FINITE-AMPLITUDE VIBRATION OF CLAMPED AND SIMPLY-SUPPORTED CIRCULAR PLATES

by

IBRAHIM MAHDI AL-KHATTAT

B.Sc., University of Wales (Britain), 1962

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1976

Approved by:

Chi-lung Huang
Major Professor

TABLE OF CONTENTS

THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (Continued)

NOMENCLATURE

r, θ, z	cylindrical coordinates used to describe the undeformed configuration of the plate
h	thickness of the plate
a	outer radius of the circular plate
ρ	mass density of the plate material
t	time variable
u, w	radial and transverse displacements of the middle plane, respectively
$\varepsilon_r, \varepsilon_\theta$	total strains in the directions indicated by subscripts
σ_r, σ_θ	stresses in the directions indicated by subscripts
E, ν	elastic modulus and Poisson's ratio, respectively
N_r, N_θ	membrane forces per unit length
K	kinetic energy of the plate
U_s, U_b	strain energy due to stretching of the middle plane and due to bending of the plate, respectively
W	work done on the plate by external forces
Ψ, ϕ	stress functions
M_r, M_θ	bending moments per unit length
$q(r,t)$	time-varying loading intensity
∇^4	biharmonic operator
∇^2	Laplacian operator, $= \frac{1}{r}\frac{\partial}{\partial r}(r \frac{\partial}{\partial r})$
D	flexural rigidity of the plate, $= \frac{Eh^3}{12(1-\nu^2)}$
ξ, τ	dimensionless space and time variables, respectively
χ	dimensionless transverse displacement
$Q(\xi), Q^*(\xi)$	dimensionless loading distributions
$g(\xi), f(\xi)$	shape functions of vibration

A,α	amplitude parameters
λ	nondimensional nonlinear eigenvalue
ω	nondimensional angular frequency, $= (\lambda)^{1/2}$
ω_ℓ	linear nondimensional angular frequency
$\{\ \}$	indicates a column vector
$\bar{Y},\bar{Z},\bar{H}$	(6x1) vector functions
M,N	coefficient matrices
$\bar{0}$	(3x1) null vector
$\bar{\eta}$	adjustable data in the related initial-value problem
J_n,Y_n	n^{th} order Bessel functions of the first and second kinds, respectively
T	nonlinear period of vibration, $= \frac{2\pi}{\omega}$
T_ℓ	linear period of vibration, $= \frac{2\pi}{\omega_\ell}$
w_o	defection at $\xi = 0.0$
CI	Clamped-immovable
CM	Clamped-movable
HI	Hinged-immovable
HM	Hinged-movable

INTRODUCTION

The study of vibration of plates is the dynamical analogue of load-deflection analysis of plates in the static case. In fact static loading is a special case of the more general dynamical problem of plate vibration.

When the amplitude of vibration, or displacement, is of the order of the thickness of the plate the deformation of the middle plane of the plate is no more negligible as in the case of small displacements, and the basic equations of motion, known as the dynamical von Kármán's equations, are nonlinear and coupled.

These equations, together with the associated boundary conditions at the center and edge of the plate constitute a two-point boundary-value problem.

The boundary-value problem becomes an eigenvalue problem by separation of the variables.

No exact solution is known for the nonlinear boundary-value problem and hence approximate methods must be used. The essence of such methods is to approximate the continuous system by a discrete one having a finite number of degrees of freedom. The discrete representation is usually achieved through an assumed space mode. Substituting this space mode in the differential equations and requiring that some measure of the error is minimized, results in the elimination of the assumed space mode. The problem then reduces to a nonlinear ordinary differential equation with time, t, as the independent variable. This equation is similar to a one-degree-of-freedom Duffing equation [13].

As an alternative method of solution for the boundary-value problem, Sandman [2] and Huang [6], assumed a time-mode function which was then eliminated by a time averaging method. The problem is reduced to an eigenvalue problem comprizing two nonlinear ordinary differential equations in the space coordinate functions, together with a suitably-reformulated set of boundary conditions. Newton's method and the principle of analytical continuation were used to solve the eigenvalue problem.

Solutions to the boundary-value problem were obtained [17,18,20] by employing the Berger assumption [16], which simplifies the equations of motion by neglecting the second strain invariant in the calculation of the strain energy of the plate. This assumption was first used, in the static case of deflections, but was later found to give unsatisfactory results when the edge of the plate was not restrained against radial displacement [19].

In this work, the solution in [2] of the clamped-immovable circular plate is extended to three other edge conditions: clamped-movable, hinged-immovable and hinged-movable, both for free and forced vibration. While all four cases are strictly theoretical, their practical importance is that they represent the limiting boundaries between which practical cases fall, depending on the extent of radial restraint. The experiments referred to in Ref. [2] are used as an example.

A comparison is made between the present solutions and the solutions in [17] and [20] which use the Berger assumption. The comparison shows that in the radially-restrained cases the Berger assumption gives good approximations to the plate responses at low amplitudes of vibrations, but the accuracy of such approximations decreases with the amplitude increasing. Comparing with the radially movable cases it is evident that the Berger assumption is entirely unsuitable for such cases.

Even in the radially-restrained cases the Berger assumption produces a linear pattern of bending stresses, contrary to the nonlinear pattern of stresses established by the present solution for all the cases considered.

CHAPTER I. THE BASIC DIFFERENTIAL EQUATIONS

Consider a thin circular plate of radius a and constant thickness h located by a cylindrical system of coordinates r, θ, and z, as shown in Fig. (1). The material of the plate is assumed to be elastic, homogeneous, and isotropic.

The plate is excited by an external force, and the resulting motion is studied, on the assumption that the amplitude of the resulting flexural vibration is of the order of the thickness of the plate.

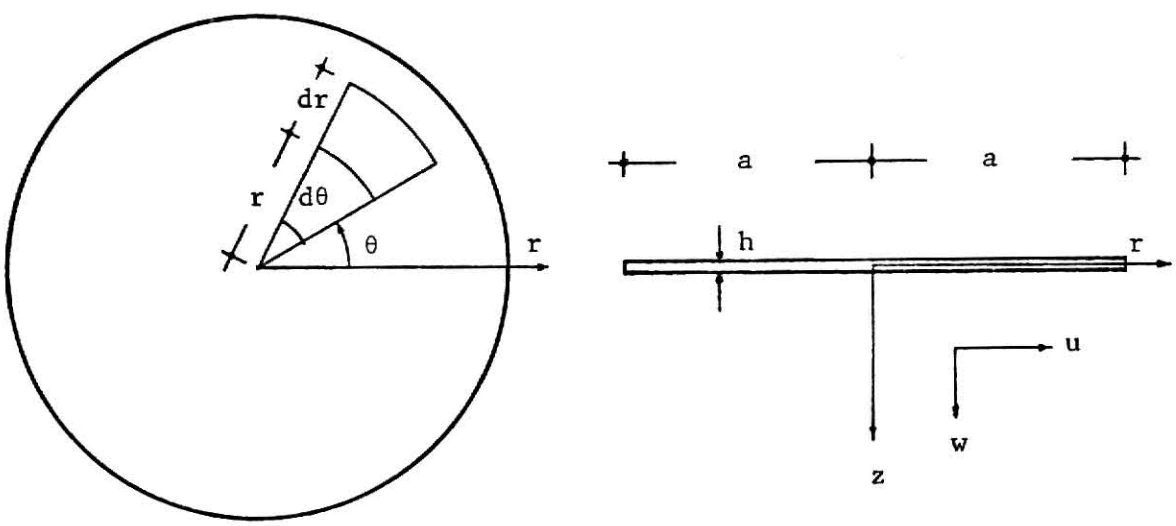

Fig. (1). The Circular Plate and the Polar Coordinate System.

a. Finite-Amplitude Displacement Theory

In the linear theory of "small displacements" of plates only flexture is accounted for and the middle plane of the plate is assumed inentensible. In contrast, membranes are assumed to have no flextural stiffness and only the "membrane" effect due to the in-plane extensions is considered.

THIS BOOK CONTAINS NUMEROUS PAGES THAT HAVE INK SPLOTCHES IN THE MIDDLE OF THE TEXT. THIS IS AS RECEIVED FROM CUSTOMER.

THESE ARE THE BEST IMAGES AVAILABLE.

"Finite-amplitude displacements" are displacements of the order of the thickness of the plate. In this case, although the displacements are still small relative to the planar dimensions of the plate, the membrane effect cannot be neglected.

For "large displacements," further modifications to the finite amplitude theory are necessary.

In mathematical terms, qualitative definitions of the three categories of displacements may be drawn from the Green's strain tensor [1]:

$$E_{ij} = \frac{1}{2} [u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}]^{*} \qquad i,j = 1,2,3 .$$

Assuming axial symmetry of the displacements u_i, the following components, in polar coordinates result:

	radial membrane strain, ε_{rr}	circumferential membrane strain, $\varepsilon_{\theta\theta}$
small displacements	u_r	$\frac{u}{r}$
finite amplitude displ.	$u_r + \frac{1}{2} (w_r)^2$	$\frac{u}{r}$
"large displacements"	$u_r + \frac{1}{2} [(w_r)^2 + (u_r)^2]$	$\frac{u}{r}$

where u and w are, respectively, the radial and transverse displacement components of the middle plane, r is the radial coordinate, u_r and w_r are the partial derivatives of u and w with respect to r.

The finite-amplitude displacements theory may therefore be summarized by making the following assumptions, the first two of which are retained from the small displacements theory:

---

*In the index notation used here, i, in u_i, is a suffix; $u_{i,j}$ is the partial derivative of u_i w.r.t the j coordinate. This is not to be confused with the notation used elsewhere.

1. Lines normal to the middle plane of the plate in the undeformed state remain straight and normal to the middle plane in the deformed state.

2. Normal stress, σ_z, is small compared with in-plane stresses and may be neglected in the stress-strain relations.

3. The only non-zero components of the strain tensor are:

$$\varepsilon_{rr} = u_r + \frac{1}{2} (w_r)^2$$

$$\varepsilon_{\theta\theta} = \frac{u}{r}$$

The radial and circumferential components of the bending strain are obtained directly from the geometry of the plate.

$$\varepsilon_{rr} = -z \, w_{rr}$$

$$\varepsilon_{\theta\theta} = -z \, w_r \quad .$$

Adding the membrane strains as derived before, the total radial and circumferential strain components are, respectively

$$\varepsilon_r = u_r + \frac{1}{2} w_r^2 - z \, w_{rr} \qquad (1a)$$

$$\varepsilon_\theta = \frac{u}{r} - \frac{z}{r} w_r \quad . \qquad (1b)$$

The stress-strain relations are derived from Hooke's Law:

$$\sigma_r = \frac{E}{1-\nu^2} (\varepsilon_r + \nu\varepsilon_o) \qquad (2a)$$

$$\sigma_\theta = \frac{E}{1-\nu^2} (\varepsilon_\theta + \nu\varepsilon_r) \qquad (2b)$$

where σ_r and σ_θ are the radial and circumferential stresses, respectively, E is Young's modulus, and ν is the Poisson's ratio of the material of the plate.

Expressions for the radial and circumferential forces per unit length, N_r and N_θ, are obtained by integrating the respective stresses across the thickness of the plate

$$N_r = \int_{-h/2}^{h/2} \sigma_r \, dz = \frac{12D}{h^2} [u_r + \frac{1}{2} w_r^2 + \nu \frac{u}{r}] \tag{3a}$$

$$N_\theta = \int_{-h/2}^{h/2} \sigma_\theta \, dz = \frac{12D}{h^2} [\frac{u}{r} + \nu u_r + \frac{\nu}{2} w_r^2] \tag{3b}$$

where $D = \dfrac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity of the plate.

Radial and circumferential moments per unit length, M_r and M_θ, respectively, are obtained by integrating across the thickness of the plate the moments of the forces about the middle plane.

$$M_r = \int_{-h/2}^{h/2} \sigma_r \, zdz = -D[w_{rr} + \frac{\nu}{r} w_r] \tag{4a}$$

$$M_\theta = \int_{-h/2}^{h/2} \sigma_\theta \, zdz = -D[\frac{1}{r} w_r + \nu w_{rr}] \tag{4b}$$

b. <u>The Energy Method</u>

The extended form of Hamilton's principle, [4], is used to derive the governing differential equations and the related boundary conditions. It states that between two instants of time, t_1 and t_2, the first variation of the Action Integral is equal to zero; i.e.:

$$\delta \int_{t_1}^{t_2} I dt = 0 \tag{5}$$

In the present case, $I = K - U_s - U_b + W$

where K = kinetic energy

U_s = strain energy due to stretching of the middle plane

U_b = strain energy due to bending

W = work done by the time-dependent external forces (in the case of forced vibration).

Rewriting (5),

$$\delta\int_{t_1}^{t_2} K dt - \delta\int_{t_1}^{t_2} U_s dt - \delta\int_{t_1}^{t_2} U_b dt + \delta\int_{t_1}^{t_2} W dt = 0 \ . \qquad (6)$$

The components of (6) are derived as follows:

1. $$K = \int_0^{2\pi}\int_0^a \frac{1}{2}\rho h\, w_t^2\, dr = r d\theta = \pi D\int_0^a \frac{\rho h}{D} r\, w_t^2\, dt$$

$$\delta\int_{t_1}^{t_2} K dt = \pi D\int_{t_1}^{t_2}\int_0^a \frac{2\rho h}{D} r\, w_t\, \delta w_t\, dr \ .$$

Integration by parts yields

$$\delta\int_{t_1}^{t_2} K dt = 2\pi D\int_{t_1}^{t_2}\int_0^a [-\frac{\rho h}{D} r\, w_{tt}\, \delta_w] dr dt + 2\pi D\int_0^a (\frac{\rho h}{D} r w_t\, \delta_w)\Big|_{t_1}^{t_2} dr \qquad (7)$$

where ρ is the mass per unit volume of the plate material.

The second integral in (7) vanishes since $\delta w = 0$ at t_1 and t_2.

2. $$U_s = \frac{Eh}{2(1-\nu^2)}\int_0^{2\pi}\int_0^a [\varepsilon_r^2 + \varepsilon_\theta^2 + 2\nu\varepsilon_r\varepsilon_\theta] dr\, r d\theta \qquad \text{(see [5])}$$

$$= \pi D\int_0^a \frac{12}{h^2}[re_1^2 - 2(1-\nu)re_2] dr \ .$$

where $e_1 = \varepsilon_r + \varepsilon_\theta$ = first strain invariant

$e_2 = \varepsilon_r\varepsilon_\theta$ = second strain invariant

and $$\delta\int_{t_1}^{t_2} U_s dt = \pi D\int_{t_1}^{t_2}\int_0^a \frac{12}{h^2}[2re_1\delta e_1 - 2(1-\nu)r\delta e_2]drdt$$

$$= 2\pi D\int_{t_1}^{t_2}\int_0^a \frac{12}{h^2}\Big\{[-(re_1w_r)_r\delta w - r(e_1)_r\delta u]$$

$$- (1-\nu)[-(uw_r)_r\delta w + \frac{1}{2}w_r^2\delta u]\Big\} drdt$$

$$+ 2\pi D\int_{t_1}^{t_2}\frac{12}{h^2}\Big\{[re_1w_r\delta w + re_1\delta u]$$

$$- (1-\nu)[uw_r\delta w + u\delta u]\Big\}\Big|_0^a dt \qquad (8)$$

3. $$U_b = \pi D\int_0^a [w_r^2 + \frac{1}{r^2}w_r^2 + \frac{2\nu}{r}w_rw_{rr}]rdr$$

$$= \pi D\int_0^a [r(w_{rr} + \frac{1}{r}w_r)^2 - 2(1-\nu)w_rw_{rr}]dr$$

$$\delta\int_{t_1}^{t_2} U_b dt = \pi D\int_{t_1}^{t_2}\int_0^a [2r(w_{rr} + \frac{1}{r}w_r)(\delta w_{rr} + \frac{1}{r}\delta w_r)$$

$$- 2(1-\nu)(w_r\delta w_{rr} + w_{rr}\delta w_r)]drdt$$

$$= 2\pi D\int_{t_1}^{t_2}\int_0^a [r\nabla^4 w]\delta w\, drdt$$

$$+ 2\pi D\int_{t_1}^{t_2}[-r(w_{rrr} + \frac{1}{r}w_{rr} - \frac{1}{r^2}w_r)\delta w$$

$$+ r(w_{rr} + \frac{\nu}{r}w_r)\delta w_r]\Big|_0^a dt \qquad (9)$$

where $\nabla^4 w = w_{rrrr} + \frac{2}{r} w_{rrr} - \frac{1}{r^2} w_{rr} + \frac{1}{r^3} w_r$

4. $W = \int_0^{2\pi} \int_0^a q(r,t)\ w dr\ r d\theta$

where $q(r,t)$ is the time-independent external loading intensity, assumed to be symmetrical with respect to the z-axis.

$$\delta\int_{t_1}^{t_2} W dt = 2\pi D\int_{t_1}^{t_2} \int_0^a [\frac{q(r,t)}{E}\ r\ \delta w] dr dt\ . \tag{10}$$

Using (7), (8), (9), and (10) in (6), we get:

$$\int_{t_1}^{t_2} \int_0^a \Big\{ [-\frac{\rho h}{D}\ r\ w_{tt} + \frac{12}{h^2}(re_1 w_r)_r - \frac{12}{h^2}(1-\nu)(uw_r)_r - r\ \nabla^4 w + \frac{q(r,t)}{D}]\delta w$$

$$+ \frac{12}{h^2}[r(e_1)_r + \frac{1}{2}(1-\nu)w_r^2]\delta u \Big\}\ dr dt$$

$$+ \int_{t_1}^{t_2} \Big\{ r(w_{rrr} + \frac{1}{r} w_{rr} - \frac{1}{r^2} w_r) - \frac{12}{h^2}[re_1 w_r - (1-\nu)uw_r] \Big\} \delta w \Big|_0^a\ dt$$

$$- \int_{t_1}^{t_2} r[w_{rr} + \frac{\nu}{r} w_r]\delta w_r \Big|_0^a\ dt - \int_{t_1}^{t_2} [re_1 - (1-\nu)u]\delta u \Big|_0^a\ dt \quad = 0$$

$$\tag{11}$$

For (11) to hold, the integrands in the double and single integrals have to vanish separately.

The double integral yields the Euler-Lagrange equations:

$$\nabla^4 w - \frac{12}{h^2}\frac{1}{r}[(re_1 w_r)_r - (1-\nu)(uw_r)_r] + \frac{\rho h}{D} w_{tt} = \frac{q(r,t)}{D} \tag{12a}$$

and
$$r(e_1)_r + \frac{1}{2}(1-\nu)w_r^2 = 0 \tag{12b}$$

The single integrals yield the natural boundary conditions

$$\frac{\partial}{\partial r}(\nabla^2 w) - \frac{12}{h^2}[e_1 - (1-\nu)\frac{u}{r}]w_r\Big|_0^a = 0 \ , \tag{13a}$$

where ∇^2 is the Laplacian harmonic operator,

$$w_{rr} + \frac{\nu}{r}w_r\Big|_0^a = 0 \ , \tag{13b}$$

and
$$e_1 - (1-\nu)\frac{u}{r}\Big|_0^a = 0 \ . \tag{13c}$$

In equations (12) the following substitutions are made:

$$\psi = rN_r \ , \quad \frac{\partial\psi}{\partial r} = N_\theta \ ,$$

where ψ is a stress function satisfying the equilibrium equation of the plate,

$$\left.\begin{aligned} N_r &= \frac{12D}{h^2}[e_1 - (1-\nu)\frac{u}{r}] \ , \\ \text{and} \quad N_\theta &= \frac{12D}{h^2}(\nu e_1 + (1-\nu)\frac{u}{r}] \end{aligned}\right\} \text{(from (3))}$$

Using these relations, equations (12) and (13) can be rewritten as follows:

$$D\nabla^4 w - \frac{1}{r}(\psi w_r)_r + \rho h\, w_{tt} = q(r,t) \tag{14a}$$

$$\psi_{rr} + \frac{1}{r}\psi_r - \frac{1}{r^2}\psi_{rr} = -\frac{Eh}{2r}w_r^2 \tag{14b}$$

$$D\frac{\partial}{\partial r}(\nabla^2 w) - \frac{1}{r}\psi w_r\Big|_0^a = 0 \tag{15a}$$

$$w_{rr} + \frac{\nu}{r}w_r\Big|_0^a = 0 \tag{15b}$$

$$\frac{\psi}{r}\Big|_0^a = 0 \tag{15c}$$

Equations (3) may also be used to derive an expression for the radial displacement, u, needed later in the definition of the boundary conditions

$$u = \frac{r}{Eh}\left(\frac{\partial\psi}{\partial r} - \nu\,\frac{\psi}{r}\right) . \tag{15d}$$

Using the substitutions

$$x = \chi/a$$

$$\psi = \frac{Eha}{1-\nu^2}\,\phi$$

$$q = \frac{E}{12(1-\nu^2)}\left(\frac{h}{a}\right)^3 P$$

$$r = a\xi$$

and

$$t = \left(\frac{\rho h a^4}{D}\right)^{1/2}\tau ,$$

equations (14) and (15) are converted into the non-dimensional forms:

$$\nabla^4\chi + \frac{\partial^2\chi}{\partial\tau^2} - 12\left(\frac{a}{h}\right)^2\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\phi\,\frac{\partial\chi}{\partial\xi}\right) = P(\xi,\tau) \tag{16a}$$

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\phi}{\partial\xi} - \frac{1}{\xi^2}\,\phi = -\,\frac{1-\nu^2}{2\xi}\left(\frac{\partial\chi}{\partial\xi}\right)^2 \tag{16b}$$

$$\frac{D}{a^2}\frac{\partial}{\partial\xi}(\nabla^2\chi) - \frac{Eh}{1-\nu^2}\,\phi\,\frac{\partial\chi}{\partial\xi}\Big|_0^1 = 0 \tag{17a}$$

$$\frac{\partial^2\chi}{\partial\xi^2} + \frac{\nu}{\xi}\frac{\partial\chi}{\partial\xi}\Big|_0^1 = 0 \tag{17b}$$

$$\frac{\phi}{\xi}\Big|_0^1 = 0 \tag{17c}$$

$$\frac{a\xi}{1-\nu^2}\left(\frac{\partial\phi}{\partial\xi} - \nu\,\frac{\phi}{\xi}\right)\Big|_0^1 = 0 \tag{17d}$$

c. Boundary Conditions

Depending on the type of support of the edge of the plate, the geometric boundary conditions are supplemented by the natural boundary conditions in (17a), (17b), and (17c) to complete the formation of the problem.

An edge is called immovable if it is held rigidly so as to prevent radial displacement; if radial displacement is allowed the edge is called movable.

The relevant boundary conditions for each type of support at $\xi = R$ (R being either 0 or 1 in the case of a solid circular plate) are displayed in Table (1).

Table 1. General Boundary Conditions

Type of Edge	Boundary Conditions at $\xi = R$	
Clamped-Immovable	$\chi = 0$	$\frac{\partial\phi}{\partial\xi} - \nu \frac{\phi}{\xi} = 0$
	$\frac{\partial\chi}{\partial\xi} = 0$	
Clamped-Movable	$\chi = 0$	$\frac{\phi}{\xi} = 0$
	$\frac{\partial\chi}{\partial\xi} = 0$	
Hinged-Immovable	$\chi = 0$	$\frac{\partial\phi}{\partial\xi} - \nu \frac{\phi}{\xi} = 0$
	$\frac{\partial^2\chi}{\partial\xi^2} + \frac{\nu}{\xi}\frac{\partial\chi}{\partial\xi} = 0$	
Hinged-Movable	$\chi = 0$	$\frac{\phi}{\xi} = 0$
	$\frac{\partial^2\chi}{\partial\xi^2} + \frac{\nu}{\xi}\frac{\partial\chi}{\partial\xi} = 0$	
Free	$\frac{\partial^2\chi}{\partial\xi^2} + \frac{\nu}{\xi}\frac{\partial\chi}{\partial\xi} = 0$	$\frac{\phi}{\xi} = 0$
	$\frac{\partial}{\partial\xi}(\nabla^2\chi) = 0$	

CHAPTER II. APPROXIMATE ANALYSIS

The differential equations of motion together with the associated boundary conditions constitute a boundary-value problem. The boundary-value problem becomes an eigenvalue problem when the differential equations of motion and the boundary conditions are homogeneous and depend on a parameter λ, and, moreover, a non-trivial solution is obtained only for certain values of the parameter λ. The transition from the boundary-value problem to the eigenvalue problem is effected by means of the separation of variables method [4]. In the case of large amplitude vibration problems, where exact solutions are unknown, function space methods are usually used to eliminate the space coordinate with an assumed mode shape function, thus reducing the problem to a nonlinear ordinary differential equation with time, t, as the independent variable. In this work, a time function is assumed, then a Kantorovich averaging method is used to reduce the nonlinear partial differential equations to a set of nonlinear ordinary differential equations.

The Kantorovich Averaging Method

In contrast with the "assumed-space-mode method," the Kantorovich averaging method is used [2] to find an assumed-time-mode solution to equations (16) that satisfies the boundary conditions of the types given in Table (1). The analysis in [2] is closely followed here.

A sinusoidal form is proposed for the loading intensity.

$$P(\xi,\tau) = Q(\xi)\sin\omega\tau$$

and the steady state response of the plate is assumed to have the forms

$$\chi(\xi,\tau) = Ag(\xi)\sin\omega\tau \tag{18a}$$

$$\phi(\xi,\tau) = A^2 f(\xi)\sin^2\omega\tau \tag{18b}$$

where A is an amplitude parameter and $g(\xi)$ and $f(\xi)$ are shape functions to be determined.

Equation (18) cannot satisfy (16) for all τ, but the "average" work done over one period of oscillation, $\frac{2\pi}{\omega}$, is minimized in order to eliminate the time variable.

Using this principle and substituting (18) in (16b) converts the differential equations (16) into the form:

$$\nabla^4 g - \lambda g - 9\,\frac{\alpha}{\xi}\,\frac{d}{d\xi}\left(f\,\frac{dg}{d\xi}\right) = \frac{Q^*}{\sqrt{\alpha}} \qquad (19a)$$

$$\frac{d^2 f}{d\xi^2} + \frac{1}{\xi}\,\frac{df}{d\xi} - \frac{f}{\xi^2} = -\,\frac{1-\nu^2}{2\xi}\left(\frac{dg}{d\xi}\right)^2 \qquad (19b)$$

where $\alpha = (A\,\frac{\alpha}{h})^2$, $Q^* = (\frac{a}{h})Q$, and $\lambda = \omega^2$.

Substituting (18) in Table (1) gives the boundary conditions in the final form shown in Table (2) for the different edge conditions.

Equations (19) are nonlinear and coupled. Together with a set of boundary conditions chosen from Table (2) they comprise a nonlinear two-point boundary value problem which is solved through the solution of the related initial-value problem.

Table 2. Final Form of the General Boundary Condition

Type of Edge	Boundary Conditions at $\xi = R$	
Clamped Immovable	$g = 0$	$\frac{df}{d\xi} - \nu \frac{f}{\xi} = 0$
	$\frac{dg}{d\xi} = 0$	
Clamped Movable	$g = 0$	$\frac{f}{\xi} = 0$
	$\frac{dg}{d\xi} = 0$	
Hinged Immovable	$g = 0$	$\frac{df}{d\xi} - \nu \frac{f}{\xi} = 0$
	$\frac{d^2g}{d\xi^2} + \frac{\nu}{\xi}\frac{dg}{d\xi} = 0$	
Hinged Movable	$g = 0$	$\frac{f}{\xi} = 0$
	$\frac{d^2g}{d\xi^2} + \frac{\nu}{\xi}\frac{dg}{d\xi} = 0$	
Free	$\frac{d^2g}{d\xi^2} + \frac{\nu}{\xi}\frac{dg}{d\xi} = 0$	$\frac{f}{\xi} = 0$
	$\frac{d}{d\xi}(\nabla^2 g) = 0$	

CHAPTER III. NUMERICAL ANALYSIS

Problem of the Initial-Value Method

Solutions to initial-value problems are well developed theoretically [7] and well adapted for solution on high-speed computers [8,9]. Nonlinear boundary-value problems and nonlinear eigenvalue problems, however, are more complicated. Hence solution of these problems by converting them into initial-value problems has become popular [10,11].

In [2], the resulting nonlinear initial-value problem for the case of free vibration is solved by the shooting method. To do this an associated variational problem is developed and used in a Newton-Raphson iteration scheme.

By analytical continuation the solution to the original boundary-value problem is obtained as a one-parameter family of solutions of the initial value problem.

