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HODGE-TYPE DECOMPOSITION IN THE HOMOLOGY OF LONG KNOTS

VICTOR TURCHIN

ABSTRACT. The paper describes a natural splitting in the rational homology and homotopy of the spaces of long knots. This decomposition presumably arises from the cabling maps in the same way as a natural decomposition in the homology of loop spaces arises from power maps. The generating function for the Euler characteristics of the terms of this splitting is presented. Based on this generating function it is shown that both the homology and homotopy ranks of the spaces in question grow at least exponentially. Using natural graph-complexes one shows that this splitting on the level of the bialgebra of chord diagrams is exactly the splitting defined earlier by Dr. Bar-Natan. The Appendix presents tables of computer calculations of the Euler characteristics. These computations give a certain optimism that the Vassiliev invariants of order > 20 can distinguish knots from their inverses.

CONTENTS

Plan of the paper. Main results

The paper is divided into 3 parts. The first part is introductive. It describes some general facts and constructions about the knot spaces we study. We also define there a convenient terminology for the sequel. The only new thing in this part is the definition of the Hodge decomposition in the rational homology and homotopy of knot spaces. The second part introduces natural graph-complexes computing the rational homology and homotopy of knot spaces together with their Hodge splitting. This construction generalizes an earlier construction of Bar-Natan that describes the primitives of the bialgebra of chord diagrams. To recall the latter bialgebra is intimately related to the Vassiliev knot invariants, which are also called invariants of finite type. The main result of this part is Theorem 8.2. We also formulate there Conjecture 12.1 that gives one more motivation for the study of the Hodge decomposition. It says that the Hodge decomposition of one-dimensional knots encodes information about the homology and homotopy of higher dimensional knots. Part 3 is more computational. Its main result is Theorem 13.1 that describes the generating function of the Euler characteristics of the terms of the Hodge splitting. Based on the formula for the generating function it is shown that the ranks of the rational homology and homotopy of the spaces of long knots grow at least exponentially, see Theorems 17.1, 17.2, 17.4. In the Appendix we present tables of the Euler characteristics of this splitting in the rational homology and rational homotopy.

Part 1. **Introduction**

1. Spaces of long knots modulo immersions

Denote by Emb_d, $d \geq 3$, the space of long knots, i.e. the space of smooth non-singular embeddings $\mathbb{R}^1 \hookrightarrow \mathbb{R}^d$ coinciding with a fixed linear embedding $t \mapsto (t, 0, 0, \ldots, 0)$ outside a compact subset of $\mathbb{R}^1$. Similarly Imm_d is the space of long immersions. The homotopy fiber of the inclusion $Emb_d \hookrightarrow Imm_d$ will be denoted by $\overline{Emb}_d$. This *space $\overline{Emb}_d$ of long knots modulo immersion* and its rational homology and homotopy will be the object of our study. The homology

$H_*(\overline{Emb}_d, \mathbb{Q})$, $d \geq 4$, is the Hochschild homology of the Poisson algebras operad [24, 39, 40]. On the other hand $\overline{Emb}_d$ is homotopy equivalent to the product [37, Proposition 5.17]:

$$\overline{Emb}_d \simeq Emb_d \times \Omega^2 S^{d-1}.$$

So, there is no much difference between the homology/homotopy of $\overline{Emb}_d$ and of Emb_d.

Proposition 1.1. *The spaces $\overline{Emb}_d$, $d \geq 3$, are acted on by the operad C_2 of little squares.*

Sketch of the proof. Consider the space of *framed* long knots Emb_d^{framed}. The points of this space are long knots in $\mathbb{R}^d$ endowed with a framing (trivialization of the normal bundle). The framed long knot should coincide with the fixed framed linear embedding (endowed with a standard constant framing) outside a compact subset of $\mathbb{R}^1$. Similarly one defines the space Imm_d^{framed} of framed long immersions. R. Budney constructed a model for the spaces Emb_d^{framed}, $d \geq 3$, that has a natural C_2-action [7]. His construction can be easily generalized to the space Imm_d^{framed}. The inclusion $Emb_d^{framed} \hookrightarrow Imm_d^{framed}$ turns out to be a map of C_2-algebras. But the homotopy fiber of this inclusion is homotopy equivalent to $\overline{Emb}_d$, which implies the result. $\square$

In case $d \geq 4$, there is a different construction of such an action due to D. Sinha [37]. It is an open question whether these two C_2-actions are equivalent.

Corollary 1.2. *For any field of coefficients $\mathbb{K}$ the homology $H_*(\overline{Emb}_d, \mathbb{K})$ is a graded bicommutative bialgebra.*

Corollary 1.3. *Since the spaces $\overline{Emb}_d$, $d \geq 4$, are connected, then for a field $\mathbb{K}$ of characteristics zero the homology $H_*(\overline{Emb}_d, \mathbb{K})$ is a graded polynomial bialgebra generated by $\pi_*(\overline{Emb}_d) \otimes \mathbb{K}$.*

2. COSIMPLICIAL MODEL FOR $\overline{Emb}_d$

For any $d \geq 4$, D. Sinha defined a cosimplicial model $C_d^{\bullet} = \{C_d^n \,|\, n \geq 0\}$, whose homotopy totalization is weakly homotopy equivalent to $\overline{Emb}_d$ [36]. The n-th stage C_d^n of this model is an appropriate compactification of the configuration space of n distinct points labeled by $1, 2, \ldots, n$ in $I \times \mathbb{R}^{d-1} = [0,1] \times \mathbb{R}^{d-1}$. The space C_d^n can also be viewed as (a compactification of) the space of bipointed embeddings $\{0, 1, \ldots, n, n+1\} \hookrightarrow I \times \mathbb{R}^{d-1}$ sending 0 to $(0, \bar{0})$ and $n+1$ to $(1, \bar{0})$. The codegeneracy $s_i \colon C_d^n \to C_d^{n-1}$, $i = 1 \ldots n$, is the forgetting of the i-th point in configuration, the coface $d_i \colon C_d^n \to C_d^{n+1}$, $i = 0 \ldots n+1$, is the doubling of the i-th point in the direction of the vector $(1, \bar{0})$. To be precise d_i sends C_d^n into a stratum of the compactified configuration space C_d^{n+1} with points i and $i+1$ collided. For more details, see [36].

The homology $\{H_*(C_d^n, \mathbb{K}), n \geq 0\}$, and homotopy $\{\pi_*(C_d^n) \otimes \mathbb{K}, n \geq 0\}$ form respectively a cosimplicial coalgebra that will be denoted by $A_d^{\bullet}$:

$$A_d^{\bullet} = \{A_d^n, n \geq 0\} = \{H_*(C_d^n, \mathbb{K}), n \geq 0\},$$

and a cosimplicial Lie algebra

$$L_d^{\bullet} = \{C_d^n, n \geq 0\} = \{\pi_*(C_d^n) \otimes \mathbb{K}, n \geq 0\}.$$

It will be usually assumed that $\mathbb{K} = \mathbb{Q}$.

The cosimplicial model $C_d^{\bullet}$ for $\overline{Emb}_d$ defines the Bousfield-Kan homology and homotopy spectral sequences, whose first term in the homological case is

$$E^1_{-p,*} = H_*^{Norm}(C_d^p, \mathbb{K}) = NA_d^p,$$

and in the homotopy case:
$$\mathcal{E}^1_{-p,*} = \pi_*^{Norm}(C_d{}^p) \otimes \mathbb{K} = NL_d{}^p.$$
Where the normalized part $H_*^{Norm}(C_d{}^p, \mathbb{K}) = NA_d{}^p$, $\pi_*^{Norm}(C_d{}^n) \otimes \mathbb{K} = NL_d{}^p$ is the intersection of kernels of degeneracies: $\bigcap_{i=1}^{p} \ker s_{i*}$.

Theorem 2.1 ([1, 23, 24]). *For a field $\mathbb{K}$ of characteristics zero and for $d \geq 4$, both the homology and homotopy Bousfield-Kan spectral sequences associated with $C_d{}^\bullet$ collapse at the second term:*
$$E^2_{*,*} = E^\infty_{*,*} = H_*(\overline{Emb}_d, \mathbb{K}), \qquad \mathcal{E}^2_{*,*} = \mathcal{E}^\infty_{*,*} = \pi_*(\overline{Emb}_d) \otimes \mathbb{K}.$$

This theorem means that rationally the homology and homotopy of $\overline{Emb}_d$ are computed by the normalized complexes:
$$\mathrm{Tot}\, A_d{}^\bullet = \left(\oplus_{n \geq 0} s^{-n} NA_d{}^n, d \right), \qquad \mathrm{Tot}\, C_d{}^\bullet = \left(\oplus_{n \geq 0} s^{-n} NC_d{}^n, d \right), \tag{2.1}$$
where the differential d is as usual the alternated sum of cofaces $d = \sum_{i=0}^{n+1}(-1)^i d_{i*}$ and s^{-n} denote the n-fold desuspension.

3. HODGE DECOMPOSITION

As usual it is assumed that the main field of coefficients $\mathbb{K}$ is of characteristics zero. Gerstenhaber and Schack [16] defined a Hodge-type decomposition in the Hochschild (co)homology of commutative algebras with coefficients in a symmetric bimodule M over A.
$$HH_*(A, M) = \oplus_i HH_*^{(i)}(A, M), \qquad HH^*(A, M) = \oplus_i HH^*_{(i)}(A, M). \tag{3.1}$$
The construction of Gerstenhaber and Schack [16] used the S_n-action on the components of Hochschild complexes:
$$(\oplus_{n \geq 0} M \otimes A^{\otimes n}, \delta), \qquad (\oplus_{n \geq 0} Hom(A^{\otimes n}, M), d).$$
They defined the families of orthogonal projectors $e_n^{(i)} \in \mathbb{K}[S_n]$, $1 \leq i \leq n$ if $n > 0$, and $i = 0$ if $n = 0$, satisfying the following conditions:

- $e_n^{(i)} \cdot e_n^{(j)} = \delta_{ij} e_n^{(i)}$ (they are orthogonal projectors);
- $\sum_{i=0}^{n} e_n^{(i)} = 1$ (the family of projectors $e_n^{(i)}$, $i = 0 \dots n$, is complete); $\qquad$ (3.2)
- $\delta e_n^{(i)} = e_{n-1}^{(i)} \delta$, $d e_n^{(i)} = e_{n+1}^{(i)} d$, $0 = 1 \dots n$.

In the above formula $e_n^{(i)} = 0$ if i is not in the range $1 \dots n$ (in case $n = 0$ one has $e_0^{(i)} = 0$ if $i \neq 0$). These projectors are called Eulerian idempotents.

By the last property, the Hochschild complexes computing (3.1) split into a direct sum of complexes with the i-th complex being the image of the projection $e^{(i)} = \sum_{n=0}^{\infty} e_n^{(i)}$. This splitting induces splitting in Hochschild (co)homology which is sometimes referred as "Hodge decomposition". Notice that the same construction works equally well in the differential graded setting [8, 42].

Later on J.-L. Loday gave a more general set up for this splitting [29]. He noticed that it takes place for any (co)simplicial complex that can be factored through the category Γ (resp. Γ^{op}) of

finite pointed sets. Recall that a simplicial (resp. cosimplicial) vector space is a functor from the category Δ^{op} (resp. Δ) to the category of vector spaces:

$$X_\bullet \colon \Delta^{op} \longrightarrow Vect, \qquad X^\bullet \colon \Delta \longrightarrow Vect.$$

We will also consider cosimplicial dg-vector spaces. In that case the target category is the category $dg - Vect$ of differential graded vector spaces. The objects of Δ^{op} are sets

$$[n] = \{0, 1, \ldots, n + 1\}, \ n = 0, 1, 2, \ldots$$

The morphisms are the bipointed ordered maps $[n] \to [m]$ preserving the linear order.[1] By "bipointed" one means maps sending 0 to 0, and $n + 1$ to $m + 1$. The morphisms are generated by the so called *face maps*:

$$d_i \colon [n] \to [n - 1], \quad i = 0 \ldots n,$$

$$d_i(j) = \begin{cases} j, & \text{if } j \leq i; \\ j - 1, & \text{if } j > i \end{cases}$$

(the preimage $d_i^{-1}(i)$ has two points i and $i + 1$), and *degeneracies*:

$$s_i \colon [n] \to [n + 1], \quad i = 1 \ldots n + 1,$$

$$s_i(j) = \begin{cases} j, & \text{if } j < i; \\ j + 1, & \text{if } j \geq i \end{cases}$$

(the preimage $s_i^{-1}(i)$ is empty).

The simplicial and cosimplicial complexes are defined as follows:

$$(\oplus_{n \geq 0} s^n X_n, \partial), \qquad (\oplus_{n \geq 0} s^{-n} X^n, d), \tag{3.3}$$

where the differential ∂ (respectively d) is the alternated sum of (co)faces d_i plus (in the differential graded case) the inner differential of each X_n (respectively X^n).

It is usually convenient to consider the normalized complexes:

$$\mathrm{Tot}\, X_\bullet = (\oplus_{n \geq 0} s^n N X_n, \partial), \qquad \mathrm{Tot}\, X^\bullet = (\oplus_{n \geq 0} s^{-n} N X^n, d),$$

which are quasi-isomorphic to the initial ones (3.3). The normalized part is defined as follows

$$N X_n = X_n / +_{i=1}^n \mathrm{Im}\, s_i, \qquad N X^n = \cap_{i=1}^n \ker s_i.$$

The category Γ has objects

$$\underline{n} = \{*, 1, 2, \ldots, n\}, \ n = 0, 1, 2, \ldots \tag{3.4}$$

The morphisms $Mor_\Gamma(\underline{m}, \underline{n})$ are the pointed maps $\underline{m} \to \underline{n}$ (we consider each set (3.4) to be pointed in $*$). One has a natural functor $\rho \colon \Delta^{op} \longrightarrow \Gamma$ induced by the set maps:

$$[n] \xrightarrow{\rho} \underline{n},$$

$$\rho(i) = \begin{cases} i, & 1 \leq i \leq n; \\ *, & i = 0 \text{ or } n + 1. \end{cases}$$

---

[1]This is the Joyal dual of the usual definition of Δ. As usual Δ^{op} is the category dual to Δ.

By abuse of the language $\rho(d_i)$, $\rho(s_i)$ will still be denoted by d_i, s_i and called faces and degeneracies respectively. The morphisms of Γ are generated by faces, degeneracies, and also by the isomorphisms of each object $\underline{n}$ (that are given by the S_n-group action).

Remark 3.1. Let Γ_{cycl} denote the subcategory of Γ, whose objects are the same and morphisms are the pointed maps $\underline{m} \to \underline{n}$ preserving the cyclic order. It is easy to see, that Γ_{cycl} is isomorphic to Δ^{op} via the functor ρ:

$$\rho \colon \Delta^{op} \xrightarrow{\simeq} \Gamma_{cycl}.$$

J.-L. Loday noticed [29] that if a simplicial (resp. cosimplicial) differential graded vector space $X_\bullet$ (resp. $X^\bullet$) can be factored through the category Γ (resp. Γ^{op}):

$$
\begin{array}{ccc}
\Delta^{op} \xrightarrow{\ X_\bullet\ } dg\text{--}Vect & \qquad \Delta \xrightarrow{\ X^\bullet\ } dg\text{--}Vect & \qquad (3.5) \\
{\scriptstyle\rho}\Big\downarrow \ \ \nearrow & {\scriptstyle\rho}\Big\downarrow \ \ \nearrow & \\
\Gamma & \Gamma^{op} &
\end{array}
$$

then each X_n (resp. X^n) admits an S_n-action, and moreover the projections $e_n^{(i)} \in \mathbb{K}[S_n]$ satisfy the properties (3.2), and therefore define the Hodge splitting

$$\mathrm{Tot}\, X_\bullet = \oplus_{i \geq 0} \mathrm{Tot}^{(i)} X_\bullet, \qquad \mathrm{Tot}\, X^\bullet = \oplus_{i \geq 0} \mathrm{Tot}^{(i)} X^\bullet.$$

4. Operadic point of view. Σ-cosimplicial spaces, and commutative Σ-cosimplicial spaces

Notice that the Hodge splitting for Hochschild complexes was possible only for *commutative* algebras with coefficients in a *symmetric* bimodule. Only in this case the symmetric group action is "nicely" compatible with the differential which is built out of the product. This section defines a natural formalism that generalizes this idea and will be helpful to detect Γ and Γ^{op}-modules.

Let $\mathcal{O}$ be any of the following three operads:

- non-Σ operad of associative algebras $\mathcal{ASSOC} = \{\mathcal{ASSOC}(n),\, n \geq 0\}$ with $\mathcal{ASSOC}(n) = \mathbb{K}$ for all n.
- operad of commutative algebras $\mathcal{COMM} = \{\mathcal{COMM}(n),\, n \geq 0\}$ with $\mathcal{COMM}(n) = \mathbb{K}$ being the trivial representation of S_n.
- operad of associative algebras $\mathcal{ASSOC}^\Sigma = \{\mathcal{ASSOC}^\Sigma(n),\, n \geq 0\}$, but in this Σ-case $\mathcal{ASSOC}^\Sigma(n) = \mathbb{K}[S_n]$. The subscript Σ is used to distinguish from the non-Σ case above.

Definition 4.1. A sequence of spaces $M = \{M(n), n \geq 0\}$ is a *weak bimodule* over an operad $\mathcal{O}$ if it is endowed with a series of composition maps:

$$\overline{\circ}_i \colon \mathcal{O}(n) \otimes M(k) \to M(n + k - 1),\ \ i = 1 \ldots n,\ \ \text{(left action)}; \tag{4.1}$$

$$\underline{\circ}_i \colon M(k) \otimes \mathcal{O}(n) \to M(k + n - 1),\ \ i = 1 \ldots k,\ \ \text{(right action)}, \tag{4.2}$$

satisfying natural unity and associativity conditions (1)-(6) specified below, and, in the Σ-case, compatibility with the S_n-group action (in the Σ-case it is assumed that $M(n)$ are S_n-modules.).[2]

---

[2]We say "*weak*" because the left action "is weak". The usual left action is given by a series of maps $\mathcal{O}(k) \otimes \big(M(k_1) \otimes M(k_2) \otimes \ldots M(k_n) \big) \to M(k_1 + \ldots + k_n)$.

Each element of O and of M is viewed as an object with some number of inputs and one output. The composition is obtained by inserting an output in one of the inputs, see Figure A below. The compatibility with the S_n-group action is the same as in the definition of the operad: no matter one permutes the inputs before or after taking the composition, the result will be the same. For the other conditions (1)-(6), we mention that in particular (1), (3), and (4) imply that a weak bimodule over an operad is always a right module.

