AN EFFICIENT METHOD FOR AN ILL-POSED PROBLEM —BAND-LIMITED EXTRAPOLATION BY REGULARIZATION

by

WEIDONG CHEN

B.S., Jiangsu College of Education, Nanjing, China, 1986

M.S., Hebei Institute of Technology, Tianjin, China, 1989

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the

requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics

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ABSTRACT

In this paper a regularized spectral estimation formula and a regularized iterative algorithm for band-limited extrapolation are presented. The ill-posedness of the problem is taken into account. First a Fredholm equation is regularized. Then it is transformed to a differential equation in the case where the time interval is $\mathbf{R}$. A fast algorithm to solve the differential equation by the finite differences is given and a regularized spectral estimation formula is obtained. Then a regularized iterative extrapolation algorithm is introduced and compared with the Papoulis and Gerchberg algorithm. A time-frequency regularized extrapolation algorithm is presented in the two-dimensional case. The Gibbs phenomenon is analyzed. Then the time-frequency regularized extrapolation algorithm is applied to image restoration and compared with other algorithms.

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ACKNOWLEDGEMENTS

Acknowledgment: The author would like to express appreciation to Professors Huanan Yang, Robert Burckel, Charles Moore, Mitchell Neilsen and Lige Li for their guidance in the course of this paper.

GLOSSARY OF SYMBOLS USED IN THIS PAPER

1. $\mathbf{F}$: Fourier transform on any space; $\mathbf{F}(f)$ is also denoted $\hat{f}$.

   $\mathbf{F}^{-1}$: inverse Fourier transform; $\mathbf{F}^{-1}(f)(t) := \frac{1}{2\pi}\hat{f}(-t) =: \tilde{f}(t)$.
2. $[-T, T]$: The time interval.
3. Supp F: the support of F, the closure of the complement of $F^{-1}(0)$.
4. $[-\Omega, \Omega]$: The band(=support) of Fourier transform.
5. $\mathbf{R}$: The set of real numbers.
6. $\mathbf{C}$: The set of complex numbers.
7. $f(t)$: The time domain signal.
8. $\hat{f}(\omega)$: The frequency domain signal.
9. η: The noise function on any subset of $\mathbf{R}$.
10. δ: The bound of the error energy on any subset of $\mathbf{R}$.
11. $f_\delta(t)$: The noisy time domain signal on any subset of $\mathbf{R}$.
12. α: The regularization parameter.
13. Θ: The stabilizing functional.
14. M^α: The smoothing functional.
15. $F_\alpha(\omega)$: The regularized frequency domain signal.
16. $f_\alpha(t)$: The regularized extrapolation.
17. $f_E(t)$: The exact time domain signal.
18. $f_T(t)$: The exact time domain signal restricted to $[-T, T]$.
19. $\widehat{f_E}(\omega)$: The exact frequency domain signal.
20. A: An operator from metric space D into metric space U.
21. $Az = u$, $z \in D$ and $u \in U$: The operator equation.
22. $R(u, \alpha)$: The regularizing operator for the above operator equation.
23. $d(\delta)$: The error of the regularized solution.

24. $F_{\alpha,j}$: The value of the regularized solution at ω_j.

25. $F_{\alpha,j}^h$: The value of numerical regularized solutions at ω_j with step size h.

26. $\mathbf{1}_S$: indicator function of subset S of $\mathbf{R}$.

27. P_T: The operation of multiplying a function on $\mathbf{R}$ by $\mathbf{1}_{[-T,T]}$ or $\mathbf{1}_{[-T_1,T_1]\times[-T_2,T_2]}$.

28. P_Ω: The operation of multiplying a function on $\mathbf{R}$ by $\mathbf{1}_{[-\Omega,\Omega]}$ or $\mathbf{1}_{[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]}$.

29. $A := B$ and $B =: A$ mean B is the definition of A.

30. $(*,*)$: the inner product in a Hilbert space.

31. Discrepancy: $\rho_U(Az_\alpha, u_\delta)$.

32. $\Re$: the real part of a complex number.

33. $\Im$: the imaginary part of a complex number.

34. $C_{0,\Omega}$: $\{F \in C[-\Omega,\Omega] | \text{supp} F \subset (-\Omega,\Omega)\}$.

35. $C^1[a,b]$: $\{F : F' \in C[a,b]\}$.

36. W_2^1: Sobolev space.

Chapter 1

INTRODUCTION

The extrapolation of band-limited signals, or functions has been discussed in many previous papers([1]-[10]). This procedure is widely used in many applied and theoretical fields such as Fourier analysis, spectral estimation and image restoration. The definition of band-limited functions is:

Definition: For a positive constant Ω, a function $f \in L^1(\mathbf{R})$ is said to be Ω*-band-limited* if $\hat{f}(\omega) = 0\ \forall \omega \in \mathbf{R} \setminus [-\Omega, \Omega]$. Here $\hat{f}$ is the *Fourier transform* of f:

$$\hat{f}(\omega) := \mathbf{F}(f)(\omega) := \int_{-\infty}^{+\infty} f(t)e^{i\omega t}dt,\ \omega \in \mathbf{R} \tag{1}$$

We then have the *inversion formula*:

$$f(t) := \mathbf{F}^{-1}(\hat{f})(t) := \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega)e^{-i\omega t}d\omega,\ \text{a.e. } t \in \mathbf{R} \tag{2}$$

This integral is defined for all $t \in \mathbf{C}$ and furnishes an analytic extension of f to $\mathbf{C}$.

We consider the so-called *band-limited extrapolation problem*:

Assume that $f : \mathbf{R} \to \mathbf{R}$ is an Ω-band-limited function and T is a positive constant.

$$\begin{aligned} &\text{Given} \quad f(t) \quad \text{for} \quad t \in [-T, T], \\ &\text{find} \quad f(t) \quad \text{for} \quad t \in \mathbf{R}\setminus[-T, T]. \end{aligned} \tag{3}$$

The fact, noted above, that f is effectively an entire function means that its values on $[-T, T]$ determine all its values. Hence, this problem is, in principle, solvable.

From the knowledge of f only on $[-T,T]$ we cannot construct $\hat{f}$, and so we cannot use (2) to find the values of f in $\mathbf{R}\setminus[-T,T]$. What we plan to do is to solve the Fredholm integral equation

$$AF(t) := \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega)e^{-i\omega t}d\omega = f(t),\ t\in[-T,T] \tag{4}$$

for

$$F\in C_{0,\Omega} := \{F\in C[-\Omega,\Omega]|\text{supp}F\subset(-\Omega,\Omega)\}$$

given

$$f\in L^2 := L^2[-T,T].$$

We will want F to be supported in $[-\Omega,\Omega]$ and F to lie in $L^2(\mathbf{R})$, so that we can form $\tilde{F}(t) = \frac{1}{2\pi}\hat{F}(-t)$. Equation (4) says that the function $\tilde{F}$, which lies in $L^2(\mathbf{R})$, satisfies

$$\tilde{F}(t) = f(t) \quad \text{for all} \quad t\in[-T,T].$$

Then $\tilde{F}$ on the whole of $\mathbf{R}$ is our candidate to solve the original problem (3) of recovering (extrapolating) f from knowledge only of $f|_{[-T,T]}$.

Note that A is a bounded linear operator from the max-norm into the L^2-norm, where

$$||F||_{C_{0,\Omega}} := \max_{-\Omega\le\omega\le\Omega}|F(\omega)|, \quad ||f||_{L^2} := \{\int_{-T}^{T}|f(t)|^2dt\}^{1/2}.$$

The problem of solving (4) is ill-posed on the pair of spaces $(C_{0,\Omega},L^2)$ ([11]-[13]), as we show by an example in section 2.2, after the definition of "ill-posedness" is given in section 2.1.

In 1974 and 1975, Papoulis and Gerchberg ([1], [2]) made a pioneering contribution to this important problem. They designed the following iterative algorithm for an Ω-band-limited function $f\in L^2(\mathbf{R})$ for which only $f(t)$ for $t\in[-T,T]$ is given .

Papoulis-Gerchberg algorithm:

$$f^{[0]} := P_T f.$$

For $l = 0, 1, 2, ...,$

$$f^{[l+1]} := P_T f + (I - P_T)\mathbf{F}^{-1} P_\Omega \mathbf{F} f^{[l]},$$

where

$$P_T(f) := f \cdot \mathbf{1}_{[-T,T]}, \qquad P_\Omega(\hat{f}) := \hat{f} \cdot \mathbf{1}_{[-\Omega,\Omega]}.$$

The convergence $||f^{[l]} - f||_{L^2(\mathbf{R})} \to 0$ is proven in [1] by Papoulis.

Here $\mathbf{F}$ designates the Fourier transformation in $L^2(\mathbf{R})$. Recall that $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is dense in $L^2(\mathbf{R})$ and that according to Plancherel's theorem (see, for example, [29]), $\hat{f} \in L^2(\mathbf{R})$ and $||\hat{f}||_{L^2} = \frac{1}{\sqrt{2\pi}} ||f||_{L^2}$ for every $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Consequently, the Fourier transformation on $L^1(\mathbf{R})$ can be uniquely extended to a $\frac{1}{\sqrt{2\pi}}$-multiple of a bijective isometry (= unitary operator) of $L^2(\mathbf{R})$. We will continue to designate it by $\mathbf{F}$ and $\wedge$. Instead of (1), the $f \in L^2(\mathbf{R})$ satisfy

$$\hat{f}(\omega) = \mathbf{F}(f)(\omega) = \lim_{N\to\infty} \widehat{P_N f}(\omega)) = \lim_{N\to\infty} \int_{-N}^{N} f(t) e^{i\omega t} dt, \text{ for a. e. } \omega \in \mathbf{R}, \qquad (1)_2$$

since $P_N f = \mathbf{1}_{[-N,N]} \cdot f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and converge to f in the L^2-norm.

In [8] the Papoulis-Gerchberg algorithm was generalized to signals in the wavelet subspaces of $L^2(\mathbf{R})$. There are also many other extrapolation algorithms ([3]-[7], [9]). In [3]-[6], iterative methods were presented. In [7] Sabri and Steenaart presented an extrapolation process requiring only a single matrix operation, but their proof was shown inadequate in [4]. However [4] gives an alternative proof using Toeplitz equations and the solution can be stabilized by adding a small positive constant to the diagonal terms of the coefficient matrix [4].

Most algorithms perform poorly when the *observed* data f_δ contain noise or distortion:

$$f_\delta(t) := f(t) + \eta_\delta(t) \qquad t \in [-T, T] \qquad (5)$$

where $\eta_\delta(t)$ is the noise or distortion satisfying

$$\int_{-T}^{T} |\eta_\delta(t)|^2 dt \le \delta^2. \qquad (6)$$

The integral $\int_{-T}^{T} |\eta_\delta(t)|^2 dt$ is called the *error energy.*

Remark: The use of the word "energy" in this context is inspired by its meaning in electrical circuitry. Our integral has similar properties. The circuitry situation is the following:

If $v(t)$ is the voltage at time t measured across a resistor of resistance R (which can always be normalized by proper choice of units to be 1), the instantaneous power dissipated in the resistor at time t is

$$P(t) = v(t)i(t) = \frac{v^2(t)}{R} = i^2(t)R,$$

where $i(t)$ is the current at t. The energy expended in the circuit is

$$E = \int P(t)dt = \int \frac{v^2(t)}{R}dt = \int i^2(t)Rdt.$$

To understand the difficulty which noise creates, we observe that theoretically the Ω-band-limited extrapolation problem is equivalent to the problem of solving a Fredholm integral equation which is highly ill-posed (see definition 2.1).

In [11] the next theorem is asserted:

Theorem: Consider any pair of positive numbers ϵ and θ, with $\epsilon < \theta$, and any Ω-band-limited function $f : \mathbf{R} \to \mathbf{R}$ and any f_δ satisfying (5) and (6). Let P be any real number with $|P| > T$. Then there exists an Ω-band-limited function $\Psi_{\epsilon,P} \in C(\mathbf{R}) \cap L^1(\mathbf{R})$ that satisfies

$$||\Psi_{\epsilon,P} - f_\delta||_{L^\infty[-T,T]} \leq \epsilon,$$

but

$$|\Psi_{\epsilon,P}(P) - f(P)| \geq \theta.$$

This theorem shows that small noise in the interval $[-T, T]$ of the signal may produce large errors outside the interval. Worse, the extrapolation does not exist if η_δ is not the restriction to the interval $[-T, T]$ of a band-limited function.

The author is unable to verify the proof in [11], but can offer an example of his own as follows:

Let

$$f_n(t) := \frac{n[1 - \cos\Omega(t-n)]}{(t-n)^2} \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}).$$

Then $\hat{f}_n(\omega) = e^{in\omega} \cdot n\pi \cdot \mathbf{1}_{[-\Omega,\Omega]}(\omega) \cdot (\Omega - |\omega|)$ and $f_n \to 0$ uniformly in $[-T, T]$ as $n \to \infty$.

But $f_n(n) = \Omega^2 n/2 \to \infty$ as $n \to \infty$.

Therefore, given $0 < \epsilon < \theta$, we can pick n so that $\Omega^2 n/2 > \theta$ and so that $|f_n(t)| < \epsilon$ for all $t \in [-T, T]$, and we can consider

$$\eta := f_n, \qquad P := n.$$

Then $\eta \in L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$ is Ω-bandlimited and satisfies $|\eta(t)| < \epsilon$ for all $t \in [-T, T]$ and $|\eta(P)| > \theta$.

Regularization methods were introduced to solve this ill-posed problem([13]). There are many other papers where regularization methods have also been used([14]-[17]).

One regularization method often used is to discretize the Fredholm integral equation first. Then algebraic regularization is used for the resulting system of linear equations. Chen [15] proposed a regularization method of this type for the band-limited extrapolation problem and obtained the following estimate:

$$d^2 = O(\delta_n^2) + O(\alpha(\delta_n)) + O(\Omega^2/n^2) + O(T^2/n^2)$$

in which d is the error of the regularized extrapolation, δ_n^2 is a bound for the error energy, $\alpha(\delta_n)$ is the regularizing parameter.

In this paper, we reverse the order of regularization and discretization. First, in section III, we regularize equation (4) with a variational problem and obtain its Euler equation, which is an integro-differential equation. Its direct discretization will lead to an algorithm which requires $O(n^3)$ computational steps. In section IV, we show that the Euler equation can be converted to a simple ordinary differential equation, a standard discretization of which yields a fast algorithm for the problem of computing the Fourier transform of the function sought. In section V, we obtain a regularized Fourier transform formula. In section VI, we introduce a regularized iterative extrapolation algorithm. In section VII, we present some numerical results and compare them with the Papoulis-Gerchberg algorithm. Section VIII is a brief conclusion. In the appendix we prove the main Theorems stated in section III, IV and V.

Chapter 2

THE CONCEPTS OF WELL-POSED AND ILL-POSED PROBLEMS AND THE REGULARIZATION METHOD

In this chapter we introduce the concept of ill-posed problems and regularization methods. Much of this material is taken directly from [13].

2.1 The Definition of Ill-posed Problems

The concept of well-posed problems was introduced by Hadamard.

Here we borrow the following definition from [13]:

Definition 2.1: Assume $A : D \to U$ is a continuous operator, where D and U are metric spaces with distances $\rho_D(*, *)$ and $\rho_U(*, *)$, respectively. The problem of determining a solution z in the space D from the "initial data" u in the space U to the equation

$$Az = u$$

is said to be *well-posed* on the pair of metric spaces (D, U) in the sense of Hadamard if the following three conditions are satisfied ([13] pp.7,8):

i) For every element $u \in U$ there exists a solution z in the space D; in other words, the mapping A is surjective.

ii) The solution is unique; in other words, the mapping A is injective.

iii) The problem is *stable* in the spaces (D, U): $\forall \epsilon > 0, \exists \delta > 0$, such that

$$\rho_U(u_1, u_2) < \delta \Rightarrow \rho_D(z_1, z_2) < \epsilon.$$

In other words, the inverse mapping A^{-1} is uniformly continuous.

Problems that violate any of the three conditions are said to be *ill-posed*.

Remark. It should be emphasized that the definition of an ill-posed problem is with respect to a given pair of metric spaces (D, U); the same problem may be well-posed in other metrics.

