} n(n-1) y_{n-1} f(y_{n-1}) F^{n-2}(y_{n-1}) dy_{n-1} f(x) - \frac{n(n-1)}{2} F^{n-2}(x)[1-F(x)]^2. \quad \text{(A-4)}$$

We have now obtained a closed form solution for computing both $\beta_{\text{RHS}}(x)$ and $\beta_{\text{LHS}}(x)$. Combining equations (A-3) and (A-4), and noting that $\frac{n(n-1)}{2} F^{n-2}(x)[1-F(x)]^2 > 0$, we see that

$$
\begin{aligned}
[F_n(x)\beta_{\text{LHS}}(x)]' &= \int_{-\infty}^{x} n(n-1) y_{n-1} f(y_{n-1}) F^{n-2}(y_{n-1}) dy_{n-1} df(x) \\
&\quad - \frac{n(n-1)}{2} F^{n-2}(x)[1-F(x)]^2 \\
&< \int_{-\infty}^{x} n(n-1) y_{n-1} f(y_{n-1}) F^{n-2}(y_{n-1}) dy_{n-1} f(x) \\
&= [F_n(x)\beta_{\text{RHS}}(x)]'.
\end{aligned}
$$

Because $\beta_{\text{RHS}}(0) = \beta_{\text{LHS}}(0) = \beta(0) = 0$, $F_n(0)\beta_{\text{RHS}}(0) = F_n(0)\beta_{\text{LHS}}(0) = 0$. By integrating both sides of the above equation, we obtain

$$\beta_{LHS}(x) < \beta_{RHS}(x), \forall x, \quad \text{(A-5)}$$

which implies that β does not exist for any x. This result contradicts the assumption that β is strictly increasing and continuous, having $\beta_{LHS}(x) = \beta_{RHS}(x)$. Thus, in the sequential first-price auctions with all bids revealed, there does not exist a symmetric, pure-strategic equilibrium.

Proof of Theorem 2: We prove the result by contradiction. We first assume that there exists a symmetric, pure-strategic equilibrium, β, in the sequential Vickrey auctions.

Again, we refer to the case where the bidder selects $z \geq x$ as a right hand side (RHS) deviation. Similarly, $z \leq x$ is a left hand side (LHS) deviation. In the following discussion, we replace β with β_{RHS} in $u(x, z \mid z \geq x)$, and replace β with β_{LHS} in $u(x, z \mid z \leq x)$. Furthermore, we replace $E[\beta(Y_n) \mid Y_n < z]$ with Θ_{RHS} in $u(x, z \mid z \geq x)$ and replace $E[\beta(Y_n) \mid Y_n < z]$ with Θ_{LHS} in $u(x, z \mid z \leq x)$. Our target is to solve β_{RHS} and β_{LHS} respectively from equations (6) and (7). Because we assume that β is continuous, strictly increasing, and invertible, we should find that $\beta_{\text{RHS}}(x) = \beta_{\text{LHS}}(x)$ and $\Theta_{\text{RHS}}(x) = \Theta_{\text{LHS}}(x)$.

We can rewrite equation (6) as follows.

$$
\begin{aligned}
&u(x, z \mid z \geq x) \\
&= F_n(z)[x - \Theta_{\text{RHS}}(z)] \\
&\quad + \int_{-\infty}^{x} \int_{z}^{+\infty} f_{n-2,n}(y_{n-2}, y_n) dy_n dy_{n-2} \times \\
&\quad [x - \frac{\int_{-\infty}^{x} \int_{z}^{+\infty} y_{n-2} f_{n-1,n}(y_{n-2}, y_n) dy_n dy_{n-2}}{\int_{-\infty}^{x} \int_{z}^{+\infty} f_{n-2,n}(y_{n-2}, y_n) dy_n dy_{n-2}}]
\end{aligned}
$$

$$
\begin{aligned}
&= F_n(z)[x-\Theta_{\mathrm{RHS}}(z)] \\
&\quad + \int_{-\infty}^{x}\int_{z}^{+\infty}[n(n-1)(n-2)f(y_{n-2})f(y_n)\times \\
&\quad F^{n-3}(y_{n-2})[F(y_n)-F(y_{n-2})]]dy_n dy_{n-2}x \\
&\quad - \int_{-\infty}^{x}\int_{z}^{+\infty}[n(n-1)(n-2)y_{n-2}f(y_{n-2})f(y_n)\times \\
&\quad F^{n-3}(y_{n-2})[F(y_n)-F(y_{n-2})]]dy_n dy_{n-2} \\
&= F_n(z)[x-\Theta_{\mathrm{RHS}}(z)] \\
&\quad + n(n-1)(n-2)F^{n-2}(x)[1-F(z)][\frac{1+F(z)}{2(n-2)}-\frac{F(x)}{n-1}]x \\
&\quad - \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})dy_{n-2}\,\frac{1-F^2(z)}{2} \\
&\quad + \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-2}(y_{n-2})dy_{n-2}[1-F(z)].
\end{aligned}
$$