The result of composition $\overline{\circ}_i(o, m)$, and $\underline{\circ}_i(m, o)$, for $o \in \mathcal{O}(n)$, and $m \in M(k)$, will be denoted by $o \circ_i m$, and $m \circ_i o$.

$$o \circ_3 m = \qquad ; \qquad m \circ_2 o =$$

Left action   Right action
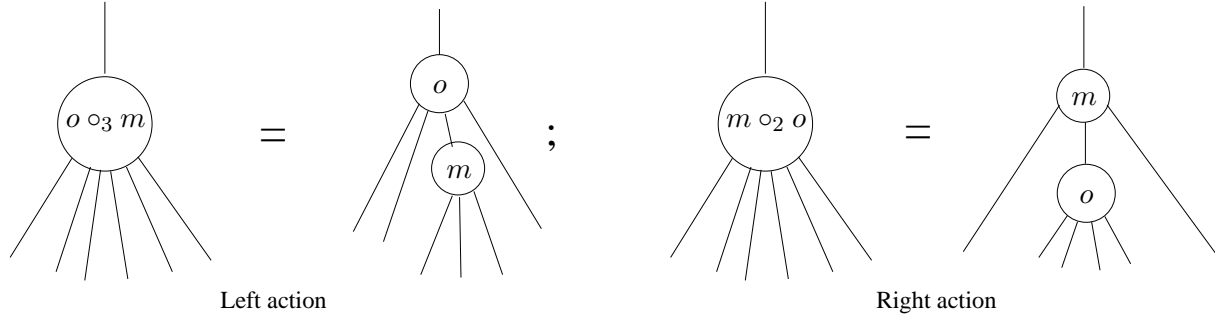
FIGURE A

We will also adopt notation using formal variables. For example, $o \circ_3 m$ from Figure A will be denoted as $o(x_1, x_2, m(x_3, x_4, x_5), x_6)$, and $m \circ_2 o$ as $m(x_1, o(x_2, x_3, x_4, x_5), x_6)$.

Let $o_1 \in O(i)$, $o_2 \in O(j)$, and $m \in M(k)$. It will be also assumed that $id \in O(1)$ is the unit of the operad O. One has the following axioms of the weak bimodule structure:

(1) Unity condition with respect to the left and right actions:
$$id \circ_1 m = m = m \circ_p id, \quad 1 \le p \le k.$$

(2) Associativity of the left action:
$$(o_1 \circ_p o_2) \circ_{p+q-1} m = o_1 \circ_p (o_2 \circ_q m), \quad 1 \le p \le i, \quad 1 \le q \le j.$$

(3) Associativity of the right action:
$$(m \circ_p o_1) \circ_{p+q-1} o_2 = m \circ_p (o_1 \circ_q o_2), \quad 1 \le p \le k, \quad 1 \le q \le i.$$

(4) Commutativity of the right action on different inputs:
$$(m \circ_p o_1) \circ_{q+i-1} o_2 = (m \circ_q o_2) \circ_p o_1, \quad 1 \le p < q \le k.$$

(5) Associativity between the left and right actions:
$$(o_1 \circ_p m) \circ_{p+q-1} o_2 = o_1 \circ_p (m \circ_q o_2), \quad 1 \le p \le i, \quad 1 \le q \le k.$$

(6) Compatibility between the bi-action and the operad composition:
$$(o_1 \circ_p m) \circ_q o_2 = (o_1 \circ_q o_2) \circ_{p+q-1} m, \quad 1 \le q < p \le i.$$
$$(o_1 \circ_p m) \circ_{q+k-1} o_2 = (o_1 \circ_q o_2) \circ_p m, \quad 1 \le p < q \le i.$$

Lemma 4.2. *The structure of a cosimplicial vector space is equivalent to the structure of a weak non-Σ bimodule over $\mathcal{ASSOC}$.*

Lemma 4.3. *The structure of a Γ^{op}-module is equivalent to the structure of a weak bimodule over $\mathcal{COMM}$.*

The same is true in the differential graded case in which the acting operads $\mathcal{ASSOC}$ and $\mathcal{COMM}$ are considered with trivial differential.

Proof of Lemmas 4.2, 4.3. Instead of giving a formal proof let us consider a few examples how cosimplicial and Γ^{op} structure maps correspond to the composition operations (4.1-4.2). Let $M = \{M(n),\, n \geq 0\}$ be a non-Σ weak bimodule over $\mathcal{ASSOC}$. Let us pick $m = 6$, $n = 3$, and some map $\alpha \in Mor_{\Delta^{op}}([6], [3])$:
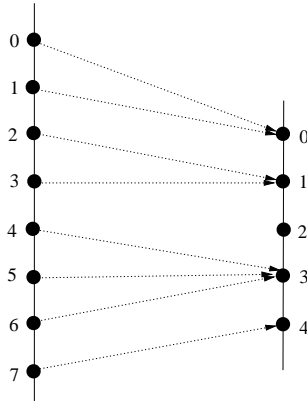
By Remark 3.1 this morphism corresponds to the following morphism in $Mor_{\Gamma_{cycl}}(\underline{6}, \underline{3})$.
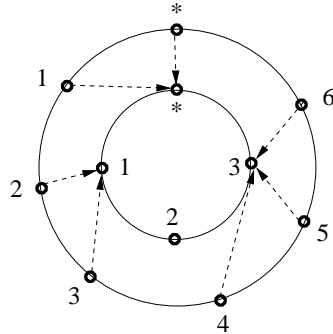
We define the map $\alpha_M \colon M(3) \to M(6)$ using the compositions (4.1), (4.2):
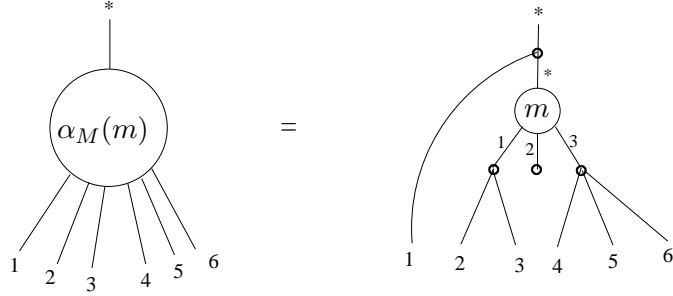
Or equivalently

$$\alpha_M(m)(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 m(x_2 x_3, 1, x_4 x_5 x_6).$$

By $\downarrow$, $\uparrow$, $\wedge$, $\wedge\!\!\wedge$, ... in the above picture one represents respectively $1 \in \mathcal{ASSOC}(0)$, $x_1 \in \mathcal{ASSOC}(1)$, $x_1 x_2 \in \mathcal{ASSOC}(2)$, $x_1 x_2 x_3 \in \mathcal{ASSOC}(3)$, ... The idea is to denote the output of m and of $\alpha(m)$ by $*$, and their inputs by $1, 2, 3, \ldots$, and then to see how the output and inputs of $\alpha_M(m)$ are connected to the output and inputs of M.

Similarly if M is a weak bimodule over $\mathcal{COMM}$, then any morphism $\alpha \colon \underline{m} \to \underline{n}$ defines a map $\alpha_M \colon M(n) \to M(m)$. For example, the map $\alpha \colon \underline{4} \to \underline{2}$:
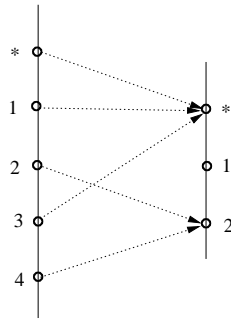
defines a map $\alpha_M \colon M(2) \to M(4)$ constructed as follows:
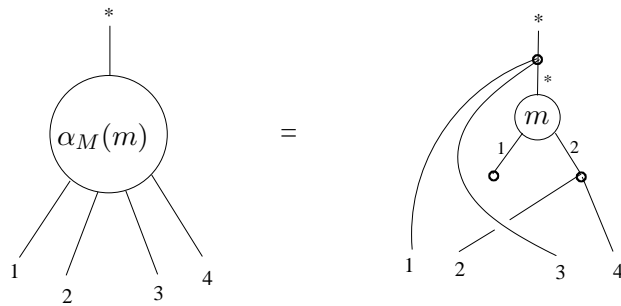
Equivalently $\alpha_M(m)(x_1, x_2, x_3, x_4) = x_1 x_3 m(1, x_2 x_4)$. $\qquad\qquad\square$

We have seen that weak bimodules over $\mathcal{COMM}$ and weak non-Σ bimodules over $\mathcal{ASSOC}$ have a very natural interpretation. Actually weak bimodules over $\mathcal{ASSOC}^\Sigma$ are also very common objects. As example the Hochschild complex of any (not necessarily symmetric) algebra A with coefficients in any A-bimodule has this structure. The category that encodes this structure is also well-known to be the category of finite pointed non-commutative sets [34].

The following definition is not standard, but will be convenient for the language of the paper.

Definition 4.4. (a) A weak bimodule over $\mathcal{ASSOC}^\Sigma$ will be called Σ-*cosimplicial space*.

(b) A weak bimodule over $\mathcal{COMM}$ or equivalently Γ^{op}-module will be called *commutative* Σ-*cosimplicial space*.

Roughly speaking Σ-cosimplicial spaces are cosimplicial spaces with S_n action on each component, and commutative Σ-cosimplicial spaces are Σ-cosimplicial spaces for which this S_n action is nicely compatible with the face maps.

4.1. **In the category of topological spaces.** Similar constructions (Lemmas 4.2-4.3, Definition 4.4) work well for any symmetric monoidal category with associative coproducts. Again abusing the notation $\mathcal{COMM} = \{\mathcal{COMM}(n), n \geq 0\} = \{*, n \geq 0\}$ will denote the topological commutative operad and $\mathcal{ASSOC}$, respectively $\mathcal{ASSOC}^\Sigma$, will denote non-Σ, respectively Σ, associative operads:

$$\text{non-}\Sigma \text{ case: } \mathcal{ASSOC} = \{\mathcal{ASSOC}(n) = *, n \geq 0\}.$$
$$\Sigma \text{ case: } \mathcal{ASSOC}^\Sigma = \{\mathcal{ASSOC}^\Sigma(n) = S_n, n \geq 0\}.$$

4.2. **Subcomplex of alternative multiderivations.** There is one important case when the Hochschild cohomology is easy to compute. This is when A is a smooth algebra, and M is a flat symmetric A-bimodule. In this case the cohomology $HH^*(A, M)$ is described by the space of alternative multiderivations.

Definition 4.5. Let $Y^\bullet$ be a commutative Σ-cosimplicial dg-vector space.

(a) An element $y \in Y^\bullet$ is called alternative if for any $\sigma \in S_n$

$$y(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) = (-1)^{|\sigma|} y(x_1, x_2, \ldots, x_n). \tag{4.3}$$

(b) An element $y \in Y^n$ is called a multiderivation if for any $i = 1 \ldots n$ one has

$$(4.4) \quad y(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{n+1}) =$$
$$x_i y(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) + y(x_1, \ldots, \hat{x}_{i+1}, \ldots, x_{n+1}) x_{i+1}.$$

Let $AM(Y^n)$ denote the subspace of alternative multiderivations in Y^n.

Lemma 4.6. *The space $AM(Y^n)$, $n \geq 0$, is a subcomplex of* $\mathrm{Tot}\, Y^\bullet$.

Proof. Let $y \in AM(Y^n)$. For any $1 \leq i \leq n$ one has

$$s_i y = y(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1}) = y(x_1, \ldots, x_{i-1}, 1 \cdot 1, x_i, \ldots, x_{n-1}) =$$
$$1 \cdot y(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1}) + y(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1}) \cdot 1 = 2 s_i y.$$

Therefore $s_i y = 0$, $1 \leq i \leq n$, and $AM(Y^n)$ lies in the normalized part of Y^n.

On the other hand the external part of the differential, which is the alternative sum of cofaces d_i, takes any multiderivation to zero, which can be checked by similar computations. One has also

that the internal part of the differential preserves the alternative and multiderivation properties since it commutes with Γ^{op} structure maps. □

The complex $\oplus_{n \geq 0} AM(Y^n)$ will be also denoted by $AM(Y^\bullet)$. The following definition is inspired by the example from the beginning of the section.

Definition 4.7. A commutative Σ-cosimplicial dg-vector space will be called *smooth* if the inclusion

$$AM(Y^\bullet) \hookrightarrow \operatorname{Tot} Y^\bullet$$

is a quasi-isomorphism.[3]

As example the Hochschild cochain complex $HC^\bullet(A, M)$ of a smooth commutative algebra A with coefficients in a flat symmetric A-bimodule is a smooth Γ^{op}-module.

Remark 4.8. For a smooth commutative Σ-cosimplicial dg-vectror space, the Hodge decomposition in the homology can be easily understood:

$$H_*^{(i)}(\operatorname{Tot} Y^\bullet) = H_*(AM(Y^i)).$$

This follows from the fact that $AM(Y^i)$ lies in the image of $e_i^{(i)} = \frac{1}{i!} \sum_{\sigma \in S_i} (-1)^{|\sigma|} \sigma$.

5. Hodge decomposition in the homology and homotopy of long knots

Let Top denote the category of simply connected topological spaces, and $hoTop$ denote the category with the same objects $Ob(hoTop) = Ob(Top)$, but with the morphisms being the homotopy classes of maps. One has a forgetful functor:

$$Top \xrightarrow{h} hoTop. \tag{5.1}$$

Proposition 5.1. *The cosimplicial space $C_d^\bullet$ (see Section 2) has the property that $h \circ C_d^\bullet$ factors through Γ^{op}:*

$$\Delta \xrightarrow{C_d^\bullet} Top \xrightarrow{h} hoTop \tag{5.2}$$
$$\rho \searrow \qquad \nearrow$$
$$\Gamma^{op}$$

or in other words $C_d^\bullet$ is a commutative Σ-cosimplicial space in $hoTop$.

Proof. D. Sinha constructed another cosimplicial model for $\overline{Emb}_d$, which is homotopy equivalent to the one that was briefly described in Section 2. The n-th stage of this model is the n-th component of some operad $\mathcal{K}_d$ which is homotopy equivalent to the operad of little d-cubes. Using a natural inclusion

$$\mathcal{ASSOC}^\Sigma \hookrightarrow \mathcal{K}_d, \tag{5.3}$$

$\mathcal{K}_d$ becomes a cosimplicial space (being a bimodule over $\mathcal{ASSOC}$, see Lemma 4.2). Moreover, according to our terminology it is a Σ-cosimplicial space. But notice that all the components

---

[3]This definition is a weaker version of [33, Definition 4.3] given by Pirashvili. [33, Theorem 4.6] implies that smooth Γ-modules in the sense of Pirashvili are always smooth in our definition.

$K_d(n)$, $n \geq 0$ are connected, therefore in the category *hoTop* the morphism (5.3) factors though $\mathcal{COMM}$:

$$\mathcal{ASSOC}^{\Sigma} \longrightarrow K_d \tag{5.4}$$

$$\mathcal{COMM}$$

As a consequence in *hoTop* the operad $\mathcal{K}_d$ is a weak bimodule over $\mathcal{COMM}$, or equivalently is a commutative Σ-cosimplicial space. $\qquad \square$

Corollary 5.2. *Since the homology functor factors through hoTop, the homology $A_d{}^\bullet = H_*(C_d{}^\bullet, \mathbb{K})$ is a commutative Σ-cosimplicial graded space. For $d \geq 3$ the same is true for the homotopy $L_d{}^\bullet = \pi_*(C_d{}^\bullet) \otimes \mathbb{K}$, since the underlying spaces are simply connected and therefore there are no base-point issues.*

There is another but similar way to see that $A_d{}^\bullet$ is a weak bimodule over $\mathcal{COMM}$. The vector spaces $\{A_d{}^n,\ n \geq 0\}$ form an operad, which is the homology operad of the little d-cubes. By [10] it is the operad of $(d-1)$-Poisson algebras: graded commutative algebras with a Lie bracket of degree $(d-1)$ compatible with the product:

$$[x_1, x_2 x_3] = [x_1, x_2]x_3 + (-1)^{|x_2| + (d-1)} x_2 [x_1, x_3].$$

This operad contains $\mathcal{COMM}$ and therefore is a weak bimodule over it.

Corollary-Definition 5.3. *The complexes* $\operatorname{Tot} A_d{}^\bullet$, $\operatorname{Tot} L_d{}^\bullet$ *admit Hodge splitting*

$$\operatorname{Tot} A_d{}^\bullet = \oplus_i \operatorname{Tot}^{(i)} A_d{}^\bullet. \tag{5.5}$$

$$\operatorname{Tot} L_d{}^\bullet = \oplus_i \operatorname{Tot}^{(i)} L_d{}^\bullet. \tag{5.6}$$

Since the above complexes compute the rational homology and homotopy of $\overline{Emb}_d$, this gives the Hodge decomposition in its homology and homotopy:

$$H_*(\overline{Emb}_d, \mathbb{Q}) = \oplus_i H_*^{(i)}(\overline{Emb}_d, \mathbb{Q}) := \oplus_i H_*(\operatorname{Tot}^{(i)} A_d{}^\bullet), \tag{5.7}$$

$$\pi_*(\overline{Emb}_d) \otimes \mathbb{Q} = \oplus_i \pi_*^{(i)}(\overline{Emb}_d, \mathbb{Q}) := \oplus_i H_*(\operatorname{Tot}^{(i)} L_d{}^\bullet). \tag{5.8}$$

6. Geometric interpretation of the Hodge decomposition. Cabling maps

Studying knot spaces is in many aspects similar to studying loop spaces. Recall that the real cohomology of the loop space ΩM of a 1-connected manifold M is naturally isomorphic to the Hochschild homology of the De Rham algebra $\Omega^*(M)$ (of differential forms on M) with trivial coefficients:

$$H^*(\Omega M, \mathbb{R}) \simeq HH_*(\Omega^*(M), \mathbb{R}). \tag{6.1}$$

Similarly the cohomology of the free loop space ΛM can be expressed as the Hochschild homology of $\Omega^*(M)$ with coefficients in itself:

$$H^*(\Lambda M, \mathbb{R}) \simeq HH_*(\Omega^*(M), \Omega^*(M)). \tag{6.2}$$

The differential algebra $\Omega^*(M)$ is graded commutative, and both bimodules $\mathbb{R}$, and $\Omega^*(M)$ are graded commutative over it. Hence it makes sense to consider the Hodge decomposition of (6.1-6.2). Let Φ_n, $n \in \mathbb{Z}$ denote the power maps

$$\Omega M \xrightarrow{\Phi_n} \Omega M, \qquad \Lambda M \xrightarrow{\Phi_n} \Lambda M.$$

induced by a degree n map of a circle into itself $S^1 \xrightarrow{n} S^1$. It was shown in [8] that the Hodge decomposition in $HH_*(\Omega^*(M), \mathbb{R})$ (resp. $HH_*(\Omega^*(M), \Omega^*(M))$) corresponds via isomorphisms (6.1), (6.2) to the decomposition into eigenspaces of the power maps in cohomology:

$$H^*(\Omega M, \mathbb{R}) \xrightarrow{\Phi_n^*} H^*(\Omega M, \mathbb{R}), \qquad H^*(\Lambda M, \mathbb{R}) \xrightarrow{\Phi_n^*} H^*(\Lambda M, \mathbb{R}).$$

More precisely the $HH_*^{(i)}$ term is always the eigenspace of Φ_n^* with eigenvalue n^i.