Again by analytical continuation and a perturbation technique, the solution to the boundary value problem for the case of forced vibration is obtained through discrete incrementation of the loading parameter.

a. Matrix Formulation

In order to solve the problem numerically, equations (19) are written as a system of six first-order differential equations

$$\frac{d\bar{Y}}{d\xi} = \bar{H}(\xi,\bar{Y};\alpha,\lambda,Q^*) \quad , \quad 0 < \xi < 1 \tag{20a}$$

where

$$\bar{Y}(\xi) = \begin{Bmatrix} g \\ g' \\ g'' \\ g''' \\ f \\ f' \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} \quad , \quad (\ )' = \frac{d}{d\xi} \quad , \quad \text{and:}$$

$$\bar{H} = \begin{Bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \\ y_6' \end{Bmatrix} = \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ -\frac{2}{\xi} y_4 + \frac{1}{\xi^2} y_3 - \frac{1}{\xi^3} y_2 + \lambda y_1 + \frac{9\alpha}{\xi} [y_3 y_5 + y_2 y_6] + \frac{Q^*(\xi)}{\sqrt{\alpha}} \\ y_6 \\ -\frac{1}{\xi} y_6 + \frac{1}{\xi^2} y_5 - \frac{1-\nu^2}{2\xi} (y_2)^2 \end{Bmatrix}$$

The parameters α and λ are additional unknowns and hence to solve six equations in eight unknowns, an additional restraint is imposed. One component of Y(0) is normalized, producing a unique solution in terms of α or λ.

Normalizing the first component of Y(0), a set of boundary conditions chosen from Table (2) may be written in the generalized form:

$$M\bar{Y}(0) = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (20b)$$

and

$$N\bar{Y}(1) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (20c)$$

where M and N are 4x6 and 3x6 coefficient matrices as shown in Table (3) for the three cases of edge conditions discussed in this work, in addition to the Clamped Immovable case treated in [2].

b. <u>The Initial-Value Problem</u>

The corresponding initial-value problem may be expressed as

$$\frac{d\bar{Z}}{d\xi} = \bar{H}(\xi, \bar{Z}; \alpha, \lambda, Q^*) \qquad (21a)$$

Table 3. Coefficient Matrices (M) and (N) of Boundary Conditions

Type of edge support	(M)	(N)
1. Clamped-Immovable	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu & 1 \end{bmatrix}$
2. Clamped-Movable	ditto	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$
3. Hinged-Immovable	ditto	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu & 1 \end{bmatrix}$
4. Hinged-Movable	ditto	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$$\bar{Z}(0) = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{Bmatrix} = \bar{\gamma} \tag{21a}$$

where z_i are identical with y_i in equations (20), $i = 1,\ldots,6$.

Since only four boundary conditions are known from $M\bar{Z}(0) = 0$, we can write

$$\bar{\gamma} = \bar{\gamma}^*(\eta_1, \eta_2) \ ,$$

where $\eta_1 = z_3$ and $\eta_2 = z_6$ are the missing boundary conditions at $\xi = 0$.

For a continuous function $Q^*(\xi)$ the existence and uniqueness of a solution $\bar{Z} = \bar{Z}(\xi;\bar{\eta},\alpha)$; $\bar{\eta} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \lambda \end{Bmatrix}$, on a closed interval [0,1] has been proved in [2].

Now, from (20c),

$$N\bar{Z}(1;\bar{\eta},\alpha) = \bar{0} \tag{22}$$

Assuming $\bar{Z}$ is continuously differentiable with respect to $\bar{\eta}$ and α then, by a well-known theorem in Matrix Theory, for the system of equations:

$$N\bar{Z}(1,\bar{\eta},\alpha) = \bar{0}$$

a necessary and sufficient condition for a unique solution, $\bar{\eta} = \bar{\eta}(\alpha)$, is that the determinant of the Jacobian matrix, $J = \frac{\partial}{\partial\bar{\eta}}[N\bar{Z}(1;\bar{\eta},\alpha)]$, is not equal to zero, i.e., $\det[N \frac{\partial}{\partial\bar{\eta}} \bar{Z}(1;\bar{\eta},\alpha)] \neq 0$.

Hence a locally unique function exists at $\xi=1$ such that:

$$N\bar{Z}(1;\eta(\alpha),\alpha) \equiv \bar{0}$$

So,

$$\bar{Y}(\xi;\alpha) = \bar{Z}(\xi;\bar{\eta}(\alpha),\alpha)$$

forms a one-parameter family of solutions to the boundary value problem (19), each of which is a solution to the initial-value problem (21).

For a fixed value of α, say α^o, the system (22) reduces to the three transcendental equations:

$$NZ(1,\bar{\eta},\alpha^o) = \bar{0} \tag{23}$$

A root $\bar{\eta}^o$ may be found by Newton's iteration method. Starting with the initial guess $\bar{\eta} = \bar{\eta}_1$ the sequence:

$$\bar{\eta}_{k+1} = \bar{\eta}_k + \Delta\bar{\eta}_k \quad , \quad k = 1,2,3,\ldots$$

is generated.

A Taylor series expansion gives the linearized correction $\Delta\bar{\eta}_k$:

$$N\bar{Z}(1;\bar{\eta}_k + \Delta\bar{\eta}_k,\alpha^o) = N\bar{Z}(1;\bar{\eta}_k,\alpha^o) + [N \frac{\partial}{\partial\bar{\eta}_k} \bar{Z}(1;\bar{\eta}_k,\alpha^o)] \cdot \Delta\bar{\eta}_k$$

$$+ O(|\Delta\bar{\eta}_k|^2) = \bar{0}$$

where $|\ |$ is the Euclidean vector norm. Retaining only the term linear in $\Delta\bar{\eta}_k$ gives:

$$\Delta\bar{\eta}_k = [N \frac{\partial}{\partial\bar{\eta}_k} Z(1;\bar{\eta}_k,\alpha^o)]^{-1} N\bar{Z}(1;\bar{\eta}_k,\alpha^o)$$

$$= [N(J_1)_k]^{-1} N\bar{Z}(1;\bar{\eta}_k,\alpha^o) \tag{24}$$

where, at the k^{th} step, the 6x3 matrix J_1 is defined as

$$J_1 = \left(\frac{\partial\bar{Z}}{\partial\bar{\eta}}\right)_{\xi=1} = \left(\frac{\partial z_i}{\partial \eta_j}\right)_{\xi=1} \quad , \quad \begin{array}{l} i = 1,\ldots,6 \\ j = 1,2,3 \end{array} \tag{25}$$

and represents the change of final values with respect to a change of $\bar{\eta}$. The expression $N\bar{Z}(1;\bar{\eta}_k,\alpha^o)$ represents the k^{th} error vector.

If $\bar{\eta}_1$ were within a sufficiently small neighborhood of the root $\bar{\eta}_o$, Newton's iteration would converge to $\bar{\eta}_o$.

c. The Associated Variational Problem

To generate the sequence $\Delta\bar{\eta}_k$ the matrix $(J_1)_k$ has to be evaluated at each step, k, of the iteration process. To do that an associated variational problem is introduced:

Formally differentiating (21) with respect to $\bar{\eta}$ produces:

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\bar{Z}}{\partial\bar{\eta}}\right) = \frac{\partial\bar{Z}}{\partial\bar{\eta}} + \left(\frac{\partial\bar{H}}{\partial\bar{Z}}\right)\left(\frac{\partial\bar{Z}}{\partial\bar{\eta}}\right) \tag{26a}$$

$$\left(\frac{\partial\bar{Z}}{\partial\bar{\eta}}\right)_{\xi=0} = \frac{\partial\gamma^*}{\partial\bar{\eta}} \tag{26b}$$

Writing (26a) explicitly, with $(\ )' \equiv \frac{d}{d\xi}$, we get:

$$\left(\frac{\partial z_1}{\partial\eta_1}\right)' = \frac{\partial z_2}{\partial\eta_1}$$

$$\left(\frac{\partial z_2}{\partial\eta_1}\right)' = \frac{\partial z_3}{\partial\eta_1}$$

$$\left(\frac{\partial z_3}{\partial\eta_1}\right)' = \frac{\partial z_4}{\partial\eta_1}$$

$$\left(\frac{\partial z_4}{\partial\eta_1}\right)' = -\frac{2}{3}\frac{\partial z_4}{\partial\eta_1} + \frac{1}{\xi^2}\frac{\partial z_3}{\partial\eta_1} - \frac{1}{\xi^3}\frac{\partial z_2}{\partial\eta_1} + \lambda\frac{\partial z_1}{\partial\eta_1} + \frac{9\alpha}{\xi}\left[z_3\frac{\partial z_5}{\partial\eta_1} + z_5\frac{\partial z_3}{\partial\eta_1} + z_2\frac{\partial z_6}{\partial\eta_1} + z_6\frac{\partial z_2}{\partial\eta_1}\right]$$

$$\left(\frac{\partial z_5}{\partial\eta_1}\right)' = \frac{\partial z_6}{\partial\eta_1}$$

$$\left(\frac{\partial z_5}{\partial \eta_1}\right)' = \frac{\partial z_6}{\partial \eta_1}$$

$$\left(\frac{\partial z_6}{\partial \eta_1}\right)' = -\frac{1}{\xi}\frac{\partial z_6}{\partial \eta_1} + \frac{1}{\xi^2}\frac{\partial z_5}{\partial \eta_1} - \frac{1-\nu^2}{\xi} z_2 \frac{\partial z_2}{\partial \eta_1}$$

$$\left(\frac{\partial z_1}{\partial \eta_2}\right)' = \frac{\partial z_2}{\partial \eta_2}$$

$$\left(\frac{\partial z_2}{\partial \eta_2}\right)' = \frac{\partial z_3}{\partial \eta_2}$$

$$\left(\frac{\partial z_3}{\partial \eta_2}\right)' = \frac{\partial z_4}{\partial \eta_2}$$

$$\left(\frac{\partial z_4}{\partial \eta_2}\right)' = -\frac{2}{\xi}\frac{\partial z_4}{\partial \eta_2} + \frac{1}{\xi^2}\frac{\partial z_2}{\partial \eta_2} - \frac{1}{\xi^3}\frac{\partial z_2}{\partial \eta_2} + \lambda\frac{\partial z_1}{\partial \eta_2} + \frac{9\alpha}{\xi}[z_3\frac{\partial z_5}{\partial \eta_2}$$

$$+ z_5\frac{\partial z_3}{\partial \eta_2} + z_2\frac{\partial z_6}{\partial \eta_2} + z_6\frac{\partial z_2}{\partial \eta_2}]$$

$$\left(\frac{\partial z_5}{\partial \eta_2}\right)' = \frac{\partial z_6}{\partial \eta_2}$$

$$\left(\frac{\partial z_6}{\partial \eta_2}\right)' = -\frac{1}{\xi}\frac{\partial z_6}{\partial \eta_2} + \frac{1}{\xi^2}\frac{\partial z_5}{\partial \eta_2} - \frac{1-\nu^2}{\xi} z_2 \frac{\partial z_2}{\partial \eta_2}$$

$$\left(\frac{\partial z_1}{\partial \lambda}\right)' = \frac{\partial z_2}{\partial \lambda}$$

$$\left(\frac{\partial z_2}{\partial \lambda}\right)' = \frac{\partial z_3}{\partial \lambda}$$

$$\left(\frac{\partial z_3}{\partial \lambda}\right)' = \frac{\partial z_4}{\partial \lambda}$$

$$\left(\frac{\partial z_4}{\partial \lambda}\right)' = -\frac{2}{\xi}\frac{\partial z_4}{\partial \lambda} + \frac{1}{\xi^2}\frac{\partial z_2}{\partial \lambda} - \frac{1}{\xi^3}\frac{\partial z_2}{\partial \lambda} + \lambda \frac{\partial z_1}{\partial \lambda} + \frac{9\alpha}{\xi}\left[z_3 \frac{\partial z_5}{\partial \lambda} + z_5 \frac{\partial z_3}{\partial \lambda} + z_2 \frac{\partial z_6}{\partial \lambda} + z_6 \frac{\partial z_2}{\partial \lambda}\right] + z_1$$

$$\left(\frac{\partial z_5}{\partial \lambda}\right)' = \frac{\partial z_6}{\partial \lambda}$$

$$\left(\frac{\partial z_6}{\partial \lambda}\right)' = -\frac{1}{\xi}\frac{\partial z_6}{\partial \lambda} + \frac{1}{\xi^2}\frac{\partial z_5}{\partial \lambda} - \frac{1-\nu^2}{\xi} z_2 \frac{\partial z_2}{\partial \lambda}$$

d. <u>The Removable Singularity at $\xi=0$</u>

It is seen that the systems in (21a) and (26a) have a singular point at $\xi=0$. Since the displacement functions z_1 and z_5 have finite derivatives and are continuous on the closed interval [0,1], they may be expanded in Taylor series about the point $\xi=0$ [12].

$$z_1 = (z_1)_o + (z_2)_o \xi + \frac{1}{2!}(z_3)_o \xi^2 + \frac{1}{3!}(z_4)_o \xi^3 + \ldots$$

$$z_5 = (z_5)_o + (z_6)_o \xi + \ldots$$

Here, $(\ )_o$ indicates that the variable inside the brackets is evaluated at $\xi=0$.

Substituting these values in (21a) and requiring by continuity that the derivatives be bounded in the limit as $\xi \to 0$ leads to the set of modified equations for the initial-value problem at $\xi=0$

$$\left.\begin{aligned}
(z_1')_o &= 0 \\
(z_2')_o &= (z_3)_o = \eta_1 \\
(z_3')_o &= 0 \\
(z_4')_o &= \frac{3}{8}\lambda + \frac{27}{4}\alpha\,\eta_1\,\eta_2 \\
(z_5')_o &= (z_6)_o = \eta_2 \\
(z_6')_o &= 0
\end{aligned}\right\} \qquad (27a)$$

Similarly, equations (26a) render the following set for the associated variational problem at ξ=0.

$$\left.\begin{aligned}
\left(\frac{\partial z_1}{\partial \lambda_1}\right)'_0 &= 0 \\
\left(\frac{\partial z_2}{\partial \eta_1}\right)'_0 &= 1 \\
\left(\frac{\partial z_3}{\partial \eta_1}\right)'_0 &= 0 \\
\left(\frac{\partial z_4}{\partial \eta_1}\right)'_0 &= \frac{27}{4}\,\alpha\,\eta_2 \\
\left(\frac{\partial z_5}{\partial \eta_1}\right)'_0 &= 0 \\
\left(\frac{\partial z_6}{\partial \eta_1}\right)'_0 &= 0 \\
\left(\frac{\partial z_1}{\partial \eta_2}\right)'_0 &= 0 \\
\left(\frac{\partial z_2}{\partial \eta_2}\right)'_0 &= 0 \\
\left(\frac{\partial z_3}{\partial \eta_2}\right)'_0 &= 0 \\
\left(\frac{\partial z_4}{\partial \eta_2}\right)'_0 &= \frac{27}{4}\,\alpha\,\eta_1 \\
\left(\frac{\partial z_5}{\partial \eta_2}\right)'_0 &= 1 \\
\left(\frac{\partial z_6}{\partial \eta_2}\right)'_0 &= 0
\end{aligned}\right\} \qquad (27b)$$

$$\left(\frac{\partial z_1}{\partial \lambda}\right)_0' = 0$$

$$\left(\frac{\partial z_2}{\partial \lambda}\right)_0' = 0$$

$$\left(\frac{\partial z_3}{\partial \lambda}\right)_0' = 0$$

$$\left(\frac{\partial z_4}{\partial \lambda}\right)_0' = \frac{3}{8}$$

$$\left(\frac{\partial z_5}{\partial \lambda}\right)_0' = 0$$

$$\left(\frac{\partial z_6}{\partial \lambda}\right)_0' = 0$$

Starting with a guessed value for the vector $\bar{\eta}$ and $\alpha = \alpha^o$, (21a) and (26a) are numerically integrated simultaneously on the interval [0,1]. The results at ξ=1 give the Jacobian $(J_1)_1$ and consequently $\Delta\bar{\eta}_1$. Repetition of the process with the corrected values of the guessed vector establishes the sequence $\bar{\eta}_k$ that converges to $\bar{\eta}^o$ within a specified error bound usually related to the accuracy of the integration method used.

Having obtained $\bar{\eta}^o$ corresponding to α^o, the value of the amplitude parameter α^o is perturbed so that

$$\alpha^1 = \alpha^o + \Delta\alpha^o$$

If $\Delta\alpha^o$ is small enough for α^o to be within the new contraction domain of Newton's method, iteration will converge to the root $\bar{\eta}^1$ corresponding to α^1. This leads to the creation of the two sequences $\{\bar{\eta}^i\}$ and $\{\alpha^i\}$ that form a discrete family of α-dependent solutions of the original boundary-value problem for the case of free vibration.

The range of α is limited by the assumptions made in the finite-amplitude theory and by physical considerations. At an assumed maximum value, $\alpha = \alpha^m$, for which $\bar{\eta} = \bar{\eta}^m$, a small load Q* is introduced. By the nature of the harmonic response of the plate, the steady state response due to free vibration will be within the contraction domain of that due to forced vibration, if the load, Q*, is small enough. Keeping Q* fixed, α is decremented from α^m and a new discrete set of solutions is obtained for the boundary-value problem of forced vibration.

Reversing the sign of Q* produces an out-of-phase response as shown by the response curves.

Using the principle of analytic continuation again, series of response curves are obtained by repeated incrementation of the loading parameter Q*.

CHAPTER IV. NUMERICAL COMPUTATIONS

A fourth-order Runge-Kutta-Gill method is used to integrate the initial-value problem and the associated variational problem over the interval [0,1]. For a stepsize of $\frac{1}{40}$ the error norm:

$$\max|NZ(1)| \leq 0.1 \times 10^{-5} ,$$

where $\max|\ \ |$ is the maximum-element norm of the error vector, is in agreement with the order of the integration method.

In order to start the integration process a reasonable guess for $\bar{\eta} = [\eta_1,\eta_2,\lambda]^T$ has to be made.

The linear case is the natural starting point and hence, setting $Q^* = 0$ and $\alpha^o = 0$, equation (19b) reduces to the linear form

$$\nabla^4 g - \lambda g = 0 \tag{28}$$

For the principal mode of vibration, (28) has the general solution [14,15],

$$g(\xi) = AJ_o(k\xi) + BY_o(k\xi) + CI_o(k\xi) + DK_o(k\xi) \tag{29}$$

where $k^4 = \lambda = \omega^2$, J_o and Y_o are Bessel functions of the first and second kinds, respectively, I_o and K_o are the modified Bessel functions of the first and second kind, respectively, and A, B, C, and D are constants depending on the boundary conditions of the plate.

For boundedness of deflections and stresses at the center of the plate, since $Y_o(0)$ and $K_o(0)$ have infinite values, B and D must be equal to zero, and

$$g(\xi) = AJ_o(k\xi) + CI_o(k\xi) \tag{30}$$

Moreover, for a supported edge (both clamped and hinged) the boundary condition $g(1) = 0$ is always valid. Using this condition, a normalized form of equation (30) is obtained, compatible with the normalization condition introduced in the numerical analysis.

$$g_n(\xi) = \frac{g(\xi)}{g(0)} = \frac{J_o(k)I_o(k\xi) - I_o(k)J_o(k\xi)}{J_o(k) - I_o(k)} \tag{31}$$

Differentiating (31) twices gives:

$$g_n''(\xi) = \frac{kJ_o(k)[\frac{1}{\xi} I_1(k\xi) + kI_2(k\xi)] + kI_o(k)[\frac{1}{\xi} J_1(k\xi) - kJ_2(k\xi)]}{J_o(k) - I_o(k)} \tag{32}$$

Using the fact that:

$$\lim_{\xi\to 0} [\frac{I_1(k\xi)}{k\xi}] = \lim_{\xi\to 0} [\frac{J_1(k\xi)}{k\xi}] = \frac{1}{2} ,$$

and if the value of k is known, a good estimate for η_1 is obtained:

$$\eta_1 = g_n''(0) = \lim_{\xi\to 0} [g_n''(\xi)] . \tag{33}$$

To determine the value of k, two edge conditions have to be considered:

a. Clamped Edge

If the boundary conditions

$$g(0) = 0,$$

$$\text{and} \qquad g'(0) = 0$$

are used, equation (30) reduces to the transcendental equation:

$$J_o(k)I_1(k) + J_o(k)J_1(k) = 0 . \tag{34}$$

From Table (2.1) of Ref. [15], $k^4 = (10.2158)^2 = \lambda$. Using this value in (32) and (33) we get $\eta_1 = -4.57$ (approximately).

b. Simple-supported (hinged) edge

The boundary conditions

$$g(1) = 0$$

and $$g''(1) + \nu\, g'(1) = 0 \; .$$

together with equation (30), produce the transcendental equation:

$$\frac{J_1(k)}{J_o(k)} + \frac{I_1(k)}{I_o(k)} = \frac{2k}{1-\nu} \tag{34}$$

For $\nu = 0.3$, Table (2.3) of Ref. [15] gives $k^4 = (4.977)^2$, and, from (32) and (33):

$$\eta_1 = -2.67 \text{ (approximately)} \; .$$

Having solved for $g(\xi)$, equation (19a) may be solved for $f(\xi)$ and hence $f'(0)$ is obtained as an estimate of η_2.

A simpler alternative, however, is to use the stipulations of the small displacement theory and neglect the membrane effect represented by $f(\xi)$ and $f'(\xi)$. Hence:

$$\eta_2 = f'(0) = 0 \; .$$

The form of the correction vector $\Delta\bar{\eta}$, equation (24), depends on the boundary conditions at the edge of the plate. Four cases are considered, the first of which is reproduced from Ref. [2] for comparison.

1. Clamped-Immovable Edge

Using the corresponding matrix (N) from Table (3) into equation (24) gives the correction vector:

$$\Delta\bar{\eta} = \begin{Bmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\lambda \end{Bmatrix} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial z_1}{\partial \eta_1} & \frac{\partial z_1}{\partial \eta_2} & \frac{\partial z_1}{\partial \lambda} \\ \frac{\partial z_2}{\partial \eta_1} & \frac{\partial z_2}{\partial \eta_2} & \frac{\partial z_2}{\partial \lambda} \\ \frac{\partial z_3}{\partial \eta_1} & \frac{\partial z_3}{\partial \eta_2} & \frac{\partial z_3}{\partial \lambda} \\ \frac{\partial z_4}{\partial \eta_1} & \frac{\partial z_4}{\partial \eta_2} & \frac{\partial z_4}{\partial \lambda} \\ \frac{\partial z_5}{\partial \eta_1} & \frac{\partial z_5}{\partial \eta_2} & \frac{\partial z_5}{\partial \lambda} \\ \frac{\partial z_6}{\partial \eta_1} & \frac{\partial z_6}{\partial \eta_2} & \frac{\partial z_6}{\partial \lambda} \end{bmatrix}_{\xi=1} \right\}^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu & 1 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{Bmatrix}_{\xi=1}$$

i.e.,

$$\begin{Bmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\lambda \end{Bmatrix} = \begin{bmatrix} \frac{\partial z_1}{\partial \eta_1} & \frac{\partial_1}{\partial \eta_2} & \frac{\partial z_1}{\partial \lambda} \\ \frac{\partial z_2}{\partial \eta_1} & \frac{\partial z_2}{\partial \eta_2} & \frac{\partial z_2}{\partial \lambda} \\ -\nu \frac{\partial z_5}{\partial \eta_1} + \frac{\partial z_6}{\partial \eta_1} & -\nu \frac{\partial z_5}{\partial \eta_2} \quad \frac{\partial z_6}{\partial \eta_2} & -\nu \frac{\partial z_5}{\partial \lambda} + \frac{\partial z_6}{\partial \lambda} \end{bmatrix}_{\xi=1}^{-1} \begin{Bmatrix} z_1 \\ z_2 \\ -\nu z_5 + z_6 \end{Bmatrix}_{\xi=1} \tag{35}$$

2. Clamped Movable Edge

The correction vector takes the form:

$$\begin{Bmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\lambda \end{Bmatrix} = \begin{bmatrix} \frac{\partial z_1}{\partial \eta_1} & \frac{\partial z_1}{\partial \eta_2} & \frac{\partial z_1}{\partial \lambda} \\ \frac{\partial z_2}{\partial \eta_1} & \frac{\partial z_2}{\partial \eta_2} & \frac{\partial z_2}{\partial \lambda} \\ \frac{\partial z_5}{\partial \eta_1} & \frac{\partial z_5}{\partial \eta_2} & \frac{\partial z_5}{\partial \lambda} \end{bmatrix}^{-1}_{\xi=1} \begin{Bmatrix} z_1 \\ z_2 \\ z_5 \end{Bmatrix}_{\xi=1} \tag{36}$$

3. Hinged Immovable Edge

$$\begin{Bmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\lambda \end{Bmatrix} = \begin{bmatrix} \frac{\partial z_1}{\partial \eta_1} & \frac{\partial z_1}{\partial \eta_2} & \frac{\partial z_1}{\partial \lambda} \\ \nu \frac{\partial z_2}{\partial \eta_1} + \frac{\partial z_3}{\partial \eta_1} & \nu \frac{\partial z_2}{\partial \eta_2} + \frac{\partial z_3}{\partial \eta_2} & \nu \frac{\partial z_2}{\partial \lambda} + \frac{\partial z_3}{\partial \lambda} \\ -\nu \frac{\partial z_5}{\partial \eta_1} + \frac{\partial z_6}{\partial \eta_1} & -\nu \frac{\partial z_5}{\partial \eta_2} + \frac{\partial z_6}{\partial \eta_2} & -\nu \frac{\partial z_5}{\partial \lambda} + \frac{\partial z_6}{\partial \lambda} \end{bmatrix}^{-1}_{\xi=1} \begin{Bmatrix} z_1 \\ \nu z_2 + z_3 \\ -\nu z_5 + z_6 \end{Bmatrix}_{\xi=1} \tag{37}$$

4. Hinged Movable Edge

$$\begin{Bmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\lambda \end{Bmatrix} = \begin{bmatrix} \frac{\partial z_1}{\partial \eta_1} & \frac{\partial z_1}{\partial \eta_2} & \frac{\partial z_1}{\partial \lambda} \\ \nu \frac{\partial z_2}{\partial \eta_1} + \frac{\partial z_3}{\partial \eta_1} & \nu \frac{\partial z_2}{\partial \eta_2} + \frac{\partial z_3}{\partial \eta_2} & \nu \frac{\partial z_2}{\partial \lambda} + \frac{\partial z_3}{\partial \lambda} \\ \frac{\partial z_5}{\partial \eta_1} & \frac{\partial z_5}{\partial \eta_2} & \frac{\partial z_5}{\partial \lambda} \end{bmatrix}^{-1}_{\xi=1} \begin{Bmatrix} z_1 \\ \nu z_2 + z_3 \\ z_5 \end{Bmatrix}_{\xi=1} \tag{38}$$

Perturbing the amplitude parameter, α, the process is re-started using the values of $\bar{\eta}$ obtained at the end of the first cycle. The stepsize used for α is 0.1 and 41 cycles are carried out, i.e., up to an amplitude of twice the thickness of the plate. Two or three iterations were needed with most values of α.

For the case of forced vibration described before, four values of Q* (the uniformly-distributed loading parameter), were used, namely $Q^* = \pm 5$ and $Q^* = \pm 10$, in a perturbation pattern similar to that used with the amplitude parameter.

Stresses

In the following expressions for the nondimensional stresses, derivable from equations (4) and (3), the meanings of the suffixes are as indicated below

σ_r	radial stress	σ^b	bending stress
σ_θ	circumferential stress	σ^m	membrane stress

$$\frac{\sigma_r^b a^2}{Eh^2} = \pm \frac{\sqrt{\alpha}}{2(1-\nu^2)} [g'' + \frac{\nu}{\xi} g'] \tag{39a}$$

$$\frac{\sigma_\theta^b a^2}{Eh^2} = \pm \frac{\sqrt{\alpha}}{2(1-\nu^2)} [\frac{1}{\xi} g' + \nu g''] \tag{39b}$$

$$\frac{\sigma_r^m a^2}{Eh^2} = \frac{\alpha}{1-\nu^2} [\frac{f}{\xi}] \tag{39c}$$

$$\frac{\sigma_\theta^m a^2}{Eh^2} = \frac{\alpha}{1-\nu^2} [f'] \tag{39d}$$

at the center of the plate the limiting values of (39) are obtained using L'Hôpital's rule:

$$\left(\frac{\sigma_r^b a^2}{Eh^2}\right)_{\xi=0} = \left(\frac{\sigma_\theta^b a^2}{Eh^2}\right)_{\xi=0} = \pm \frac{\sqrt{\alpha}}{2(1-\nu)} [g'']_{\xi=0} \qquad (40a)$$

$$\left(\frac{\sigma_r^m a^2}{Eh^2}\right)_{\xi=0} = \left(\frac{\sigma_\theta^m a^2}{Eh^2}\right)_{\xi=0} = \frac{\alpha}{1-\nu^2} [f']_{\xi=0} \qquad (40b)$$

For a value of $\nu = 0.3$ the characteristics of the plate vibration are displayed in the response, shape function, and stress patterns for each of the four cases of edge support.