2.2 Examples of ill-posed problems

A typical example is the $\mathbf{R}$-valued Fredholm integral equation of the first kind with continuous kernel K on $[c, d] \times [a, b]$ for which $\frac{\partial K}{\partial x}$ exists and is also continuous:

$$Az(x) := \int_a^b K(x,s)z(s)ds = u(x), \quad c \le x \le d \tag{$*$}$$

where z is the unknown function in a space D and u is a given function in a space U. We will measure changes in the right-hand member of the equation with the L^2-norm defined by

$$||u_1 - u_2||_{L^2} := [\int_c^d |u_1(x) - u_2(x)|^2 dx]^{1/2}$$

and measure changes in the solution in either the L^2-norm, or the C-norm defined by

$$||z_1 - z_2||_C := \max_{s \in [a,b]} |z_1(s) - z_2(s)|.$$

To see the ill-posedness of the problem, suppose z_1 satisfies (*) with $u = u_1$. Note that for any $\omega \in \mathbf{R},\ N \in \mathbf{N}$

$$z_2(s) := z_1(s) + N \sin \omega s,\ s \in [a, b]$$

is then a solution of equation (*) with right-hand member

$$u_2(x) := u_1(x) + N \int_a^b K(x,s) \sin \omega s ds,\ c \le x \le d.$$

Then

$$||u_1 - u_2||_{L^2} = |N| \{ \int_c^d | \int_a^b K(x,s) \sin \omega s ds|^2 dx \}^{1/2} \to 0$$

as $|\omega| \to \infty$. This can be seen as follows:

According to the Riemann-Lebesgue Lemma, for each $x_0 \in [c,d]$, $\int_a^b K(x_0,s)\sin\omega s ds \to 0$ as $|\omega| \to \infty$.

So $\forall \epsilon > 0$, $\exists M(x_0,\epsilon) > 0$ such that

$$|\int_a^b K(x_0,s)\sin\omega s ds| < \epsilon, \text{ whenever } |\omega| > M(x_0,\epsilon).$$

Since $|K(x,s) - K(x_0,s)| \to 0$ uniformly in s as $x \to x_0$

$$|\int_a^b K(x,s)\sin\omega s ds - \int_a^b K(x_0,s)\sin\omega s ds| \le \int_a^b |K(x,s) - K(x_0,s)| ds \to 0$$

uniformly in ω as $x \to x_0$. Consequently there exists $\delta(x_0,\epsilon) > 0$ which is independent of ω such that

$$|\int_a^b K(x,s)\sin\omega s ds| < \epsilon, \text{ whenever } |\omega| > M(x_0,\epsilon) \ \& \ x \in (x_0 - \delta(x_0,\epsilon), x_0 + \delta(x_0,\epsilon)).$$

Since $\{(x_0 - \delta(x_0,\epsilon), x_0 + \delta(x_0,\epsilon)) : x_0 \in [c,d]\}$ covers $[c,d]$, there exist $x_1, ..., x_n \in [c,d]$ such that

$$\bigcup_{i=1}^{n} (x_i - \delta(x_i,\epsilon), x_i + \delta(x_i,\epsilon)) \supset [c,d].$$

Let $M(\epsilon) = \max_{1 \le i \le n} M(x_i,\epsilon)$. Then

$$|\int_a^b K(x,s)\sin\omega s ds| < \epsilon, \ \forall \ |\omega| > M(\epsilon) \text{ and } \forall x \in [c,d].$$

However, as soon as $|\omega| \ge \pi/(b-a)$

$$||z_1 - z_2||_C = \max_{s\in[a,b]} |z_1(s) - z_2(s)| = \max_{s\in[a,b]} |N \sin\omega s| = |N|,$$

and

$$||z_1 - z_2||_{L^2} = |N| \{\int_a^b \sin^2 \omega s ds\}^{1/2}$$

$$= |N| [\frac{b-a}{2} - \frac{1}{2\omega} \sin\omega(b-a)\cos\omega(b+a)]^{1/2} \to |N| (\frac{b-a}{2})^{1/2}$$

as $|\omega| \to \infty$.

So this problem is ill-posed in each of the pairs (C, L^2) and (L^2, L^2).

2.3 The concept of regularizing operator

For the equation

$$Az = u \tag{6a}$$

suppose that the continuous operator A is such that its inverse A^{-1} exists but is *not* continuous on the set AD and $AD \neq U$.

In this case, the uniqueness condition is satisfied, but the existence condition and stability condition are not satisfied, so this problem is ill-posed.

Remark. If D and U are each Banach spaces and A is linear, $AD \neq U$ is a consequence of the fact that A^{-1} exists and is not continuous on the set AD; for the Open Map Theorem says that a continuous linear bijection between Banach spaces is necessarily bicontinuous.

Definition of regularizing operators (taken from [13], p.47):

Let D and U be metric spaces and $A : D \to U$ be an operator for which problem $(6a)$ may be ill-posed. Assume

$$Az_E = u_E, \tag{6b}$$

where $z_E \in D$ and $u_E \in U$. An operator $R(u, \alpha)$ defined for certain u and certain $\alpha > 0$ (see below) is called a *regularizing operator* for the equation $Az = u$ in *a neighborhood of* u_E if

1) there exists a positive number δ_1 such that the element $R(u, \alpha)$ of D is defined for every $\alpha > 0$ and every $u \in U$ for which

$$\rho_U(u, u_E) \leq \delta_1$$

and

2) for every $\epsilon > 0$, there exists a positive number $\delta(\epsilon) \leq \delta_1$ and for each positive $\delta \leq \delta(\epsilon)$ a number $\alpha(\delta) > 0$ such that

$$u \in U \qquad \& \qquad \rho_U(u, u_E) \leq \delta \leq \delta(\epsilon)$$

imply $z_\alpha(u) := R(u, \alpha(\delta))$ satisfies

$$\rho_D(z_\alpha(u), z_E) \leq \epsilon.$$

The procedure of finding a *computable* regularizing operator for the equation $Az = u$ is called *regularization* of the problem. The numerical parameter α is called the *regularization parameter*.

Thus the problem of finding an approximate solution z such that $z \approx z_E$ reduces to

i) finding regularizing operators,

ii) determining the regularization parameter $\alpha = \alpha(\delta)$ as a function of δ such that the condition 2) in the definition of regularizing operators is satisfied.

Remark. Of course

$$R(u, \alpha) := z_E, \qquad \forall u \in U, \qquad \forall \alpha > 0.$$

is trivially a regularizing operator. But it is useless, as z_E is the unknown being sought; hence the italicized word "computable" used above.

The approximation $z_\alpha = R(u, \alpha)$ to the exact solution z_E obtained by a method of regularization is customarily called a *regularized solution* of $(6b)$, although this is an abuse of language, since no z_α need be equal to the actual solution z_E of $(6b)$.

2.4 Methods of constructing regularizing operators

Definition of stabilizing functionals:

Assume A is continuous and 1-1. Let Θ denote a continuous nonnegative functional defined on a subset D_1 of D that is everywhere dense in D. Suppose that

i) $z_E \in D_1$

ii) $\forall c > 0$, $\{z \in D_1 : \Theta[z] \leq c\}$ is a compact subset of D_1.

Θ is then called a *stabilizing functional* or simply a *stabilizer*.

Note. It is called a *stabilizer* since we can obtain stable solutions with its help.

Suppose the right-hand member of equation $(6a)$ $u = u_\delta$ is known with an error δ; that is, $\rho_U(u_\delta, u_E) \leq \delta$.

Lemma 1 ([13], pp.51-53). Let $Q_\delta := \{z : \rho_U(Az, u_\delta) \leq \delta\}$, $D_{1,\delta} := Q_\delta \cap D_1$. Then there exists $z_\delta \in D_{1,\delta}$ that minimizes $\Theta[z]$ over $z \in D_{1,\delta}$.

Let

$$\Theta_0 := \inf_{z \in D_1} \Theta[z]$$

and

$$M_0 := \{z \in D_1 : \Theta[z] = \Theta_0\}.$$

Lemma 2 ([13], pp.55-56). If for each $\delta > 0$ there exists a $z_\delta \in M_0 \cap D_{1,\delta}$, then z_δ approaches z_E as $\delta \to 0$.

For the case $M_0 \cap D_{1,\delta} = \emptyset$, we import the concept of quasi-monotonic functionals.

Definition. The functional Θ is said to be *quasi-monotonic* if, for every element $z_0 \in D_1 \setminus M_0$, every neighborhood of z_0 includes an element $z_1 \in D_1$ such that $\Theta[z_1] < \Theta[z_0]$.

Lemma 3 ([13], p.56). Let u_δ, $D_{1,\delta}$, Θ and M_0 be as in in Lemma 1, and suppose $M_0 \cap D_{1,\delta} = \emptyset$. The greatest lower bound of Θ on $D_{1,\delta}$ is attained for an element z_δ for which $\rho_U(Az_\delta, u_\delta) = \delta$.

Remark: We can use this lemma to solve the problem of minimizing the functional Θ, not just within the set $D_{1,\delta}$, but within the set D_1 under the condition that the minimizing element z being sought satisfies

$$\rho_U(Az, u_\delta) = \delta.$$

This is a conditional extremum problem. So we can solve it by the method of undetermined Lagrange multipliers; that is, for each $\alpha > 0$ we minimize the functional

$$M^\alpha[z, u_\delta] := \rho_U^2(Az, u_\delta) + \alpha\Theta[z]$$

over all $z \in D_1$.We will call $M^\alpha[z, u_\delta]$ a *smoothing functional.*

2.5 The construction of regularizing operators by minimization of a smoothing functional

Theorem 2.1 ([13], p.63). Let A denote a continuous operator from D into U, Θ a stabilizing functional on a dense subset D_1 of D. For every element $u \in U$, and every $\alpha > 0$, $\exists z_\alpha(u) \in D_1$ for which the functional

$$M^\alpha[z, u] = \rho_U^2(Az, u) + \alpha\Theta[z]$$

attains its greatest lower bound:

$$\inf_{z \in D_1} M^\alpha[z, u] = M^\alpha[z_\alpha(u), u].$$

Remark. By this theorem, for every $\alpha > 0$, an operator $R_1(*, \alpha)$ is defined from U to D_1 so that for each $u \in U$ the element

$$z_\alpha(u) =: R_1(u, \alpha)$$

minimizes the function $M^\alpha[z, u]$ over all $z \in D_1$. For notational simplicity, we will use z_α instead of $z_\alpha(u)$. We plan to show that $R_1(*, \alpha)$ is a regularizing operator according to the definition in 2.3. So we use the same symbol $z_\alpha(u)$ here as in the definition in 2.3.

For any positive number δ_1, denote by T_{δ_1} the class of functions that are non-negative, nondecreasing and continuous on an interval $[0, \delta_1]$.

Theorem 2.2 ([13], p.65). Let z_E denote a solution of equation $(6a)$ with right-hand member $u = u_E$; that is,

$$Az_E = u_E. \tag{6b}$$

Then, for any $\delta_1 > 0$, $\epsilon > 0$ and any functions β_1 and β_2 in the class of T_{δ_1}, such that $\beta_2(0) = 0$ and $\delta^2/\beta_1(\delta) \leq \beta_2(\delta)$ for all sufficiently small $\delta > 0$, there exists a positive number

$$\delta_0 = \delta_0(\epsilon, \beta_1, \beta_2) \leq \delta_1$$

such that for $u \in U$ and $\delta \leq \delta_0$ the inequality $\rho_U(u, u_E) \leq \delta$ implies the inequality $\rho_D(z_\alpha, z_E) \leq \epsilon$, where $z_\alpha := R_1(u, \alpha)$ for all α satisfying the inequalities

$$\frac{\delta^2}{\beta_1(\delta)} \leq \alpha \leq \beta_2(\delta).$$

2.6 The choice of stabilizing functionals

Suppose that a dense subset Φ of a metric space D admits a metric ρ_Φ which in general is different from the metric ρ_D that D comes with. If for some $z_0 \in \Phi$ and every $d > 0$ the ball

$$\{z \in \Phi : \ \rho_\Phi(z, z_0) \leq d\}$$

is compact in D according to the original metric of D, then the function defined by

$$\Theta[z] := \rho_\Phi^2(z, z_0), \quad z \in \Phi$$

is a stabilizing function; consequently Theorem 2.1 is valid for it. That is, for each fixed $u \in U$, there exists an element $z_\alpha \in \Phi$ minimizing the functional

$$M^\alpha[z, u] := \rho_U^2(Az, u) + \alpha\Theta[z]$$

over $z \in \Phi$ ([13], p.68).

If the exact solution z_E of $(6a)$ belongs to the set Φ, then by Theorem 2.1, the operator $R_1(u, \alpha)$ which provides, for every $\alpha > 0$ and $u \in U$, an element z_α minimizing the functional $M^\alpha[z, u]$ is a regularizing operator.

In [13] the following examples are offered:

Example 1. If $D := C[a, b]$ with metric

$$\rho_D(z_1, z_2) := \sup_{x \in [a,b]} |z_1(x) - z_2(x)|,$$

then we take $\Phi := C^1[a,b]$, the dense subspace of continuously differentiable real-valued functions on the interval $[a,b]$ with metric

$$\rho_\Phi(z_1,z_2) := \sup_{x\in[a,b]} \{|z_1(x) - z_2(x)| + |z_1'(x) - z_2'(x)|\}.$$

By Arzela's theorem, every ball $\{z \in \Phi : \rho_\Phi(z,z_0) \leq d\}$ is ρ_D-compact in $C[a,\ b]$.

Example 2. D is still $C[a,b]$ with the same metric as in Example 1. Φ is the Sobolev space W_2^p. The metric ρ_Φ is defined by

$$\rho_\Phi(z_1,z_2) := \{\int_a^b \sum_{r=0}^{p} q_r(x)(\frac{d^r z}{dx^r})^2 dx\}^{1/2},\ z := z_1 - z_2,$$

where $q_0,\ q_1, ...,\ q_{p-1},\ q_p$ are given positive continuous functions on $[a,\ b]$. These functions are fixed for the sequel.

Here, for every p, the space W_2^p is a Hilbert space and any closed ball in it is ρ_D-compact in $C[a,\ b]$. Consequently, if we seek regularized solutions of $(6a)$ in the space W_2^p, Theorem 2.1 and 2.2 are also valid for them. A stabilizing functional is

$$\Theta[z] := \int_a^b \sum_{r=0}^{p} q_r(x)(\frac{d^r z}{dx^r})^2 dx. \qquad (**)$$

and p is called the *order* of the stabilizer ([13], p.70).

2.7 Application of the regularization method to the approximate solution of integral equations of the first kind

Suppose that we are required to find a regularized solution of a Fredholm integral equation of the first kind

$$(Az)(x) := \int_a^b K(x,s)z(s)ds = u(x)$$

where $c \leq x \leq d,\ K(x,s)$ is continuous on $[c,d] \times [a,b],\ z \in C[a,b]$ and $u \in L^2[c,d] =: U$ is given and a solution z in the space $D := C[a,b]$ is sought.

Let us use the first-order stabilizer (**). Thus, we seek a regularized solution $z_\alpha \in W_2^1[-\Omega, \Omega]$. It will minimize the functional

$$M^\alpha[z,u] = \int_c^d \{\int_a^b K(x,s)z(s)ds - u(x)\}^2 dx + \alpha \int_a^b \{q_0(s)z^2(s) + q_1(s)(\frac{dz}{ds})^2\} ds$$

over $z \in W_2^1$. This minimizing z can be found by solving the Euler equation corresponding to the functional $M^\alpha[z,u]$ ([13], p.74): For all $v \in W_2^1$

$$\int_a^b \left(-\alpha\{\frac{d}{ds}[q_1(s)\frac{dz}{ds}] - q_0(s)z(s)\} + \int_a^b \bar{K}(s,t)z(t)dt - \bar{b}(s)\right) v(s)ds$$
$$+[\alpha q_1(s)z'(s)v(s)]_a^b = 0,$$

which is a necessary condition for the minimizer. Here

$$\bar{K}(s,t) := \int_c^d K(\xi,s)K(\xi,t)d\xi,$$

and

$$\bar{b}(s) := \int_c^d K(\xi,s)u(\xi)d\xi.$$

In this paper, we assume the values of the desired solution z of $(6a)$ at both ends of the interval $[a,b]$ are known. That is,

$$z(a) = \bar{z}_1,\ z(b) = \bar{z}_2$$

where $\bar{z}_1$ and $\bar{z}_2$ are known numbers. This restricts the generality of the solution that we will ultimately find. In this case, since $v + u$ is supposed to be a solution candidate as well as u, we also require

$$v(a) = 0,\ v(b) = 0.$$

The preceding necessary condition now reads: for all such $v \in W_2^1$

$$\int_a^b \left(-\alpha\{\frac{d}{ds}[q_1(s)\frac{dz}{ds}] - q_0(s)z(s)\} + \int_a^b \bar{K}(s,t)z(t)dt - \bar{b}(s)\right) v(s)ds = 0. \qquad (\dagger)$$

From $(\dagger)$ holding for all v follows an equation which any minimizer z must satisfy ([13], p. 75):

$$\int_a^b \bar{K}(s,t)z(t)dt - \alpha\{\frac{d}{ds}[q_1(s)\frac{dz}{ds}] - q_0(s)z(s)\} = \bar{b}(s),\ \forall s \in [a,\ b]. \qquad (\ddagger)$$

Remark. If the solution of (‡) is unique, and if a minimizer exists at all, then this solution must be the minimizer. When the regularization method is used for the extrapolation problem in chapter 3, we will prove the existence of the minimizer and the uniqueness of the solution of (‡).