We believe that a similar assertion holds for the homotopy and homology of $\overline{Emb}_d$. But instead of power maps one has to use cabling maps (abusing notation they will be denoted similarly):

$$\Phi_n \colon \overline{Emb}_d \to \overline{Emb}_d.$$

The construction of Φ_n is similar to the proof of Proposition 1.1 and goes in two steps. First one constructs such maps for the spaces of long framed embeddings Emb_d^{framed} and long framed immersions Imm_d^{framed}:

$$
\begin{array}{ccc}
Emb_d^{framed} & \xrightarrow{\Phi_n} & Emb_d^{framed} \\
\cap \downarrow & & \cap \downarrow \\
Imm_d^{framed} & \xrightarrow{\Phi_n} & Imm_d^{framed}.
\end{array}
\tag{6.3}
$$

Since the diagram (6.3) commutes, one obtains the induced "cabling" map Φ_n on the homotopy fiber of the inclusion $Emb_d^{framed} \hookrightarrow Imm_d^{framed}$. But this homotopy fiber is exactly $\overline{Emb}_d$.

Below it is shown how to construct an n-cabling of a framed long knot (for framed immersions the construction works in the same way). One will be using a slightly different model for the space of long knots. Namely, it will assumed that a long knot is an embedding

$$f \colon [0,1] \hookrightarrow [0,1]^d,$$

coinciding with a fixed linear embedding in a neighborhood of the boundary points 0 and 1. The image of the endpoints will be denoted by $a = f(0) = (0, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ and $b = f(1) = (1, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. A framed knot is a knot f together with an orthonormal trivialization of its normal bundle $e_1(t), \dots, e_{d-1}(t)$. Moreover one has $e_i(t) = \frac{\partial}{\partial x_{i+1}}$, $1 \leq i \leq d-1$, for t close to the endpoints.

Let $n > 0$, define a string link

$$f_{n,\varepsilon} \colon [0,1] \times \{0, 1, \dots, n-1\} \hookrightarrow [0,1]^d,$$

$$f_{n,\varepsilon}(t, i) = f(t) + i\varepsilon e_1(t).$$

For example for $n = 2$ one gets:
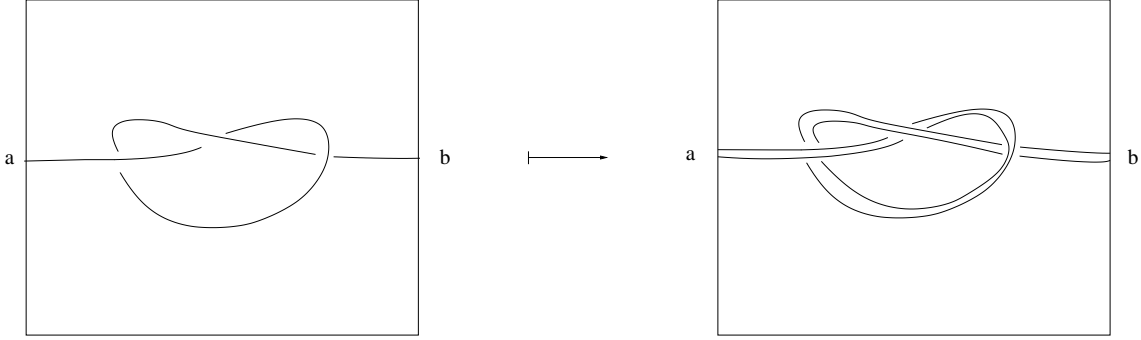
The number ε depends on the knot f. It should be such that any ε'-wave front of the knot, with $0 < \varepsilon' \leq \varepsilon$, does not have self-intersections. This ε can be chosen continuously on f. (For an immersion this ε can be chosen to be zero, but one should still make this choice in the way to make the diagram (6.3) commute.)

Then one shrinks the string link into the cube $[\frac{1}{4}, \frac{3}{4}]^d$. Let $a_i = f_{n,\varepsilon}(0, i)$, $b_i = f_{n,\varepsilon}(1, i)$ be the starting and ending point of the i-th component of the shrunk string link, $0 \leq i < n$. To finish the construction one joins the points a with a_0, b_i with a_{i+1}, $i = 0 \ldots n - 2$, and b_{n-1} to b, by embedded lines (on which one puts appropriate framing).
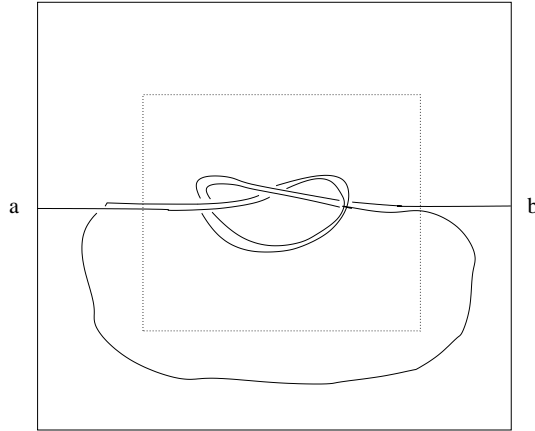
In our construction the picture outside $[\frac{1}{4}, \frac{3}{4}]^d$ does not depend on the initial knot f.

The 0-cabling is defined as a constant map $\Phi_0 \colon Emb_d^{framed} \to Emb_d^{framed}$ sending any knot to the fixed linear embedding. The negative cablings Φ_{-n} are defined similarly to Φ_n. First one constructs the string link $f_{n,\varepsilon}$ which we shrink into the cube $[\frac{1}{4}, \frac{3}{4}]^d$. Then one joins a with b_0, a_i with b_{i+1}, $i = 0 \ldots n - 2$, a_{n-1} with b. For example, (-1)-cabling of a knot is shown on Figure B.

Notice that for this construction only the first section e_1 of the framing was used. This means that instead of framed knots/immersions one can use long knots/immersions endowed with only one non-trivial section of the normal bundle. A model for such space is the space of embeddings/immersions

$$[0, 1] \times [0, 1] \longrightarrow [0, 1]^d,$$
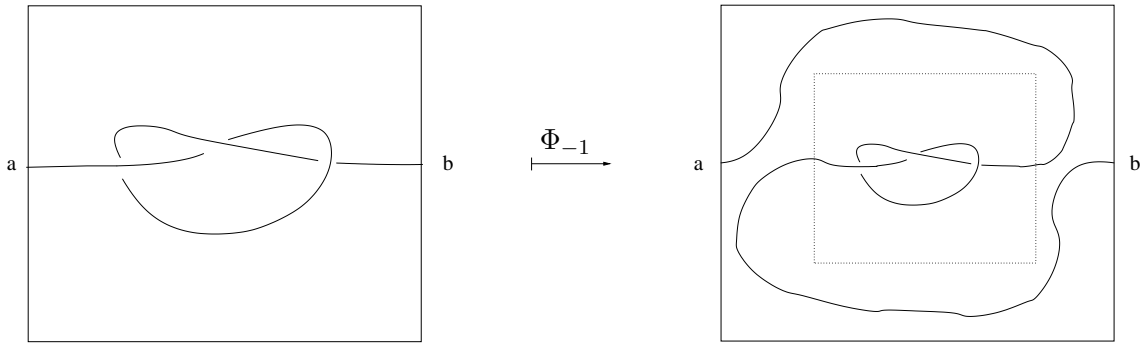
FIGURE B. (-1)-cabling.

that have a fixed behavior in a neighborhood of $[0,1] \times \{0,1\}$. For such spaces cabling maps can also be easily defined (moreover one can do it without caring about ε). The construction of cabling maps works also nicely if one uses Budney's model for the space of framed long knots [7] (the space denoted by $EC(1, D^{d-1})$), and its analogue for the space of framed immersions.

It is easy to see that for $d \geq 4$, one has that $\Phi_n \circ \Phi_m$ is homotopy equivalent to Φ_{nm} for any pair of integers n and m. In particular it means that the induced maps in the homology/homotopy commute. On the other hand, the totalization of any commutative Σ cosimplicial space (Γ^{op}-module) is endowed with so called Adams operations Ψ_n, $n \in \mathbb{Z}$, commuting with the differential, see [16, 29, 31]. In each degree they are defined as a sum of shuffle permutations with appropriate signs. The operation Ψ_n acts by multiplication by n^{i+1} on the degree i component $\mathrm{Tot}^{(i)}$ of the totalization.

Conjecture 6.1. *For $d \geq 4$, the maps induced in the rational homology and rational homotopy by the cablings intertwine with the Adams operations in the homology of totalizations:*

$$\begin{array}{ccc}
H_*(\overline{Emb}_d, \mathbb{Q}) \xrightarrow{\simeq} H_*(\mathrm{Tot}\, A_d^{\bullet}) \ , & \qquad & \pi_*(\overline{Emb}_d) \otimes \mathbb{Q} \xrightarrow{\simeq} H_*(\mathrm{Tot}\, L_d^{\bullet}) \\
\Phi_{n*} \downarrow \qquad\quad \downarrow \frac{1}{n}\Psi_{n*} & & \Phi_{n*} \downarrow \qquad\quad \downarrow \frac{1}{n}\Psi_{n*} \\
H_*(\overline{Emb}_d, \mathbb{Q}) \xrightarrow{\simeq} H_*(\mathrm{Tot}\, A_d^{\bullet}), & & \pi_*(\overline{Emb}_d) \otimes \mathbb{Q} \xrightarrow{\simeq} H_*(\mathrm{Tot}\, L_d^{\bullet}),
\end{array} \qquad (6.4)$$

This conjecture would imply that the degree i component $H_*^{(i)}(\overline{Emb}_d, \mathbb{Q})$, $\pi_*^{(i)}(\overline{Emb}_d, \mathbb{Q})$ of the Hodge decomposition can be defined as the eigen space with eigen value n^i of any cabling map Φ_{n*}, $n \neq \pm 1$ or 0.

One can try to prove this conjecture using (co)simplicial techniques. In this approach the main technical difficulty would be to have an appropriate compactification of the configuration spaces. Recently Munson and Volić defined a cosimplical model for the space of string links [30]. One can try to use their model to go through the first step of the cabling construction, where one defines a string link from a given long knot. To define such a map between the cosimplicial models (of long knots and of string links) one can double all points in the configuration in the direction different to the coface maps. But unfortunately this "doubling of each point" map does not respect the cosimplicial structure: it does not commute with the coface maps.

7. Relation between the Hodge decomposition in the homology and homotopy

The rational homotopy of $\overline{Emb}_d$ is isomorphic to the primitive part of the rational homology, see Corollary 1.3. By Theorem 2.1,

$$H_*(\overline{Emb}_d, \mathbb{Q}) = H_*(\operatorname{Tot} A_d{}^\bullet), \qquad \pi_*(\overline{Emb}_d) \otimes \mathbb{Q} = H_*(\operatorname{Tot} L_d{}^\bullet).$$

Therefore the space of primitives of $H_*(\operatorname{Tot} A_d{}^\bullet)$ is isomorphic to $H_*(\operatorname{Tot} L_d{}^\bullet)$:

$$Prim(H_*(\operatorname{Tot} A_d{}^\bullet)) \simeq H_*(\operatorname{Tot} L_d{}^\bullet) \tag{7.1}$$

A pure algebraic proof for this isomorphism is given in [23].

Proposition 7.1. *(i) The coproduct structure of* $\operatorname{Tot} A^\bullet$ *respects the Hodge degree:*

$$\Delta \operatorname{Tot}^{(k)} A_d{}^\bullet \subset \bigoplus_{i=0}^{k} \operatorname{Tot}^{(i)} A_d{}^\bullet \otimes \operatorname{Tot}^{(k-i)} A_d{}^\bullet.$$

(ii) The isomorphism (7.1) *respects the Hodge decomposition.*

The above proposition means that the study of the Hodge decomposition in the homology or in the homotopy are two equivalent problems, since the homology $H_*(\operatorname{Tot} A_d{}^\bullet)$ is a free graded cocommutative coalgebra over $Prim(H_*(\operatorname{Tot} A_d{}^\bullet)) \simeq H_*(\operatorname{Tot} L_d{}^\bullet)$.

Proof. The proof of (i) follows from the following lemma:

Lemma 7.2. *Let* $B^\bullet$ *and* $C^\bullet$ *be two commutative* Σ*-cosimplicial spaces, then* $B^\bullet \otimes C^\bullet$ *is also a commutative* Σ*-cosimplicial space and moreover the Eilenberg-Mac Lane map*

$$\operatorname{Tot}(B^\bullet \otimes C^\bullet) \to \operatorname{Tot} B^\bullet \otimes \operatorname{Tot} C^\bullet$$

respects the Hodge degree.

Proof. This fact is well known. For example, one can see it from the geometric approach of F. Patras to describe Adams operations [31, 32] (see in particular Proposition 1.3 of [32] and the remark that follows.) $\qquad\square$

To prove (i) one can notice that the coproduct on $\operatorname{Tot} A_d{}^\bullet$ is obtained as a composition of two maps:

$$\operatorname{Tot} A_d{}^\bullet \xrightarrow{\Delta^\bullet} \operatorname{Tot}(A_d{}^\bullet \otimes A_d{}^\bullet) \xrightarrow{EM} \operatorname{Tot} A_d{}^\bullet \otimes \operatorname{Tot} A_d{}^\bullet.$$

The first map $\Delta^\bullet$ is induced by a degree-wise coproduct (in the homology of configuration spaces), which is a morphism of commutative Σ-cosimplicial spaces. The second one respects the Hodge degree by Lemma 7.2.

(ii) Recall that the proof of (7.1) in [23] was given by the following sequence of quasi-isomorphisms:

$$\operatorname{Tot} A_{d\bullet}/(\operatorname{Tot} A_{d\bullet})^2 \xleftarrow[\alpha]{\simeq} \mathcal{L}(\operatorname{Tot} A_{d\bullet}) \xrightarrow[EM_\mathcal{L}]{\simeq} \operatorname{Tot} \mathcal{L}(A_{d\bullet}) \xleftarrow{\simeq} \operatorname{Tot}(L_{d\bullet}), \tag{7.2}$$

where $A_{d\bullet}$ is the simplicial graded commutative algebra dual to $A_d{}^\bullet$ (i.e. $A_{dn} = H^*(C_d{}^n, \mathbb{Q})$), and $L_{d\bullet}$ is the simplicial graded Lie coalgebra dual to $L_d{}^\bullet$ (i.e. $L_{dn} = Mor(\pi_*(C_d{}^n), \mathbb{Q})$). The algebra $\operatorname{Tot} A_{d\bullet}$ is polynomial, $\operatorname{Tot} A_{d\bullet}/(\operatorname{Tot} A_{d\bullet})^2$ describes the quotient complex of indecomposables. The homology of $\operatorname{Tot} A_{d\bullet}/(\operatorname{Tot} A_{d\bullet})^2$ is $Prim(H_*(\operatorname{Tot} A_{d\bullet}))$. The functor $\mathcal{L}(-)$ is the cobar construction that assigns to any commutative *dg*-algebra a quasi-free *dg*-Lie coalgebra. The

first, and the second quasi-isomorphisms, α, and $EM_{\mathcal{L}}$, respect the Hodge decomposition by Lemma 7.2. The last one is induced by a morphism of differential graded Γ-modules and therefore respects the Hodge degree. □

Part 2. Graph-complexes

8. Bialgebra of chord diagrams. Generalizing a result of Bar-Natan

The bialgebra of chord diagrams is a well-known object in Low Dimensional Topology which encodes the so called Vassiliev invariants of knots. Bar-Natan has shown that this bialgebra is isomorphic to the bialgebra of unitrivalent graphs attached to a line modulo STU, IHX, and AS relations [6]. Theorem 8.6 of [23] generalizes and also gives another proof of this result using graph-complexes. In this paper we will go a step further and will generalize the following result of Bar-Natan:

Theorem 8.1 (Bar-Natan [6])**.** *The vector space of primitive elements of the bialgebra of chord diagrams is isomorphic to the space of connected unitrivalent graphs (with at least one univalent vertex) modulo IHX, and AS relations.*

FIGURE C. Examples of uni-trivalent graphs

The bialgebra of chord diagrams is naturally graded by *complexity* - number of chords. The complexity of a unitrivalent graph is the Betti number of the graph obtained by gluing together all univalent vertices, see Figure D.
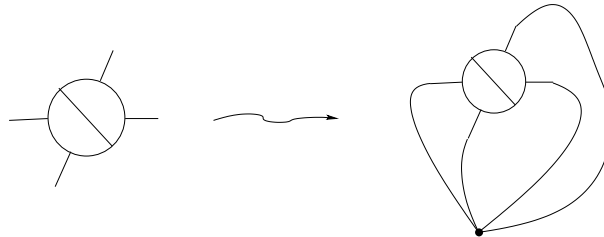
FIGURE D

From this theorem one can see that the space of primitives has another grading which is the number of univalent vertices.

Section 8.1 defines graph-complexes $AM(P_d{}^\bullet)$, $AM(D_d{}^\bullet)$. The first graph-complex is spanned by connected graphs whose vertices are either univalent or ≥ 3-valent. The differential is the sum of expansions of vertices. The above vector space of unitrivalent graphs modulo IHX and AS relations is a subspace of the homology of $AM(P_d{}^\bullet)$ in the lowest non-trivial degree. The IHX relations arise as a sum of 3 possible expansions of a quadrivalent vertex, i.e. these relations

appear as the image of the differential. The AS relations are equivalent to the orientation relations described in Section 8.1.

The complex $AM(D_d{}^\bullet)$ is similar to $AM(P_d{}^\bullet)$, except that it is spanned by graphs with any number of connected components.

Theorem 8.2. *The homotopy $\pi_*(Emb_d) \otimes \mathbb{Q}$, resp. the homology $H_*(\overline{Emb}_d, \mathbb{Q})$ is computed by the graph-complex $AM(P_d{}^\bullet)$, resp. $AM(D_d{}^\bullet)$ (that are defined below) of connected, resp. possibly disconnected or empty uni-$\geq$ 3-valent graphs. Moreover via this isomorphism the complexity grading corresponds to the first Betti number of the graph obtained by gluing together all the univalent vertices, and the Hodge degree corresponds to the number of univalent vertices.*

This theorem is proved in Section 10.

8.1. **Definition of** $AM(P_d{}^\bullet)$, $AM(D_d{}^\bullet)$. Let us first define the complex $AM(P_d{}^\bullet)$. By a connected uni-$\geq$ 3-valent graph we mean an abstract connected graph with a non-empty set of univalent vertices (which are called external) and some (possibly empty) set of vertices of valence at least 3 (those vertices are called external). The graphs are allowed to have both multiple edges and loops — edges connecting a vertex to itself:
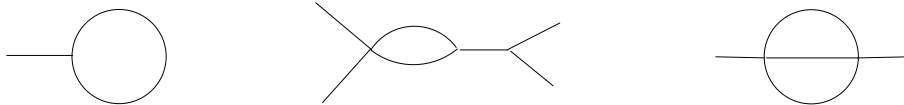
FIGURE E. Examples of connected uni-$\geq$ 3-valent graphs.