For all the cases considered the plate response curves display the jump phenomena associated with the nonlinear vibration of a hard spring [13].

It is also evident from the curves that the absence of radial restraint at the edge of the plate causes a fundamental change in the response of the plate, and the pattern and nature of membrane stresses for both the clamped and hinged cases. The effect of radial restraint on bending stresses is, however, negligible.

The maximum change in the shape function with respect to amplitude, is noticed in the clamped-immovable case. The least amount of change is in the hinged immovable case, which indicates that the change in the shape function increases with the amount of edge fixity.

The edge stresses, in all the cases considered, are similar to those of a hard spring, having an increasing rate of change with respect to the amplitude. In the center, membrane stresses behave similarly, but bending stresses are similar to stresses induced in a soft spring, having a decreasing rate of change with respect to the amplitude.

Comparison of the results obtained for the clamped-immovable case with those in Ref. [2], where a value $\nu = 1/3$ was used, indicates negligible differences.

In the Figures (...-2) to (...-9) each of the clamped-immovable and hinged-immovable cases is followed by its counterpart of the clamped-movable and hinged-movable cases, respectively. Such pairing is hoped to facilitate the designer's task of locating his range between the two theoretical extremes of a radially-restrained and a radially-free edge condition.

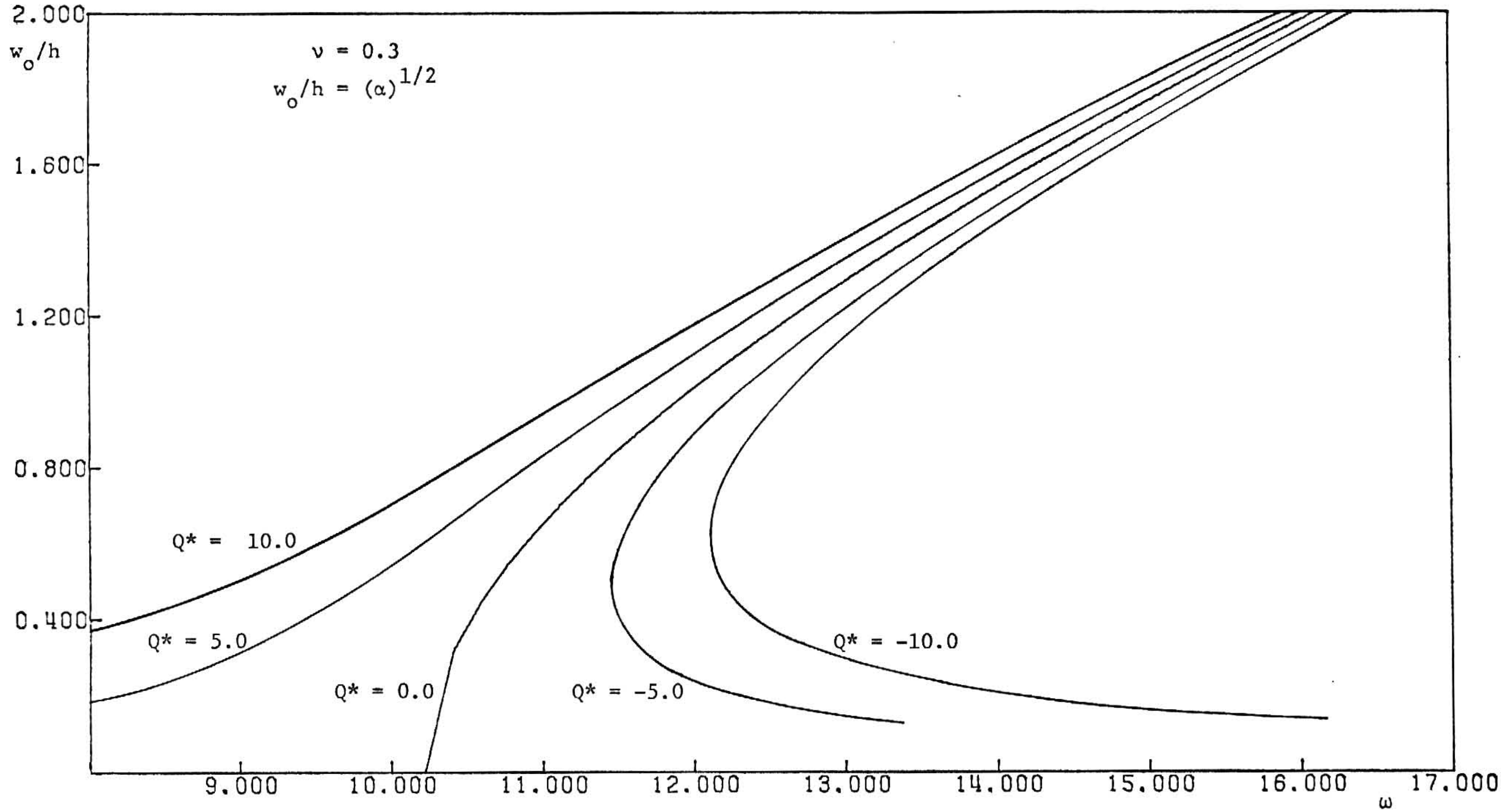

Fig. (CI-2). Harmonic Response of a Clamped-Immovable Plate.

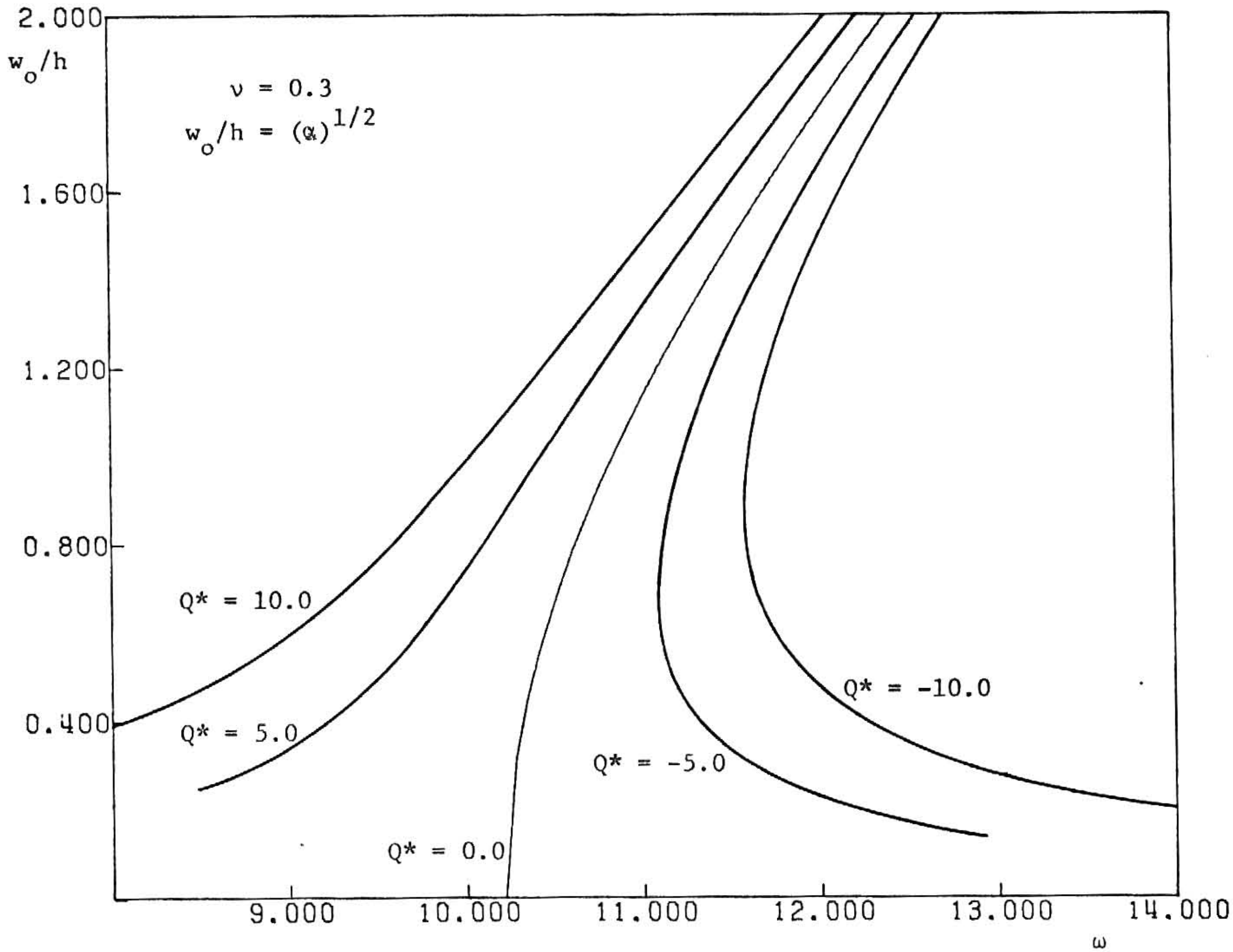

Fig. (CM-2). Harmonic Response of a Clamped-Movable Plate.

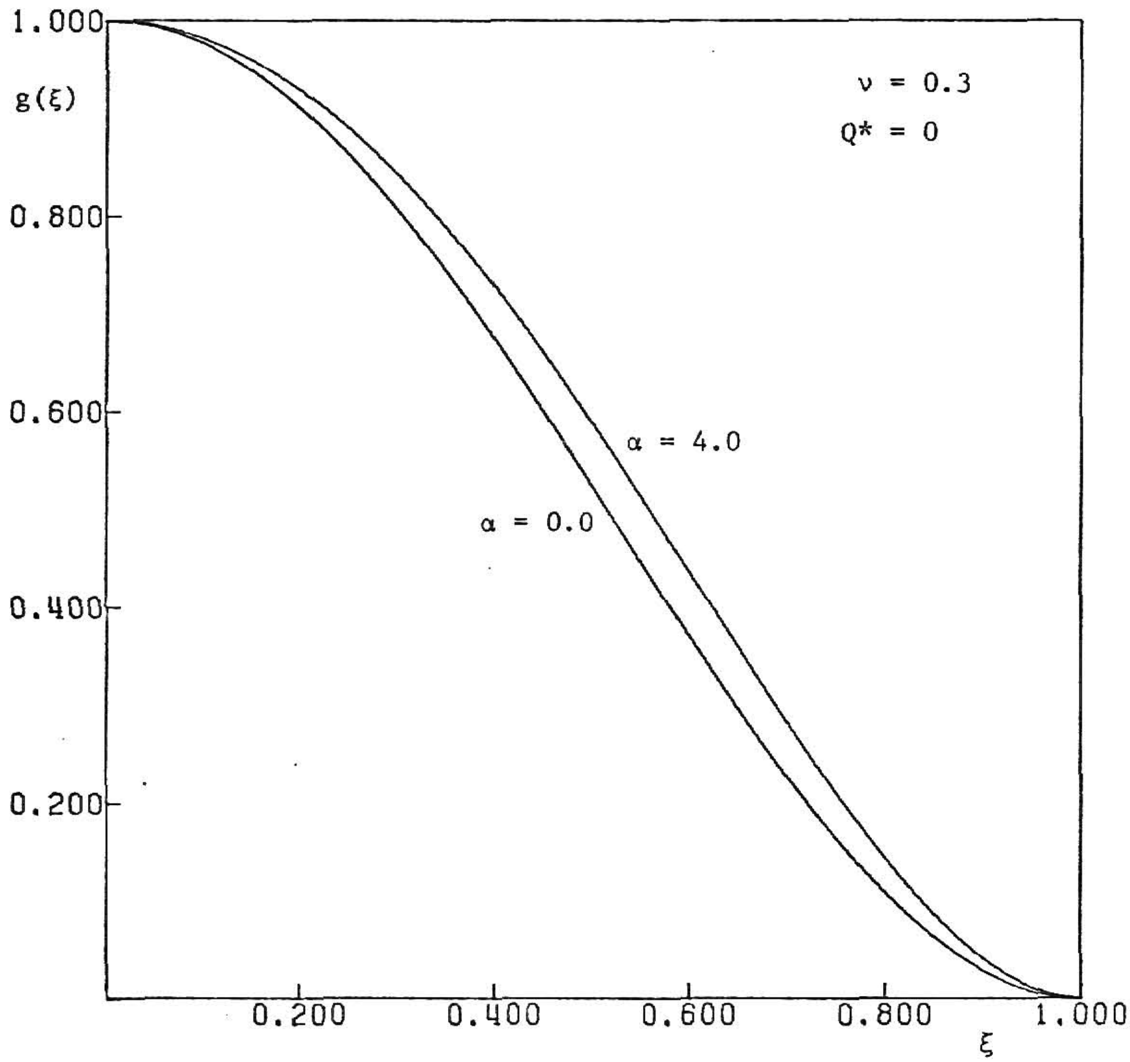

Fig. (CI-3). Shape Function for a Clamped-Immovable Plate.

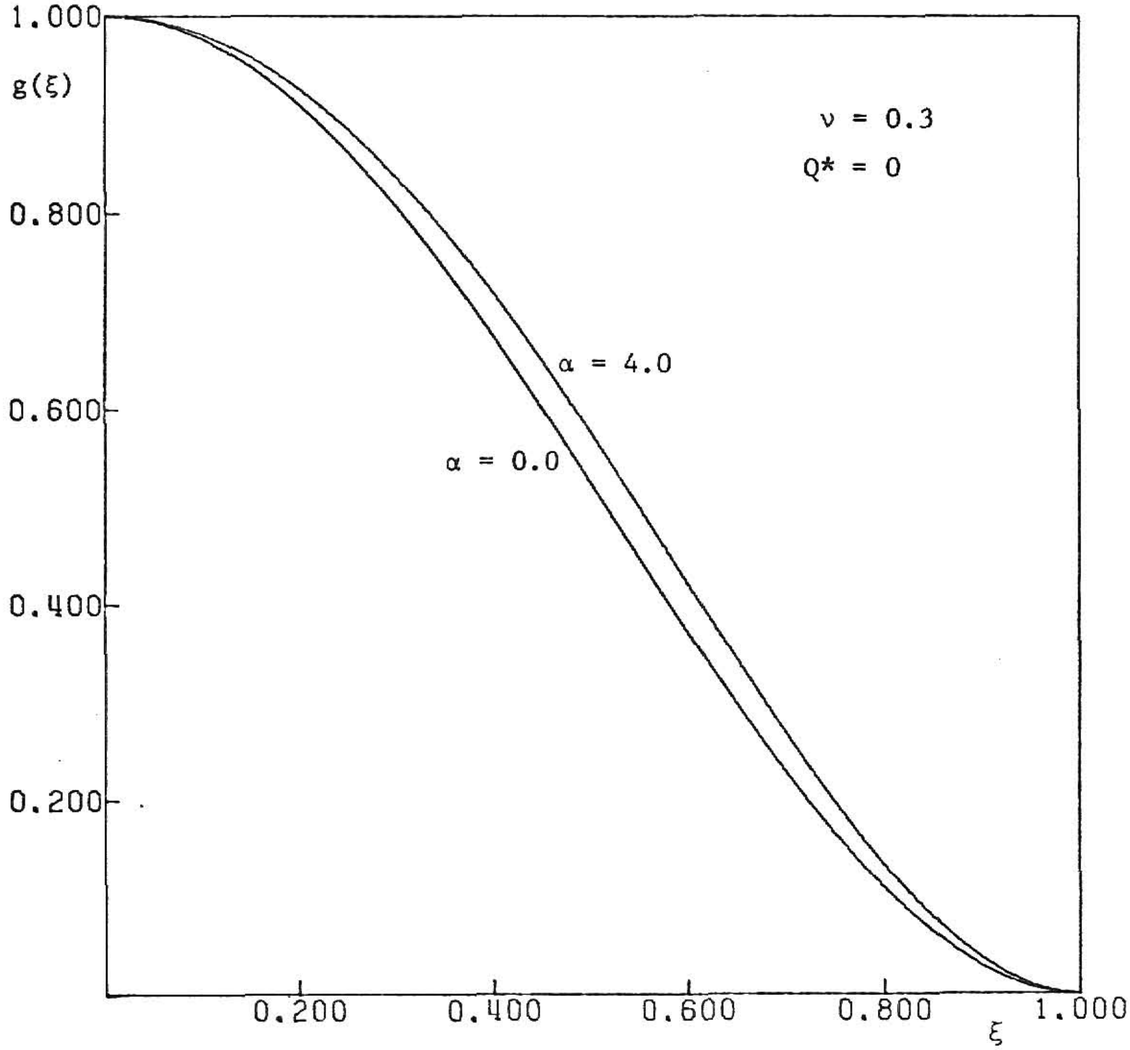

Fig. (CM-3). Shape Function for a Clamped-Movable Plate.

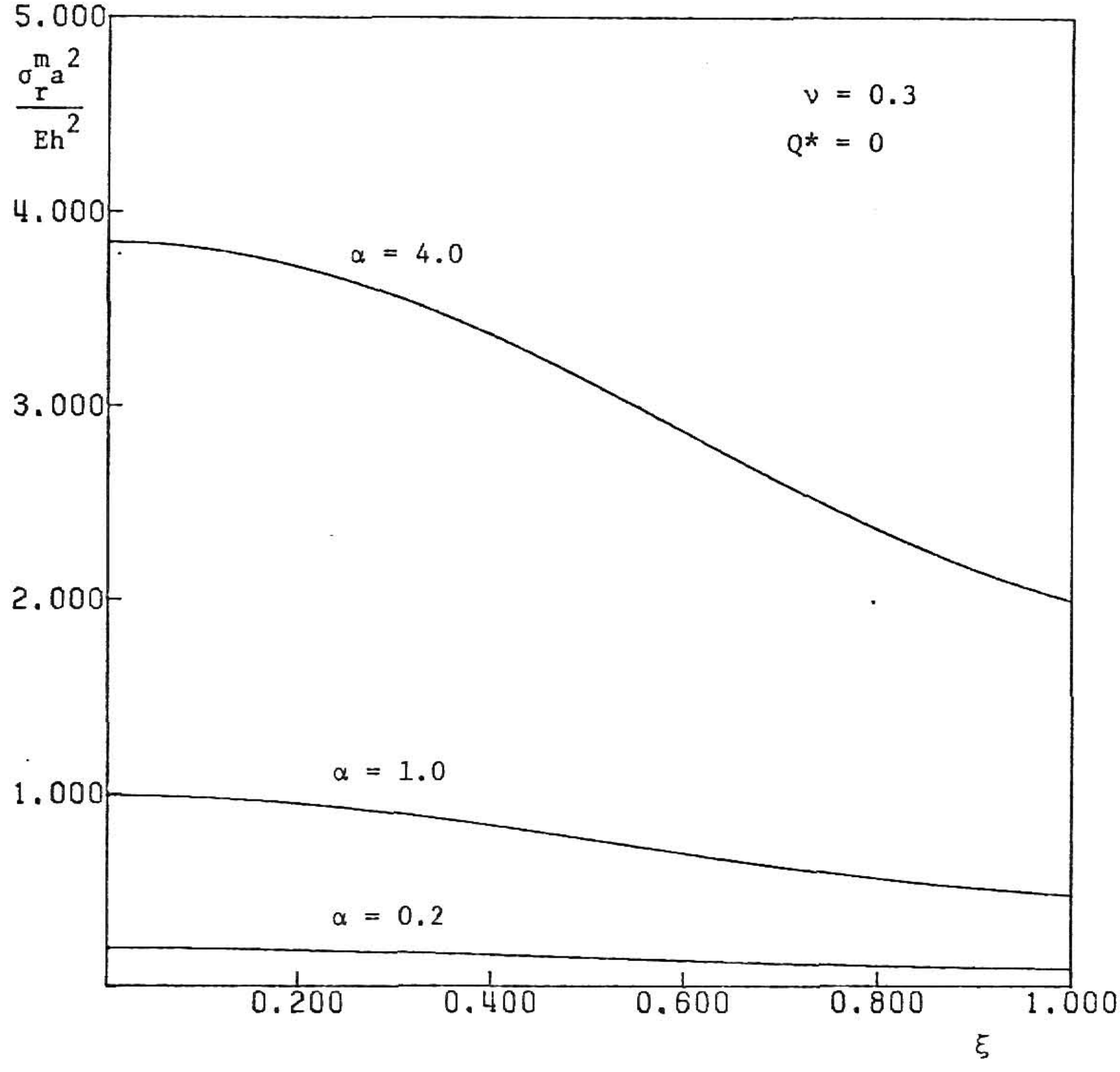

Fig. (CI-4). Radial Membrane Stress.

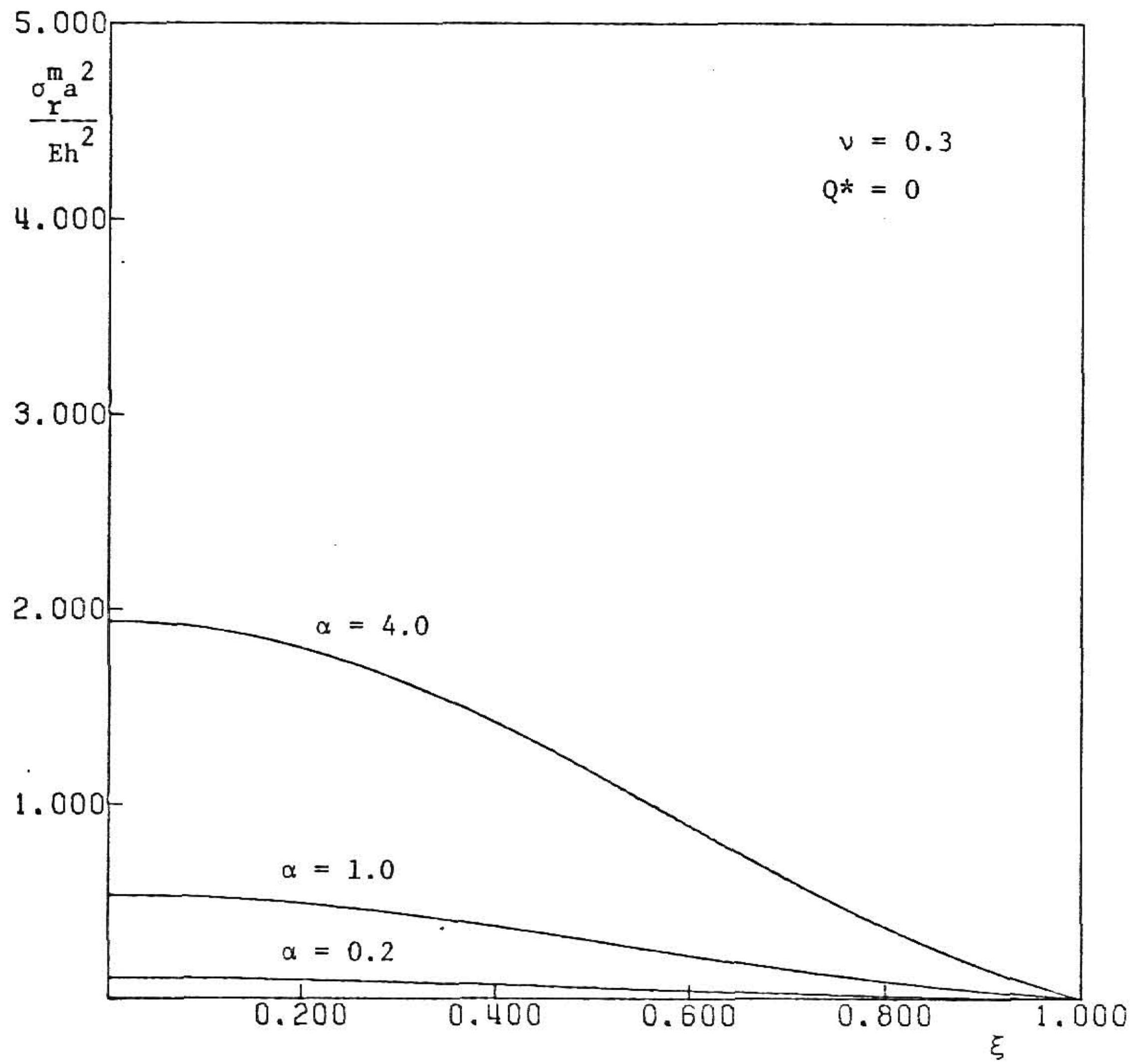

Fig. (CM-4). Radial Membrane Stress.

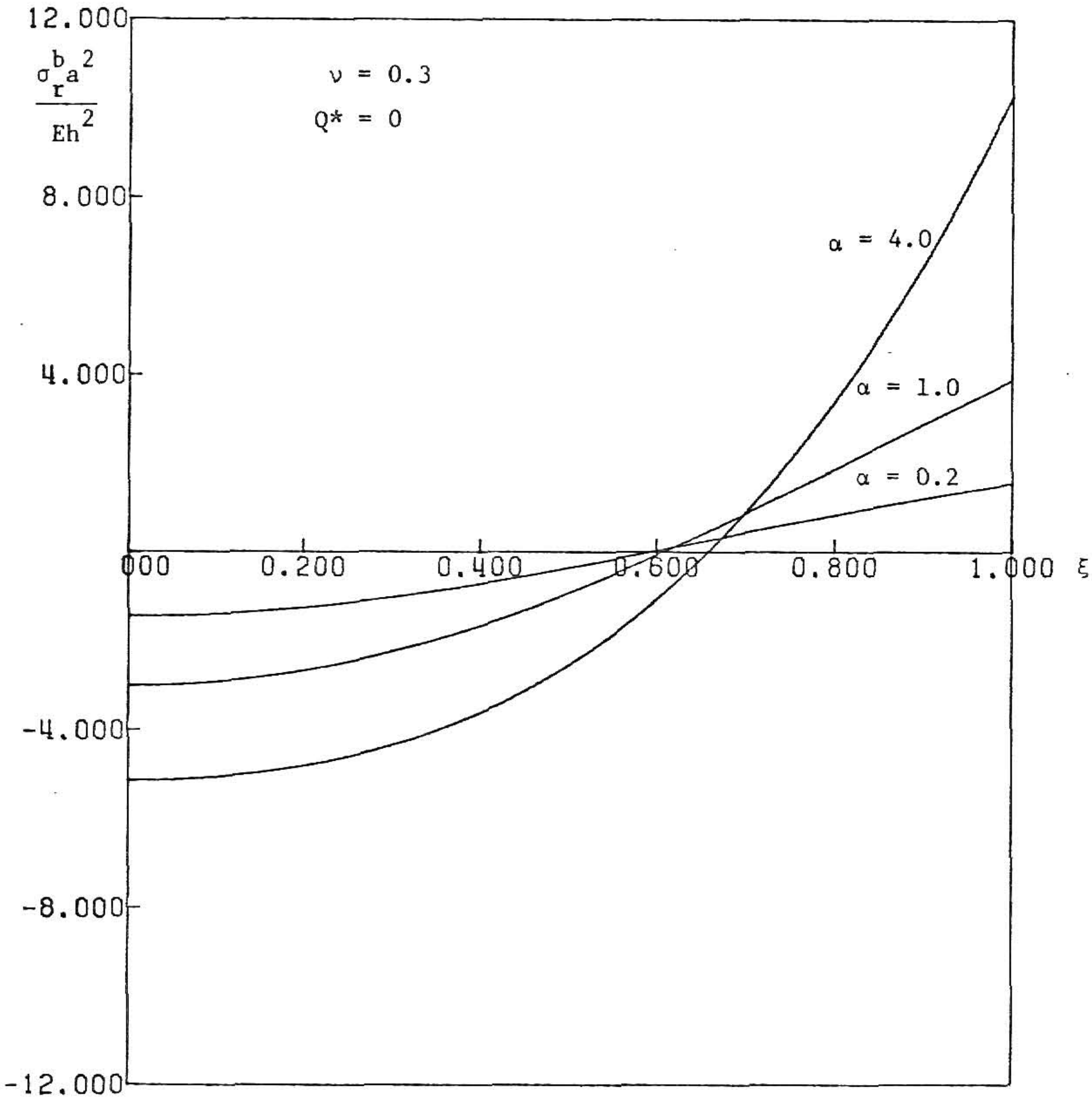

Fig. (CI-5). Radial Bending Stress.