In such a case this equation can be replaced with its finite-difference approximation on a given grid in order to find a numerical solution. Here we take a uniform grid with step $h := \frac{b-a}{n}$. The equation (‡) is replaced with a system of finite-difference equations of the form

$$-\frac{\alpha}{h^2}\{q_{1,k-1}z_{k-1}+q_{1,k}z_{k+1}-(q_{1,k}+q_{1,k-1})z_k-h^2q_{0,k}z_k\}+\sum_{r=0}^{n}\bar{K}_{k,r}z_rh=\bar{b}_k;\ 1\leq k<n.$$

Here

$$q_{1,k} := q_1(s_k),\ q_{0,k} := q_0(s_k),\ z_k := z(s_k),\ \bar{b}_k := \bar{b}(s_k),$$

$$s_k := kh + a,$$

and the $\bar{K}_{k,r} := \bar{K}(s_k, s_r)$ are the coefficients in the quadrature formula used to replace the integral in the equation with a finite sum.

Since the solution of the system of equations must satisfy the boundary condition, then, in the system we set

$$z_0 := \bar{z}_1,\ z_n := \bar{z}_2.$$

We obtain an $(n-1)\times(n-1)$ system of equations, which is easy to solve numerically when the regularization method is used for the equation (4), since it is a system of linear equations and the coefficient matrix is positive-definite. The proof will be given in chapter III.

2.8 Determination of the regularization parameter

The regularization parameter α can be determined from the discrepancy, that is, from

$$\rho_U(Az_\alpha, u_\delta) = \delta$$

under certain restrictions. Let us abbreviate

$$m(\alpha) := M^{\alpha}[z, u],$$

$$\phi(\alpha) := \rho_U^2(Az_\alpha, u),$$

$$\psi(\alpha) := \Theta[z_\alpha].$$

Lemma 1 ([13], p.88). The functions m and ϕ are nondecreasing; ψ is a non-increasing function.

Lemma 2 ([13], p.89). Let $\{\alpha_n\}$ denote a positive sequence that converges to a positive number α_0. If the sequence $\{z_{\alpha_n}\}$ of minimizers in Theorem 2.1 converges,

$$\lim z_{\alpha_n} =: \bar{z},$$

then $\bar{z}$ minimizes $M^{\alpha_0}[z, u]$ over $z \in D_1$, where D_1 is the domain of the stabilizing functional Θ.

Lemma 3 ([13], p.91). The function m is continuous and nondecreasing.

Theorem ([13], p.91). For any $u \in U$ and every positive number

$$\delta < \rho_U(Az_0, u)$$

where

$$z_0 \in \{z : \Theta[z] = \Theta_0 = \inf_{Y \in D_1} \Theta[Y]\},$$

there exists an $\alpha(\delta)$ such that

$$\rho_U(Az_{\alpha(\delta)}, u) = \delta.$$

Chapter 3

THE ILL-POSEDNESS OF EXTRAPOLATION AND REGULARIZED EXTRAPOLATION

3.1 THE ILL-POSEDNESS OF THE PROBLEM

The problem of solving equation (4) on the pair of spaces $(C_{0,\Omega}, L^2)$ is ill-posed since it violates conditions (i) and (iii) in definition 2.1:

i) The existence condition is not satisfied since it is possible that $f \in L^2$ but $f \notin AC_{0,\Omega}$. In fact, *every* $f \in L^2[-T,T] \backslash C[-T,T]$ is not in the range of A. That is, the A in (4) is not surjective.

ii) It is easy to see that the problem satisfies ii);

iii) The stability condition is not satisfied. That is, A^{-1} is not continuous.

This can be seen from the next example.

Example 3. For positive integers n define functions in $C_{0,\Omega}$ by

$$F_n(\omega) := \sin n\omega, \;\; F(\omega) := 0 \quad \text{if} \quad |\omega| \le \Omega := \pi.$$

Then (see [p.1050 of [15] for details)

$$||AF_n - AF||^2_{L^2[-T,T]} \to 0 \; (n \to \infty).$$

But for all n

$$||F_n - F||_{C_{0,\Omega}} = 1.$$

3.2 REGULARIZATION FOR EXTRAPOLATION

For the integral equation of the first type (4), we employ the smoothing functional

$$M^\alpha[F, f_\delta] := ||AF - f_\delta||^2_{L^2} + \alpha ||F||^2_{W_2^1} \tag{7}$$

where $\alpha > 0$, $f_\delta \in L^2[-T, T]$, $F \in W_2^1 := W_2^1[-\Omega,\ \Omega]$, and

$$||F||^2_{W_2^1} := \int_{-\Omega}^{\Omega} (|F(\omega)|^2 + |F'(\omega)|^2) d\omega$$

is the square of the Sobolev norm ([20]).

The regularized extrapolation to be found is the inverse Fourier transform of a minimizer $F_\alpha \in W_2^1$ of the smoothing functional:

$$M^\alpha[F_\alpha, f_\delta] = \inf_{F \in W_2^1} M^\alpha[F, f_\delta].$$

We will prove the existence and uniqueness of F_α in the Appendix and the following result concerning the minimizer F_α:

In Theorem 3.2, we will prove F_α converges uniformly on $[-\Omega, \Omega]$ to the true frequency-domain signal $\hat{f}$ as $\delta \to 0$ provided that $\alpha = \alpha(\delta)$ is a suitably defined function of δ. Hence, the regularized extrapolation

$$f_\alpha(t) := \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F_\alpha(\omega) e^{-i\omega t} d\omega =: AF_\alpha(t) \tag{7a}$$

will converge uniformly to the true extrapolation $f(t)$ over $t \in \mathbf{R}$, namely, to the right-hand side of (2). Here is the reason: The functions f_α do converge to a function g which agrees with f in $[-T, T]$. From the uniqueness of the (analytic!) extrapolation (see page 3) it follows that $g = f$ in $\mathbf{R}$.

Theorem 3.1. For $\alpha > 0$, and $f \in L^2[-T,T]$, there exists a function $F_\alpha \in W_2^1$ such that

i) $M^\alpha[F_\alpha, f] = \inf_{F\in W_2^1} M^\alpha[F,f]$.

ii) F''_α exists and F_α satisfies the Euler equation ([19], pp.183-214): for all $\omega \in [-\Omega,\Omega]$

$$\frac{1}{2\pi^2}\int_{-\Omega}^{\Omega}\frac{\sin(s-\omega)T}{s-\omega}F_\alpha(s)ds + \alpha[F_\alpha(\omega) - F''_\alpha(\omega)] = \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{i\omega t}dt. \qquad (7b)$$

The proof is in the Appendix.

Suppose $F \in W_2^1[-\Omega,\Omega]$ satisfies the integral equation

$$AF(t) := \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega)e^{-i\omega t}d\omega = f(t) \quad a.\ e.\ t\in[-T,T] \qquad (4)'$$

Since $F \in L^2[-\Omega,\Omega] \subset L^1(\mathbf{R})\cap Ł^2(\mathbf{R})$, the function $\tilde{F}$ lies in $L^1(\mathbf{R})\cap C(\mathbf{R})$. According to $(4)'$ it satisfies

$$\tilde{F}(t) = f(t) \quad a.\ e.\ t\in[-T,T].$$

Therefore $\tilde{F}$ on the whole of $\mathbf{R}$ is our candidate to solve the original problem (p.3) of recovering (extrapolating) f from knowledge only of $f|_{[-T,T]}$.

Let f_E denote the function defined for all $t\in\mathbf{R}$ by the left side of $(4)'$. Thus $f_E \in L^2(\mathbf{R})\cap C(\mathbf{R})$ and f_E interpolates $f|_{[-T,T]}$ and $\widehat{f_E} = F$ on $[-\Omega,\Omega]$, $\widehat{f_E} = 0$ on $\mathbf{R}\setminus[-\Omega,\Omega]$.

Theorem 3.2. Let $\{f_\delta\}\subset L^2[-T,T]$ be such that

$$||f_\delta - f_E||^2_{L^2} = \int_{-T}^{T}|f_\delta(t) - f_E(t)|^2 dt \le \delta^2.$$

If we choose $\alpha = \alpha(\delta)$ such that $\alpha(\delta)\to 0$ and $\delta^2/\alpha(\delta)$ is bounded as $\delta\to 0$, then

$$\lim_{\delta\to 0}||F_{\alpha(\delta)} - \widehat{f_E}||_{C_{0,\Omega}} = \lim_{\delta\to 0}\max_{-\Omega\le\omega\le\Omega}|F_{\alpha(\delta)}(\omega) - \widehat{f_E}(\omega)| = 0$$

where $F_{\alpha(\delta)}$ is the solution of the Euler equation $(7b)$ in which f is f_δ.

The proof is in the Appendix.

From this theorem we can see that the solution of the Euler equation $F_{\alpha(\delta)}$ approaches the exact solution $\widehat{f_E}$ of (4) in $C_{0,\Omega}$ as $\delta \to 0$. We can also give an error bound for the regularized extrapolation. We will use the notation

$$d(\delta) := ||F_{\alpha(\delta)} - \widehat{f_E}||_{C_{0,\Omega}},$$

which the next theorem concerns:

Theorem 3.3. Under the conditions of theorem 3.2, we have

$$d(\delta)^2 = O(\alpha(\delta)) + O(\delta^2).$$

The proof is in the Appendix.

From this theorem 3.3 we have the next corollaries for f_α defined in $(7a)$.

Corollary 1.

$$\sup_{-\infty<t<+\infty} |f_{\alpha(\delta)}(t) - f_E(t)|^2 = O(\delta^2) + O(\alpha(\delta)).$$

Since F_α and $\widehat{f_E}$ are supported in $[-\Omega, \Omega]$, using Parseval's identity and $(7a)$, we have

Corollary 2.

$$\int_{-\infty}^{+\infty} |f_{\alpha(\delta)}(t) - f_E(t)|^2 dt = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |F_{\alpha(\delta)}(\omega) - \widehat{f_E}(\omega)|^2 d\omega = O(\delta^2) + O(\alpha(\delta)).$$

From theorem 3.2 and theorem 3.3, we can conclude that the error of the regularized solution depends on the regularization parameter α. Therefore it is important to determine the parameter α. In general α should be small if δ is small. Here we suggest an algorithm to choose the regularization parameter by solving ([13], pp.87-93)

$$||AF_{\alpha(\delta)} - f_\delta||^2_{L^2[-T,T]} = \delta^2$$

for $\alpha(\delta)$. This equation can be solved for α by using Newton's method. Other methods to choose the regularization parameter are discussed in [16] and [17].

We can solve the Euler equation (7b) by the following finite-difference method:

$$\frac{1}{2\pi^2}\sum_{j=-n+1}^{n-1}\frac{\sin(s_j-\omega_k)T}{s_j-\omega_k}F_j^h+\alpha[F_k^h-\frac{F_{k+1}^h-2F_k^h+F_{k-1}^h}{h^2}]$$

$$=\frac{1}{2\pi}\int_{-T}^{T}f(t)e^{i\omega_k t}dt, \qquad k=-n+1,...,n-1, \qquad (\dagger\dagger)$$

in which $h:=\Omega/n,\ s_j=\omega_j:=jh,\ j=-n+1,...,n-1$, and $F_j^h,\ j=-n+1,...,n-1$ are the unknowns. We set $F_{-n}^h:=F_n^h:=0$ to ensure the Ω-band-limitedness.

Now we can prove the coefficient matrix is positive-definite. The coefficient matrix is

$$C:=C_1+\alpha I+\frac{1}{h^2}H$$

where $C_1:=\left(\frac{\sin(s_j-\omega_k)T}{s_j-\omega_k}\right)$, I the $(2n-1)$ by $(2n-1)$ identity matrix and

$$H:=[-1,2,-1]=\begin{bmatrix}2 & -1 & 0 & ... & 0\\ -1 & 2 & -1 & ... & 0\\ ...... & & & & \\ 0 & ... & 0 & -1 & 2\end{bmatrix}$$

in which all diagonal entries are 2, all super-diagonal entries are -1, all sub-diagonal entries are -1 and all other entries are 0. Then for any real vector $z=(z_{-n+1},...,z_{n-1})$

$$z^TC_1z=\sum_{j=-n+1}^{n-1}\sum_{k=-n+1}^{n-1}\frac{\sin(s_j-\omega_k)T}{s_j-\omega_k}z_jz_k$$

$$=\frac{1}{2}\sum_{j=-n+1}^{n-1}\sum_{k=-n+1}^{n-1}\int_{-T}^{T}e^{i(s_j-\omega_k)t}z_jz_kdt=\frac{1}{2}\sum_{j=-n+1}^{n-1}\left|\int_{-T}^{T}e^{-i\omega_jt}z_jdt\right|^2\geq 0,$$

and

$$z^THz=\sum_{j=-n+1}^{n-1}(z_j-z_{j-1})^2+z_{-n+1}^2+z_{n-1}^2\geq 0.$$

The amount of computation time needed to solve the system of equations $(\dagger\dagger)$ is of the order $O(n^3)$ ([21]). Hence, it will be useful to find a more effective algorithm for extrapolation.

3.3 A FAST ALGORITHM FOR COMPUTING FOURIER TRANSFORMS

To find a fast algorithm to solve the Euler equation for computing Fourier transforms, we first write the Euler equation in the form

$$A^*AF_\alpha + \alpha[F_\alpha - F_\alpha''] = A^*f \tag{7c}$$

where A^* is the adjoint operator of A when A is the operator from $L^2[-\Omega,\Omega]$ into $L^2[-T,T]$ in (4) Since

$$A^*AF_\alpha = \frac{1}{2\pi^2}\int_{-\Omega}^{\Omega}\frac{\sin(s-\omega)T}{s-\omega}F_\alpha(s)ds \text{ and } A^*f = \frac{1}{2\pi}\int_{-T}^{T}f(t)e^{i\omega t}dt.$$

Now we can see the uniqueness of the solution of $(7c)$ by

$$(A^*AF + \alpha[F - F''], F) = (AF, AF) + \alpha(F,F) - \alpha(F'', F)$$

$$= (AF, AF) + \alpha(F,F) + \alpha(F', F') > 0, \text{ for } F \neq 0 \text{ in } C_{0,\Omega}.$$

The operator in the right-hand side of equation $(7c)$ is therefore positive, which implies that the solution of Euler equation $(7b)$ is unique. In the sequel A will in fact be $\mathbf{F}^{-1}$ from $L^2[-\Omega,\Omega]$ into $L^2[-T,T]$. When $T=\infty,\ A=\mathbf{F}^{-1}$.

Lemma. If a linear transform S on a Hilbert space is surjective and isometric, then $S^* = S^{-1}$.

In previous sections $T < +\infty$; now we consider the case $T = +\infty$. Then $\sqrt{2\pi}A = \sqrt{2\pi}\mathbf{F}^{-1}$ is surjective and isometric on $L^2(\mathbf{R})$. By the lemma:

$$(\sqrt{2\pi}A)^* = (\sqrt{2\pi}\mathbf{F}^{-1})^{-1}.$$

We can calculate

$$A^* = (\mathbf{F}^{-1})^* = \frac{1}{2\pi}\mathbf{F}$$

and

$$(A^*)^{-1} = 2\pi \mathbf{F}^{-1} = 2\pi A.$$

Then for $f \in L^2(\mathbf{R}) \cap L^1(\mathbf{R})$, the Euler equation (7c) can be transformed to the following forms successively:

$$AF_\alpha + \alpha(A^*)^{-1}(F_\alpha - F''_\alpha) = f$$

$$AF_\alpha + 2\pi\alpha A(F_\alpha - F''_\alpha) = f$$

$$(1+2\pi\alpha)F_\alpha - 2\pi\alpha F''_\alpha = A^{-1}f = \mathbf{F}[f]. \qquad (7d)$$

We use the finite-difference method to solve this differential equation:

$$(1+2\pi\alpha)F^h_{\alpha,j} - 2\pi\alpha \frac{F^h_{\alpha,j+1} - 2F^h_{\alpha,j} + F^h_{\alpha,j-1}}{h^2} = \hat{g}_j := \int_{-\infty}^{\infty} f(t)e^{i\omega_j t}dt \qquad (8)$$

where $h := \Omega/n,\ \omega_j := jh\ (j = -n+1, ..., n-1)$. Here we write $F^h_{\alpha,j}$ for the solutions of the linear system (8), since they depend on α.