Neither external, nor internal vertices are labeled. The orientation set of a graph is the set of its external (univalent) vertices (those vertices are considered to have degree -1), its internal vertices (which are considered to have degree $-d$), and its edges (considered to have degree $(d-1)$). The total grading of a graph is the sum of gradings of its edges, and vertices. An *orientation* of a graph is an ordering of its orientation set together with fixing of orientation of each edge. Two graphs are considered to be equivalent if there is a bijection between their sets of vertices and edges respecting the adjacency structure of the graphs, orientation of edges, and the order of orientation sets.

Definition 8.3. The space of $AM(P_d{}^\bullet)$ is a graded vector space over $\mathbb{K}$ spanned by the above connected uni-$\geq$ 3-valent graphs modulo the orientation relations:
(1) If Γ_1 and Γ_2 differ only by an orientation of an edge, then
$$\Gamma_1 = (-1)^d \Gamma_2.$$
(2) If Γ_1 and Γ_2 differ only by a permutation of an orientation set, then
$$\Gamma_1 = \pm\Gamma_2,$$
where the sign is the Koszul sign of permutation (taking into account the degrees of elements).

The differential in $AM(P_d{}^\bullet)$ is a sum of expansions of internal vertices, see example below.

The orientation of the graph in the boundary is defined as follows: the new edge and the new vertex are added as the first and the second elements in the orientation set, and the edge is oriented from the old vertex to the new one. There is a choice which of the vertices of the new edge is considered as new and which as old, but regardless of this choice, the obtained orientation is the same.

Together with the main grading which is the sum of gradings of edges and vertices, one will consider two additional gradings. The first one, *complexity*, is defined as the first Betti number of the graph obtained by gluing all univalent vertices together, see Figure D. The second one, *Hodge degree*, is defined as the number of external vertices (we will later on see that indeed this grading corresponds to to the Hodge degree of some Hochschild complex). The differential in $AM(P_d{}^\bullet)$ preserves both the complexity and the Hodge degree, therefore the complex $AM(P_d{}^\bullet)$ splits into a direct sum of complexes:

$$AM(P_d{}^\bullet) = \oplus_{i,j} AM_j(P_d{}^i)$$

by complexity j, and Hodge degree i.

Remark 8.4. Similar graph-complexes appear in the study of the homology of the outer spaces [12, 13, 18, 20, 26]. The difference is that in our case the graphs have univalent vertices.

The second complex $AM(D_d{}^\bullet)$ can be defined as a free differential graded cocommutative coalgebra cogenerated by the complex $AM(P_d{}^\bullet)$. This complex can also be viewed as a graph-complex of possibly empty or disconnected uni-$\geq$ 3-valent graphs, each connected component being from the space $AM(P_d{}^\bullet)$.
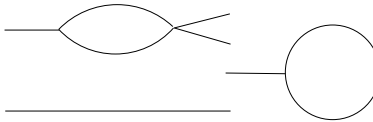
FIGURE F. Example of a graph from $AM(D_d{}^\bullet)$. It has 3 connected components and has in total 6 external, and 3 internal vertices.

The notation $AM(-)$ comes from "Alternative Multiderivations" (see Section 4.2). One will see in Section 10 that they are indeed isomorphic to the complexes of alternative multiderivations of the commutative Σ-cosimplicial dg-vector spaces $P_d{}^\bullet$, and $D_d{}^\bullet$, that will appear in the next section.

9. Operadic graph-complexes

Recall that the components of the cosimplicial space $A_d{}^\bullet = \{A_d{}^n,\ n \geq 0\} = \{H_*(C_d{}^n),\ n \geq 0\}$ form the operad of $(d-1)$-Poisson algebras, that will also be denoted by A_d.[4] This section defines a series of graph-complexes $D_d = \{D_d(n),\ n \geq 0\}$, that form an operad quasi-isomorphic to the operad A_d. Moreover the quasi-isomorphism is given by a natural inclusion. The construction of D_d takes roots in [22] and had a number of remakes [23, 25]. The difference of our construction is that we obtain an operad, and not a cooperad like in [22, 23, 25]. So our construction is

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[4]In [24, 39, 40] this operad was denoted by $\mathcal{P}oiss_{d-1}$. We switched the notation for a convenience of presentation.

dual to the previous ones. Another difference is that we also allow more admissible graphs: we permit graphs to have multiple edges (contrary to [22, 25], but similarly to [23]), and loops — edges connecting a vertex to itself (contrary to all previous versions). The n-th component $D_d(n)$ is spanned by *admissible graphs* that have n external vertices labeled by $1, 2, \ldots, n$, and some number of non-labeled internal vertices. The external vertices can be of any non-negative valence, the internal ones should be of valence at least 3. The only condition we put on graphs – each connected component should contain at least one external vertex (no pieces flying in air).
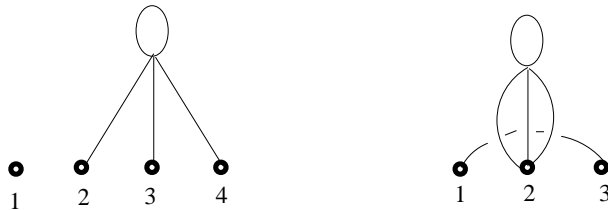
FIGURE G. Examples of graphs in $D_d(4)$ and $D_d(3)$.

The *orientation set* of such a graph is the union of the set of internal vertices (those elements are considered to be of degree $-d$), and the set of edges (such elements are of degree $(d-1)$). Similarly to Section 8 one will say that a graph is *oriented* if one fixes orientation of all its edges, and an ordering of its orientation set. The space $D_d(n)$ is defined to be spanned by all oriented admissible graphs (with n external vertices) modulo the orientation relations (1)-(2) similar to those of Definition 8.3.

Remark 9.1. For even d the orientation relations kill the graphs with multiple edges, and for odd d — the graphs with loops.

The differential is the sum of expansions of vertices (being dual to to the sum of contractions of edges). It diminishes the total degree by 1. One should be a little bit careful with the external vertices. An expansion of an external vertex produces one external vertex (with the same label) and one internal one.

The orientation of the graph in the boundary is defined as follows: the new edge and the new vertex are added as the first and the second elements in the orientation set, and the edge is oriented from the old vertex to the new one. There is a choice which of the vertices of the new edge is considered as new and which as old, but regardless of this choice, the obtained orientation is the same.

The operadic structure defined by compositions

$$\circ_i \colon D_d(n) \otimes D_d(m) \to D_d(n + m - 1)$$

is dual to the cooperadic structure in [25]. If $\Gamma_1 \in D_d(n)$, $\Gamma_2 \in D_d(m)$ are two graphs, then $\Gamma_1 \circ_i \Gamma_2$ is the sum of graphs obtained by making Γ_2 very small and inserting it in the i-th external point of Γ_1. The edges adjacent to the external vertex i in Γ_1 become adjacent to one of the vertices of Γ_2. The sum is taken by all such insertions, see Figure H. The orientation set of each graph in the sum is obtained by concatenation of the orientation sets of Γ_1 and Γ_2. With this definition of orientation, all the signs in Figure H are positive.
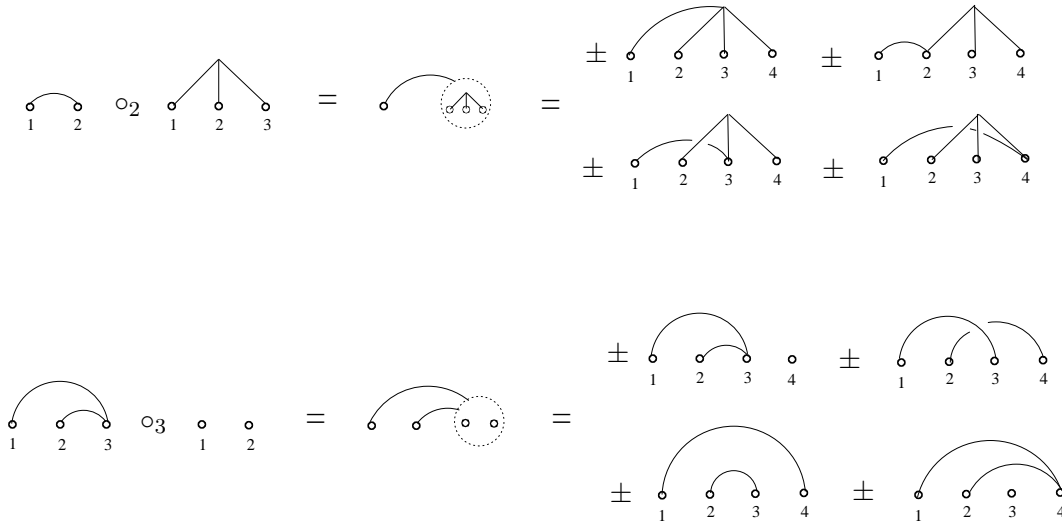
FIGURE H. Examples of composition

Proposition 9.2. *The assignment*

$$x_1 x_2 \mapsto \underset{1}{\bullet} \quad \underset{2}{\bullet}, \qquad [x_1, x_2] \mapsto \underset{1}{\frown} \underset{2}{\,},$$

where $x_1 x_2$, $[x_1, x_2] \in A_d(2)$ are the product and the bracket of the operad A_d of $(d-1)$-Poisson algebras, defines an inclusion of operads

$$A_d \overset{\simeq}{\hookrightarrow} D_d \tag{9.1}$$

that turns out to be a quasi-isomorphism (A_d is considered to have a zero differential).

Proof. This assertion is dual to [25, Theorem 9.1]. The graph-complexes used in [25] are slightly different, but the proof is the same. $\qquad\square$

Notice that the operad A_d contains the operad $\mathcal{COMM}$. Therefore the inclusion (9.1) can be viewed as a morphism of weak bimodules over $\mathcal{COMM}$. Denote by $D_d{}^\bullet = \{D_d{}^n, \, n \geq 0\} = \{D_d(n), \, n \geq 0\}$ the corresponding commutative Σ-cosimplicial space.

Corollary 9.3. *Inclusion* (9.1) *defines a quasi-isomorphism of complexes*

$$\mathrm{Tot}\, A_d{}^\bullet \overset{\simeq}{\hookrightarrow} \mathrm{Tot}\, D_d{}^\bullet$$

that respects the Hodge splitting.

The complex $\mathrm{Tot}\, D_d{}^\bullet$ is a graph-complex spanned by admissible graphs whose external vertices are of positive valence. The differential is the sum of expansions of vertices. An expansion of an external vertex can produce either two external vertices, or one external vertex and one internal one:

Corollary 9.4. *The graph-complex* $\mathrm{Tot}\, D_d{}^\bullet$ *computes the rational homology of the space* $\overline{Emb}_d$ *together with its Hodge splitting: for* $\mathbb{K} = \mathbb{Q}$ *one has*

$$H_*(\mathrm{Tot}^{(i)} D_d{}^\bullet) = H_*^{(i)}(\overline{Emb}_d, \mathbb{Q}).$$

The operad D_d is actually an operad in the category of graded cocommutative coalgebras.[5] The coalgebra structure in each component $D_d(n)$ is given by a cosuperimposing:

Let $P_d{}^n$ denote the primitive part of $D_d{}^n = D_d(n)$. The space $P_d{}^n$ is spanned by the graphs with n external vertices that become connected if one removes all the external vertices together with their small vicinities.

The family of spaces $P_d{}^\bullet = \{P_d{}^n, n \geq 0\}$ is preserved by the commutative Σ-cosimplicial structure maps, simply because these maps respect the coalgebra structure of $D_d{}^n$, $n \geq 0$.

Proposition 9.5. *The weak Σ-cosimplicial dg-vector spaces $L_d{}^\bullet$ and $P_d{}^\bullet$ are quasi-isomorphic (by a zig-zag of quasi-isomorphisms). As a consequence* $\mathrm{Tot}\, P_d{}^\bullet$ *computes the rational homotopy of* $\overline{Emb}_d$ *together with its Hodge splitting: for* $\mathbb{K} = \mathbb{Q}$ *one has*

$$H_*(\mathrm{Tot}^{(i)} P_d{}^\bullet) = \pi_*^{(i)}(\overline{Emb}_d, \mathbb{Q}).$$

Proof. The proof of Theorem 9.4 in [23] gives the necessary zig-zag. □

---

[5]Moreover (9.1) is a morphism of such operads.

10. $D_d{}^\bullet$ AND $P_d{}^\bullet$ ARE SMOOTH

Recall Definition 4.7 of a smooth commutative Σ-cosimplicial dg-vector space.

Theorem 10.1. *The commutative Σ-cosimplicial spaces $D_d{}^\bullet$ and $P_d{}^\bullet$ are smooth. Moreover their complexes of alternative multiderivations are isomorphic to the graph-complexes described in Section 8.1.* [6]

An immediate corollary of this theorem (and of Corollary 9.4, and Proposition 9.5) is Theorem 8.2.

Proof. Let us first show that the complexes from Section 8.1 are indeed isomorphic to $AM(D_d{}^\bullet)$, $AM(P_d{}^\bullet)$. An element of $D_d{}^\bullet$ (resp. $P_d{}^\bullet$) is a multiderivation if and only if it is a linear combination of graphs whose all external vertices are univalent. If the valence of the i-th vertex is 1, then one has:

which is exactly the equation (4.4). To see that the converse is true (i.e. that equation (4.4) implies that the valence of the i-th vertex for all the graphs in the sum is one), we define a map $r_i \colon D_d{}^{n+1} \to D_d{}^n$ that glues together the i-th and $(i+1)$-st external vertices:
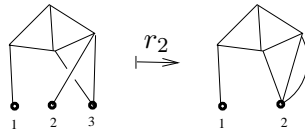
It is easy to see that if the valence of the i-th external vertex of a graph $\Gamma \in D_d{}^n$ is j, then $r_i d_i(\Gamma) = 2^j \Gamma$:

---

[6]One abused the notation: in Section 8.1 one denoted the defined complexes by $AM(D_d{}^\bullet)$, $AM(P_d{}^\bullet)$. We did it deliberately since they are naturally isomorphic to the complexes of alternative multiderivations that we consider in this section.
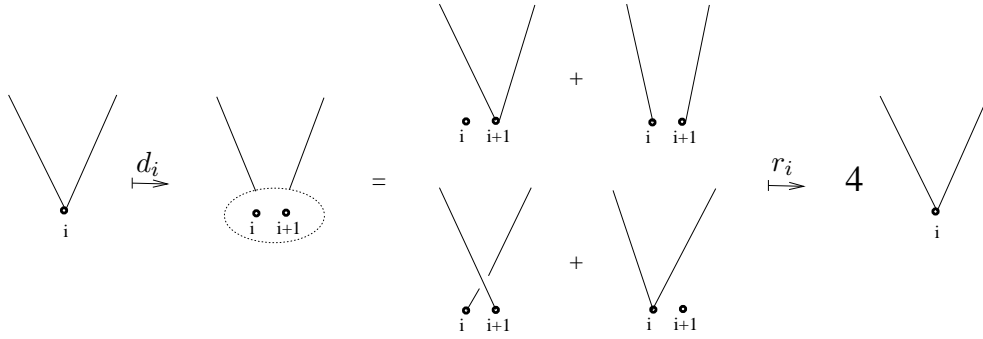
But if $y \in D_d{}^n$ satisfies (4.4), then $r_i d_i y = 2y$, which finally implies that any multiderivation is a linear combination of graphs with univalent external vertices.

Denote by $M(D_d{}^\bullet) = \oplus_n M(D_d{}^n)$, and $M(P_d{}^\bullet) = \oplus_n M(P_d{}^n)$ the subcomplexes of multiderivations of $\operatorname{Tot} D_d{}^\bullet$, $\operatorname{Tot} P_d{}^\bullet$. As we have seen they are spanned by the graphs whose all external vertices are univalent. It is easy to see that the complexes defined in Section 8.1 are isomorphic to $\oplus_n M(P_d{}^n) \otimes_{S_n} sign_n$, and $\oplus_n M(D_d{}^n) \otimes_{S_n} sign_n$, where $sign_n$ is the sign representation of S_n. This follows from the fact that in Section 8.1 the univalent vertices of graphs were not labeled and appear as elements of degree -1 of their orientetion sets. On the other hand, in characteristic zero the invariants and coinvariants of a group action coincide, and one has:

$$\oplus_n M(D_d{}^n) \otimes_{S_n} sign_n \simeq \oplus_n AM(D_d{}^n), \qquad \oplus_n M(P_d{}^n) \otimes_{S_n} sign_n \simeq \oplus_n AM(P_d{}^n).$$

Now let us show that $D_d{}^\bullet$, $P_d{}^\bullet$ are indeed smooth. Consider the free commutative algebra generated by $x_1, x_2, \ldots, x_n$. Let us take its normalized Hochschild complex with coefficients in the trivial bimodule $\mathbb{K}$. Let K_n denote the subcomplex of this Hochschild complex spanned by the elements of degree 1 in each variable x_i, $i = 1 \ldots n$. Notice that K_n is finite-dimensional. For example, K_2 is spanned by three elements: $x_1 \otimes x_2$, $x_2 \otimes x_1$, and $x_1 x_2$. Let $K_n^\vee$ denote the dual of K_n. In $\operatorname{Tot} P_d{}^\bullet$, $\operatorname{Tot} D_d{}^\bullet$ (and also in their subcomplexes $AM(P_d{}^\bullet)$, $AM(P_d{}^\bullet)$) consider the filtration by the number of internal vertices. The differential d_0 of the associated spectral sequence is the alternated sum of faces (external part of the differential in totalization). One can easily see that the term E_0 of the associated spectral sequence is isomorphic as a complex respectively to $\oplus_{n \geq 0} K_n^\vee \otimes_{S_n} M(P_d{}^n)$, and $\oplus_{n \geq 0} K_n^\vee \otimes_{S_n} M(D_d{}^n)$, where $M(P_d{}^n)$, $M(D_d{}^n)$ are taken with zero differential. Below there are two examples illustrating how this isomorphism works:
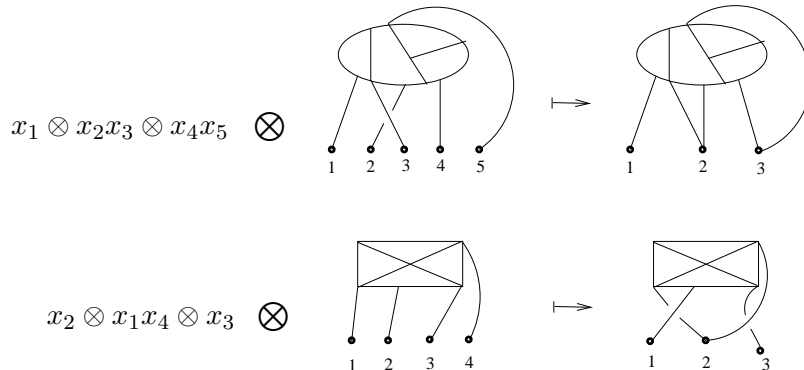
The idea of this isomorphism is that any graph in $\operatorname{Tot} P_d^\bullet$ (resp. $\operatorname{Tot} D_d^\bullet$) can be obtained from a graph with all external vertices of valence 1 by gluing together consecutive external vertices. This correspondence is not unique that's why one takes the tensor product over the symmetric group S_n.