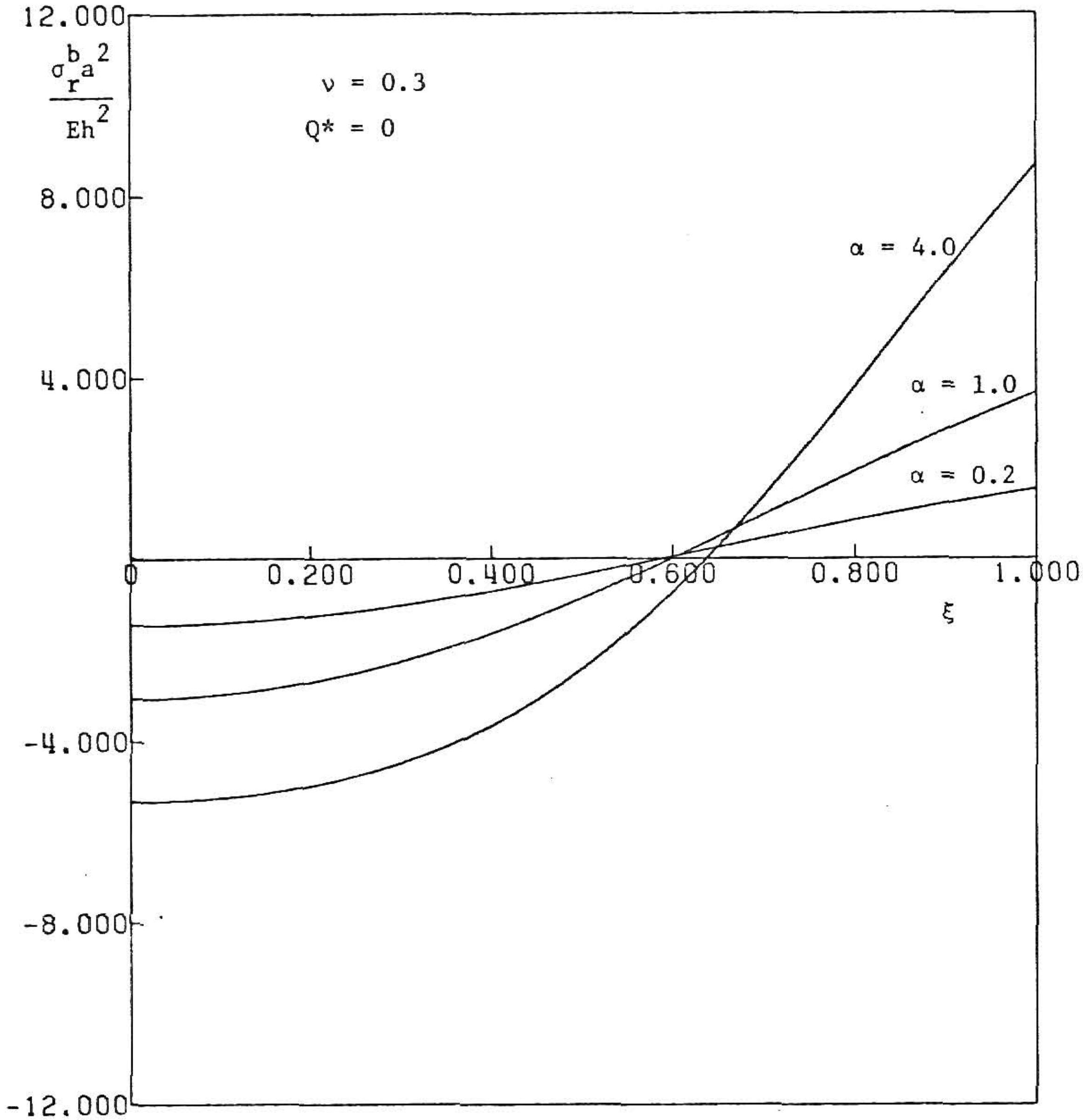

Fig. (CM-5). Radial Bending Stress.

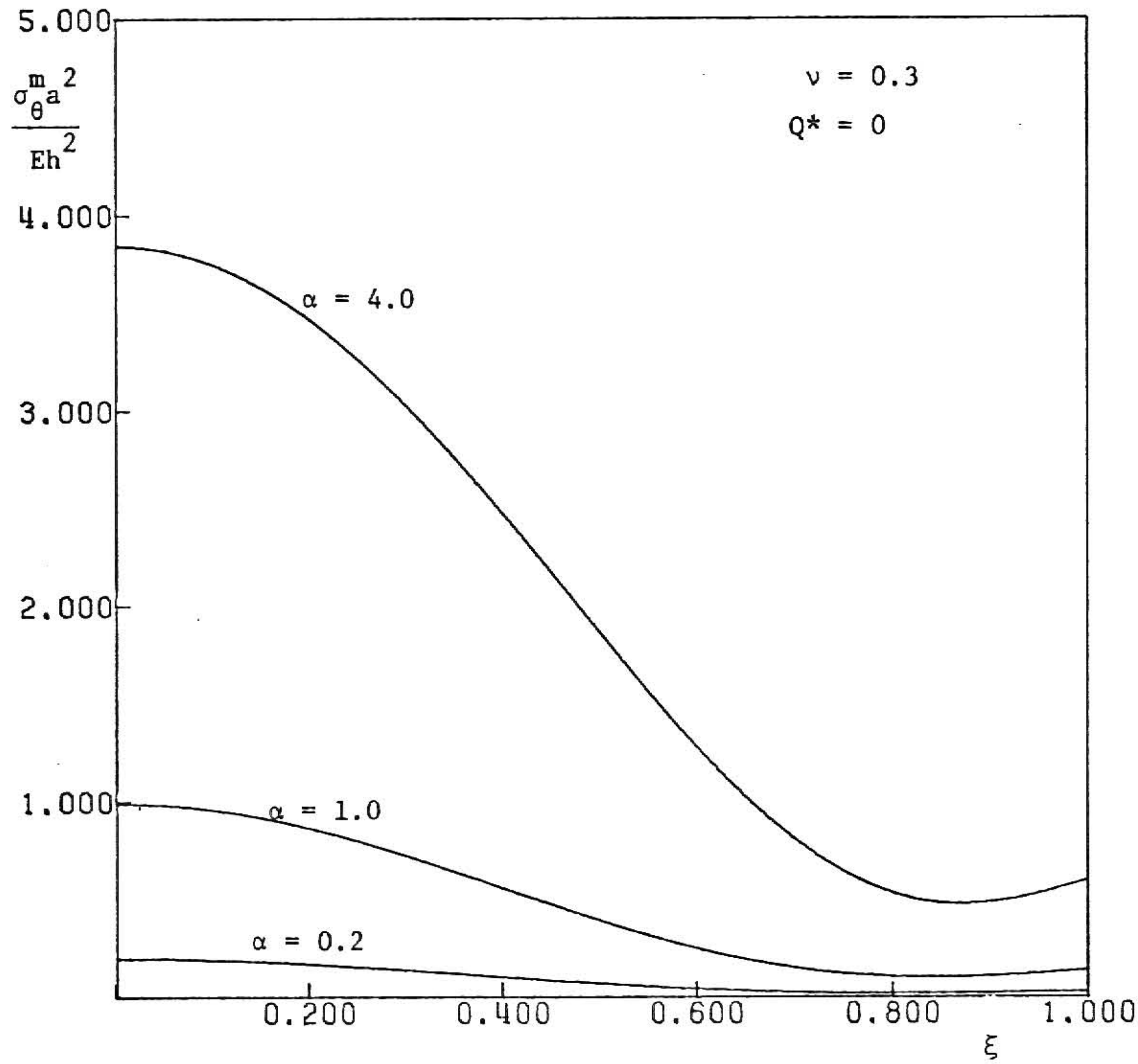

Fig. (CI-6). Circumferential Membrane Stress.

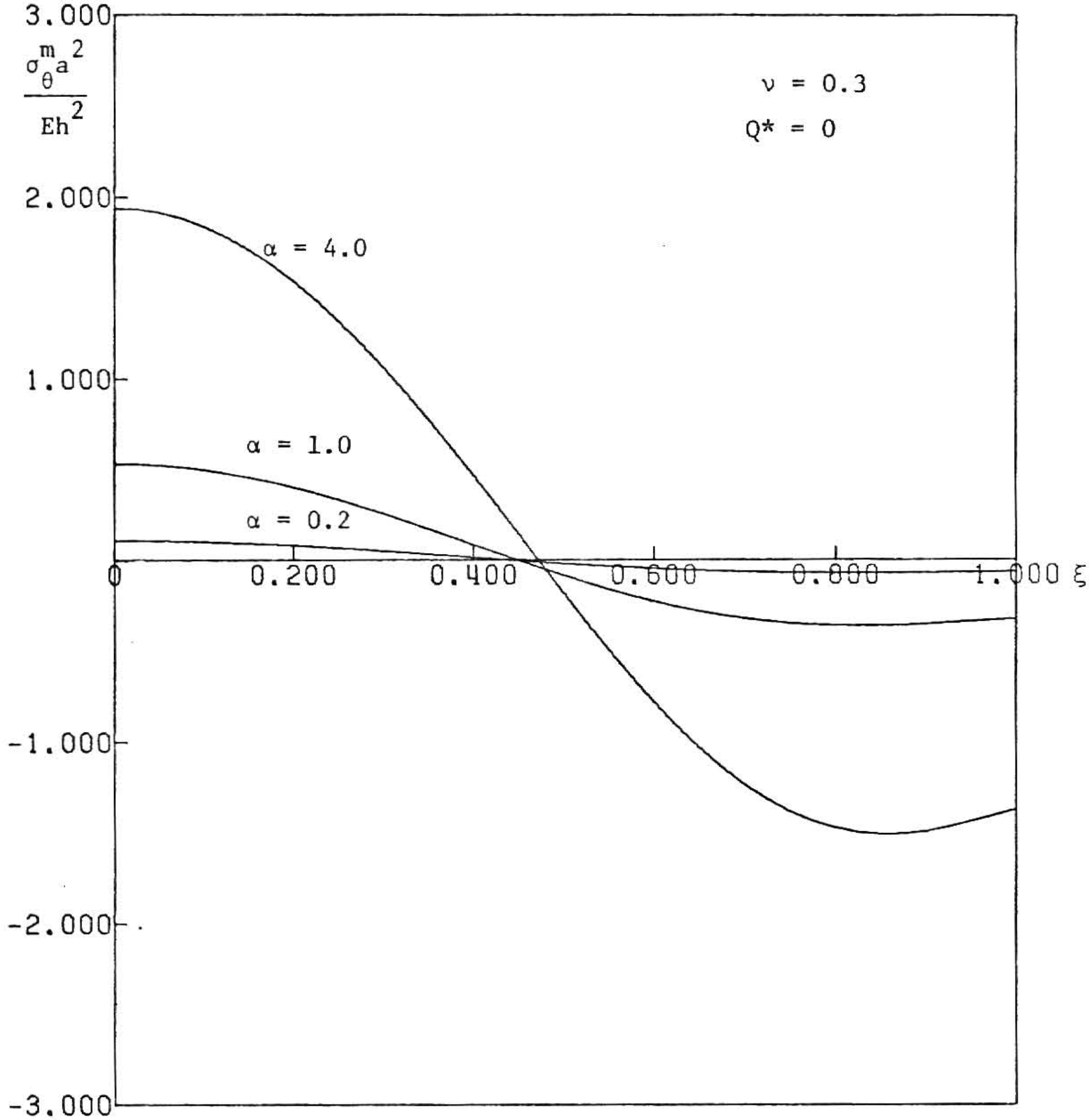

Fig. (CM-6). Circumferential Membrane Stress.

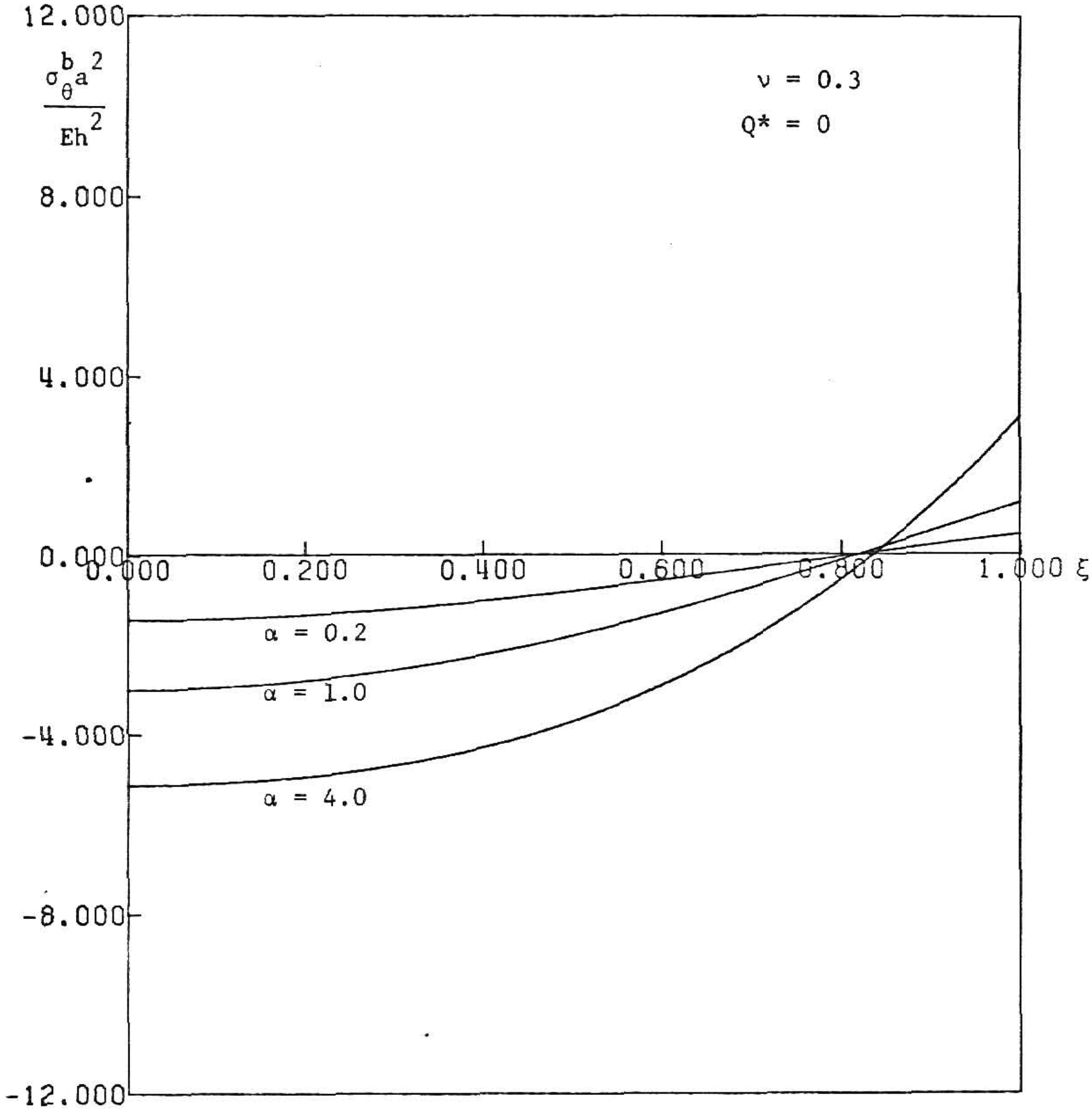

Fig. (CI-7). Circumferential Bending Stress.

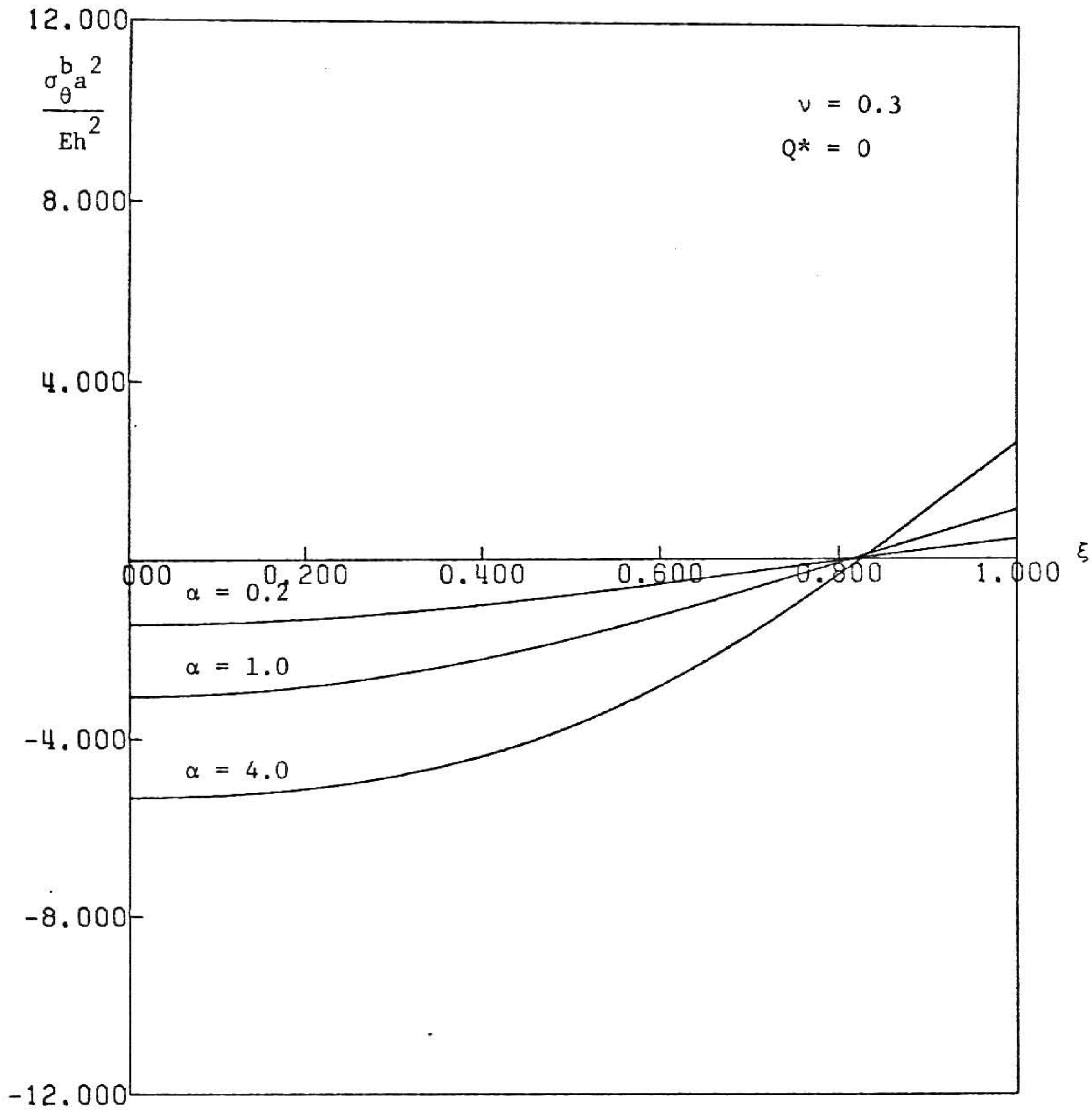

Fig. (CM-7). Circumferential Bending Stress.

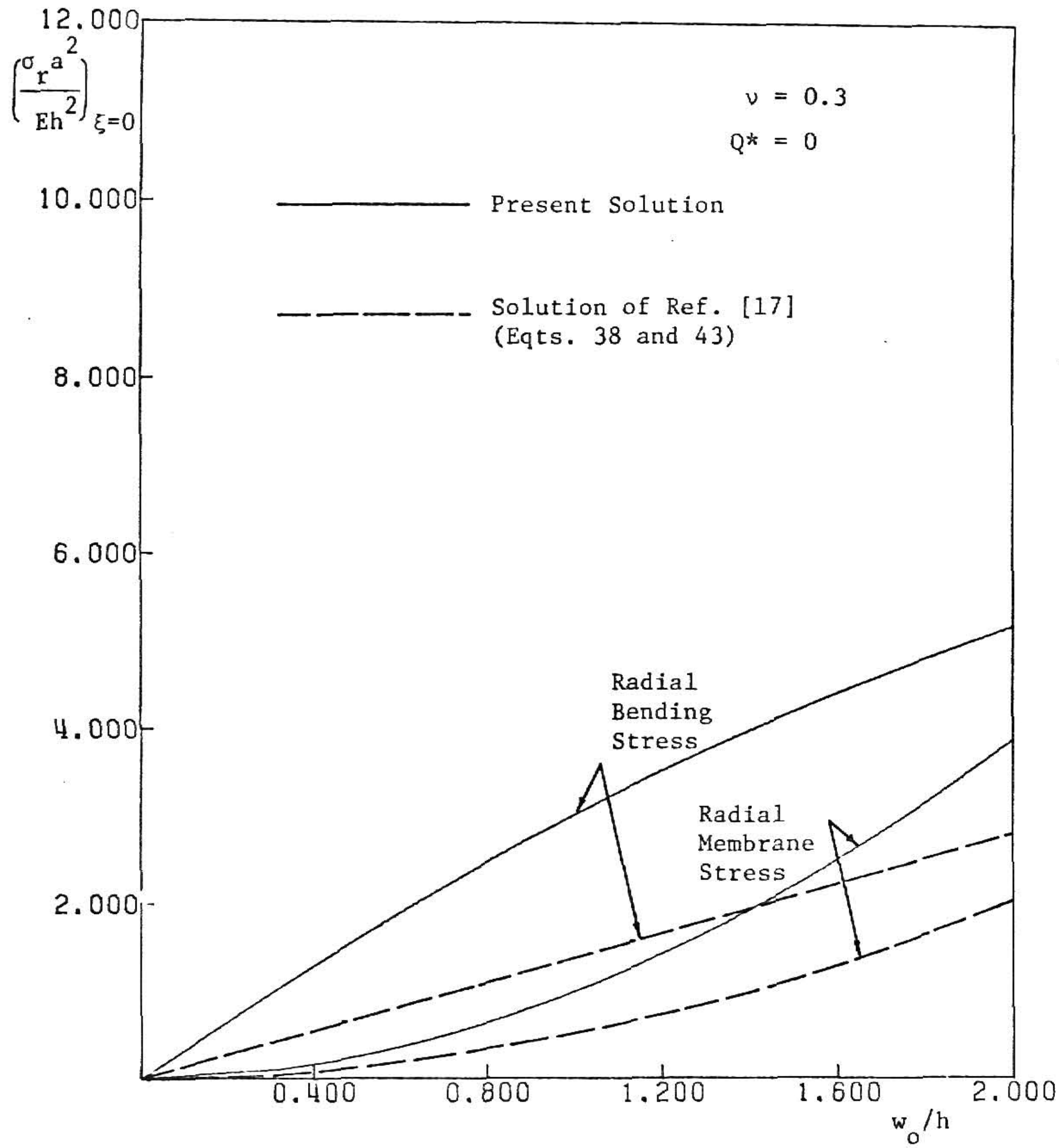

Fig. (CI-8). Radial Stresses at the Center of the Plate.

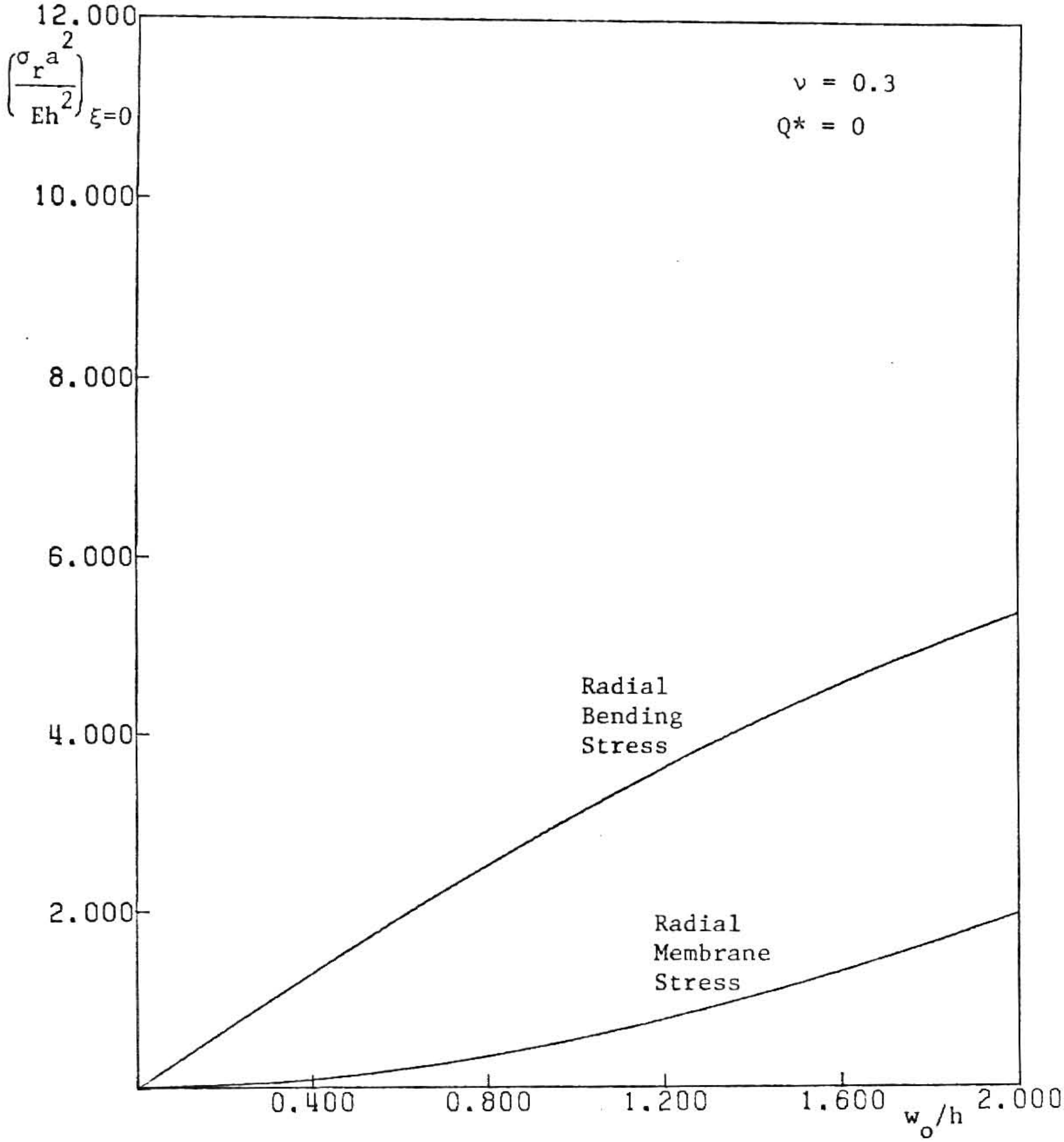

Fig. (CM-8). Radial Stresses at the Center of the Plate.