Solving the first order condition, we obtain

$$
\begin{aligned}
&\frac{\partial u(x,z\mid z\ge x)}{\partial z} \\
&= f_n(z)[x-\Theta_{\mathrm{RHS}}(z)]-F_n(z)\Theta'_{\mathrm{RHS}}(z) \\
&\quad + n(n-1)(n-2)F^{n-2}(x)[\frac{-2F(z)f(z)}{2(n-2)}+\frac{F(x)f(z)}{n-1}]x \\
&\quad - \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})dy_{n-2}\times\frac{[-2F(z)f(z)]}{2} \\
&\quad + \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-2}(y_{n-2})dy_{n-2}\times[-f(z)] \\
&= -[F_n(z)\Theta_{\mathrm{RHS}}(z)]' + nf(z)F^{n-1}(z)x \\
&\quad + n(n-1)(n-2)F^{n-2}(x)[\frac{-2F(z)f(z)}{2(n-2)}+\frac{F(x)f(z)}{n-1}]x \\
&\quad + F(z)f(z)\int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})dy_{n-2} \\
&\quad - f(z)\int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-2}(y_{n-2})dy_{n-2} \\
&= -[F_n(z)\Theta_{\mathrm{RHS}}(z)]' + nf(z)F^{n-1}(z)x \\
&\quad - n(n-1)(n-2)F^{n-2}(x)[\frac{F(z)}{n-2}-\frac{F(x)}{n-1}]f(z)x \\
&\quad + \int_{-\infty}^{x}[n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})\times \\
&\quad [F(z)-F(y_{n-2})]]dy_{n-2}f(z) \\
&\le 0.
\end{aligned}
$$

From the definition of equilibrium, we know that $u(x,z\mid z\ge x)$ reaches its optimal point when $z=x$. By setting $z=x$, the above equation reduces to

$$
\begin{aligned}
&\frac{\partial u(x,z\mid z\ge x)}{\partial z}\Big|_{z=x} \\
&= -[F_n(x)\Theta_{\mathrm{RHS}}(x)]' \\
&\quad + \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})\times \\
&\quad F^{n-3}(y_{n-2})[F(x)-F(y_{n-2})]dy_{n-2}f(x) \\
&= 0.
\end{aligned}
$$

As a result,

$$
[F_n(x)\Theta_{\mathrm{RHS}}(x)]' = \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})\times[F(x)-F(y_{n-2})]dy_{n-2}f(x). \quad \text{(A-6)}
$$