The homology of $K_n^\vee$ is concentrated in the top degree and is isomorphic to the sign representation $sign_n$ of S_n. One can define an explicit inclusion

$$sign_n \hookrightarrow K_n^\vee$$

that sends the generator to $\frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{|\sigma|} x_{\sigma_1} \otimes \ldots \otimes x_{\sigma_n}$ and defines a quasi-isomorphism of dg-S_n-modules. As a consequence the E_1 term is isomorphic to

$$\oplus_n M(P_d{}^n) \otimes_{S_n} sign_n, \qquad \oplus_n M(D_d{}^n) \otimes_{S_n} sign_n.$$

respectively. But the above complexes are exactly $AM(D_d^\bullet)$, $AM(P_d^\bullet)$. Therefore the inclusions $AM(P_d^\bullet) \hookrightarrow \operatorname{Tot} P_d^\bullet$, $AM(D_d^\bullet) \hookrightarrow \operatorname{Tot} D_d^\bullet$ induce an isomorphism of the spectral sequences (associated with the above filtration) after the first term. Therefore these inclusions are quasi-isomorphisms. $\square$

Remark 10.2. Recall Remark 9.1 that the loops are possible only if d is even. But even in this case if we quotient out $P_d^\bullet$ by the graphs with loops, then $P_d^\bullet$ is no more smooth only in complexity 1. Indeed, the isomorphism $E_0 \simeq \oplus_{n \geq 0} K_n^\vee \otimes_{S_n} M(P_d{}^n)$ fails to be true only in complexity 1, since there is only one graph $\overset{\frown}{_1 \ _2}$ in $M(P_d^\bullet)$ that can produce a loop by gluing consecutive external vertices.

This remark will be used in the next section that studies $AM(P_d^\bullet)$ in small degrees.

11. $AM(P_d^\bullet)$ IN SMALL COMPLEXITIES

This section presents the results of computations of the homology of $AM(P_d^\bullet)$ for complexities $j = 1$, 2, and 3.

Complexity $j = 1$

Odd d. There is only one graph which is not canceled by the orientation relations:

$$(11.1)$$

It defines a rational homotopy class of $\overline{Emb}_d$ of dimension $d - 3$. Its Hodge degree is 2.

Even d. Again one has only one non-trivial graph:

$$(11.2)$$

that defines a rational homotopy class of dimension $d - 3$. Its Hodge degree is 1.

Complexity $j = 2$

Odd d. There are only 2 non-trivial graphs

$$(11.3)$$

and

$$(11.4)$$

The first one has the degree $2d - 6$ (its Hodge degree is 2), the second one has the degree $2d - 5$ (its Hodge degree is 1).

Remark 11.1. The cycles (11.1), (11.2), and (11.4) are exactly those that arise from the rational homotopy of the factor $\Omega^2 S^{d-1}$ in $\overline{Emb}_d = Emb_d \times \Omega^2 S^{d-1}$.
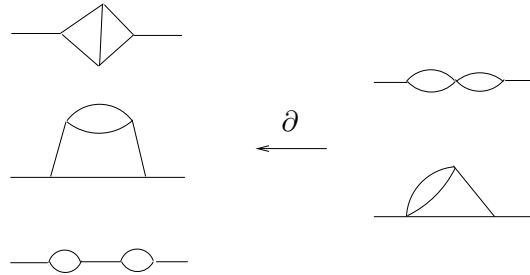
Even d. Recall from Remark 10.2, that starting from complexity 2 one can consider a quasi-isomorphic complex spanned by the graphs in $AM(D_{even}{}^\bullet)$ without loops. In complexity 2 one has only one such graph:

$$(11.5)$$

It defines a cycle of dimension $2d - 6$. Its Hodge degree is 3.

Complexity $j = 3$

Odd d. The case of odd d is harder because of the presence of multiple edges. In the Hodge degrees 4 and 3 all the graphs are trivial modulo the orientation relations. In the Hodge degree 2 one has a complex spanned by 5 graphs:

The homology is one-dimensional and concentrated in the lowest degree $3d - 9$. Its generator is represented by any of the 3 graphs:

$$(11.6)$$

One brings attention of the reader that pairing with the dual graph-complex (whose differential is a sum of contractions of edges) is given by the following rule: if a non-trivial graph in the dual graph complex is not isomorphic to a graph Γ in $AM(P_d{}^\bullet)$ then the pairing is zero, otherwise it is the order of the group of symmetries of Γ. In particular a homology generator dual to (11.6) is represented by the sum:

The order of the group of symmetries for the first graph is 8, for the second and the third ones it is 4. Therefore the pairing between (11.6) and the above cycle is 8.

In Hodge degree 1, one has a complex spanned by 9 graphs:
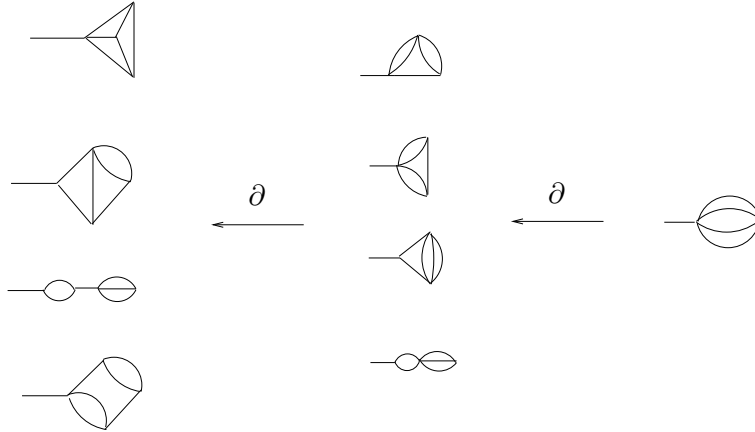
Easy computations show that up to a non-zero multiple there is only one non-trivial homology generator which lies in degree $3d - 8$. It is given by any of 2 graphs:

$$(11.7)$$

The other 2 graphs in this degree lie in the boundary. The dual homology generator is the sum:

Even d. The case of even d is easier since the graphs are without multiple edges (and also without loops due to Remark 10.2).

In Hodge degree 4 there are only 2 non-trivial graphs that cancel each other:

In Hodge degree 3 there are no non-trivial graphs. In Hodge degree 2 there is only 1 graph:

$$(11.8)$$

that defines a homology generator of degree $3d - 9$.

In Hodge degree 1 one has only 1 non-trivial graph:
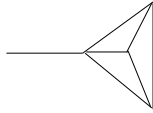
$$(11.9)$$

It defines a generator of degree $3d - 8$.

Remark 11.2. In complexity 3 the rational homotopy classes for both even and odd d are given by the same graphs:

                and
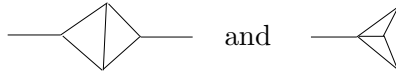
Remark 11.3. The computations in this section are consistent with the computations of the Euler characteristics of the Hodge splitting in rational homotopy, see Tables 1 and 3, and also with the previous computation of the rational homotopy of $\overline{Emb}_d$, see [38, 40].

12. Homotopy graph-complexes for spaces of higher dimensional long embeddings

Theorem 8.1 seems unnatural since it forgets the natural "linear" topology of the real line $\mathbb{R}^1$. From this point of view Theorem 8.2 seems to be even more strange and unbelievable. It turns out that these complexes $AM(P_d{}^\bullet)$ are conjecturally related to the topology of embedding spaces of higher dimensional affine spaces for which the complexes in question do seem naturally appropriate.

Let $Emb(\mathbb{R}^k, \mathbb{R}^d)$ denote the space of smooth non-singular embeddings $\mathbb{R}^k \hookrightarrow \mathbb{R}^d$ with a fixed linear behavior outside some compact subset of $\mathbb{R}^k$. Similarly let $Imm(\mathbb{R}^k, \mathbb{R}^d)$ denote the space of immersions with the same behavior at infinity, and let $\overline{Emb}(\mathbb{R}^k, \mathbb{R}^d)$ denote the homotopy fiber of the inclusion $Emb(\mathbb{R}^k, \mathbb{R}^d) \hookrightarrow Imm(\mathbb{R}^k, \mathbb{R}^d)$.

Let $\mathcal{E}_{k,d}$ denote a graph-complex whose definition is the same as the one given for $AM(P_d{}^\bullet)$ in Section 8.1 with the only exception that one assigns degree $-k$ instead of -1 to the external vertices of the uni-$\geq$ 3-valent graphs. In particular $\mathcal{E}_{1,d} = AM(P_d{}^\bullet)$. Up to a shift of gradings $\mathcal{E}_{k,d}$ depends on the parities of k, and d only.

Conjecture 12.1. *The rational homotopy of $\overline{Emb}(\mathbb{R}^k, \mathbb{R}^d)$, $d \geq 2k + 2$, is naturally isomorphic to the homology of $\mathcal{E}_{k,d}$:*

$$\pi_*(\overline{Emb}(\mathbb{R}^k, \mathbb{R}^d)) \otimes \mathbb{Q} \simeq H_*(\mathcal{E}_{k,d}).$$

This conjecture would imply that up to a shift of gradings the rational homotopy and homology of $\overline{Emb}(\mathbb{R}^k, \mathbb{R}^d)$, $d \geq 2k + 2$, depends on the parities of k and d only. We stress again the fact that surprisingly this biperiodicity starts already from $k = 1$!

The above conjecture is equivalent to a collapse at E^2 of a spectral sequence arising naturally from the Goodwillie-Weiss embedding calculus and computing the rational homology of $\overline{Emb}(\mathbb{R}^k, \mathbb{R}^d)$. To be precise $H_*(\mathcal{E}_{k,d})$ is exactly the primitive part of E^2. The E^2 term itself is isomorphic to the higher order Hochschild homology [33] of the Γ^{op}-module $A_d{}^\bullet$:

$$E^2 \simeq HH_*^{[k]}(A_d{}^\bullet) \simeq HH_*^{[k]}(D_d{}^\bullet).$$

The last isomorphism is due to the fact that $D_d{}^\bullet$ is quasi-isomorphic to $A_d{}^\bullet$ as a Γ^{op}-module. Finally one can show that $D_d{}^\bullet$ is smooth in the sense of Pirashvili [33, Definition 4.3] (the proof is similar to that of Theorem 10.1) and therefore a generalization of the Hochschild-Kostant-Rosenberg theorem [33, Theorem 4.6] can be applied. Which finally shows that $Prim(E^2) \simeq H_*(\mathcal{E}_{k,d})$. More details will be given in a future work of the author and G. Arone where we hope to prove the above conjecture together with the collapse result.

One should mention that besides the long knots (Theorem 2.1) this rational homology collapse result holds also for the spaces of embeddings modulo immersions $\overline{Emb}(M, \mathbb{R}^d)$ of any compact manifold into an affine space of a sufficiently high dimension d [2].

Part 3. Euler characteristics of the Hodge splitting

13. GENERATING FUNCTION OF EULER CHARACTERISTICS

Recall that the rational homology of $\overline{Emb}_d$ is computed by the complex

$$\text{Tot } A_d{}^\bullet = (\oplus_{n \geq 0} s^{-n} NA_d{}^n, \partial) = (\oplus_{n \geq 0} s^{-n} H_*^{Norm}(C_d{}^n, \mathbb{Q}), \partial).$$

The homology of $C_d{}^n$ is concentrated in the gradings that are multiples of $(d-1)$:

$$* = j \cdot (d-1), \qquad 0 \leq j \leq n-1.$$

For the normalized part one has the restrictions:

$$\frac{n}{2} \leq j \leq n-1.$$

The lower bound happens for the so called "chord diagrams", see [39]. The differential ∂ preserves this grading j that will be called *complexity*. Since ∂ also preserves the Hodge degree, one has a double splitting:

$$\text{Tot } A_d{}^\bullet = \oplus_{i,j} \text{Tot}^{(i,j)} A_d{}^\bullet = \oplus_{i,j} (\oplus_n s^{-n} e_n^{(i)} (H_{j(d-1)}^{Norm}(C_d{}^n, \mathbb{Q})), \partial). \tag{13.1}$$

Here $e_n^{(i)}$ is the i-th projector of the Hodge decomposition, see Section 3, s^{-n} denotes the n-fold desuspension.

Denote by $H_*^{(i,j)}(\overline{Emb}_d, \mathbb{Q}) = H_*(\text{Tot}^{(i,j)} A_d{}^\bullet)$. Let $\chi_{i,j}$ denote the Euler characteristics in bigrading (i,j):

$$\chi_{i,j} = \sum_{n=j(d-3)}^{j(d-2)-1} (-1)^n \dim(H_n^{(i,j)}(Emb_d, \mathbb{Q})),$$

and let $F_d(x, u)$ be the corresponding generating function

$$F_d(x, u) = \sum_{i,j} \chi_{i,j} x^i u^j.$$

The variable x is responsible for the Hodge degree and the variable u is responsible for the complexity. Since up to an (even) shift of gradings the complexes $\text{Tot } A_d{}^\bullet$ depend on the parity of d only, one has that $F_d(x, u)$ are the same for d's of the same parity. The corresponding generating functions will be denoted by $F_{odd}(x, u)$, and $F_{even}(x, u)$.

Let $E_\ell(y)$ be the polynomial $\frac{1}{\ell} \sum_{d | \ell} \mu(d) y^{\ell/d}$ (where $\mu(-)$ is the standard Möbius function), and let $\Gamma(y)$ be the usual gamma function which is $(y-1)!$ for positive integers y.

Theorem 13.1. *(a)*

$$F_{odd}(x, u) = \prod_{\ell \geq 1} \frac{\Gamma(E_\ell(\frac{1}{u}) - E_\ell(x))}{(\ell u^\ell)^{E_\ell(x)} \Gamma(E_\ell(\frac{1}{u}))}, \tag{13.2}$$

where each factor in the product is understood as the asymptotic expansion (see Definition 14.1) of the underlying function when $u \to +0$, and x is considered as a fixed parameter.

(b)

$$F_{even}(x, u) = \prod_{\ell \geq 1} \frac{\Gamma(-E_\ell(\frac{1}{u}) - E_\ell(x))}{(-\ell u^\ell)^{E_\ell(x)} \Gamma(-E_\ell(\frac{1}{u}))}, \tag{13.3}$$

where each factor in the product is understood as the asymptotic expansion of the underlying function when u is complex and $u^\ell \to -0$. Again x is considered as a fixed parameter.

The next section gives a better understanding of this formula. We warn the reader that the series corresponding to each factor can be divergent depending on x. We mention that both $F_{odd}(x, u)$, $F_{even}(x, u)$ have the form $\sum_{j=0}^{+\infty} P_j(x) u^j$, for $P_j(x)$, $j = 0, 1, 2, \ldots$, being a sequence of polynomials with integer coefficients. Each factor of (13.2)-(13.3) has a similar *polynomial* asymptotic expansion, but the polynomials in their expansions have rational coefficients.[7] For small complexities j one has

$$F_{odd}(x, u) = 1 + x^2 u + (x^4 + x^2 - x)u^2 + (x^6 + x^4 - x^3 + x^2 - x)u^3 + (x^8 + x^6 - x^5 + 3x^4 - 3x^3 + x^2 - x)u^4 + \ldots,$$

$$F_{even}(x, u) = 1 - xu + x^3 u^2 + (-x^4 - x^2 + x)u^3 + (x^6 + x^3 - x^2)u^4 + \ldots,$$

see Tables 2 and 4.

14. UNDERSTANDING FORMULAE (13.2)-(13.3)

14.1. **Looking at the first factor.**

Definition 14.1. A function $f(u)$ is said to have an *asymptotic expansion* $\sum_{j=0}^{+\infty} a_j u^j$ when $u \to +0$ if for any $n \geq 0$ one has

$$f(u) = \sum_{j=0}^{n} a_j u^j + o(u^n),$$

when $u \to +0$.

Notice that the series $\sum_{j=0}^{+\infty} a_j u^j$ is considered as a formal series which is not necessary convergent, or even if it is convergent it is not supposed to converge to $f(u)$ for $u \neq 0$.

Now consider the first factor

$$\frac{\Gamma(\frac{1}{u} - x)}{u^x \Gamma(\frac{1}{u})} \tag{14.1}$$

---

[7]The polynomials in these expansions belong to the class of polynomials that take integer values on integer points. Which obviously does not guarantee that the coefficients are integer.

of the product (13.2). The variable x is a parameter. Let x be a positive integer n. Applying the identity $\Gamma(z + 1) = z\Gamma(z)$, one obtains:

$$\frac{\Gamma(\frac{1}{u} - n)}{u^n \Gamma(\frac{1}{u})} = \frac{1}{(1 - u)(1 - 2u) \ldots (1 - nu)} = \sum_j \gamma_j(n) u^j, \tag{14.2}$$

where $\gamma_j(n) = \sum_{1 \leq i_1 \leq \ldots \leq i_j \leq n} i_1 i_2 \ldots i_j$. It is easy to see that $\gamma_j(n)$ is a polynomial function of n. Now let x be a negative integer $-n$. Similarly one gets

$$\frac{\Gamma(\frac{1}{u} + n)}{u^{-n} \Gamma(\frac{1}{u})} = (1 + u)(1 + 2u) \ldots (1 + (n - 1)u) = \sum_{j=0}^{+\infty} \tilde{\gamma}_j(n) u^j, \tag{14.3}$$

where $\tilde{\gamma}_j(n) = \sum_{1 \leq i_1 < \ldots < i_j \leq n-1} i_1 \ldots i_j$ again are polynomials of n.

Lemma 14.2. *The polynomials $\gamma_j(x)$, $\tilde{\gamma}_j(x)$, $j = 0, 1, 2, \ldots$, are related to each other:*

$$\tilde{\gamma}_j(x) = \gamma_j(-x).$$

Moreover the function (14.1) for any real parameter x has the asymptotic expansion

$$\sum_{j=0}^{+\infty} \gamma_j(x) u^j,$$

when $u \to +0$.

In the sequel the following definition will be used.

Definition 14.3. Let $F(x, u)$ be a function of two variables that has an asymptotic expansion $\sum_{j=0}^{\infty} f_j(x) u^j$ at $u = 0$ defined for any real x. We will say that this expansion is *polynomial* if each function $f_j(x)$ in the expansion is a polynomial of x.