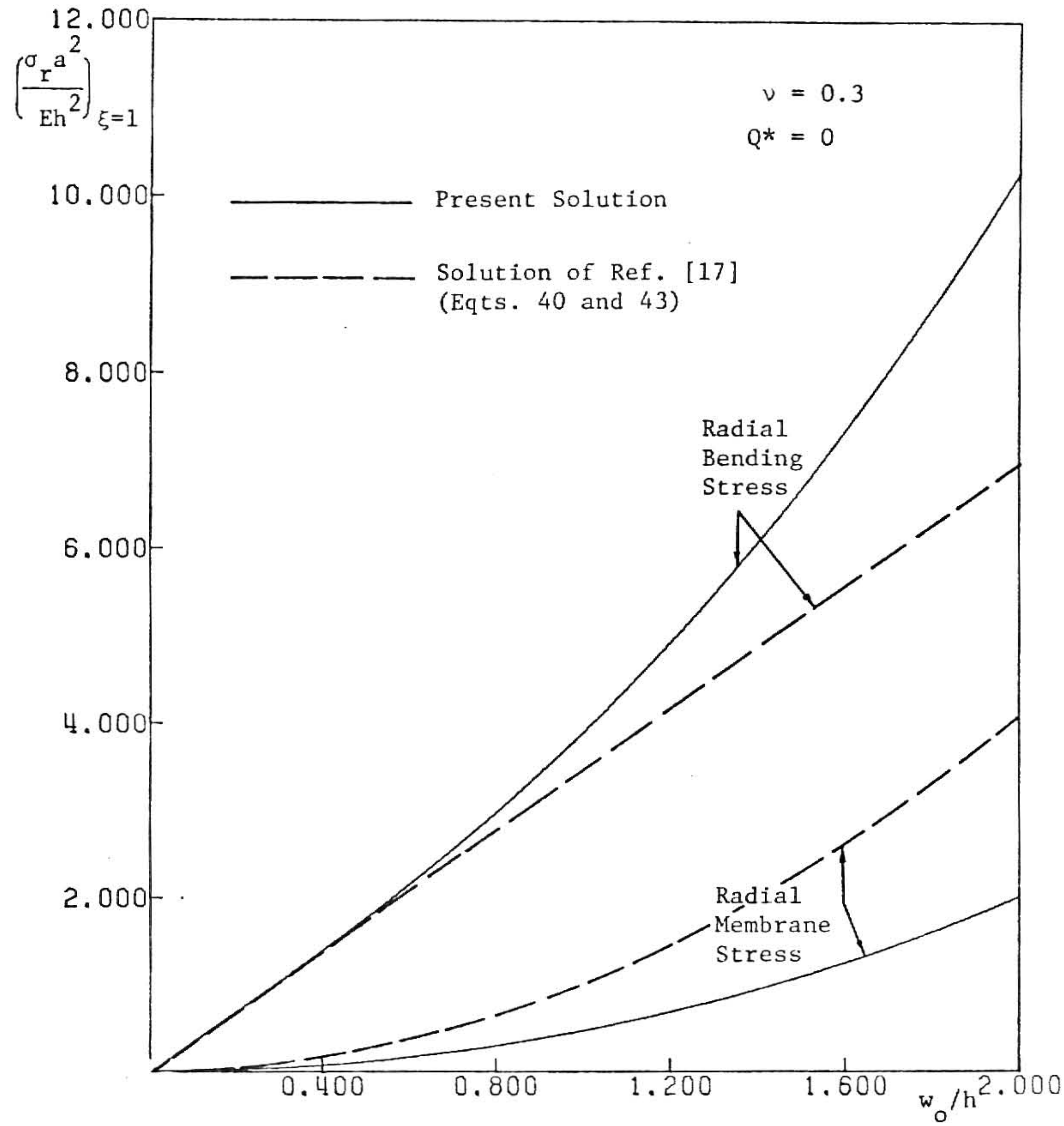

Fig. (CI-9). Radial Stresses at the Edge of the Plate.

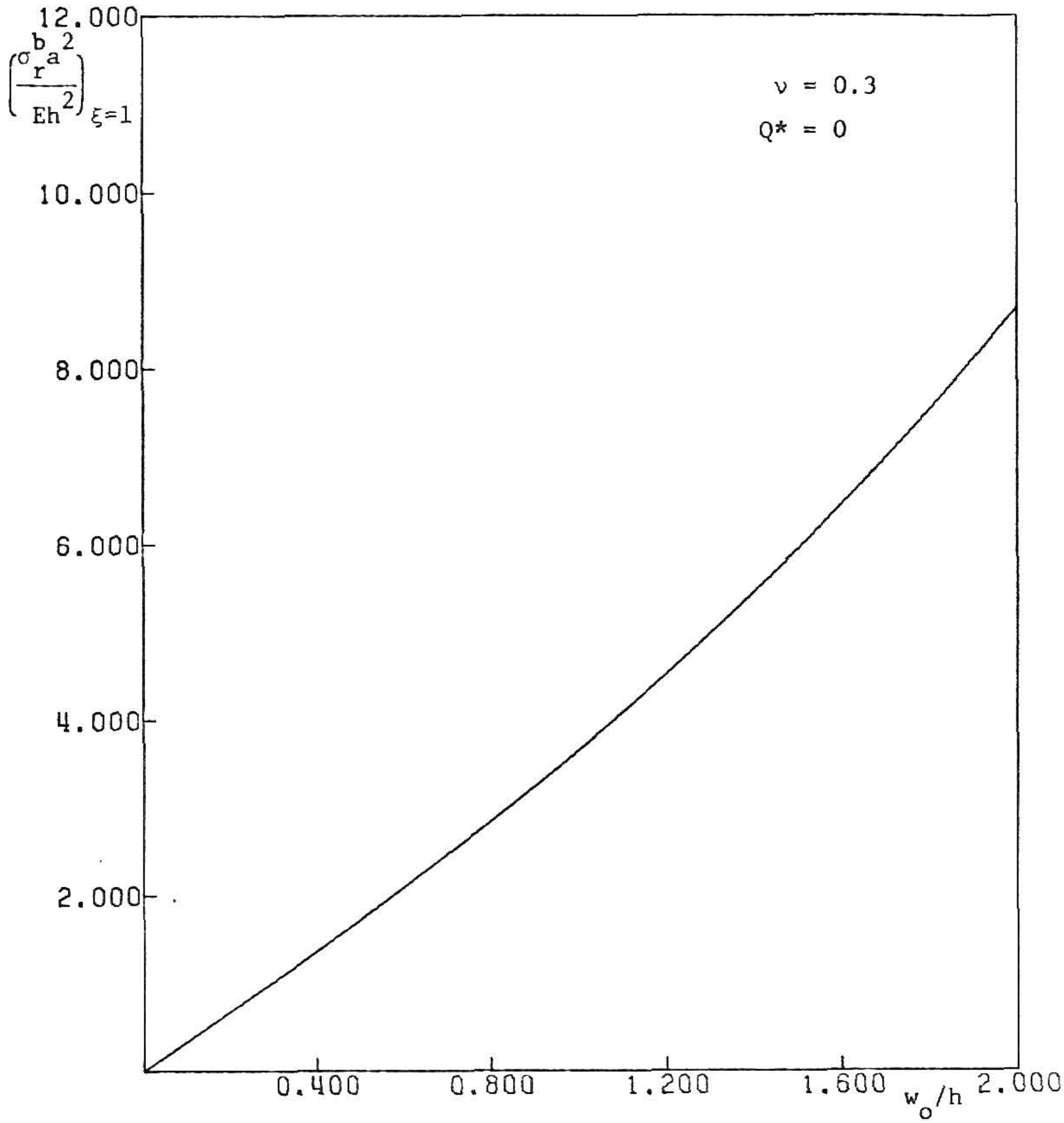

Fig. (CM-9). Radial Bending Stress at the Edge of the Plate.

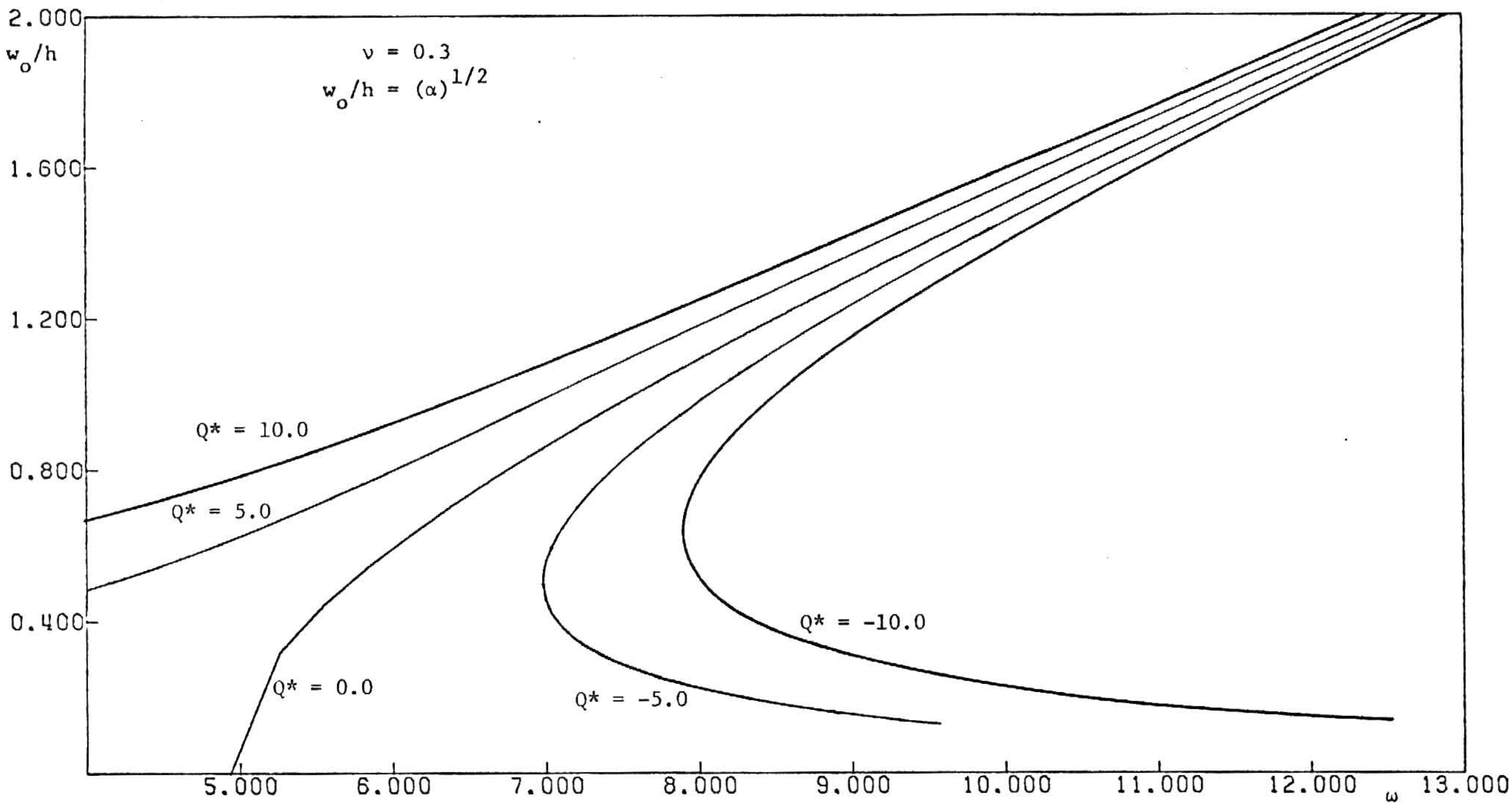

Fig. (HI-2). Harmonic Response of a Hinged Immovable Plate.

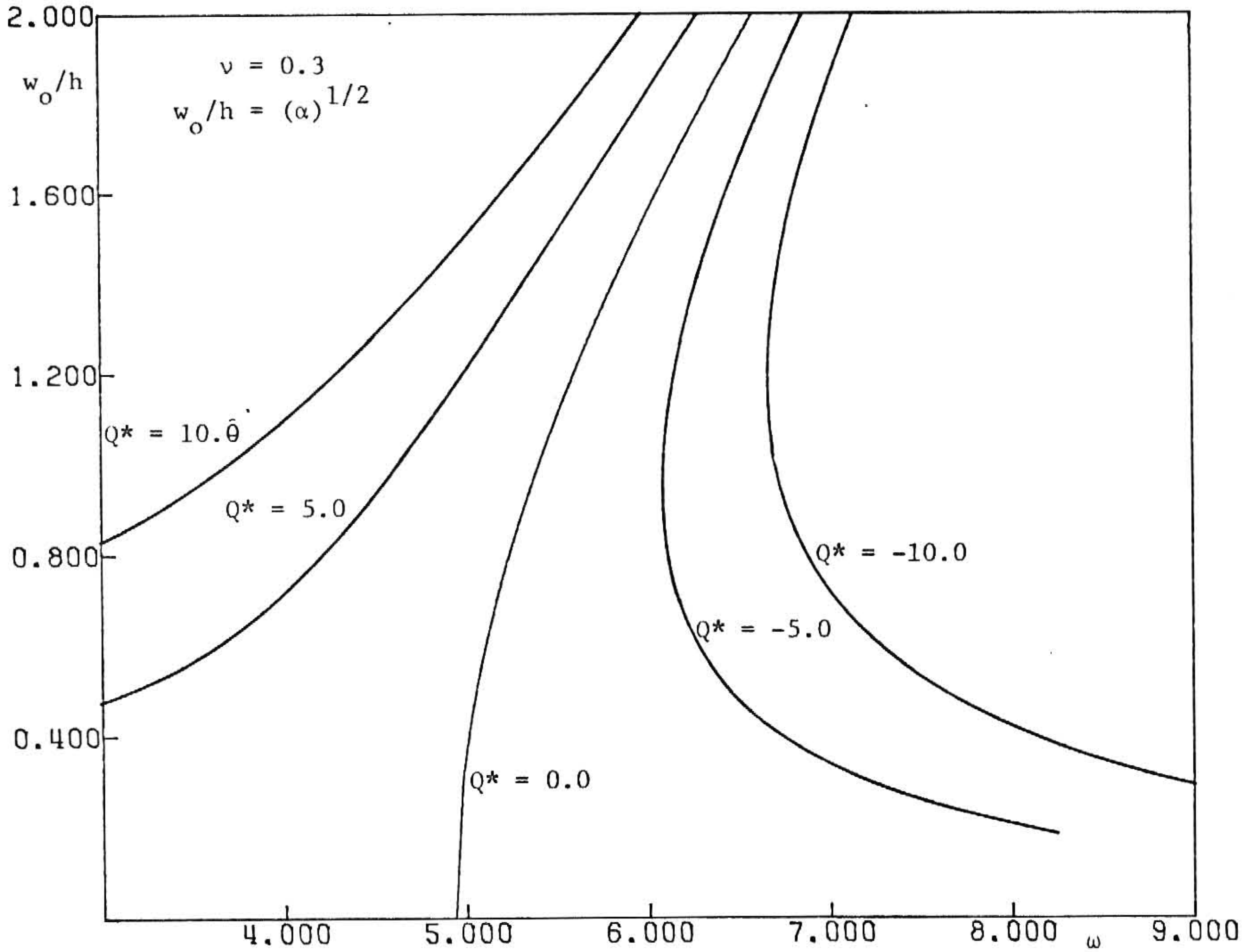

Fig. (HM-2). Harmonic Response of a Hinged-Movable Plate.

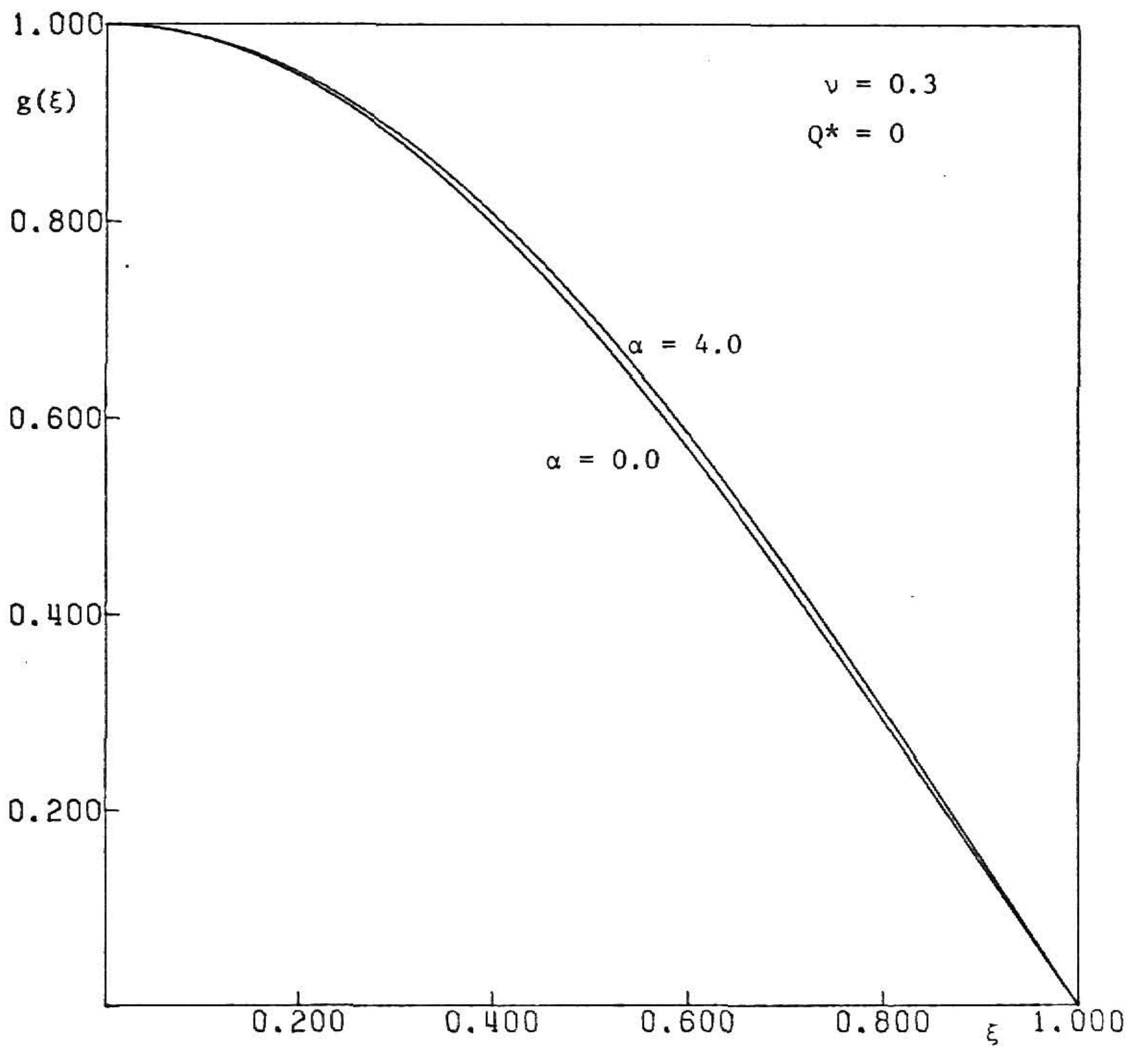

Fig. (HI-3). Shape Function for a Hinged-Immovable Plate.

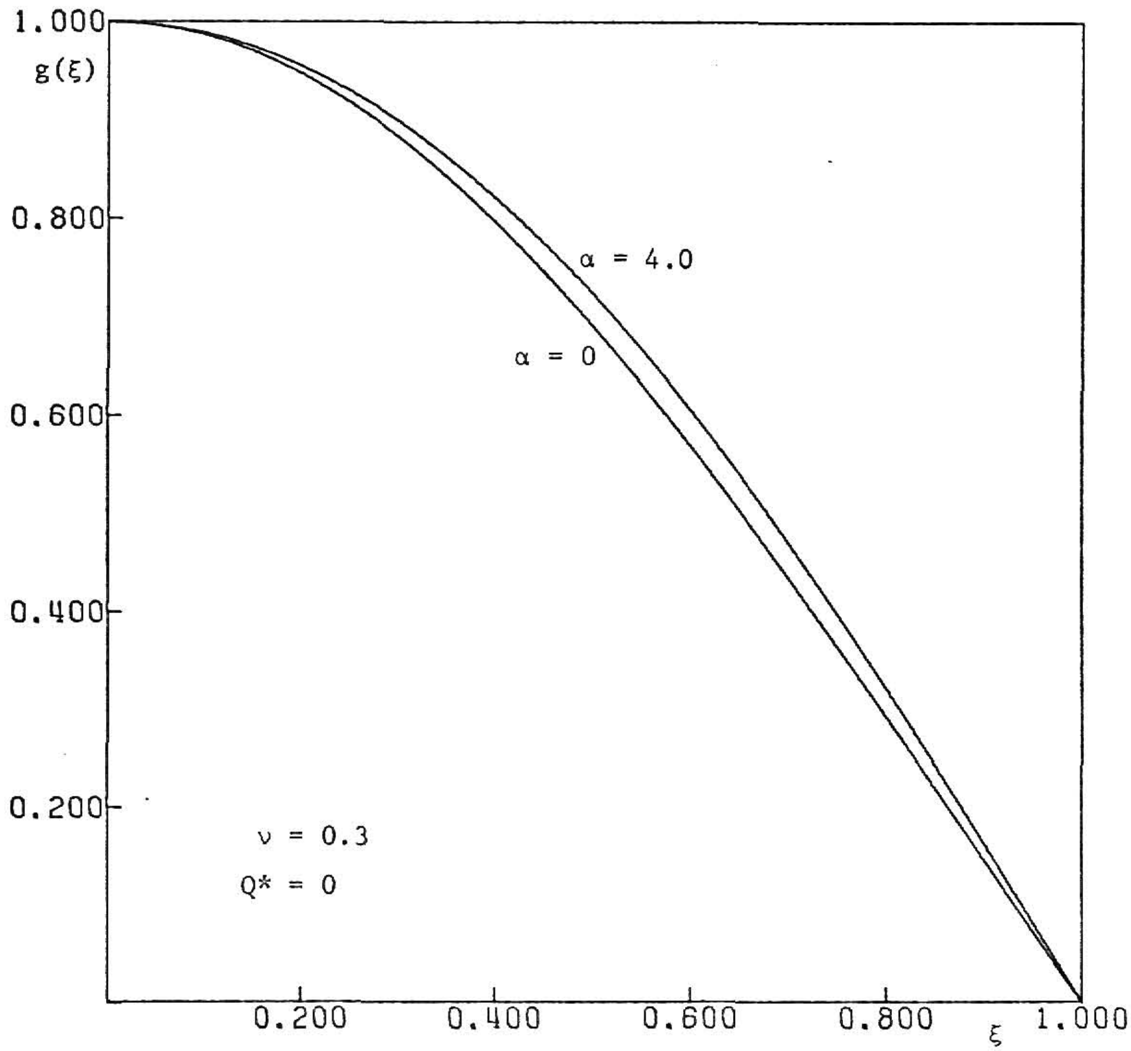

Fig. (HM-3). Shape Function for a Clamped Movable Plate.

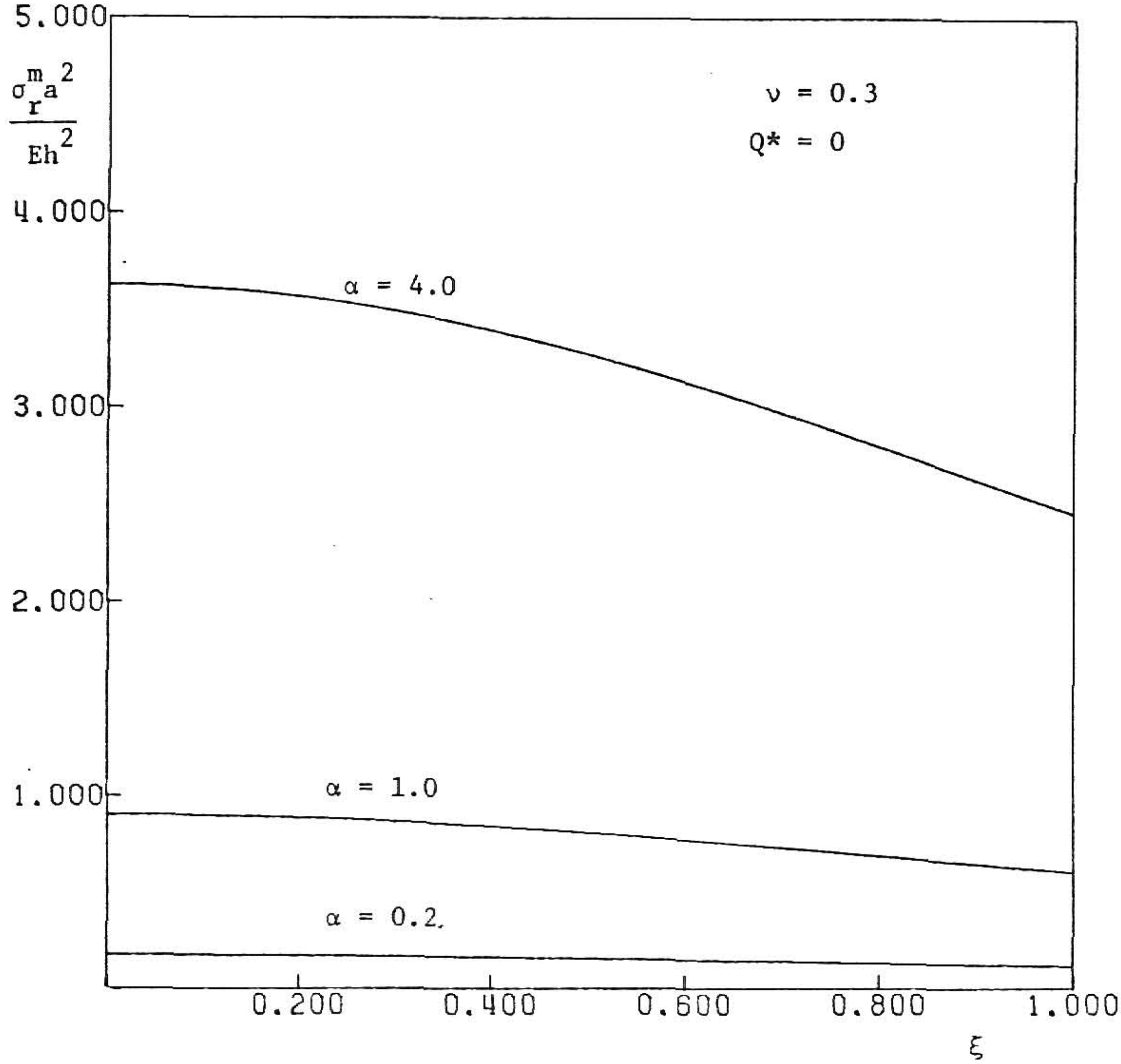

Fig. (HI-4). Radial Membrane Stress.

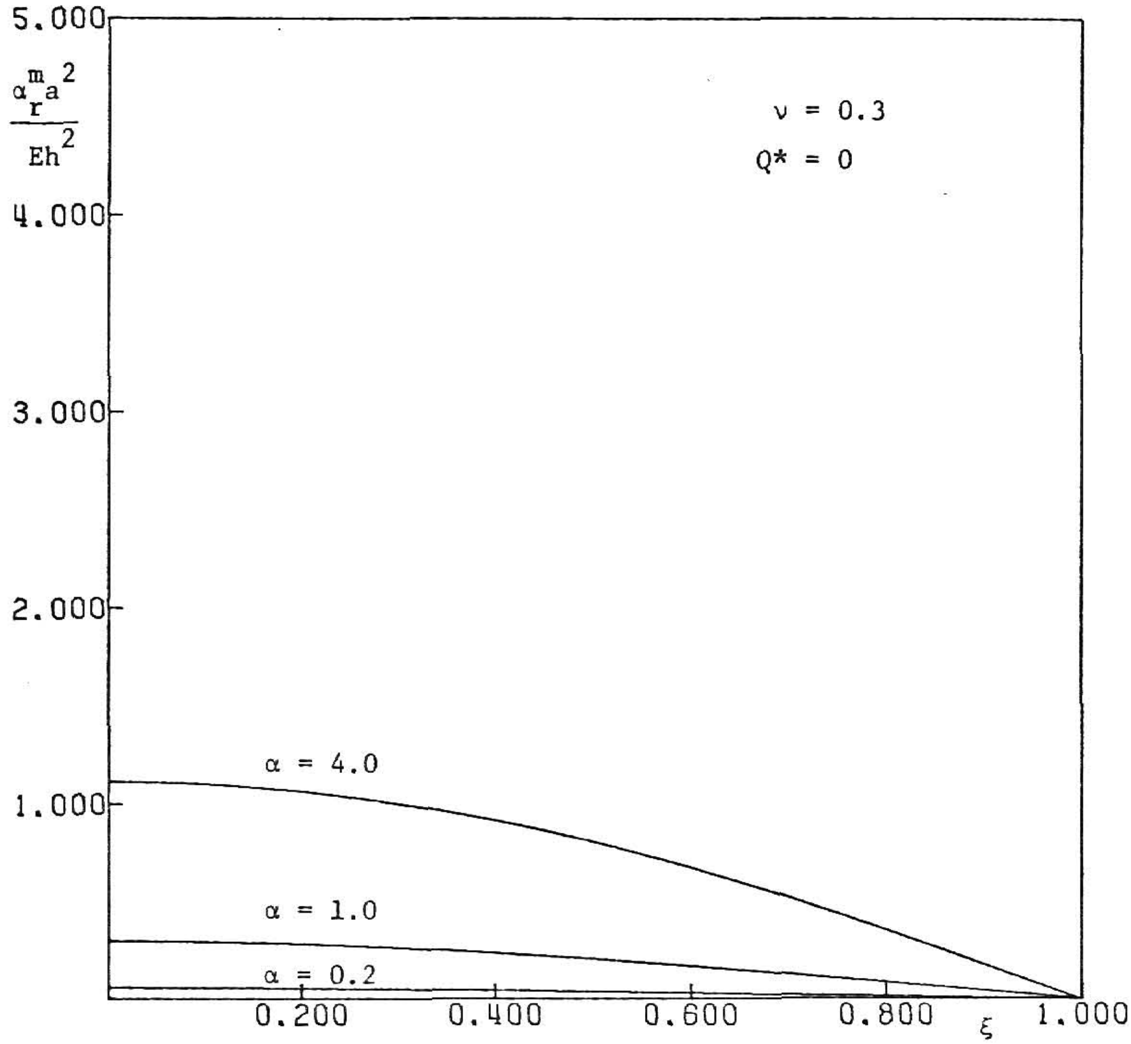

Fig. (HM-4). Radial Membrane Stress.