Now, let us consider the case where $z\le x$. Equation (7) can be rewritten as

$$
\begin{aligned}
&u(x,z \mid z \le x) \\
&\quad = F_n(z)[x-\Theta_{\mathrm{LHS}}(z)] \\
&\qquad + \int_{-\infty}^{z}\int_{z}^{+\infty} f_{n-2,n}(y_{n-2},y_n)dy_n dy_{n-2} \times \\
&\qquad [x-\frac{\int_{-\infty}^{z}\int_{z}^{+\infty} y_{n-2} f_{n-1,n}(y_{n-2},y_n)dy_n dy_{n-2}}{\int_{-\infty}^{x}\int_{z}^{+\infty} f_{n-2,n}(y_{n-2},y_n)dy_n dy_{n-2}}] \\
&\qquad + \int_{-\infty}^{z}\int_{z}^{+\infty} \frac{n!}{(n-4)!2!} f(y_{n-3}) f(y_{n-2}) F^{n-4}(y_{n-3}) \times \\
&\qquad [1-F(y_{n-2})]^2 dy_{n-3} dy_{n-2}(x-z) \\
&\qquad + \int_{z}^{x} \frac{n!}{(n-4)!3!} f(y_{n-3}) F^{n-4}(y_{n-3})[1-F(y_{n-2})]^3 dy_{n-3} x \\
&\qquad - \int_{z}^{x} \frac{n!}{(n-4)!3!} y_{n-3} f(y_{n-3}) F^{n-4}(y_{n-3})[1-F(y_{n-2})]^3 dy_{n-3} \\
&\quad = F_n(z)[x-\Theta_{\mathrm{LHS}}(z)] \\
&\qquad + \int_{-\infty}^{z}\int_{z}^{+\infty} n(n-1)(n-2) f(y_{n-2}) f(y_n) F^{n-3}(y_{n-2}) \times \\
&\qquad [F(y_n)-F(y_{n-2})]dy_n dy_{n-2} x \\
&\qquad - \int_{-\infty}^{z}\int_{z}^{+\infty} n(n-1)(n-2) y_{n-2} f(y_{n-2}) f(y_n) \times \\
&\qquad F^{n-3}(y_{n-2})[F(y_n)-F(y_{n-2})]dy_n dy_{n-2} \\
&\qquad + \frac{n!}{(n-4)!2!}\int_{-\infty}^{z} f(y_{n-3}) F^{n-4}(y_{n-3}) dy_{n-3} \times \\
&\qquad \int_{z}^{+\infty} f(y_{n-2})[1-F(y_{n-2})]^2 dy_{n-2}(x-z) \\
&\qquad + \frac{n!}{(n-4)!3!}\int_{z}^{x} [F^{n-4}(y_{n-3}) - 3F^{n-3}(y_{n-3}) \\
&\qquad + 3F^{n-2}(y_{n-3}) - F^{n-1}(y_{n-3})]dF(y_{n-3}) x \\
&\qquad - \int_{z}^{x} \frac{n!}{(n-4)!3!} y_{n-3} f(y_{n-3}) F^{n-4}(y_{n-3})[1-F(y_{n-2})]^3 dy_{n-3} \\
&= F_n(z)[x-\Theta_{\mathrm{LHS}}(z)] \\
&+ n(n-1)(n-2)F^{n-2}(z)[1-F(z)][\frac{1+F(z)}{2(n-2)} - \frac{F(z)}{n-1}]x \\
&- \int_{-\infty}^{z} n(n-1)(n-2) y_{n-2} f(y_{n-2}) F^{n-3}(y_{n-2}) dy_{n-2}\ \frac{1-F^2(z)}{2} \\
&+ \int_{-\infty}^{z} n(n-1)(n-2) y_{n-2} f(y_{n-2}) F^{n-2}(y_{n-2}) dy_{n-2}[1-F(z)] \\
&+ \frac{n!}{(n-3)!3!} F^{n-3}(z)[1-F(z)]^3(x-z) \\
&+ \frac{n!}{(n-4)!3!}[\frac{F^{n-3}(x)-F^{n-3}(z)}{n-3} - \frac{3[F^{n-2}(x)-F^{n-2}(z)]}{n-2} + \\
&\frac{3[F^{n-1}(x)-F^{n-1}(z)]}{n-1} - \frac{F^n(x)-F^n(z)}{n}]x \\
&- \int_{z}^{x} \frac{n!}{(n-4)!3!} y_{n-3} f(y_{n-3}) F^{n-4}(y_{n-3})[1-F(y_{n-2})]^3 dy_{n-3}.
\end{aligned}
$$

The first order condition is:

$$
\begin{aligned}
&\frac{\partial u(x,z \mid z \le x)}{\partial z} \\
&= -[F_n(z)\Theta_{\text{LHS}}(z)]' + nf(z)F^{n-1}(z)x \\
&\quad + n(n-1)(n-2)^2 F^{n-3}(z)[\frac{1+F(z)}{2(n-2)} - \frac{F(z)}{n-1}][1-F(z)]f(z)x \\
&\quad + n(n-1)(n-2)F^{n-2}(z)[\frac{-2F(z)f(z)}{2(n-2)} - \frac{f(z)[1-F(z)]-F(z)f(z)}{n-1}]f(z)x \\
&\quad + n(n-1)(n-2)zf(z)F^{n-2}(z)[1-F(z)] \\
&\quad - n(n-1)(n-2)zf(z)F^{n-3}(z)\frac{1-F^2(z)}{2} \\
&\quad - \frac{-2F(z)f(z)}{2}\int_{-\infty}^{z} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})dy_{n-2} \\
&\quad - f(z)\int_{-\infty}^{z} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-2}(y_{n-2})dy_{n-2} \\
&\quad + \frac{n!}{(n-3)!3!}F^{n-4}f(z)[1-F(z)]^3(x-z) \\
&\quad - \frac{n!}{(n-3)!3!}F^{n-3}(z)3[1-F(z)]^2 f(z)(x-z) \\
&\quad - \frac{n!}{(n-3)!3!}F^{n-3}(z)[1-F(z)]^3 \\
&\quad + \frac{n!}{(n-4)!3!}F^{n-4}(z)[-1+3F(z)-3F^2(z)+F^3(z)]f(z)x \\
&\quad + \frac{n!}{(n-4)!3!}F^{n-4}(z)[1-F(z)]^3 f(z)z \\
&= -[F_n(z)\Theta_{\text{LHS}}(z)]' + nf(z)F^{n-1}(z)x \\
&\quad - n(n-1)(n-2)F^{n-2}(z)[\frac{F(z)f(z)}{n-2} - \frac{F(z)f(z)}{n-1}]f(z)x \\
&\quad + n(n-1)(n-2)F^{n-3}(z)\times[\frac{1-F(z)^2}{2} - \frac{(n-2)F(z)[1-F(z)]}{n-1} - \frac{F(z)[1-F(z)]}{n-1}]f(z)x \\
&\quad + n(n-1)(n-2)zf(z)F^{n-3}(z)[F(z)(1-F(z)) - \frac{1-F^2(z)}{2}] \\
&\quad + \int_{-\infty}^{z} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})[F(z)-F(y_{n-2})]dy_{n-2}f(z) \\
&\quad + \frac{n!}{(n-3)!3!}F^{n-4}(z)[1-F(z)]^2[(n-3)(x-z)f(z) - \\
&\quad (n-3)(x-z)f(z)F(z) - 3(x-z)f(z)F(z) - F(z)(1-F(z))] \\
&\quad - \frac{n!}{(n-4)!3!}F^{n-4}(z)[1-F(z)]^3 f(z)x + \frac{n!}{(n-4)!3!}F^{n-4}(z)[1-F(z)]^3 f(z)x] \\
&= -[F_n(z)\Theta_{\text{LHS}}(z)]' + nf(z)F^{n-1}(z)x - nf(z)F^{n-1}(z)x \\
&\quad + n(n-1)(n-2)F^{n-3}(z)\frac{[1-F(z)]^2}{2}f(z)x \\
&\quad - n(n-1)(n-2)F^{n-3}(z)\frac{[1-F(z)]^2}{2}f(z)z \\
&\quad + \int_{-\infty}^{z} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})\times \\
&\quad [F(z)-F(y_{n-2})]dy_{n-2}f(z) \\
&\quad + \frac{n!}{(n-3)!3!}F^{n-4}(z)[1-F(z)]^2\times \\
&\quad [(n-3)(x-z)f(z) - n(x-z)f(z)F(z) - F(z)(1-F(z))] \\
&\quad + \frac{n!}{(n-4)!3!}F^{n-4}(z)[1-F(z)]^3 f(z)(z-x) \\
&\ge 0.
\end{aligned}
$$

At $z = x$, the above equation reduces to

$$
\begin{aligned}
&\frac{\partial u(x,z \mid z \le x)}{\partial z}\Big|_{z=x} \\
&= -[F_n(x)\Theta_{\text{LHS}}(x)]' \\
&\quad + \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})\times \\
&\quad [F(x)-F(y_{n-2})]dy_{n-2}f(x) \\
&\quad - \frac{n!}{(n-3)!3!}F^{n-3}(x)[1-F(x)]^3 \\
&= 0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
[F_n(x)\Theta_{\text{LHS}}(x)]' &= \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})\times[F(z)-F(y_{n-2})]dy_{n-2}f(x) \\
&\quad - \frac{n!}{(n-3)!3!}F^{n-3}(x)[1-F(x)]^3. \qquad \text{(A-7)}
\end{aligned}
$$

We have now obtained a closed form solution for computing both $\Theta_{\text{RHS}}(x)$ and $\Theta_{\text{LHS}}(x)$. Combining equations (A-6) and (A-7), and noting that $\frac{n!}{(n-3)!3!}F^{n-3}(x)[1-F(x)]^3 > 0$, we see that

$$\begin{aligned}
[F_n(x)\Theta_{\text{LHS}}(x)]' &= \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})\times[F(z)-F(y_{n-2})]dy_{n-2}f(x) \\
&\quad -\frac{n!}{(n-3)!3!}F^{n-3}(x)[1-F(x)]^3 \\
&< \int_{-\infty}^{x} n(n-1)(n-2)y_{n-2}f(y_{n-2})F^{n-3}(y_{n-2})\times[F(z)-F(y_{n-2})]dy_{n-2}f(x) \\
&= [F_n(x)\Theta_{\text{RHS}}(x)]'.
\end{aligned}$$

Because $\beta_{\text{RHS}}(0)=\beta_{\text{LHS}}(0)=\beta(0)=0$, $F_n(0)\Theta_{\text{RHS}}(0)=F_n(0)\Theta_{\text{LHS}}(0)=0$. Thus, integrating both side of the above equation, we obtain:

$$\Theta_{LHS}(x) < \Theta_{RHS}(x), \forall x,$$

and thus

$$\beta_{LHS}(x) < \beta_{RHS}(x), \forall x,$$

which implies that β does not exist for any x. This result contradicts the assumption that β is strictly increasing and continuous with $\beta_{LHS}(x)=\beta_{RHS}(x)$. Thus, in the sequential Vickrey auctions with all bids revealed, there does not exist a symmetric, pure-strategic equilibrium in general.