The coefficient matrix is a $(2n-1)$ by $(2n-1)$ symmetric matrix which is tridiagonal with diagonal entries:

$$b_j := b = 1 + 2\pi\alpha + 4\pi\alpha/h^2,\ (j = -n+1, ..., n-1),$$

sub-diagonal entries

$$a_j := a = -2\pi\alpha/h^2,\ (j = -n+1, ..., n-1),$$

and super-diagonal entries

$$c_j := c = -2\pi\alpha/h^2,\ (j = -n+1, ..., n-1).$$

Since $\alpha > 0$, the matrix is strictly diagonally dominant. Hence, the system of equations (8) can be solved with the Thomas algorithm ([21]):

1. $u_{-n+1} := c/b,\ u_j := c/(b - au_{j-1}),\ j = -n+2, ..., n-1.$

2. $y_{-n+1} := \hat{g}_{-n+1}/b,\ y_j := (\hat{g}_j - ay_{j-1})/(b - au_{j-1}),\ j = -n+2, ..., n-1.$

3. $F^h_{\alpha,n-1} := y_{n-1},\ F^h_{\alpha,j} := y_j - u_j F^h_{\alpha,j+1},\ j = n-2, ..., -n+1.$

The time of computation needed to solve the system of equations (8) is of the order $O(n)$ ([21]). We construct a piecewise-linear approximation to F_α:

$$F^h_\alpha(\omega) := F^h_{\alpha,j} + \frac{F^h_{\alpha,j+1} - F^h_{\alpha,j}}{h}(\omega - \omega_j),\ \omega \in [\omega_j, \omega_{j+1}],\ j = -n, ..., n-1.$$

Since there are errors in the discretizing process, we must also estimate the error bound of this discretization.

By the Remark following the lemma in the appendix, if $f(t),\ tf(t),\ t^2 f(t) \in L^1(\mathbf{R})$, the regularized solution $F_\alpha \in C^4[-\Omega, \Omega]$. So by the Taylor formula

$$F''_\alpha(\omega_j) = \frac{F_\alpha(\omega_{j+1}) - 2F_\alpha(\omega_j) + F_\alpha(\omega_{j-1})}{h^2} + O(h^2).$$

With the formula above we will obtain an error estimation for the finite-difference method (8). First the solution of Euler equation $(7d)$ satisfies, for $j = -n+1, ..., n-1$

$$(1+2\pi\alpha)F_\alpha(\omega_j) - 2\pi\alpha\frac{F_\alpha(\omega_{j+1}) - 2F_\alpha(\omega_j) + F_\alpha(\omega_{j-1})}{h^2} = \hat{g}_j + O(h^2). \tag{9}$$

Then we have the next theorem.

Theorem 3.4. We use the abbreviation $F_{\alpha,j}$ to denote $F_\alpha(\omega_j)$ and suppose

$$F^h_\alpha := (F^h_{\alpha,-n+1}, F^h_{\alpha,-n+2}, ..., F^h_{\alpha,n-1}),\ F_\alpha := (F_{\alpha,-n+1}, F_{\alpha,-n+2}, ..., F_{\alpha,n-1})$$

are the solutions of (8), (9) respectively. If $f(t),\ tf(t),\ t^2 f(t) \in L^1(\mathbf{R})$, then

$$\sum_{j=-n+1}^{n-1} |F^h_{\alpha,j} - F_{\alpha,j}|^2 = \frac{O(h^3)}{1 + 2\pi\alpha + \frac{2\pi\alpha}{h^2}\lambda_{\min}}$$

where $\lambda_{\min} := 2 + 2\cos(\pi(2n-1)/2n)$ is the minimal eigenvalue of the positive-definite matrix $H = [-1, 2, -1]$ ([22]).

The proof is in the Appendix.

We can construct a function f_α^h which is called a *regularized restoration* by:

$$f_\alpha^h(t) := \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F_\alpha^h(\omega)e^{-i\omega t}d\omega = \frac{1}{2\pi}\sum_{j=-n}^{n-1}\int_{\omega_j}^{\omega_{j+1}}\left[F_{\alpha,j}^h + \frac{F_{\alpha,j+1}^h - F_{\alpha,j}^h}{h}(\omega-\omega_j)\right]e^{-i\omega t}d\omega$$

$$= -\frac{1}{2\pi it}\sum_{j=-n}^{n-1}\int_{\omega_j}^{\omega_{j+1}}\left[F_{\alpha,j}^h + \frac{F_{\alpha,j+1}^h - F_{\alpha,j}^h}{h}(\omega-\omega_j)\right]de^{-i\omega t}$$

$$= -\frac{1}{2\pi it}\sum_{j=-n}^{n-1}\left(\left[(F_{\alpha,j}^h + \frac{F_{\alpha,j+1}^h - F_{\alpha,j}^h}{h}(\omega-\omega_j))e^{-i\omega t}\right]_{\omega_j}^{\omega_{j+1}} - \int_{\omega_j}^{\omega_{j+1}}\frac{F_{\alpha,j+1}^h - F_{\alpha,j}^h}{h}e^{-i\omega t}d\omega\right)$$

$$= -\frac{1}{2\pi it}\sum_{j=-n}^{n-1}\left((F_{\alpha,j+1}^h e^{-i\omega_{j+1}t} - F_{\alpha,j}^h e^{-i\omega_j t}) + \left[\frac{F_{\alpha,j+1}^h - F_{\alpha,j}^h}{hti}e^{-i\omega t}\right]_{\omega_j}^{\omega_{j+1}}\right)$$

$$= \frac{1}{2\pi t^2}\sum_{j=-n}^{n-1}\frac{F_{\alpha,j+1}^h - F_{\alpha,j}^h}{h}(e^{-i\omega_{j+1}t} - e^{-i\omega_j t}), \quad t \in \mathbf{R}.$$

Remark. The *apparent* singularity at $t=0$ in the last formula is only apparent, since we have set $F_{-n}^h := F_n^h := 0$ to ensure the Ω-band-limitedness.

From theorem 3.4 we have the following corollaries.

Corollary 3.

$$\max_{-n+1\le j\le n-1} |F_{\alpha j}^h - (\widehat{f_E})_j|^2 = O(\delta^2) + O(\alpha(\delta^2)) + \frac{O(h^3)}{1+2\pi\alpha+\frac{2\pi\alpha}{h^2}\lambda_{\min}}.$$

Corollary 4.

$$\sup_{-\infty<t<+\infty} |f_\alpha^h(t) - f_E(t)|^2 = O(\delta^2) + O(\alpha(\delta^2)) + \frac{O(h^3)}{1+2\pi\alpha+\frac{2\pi\alpha}{h^2}\lambda_{\min}}.$$

Using Parseval's identity, we have, since F_α^h and $\widehat{f_E}$ are supported in $[-\Omega,\Omega]$

Corollary 5.

$$\int_{-\infty}^{+\infty}|f_\alpha^h(t) - f_E(t)|^2dt = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}|F_\alpha^h(\omega) - \widehat{f_E}(\omega)|^2d\omega$$

$$= O(\delta^2) + O(\alpha(\delta^2)) + \frac{O(h^3)}{1+2\pi\alpha+\frac{2\pi\alpha}{h^2}\lambda_{\min}}.$$

3.4 A REGULARIZED FORMULA FOR SPECTRAL ANALYSIS

By inverse Fourier transforming both sides of (7d), we obtain

$$F_\alpha(\omega) = \int_{-\infty}^{\infty} \frac{f(t)}{1 + 2\pi\alpha + 2\pi\alpha t^2} e^{i\omega t} dt, \qquad \omega \in [-\Omega, \Omega]. \tag{10}$$

For this formula, the convergence property and the error estimation in section III are also true. In general, to guarantee the convergence property of the regularized solution it is necessary for the exact solution to belong to a subset of the metric space D which is compact in the topology of D([13], [18]). In this paper, we solved the integral equation $(4)'$ in the class W_2^1, which resulted in $\widehat{f_E}$ being in W_2^1 (since $(4)'$ and Fourier Inversion imply $\hat{f} = F$), which means $\widehat{f_E}$ lies in a compact subset of $C_{0,\Omega}$. Now in the next theorem we can prove the regularized solutions F_α converge to the exact solution without the condition $\widehat{f_E} \in W_2^1$.

Theorem 3.5. The regularized solution (10) satisfies

$$||F_{\alpha(\delta)} - \widehat{f_E}||_{C_{0,\Omega}} = O(\frac{\delta}{\alpha^{1/4}}) + \varphi(\alpha)$$

where

$$\varphi(\alpha) := \int_{-\infty}^{\infty} \frac{2\pi\alpha + 2\pi\alpha t^2}{1 + 2\pi\alpha + 2\pi\alpha t^2} |f_E(t)| dt.$$

and

$$\varphi(\alpha) \to 0 \quad \text{as} \quad \alpha \to 0.$$

The proof is in the appendix.

3.5 REGULARIZED ITERATIVE EXTRAPOLATION ALGORITHM

Based on the regularized spectral estimation formula (10), we present the next regularized Iterative Extrapolation Algorithm:

$$f^{[0]}(t) := \frac{P_T f(t)}{1 + 2\pi\alpha + 2\pi\alpha t^2}.$$

For $l = 0, 1, 2, \ldots,$

$$f^{[l+1]}(t) := \frac{P_T f(t)}{1 + 2\pi\alpha + 2\pi\alpha t^2} + \frac{(I - P_T)\mathbf{F}^{-1} P_\Omega \mathbf{F} f^{[l]}(t)}{1 + 2\pi\alpha + 2\pi\alpha t^2},$$

where

$$P_T f(t) := \begin{cases} f(t), \ t \in [-T, T] \\ 0, \ t \in \mathbf{R} \setminus [-T, T] \end{cases}$$

and

$$P_\Omega \hat{f}(\omega) := \begin{cases} \hat{f}(\omega), \ \omega \in [-\Omega, \Omega] \\ 0, \ \omega \in \mathbf{R} \setminus [-\Omega, \Omega]. \end{cases}$$

3.6 EXPERIMENTAL RESULTS

In this section, we give some examples to show that the regularized iterative extrapolation algorithm is more effective in controlling the Gibbs phenomena and noise than the Papoulis-Gerchberg algorithm.

Suppose

$$f_E(t) := \frac{\sin t}{\pi t}, \quad t \in \mathbf{R}.$$

Then $f_E \in L^2(\mathbf{R})$ and

$$\widehat{f_E}(\omega) = \begin{cases} 1, \ \omega \in [-1, 1] \\ 0, \ \omega \in \mathbf{R} \setminus [-1, 1]. \end{cases}$$

Example 1. Let $[-T, T] = [-\pi/5, \pi/5]$, The numerical results concerning $\hat{f}$ gotten by the Papoulis-Gerchberg algorithm and the regularized iterative extrapolation algorithm for the iterations 1,2,3,4 are in fig.1 and fig.2. And $\alpha = 0.0001$ in the regularized iterative extrapolation algorithm.

Example 2. Suppose $f_{\delta_n}(t) := f_E(t) + \eta_{\delta_n}(t), \ t \in [-T, T] = [-20, 20]$, where

$$\eta_{\delta_n}(t) := (t/T)^{n^4}, \ t \in [-T, T].$$

In this example, η_{δ_n} is not the restriction to $[-T, T]$ of a band-limited function. But

$$\int_{-T}^{T} |\eta_{\delta_n}(t)|^2 dt \le \delta_n^2 := \frac{2T}{2n^4 + 1} \to 0 \ (n \to \infty).$$

We choose $n = 10$ and $\alpha = 1/n^3$. The numerical results by the Papoulis-Gerchberg algorithm for the iterations 1,2,3,4 are in fig.3, fig.4, fig.5, fig.6, fig.7. The numerical results by the regularized iterative extrapolation algorithm for the iterations 1,2,3,4 are in fig.8, fig.9, fig.10, fig.11, fig.12.

Example 3. Suppose $f_{\delta_n}(t) := f_E(t) + \eta_{\delta_n}(t),\ t \in [-T, T] = [-80, 80]$, where

$$\eta_{\delta_n}(t) := \frac{1}{2\pi}\left[\frac{\sin(n^2 - t)\pi}{n^2 - t} + \frac{\sin(n^2 + t)\pi}{n^2 + t}\right].$$

In this case

$$(A^{-1}\eta_{\delta_n})(\omega) = \cos n^2\omega$$

so the stability condition is not satisfied, as the example in section II shows. However

$$\int_{-T}^{T} |\eta_{\delta_n}(t)|^2 dt \le \delta_n^2 := \frac{2Tn^4}{\pi^2(n^4 - T^2)^2} \to 0\ (n \to \infty).$$

For $n = 3$, the numerical results of the Fourier transform and regularized Fourier transform are in fig.13 and fig.14 ($\alpha = 0.1/n^4$).

3.7 CONCLUSION

Regularization is used to extrapolate band-limited signals in this paper. The reason for using regularization is to get the Euler equation——a differential-integral equation first——then a differential equation for the Fourier transform. A fast algorithm to solve the differential equation is presented by using finite differences, and a regularized Fourier transform formula is given by the differential equation. A regularized iterative extrapolation algorithm is presented based on the regularized Fourier transform formula. Moreover, the precondition $\widehat{f_E} \in W_2^1$ for using the regularization method can be dispensed with, by theorem 3.5 in section 3.6.

Chapter 4

THE APPLICATION OF TWO-DIMENSIONAL EXTRAPOLATION IN IMAGE RESTORATION

In this chapter we extend the extrapolation model to the two-dimensional case and then apply it in image processing.

4.1 The Model

Definition: For two positive $\Omega_1,\ \Omega_2 \in \mathbf{R}$, a function $f \in L^1(\mathbf{R}^2)$ is said to be *band-limited* if $\hat{f}(\omega_1,\ \omega_2) = 0\ \forall(\omega_1,\ \omega_2) \in \mathbf{R}^2 \backslash [-\Omega_1, \Omega_1] \times [-\Omega_2, \Omega_2]$.

Here $\hat{f}$ is the *Fourier transform* of f:

$$\mathbf{F}(f)(\omega_1, \omega_2) = \hat{f}(\omega_1, \omega_2)$$

$$:= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t_1, t_2) e^{i(\omega_1 t_1 + \omega_2 t_2)} dt_1 dt_2,\ (\omega_1,\ \omega_2) \in \mathbf{R}^2. \tag{11}$$

We then have the inversion formula:

$$\mathbf{F}^{-1}(\hat{f})(t_1,\ t_2) = f(t_1,\ t_2)$$

$$= \frac{1}{(2\pi)^2} \int_{-\Omega_1}^{\Omega_1} \int_{-\Omega_2}^{\Omega_2} \hat{f}(\omega_1, \omega_2) e^{-i(\omega_1 t_1 + \omega_2 t_2)} d\omega_1 d\omega_2,\ \text{a.e. } (t_1,\ t_2) \in \mathbf{R}^2. \tag{12}$$

This integral is meaningful for all $(t_1, t_2) \in \mathbf{C}^2$ and furnishes an analytic extension of f to $\mathbf{C}^2$.

We consider the so-called *band-limited extrapolation problem*:

Assume that $f:\mathbf{R}^2 \to \mathbf{C}$ is a band-limited function and T_1, T_2 are two positive constants.

$$\begin{aligned} &\text{Given} \quad f(t_1,t_2) \quad \text{for} \quad (t_1,t_2) \in [-T_1,T_1]\times[-T_2,T_2], \\ &\text{find} \quad f(t_1,t_2) \quad \text{for} \quad (t_1,t_2) \in \mathbf{R}^2 \backslash [-T_1,T_1]\times[-T_2,T_2]. \end{aligned} \tag{13}$$

We introduce the following regularized iterative extrapolation algorithm:

$$f^{[0]}(t_1,t_2) := \frac{P_T f(t_1,t_2)}{(1+2\pi\alpha+2\pi\alpha t_1^2)(1+2\pi\alpha+2\pi\alpha t_2^2)}.$$

For $l = 0, 1, 2, \ldots,$

$$f^{[l+1]}(t_1,t_2) := \frac{P_T f(t_1,t_2)}{(1+2\pi\alpha+2\pi\alpha t_1^2)(1+2\pi\alpha+2\pi\alpha t_2^2)} + \frac{(I-P_T)\mathbf{F}^{-1}P_\Omega \mathbf{F} f^{[l]}(t_1,t_2)}{(1+2\pi\alpha+2\pi\alpha t_1^2)(1+2\pi\alpha+2\pi\alpha t_2^2)},$$

where

$$P_T f(t_1,t_2) := \begin{cases} f(t), \ (t_1,t_2) \in [-T_1,T_1]\times[-T_2,T_2] \\ 0, \ (t_1,t_2) \in \mathbf{R}^2\backslash[-T_1,T_1]\times[-T_2,T_2] \end{cases}$$

and

$$P_\Omega \hat{f}(\omega_1,\omega_2) := \begin{cases} \hat{f}(\omega_1,\omega_2), \ (\omega_1,\omega_2) \in [-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2] \\ 0, \ (\omega_1,\omega_2) \in \mathbf{R}^2\backslash[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]. \end{cases}$$

Remark. In application to image restoration in $\mathbf{R}^2\backslash[-T_1,T_1]\times[-T_2,T_2]$, $f(t_1,t_2)$ will be the brightness of the image at the point (t_1,t_2).

4.2 Application of two-dimensional extrapolation in image restoration

In this section we give an example of an application of two-dimensional extrapolation in image processing([28]). We compare the regularized extrapolation with the Papoulis-Gerchberg (=PG) algorithm in this case.