The lemma above says that the function (14.1) has a polynomial expansion when $u \to +0$. If two functions $F(x, u)$ and $G(x, u)$ have a polynomial expansion, then in order to prove or disprove that their expansions coincide it is sufficient to compare their expansions only for a countable set of x's. This advantage of having a polynomial expansion will be used in the proof of Proposition 15.7.

Proof. It is sufficient to show that the function (14.1) has a polynomial asymptotic expansion. Since a polynomial is uniquely determined by its values on the set of positive (resp. negative) integers, the result will follow.

By the generalized Stirling formula [3, p. 24]

$$\Gamma(z) = \sqrt{2\pi} z^{z - \frac{1}{2}} e^{-z + \nu(z)}, \tag{14.4}$$

where $\nu(z)$ has the asymptotic expansion at $z \to +\infty$:

$$\sum_{j=1}^{+\infty} h_j \frac{1}{z^{2j-1}} \tag{14.5},$$

where $h_j = 2(-1)^{j-1}(2j - 2)! \sum_{k=1}^{+\infty} \frac{1}{(2\pi k)^{2j-2}} = (-1)^{j-1}\frac{B_j}{2j(2j-1)}$, where B_j are the Bernoulli numbers:

$$B_1 = 1/6, \quad B_2 = 1/30, \quad B_3 = 1/42, \quad B_4 = 1/30, \quad B_5 = 5/66, \quad \text{etc.}$$

The formal series (14.5) is divergent for any complex z, since the coefficients h_j has a faster then exponential growth.

Applying (14.4), one gets

$$(14.6) \quad \frac{\Gamma(\frac{1}{u} - x)}{u^x \Gamma(\frac{1}{u})} = e^x \cdot (1 - ux)^{\frac{1}{u}} \cdot (1 - xu)^{-x - \frac{1}{2}} \cdot e^{\nu(\frac{1}{u} - x) - \nu(\frac{1}{u})} =$$

$$e^{\left(\frac{x^2 u}{2} - \frac{x^3 u^2}{3} + \frac{x^4 u^3}{4} - \cdots \right)} \cdot (1 - xu)^{-x - \frac{1}{2}} \cdot e^{\nu(\frac{1}{u} - x) - \nu(\frac{1}{u})}.$$

Both the first and the second factors of the second line have a polynomial asymptotic expansion. The asymptotic expansion (14.5) of $\nu(z)$ implies the following asymptotic expansion of $\nu(\frac{1}{u} - x) - \nu(\frac{1}{u})$:

$$\sum_{j=1}^{+\infty} h_j \frac{1}{(\frac{1}{u} - x)^{2j-1}} - \sum_{j=1}^{+\infty} h_j \frac{1}{(\frac{1}{u})^{2j-1}} = \sum_{j=1}^{+\infty} h_j \frac{u^{2j-1}}{(1 - xu)^{2j-1}} - \sum_{j=1}^{+\infty} h_j u^{2j-1},$$

which can be rewritten as $\sum_{j=1}^{+\infty} \rho_j(x) u^j$ for some polynomials $\rho_j(x)$. As a consequence $e^{\nu(\frac{1}{u} - x) - \nu(\frac{1}{u})}$ has also a polynomial asymptotic expansion when $u \to +0$. $\qquad \square$

Notation 14.4. One will denote by

$$\Gamma(x, u) = \sum_{j=0}^{+\infty} \gamma_j(x) u^j \qquad (14.7)$$

the asymptotic expansion of $\frac{\Gamma(\frac{1}{u} - x)}{u^x \Gamma(\frac{1}{u})}$ when $u \to +0$.

Our next goal is to show that the series $\Gamma(x, u)$ does not have nice convergency properties when x is not an integer.

Proposition 14.5. *For any $x \in \mathbb{C} \setminus \mathbb{Z}$ the series $\Gamma(x, u)$ has zero radius of convergence in u.*

This result is classical: it is mentioned in [15]. The argument below is a modification of an argument given in [15], which is given for a completeness of exposition. Notice that the main reason for it is that the function $\frac{\Gamma(\frac{1}{u} - x)}{u^x \Gamma(\frac{1}{u})}$ is not holomorphic in a neighborhood of 0 if x is not an integer. Indeed, this function has poles at $u = \frac{1}{x - n}$, $n \in \mathbb{N}$, concentrating at $u = 0$, and moreover its Riemann surface has a ramification at $u = 0$ due to the factor u^x in the denominator.

Proof. Consider the series

$$\ln \Gamma(x, u) = \sum_{k=1}^{+\infty} \frac{S_k(x)}{k} u^k.$$

It has zero radius of convergence if and only if $\Gamma(x, u)$ has zero radius of convergence. By Lemma 14.2, $S_k(x)$ are polynomials in x. One has to show that

$$\varlimsup_{k \to +\infty} \sqrt[k]{S_k(x)} = +\infty \qquad (14.8)$$

for any $x \in \mathbb{C}/\mathbb{Z}$. It follows from (14.2) that for a positive integer $x = n$, one has $S_k(n) = \sum_{j=0}^{n} j^k$ for $k \geq 1$, which in its turn implies:

$$S(x,t) := \sum_{k=0}^{+\infty} \frac{S_k(x)t^k}{k!} = \frac{e^{(x+1)t} - 1}{e^t - 1}.$$

When x is not integer the above function has poles at $t = \pm 2\pi i$, which means that the radius of convergence of $S(x,t)$ in $t = 0$ is 2π. Therefore

$$\overline{\lim_{k \to +\infty}} \sqrt[k]{\frac{S_k(x)}{k!}} = \frac{1}{2\pi}.$$

And equation (14.8) is proved. □

14.2. **Understanding all factors of** (13.2)-(13.3). It turns out that all the factors of (13.2)-(13.3) can be easily expressed via the first one considered in the previous subsection. Consider an arbitrary factor of (13.2). We are interested in its asymptotic expansion when $u \to +0$. One can rewrite it as follows:

$$\frac{\Gamma(E_\ell(\frac{1}{u}) - E_\ell(x))}{(\ell u^\ell)^{E_\ell(x)}\Gamma(E_\ell(\frac{1}{u}))} = \frac{\Gamma(E_\ell(\frac{1}{u}) - E_\ell(x))}{(E_\ell(\frac{1}{u}))^{-E_\ell(x)}\Gamma(E_\ell(\frac{1}{u}))} \cdot \frac{1}{\left(\ell u^\ell E_\ell\left(\frac{1}{u}\right)\right)^{E_\ell(x)}} \tag{14.9}$$

The first factor has the asymptotic expansion $\Gamma\left(E_\ell(x), \frac{1}{E_\ell(1/u)}\right)$ (see Notation 14.4). The second factor is holomorphic in a neighborhood of $u = 0$ (for all complex x).

Let $F_\ell(y)$ denote the polynomial

$$F_\ell(u) = \ell u^\ell E_\ell(1/u) = \sum_{d|\ell} \mu(d)u^{\ell-\ell/d} = 1 - u^{\ell-\ell/p_1} - u^{\ell-\ell/p_2} + u^{\ell-\ell/p_1 p_2} + \ldots,$$

where p_1, p_2 are first two prime factors of ℓ.

One can rewrite (13.2) and (13.3) as follows:

Lemma 14.6.

$$(i) \quad F_{odd}(x,u) = \frac{\prod_{\ell \geq 1} \Gamma\left(E_\ell(x), \frac{1}{E_\ell(\frac{1}{u})}\right)}{\prod_{\ell \geq 1} [F_\ell(u)]^{E_\ell(x)}} = \frac{\prod_{\ell \geq 1} \Gamma\left(E_\ell(x), \frac{\ell u^\ell}{F_\ell(u)}\right)}{\prod_{\ell \geq 1} [F_\ell(u)]^{E_\ell(x)}},$$

$$(ii) \quad F_{even}(x,u) = \frac{\prod_{\ell \geq 1} \Gamma\left(E_\ell(x), -\frac{1}{E_\ell(\frac{1}{u})}\right)}{\prod_{\ell \geq 1} [F_\ell(u)]^{E_\ell(x)}} = \frac{\prod_{\ell \geq 1} \Gamma\left(E_\ell(x), -\frac{\ell u^\ell}{F_\ell(u)}\right)}{\prod_{\ell \geq 1} [F_\ell(u)]^{E_\ell(x)}}.$$

Proof. (i) follows from (14.9). (ii) is analogous — we only mention that we use that $u^\ell \to -0$ implies $-\frac{1}{E_\ell(\frac{1}{u})} \to +0$. □

15. Proof of Theorem 13.1

The proof of Theorem 13.1 is based on the character computations of the symmetric group action on the homology of configuration spaces [27], and on the components of the Hodge decomposition [17].

Recall that the Hodge splitting appear through the action of the symmetric group (13.1):

$$\text{Tot}^{(i,j)} A_d^\bullet = \oplus_{n=j+1}^{2j} s^{-n} e_n^{(i)}(H_{j(d-1)}^{Norm}(C_d{}^n, \mathbb{Q})), \tag{15.1}$$

where $e_n^{(i)}$, $0 \leq i \leq n$, are the Eulerian idempotents of the symmetric group action. Define a graded symmetric sequence $\chi(-)$ corresponding to the Hodge decomposition:

$$\chi(n) = \oplus_{i=0}^n \chi_i(n) = \oplus_{i=0}^n e_n^{(i)} \cdot \mathbb{K}[S_n]. \tag{15.2}$$

One has

$$e_n^{(i)}(H_{j(d-1)}^{Norm}(C_d{}^n, \mathbb{Q})) \simeq Hom\left(\chi_i(n), H_{j(d-1)}^{Norm}(C_d{}^n, \mathbb{Q})\right)^{S_n}.$$

Define the generating function

$$\Phi_d(x, u, z) = \sum_{i,j,n} \dim\ Hom\left(\chi_i(n), H_{j(d-1)}^{Norm}(C_d{}^n, \mathbb{Q})\right)^{S_n} x^i u^j z^{j(d-1)-n}. \tag{15.3}$$

Since the complex $\text{Tot}^{(i,j)} A_d^\bullet$ for each i and j is finite dimensional, the generating function of the Euler characteristics of the double splitting is

$$F_d(x, u) = \Phi_d(x, u, -1).$$

To compute $\Phi_d(x, u, z)$ we will be using methods of the character computations for symmetric sequences, which are outlined in the next subsection.

15.1. Character computations for symmetric sequences.
The ground field, denoted by $\mathbb{K}$, is as usual of characteristic zero. This section introduces some standard notation which will be used in the sequel.

For each permutation $\sigma \in S_n$ define $Z(\sigma)$, the *cycle indicator* of σ, by

$$Z(\sigma) = \prod_\ell a_\ell^{j_\ell(\sigma)},$$

where $j_\ell(\sigma)$ is the number of ℓ-cycles of σ and where a_1, a_2, a_3, ... is an infinite family of commuting variables.

Remark 15.1. Notice that $Z(\sigma') = Z(\sigma)$ for $\sigma' \in S_{n'}$, $\sigma \in S_n$ if and only if $n' = n$ and moreover σ' is conjugate to σ. For σ with $Z(\sigma) = \prod_\ell a_\ell^{j_\ell(\sigma)}$, there are exactly $\frac{n!}{\prod_\ell(\ell^{j_\ell} j_\ell!)}$ elements σ' conjugate to σ.

Let $\rho^V \colon S_n \to GL(V)$ be a representation of S_n. Define $Z_V(a_1, a_2, \ldots)$ the *cycle index* of V, by

$$Z_V(a_1, a_2, \ldots) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{tr}\, \rho^V(\sigma) \cdot Z(\sigma).$$

Similarly for a symmetric sequence $W = \{W(n), n \geq 0\}$ — sequence of S_n-modules $W(n)$, $n = 0, 1, 2, \ldots$, one defines its *cycle index sum* Z_W by

$$Z_W(a_1, a_2, \ldots) = \sum_{n=0}^{+\infty} Z_{W(n)}(a_1, a_2, \ldots).$$

Definition 15.2. The external tensor product of two symmetric sequences V, and W is a symmetric sequence $V \hat{\otimes} W$ given by

$$V \hat{\otimes} W(n) := \bigoplus_{i=0}^{n} \mathrm{Ind}_{S_i \times S_{n-i}}^{S_n} V(i) \otimes W(n-i) = \bigoplus_{i=0}^{n} \big(V(i) \otimes W(n-i) \big) \otimes_{S_i \times S_{n-i}} \mathbb{K}[S_n].$$

Proposition 15.3. *For any finite symmetric sequences (finite-dimensional in each component) V and W, one has:*

$$Z_{V \hat{\otimes} W}(a_1, a_2, \ldots) = Z_V(a_1, a_2, \ldots) \cdot Z_W(a_1, a_2, \ldots).$$

Proof. This result is standard, the idea is that if N is a subgroup of S_n, and V is a representation of N, then

$$\frac{1}{|N|} \sum_{\sigma \in N} \mathrm{tr}\, \rho^V(\sigma) Z(\sigma) = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \mathrm{tr}\, \rho^{\mathrm{Ind}_N^{S_n} V}(\sigma) Z(\sigma),$$

see [14]. The right hand-side is exactly $Z_{\mathrm{Ind}_N^{S_n} V}(a_1, a_2, \ldots)$. $\square$

Another important property is given by the following lemma:

Lemma 15.4. *Let V and W be two S_n-modules, then*

$$\dim Hom(V, W)^{S_n} = \big(Z_V(a_\ell \leftarrow \partial/\partial a_\ell, \ell \in \mathbb{N})\ Z_W(a_\ell \leftarrow \ell a_\ell, \ell \in \mathbb{N}) \big) \Big|_{\substack{a_\ell = 0, \\ \ell \in \mathbb{N}}}.$$

In the above formula $Z_W(a_\ell \leftarrow \ell a_\ell, \ell \in \mathbb{N})$ is a polynomial obtained from $Z_W(a_1, a_2, \ldots)$ by replacing each variable a_ℓ by ℓa_ℓ. The expression $Z_V(a_\ell \leftarrow \partial/\partial a_\ell, \ell \in \mathbb{N})$ is a differential operator obtained from the polynomial $Z_V(a_1, a_2, \ldots)$ by replacing each a_ℓ by $\partial/\partial a_\ell$. The differential operator is applied to the polynomial and at the end one takes all the variables a_ℓ, $\ell \in \mathbb{N}$, to be zero.[8]

Proof. This follows from the formula

$$\dim Hom(V, W)^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}\, \rho^V(g) \cdot \mathrm{tr}\, \rho^W(g^{-1}),$$

that holds for any finite group G and any its finite-dimensional representations V, W.

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[8]It is easy to see that by a linear change of variables the right-hand side of the above formula is equal to

$$\big(Z_V(a_\ell \leftarrow \ell \partial/\partial a_\ell, \ell \in \mathbb{N})\ Z_W(a_1, a_2, \ldots) \big) \Big|_{\substack{a_\ell = 0, \\ \ell \in \mathbb{N}}},$$

or more symmetrically to

$$\Big(Z_V(a_\ell \leftarrow \sqrt{\ell}\, \partial/\partial a_\ell, \ell \in \mathbb{N})\ Z_W(a_\ell \leftarrow \sqrt{\ell}\, a_\ell, \ell \in \mathbb{N}) \Big) \Big|_{\substack{a_\ell = 0, \\ \ell \in \mathbb{N}}}.$$

Since in the symmetric group any element is conjugate to its inverse, one has

$$\dim Hom(V,W)^{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} tr\rho^V(\sigma) \cdot tr\rho^W(\sigma).$$

The rest follows by a direct computation that uses Remark 15.1. $\qquad\square$

In case $V = \oplus_i V_i$, $W = \oplus_i W_i$ are graded S_n-modules, and $\dim Hom(V,W)^{S_n}$ is the graded dimension:

$$\dim Hom(V,W)^{S_n} = \sum_{i,j \in \mathbb{Z}} \dim Hom(V_i, W_j)^{S_n} z^{j-i},$$

and Z_V, Z_W are *graded* cycle indices:

$$Z_V(z; a_1, a_2, \ldots) = \sum_{i \in \mathbb{Z}} Z_{V_i}(a_1, a_2, \ldots) z^i,$$

$$Z_W(z; a_1, a_2, \ldots) = \sum_{i \in \mathbb{Z}} Z_{W_i}(a_1, a_2, \ldots) z^i.$$

Then

$$\dim Hom(V,W)^{S_n} = Z_V(1/z; a_\ell \leftarrow \partial/\partial a_\ell, \ell \in \mathbb{N}) \ Z_W(z; a_\ell \leftarrow \ell a_\ell, \ell \in \mathbb{N})\Big|_{\substack{a_\ell=0, \\ \ell \in \mathbb{N}}}.$$

Corollary 15.5. *Let $V = \{V(n), n \geq 0\}$, $W = \{W(n), n \geq 0\}$ be a pair of symmetric sequences of graded S_n-modules. Then*

$$(15.4) \quad \dim Hom(V,W) = \dim \big(\oplus_n Hom(V(n), W(n))^{S_n}\big) =$$

$$= Z_V(1/z; a_\ell \leftarrow \partial/\partial a_\ell, \ell \in \mathbb{N}) \ Z_W(z; a_\ell \leftarrow \ell a_\ell, \ell \in \mathbb{N})\Big|_{\substack{a_\ell=0, \\ \ell \in \mathbb{N}}}.$$

Later on we will also consider bigraded symmetric sequences. Similarly in our computations we will add one more variable x or u responsible for the second grading.

15.2. **Symmetric sequences $A_d{}^\bullet$, $NA_d{}^\bullet$.** The symmetric group action on the homology of configuration spaces $C_d(n) = F(n, \mathbb{R}^d)$ is well studied [11, 27, 28].

Proposition 15.6. *The graded cycle index sum for the symmetric sequence*

$$A_d{}^\bullet = \{A_d{}^n | n \geq 0\} = \{H_*(C_d(n), \mathbb{K}) | n \geq 0\}$$

is given by the following formula:

$$Z_{A_d{}^\bullet}(z; a_1, a_2, \ldots) = \prod_{\ell=1}^{+\infty} \left(1 + (-1)^d (-z)^{(d-1)\ell} a_\ell\right)^{(-1)^d E_\ell\left(\frac{1}{(-z)^{d-1}}\right)}, \qquad (15.5)$$

where $E_\ell(y) = \frac{1}{\ell} \sum_{d|\ell} \mu(d) y^{\frac{\ell}{d}}$.

Proof. It is an easy consequence of [27, Theorem B] and Remark 15.1. $\qquad\square$

The author failed to find this formula in the literature, however one can find similar formulae in the study of S_n-modules closely related to the homology of configuration spaces [9, 17].