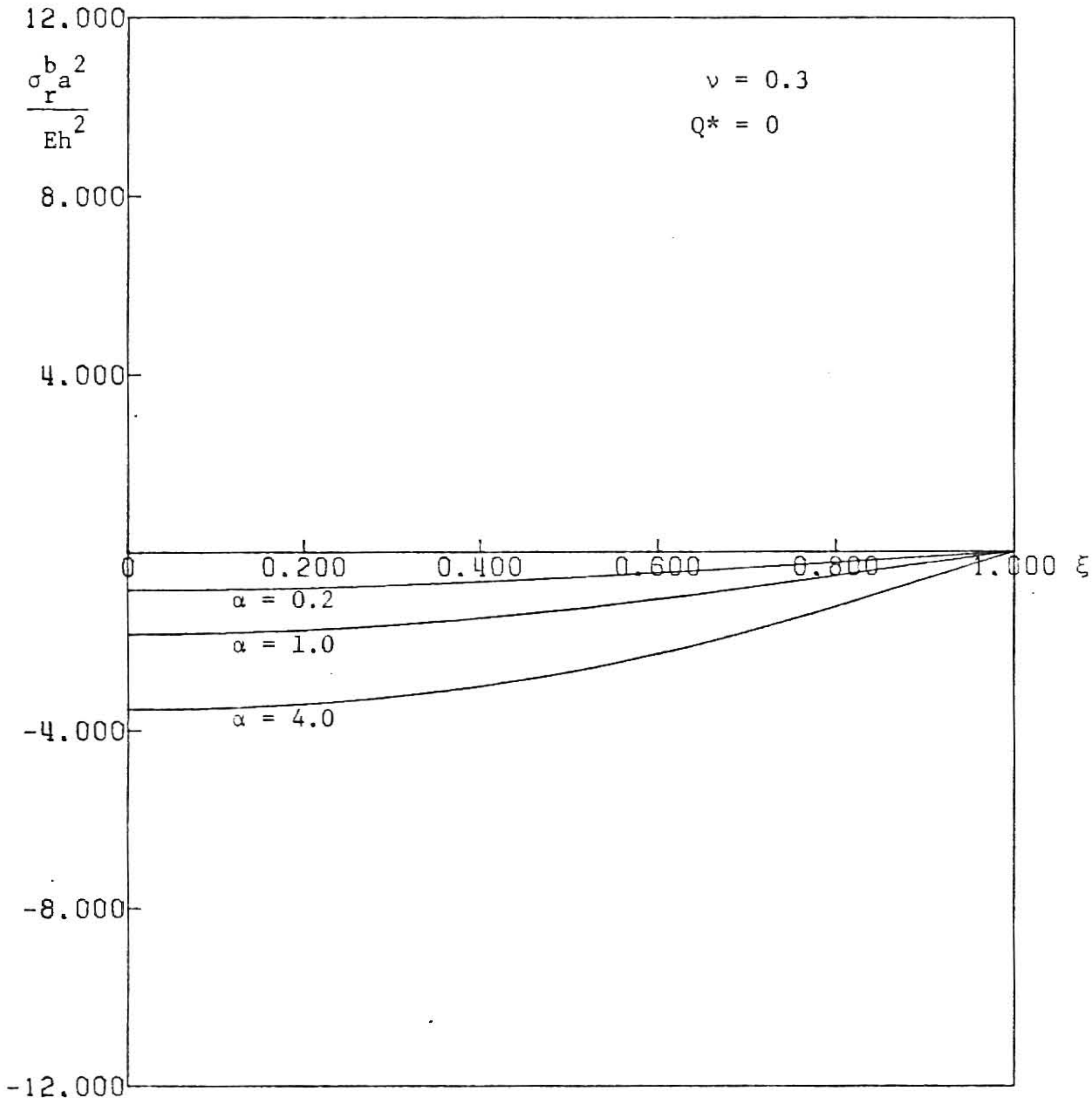

Fig. (HI-5). Radial Bending Stress.

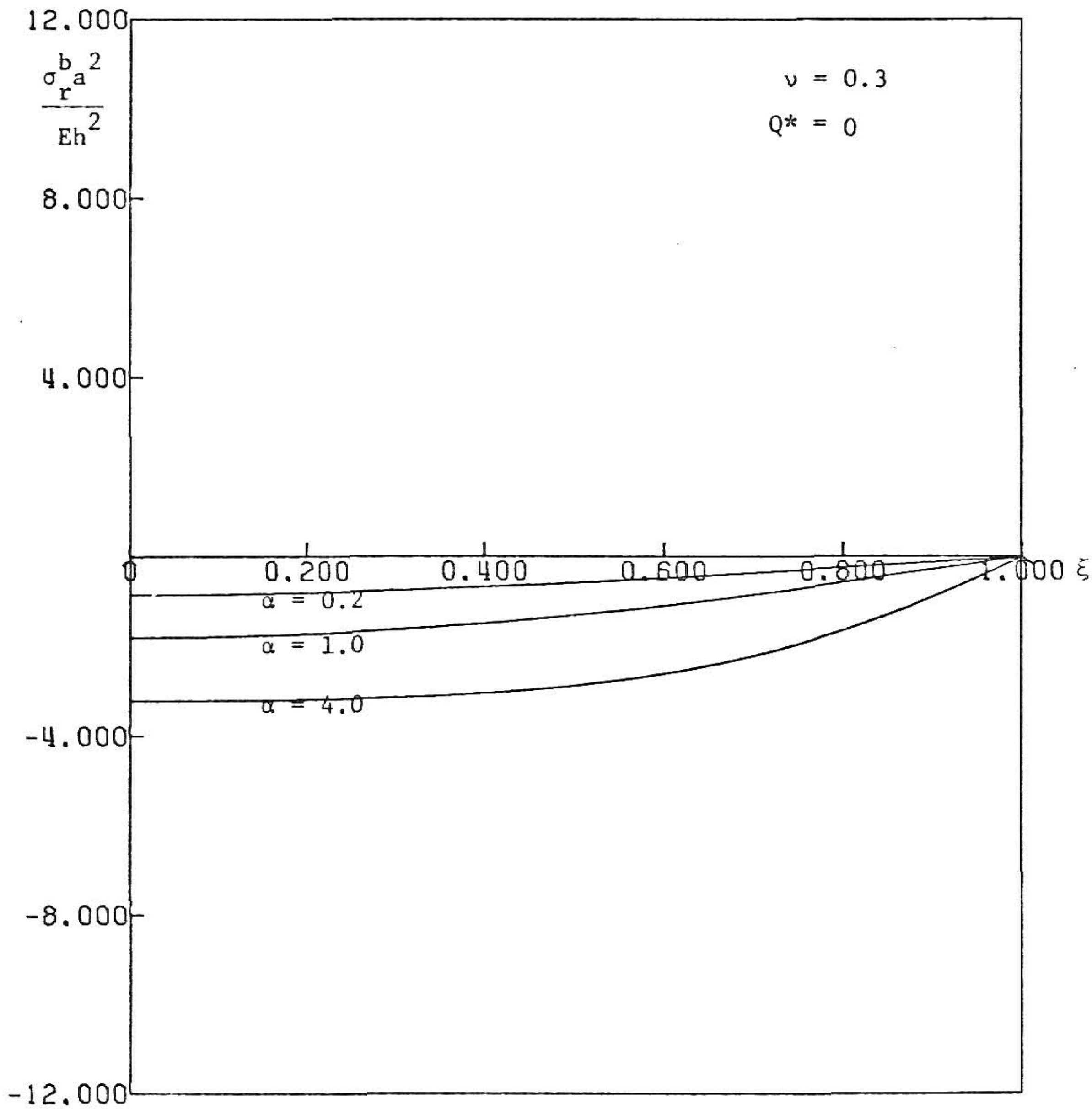

Fig. (HM-5). Radial Bending Stress.

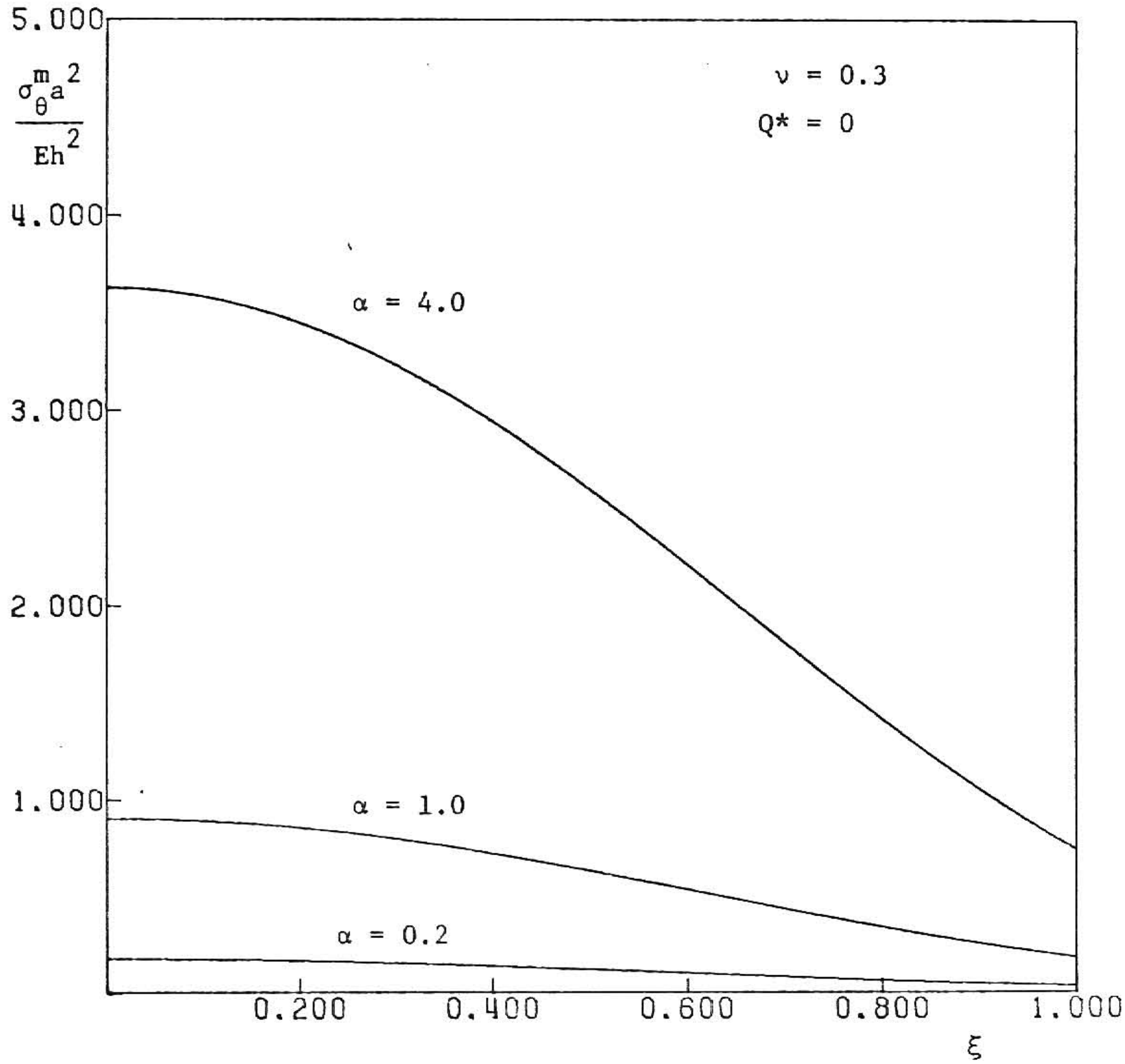

Fig. (HI-6). Circumferential Membrane Stress.

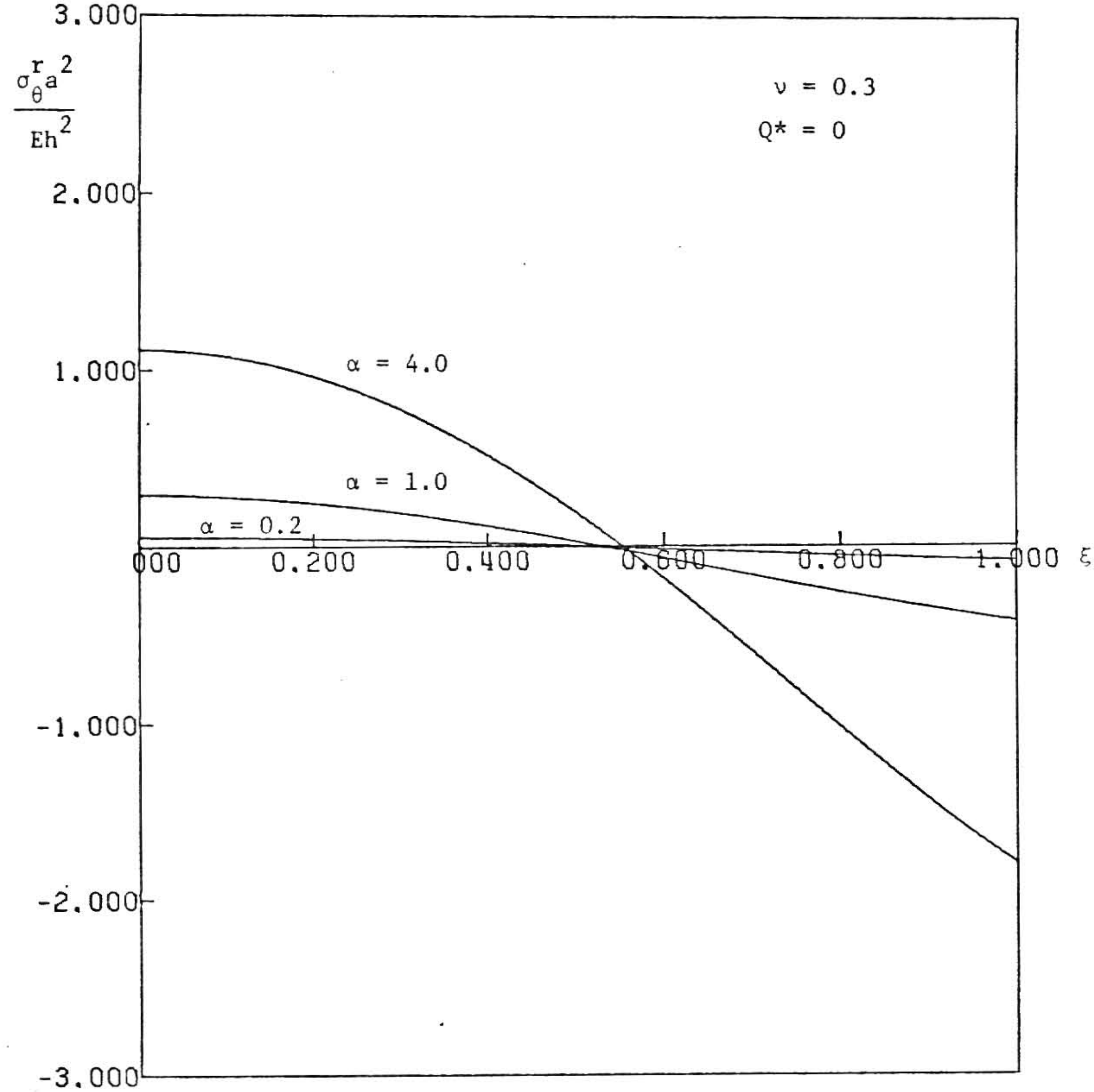

Fig. (HM-6). Circumferential Membrane Stress.

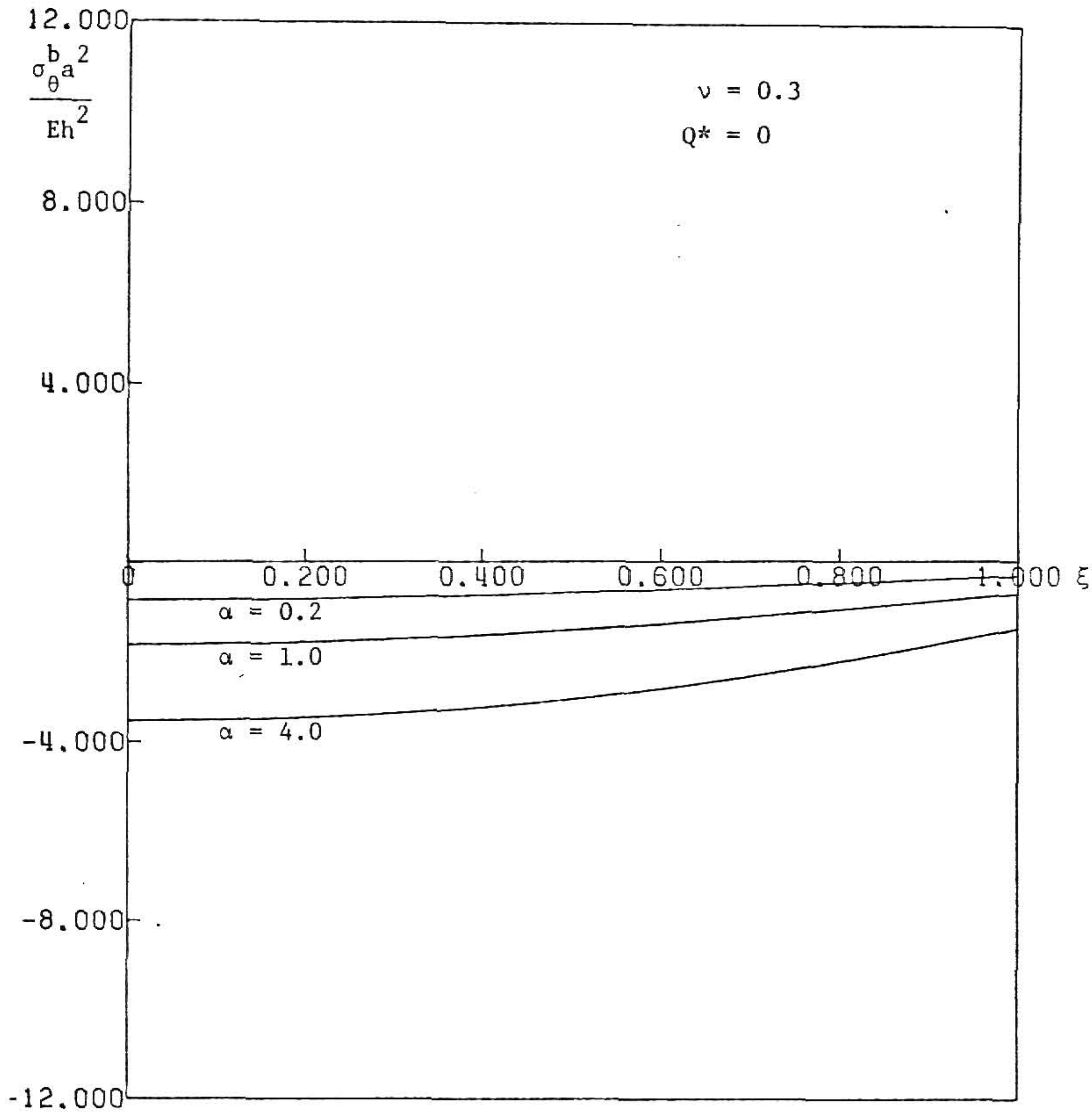

Fig. (HI-7). Circumferential Bending Stress.

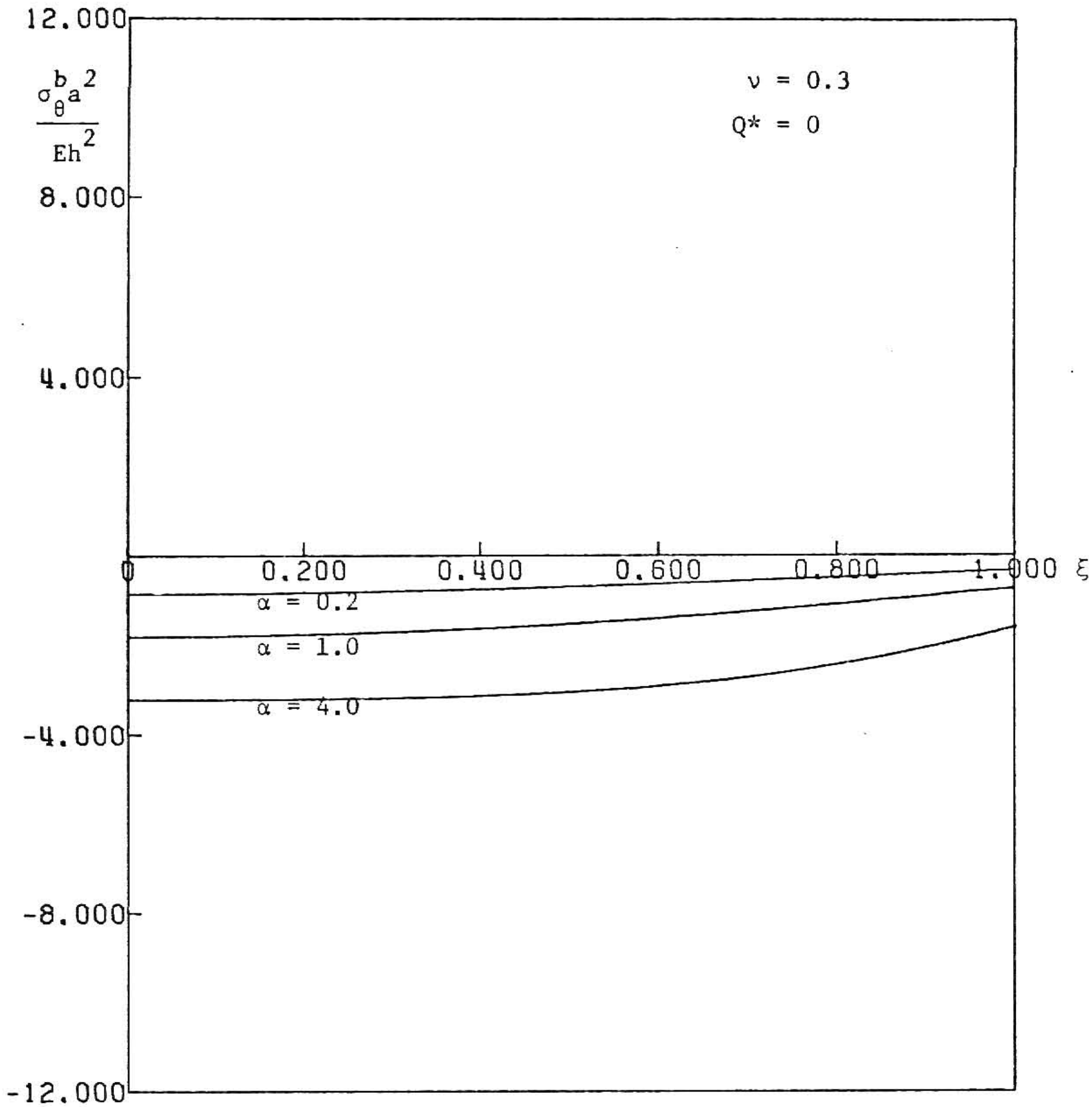

Fig. (HM-7). Circumferential Bending Stress.

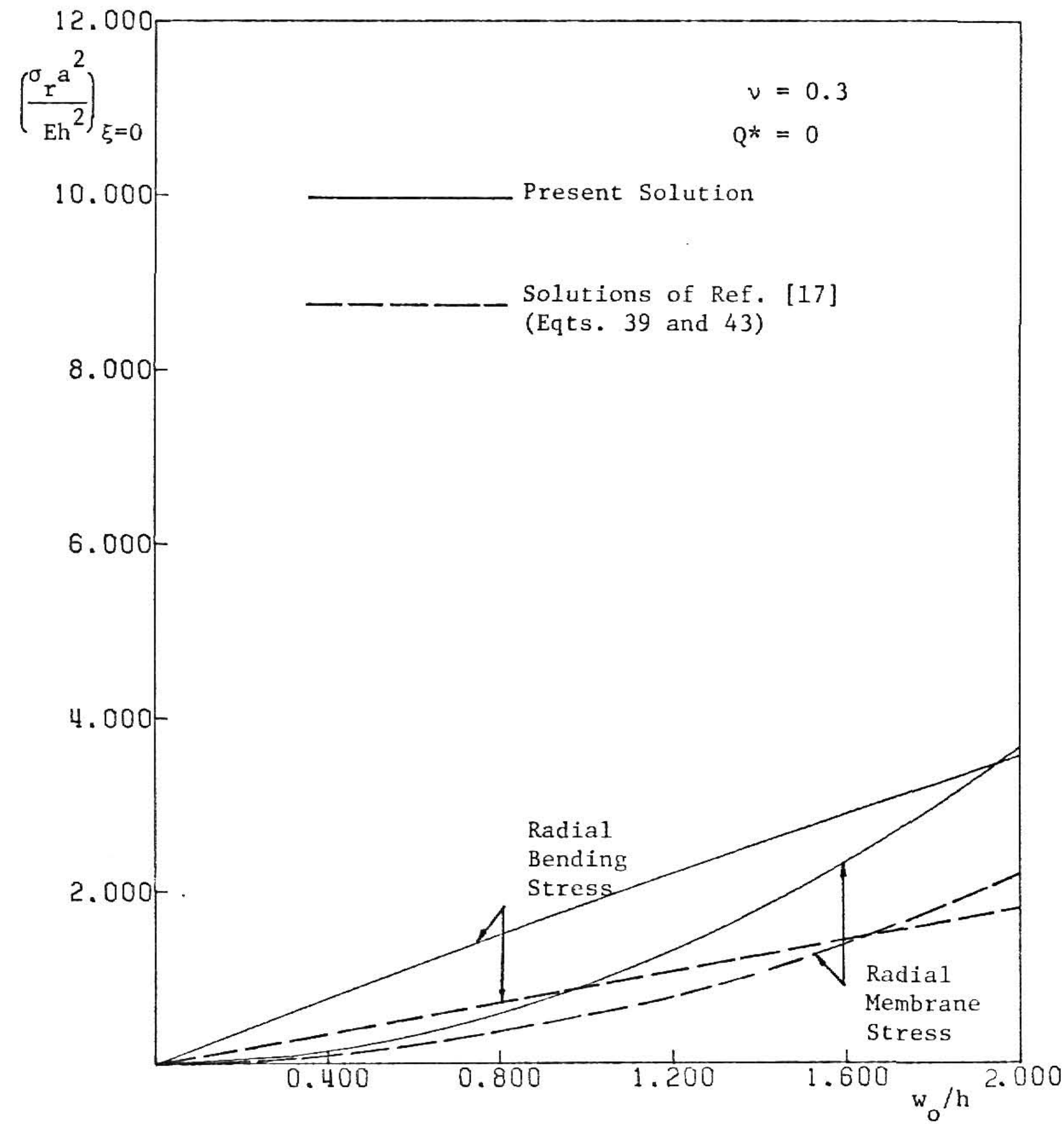

Fig. (HI-8). Radial Stresses at the Center of the Plate.