We choose the image in figure 7. And we assume the image to be extrapolated is figure 8. The known section is the square in which the cross is imbedded. We can use extrapolation algorithms to compute the unknown part of the image that is outside of the square.

The result of five iterations of the PG algorithm is in figure 9. The result of five iterations of the regularized algorithm with $\alpha = 0.00000001$ is in figure 10.

We can see many stripes in the image by the extrapolation algorithms. This is due to the Gibbs phenomena.

The image signal is not exactly band-limited, it is only approximately band-limited. But in computation we just use the part of the frequency domain in a finite interval. This gives rise to the Gibbs phenomena in the time domain.

This can be seen from the next demonstration. Let

$$f(t) := \begin{cases} 1, \ t \in [-\pi, \pi] \\ 0, \ t \in \mathbf{R} \backslash [-\pi, \pi]. \end{cases}$$

Then

$$f_\Omega(t) := \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \int_{-\pi}^{\pi} e^{i\omega s} ds e^{-i\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\Omega}^{\Omega} e^{i\omega(s-t)} d\omega ds = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin \Omega(s-t) ds}{s-t}.$$

Let $\Omega := N, \ t := \pi - \frac{k\pi}{N}$, where $k > 0$ is a constant integer. Then

$$f_N(\pi - \frac{k\pi}{N}) = \frac{(-1)^{N-k}}{\pi} \int_{-\pi}^{\pi} \frac{\sin N s ds}{s - (\pi - \frac{k\pi}{N})}.$$

Let $u := Ns - (N-k)\pi$. Then

$$f_N(\pi - \frac{k\pi}{N}) = \frac{1}{\pi} \int_{-2N\pi+\pi}^{k\pi} \frac{\sin u}{u} du \to \frac{1}{\pi} \int_{-\infty}^{k\pi} \frac{\sin u}{u} du,$$

as $N \to \infty$.

If we choose $k = 1, 2, ...,$ the values are different.

4.3. Time-Frequency Regularized Extrapolation Algorithm

In this section we introduce a generalized extrapolation model.

Assume $f \in L^1 \cap L^2(\mathbf{R})$, and $\hat{f} \in L^1(\mathbf{R})$.

Then $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega t} d\omega$ for $a.\ e.\ t \in \mathbf{R}$, thanks to the Inversion Theorem.

Here f is not necessarily band-limited.

But $\forall \epsilon > 0,\ \exists \Omega > 0$, such that

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{-i\omega t} d\omega + \frac{1}{2\pi}\int_{|\omega|>\Omega} \hat{f}(\omega) e^{-i\omega t} d\omega,\ a.\ e.\ t \in \mathbf{R},$$

where

$$|\frac{1}{2\pi}\int_{|\omega|>\Omega} \hat{f}(\omega) e^{-i\omega t} d\omega| \leq \frac{1}{2\pi}\int_{|\omega|>\Omega} |\hat{f}(\omega)| d\omega < \epsilon.$$

This uses the hypothesis that $\hat{f} \in L^1(\mathbf{R})$.

Therefore f can be approximated in the maximum norm to any desired accuracy by a band-limited function with a sufficiently long band width 2Ω. However, Gibbs phenomena may pollute the restored image even if we use the regularized algorithm presented in Section II. To alleviate this drawback, we present a revised regularized extrapolation. In the frequency domain we multiply the kernel by a factor similar to that used in the time domain:

$$\frac{1}{(1+2\pi\beta+2\pi\beta\omega_1^2)(1+2\pi\beta+2\pi\beta\omega_2^2)},$$

where $\beta > 0$ is a regularization parameter like the earlier α.

First we define the weighted Fourier inverse formula

$$\mathbf{F}_\beta^{-1}(\hat{f})(t_1,t_2) := \frac{1}{(2\pi)^2}\int_{-\Omega_1}^{\Omega_1}\int_{-\Omega_2}^{\Omega_2} \frac{\hat{f}(\omega_1,\omega_2)e^{-i(\omega_1 t_1+\omega_2 t_2)}d\omega_1 d\omega_2}{(1+2\pi\beta+2\pi\beta\omega_1^2)(1+2\pi\beta+2\pi\beta\omega_2^2)},\ \text{a.e.}\ (t_1,\ t_2) \in \mathbf{R}^2.$$

Next we define the Time-Frequency regularized iterative extrapolation algorithm:

$$f^{[0]}(t_1,t_2) := \frac{P_T f(t_1,t_2)}{(1+2\pi\alpha+2\pi\alpha t_1^2)(1+2\pi\alpha+2\pi\alpha t_2^2)}.$$

For $l = 0, 1, 2, \ldots,$

$$f^{[l+1]}(t_1,t_2) := \frac{P_T f(t_1,t_2)}{(1+2\pi\alpha+2\pi\alpha t_1^2)(1+2\pi\alpha+2\pi\alpha t_2^2)} + \frac{(I-P_T)\mathbf{F}_\beta^{-1} P_\Omega \mathbf{F} f^{[l]}(t_1,t_2)}{(1+2\pi\alpha+2\pi\alpha t_1^2)(1+2\pi\alpha+2\pi\alpha t_2^2)},$$

Since a similar multiplicative factor is involved in both Time and Frequency domain, we call this the Time-Frequency Regularized Extrapolation algorithm.

This algorithm is based on the next theorem.

Theorem 4.1. Assume $f_E \in L^1 \cap L^2$, $\hat{f}_E \in L^1$,

$$||\hat{f}_\delta - \hat{f}_E||_{L^2} \le \delta,$$

$$f_{\Omega,\beta}(t) := \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{\hat{f}_\delta(\omega) e^{-i\omega t} d\omega}{1 + 2\pi\beta + 2\pi\beta\omega^2},$$

where $\beta(\delta) \to 0$, $\frac{\delta}{\beta^{\frac{1}{4}}(\delta)} \to 0$, as $\delta \to 0$.

Then for each $\epsilon > 0$, there exists $\Omega(\epsilon) > 0$ and $\delta(\epsilon) > 0$ such that

$$||f_{\Omega(\epsilon),\beta} - f_E||_{C(-\infty,\, \infty)} < \epsilon$$

whenever

$$0 < \delta < \delta(\epsilon).$$

The proof can be seen in the appendix.

4.4 EXPERIMENTAL RESULTS

In this section we use the Time-Frequency Regularized Extrapolation algorithm. The result of j iterations of the algorithm with $\alpha = 0.00000001$ and $\beta = 0.00004/2^j$, $j = 1, 2, 3, 4, 5$ is in figures 11-15.

4.5. CONCLUSION

A two-dimensional band-limited extrapolation model is introduced in this chapter. In considering its application in image-processing we presented a generalized extrapolation model in which the signal is not necessarily band-limited. A time-frequency regularized extrapolation algorithm is presented to control the Gibbs phenomena.

APPENDIX

Proof of Theorem 3.1: Since every $M^\alpha[F,f] \geq 0$, there exists

$$M_\alpha := \inf_{F\in W_2^1} M^\alpha[F,f].$$

Here the spaces in section 2.4 are $U := L^2[-T,T]$, $D := C_{0,\Omega}$, $D_1 := W_2^1[-\Omega,\Omega]$, M^α is defined in (7), and f is a fixed function in U.

Let F_m denote a minimizing sequence for M^α, specifically, one such that

$$M_\alpha \leq M^\alpha[F_m,f] \leq M_\alpha + \frac{1}{m}.$$

The inequality

$$(a+b)^2 \leq 2(a^2+b^2)$$

implies that

$$||\frac{AF_m - f}{2} + \frac{AF_{m+p} - f}{2}||^2 \leq (||\frac{AF_m - f}{2}|| + ||\frac{AF_{m+p} - f}{2}||)^2$$
$$\leq 2(||\frac{AF_m - f}{2}||^2 + ||\frac{AF_{m+p} - f}{2}||^2).$$

where A is defined in $(7a)$.

Consequently we have, by the parallelogram law in Hilbert space and the definition of M^α in equation (7) in section 3.2

$$\alpha||\frac{F_m - F_{m+p}}{2}||^2_{W_2^1} = -\alpha||\frac{F_m + F_{m+p}}{2}||^2_{W_2^1} + \frac{\alpha}{2}||F_m||^2_{W_2^1} + \frac{\alpha}{2}||F_{m+p}||^2_{W_2^1}$$
$$= -M^\alpha[\frac{F_m + F_{m+p}}{2}, f] + \frac{1}{2}M^\alpha[F_m,f] + \frac{1}{2}M^\alpha[F_{m+p},f]$$
$$+||A\frac{F_m + F_{m+p}}{2} - f||^2_{L^2} - \frac{1}{2}||AF_m - f||^2_{L^2} - \frac{1}{2}||AF_{m+p} - f||^2_{L^2}$$
$$\leq -M_\alpha + \frac{1}{2}(M_\alpha + \frac{1}{m}) + \frac{1}{2}(M_\alpha + \frac{1}{m}) = \frac{1}{m} \to 0 \text{ as } m \to \infty.$$

It follows from the completeness of the space W_2^1 that the sequence $\{F_m\}$ converges in it. Let us define

$$F_\alpha := \lim_{m\to\infty} F_m.$$

Since A is continuous in L^2-norm, M^α is also (see eq (7)), and therefore

$$\lim_{m\to\infty} M^\alpha[F_m, f] = M^\alpha[F_\alpha, f].$$

Consequently we have

$$M^\alpha[F_\alpha, f] = M_\alpha.$$

The uniqueness of the element F_α follows from the fact that $M^\alpha[F, f]$ is a nonnegative quadratic functional, and it cannot attain its least value at two distinct elements. By a variational principle ([19], pp.183-214),

$$\frac{dM^\alpha[F+\epsilon\eta]}{d\epsilon}|_{\epsilon=0} = 0, \qquad \forall \eta \in W_2^1[-\Omega, \Omega].$$

Then upon expanding $M^\alpha[F+\epsilon\eta]$ using definition (7) and linearity, we find for the real part of this derivative

$$\Re\{\int_{-\Omega}^{\Omega}\left[\frac{1}{2\pi^2}\int_{-\Omega}^{\Omega}\frac{\sin(s-\omega)T}{s-\omega}F_\alpha(s)ds - \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{i\omega t}dt\right]\eta(\omega)d\omega$$

$$+\alpha\int_{-\Omega}^{\Omega} F_\alpha(\omega)\eta(\omega)d\omega + \alpha\int_{-\Omega}^{\Omega} F_\alpha'(\omega)\eta'(\omega)d\omega\} = 0. \tag{20}$$

If we choose η to be a real-valued function, this becomes

$$\int_{-\Omega}^{\Omega}\left[\frac{1}{2\pi^2}\int_{-\Omega}^{\Omega}\frac{\sin(s-\omega)T}{s-\omega}\Re F_\alpha(s)ds - \frac{1}{2\pi}\Re\int_{-T}^{T} f(t)e^{i\omega t}dt\right]\eta(\omega)d\omega$$

$$+\alpha\int_{-\Omega}^{\Omega}\Re F_\alpha(\omega)\eta(\omega)d\omega + \alpha\int_{-\Omega}^{\Omega}\Re F_\alpha'(\omega)\eta'(\omega)d\omega = 0.$$

So

$$\alpha\int_{-\Omega}^{\Omega}\Re F_\alpha'(\omega)\eta'(\omega)d\omega$$

$$= -\int_{-\Omega}^{\Omega}\left[\frac{1}{2\pi^2}\int_{-\Omega}^{\Omega}\frac{\sin(s-\omega)T}{s-\omega}\Re F_\alpha(s)ds - \frac{1}{2\pi}\Re\int_{-T}^{T} f(t)e^{i\omega t}dt + \alpha\Re F_\alpha(\omega)\right]\eta(\omega)d\omega$$

Hence $\Re F_\alpha''$ exists in the weak sense. For the same reason $\Im F_\alpha''$ exists in the weak sense. (We will prove that the derivative exists pointwise in the remark following this proof.) Therefore F_α'' exists in the weak sense and from (20) we can see that

$$\Re\{\int_{-\Omega}^{\Omega}\left[\frac{1}{2\pi^2}\int_{-\Omega}^{\Omega}\frac{\sin(s-\omega)T}{s-\omega}F_\alpha(s)ds - \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{i\omega t}dt\right]\eta(\omega)d\omega +$$

$$+\alpha \int_{-\Omega}^{\Omega} F_\alpha(\omega)\eta(\omega)d\omega - \alpha \int_{-\Omega}^{\Omega} F''_\alpha(\omega)\eta(\omega)d\omega\} = 0.$$

The preceding equation must hold for all $\eta \in W_2^1[-\Omega,\Omega]$. Since $W_2^1[-\Omega,\Omega]$ is dense in $L^2[-\Omega,\Omega]$, it also holds for all $\eta \in L^2[-\Omega,\Omega]$. Let us take for η the function whose complex conjugate is

$$\overline{\eta(\omega)} := \frac{1}{2\pi^2}\int_{-\Omega}^{\Omega} \frac{\sin(s-\omega)T}{s-\omega} F_\alpha(s)ds - \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{i\omega t}dt + \alpha F_\alpha(\omega) - \alpha F''_\alpha(\omega).$$

Note that this function does indeed lie in $L^2[-\Omega,\Omega]$. The preceding equation for this η reads

$$\int_{-\Omega}^{\Omega} \left| \frac{1}{2\pi^2}\int_{-\Omega}^{\Omega} \frac{\sin(s-\omega)T}{s-\omega} F_\alpha(s)ds - \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{i\omega t}dt + \alpha F_\alpha(\omega) - \alpha F''_\alpha(\omega)\right|^2 d\omega = 0.$$

We thus see that F_α must satisfy the Euler equation $(7b)$ for a.e. ω.

Remark. Here the derivatives are weak derivatives. So the Euler equation $(7b)$ is true almost everywhere in $[-\Omega,\Omega]$. We choose the solution to be the representative such that the Euler equation is satisfied pointwise in $[-\Omega,\Omega]$. To see that this is possible we need next theorem

Theorem. If $u \in L^1[-\Omega,\Omega]$ is weakly differentiable on $[-\Omega,\Omega]$ and the weak derivative Du is continuous then there exists $v^* \in C^1[-\Omega,\Omega]$ such that $u = v^*$ a.e..

Proof. Define

$$w(x) := ke^{-\frac{1}{1-x^2}} \cdot \mathbf{1}_{[-1,1]}(x),\ x \in \mathbf{R},$$

where k is a constant such that $\int_{\mathbf{R}} w(x)dx = 1$.

Let $w_h(x) := \frac{1}{h}w(\frac{x}{h})$ and $u_h := u * w_h$, $x \in \mathbf{R}$, $h > 0$.

Then $D(u_h) = (Du)_h$. In general Du, $D(u_h)$, $(Du)_h$ are only distributions but under the present hypthesis they are continuous functions and can consequently be evaluated at points. Hence for any $x, y \in \mathbf{R}$

$$|D(u_h)(x) - D(u_h)(y)| = |(Du)_h(x) - (Du)_h(y)|.$$

$$= \left| \int_{\mathbf{R}} w_h(z)[(Du)(x+z) - (Du)(y+z)]dz \right|.$$

So the family $\{D(u_h) : h > 0\}$ is equicontinuous, since Du is continuous.

On the other hand, $D(u_h)$ is also uniformly bounded.

By the Ascoli-Arzela Theorem $D(u_h) \to v \in C[-\Omega, \Omega]$ as $h \to 0$.

This implies $u_h \to v^* \in C^1[-\Omega, \Omega]$ as $h \to 0$, where v^* is an anti-derivative of v.

However $u_h \to u \in L^1[-\Omega, \Omega]$ as $h \to 0$.

Therefore $u = v^* \in C^1[-\Omega, \Omega]$, a. e. in $[-\Omega, \Omega]$.

Note. Based on the theorem above, since F''_α is continuous, there exists $F^*_\alpha \in C^2[-\Omega, \Omega]$ such that $F^*_\alpha = F_\alpha$ a.e. in $[-\Omega, \Omega]$. Then for F^*_α the Euler equation $(7b)$ is pointwise true. In the sequel we just take F_α to be F^*_α.

Proof of Theorem 3.2: Since F_α is the minimizer in Theorem 3.1 and $\widehat{f_E} \in W^1_2$ (see p.23)

$$\alpha(\delta)||F_{\alpha(\delta)}||^2_{W^1_2} \leq M^\alpha[F_{\alpha(\delta)}, f_\delta] \leq M^\alpha[\widehat{f_E}, f_\delta]$$

$$= ||A\widehat{f_E} - f_\delta||^2_{L^2} + \alpha(\delta)||\widehat{f_E}||^2_{W^1_2} \leq \delta^2 + \alpha(\delta)||\widehat{f_E}||^2_{W^1_2}$$

where $A = \mathbf{F}^{-1}$ from $C_{0,\Omega}$ into $L^2[-T, T]$ according to $(4)'$. Hence dividing by $\alpha(\delta)$,

$$||F_{\alpha(\delta)}||^2_{W^1_2} \leq \delta^2/\alpha(\delta) + ||\widehat{f_E}||^2_{W^1_2} \leq C < \infty,$$

where C is a constant independent of δ. The Sobolev imbedding theorem implies that every sequence of the elements of the set $\{F_{\alpha(\delta)}\}$ has a subsequence $\{F_{\alpha(\delta_n)}\}$ that converges to some element $F_0 \in C_{0,\Omega}$ in the max norm([20]).