Let us add another variable u that will be responsible for the complexity, which is the homology degree divided by $(d-1)$. The u-degree is the z-degree divided by $(d-1)$:

$$Z_{A_d}\bullet(z, u; a_1, a_2, \ldots) = \prod_{\ell=1}^{+\infty} \left(1 + (-1)^d((-z)^{(d-1)}u)^\ell a_\ell\right)^{(-1)^d E_\ell\left(\frac{1}{(-z)^{d-1}u}\right)}.$$

Consider the symmetric sequence $NA_d^\bullet = \{NA_d{}^n | n \geq 0\} = \{H_*^{Norm}(C_d(n), \mathbb{K}) | n \geq 0\}$. It is easy to see that

$$(15.6) \qquad A_d{}^n \simeq \bigoplus_{i=0}^{n} \mathrm{Ind}_{S_i \times S_{n-i}}^{S_n} NA_d{}^i,$$

Where the S_i-module $NA_d{}^i$ is considered as an $S_i \times S_{n-i}$-module being acted on trivially by the second factor S_{n-i}.

According to Definition 15.2 formula (15.6) means that

$$A_d{}^\bullet \simeq NA_d{}^\bullet \hat{\otimes} I,$$

where $I = \{I(n), n \geq 0\}$ is a sequence of trivial 1-dimensional representations. Using Remark 15.1 it is easy to check that

$$Z_I(a_1, a_2, \ldots) = \prod_{\ell \geq 1} e^{\frac{a_\ell}{\ell}}.$$

It follows from Proposition 15.3

$$(15.7) \qquad Z_{NA_d}\bullet(z, u; a_1, a_2, \ldots) = \prod_{\ell=1}^{+\infty} e^{-\frac{a_\ell}{\ell}} \left(1 + (-1)^d((-z)^{(d-1)}u)^\ell a_\ell\right)^{(-1)^d E_\ell\left(\frac{1}{(-z)^{d-1}u}\right)}.$$

15.3. **Symmetric sequence of the Hodge decomposition.** Recall that by $\chi(-) = \{\chi(n) | n \geq 0\}$ one denotes the graded (by i) symmetric sequence corresponding to the Hodge decomposition:

$$\chi(n) = \oplus_i e_n^{(i)} \cdot \mathbb{K}[S_n]. \qquad (15.8)$$

It was shown by Hanlon [17, equation (6.1)] that the graded cycle index sum of $\chi(-)$ is given by the following formula:

$$Z_{\chi(-)} = \prod_{\ell}(1 + (-1)^\ell a_\ell)^{-E_\ell(x)}, \qquad (15.9)$$

where the variable x is responsible for the Hodge degree i, and $E_\ell(x) = \frac{1}{\ell}\sum_{d|\ell} \mu(d) x^{\ell/d}$. Notice that this cycle index sum is actually very similar to (15.5).

15.4. **Proof of Theorem 13.1.** First notice that Corollary 15.5 together with the formulae (15.7), and (15.9) produce the following formula for the generating function $\Phi_d(x, u, z)$ of the dimensions of the complex computing $H_*(\overline{Emb}_d, \mathbb{Q})$, $d \geq 4$:

$$(15.10) \quad \Phi_d(x, u, z) =$$

$$\left(\prod_{\ell=1}^{+\infty} \left(1+(-1/z)^\ell \partial/\partial a_\ell\right)^{-E_\ell(x)} \prod_{\ell=1}^{+\infty} e^{-a_\ell} \left(1+(-1)^d \ell((-z)^{(d-1)}u)^\ell a_\ell\right)^{(-1)^d E_\ell\left(\frac{1}{(-z)^{d-1}u}\right)} \right) \Bigg|_{\substack{a_\ell=0 \\ \ell \in \mathbb{N}}} =$$

$$\prod_{\ell=1}^{+\infty} \left(\left(1+(-1/z)^\ell \partial/\partial a_\ell\right)^{-E_\ell(x)} e^{-a_\ell} \left(1+(-1)^d \ell((-z)^{(d-1)}u)^\ell a_\ell\right)^{(-1)^d E_\ell\left(\frac{1}{(-z)^{d-1}u}\right)} \right) \Bigg|_{a_\ell=0}.$$

In the above formula one has $\left(1+(-1/z)^\ell \partial/\partial a_\ell\right)$ instead of $\left(1+(-1)^\ell \partial/\partial a_\ell\right)$ because of the desuspensions s^{-n} in the definition of the totalized complex (2.1).

Since $F_d(x, u) = \Phi_d(x, u, -1)$ one gets

$$F_d(x, u) = \prod_{\ell=1}^{+\infty} \left((1+\partial/\partial a)^{-E_\ell(x)} e^{-a} \left(1+(-1)^d \ell u^\ell a\right)^{(-1)^d E_\ell\left(\frac{1}{u}\right)} \right) \Bigg|_{a=0}.$$

Notice that in the above formula we replaced a_ℓ by a. One could do so because each factor uses only one variable a_ℓ which is anyway taken to be zero.

Theorem 13.1 follows immediately from the following proposition:

Proposition 15.7.

$$(15.11) \quad \left((1+\partial/\partial a)^{-E_\ell(x)} e^{-a} \left(1+(-1)^d \ell u^\ell a\right)^{(-1)^d E_\ell\left(\frac{1}{u}\right)} \right) \Bigg|_{a=0} =$$

$$= \frac{\Gamma\left((-1)^{d-1} E_\ell\left(\frac{1}{u}\right) - E_\ell(x)\right)}{\left((-1)^{d-1} \ell u^\ell\right)^{E_\ell(x)} \Gamma\left((-1)^{d-1} E_\ell\left(\frac{1}{u}\right)\right)},$$

where the right-hand side is understood as the asymptotic expansion of the underlying function when $(-1)^{d-1} u^\ell \to +0$.

Proof. For simplicity consider the case of odd d, and $\ell = 1$. Other cases are absolutely analogous. In this situation the left-hand side becomes

$$\left((1+\partial/\partial a)^{-x} e^{-a} (1-ua)^{-\frac{1}{u}} \right) \Bigg|_{a=0}, \tag{15.12}$$

the right-hand side is

$$\frac{\Gamma\left(\frac{1}{u}-x\right)}{u^x \Gamma\left(\frac{1}{u}\right)} = \Gamma(x, u). \tag{15.13}$$

Notice that both (15.12), and (15.13) have a polynomial asymptotic expansion. Indeed, (15.12) has such expansion because the normalized complex Tot $A_d^\bullet$ is finite in each complexity j, the expression (15.13) has this form by Lemma 14.2. We conclude that it suffices to check the equality when x is any negative integer number: $x = -n$. In this case

$$\Gamma(-n, u) = (1+u)(1+2u)\ldots(1+(n-1)u).$$

One can also prove by induction over n that

$$(1 + \partial/\partial a)^n \, e^{-a} \, (1 - ua)^{-\frac{1}{u}} = \Gamma(-n, u) \cdot e^{-a}(1 - ua)^{-\frac{1}{u} - n}.$$

Taking $a = 0$ implies the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

15.5. **Alternative proof of Proposition 15.7.** There is another proof which makes more natural the appearance of the gamma function. The proof is more technical, so we give only its idea. It uses the following lemma:

Lemma 15.8. *Let X be a complex number with a positive real part, and $f(a)$ be any polynomial, then*

$$(1 + \partial/\partial a)^{-X} f(a)\big|_{a=0} = \frac{1}{\Gamma(X)} \int_{-\infty}^{0} (-a)^{X-1} e^a f(a) \, da.^9$$

To prove the lemma we notice that for $f(a) = a^n$ both sides are equal to $(-1)^n X(X + 1)(X + 2) \ldots (X + n - 1)$.

Now we apply the above lemma for $X = E_\ell(x)$ and taking instead of $f(a)$ the generating function of a sequence of polynomials $\sum_j f_j(a) u^j = e^{-a} \left(1 - (-1)^{d-1} \ell u^\ell a\right)^{-(-1)^{d-1} E_\ell\left(\frac{1}{u}\right)}$:

$$\left((1 + \partial/\partial a)^{-E_\ell(x)} \, e^{-a} \left(1 - (-1)^{d-1} \ell u^\ell a\right)^{-(-1)^{d-1} E_\ell\left(\frac{1}{u}\right)} \right)\Bigg|_{a=0} =$$

$$\frac{1}{\Gamma(E_\ell(x))} \int_{-\infty}^{0} (-a)^{E_\ell(x)-1} \left(1 - (-1)^{d-1} \ell u^\ell a\right)^{-(-1)^{d-1} E_\ell\left(\frac{1}{u}\right)} da. \qquad (15.14)$$

Assuming that u is a small complex number such that $(-1)^{d-1} u^\ell$ is real positive one makes a change of variables

$$a = -\frac{(-1)^{d-1}}{\ell u^\ell} \cdot \frac{t}{1 - t},$$

that gives that the above expression is equal to the following:

$$\frac{1}{((-1)^{d-1} \ell u^\ell)^{E_\ell(x)} \, \Gamma(E_\ell(x))} \int_0^1 t^{E_\ell(x)-1} (1 - t)^{(-1)^{d-1} E_\ell\left(\frac{1}{u}\right) - E_\ell(x) - 1} \, dt =$$

$$= \frac{1}{((-1)^{d-1} \ell u^\ell)^{E_\ell(x)} \, \Gamma(E_\ell(x))} \times \frac{\Gamma(E_\ell(x)) \cdot \Gamma((-1)^{d-1} E_\ell\left(\frac{1}{u}\right) - E_\ell(x))}{\Gamma((-1)^{d-1} E_\ell\left(\frac{1}{u}\right))} =$$

$$= \frac{\Gamma((-1)^{d-1} E_\ell\left(\frac{1}{u}\right) - E_\ell(x))}{((-1)^{d-1} \ell u^\ell)^{E_\ell(x)} \, \Gamma((-1)^{d-1} E_\ell\left(\frac{1}{u}\right))}.$$

However the proof has a serious analytical gap since the series $\sum_j f_j(a) u^j$ does not converge to $e^{-a} \left(1 - (-1)^{d-1} \ell u^\ell a\right)^{-(-1)^{d-1} E_\ell\left(\frac{1}{u}\right)}$ when $|a| > \frac{1}{|\ell u^\ell|}$. To make this proof work one needs to split the integral (15.14) as a sum $\int_{-\infty}^{-\frac{(-1)^{d-1}}{\ell u^\ell}} + \int_{-\frac{(-1)^{d-1}}{\ell u^\ell}}^{0}$. It is easy to see that the first integral has

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[9]This formula was obtained by using Fourier transform which permitted to rewrite the differential operator $(1 + \partial/\partial a)^{-X}$ as an integral operator.

zero asymptotic expansion with respect to u when $(-1)^{d-1}u^\ell \to +0$. The second integral can now be replaced by a series of integrals, whose expansion in u has to be studied.

16. Results of computations

In the Appendix the results of computations of the Euler characteristics are presented. These computations were produced using Maple. Let h_{ijk} denote the rank of the (i,j)-component of $H_k(\overline{Emb}_d, \mathbb{Q})$. Similarly let π_{ijk} denote the rank of the (i,j)-component of $\pi_k(\overline{Emb}_d) \otimes \mathbb{Q}$. The homotopy Euler characteristics will be denoted by χ_{ij}^π:

$$\chi_{ij}^\pi = \sum_k (-1)^k \pi_{ijk}.$$

The following lemma describes how the homotopy Euler characteristics can be obtained from the homology Euler characteristics.

Lemma 16.1.

$$F_d(x,u) = \sum_{ij} \chi_{ij} x^i u^j = \prod_{ij} \frac{1}{(1 - x^i u^j)^{\chi_{ij}^\pi}}. \tag{16.1}$$

Proof. By Proposition 7.1 one has

$$\sum_{ijk} h_{ijk} x^i u^j z^k = \prod_{\substack{ij \\ k \text{ odd}}} (1 + x^i u^j z^k)^{\pi_{ijk}} \Big/ \prod_{\substack{ij \\ k \text{ even}}} (1 - x^i u^j z^k)^{\pi_{ijk}} .$$

Taking $z = -1$, one obtains that the left-hand side is $F(x,u) = \sum_{ij} \chi_{ij} x^i u^j$, and the right-hand side is

$$\prod_{\substack{ij \\ k \text{ odd}}} (1 - x^i u^j)^{\pi_{ijk}} \Big/ \prod_{\substack{ij \\ k \text{ even}}} (1 - x^i u^j)^{\pi_{ijk}} = \prod_{ij} \frac{1}{(1 - x^i u^j)^{\chi_{ij}^\pi}} .$$

$\square$

The formula (16.1) was used to fill the Tables 1 and 3 in the Appendix. The column "total" in the tables states for the sum of absolute values of χ_{ij}^π for a fixed complexity j:

$$total = \sum_i |\chi_{ij}^\pi|.$$

It gives a lower bound estimate for the rank of rational homotopy in a given complexity j. We did not make this column for the homology tables since a better lower bound estimate of the homology rank in a given complexity can be obtained using the Hodge decomposition in homotopy.

It is interesting to compare the first table with the table of primitive elements in the bialgebra of chord diagrams [19] which is copied below:

rk_{ij}	$i=2$	$i=4$	$i=6$	$i=8$	$i=10$	$i=12$	total
$j=1$	1						1
$j=2$	1						1
$j=3$	1						1
$j=4$	1	1					2
$j=5$	2	1					3
$j=6$	2	2	1				5
$j=7$	3	3	2				8
$j=8$	4	4	3	1			12
$j=9$	5	6	5	2			18
$j=10$	6	8	8	4	1		27
$j=11$	8	10	11	8	2		39
$j=12$	9	13	15	12	5	1	55

Recall that Bar-Natan [6] described the space of primitives of the bialgebra of chord diagrams as the space of uni-trivalent graphs modulo STU and IHX relations (Theorem 8.1). The latter space has a natural double grading. The first grading complexity j – for any graph it is the first Betti number of the graph obtained by gluing all the univalent vertices together – this grading corresponds to the number of chords in chord diagrams. The second grading is the number i of univalent vertices. It turns out that the last grading is exactly our Hodge degree, see Theorem 8.2. In our terms the above table describes the rank of the $j(d-3)$-dimensional rational homotopy $\pi_{j(d-3)}^{(i,j)}(\overline{Emb}_d, \mathbb{Q})$ (d being odd) in complexity j and Hodge degree i. Notice that there is no non-trivial generators in odd Hodge degree. From the point of view of knot theory this means that up to the order 12 Vassiliev invariants are orientation insensitive.[10] It rises the question whether Vassiliev invariants can distinguish a knot from its inverse. More generally looking at Table 1 one can ask whether even cycles are all in even Hodge degrees and all odd cycles are all in odd Hodge degrees? Comparing the above table with Table 1 one can see that there must be at least one odd cycle in complexity 10 and of Hodge degree 2. Indeed, in this bigrading one has $\chi_{2,10}^{\pi} = 5$, but the rank of the primitives of the bialgebra of chord diagrams is 6. Even more dramatically it turns out that the sign of χ_{ij}^{π} can be different from $(-1)^i$. The first counter example appears in complexity 20:

$$\chi_{1,20}^{\pi} = 12 > 0,$$

see Table 1.

It would be interesting to understand the geometrical reason for this phenomenon of sign alternation for small complexities. Notice that this happens only when d is odd (in other Tables 3-4 the signs of entries look rather random). One should also try to compute the Table 1 for higher complexities j to check wether this almost alternation of signs keeps take place or completely disappears. (Using Maple we could do it only up to $j = 23$). As a conclusion one should say that

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[10]Recall that a cycle is of a Hodge degree i if it is an eigen vector with eigen value n^i with respect to any cabling Φ_n. In particular for a (-1)-cabling the eigen value is $(-1)^i$. For the dimension 3 one should be more careful, because the definition of cabling maps is not unique up to homotopy, but still cablings produce uniquely defined maps on the level of the bialgebra of chord diagrams – the dual to the graded quotient of the space of finite type invariants. Obviously, the (-1)-cabling on the level of the isotopy classes of knots is the change of orientation, see Figure B. Notice also that any rational knot invariant is a sum of an orientation insensitive invariant and an invariant that changes its sign when orientation changes. Therefore existence of odd Hodge degree elements in the bialgebra of chord diagrams is equivalent to existence of orientation sensitive Vassiliev invariants.

these results give some optimism for finding Vassiliev invariants that can distinguish orientation of a knot. Personally I would try with the complexity 22 and Hodge degree 5!

17. Exponential growth of the homology and homotopy of $\overline{Emb}_d$ or taking $F(\pm 1, u)$

One can get a lower bound estimation for the rank of the homology groups in a given complexity j by taking $x = \pm 1$ in the formula for $F(x, u) = \sum_{i,j} \chi_{ij} x^i u^j$. We notice first that

$$E_\ell(1) = \begin{cases} 1, & \text{if } \ell = 1 \\ 0, & \ell \geq 2; \end{cases} \qquad E_\ell(-1) = \begin{cases} (-1)^\ell, & \text{if } \ell = 1 \text{ or } 2 \\ 0, & \ell \geq 3. \end{cases}$$

This means that for $x = 1$ only the first factor of the product (13.2)-(13.3) can be different from 1; and for $x = -1$ only the first two factors can differ from 1.

Easy computations show that

$$F_{odd}(1, u) = \frac{1}{1 - u}, \qquad F_{odd}(-1, u) = \frac{1}{1 - u - 2u^2}.$$

$$F_{even}(1, u) = \frac{1}{1 + u}, \qquad F_{even}(-1, u) = \frac{1}{1 - u + 2u^2}.$$

From the above formulas one will derive the following result.

Theorem 17.1. *The rank of the rational homology of $\overline{Emb}_d$ in a given complexity j grows at least exponentially with j.*

Proof. Consider first the case when d is odd. The formula $F_{odd}(-1, u) = \frac{1}{1-u-2u^2} = \frac{1}{(1+u)(1-2u)} = \frac{1/3}{1+u} + \frac{2/3}{1-2u}$ implies at least exponential growth $\approx \frac{2}{3} 2^j$ of the rank of the homology groups in complexity j in this case.

Similarly in the case of even d one has $F_{even}(-1, u) = \frac{1}{1-u+2u^2} = \frac{1}{(1 - \frac{1+\sqrt{-7}}{2} u)(1 - \frac{1-\sqrt{-7}}{2} u)} = \sum_j a_j u^j$ with $a_j = \frac{2}{\sqrt{7}} \text{Im} \left(\frac{1+\sqrt{-7}}{2} \right)^{j+1}$. Using Baker's theorem [4, 5] one can get that a_j has also exponential growth,[11] more precisely for some $C_1 > 0$ and $C_2 > 0$ one has

$$|a_j| > C_1 \left| \frac{1 + \sqrt{-7}}{2} \right|^j \bigg/ j^{C_2} = C_1 \frac{2^{j/2}}{j^{C_2}}.$$

Indeed, Baker's theorem can be formulated as follows [4, III].

Theorem. *(A. Baker [4, III]) If $\alpha_1, \alpha_2, \ldots, \alpha_n$, and $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are algebraic of degree at most D and heights at most A and B (assuming that $B \geq 2$) respectively, then*

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n$$

has $\Lambda = 0$ or $|\Lambda| > B^{-C}$ where C is a constant depending only on n, D, and A.

To recall the *height* of an algebraic number is the maximum of the absolute values of the relatively prime integer coefficients in the minimal defining polynomial.