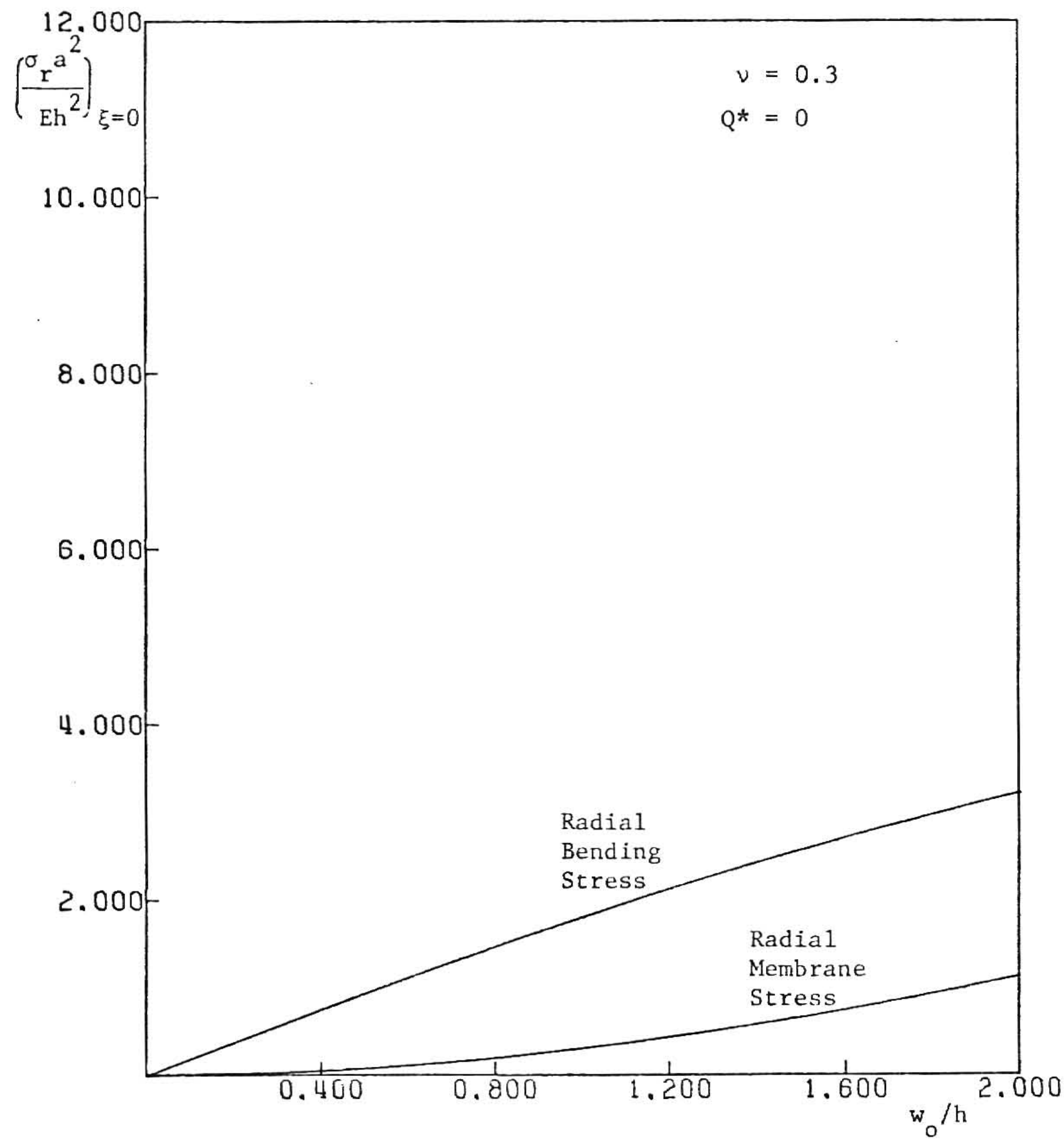

Fig. (HM-8). Radial Stresses at the Center of the Plate.

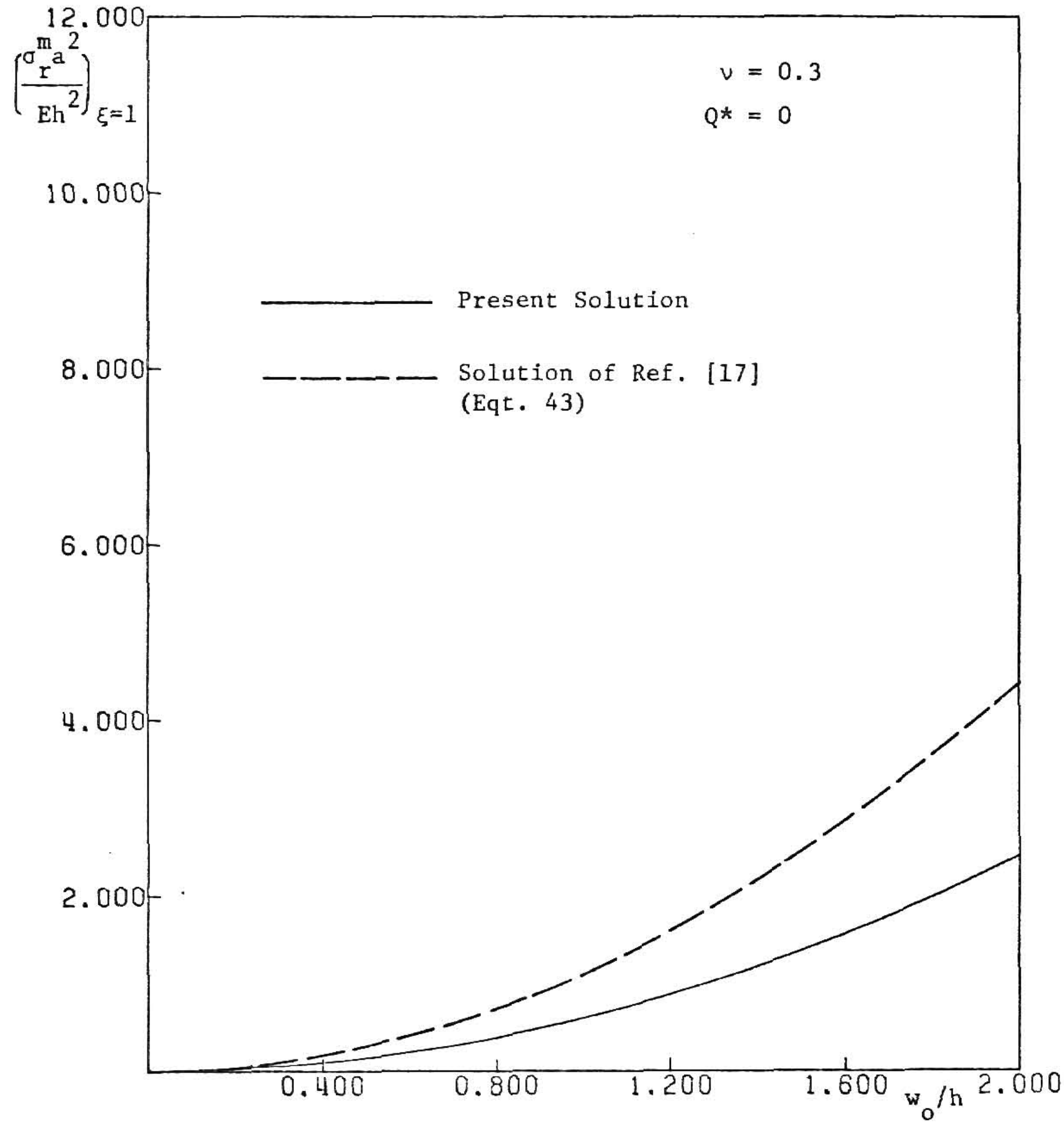

Fig. (HI-9). Radial Membrane Stress at the Edge of the Plate.

CHAPTER V. THE BERGER ASSUMPTION

In his analysis of large static deflections of plates, Berger, [16], proposed that the strain energy due to the second invariant of the middle-surface strains may be neglected. This reduces the differential equation governing the transverse displacement, w, to a linear form solvable in terms of Bessel functions. Wah [17] and Nash and Modeer [18], extended Berger's assumption to the dynamic case of vibration of plates.

In Chapter I, if the Berger assumption is applied the second term in equation (12b) drops out and we obtain the solution:

$$e_1 = \text{a constant} \tag{41}$$

In equation (12a) the second term in the brackets, $-(1-\nu)(Uw_r)_r$, also disappears and the equation takes the form

$$\nabla^4 w - \left(\frac{12e_1}{h^2}\right) \nabla^2 w + \frac{\rho h}{D} w_{tt} = \frac{q}{D}(r,t) \tag{42}$$

Expressing the condition of vanishing boundary displacement in terms of a double integral, Wah solved (41) and (42), using a modified Galerkin method, in terms of elliptic integrals.

Wah's solution results in linear patterns of bending stresses both at the center and at the edge of the plate contrary to the obvious nonlinearity of bending stresses obtained here. His values for bending and membrane stresses are superimposed on the values obtained in the present work in Figs. CI-8, CI-9, HI-8, and HI-9.

Srinivasan [20], also used the Berger assumption to obtain steady state response curves for forced vibration of immovable clamped and hinged circular plates. Using the Ritz method he reduced the nonlinear partial differential equations to nonlinear algebraic equations. His results for the special case of free vibration are similar to those in Ref. [17]. The response curves based on Ref. [20] are superimposed on those obtained in this work in Figs. C-10, H-10, C-11, and H-11. Comparison of the two sets indicates good agreement only with the immovable edge cases at low amplitudes. This observation is the dynamical parallel of Nowinski and Ohnabe's conclusion [19] for the static case of large deflection of plates.

In both cases of free and forced vibration the deviation between the present solutions for the immovable edge cases, and those of Wah and Srinivasan increases with amplitude.

Larger deviations are obtained in the hinged case as compared with the clamped case. This is because the accuracy of the approximate Berger solution decreases as the order of the differential operators appearing in the boundary conditions increases. As an example, Fig. C-10 shows that in the clamped case, with an amplitude/thickness ratio, $\alpha = 0.5$, the Berger solution deviates by roughly 2.8% (6.6%) in the immovable case, and 3.4% (16.7%) in the movable case. With $\alpha = 2.0$, the deviations become 3.7% (8.7%) in the immovable case, and 34.7% (108.3%) in the movable case. The values in the brackets belong to the corresponding deviations in the hinged case, Fig. H-10.

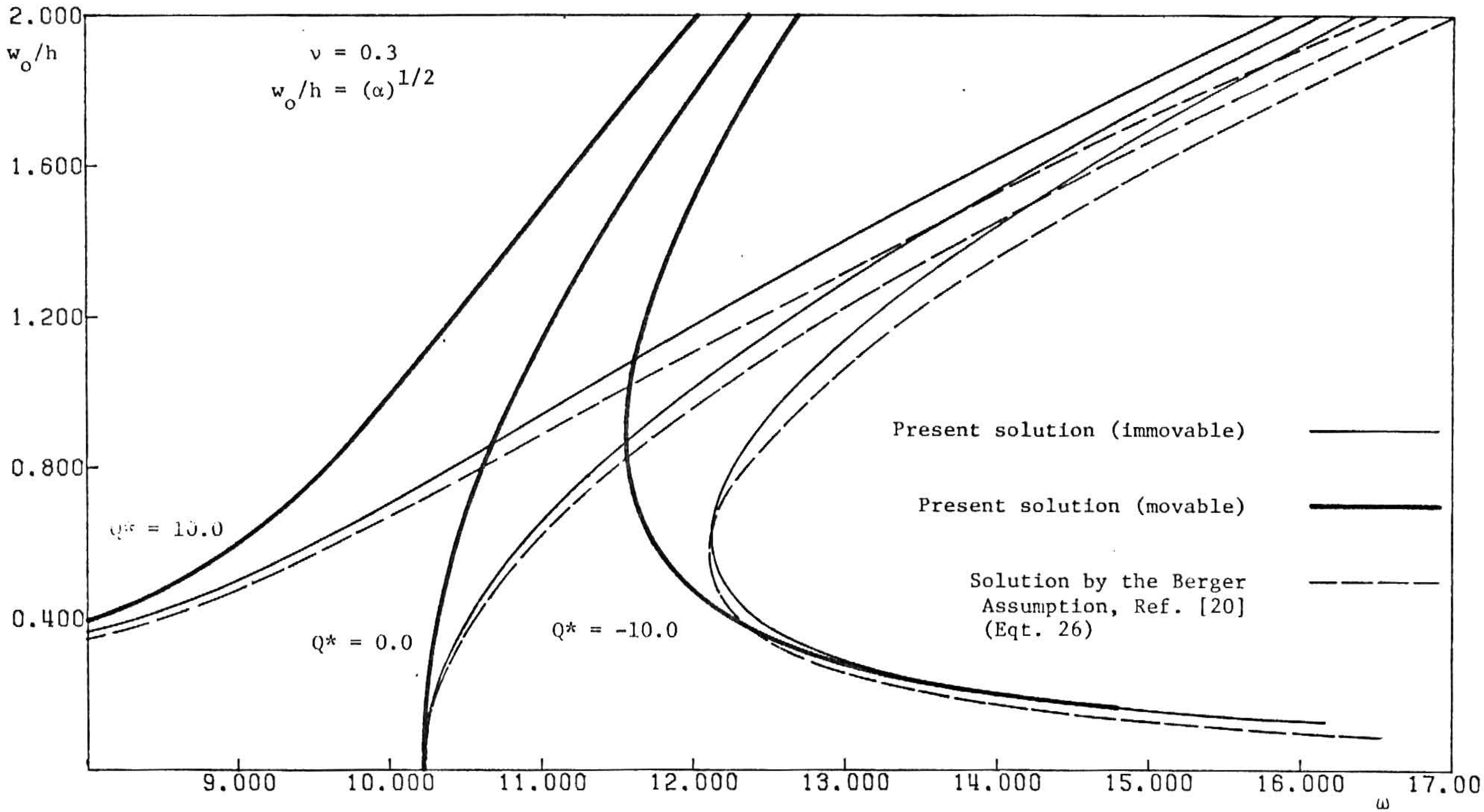

Fig. (C-10). Harmonic Response of a Clamped Plate by the Berger Assumption in Comparison with the Present Solutions.

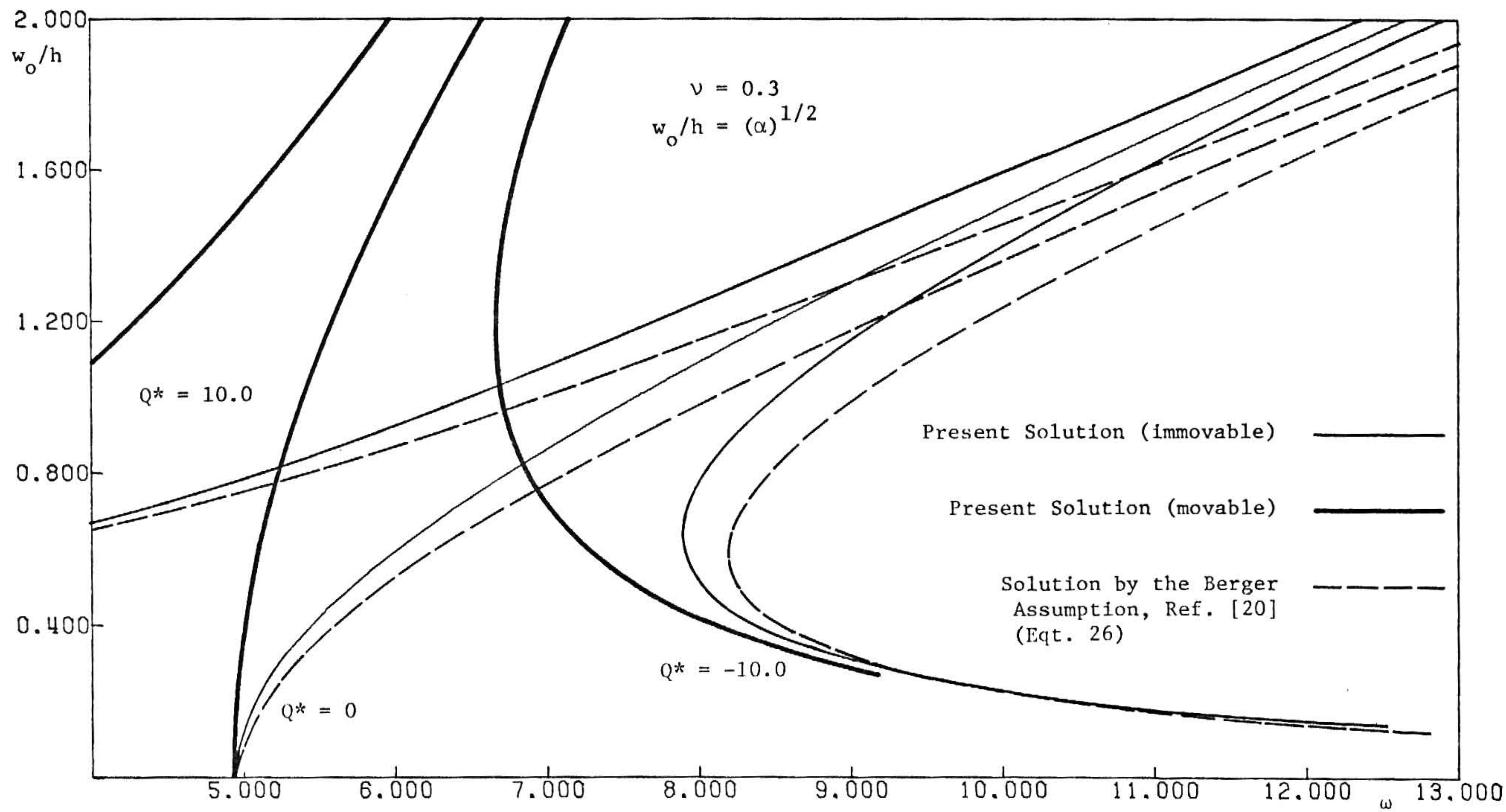

Fig. (H-10). Harmonic Response of a Hinged Plate by the Berger Assumption in Comparison with the Present Solutions.

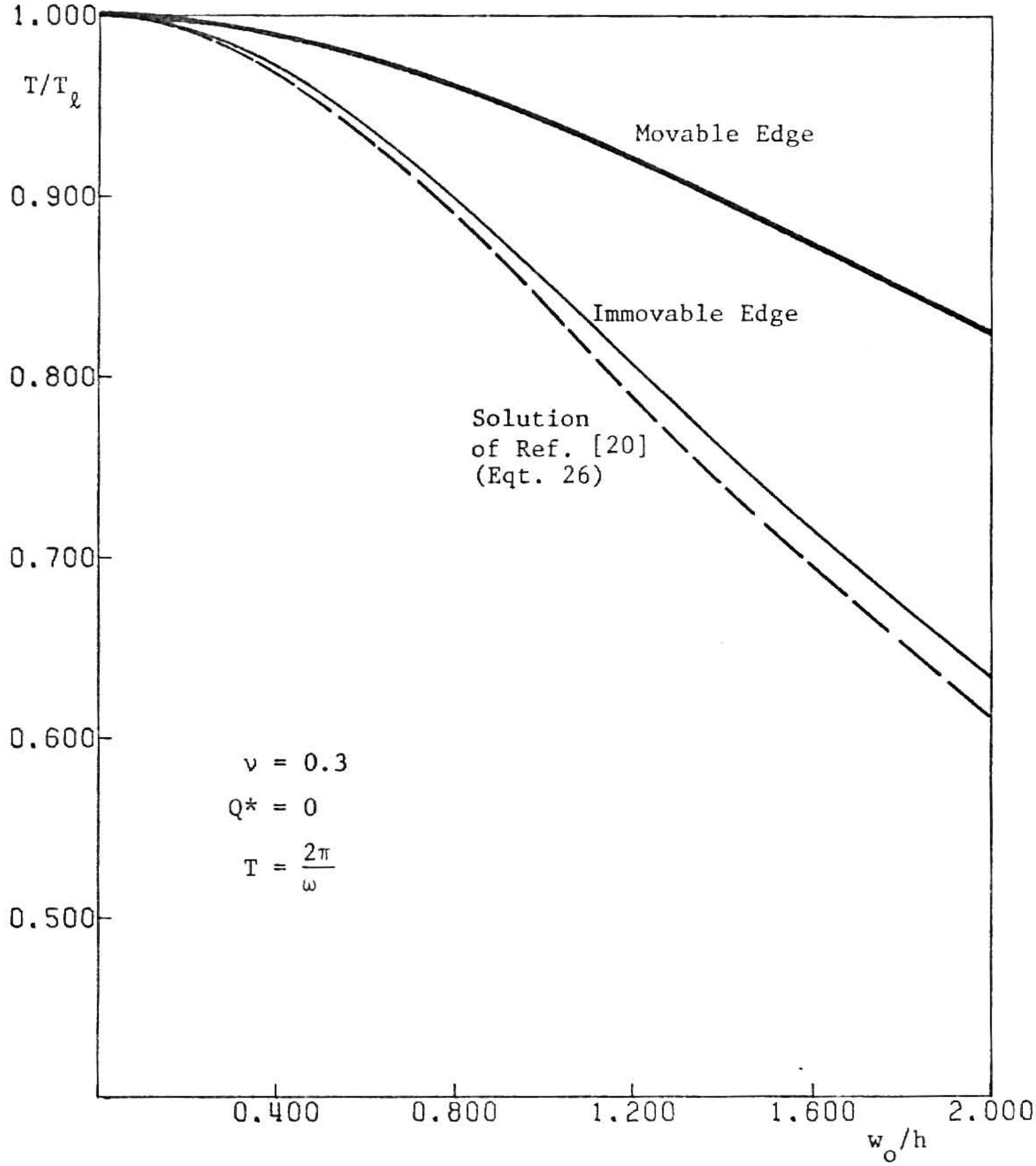

Fig. (C-11). Nonlinear Period of Free Vibration for a Clamped Circular Plate.

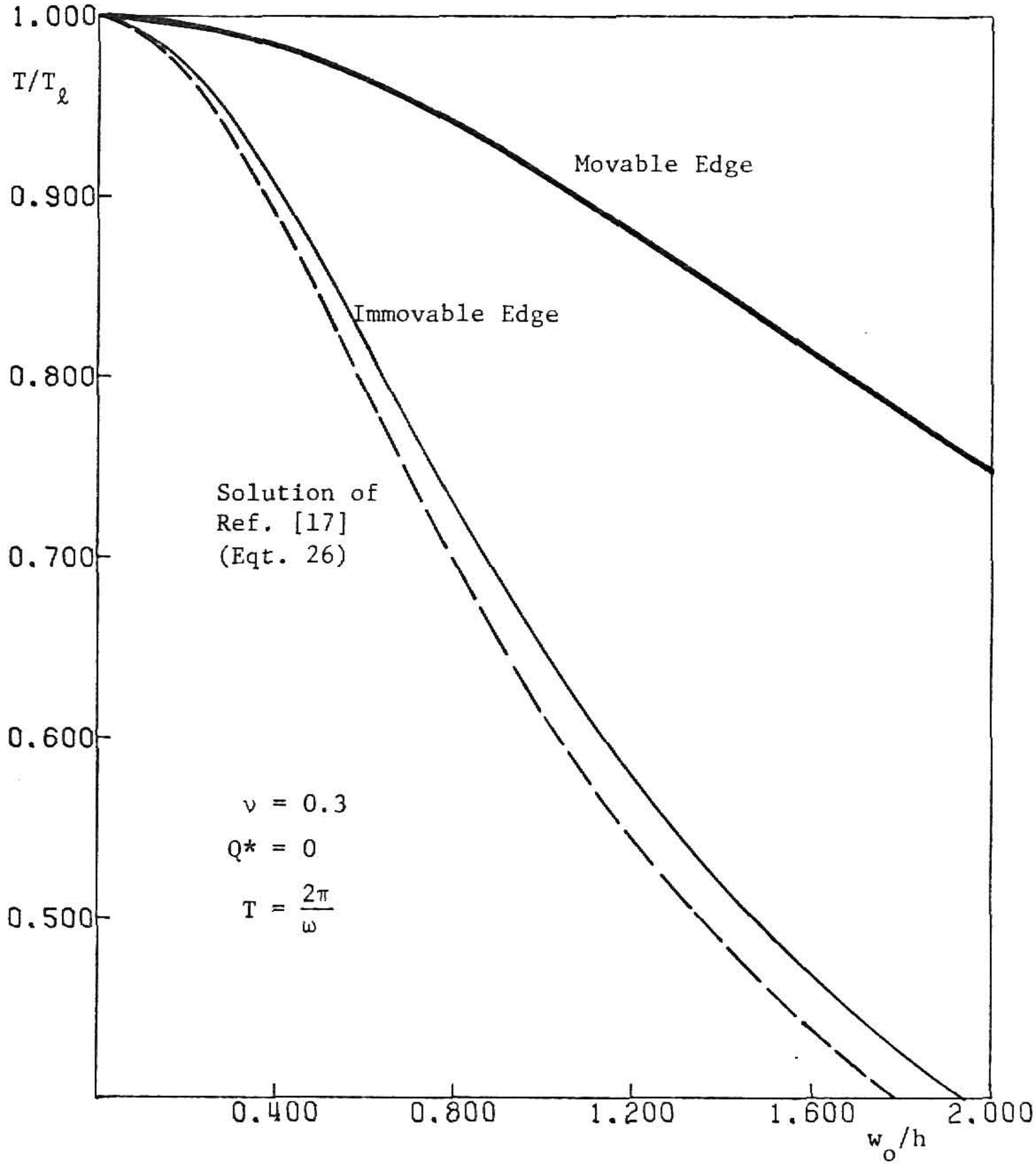

Fig. (H-11). Nonlinear Period of Free Vibration for a Hinged Circular Plate.

CHAPTER VI. CONCLUSIONS

By elimination of the time variable from the equations of motion, (16), an infinite number of degrees of freedom in the space coordinate functions is achieved. By the nature of the approximate numerical integration process the solution of the continuous system is obtained at a number of discrete points along the radius of the plate, hence reducing the number of degrees of freedom to the number of points considered.

In all the cases of edge support considered, a behavior similar to that of a hard spring is exhibited by the responses of the plate.

As noted in Ref. [2] for the clamped immovable case, the mode shape function, bending stresses, and membrane stresses are nonlinear functions of the amplitude of vibration.

Comparison of the solutions obtained here with those based on the Berger assumption [17,20], clearly indicates that, while good agreement exists for small amplitudes of vibration in the radially restrained (immovable) cases, the Berger assumption leads to inaccurate solutions in the absence of radial restraint.

Even with radial restraint the Berger assumption results in linear distributions of bending stresses contrary to the nonlinear patterns established above.

Finally, the cases of edge conditions analyzed here are really mathematical idealizations of the cases met in practice. Their importance to the designer is that they define range envelopes for the variables sought.

As an example, the normalized results of Experiments A, B, and C in Ref. [2] are plotted in Fig. (C-12). They all fall within the response envelope defined by the "clamped immovable" and the "clamped-movable" cases.

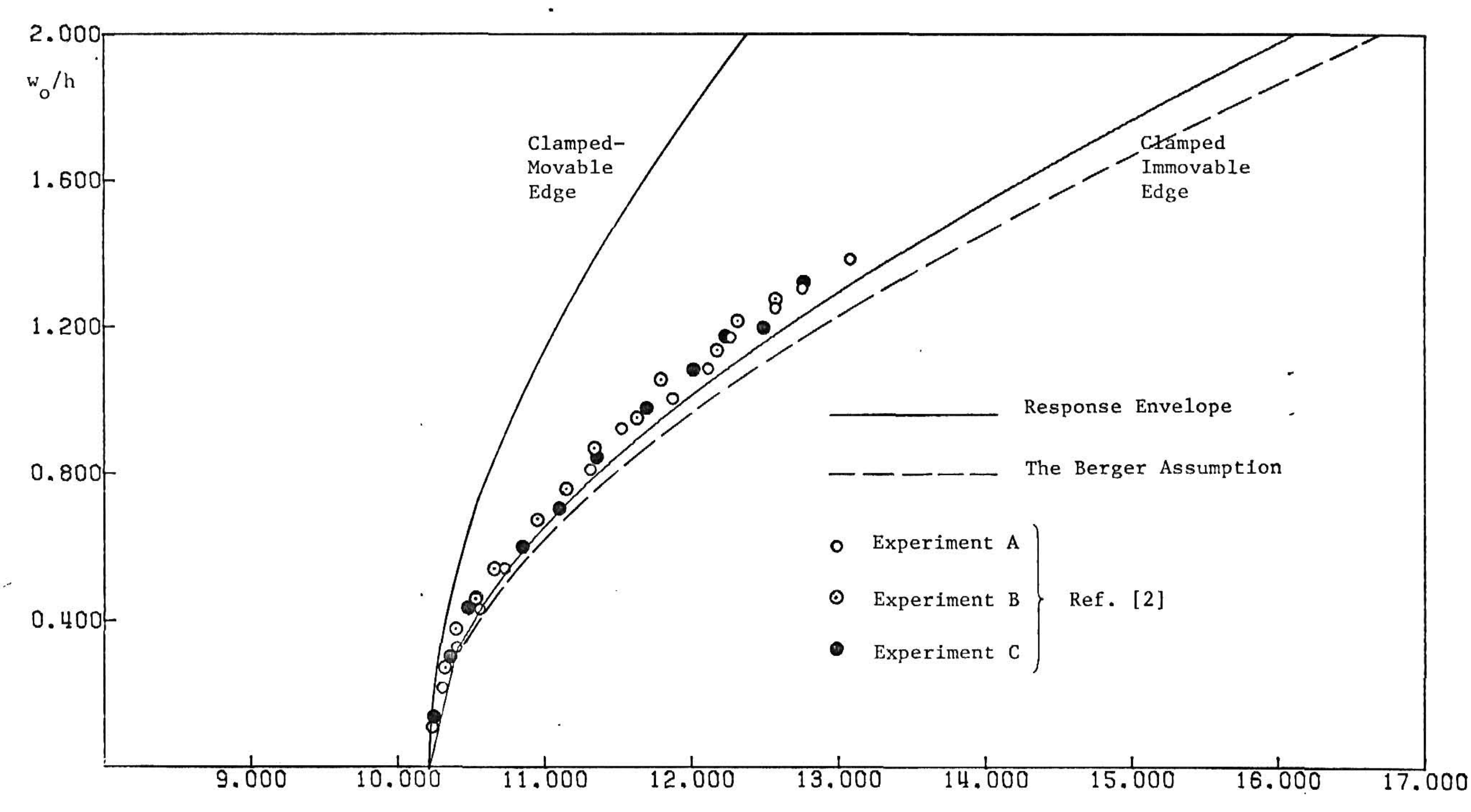

Fig. (C-12). Harmonic Response Envelope for Free Vibration of a Clamped Circular Plate.