We now show $F_0 = \widehat{f_E}$, thus proving that every convergent subsequence of $\{F_{\alpha(\delta)}\}$ converges to $\widehat{f_E}$.

$$\int_{-T}^{T} |(AF_0)(t) - f_E(t)|^2 dt \leq 3\int_{-T}^{T} |(AF_0)(t) - (AF_{\alpha(\delta_n)})(t)|^2 dt$$

$$+3\int_{-T}^{T} |(AF_{\alpha(\delta_n)})(t) - f_{\delta_n}(t)|^2 dt + 3\int_{-T}^{T} |f_{\delta_n}(t) - f_E(t)|^2 dt.$$

By the hypothesis of Theorem 3.2.

$$\int_{-T}^{T} |f_{\delta_n}(t) - f_E(t)|^2 dt \to 0.$$

Since A is continuous from $C_{0,\Omega}$ into $L^2[-T,T]$

$$\int_{-T}^{T} |(AF_0)(t) - (AF_{\alpha(\delta_n)})(t)|^2 dt \to 0.$$

Also

$$\int_{-T}^{T} |(AF_{\alpha(\delta_n)})(t) - f_{\delta_n}(t)|^2 dt \le M^\alpha[F_{\alpha(\delta_n)}, f_{\delta_n}] \le \delta_n^2 + \alpha(\delta_n)||\widehat{f_E}||^2_{W_2^1}.$$

Therefore

$$\int_{-T}^{T} |(AF_0)(t) - f_E(t)|^2 dt = 0,$$

whence

$$AF_0 = f_E.$$

The invertibility of A implies that

$$F_0 = \widehat{f_E}.$$

Proof of Theorem 3.3: Since $\widehat{f_E} \in W_2^1$ and $||F_{\alpha(\delta)}||^2_{W_2^1} \le C$, by the Sobolev embedding theorem, $F_{\alpha(\delta)}$ and $\widehat{f_E}$ belong to the same compact subset

$$W^1_{2,M} := \{F \in C_{0,\Omega} : \int_{-\Omega}^{\Omega} (|F(\omega)|^2 + |F'(\omega)|^2) d\omega \le M\}$$

of the space $C_{0,\Omega}$, where M is a finite constant. It follows from the continuity of A^{-1} $(= \mathbf{F})$ from $AW^1_{2,M}$ into $C_{0,\Omega}$ ([13],p.29) that

$$d^2(\delta) := ||F_{\alpha(\delta)} - \widehat{f_E}||^2_{C_{0,\Omega}} \le ||A^{-1}||^2_{AW^1_{2,M}} ||AF_{\alpha(\delta)} - f_E||^2_{L^2}$$

where

$$||AF_{\alpha(\delta)} - f_E||^2_{L^2} \le 2||AF_{\alpha(\delta)} - f_\delta||^2_{L^2} + 2||f_\delta - f_E||^2_{L^2}$$

$$\le 2M^\alpha[F_\alpha, f_\delta] + 2\delta^2 \le 2M^\alpha[\widehat{f_E}, f_\delta] + 2\delta^2 \le 2(\delta^2 + \alpha(\delta)||\widehat{f_E}||^2_{W_2^1}) + 2\delta^2.$$

Hence

$$d^2(\delta) = O(\delta^2) + O(\alpha(\delta)).$$

In order to prove Theorem 3.4, we need a lemma.

Lemma. If $f \in L^1(\mathbf{R})$, the regularized solution F_α of $(7d)$ lies in the set $C^2[-\Omega, \Omega] := \{F \in C_{0,\Omega} : F'' \text{ is continuous}\}$.

Proof: Since every function in $W_2^1[-\Omega, \Omega]$ has an a.e. representative in $C[-\Omega, \Omega]$ we may suppose, as in the preceding Note, that F_α is continuous. F_α is a solution of $(7d)$ and $\mathbf{F}[f]$ is continuous since $f \in L^1(\mathbf{R})$. All other terms in $(7d)$ except for F''_α lie in the set $C[-\Omega, \Omega]$. Therefore F''_α is continuous, so F_α lies in the set $C^2[-\Omega, \Omega]$.

Remark. If $f(t),\ tf(t),\ t^2 f(t) \in L^1(\mathbf{R})$, then $\mathbf{F}(f) \in C^2(\mathbf{R})$, and this implies that the regularized solution F_α of $(7d)$ lies in the set $C^4[-\Omega, \Omega] := \{F \in C_{0,\Omega} : F^{(4)}$ is continuous$\}$. This can be seen if we calculate derivatives twice in $(7d)$:

$$(1 + 2\pi\alpha)F''_\alpha - 2\pi\alpha F^{(4)}_\alpha = [\mathbf{F}(f)]'' \in C(\mathbf{R}).$$

Proof of Theorem 3.4:

Since $f(t),\ tf(t),\ t^2 f(t) \in L^1(\mathbf{R})$, the discussion preceding equation (9) shows its validity.

From (8) and (9), we have

$$[(1 + 2\pi\alpha)I + \frac{2\pi\alpha}{h^2}H](F^h_\alpha - F_\alpha) = Q(h^2)$$

where H is defined on p.25, $Q(h^2) := (O_{-n+1}(h^2), ..., O_{n-1}(h^2))$ in which $O_j(h^2)$ is a function that converges to 0 like h^2 for $j = -n+1, ..., n-1$.

Since H is positive-definite, $H = SDS^*$ where S is a unitary matrix and S^* is it conjugate transpose such that $S^*S = I$, $D = [\lambda_{-n+1}, \lambda_{-n+2}, ..., \lambda_{n-1}]$ $(\lambda_j > 0,\ j = -n+1, -n+2, ..., n-1)$.

Hence

$$F_\alpha^h - F_\alpha = S[(1+2\pi\alpha)I + \frac{2\pi\alpha}{h^2}D]^{-1}S^*Q(h^2).$$

Therefore

$$\sum_{j=-n+1}^{n-1} |F_{\alpha j}^h - F_{\alpha j}|^2$$

$$= Q(h^2)^*S[(1+2\pi\alpha)I + \frac{2\pi\alpha}{h^2}D]^{-1}S^*S[(1+2\pi\alpha)I + \frac{2\pi\alpha}{h^2}D]^{-1}S^*Q(h^2)$$

$$= Q(h^2)^*S[(1+2\pi\alpha)I + \frac{2\pi\alpha}{h^2}D]^{-2}S^*Q(h^2)$$

$$= V^*[(1+2\pi\alpha)I + \frac{2\pi\alpha}{h^2}D]^{-2}V \qquad (V = S^*Q(h^2))$$

$$= \sum_{j=-n+1}^{n-1} \frac{|V_j|^2}{(1+2\pi\alpha+\frac{2\pi\alpha}{h^2}\lambda_j)^2} \qquad (V = (V_{-n+1}, V_{-n+2}, ..., V_{n-1})^*)$$

$$\leq \frac{1}{(1+2\pi\alpha+\frac{2\pi\alpha}{h^2}\min\lambda_j)^2} \sum_{j=-n+1}^{n-1} |V_j|^2$$

$$= \frac{1}{(1+2\pi\alpha+\frac{2\pi\alpha}{h^2}\min\lambda_j)^2} \sum_{j=-n+1}^{n-1} O_j(h^4)$$

in which $\min\lambda_j = 2 + 2\cos(\pi(2n-1)/2n)$ ([22]).

Proof of Theorem 3.5:

$$F_{\alpha(\delta)}(\omega) - \widehat{f_E}(\omega) = \int_{-\infty}^{\infty} \frac{f_\delta(t) - f_E(t)}{1+2\pi\alpha+2\pi\alpha t^2} e^{i\omega t}dt - \int_{-\infty}^{\infty} \frac{2\pi\alpha + 2\pi\alpha t^2}{1+2\pi\alpha+2\pi\alpha t^2} f_E(t)e^{i\omega t}dt$$

$$=: I_1 + I_2.$$

$$|I_1| \leq (\int_{-\infty}^{\infty} |f_\delta(t) - f_E(t)|^2 dt)^{1/2} [\int_{-\infty}^{\infty} \frac{dt}{(1+2\pi\alpha+2\pi\alpha t^2)^2}]^{1/2}$$

$$\leq \frac{\delta}{2\pi\alpha}[\int_{-\infty}^{\infty} \frac{dt}{(a^2+t^2)^2}]^{1/2} = \frac{\delta}{2\pi\alpha}(\frac{\pi}{2a^3})^{1/2} \ (a^2 = \frac{1+2\pi\alpha}{2\pi\alpha})$$

$$= \frac{\delta}{\alpha^{1/4}} \frac{(2\pi)^{1/4}}{2(1+2\pi\alpha)^{3/4}} = O(\frac{\delta}{\alpha^{1/4}}).$$

Next

$$|I_2| \leq \alpha \int_{|t|\leq T} (2\pi + 2\pi t^2)|f_E(t)|dt + \int_{|t|\geq T} |f_E(t)|dt.$$

Now, given $\forall\epsilon > 0\ \exists T = T(\epsilon)$ such that $\int_{|t|\geq T} |f_E(t)|dt < \epsilon/2$ and $\exists\alpha(\epsilon)$ such that

$$\alpha \int_{|t|\leq T} (2\pi + 2\pi t^2)|f_E(t)|dt < \epsilon/2 \text{ whenever } 0 < \alpha < \alpha(\epsilon).$$

Therefore

$$|I_2| < \epsilon \text{ whenever } 0 < \alpha < \alpha(\epsilon).$$

Proof of Theorem 4.1: For for every $t \in \mathbf{R}$

$$f_{\Omega,\beta(\delta)}(t) - f_E(t) = \int_{-\Omega}^{\Omega} \frac{\widehat{f_\delta}(\omega)}{1 + 2\pi\beta + 2\pi\beta\omega^2} e^{-i\omega t} d\omega - \int_{-\infty}^{\infty} \widehat{f_E}(\omega) e^{-i\omega t} d\omega$$

$$= \int_{-\Omega}^{\Omega} \frac{\widehat{f_\delta}(\omega) - \widehat{f_E}(\omega)}{1 + 2\pi\beta + 2\pi\beta\omega^2} e^{-i\omega t} d\omega - \int_{-\Omega}^{\Omega} \frac{2\pi\beta + 2\pi\beta\omega^2}{1 + 2\pi\beta + 2\pi\beta\omega^2} \widehat{f_E}(\omega) e^{-i\omega t} d\omega$$

$$+ \int_{|\omega|>\Omega} \widehat{f_E}(\omega) e^{-i\omega t} d\omega =: I_1 + I_2 + I_3.$$

$$|I_1| \leq (\int_{-\Omega}^{\Omega} |\widehat{f_\delta}(\omega) - \widehat{f_E}(\omega)|^2 d\omega)^{1/2} [\int_{-\Omega}^{\Omega} \frac{d\omega}{(1 + 2\pi\beta + 2\pi\beta\omega^2)^2}]^{1/2}$$

$$\leq \frac{\delta}{2\pi\beta} [\int_{-\infty}^{\infty} \frac{d\omega}{(a^2 + \omega^2)^2}]^{1/2} = \frac{\delta}{2\pi\beta} (\frac{\pi}{2a^3})^{1/2} \ (a^2 := \frac{1 + 2\pi\beta}{2\pi\beta})$$

$$= \frac{\delta}{\beta^{1/4}} \frac{(2\pi)^{1/4}}{2(1 + 2\pi\beta)^{3/4}} = O(\frac{\delta}{\beta^{1/4}}).$$

Next

$$|I_2| \leq \beta \int_{|\omega|\leq\Omega} (2\pi + 2\pi\omega^2)|\widehat{f_E}(\omega)| d\omega.$$

Now, given $\epsilon > 0,\ \exists\Omega(\epsilon)$ such that

$$|I_3| < \epsilon/3, \text{ when } \Omega = \Omega(\epsilon)$$

and $\exists\delta(\epsilon)$ such that

$$|I_1| < \epsilon/3,\ |I_2| < \epsilon/3 \text{ whenever } 0 < \delta < \delta(\epsilon).$$

Therefore for each $t \in \mathbf{R}$

$$|f_{\Omega(\epsilon),\beta(\delta)}(t) - f_E(t)| < \epsilon \text{ whenever } 0 < \delta < \delta(\epsilon).$$

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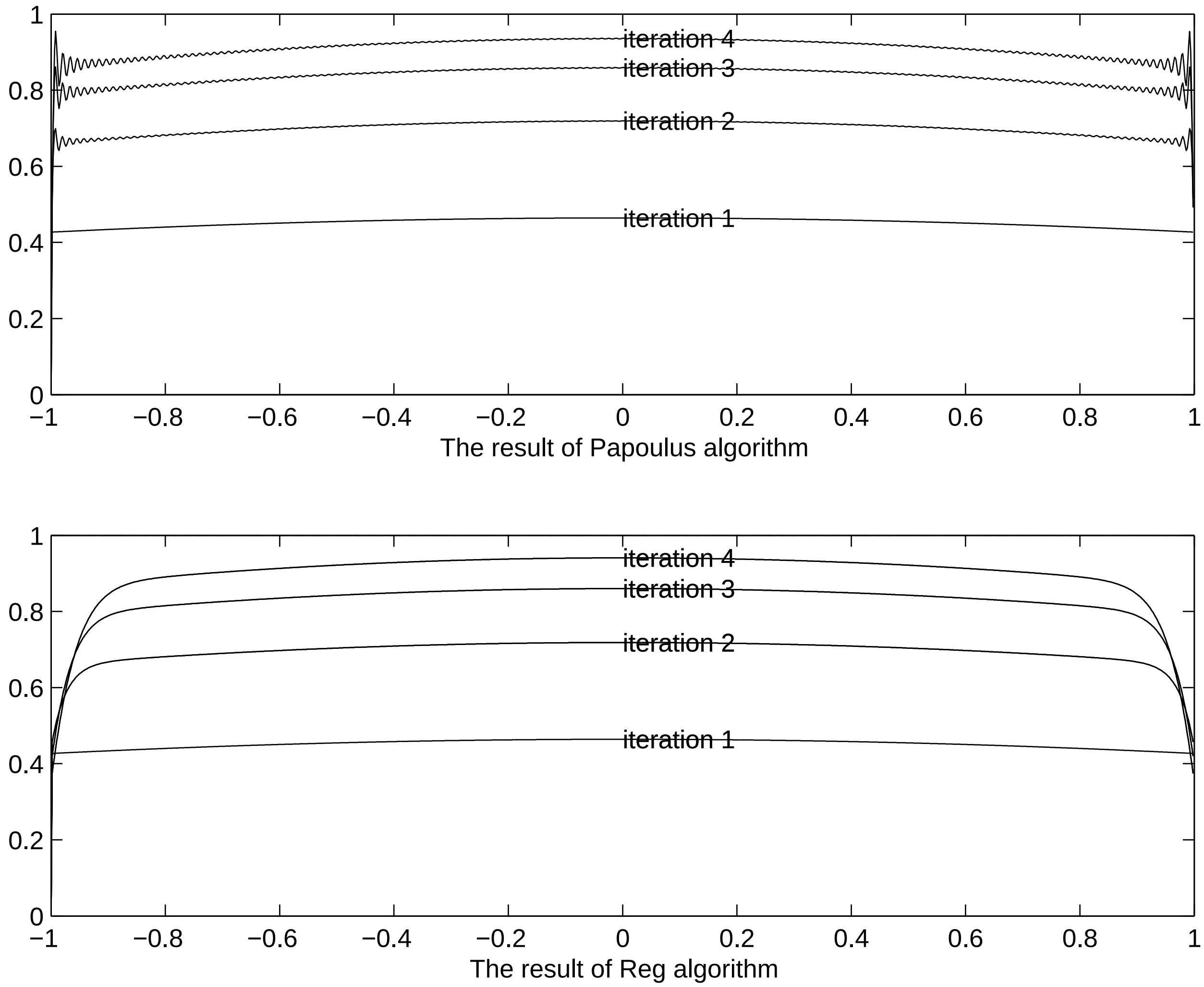

Figure 1: The numerical results of $\hat{f}$ by the Papoulis-Gerchberg algorithm and regularized iterative extrapolation for the iterations 1,2,3,4.