One will need this theorem only when $n = 2$ (and also when $n = 3$ for the proof of Theorem 17.2).

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[11]I am gratefull to C. Pinner for the argument that follows.

Take $\alpha_1 = \frac{1+\sqrt{7}i}{2\sqrt{2}} = e^{i\pi\theta}$, where $0 < \theta < \frac{1}{2}$, and $\alpha_2 = -1 = e^{i\pi}$. So one has $\log\alpha_1 = i\pi\theta$, $\log\alpha_2 = i\pi$. One has

$$|a_j| = \left| \frac{2}{\sqrt{7}}\text{Im}\left(\frac{1+i\sqrt{7}}{2}\right)^{j+1} \right| = \frac{2}{\sqrt{7}}\sqrt{2}^{j+1}|\sin(\pi(j+1)\theta)| =$$

$$= \frac{2}{\sqrt{7}}\sqrt{2}^{j+1}\sin\pi||(j+1)\theta|| \geq \frac{2}{\sqrt{7}}\sqrt{2}^{j+1}\frac{2}{\pi}\pi||(j+1)\theta||,$$

where $||x||$ denote the distance to the nearest integer. Let $n = \lfloor(j+1)\theta\rfloor$ or $\lfloor(j+1)\theta\rfloor+1$ be this nearest integer. Obviously, $n \leq j+1$ since $\theta < \frac{1}{2}$. The right-hand side is

$$\frac{4}{\pi\sqrt{7}}\sqrt{2}^{j+1}|\pi((j+1)\theta-n)| = \frac{4}{\pi\sqrt{7}}\sqrt{2}^{j+1}|(j+1)\log\alpha_1 - n\log\alpha_2| > \frac{4}{\sqrt{7}\pi}\sqrt{2}^{j+1}(j+1)^{-C}.$$

The last inequality uses Baker's theorem. The only thing one needs to check is that $(j+1)\theta \neq n$, or in other words that α_1 is not a root of unity. But α_1 is not even an algebraic integer since its minimal polynomial is

$$\prod\left(x \pm \left(\frac{1\pm\sqrt{7}i}{2\sqrt{2}}\right)\right) = x^4 + \frac{3}{2}x^2 + 1.$$

$\square$

A similar result holds for the rational homotopy of these spaces.

Theorem 17.2. *The rank of the rational homotopy of $\overline{Emb}_d$ in a given complexity j grows at least exponentially with j.*

The proof is based on a few observations. Let

$$\chi_j = \sum_i \chi_{ij}, \qquad \chi_j^\pi = \sum_i \chi_{ij}^\pi$$

denote the Euler characteristic of the homology, resp. homotopy of $\overline{Emb}_d$ in complexity j. One has

$$F_d(1,u) = \sum_j \chi_j u^j = \prod_j \frac{1}{(1-u^j)^{\chi_j^\pi}}.$$

The last equality is a consequence of Lemma 16.1. For the proof of Theorem 17.2 one will need the following lemma.

Lemma 17.3 (Scannell, Sinha [38]). *The Euler characteristic of the rational homotopy of $\overline{Emb}_d$ in complexity j for odd d is*

$$\chi_j^\pi = \begin{cases} 1, & \text{if } j = 1; \\ 0, & j \geq 2; \end{cases}$$

for even d is

$$\chi_j^\pi = \begin{cases} (-1)^j, & \text{if } j = 1 \text{ or } 2; \\ 0, & j \geq 3. \end{cases}$$

Proof. The first assertion is true since

$$F_{odd}(1, u) = \frac{1}{1 - u} = \frac{1}{(1 - u)^1}.$$

The second one is true since

$$F_{even}(1, u) = \frac{1}{1 + u} = \frac{1}{(1 - u)^{-1}} \cdot \frac{1}{(1 - u^2)^1}.$$

$\square$

Proof of Theorem 17.2. Denote by $\chi_{E,j}^{\pi}$, resp. $\chi_{O,j}^{\pi}$ the sum of Euler characteristics over even, resp. odd Hodge degrees:

$$\chi_{E,j}^{\pi} = \sum_{i \text{ even}} \chi_{i,j}^{\pi}, \qquad \chi_{O,j}^{\pi} = \sum_{i \text{ odd}} \chi_{i,j}^{\pi}.$$

It follows from Lemma 16.1 that

$$F_d(-1, u) = \prod_i \frac{(1 + u^j)^{-\chi_{O,j}^{\pi}}}{(1 - u^j)^{\chi_{E,j}^{\pi}}}.$$

Since $\chi_{E,j}^{\pi} + \chi_{O,j}^{\pi} = \chi_j^{\pi} = 0$ for $j \geq 2$ in case of odd d (and for $j \geq 3$ in case of even d, see Lemma 17.3), one has:

$$F_{odd}(-1, u) = \frac{1}{1 - u} \prod_{j \geq 2} \left(\frac{1 + u^j}{1 - u^j} \right)^{\chi_{E,j}^{\pi}} = \frac{1}{1 - u - 2u^2} = \frac{1}{(1 + u)(1 - 2u)}.$$

(We used that $\chi_{E,1}^{\pi} = 1$ and $\chi_{O,1}^{\pi} = 0$, see the first row of Table 1.) Or, equivalently,

$$\prod_{j \geq 1} \left(\frac{1 + u^j}{1 - u^j} \right)^{\chi_{E,j}^{\pi}} = \frac{1}{1 - 2u}.$$

Applying logarithmic derivative, and multiplying each side by u, one has:

$$\sum_{j \geq 1} 2j \chi_{E,j}^{\pi} \left(\frac{u^j}{1 - u^{2j}} \right) = \frac{2u}{1 - 2u}.$$

Therefore,

$$\sum_{\substack{k|j \\ k \text{ odd}}} 2\frac{j}{k} \chi_{E, \frac{j}{k}}^{\pi} = 2^j.$$

Using Möbius transformation one obtains:

$$\chi_{E,j}^{\pi} = \frac{1}{2j} \sum_{\substack{k|j \\ k \text{ odd}}} \mu(k) 2^{j/k} = \frac{1}{2j} 2^j + \frac{1}{2j} \sum_{\substack{k|j \\ k > 1 \text{ odd}}} \mu(k) 2^{j/k}.$$

The above formula implies that, in case of odd d, $\chi_{E,j}^{\pi}$ has asymptotics $2^j/2j$. Indeed, since the number of divisors is always less than the number itself, the absolute value of the right summand is less than $\frac{1}{2} 2^{j/2}$ which is infinitely small compared to $2^j/2j$. Since the rank of rational homotopy in complexity j is greater than $\chi_{E,j}^{\pi}$, one gets the result of the theorem for odd d.

The case of even d is obtained in a similar way. One has

$$F_{even}(-1, u) = \frac{1+u}{1} \cdot \frac{(1+u^2)^{-1}}{1} \cdot \prod_{j \geq 3} \left(\frac{1+u^j}{1-u^j} \right)^{\chi^{\pi}_{E,j}} = \frac{1}{1-u+2u^2}$$

(see the first and the second rows of Table 3.). Equivalently,

$$\prod_{j \geq 1} \left(\frac{1+u^j}{1-u^j} \right)^{\chi^{\pi}_{E,j}} = \frac{1+u^2}{(1+u)(1-u+2u^2)}.$$

Taking logarithmic derivative and multiplying each side by u, one obtains:

$$\sum_{j \geq 1} 2j\chi^{\pi}_{E,j} \left(\frac{u^j}{1-u^{2j}} \right) = \frac{2u^2}{1+u^2} - \frac{u}{1+u} + \frac{u-4u^2}{1-u+2u^2} = \sum_j A_j u^j.$$

Using an argument similar to the proof of Theorem 17.1, one can show that the sequence $|A_j|$ starting from some j is bounded by

$$\frac{C_1 2^{j/2}}{j^\alpha} < |A_j| < C_2 2^{j/2} \tag{17.1}$$

for some positive constants C_1, C_2, α. Using Möbius transformation, one has

$$\chi^{\pi}_{E,j} = \frac{1}{2j} \sum_{\substack{k|j \\ k \text{ odd}}} \mu(k) A_{j/k} = \frac{1}{2j} A_j + \frac{1}{2j} \sum_{\substack{k|j \\ k > 1 \text{ odd}}} \mu(k) A_{j/k}.$$

Using the lower bound of (17.1) for the first summand, and the upper bound of (17.1) to estimate the second one, we see that $\chi^{\pi}_{E,j}$ in case of even d again has an exponential growth. $\qquad \square$

An immediate consequence of Theorems 17.1, 17.2 is the following.

Theorem 17.4. *The cumulative ranks of the rational homology and of rational homotopy*

$$\text{rank}(H_{\leq n}(\overline{Emb}_d, \mathbb{Q})), \qquad \text{rank}(\pi_{\leq n}(\overline{Emb}_d) \otimes \mathbb{Q})$$

grow at least exponentially with n.

Proof. The idea is that the homology/homotopy in complexity j has the total degree less than $j(d-2)$, see Section 13. Therefore for a given n the cumulative homology/homotopy is sure to contain the whole homology/homotopy of complexity $\lfloor \frac{n}{d-2} \rfloor$. $\qquad \square$

Appendix A. Tables

									Hodge degree i															
j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total
1		1																							1
2	-1	1																							2
3	-1	1																							2
4	-1	1	-1	1																					4
5	-2	2	-1	1																					6
6	-1	2	-3	2	-1	1																			10
7	-2	3	-4	4	-3	2																			18
8	-2	4	-6	7	-6	4	-2	1																	32
9	-2	5	-10	12	-11	9	-5	2																	56
10	-1	5	-14	20	-22	19	-12	6	-2	1															102
11	-2	7	-17	30	-39	38	-29	16	-6	2															186
12		5	-22	45	-66	72	-60	40	-20	7	-2	1													340
13		4	-25	60	-104	133	-125	91	-52	24	-9	3													630
14	-1	2	-22	79	-155	221	-244	203	-130	68	-30	11	-3	1											1170
15	-1	3	-17	81	-217	368	-445	413	-308	186	-91	37	-12	3											2182
16	-4	3	-12	83	-275	549	-769	823	-685	455	-255	121	-45	13	-3	1									4096
17	-18	19	-12	79	-307	751	-1258	1528	-1422	1071	-672	351	-152	53	-14	3									7710
18	-20	59	-83	65	-257	964	-1943	2651	-2781	2369	-1666	969	-465	186	-62	16	-3	1							14560
19	-13	124	-188	59	-298	1234	-2646	4224	-5203	4983	-3850	2486	-1353	613	-228	70	-18	4							27594
20	12	115	-225	442	-807	1202	-3068	6527	-9208	9707	-8379	6075	-3672	1847	-781	278	-82	20	-4	1					52452
21	158	-281	-607	1998	-1171	378	-4068	9921	-14656	17558	-17437	14053	-9307	5204	-2483	999	-336	95	-22	4					100736
22	638	-457	-2294	2080	613	3026	-8531	11888	-20247	30923	-34589	30247	-22186	13900	-7367	3287	-1248	406	-110	24	-4	1			194066
23	480	1706	967	-7614	-6392	20835	-8447	5974	-31163	54026	-61977	60522	-50681	35146	-20337	10068	-4302	1570	-484	124	-25	4			382844

TABLE 1. Table of Euler characteristics χ_{ij}^{π} by complexity j and Hodge degree i of $\pi_*(\overline{Emb_d}) \otimes \mathbb{Q}$ for odd d.

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	Hodge degree i																						
j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1		1																					
2	-1	1		1																			
3	-1	1	-1	1		1																	
4	-1	1	-3	3	-1	1		1															
5	-2	3	-4	3	-3	3	-1	1		1													
6	-1	3	-8	8	-8	6	-3	3	-1	1		1											
7	-2	6	-11	13	-17	13	-8	6	-3	3	-1	1		1									
8	-2	7	-16	25	-30	25	-25	18	-8	6	-3	3	-1	1		1							
9	-2	10	-25	41	-55	57	-51	34	-25	18	-8	6	-3	3	-1	1		1					
10	-1	11	-36	67	-95	108	-107	86	-65	41	-25	18	-8	6	-3	3	-1	1		1			
11	-2	15	-45	101	-166	207	-217	188	-150	102	-65	41	-25	18	-8	6	-3	3	-1	1		1	
12		14	-61	153	-267	367	-422	405	-340	244	-173	113	-65	41	-25	18	-8	6	-3	3	-1	1	
13		15	-74	210	-421	648	-795	820	-743	584	-422	271	-173	113	-65	41	-25	18	-8	6	-3	3	-1
14	-1	12	-83	290	-633	1063	-1449	1629	-1557	1307	-1003	692	-460	286	-173	113	-65	41	-25	18	-8	6	-3
15	-1	13	-88	355	-919	1730	-2529	3065	-3190	2882	-2308	1673	-1152	736	-460	286	-173	113	-65	41	-25	18	-8
16	-4	14	-97	435	-1256	2628	-4264	5662	-6288	6027	-5164	3991	-2826	1862	-1211	758	-460	286	-173	113	-65	41	-25
17	-18	29	-105	510	-1646	3858	-6917	9960	-11959	12375	-11226	9101	-6791	4701	-3082	1930	-1211	758	-460	286	-173	113	-65
18	-20	72	-205	600	-2016	5418	-10793	16931	-21970	24380	-23612	20356	-15886	11412	-7761	5026	-3172	1960	-1211	758	-460	286	-173
19	-13	155	-348	726	-2645	7467	-15913	27513	-39089	46618	-48051	43930	-36222	27330	-19176	12706	-8187	5131	-3172	1960	-1211	758	-460
20	12	164	-461	1415	-3897	9561	-22616	43567	-66750	85597	-94842	92232	-80218	63394	-46401	31900	-20912	13236	-8322	5173	-3172	1960	-1211
21	158	-217	-956	3381	-5196	11975	-32592	66403	-109135	152157	-181121	187515	-172652	143782	-109800	78122	-52716	34181	-21600	13392	-8322	5173	-3172
22	638	-391	-2652	3573	-5265	20670	-48875	94966	-171890	262183	-334105	368926	-360657	316730	-253621	187909	-130721	86694	-55715	35034	-21798	13448	-8322
23	480	1619	1446	-5648	-17669	45575	-61584	130666	-271978	436507	-591664	702289	-731813	678875	-570539	441262	-318509	217419	-142337	90568	-56797	35263	-21798

TABLE 2. Table of Euler characteristics χ_{ij} by complexity j and Hodge degree i of $H_*(\overline{Emb}_d, \mathbb{Q})$ for odd d.

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	Hodge degree i																							
j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total
1	-1																							1
2			1																					1
3	1	-1																						2
4																								0
5	1		-1	1	-1																			4
6	-1	1																						2
7	1		-1		1	-1																		4
8	-1	1	1	-2	2	-1																		8
9			2	-1			-1	1	-1															6
10	-2		3	-2		3	-4	2																16
11	2	-3	1	3	-6	5	-1	-1	1	-1														24
12		-1		4	-6	4	2	-6	5	-3	1													32
13	3	-2	-3	12	-10	-6	15	-13	5	1	-2	1	-1											74
14		3	-13	8	10	-20	19	-8	-4	10	-8	4	-1											108
15		5	-15	5	23	-36	23	11	-32	25	-11	1	3	-2										192
16	-7	18	-9	-30	59	-38	-21	68	-59	17	11	-16	11	-5	1									370
17	-14	19	23	-82	64	34	-100	96	-43	-21	56	-49	21	-2	-3	2	-1							630
18	-16	-1	52	-120	78	152	-268	122	85	-168	126	-45	-12	26	-16	6	-1							1294
19	-12	-88	176	-8	-186	290	-265	-30	346	-339	100	76	-108	75	-33	5	3	-2						2142
20	7	-167	393	145	-937	558	327	-611	531	-312	-98	363	-294	101	17	-40	23	-7	1					4932
21	168	-37	13	-108	-1151	1472	1007	-2404	871	718	-984	667	-238	-105	206	-132	45	-6	-3	2	-1			10338
22	638	-241	-2676	1806	2506	-1378	-349	-2171	1510	2159	-2672	707	580	-748	492	-187	-6	53	-31	9	-1			20920
23	468	-2644	-2607	12686	1016	-18755	5351	8867	-4274	1079	-2353	228	2389	-2042	537	261	-351	201	-66	8	3	-2		66188

TABLE 3. Table of Euler characteristics χ^{π}_{ij} by complexity j and Hodge degree i of $\pi_*(\overline{Emb_d}) \otimes \mathbb{Q}$ for even d.

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j	Hodge degree i																						
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	-1																						
2			1																				
3	1	-1		-1																			
4		-1	1			1																	
5	1		-1	2	-2		-1																
6	-1	1	-1	1	-2	2			1														
7	1	1	-3	2	1	-2	2	-2		-1													
8	-1	1		-3	5	-5	3	-2	2			1											
9			3	-2	4	-7	4	1	-3	2	-2		-1										
10	-2	2	3	-10	6	3	-10	11	-7	4	-2	2			1								
11	2	-3	3	1	-8	14	-12	12	-11	4	1	-3	2	-2			-1						
12		-2	3	1	-11	21	-23	6	11	-18	15	-7	4	-2	2			1					
13	3	-6	3	20	-33	14	15	-36	41	-28	19	-13	3	1	-3	2	-2		-1				
14		4	-16	15	-5	-16	45	-59	51	-30		19	-22	17	-7	4	-2	2			1		
15		2	-20	33	3	-60	105	-83	2	65	-88	72	-38	20	-14	3	1	-3	2	-2		-1	
16	-7	25	-27	-11	88	-139	67	60	-147	165	-127	75	-30	-9	25	-22	17	-7	4	-2	2		
17	-14	22	10	-86	126	-74	-33	190	-295	257	-91	-79	148	-135	90	-42	19	-14	3	1	-3	2	-2
18	-16	20	31	-188	266	29	-425	526	-326	-42	332	-413	330	-188	80	-22	-13	27	-22	17	-7	4	-2
19	-12	-84	216	-136	-38	368	-761	556	184	-721	814	-622	318	-4	-193	219	-161	94	-43	19	-14	3	1
20	7	-165	519	-77	-991	1031	-444	-27	951	-1739	1332	-232	-616	871	-725	452	-210	73	-16	-13	27	-22	17
21	168	-74	245	-495	-1705	3209	117	-3475	3050	-1416	-96	1624	-2223	1691	-832	214	141	-285	255	-169	93	-43	19
22	638	-425	-2710	2145	2011	404	-1311	-5578	6395	632	-4769	4616	-3318	1372	602	-1566	1455	-944	498	-209	69	-13	-13
23	468	-3290	-2544	16300	-1620	-21938	8761	5388	1424	1676	-11716	9157	-435	-4214	4986	-3974	2252	-794	35	259	-329	263	-171

TABLE 4. Table of Euler characteristics χ_{ij} by complexity j and Hodge degree i of $H_*(\overline{Emb}_d, \mathbb{Q})$ for even d.

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