References

1. Fung, Y. C., A First Course in Continuum Mechanics, Prentice Hall, 1969, pp. 97.

2. Sandman, B. E., Harmonic Oscillations of Circular and Annular Plates at Finite Amplitude, Ph.D. Dissertation, Kansas State University, Manhattan, Kansas, 1970.

3. Langhaar, H. L., Energy Methods in Applied Mechanics, John Wiley and Sons, Inc., 1962, pp. 159-198.

4. Meirovitch, L., Analytical Methods in Vibrations, Macmillan, 1967, pp. 42-45 and pp. 126-200.

5. Timoshenko, S. and Woinowsky-Krieger, S., Theory of Plates and Shells, Second Edition, McGraw-Hill, 1968, pp. 396-428.

6. Huang, C. L., A Direct Method for Solving Nonlinear Eigenvalue Problems, International Conference on Computational Methods in Nonlinear Mechanics, Austin, Texas, 1974.

7. Ince, E. L., Ordinary Differential Equations, Dover, New York, 1956.

8. Gear, C. W., Numerical Initial-Value Problems in Ordinary Differential Equations, Prentice Hall, 1971.

9. Hindmarsh, A. C., Numerical Solution of Ordinary Differential Equations: Lecture Notes, Report No. UCID-16558, Lawrence Livermore Laboratory, Livermore, California, June 1974.

10. Keller, H. B., Numerical Methods for Two-Point Boundary-Value Problems, Ginn Blaisdell, 1968.

11. Roberts, S. M. and Shipman, J. S., Two-Point Boundary-Value Problems: Shooting Methods, American Elsevier, 1972.

12. Apostol, T. M., Mathematical Analysis, Second Edition, Addison Wesley, 1974, pp. 93, 113, and 240-241.

13. Stoker, J. J., Nonlinear Vibrations, Interscience Publishers Inc., 1950, pp. 81-95.

14. McLachlan, N. W., Bessel Functions for Engineers, second edition, Oxford Engineering Science Series, 1961, pp. 190.

15. Leissa, A. W., Vibration of Plates, NASA SP-160, 1969, pp. 7-9.

16. Berger, H. M., A New Approach to the Analysis of Large Deflections of Plates, J. Appl. Mech., Vol. 20-22, 1955, pp. 465-472.

17. Wah, T., Vibration of Circular Plates at Large Amplitudes, J. of Engg. Mech. Div., Proc. ASCE, Vol. 89, 1963, p. 1-15.

18. Nash, W. A., and Modeer, H. R., Certain Approximate Analyses of the Nonlinear Behavior of Plates and Shallow Shells, Proc. Symp. on the Theory of Thin Elastic Shells, Int. Union of Theoretical and Applied Mechanics, North-Holland Publ. Co., 1959.

19. Nowinski, J. L. and Ohnabe, H., On Certain Inconsistencies in Berger Equations for Large Deflections of Plastic Plates, Int. J. Mech. Sci., Vol. 14, 1972, pp. 165-170.

20. Srinivasan, A. V., Nonlinear Vibrations of Beams and Plates, Int. J. Non-linear Mechanics, Vol. 1, 1966, pp. 179-191.

APPENDIX A

Computer Program for Free Vibration of a Hinged-Immovable Plate

This program is a modified form of the one given in Ref. [2] for the free vibration of a clamped-immovable plate. The modifications include: the initial values of the vector $\bar{\eta}$, the error vector, the Jacobian matrix, and the correction vector. The program is also adapted for plotting purposes.

The correspondence between the equations given in the text and the symbols appearing in the program is as follows:

$$Y(I) = z_I$$

$$Y(I + 6) = \frac{\partial z_I}{\partial \eta_1}$$

$$Y(I + 12) = \frac{\partial z_I}{\partial \eta_2} \qquad I = 1,\ldots,6$$

$$Y(I + 18) = \frac{\partial z_I}{\partial \lambda}$$

```
C      INITIAL VALUE METHOD - FREE VIBRATION OF A
C      SOLID CIRCULAR PLATE WITH A HINGED  IMMOVABLE BOUNDARY
C
C    V=POISSON'S RATIO
C    QL=NONDIMENSIONAL TRANSVERSE LOADING
C    A=AMPLITUDE PARAMETER
C    DA=INCREMENT IN A
C    P=NONLINEAR EIGENVALUE
C    H=STEP SIZE FOR NUMERICAL INTEGRATION
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION Y(24),Q(24),TP(3,3),D(6,81)
       DIMENSION C(3),LW(3),MW(3),ER(3)
       DIMENSION YV1(43,43),YV2(43,43),YV3(43,43),YV4(43,43),YV5(43,43),
      1YV6(43,43),SPV(43),SRAV(43),XV(43),YYV1(43),YYV41(43),IBUF(4000)
  112 FORMAT(5X,'AMP=',D22.14,3X,'FREQ=',D22.14,5X,'ITER=',I2)
  113 FORMAT (9X,'W',19X,'DW',18X,'DDW',17X,'DDDW')
  114 FORMAT(4D22.14)
  115 FORMAT(//9X,'F',19X,'DF')
  117 FORMAT (1H )
      V=0.3
      QL=0.0
      IK=1
      A=0.0D-0
      DA=0.1D-0
      P=4.977**2
      LL=41
      DIV=LL-1
      H=1./DIV
C     CONSTRUCT INITIAL VALUES
  500 ITER=0
      DO 9  I=1,24
    9 Y(I)=0.0D-0
      Y(1)=1.0D-0
      Y(3)=-4.6D-0
      Y(6)=0.82D-0
      Y(9)=1.0
      Y(18)=1.0
      IF(IK.EQ.1)  GO TO 600
      DO 10  I=1,6
   10 Y(I)=D(I,1)
C     X=INDEPENDENT VARIABLE
  600 X=0.0D-0
      DO 23  I=1,24
   23 Q(I)=0.0D-0
      DO 20  I=1,6
   20 D(I,1)=Y(I)
      YV1(IK,1)=D(1,1)
      YV2(IK,1)=D(2,1)
      YV3(IK,1)=D(3,1)
      YV4(IK,1)=D(4,1)
      YV5(IK,1)=D(5,1)
      YV6(IK,1)=D(6,1)
      XV(1)=X
C     RKG INTEGRATION
      DO 25  I=2,LL
      CALL RKG(X,H,Y,Q,P,A,QL,V)
      XV(I)=X
      DO 30  J=1,6
   30 D(J,I)=Y(J)
```

```
      YV1(IK,I)=D(1,I)
      YV2(IK,I)=D(2,I)
      YV3(IK,I)=D(3,I)
      YV4(IK,I)=D(4,I)
      YV5(IK,I)=D(5,I)
      YV6(IK,I)=D(6,I)
   25 CONTINUE
C     ER(I)=ERROR VECTOR FOR BOUNDARY CONDITIONS AT X=1.0
      ER(1)=D(1,LL)
      ER(2)=D(2,LL)*V +D(3,LL)
      ER(3)=-D(5,LL)*V +D(6,LL)
      DO 26  I=1,3
      DER=DABS(ER(I))
      IF(DER.GT.0.1D-5) GO TO 28
   26 CONTINUE
      GO TO 900
   28 CONTINUE
C     TP(I,J) IS THE JACOBIAN OF THE MAPPING OF INITIAL VALUES
C     TO FINAL VALUES
      TP(1,1)=Y(7)
      TP(2,1)=Y(8)*V +Y(9)
      TP(3,1)=Y(12)-Y(11)*V
      TP(1,2)=Y(13)
      TP(2,2)=Y(14)*V +Y(15)
      TP(3,2)=Y(18)-Y(17)*V
      TP(1,3)=Y(19)
      TP(2,3)=Y(20)*V +Y(21)
      TP(3,3)=Y(24)-Y(23)*V
      DET=0.0
      CALL DMINV(TP,3,DET,LW,MW)
C     C(I)=CORRECTION VECTOR
      DO 75  I=1,3
      C(I)=0.0
      DO 75  J=1,3
   75 C(I)=C(I)-TP(I,J)*ER(J)
      DO 76  I=1,6
   76 Y(I)=D(I,1)
      Y(3)=Y(3)+C(1)
      Y(6)=Y(6)+C(2)
      P=P+C(3)
      ITER=ITER+1
      DO 77  I=7,24
   77 Y(I)=0.0
      Y(9)=1.0
      Y(18)=1.0
      CALL ERRSET(207,256,-1,1)
      GO TO 600
  900 SRA=DSQRT(A)
      SP=DSQRT(P)
      WRITE(6,117)
      WRITE(6,112)  SRA,SP,ITER
      WRITE(6,117)
      WRITE(6,113)
      DO 901  J=1,LL
  901 WRITE(6,114)  (D(I,J),I=1,4)
      WRITE(6,115)
      DO 902 J=1,LL
  902 WRITE(6,114)  (D(L,J),L=5,6)
      WRITE(6,117)
      SPV(IK)=SP
```

```
      SRAV(IK)=SRA
      A=A+DA
      IK=IK+1
      IF(IK.GT.41)  GO TO 550
      GO TO 500
  550 WRITE(15) YV1,YV2,YV3,YV4,YV5,YV6,XV,SPV,SRAV
      STOP
      END
      SUBROUTINE RKG(X,H,Y,Q,P,AP,QL,V)
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(24),Q(24),DY(24),A(2)
      A(1)=0.29289321881345247
      A(2)=1.7071067811865475
      H2=0.5*H
      CALL DERIV(X,H,Y,DY,P,AP,QL,V)
      DO 13  I=1,24
      R=H2*DY(I)-Q(I)
      Y(I)=Y(I)+R
   13 Q(I)=Q(I)+3.0*R-H2*DY(I)
      X=X+H2
      DO 60  J=1,2
      CALL DERIV(X,H,Y,DY,P,AP,QL,V)
      DO 20  I=1,24
      R=A(J)* (H*DY(I)-Q(I))
      Y(I)=Y(I)+R
   20 Q(I)=Q(I)+3.0*R-A(J)*H*DY(I)
   60 CONTINUE
      X=X+H2
      CALL DERIV(X,H,Y,DY,P,AP,QL,V)
      DO 26  I=1,24
      R=(H*DY(I)-2.0*Q(I))/6.0
      Y(I)=Y(I)+R
   26 Q(I)=Q(I)+3.0*R-H2*DY(I)
      RETURN
      END
      SUBROUTINE DERIV(X,H,Y,DY,P,AP,QL,V)
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(24),DY(24)
      VV=1.-V**2
      DO 10  I=1,3
   10 DY(I)=Y(I+1)
      DY(5)=Y(6)
      DO 12  I=7,9
   12 DY(I)=Y(I+1)
      DY(11)=Y(12)
      DO 14  I=13,15
   14 DY(I)=Y(I+1)
      DY(17)=Y(18)
      DO 16  I=19,21
   16 DY(I)=Y(I+1)
      DY(23)=Y(24)
      IF(X.GE.0.1D-2) GO TO 50
      DY(4)=(3.*P*Y(1))/8.+(27.*AP*Y(3)*Y(6))/4.
      DY(10)=(3.*P*Y(7))/8.+(27.*AP*(Y(9)*Y(6)+Y(12)*Y(3)))/4.
      DY(16)=(3.*P*Y(13))/8.+(27.*AP*(Y(3)*Y(18)+Y(6)*Y(15)))/4.
      DY(22)=(3.*Y(1))/8.+(3.*P*Y(19))/3.
      DY(22)=DY(22)+(27.*AP*(Y(21)*Y(6)+Y(24)*Y(3)))/4.
      DY(6)=0.0
      DY(12)=0.0
      DY(18)=0.0
```

```
      DY(24)=0.0
      GO TO 70
   50 DY(4)=-2.*(Y(4)/X)+Y(3)/(X**2)-Y(2)/(X**3)+P*Y(1)
      DY(4)=DY(4)+9.*AP*(Y(3)*Y(5)+Y(2)*Y(6))/X
      DY(6)=-Y(6)/X+Y(5)/(X**2)-(VV*(Y(2)**2))/(2.*X)
      DY(10)=-2.*(Y(10)/X)+Y(9)/(X**2)-Y(8)/(X**3)+P*Y(7)
     1+9.*AP*(Y(5)*Y(9)+Y(3)*Y(11)+Y(2)*Y(12)+Y(6)*Y(8))/X
      DY(12)=-Y(12)/X+Y(11)/X**2-(VV*Y(2)*Y(8))/X
      DY(16)=-2.*(Y(16)/X)+Y(15)/X**2-Y(14)/X**3+P*Y(13)
     1+9.*AP*(Y(3)*Y(17)+Y(5)*Y(15)+Y(2)*Y(18)+Y(6)*Y(14))/X
      DY(18)=-Y(18)/X+Y(17)/X**2-(VV*Y(2)*Y(14))/X
      DY(22)=-2.*(Y(22)/X)+Y(21)/X**2-Y(20)/X**3+P*Y(19)+Y(1)
     1+9.*AP*(Y(3)*Y(23)+Y(5)*Y(21)+Y(2)*Y(24)+Y(6)*Y(20))/X
      DY(24)=-Y(24)/X+Y(23)/X**2-(VV*Y(2)*Y(20))/X
   70 RETURN
      END
```

APPENDIX B

Computer Plotting Program for Free Vibration of a Hinged-Immovable Plate

This program uses the KSU CalComp model 663 digital incremental drum plotter facility.

The 9-track tape created by the program in Appendix A is read by the present program. The explanation of the symbols particular to this program is as follows:

CRM = Radial membrane stress at the center of the plate

CRB = Radial bending stress at the center of the plate

ERM = Radial membrane stress at the edge of the plate

ERB = Radial bending stress at the edge of the plate

TM045/RM045 = Circumferential/radial/membrane stress for α = 0.2

TM1/RM1 = Circumferential/radial/membrane stress for α = 1.0

TM2/RM2 = Circumferential/radial/membrane stress for α = 4.0

TB045/RB045 = Circumferential/radial/bending stress for α = 0.2

TB1/RB1 = Circumferential/radial/bending stress for α = 1.0

TB2/RB2 = Circumferential/radial/bending stress for α = 4.0

```
C    FINITE-AMPLITUDE VIBRATION OF CLAMPED AND HINGED
C    CIRCULAR PLATES
C    THE CASE OF A HINGED-IMMOVABLE EDGE
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION YV1(43,43),YV2(43,43),YV3(43,43),YV4(43,43),YV5(43,43),
     1YV6(43,43),SPV(43),SRAV(43),XV(43),YYV1(43),YYV41(43),IBUF(4000)
      DIMENSION SPVEC(43),CRM(43),CRB(43),ERM(43),ERB(43)
      DIMENSION RM045(43),RM1(43),RM2(43),RB045(43),RB1(43),RB2(43)
      DIMENSION TM045(43),TM1(43),TM2(43),TB045(43),TB1(43),TB2(43)
      READ(15)  YV1,YV2,YV3,YV4,YV5,YV6,XV,SPV,SRAV
C
      V=0.3
      VV=1.-V**2
      DO 1  I=1,41
      SPVEC(I)=SPV(1)/SPV(I)
      CRM(I)=DABS(SRAV(I)*SRAV(I)*YV6(I,1)/VV)
      CRB(I)=DABS(SRAV(I)*YV3(I,1)/(2.-2.*V))
      ERM(I) =DABS(SRAV(I)*SRAV(I)*YV5(I,41)/VV)
      ERB(I)  =DABS(SRAV(I)*(YV3(I,41)+V*YV2(I,41))/(2.*VV))
    1 CONTINUE
C
      DO 2  I=1,41
      YYV1(I)=YV1(1,I)
      YYV41(I)=YV1(41,I)
      TM045(I) =       SRAV( 3)*SRAV( 3)*YV6( 3,I)/VV
      TM1(I)   =       SRAV(11)*SRAV(11)*YV6(11,I)/VV
      TM2(I)   =       SRAV(41)*SRAV(41)*YV6(41,I)/VV
      IF(I.NE.1) GO TO 3
      RM045(I) =       SRAV( 3)*SRAV( 3)*YV6( 3,I)/VV
      RM1(I)   =       SRAV(11)*SRAV(11)*YV6(11,I)/VV
      RM2(I)   =       SRAV(41)*SRAV(41)*YV6(41,I)/VV
      RB045(I) =     SRAV( 3)*YV3( 3,I)/(2.-2.*V)
      RB1(I)   =     SRAV(11)*YV3(11,I)/(2.-2.*V)
      RB2(I)   =     SRAV(41)*YV3(41,I)/(2.-2.*V)
      TB045(I) =     SRAV( 3)*YV3( 3,I)/(2.-2.*V)
      TB1(I)   =     SRAV(11)*YV3(11,I)/(2.-2.*V)
      TB2(I)   =     SRAV(41)*YV3(41,I)/(2.-2.*V)
      GO TO 2
    3 RM045(I) = SRAV( 3)*SRAV( 3)*YV5( 3,I)/(VV*XV(I))
      RM1(I)   = SRAV(11)*SRAV(11)*YV5(11,I)/(VV*XV(I))
      RM2(I)   = SRAV(41)*SRAV(41)*YV5(41,I)/(VV*XV(I))
      RB045(I) =     SRAV( 3)*(YV3( 3,I)+V*YV2( 3,I)/XV(I))/(2.*VV)
      RB1(I)   =     SRAV(11)*(YV3(11,I)+V*YV2(11,I)/XV(I))/(2.*VV)
      RB2(I)   =     SRAV(41)*(YV3(41,I)+V*YV2(41,I)/XV(I))/(2.*VV)
      TB045(I) =     SRAV( 3)*(YV3( 3,I)*V+YV2( 3,I)/XV(I))/(2.*VV)
      TB1(I)   =     SRAV(11)*(YV3(11,I)*V+YV2(11,I)/XV(I))/(2.*VV)
      TB2(I)   =     SRAV(41)*(YV3(41,I)*V+YV2(41,I)/XV(I))/(2.*VV)
    2 CONTINUE
C
      CALL LIMITS(150.,11.0,25,6,+3)
      CALL PLOTS(IBUF,4000)
      CALL PLOT(0.,-11.,23)
C    FIG.(HI-2)  HARMONIC RESPONSE OF FREE VIBRATION
      CALL PLOT(2.,3.,23)
      SPV(42)=4.
      SPV(43)=1.
      CALL DAXIS(0.,0.,9.,0.,1.,+1)
      CALL SAXIS(4.,1.0,3,+1,-1,'SP',0)
      SRAV(42)=0.
      SRAV(43)=0.4
```

```
      CALL DAXIS(0.,0.,5.,90.,1.0,-1)
      CALL SAXIS(0.,0.4,3,3,1,'SRA',0)
      CALL PLOT(0.,5.,3)
      CALL PLOT(9.,5.,2)
      CALL PLOT(9.,0.,2)
      CALL LINE(SPV ,SRAV ,41,2,0,0)
C    FIG.(HI-3)  SHAPE FUNCTION
      CALL PLOT(14.,0.,-3)
      XV(42)=0.
      XV(43)=0.2
      CALL DAXIS(0.,0.,5.,0.,1.,+1)
      CALL SAXIS(0.,0.2,3,+1,-1,'X',0)
      YYV1(42)=0.
      YYV1(43)=0.2
      YYV41(42)=0.
      YYV41(43)=0.2
      CALL DAXIS(0.,0.,5.,90.,1.,-1)
      CALL SAXIS(0.,0.2,3,3,1,'Y(1)',0)
      CALL PLOT(0.,5.,3)
      CALL PLOT(5.,5.,2)
      CALL PLOT(5.,0.,2)
      CALL LINE(XV   ,YYV1 ,41,2,0,0)
      CALL LINE(XV   ,YYV41,41,2,0,0)
C    FIG.(HI-4)  RADIAL MEMBRANE STRESS
      CALL PLOT(10.,0.,-3)
      CALL DAXIS(0.,0.,5.,0.,1.,+1)
      CALL SAXIS(0.,0.2,3,+1,-1,'X',0)
      RM045(42)=0.
      RM045(43)=1.
      RM1(42)=0.
      RM1(43)=1.
      RM2(42)=0.
      RM2(43)=1.
      CALL DAXIS(0.,0.,5.,90.,1.,-1)
      CALL SAXIS(0.,1.0,3,3,1,'RADIAL MEMB. STRESS',0)
      CALL PLOT(0.,5.,3)
      CALL PLOT(5.,5.,2)
      CALL PLOT(5.,0.,2)
      CALL LINE(XV,RM045,41,2,0,0)
      CALL LINE(XV,RM1  ,41,2,0,0)
      CALL LINE(XV,RM2  ,41,2,0,0)
C    FIG.(HI-5)  RADIAL BENDING STRESS
      CALL PLOT(10.,0.,-3)
      CALL DAXIS(0.,3.,5.,0.,1.,+1)
      CALL SAXIS(0.,0.2,3,+1,-1,'X',0)
      RB045(42)=-12.
      RB045(43)=4.
      RB1(42)=-12.
      RB1(43)=4.
      RB2(42)=-12.
      RB2(43)=4.
      CALL DAXIS(0.,0.,6.,90.,1.,-1)
      CALL SAXIS(-12.,4.,3,3,1,'RADIAL BEND. STRESS',0)
      CALL PLOT(0.,6.,3)
      CALL PLOT(5.,6.,2)
      CALL PLOT(5.,0.,2)
      CALL PLOT(0.,0.,2)
      CALL LINE(XV,RB045,41,2,0,0)
      CALL LINE(XV,RB1  ,41,2,0,0)
      CALL LINE(XV,RB2  ,41,2,0,0)
```

```
C    FIG.(HI-6)  CIRCUMFERENTIAL MEMBRANE STRESS
      CALL PLOT(10.,0.,-3)
      CALL DAXIS(0.,0.,5.,0.,1.,+1)
      CALL SAXIS(0.,0.2,3,+1,-1,'X',0)
      TM045(42)=0.
      TM045(43)=1.
      TM1(42)=0.
      TM1(43)=1.
      TM2(42)=0.
      TM2(43)=1.
      CALL DAXIS(0.,0.,5.,90.,1.,-1)
      CALL SAXIS(0.,1.0,3,3,1,'CIRCUM MEMB. STRESS',0)
      CALL PLOT(0.,5.,3)
      CALL PLOT(5.,5.,2)
      CALL PLOT(5.,0.,2)
      CALL LINE(XV,TM045,41,2,0,0)
      CALL LINE(XV,TM1   ,41,2,0,0)
      CALL LINE(XV,TM2   ,41,2,0,0)
C    FIG.(HI-7)  CIRCUMFERENTIAL BENDING STRESS
      CALL PLOT(10.,0.,-3)
      CALL DAXIS(0.,3.,5.,0.,1.,+1)
      CALL SAXIS(0.,0.2,3,+1,-1,'X',0)
      TB045(42)=-12.
      TB045(43)=4.
      TB1(42)=-12.
      TB1(43)=4.
      TB2(42)=-12.
      TB2(43)=4.
      CALL DAXIS(0.,0.,6.,90.,1.,-1)
      CALL SAXIS(-12.,4.,3,3,1,'CIRCUM BEND. STRESS',0)
      CALL PLOT(0.,6.,3)
      CALL PLOT(5.,6.,2)
      CALL PLOT(5.,0.,2)
      CALL PLOT(0.,0.,2)
      CALL LINE(XV,TB045,41,2,0,0)
      CALL LINE(XV,TB1   ,41,2,0,0)
      CALL LINE(XV,TB2   ,41,2,0,0)
C    FIG.(HI-8)  RADIAL STRESSES AT THE CENTER OF THE PLATE
      CALL PLOT(10.,0.,-3)
      CALL DAXIS(0.,0.,5.,0.,1.,+1)
      CALL SAXIS(0.,0.4,3,+1,-1,'SRA',0)
      CRM(42)=0.
      CRM(43)=2.
      CRB(42)=0.
      CRB(43)=2.
      CALL DAXIS(0.,0.,6.,90.,1.,-1)
      CALL SAXIS(0.,2.0,3,3,1,'CENTER STRESSES',0)
      CALL PLOT(0.,6.,3)
      CALL PLOT(5.,6.,2)
      CALL PLOT(5.,0.,2)
      CALL LINE(SRAV,CRM   ,41,2,0,0)
      CALL LINE(SRAV,CRB   ,41,2,0,0)
C    FIG.(HI-9)  RADIAL MEMBRANE STRESS AT THE EDGE OF THE PLATE
      CALL PLOT(10.,0.,-3)
      CALL DAXIS(0.,0.,5.,0.,1.,+1)
      CALL SAXIS(0.,0.4,3,+1,-1,'SRA',0)
      ERM(42)=0.
      ERM(43)=2.
      ERB(42)=0.
      ERB(43)=2.
```

```
      CALL DAXIS(0.,0.,6.,90.,1.,-1)
      CALL SAXIS(0.,2.0,3,3,1,'EDGE STRESSES',0)
      CALL PLOT(0.,6.,3)
      CALL PLOT(5.,6.,2)
      CALL PLOT(5.,0.,2)
      CALL LINE(SRAV, ERM ,41,2,0,0)
C   FIG.(H-11)  NONLINEAR PERIOD OF FREE VIBRATION
      CALL PLOT(10.,0.,-3)
      CALL DAXIS(0.,0.,5.,0.,1.,+1)
      CALL SAXIS(0.,0.4,3,+1,-1,'SRA',0)
      SPVEC(42)=0.4
      SPVEC(43)=0.1
      CALL DAXIS(0.,0.,6.,90.,1.,-1)
      CALL SAXIS(0.4,0.1,3,3,1,'RELATIVE FREQUENCY',0)
      CALL PLOT(0.,6.,3)
      CALL PLOT(5.,6.,2)
      CALL PLOT(5.,0.,2)
      CALL LINE(SRAV,SPVEC,41,2,0,0)
C
      CALL PLOT(0.,0.,999)
      STOP
      END
```

ACKNOWLEDGEMENT

In recognition of his continual support and guidance, I would like to express my gratitude to Professor Chi-lung Huang, my major advisor.

Thanks are also due to the other members of my committee: Professor Philip G. Kirmser, Professor Frank J. McCormick, and Professor Hugh S. Walker.

I would also like to thank the Iraqi Ministry of Higher Education and Scientific Research for granting me a state scholarship.

I am grateful to Professor P. G. Kirmser, the then head of the Applied Mechanics Department and Professor P. L. Miller, head of the Mechanical Engineering Department for respectively arranging and extending my graduate research assistantship up to the award of the scholarship.

FINITE-AMPLITUDE VIBRATION OF CLAMPED AND SIMPLY-SUPPORTED CIRCULAR PLATES

by

IBRAHIM MAHDI AL-KHATTAT

B.Sc., University of Wales (Britain), 1962

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1976

ABSTRACT

The problem of finite-amplitude, axisymmetric free and forced vibration of a clamped radially-immovable circular plate is extended to other edge conditions. The cases of clamped-movable, hinged-immovable, and hinged-movable are solved by the Method of Huang and Sandman. A Kantorovich averaging technique is applied to convert the nonlinear boundary-value problem into the corresponding eigenvalue problem by elimination of the time variable. Then by a Newton-Raphson iteration scheme, and the concept of analytical continuation, the solution to the nonlinear eigenvalue problem for free vibration is obtained in the form of a discrete one-parameter family of solutions to the related initial-value problem. Also by analytical continuation, the solution is extended to the case of forced vibration.

The hard-spring behavior and the nonlinearity of bending and membrane stresses of the clamped-immovable plate, are also exhibited in the other three cases. It is seen that removal of radial restraint causes drastic changes in the plate responses and the patterns of membrane stresses.

Comparison with solutions which use the Berger assumption reveals the unsuitability of the assumption when the plate is not radially restrained. Even for the radially-restrained cases the accuracy of such solutions diminish as the amplitude of vibration increases. The linear bending stress patterns produced by these solutions are in obvious contradiction with the nonlinear distributions obtained here.