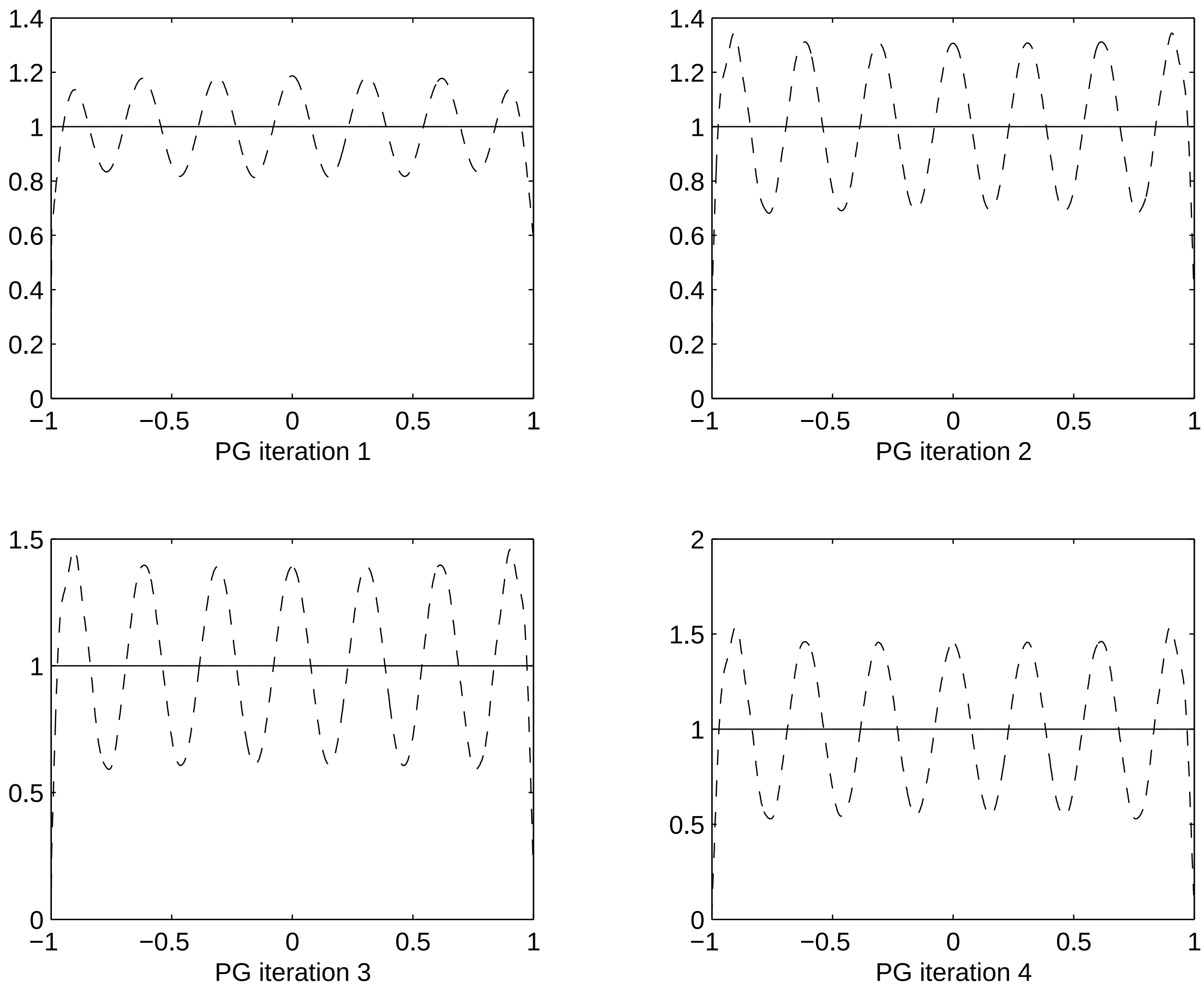

Figure 2: The numerical results of $\hat{f}$ by the Papoulis-Gerchberg algorithm for the iterations 1,2,3,4. (solid–exact solution, dashed–approximate solution)

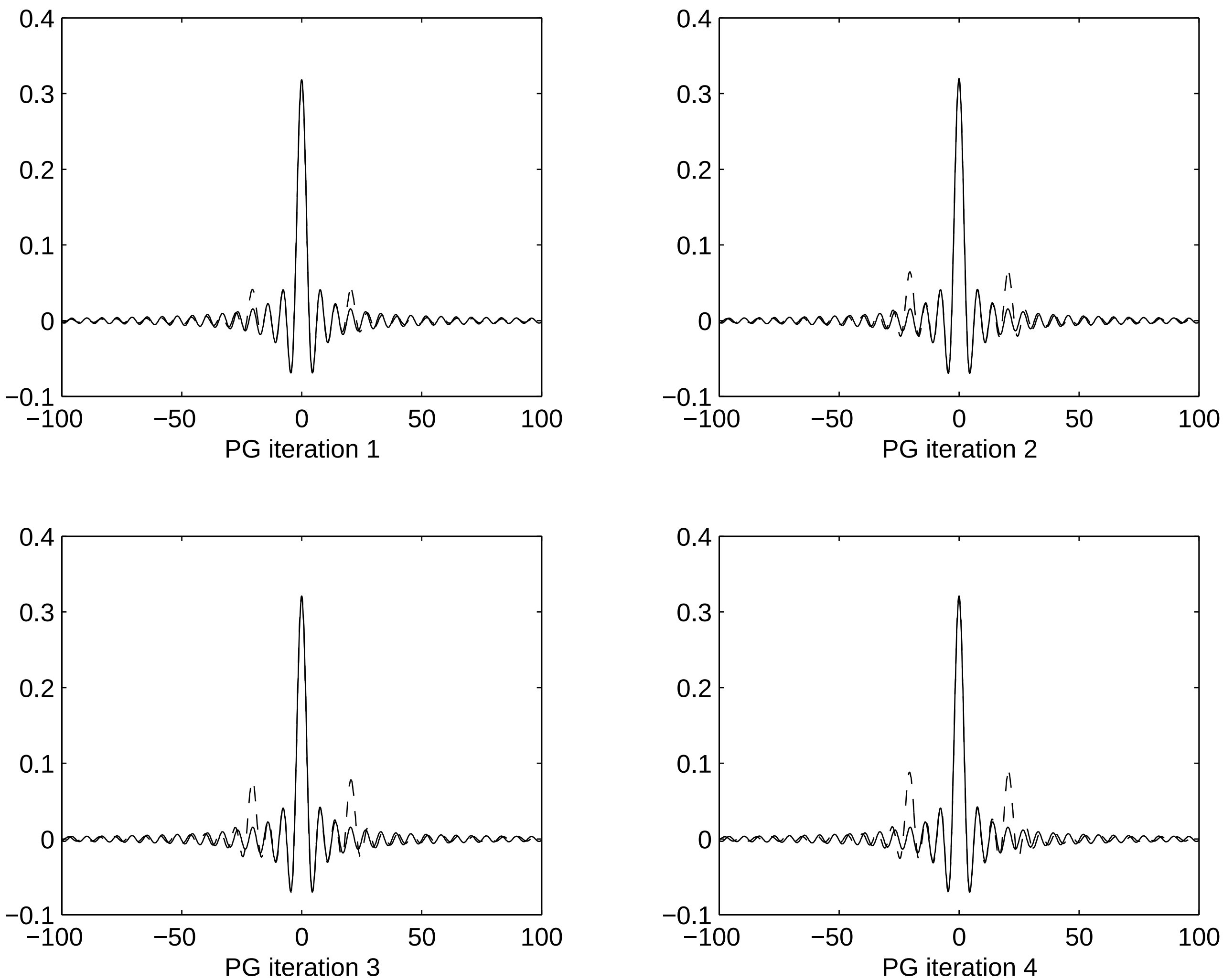

Figure 3: The extrapolation by the Papoulis-Gerchberg algorithm for the iteration 1,2,3,4. (solid–exact solution, dashed–approximate solution)

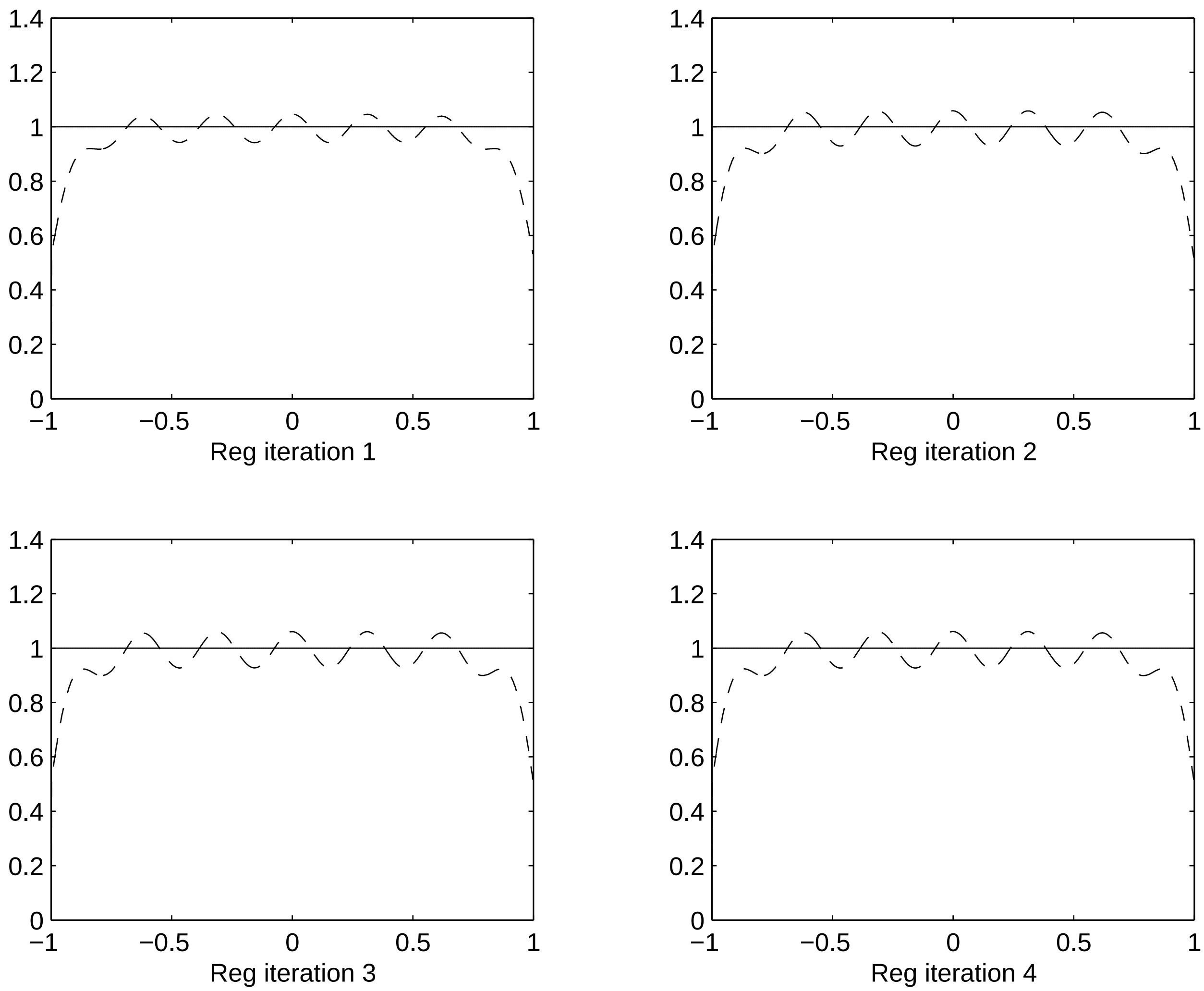

Figure 4: The numerical results of $\hat{f}$ by regularized iterative extrapolation for the iterations 1,2,3,4. (solid–exact solution, dashed–approximate solution)

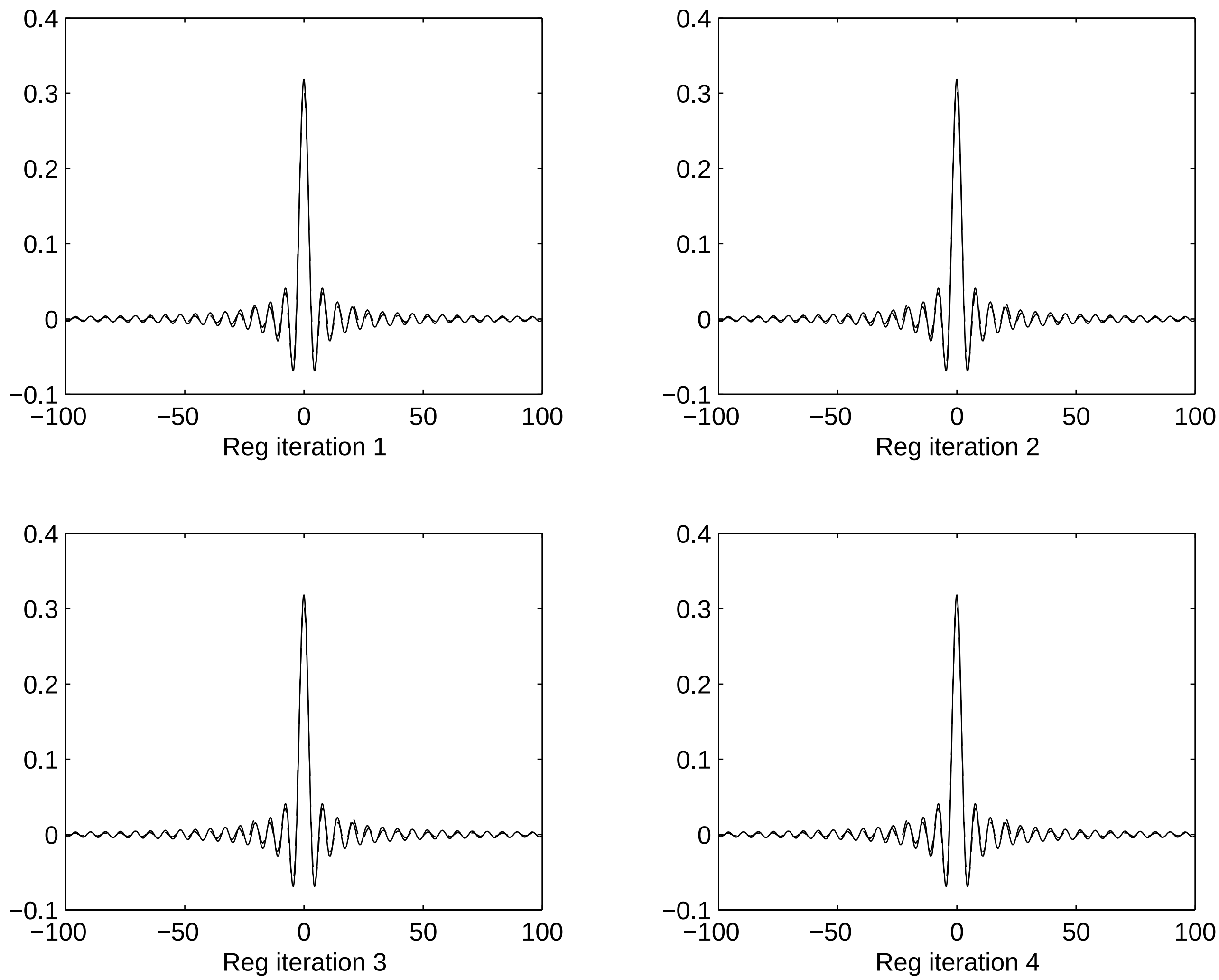

Figure 5: The extrapolation by regularized iterative extrapolation for the iteration 1,2,3,4. (solid–exact solution, dashed–approximate solution)

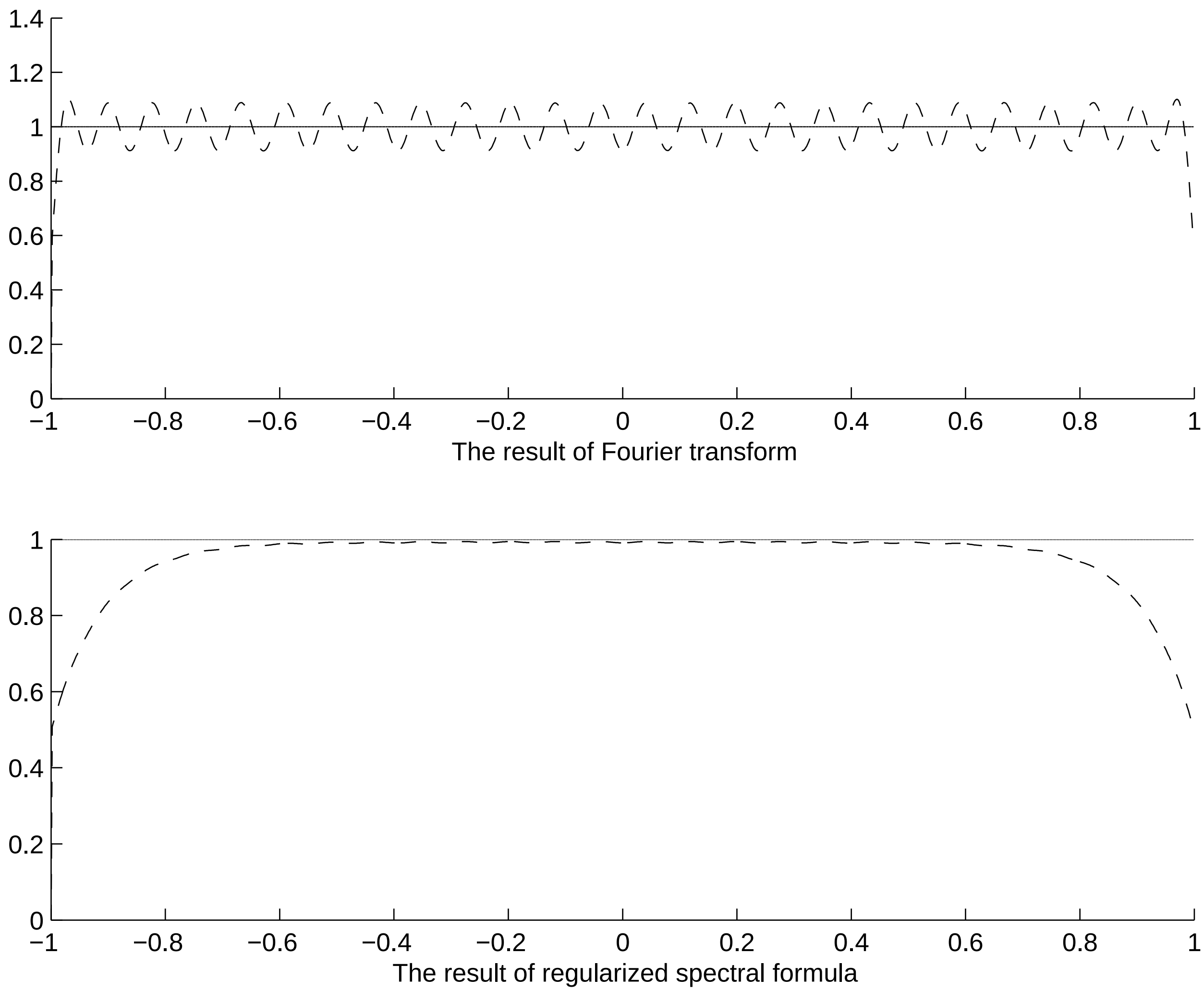

Figure 6: The numerical results of $\hat{f}$ by Fourier transform and the regularized spectral formula.(solid–exact solution, dashed–approximate solution)

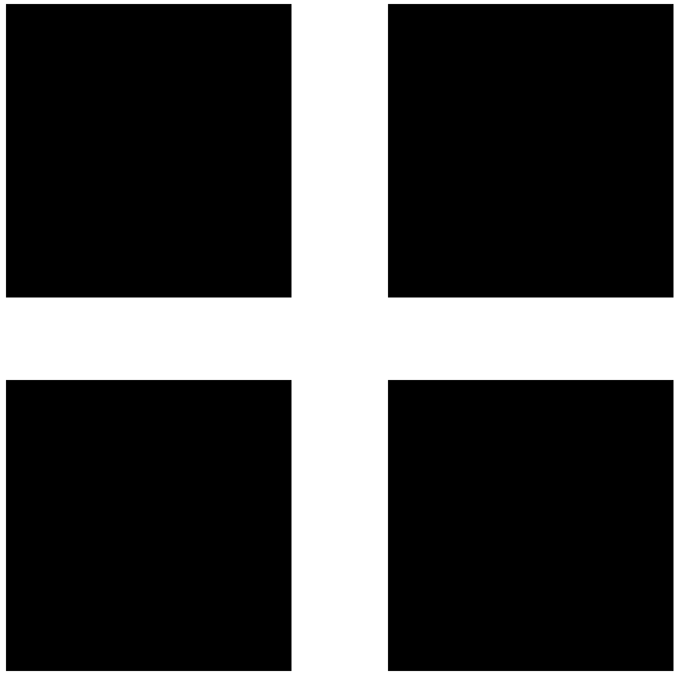

Figure 7: The original image

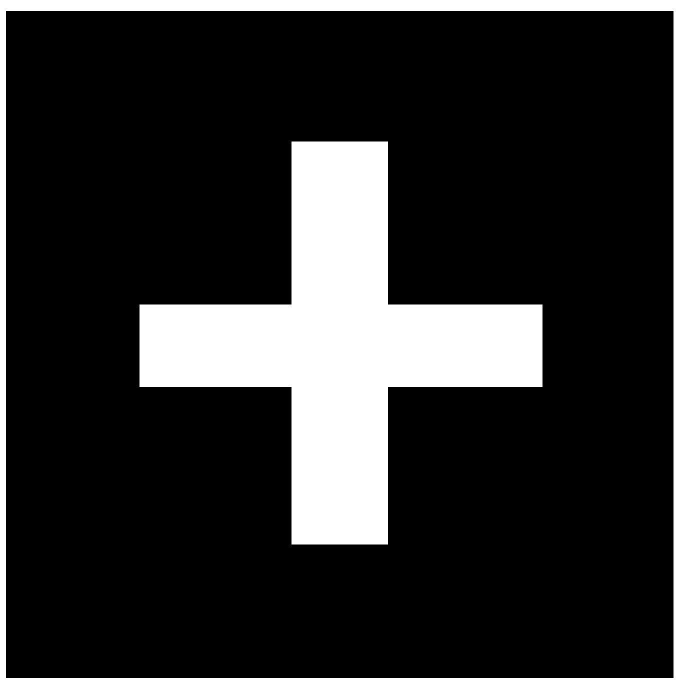

Figure 8: The image to be extrapolated

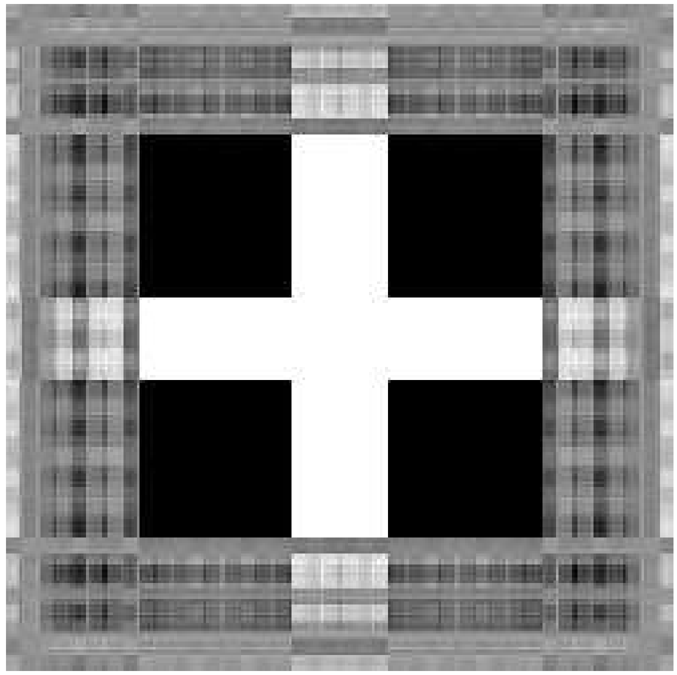

Figure 9: The extrapolation by 5 iterations of the Papoulis-Gerchberg algorithm

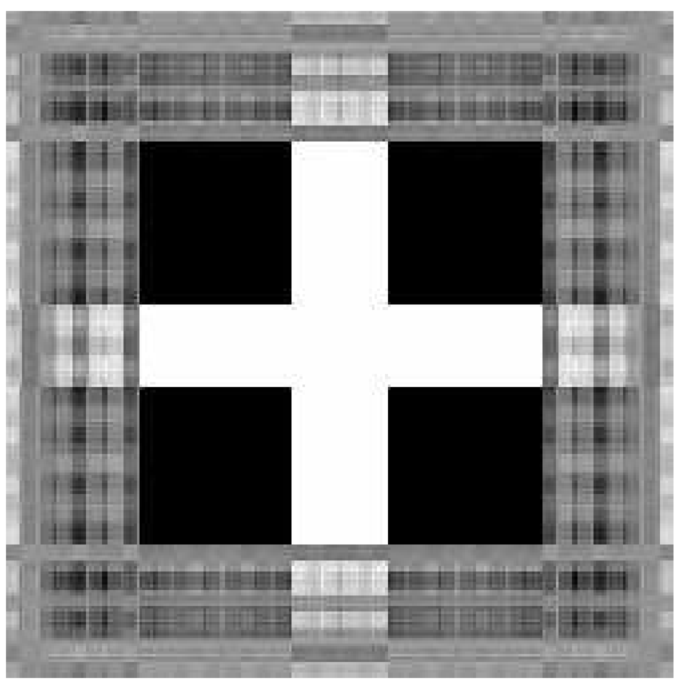

Figure 10: The extrapolation by 5 iterations of the regularized algorithm

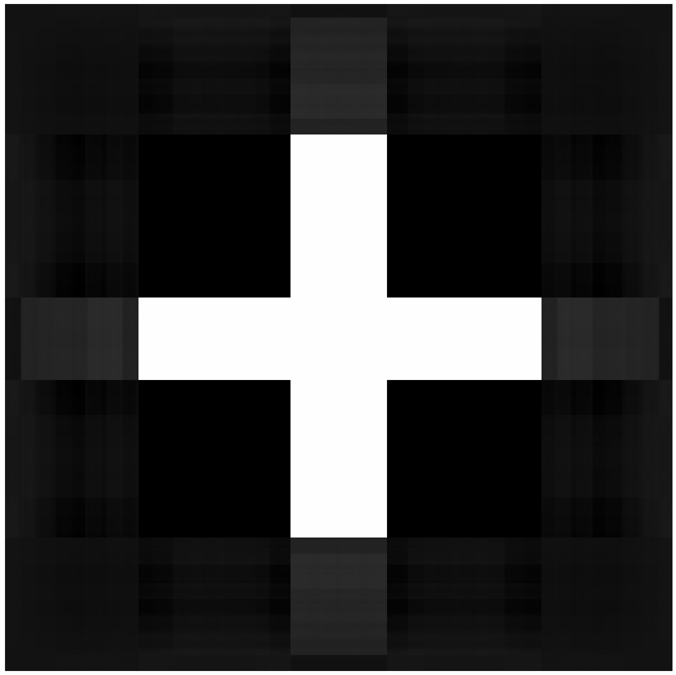

Figure 11: The extrapolation by 1 iteration of the T-F regularized algorithm

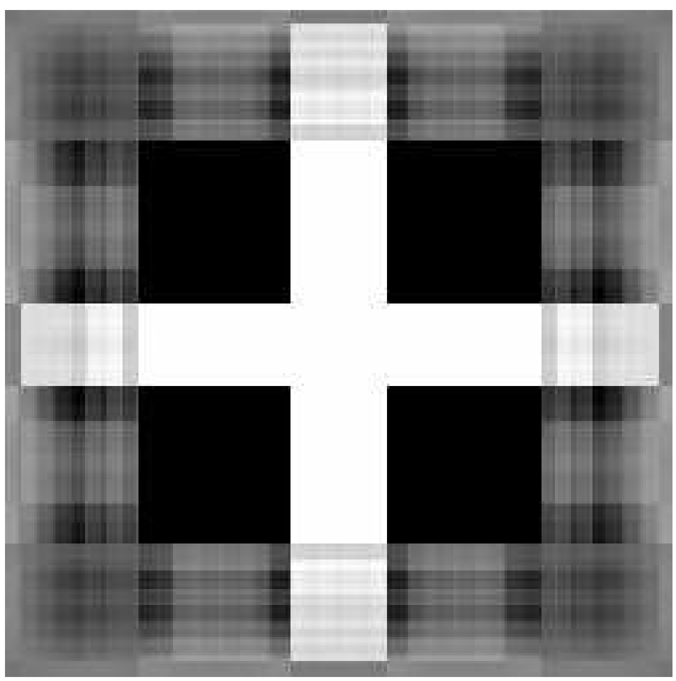

Figure 12: The extrapolation by 2 iterations of the T-F regularized algorithm

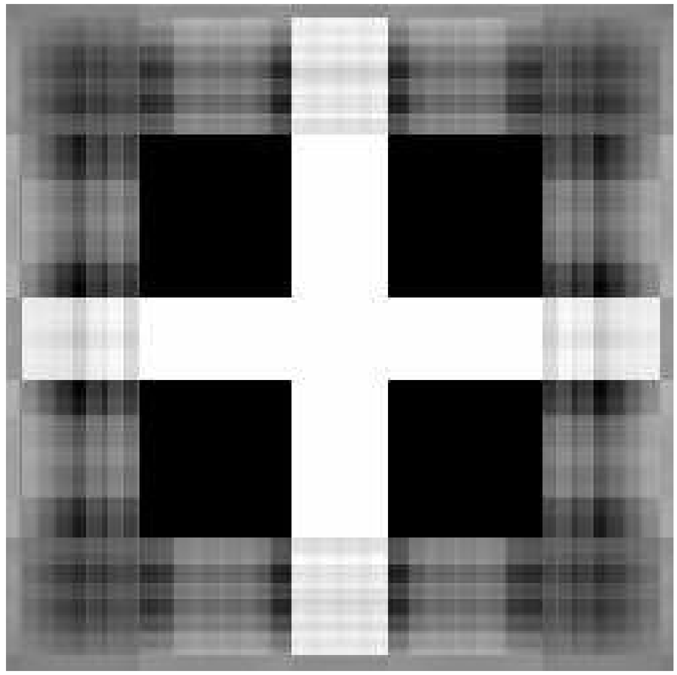

Figure 13: The extrapolation by 3 iterations of the T-F regularized algorithm

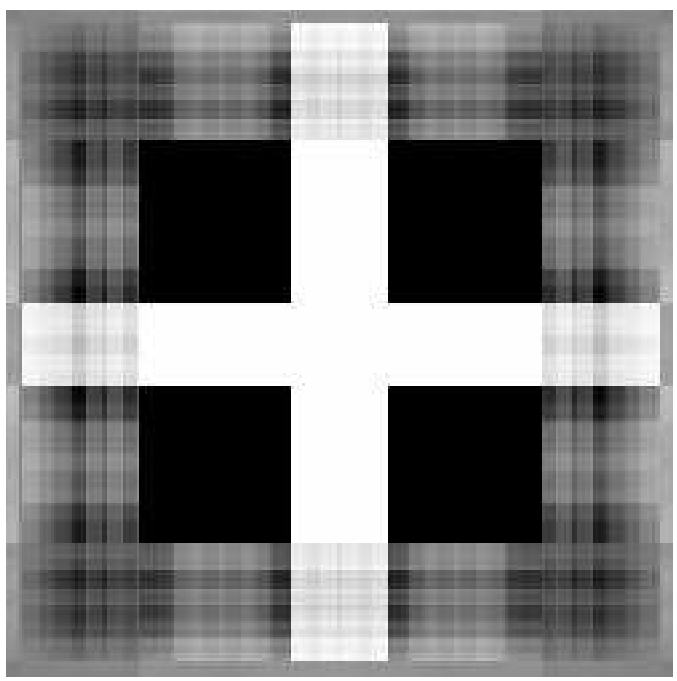

Figure 14: The extrapolation by 4 iterations of the T-F regularized algorithm

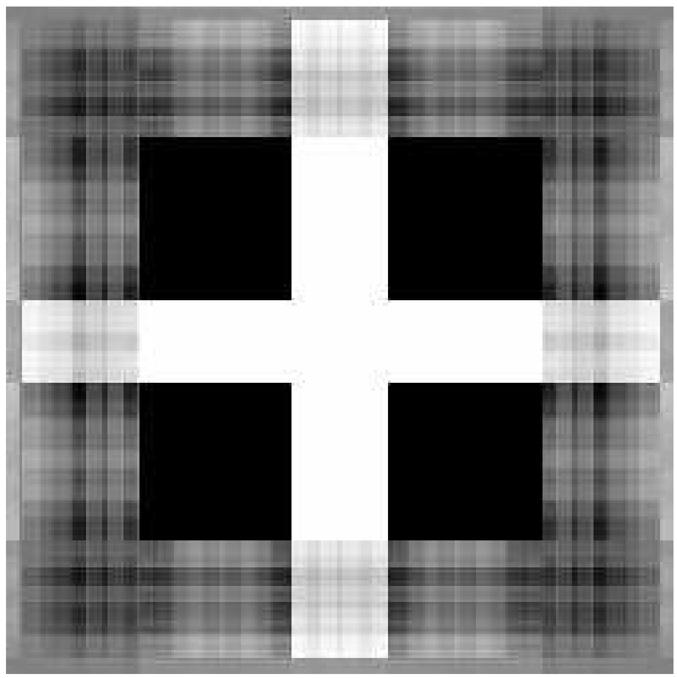

Figure 15: The extrapolation by 5 iterations of the T-F regularized algorithm