Electromagnetic wave scattering by many small bodies and creating materials with a desired refraction coefficient.

A.G. Ramm

How to cite this manuscript

If you make reference to this version of the manuscript, use the following information:

Ramm, A.G. (2010). Electromagnetic wave scattering by many small bodies and creating materials with a desired refraction coefficient. Retrieved from http://krex.ksu.edu

Published Version Information

Citation: Ramm, A.G. (2010). Electromagnetic wave scattering by many small bodies and creating materials with a desired refraction coefficient. Progress In Electromagnetics Research M, 13, 203-215.

Digital Object Identifier (DOI): doi :10.2528/PIERM10072307

Publisher's Link: http://www.jpier.org/PIERM/pier.php?paper=10072307

Progress in Electromag Research, M, (PIER M), 13, (2010), 203-215.

*Email: ramm@math.ksu.edu

Electromagnetic wave scattering by many small bodies and creating materials with a desired refraction coefficient

A. G. Ramm†*

†Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA

Abstract

Electromagnetic wave scattering by many small particles is studied. An integral equation is derived for the self-consistent field E in a medium, obtained by embedding many small particles into a given region D.

The derivation of this integral equation uses a lemma about convergence of certain sums. These sums are similar to Riemannian sums for the integral equation for E.

Convergence of these sums is essentially equivalent to convergence of a collocation method for solving this integral equation.

By choosing the distribution law for embedding the small particles and their physical properties one can create a medium with a desired refraction coefficient. This coefficient can be a tensor. It may have desired absorption properties.

Keywords electromagnetic waves; scattering theory; many-body scattering problem; singular integral equations.

MSC 78A25; 78A45; 45E05; **PACS** 41.20.Jb; 42.25.Bs; 42.25.Gy

1 Introduction

The problem of scalar wave scattering by many small bodies is reduced in [5] and [8] to solving an integral equation for the effective (self-consistent) field in the medium.

There is a large literature on homogenization for partial differential equations and for problems of physics and material science, which includes many books and dozens of papers (see [1], [3], and references therein). In most cases a periodic structure of the medium is assumed and selfadjointness of the operators involved is used in the mathematical literature. The homogenization in material science literature often leads to a homogeneous limiting medium. None of the above assumptions are used in this paper. The limiting medium in this paper is not homogeneous, the operators involved are not necessarily selfadjoint. Our methods differ from the usual methods in the homogenization theory.

The problem we pose was not studied in the literature, to our knowledge. This problem is:

How can one create a material with a desired refraction coefficient by embedding in a given material many small particles or many small inhomogeneities?

A study of this problem was initiated in the works [5]-[11].

In this paper an equation is derived for effective (self-consistent) field in the medium for electromagnetic wave scattering. Convergence of certain sums to the solution to this equation can be considered as converegence of a collocation method for solving a limiting equation for the effective field in the medium, see paper [9] where the convergence of this collocation method is proved for a wide class of integral equations.

By choosing the distribution law for embedding small particles into a given domain D, filled with a material with known properties, namely, with known dielectric parameter, conductivity, and a constant magnetic parameter, and by choosing the physical properties of the embedded small particles, namely, their dielectric parameters and conductivities, one can create a new material with a desired refraction coefficient. This refraction coefficient can be a tensor, and it may have desired absorption properties.

In [5] and [8] similar theory was developed for the scalar wave scattering by many small particles embedded in an inhomogeneous medium.

In Section 1 we prove an auxiliary result, formulated as Theorem 1. It deals with convergence of certain sums. Our derivation is simple and is not based on any results concerning weak convergence of measures.

In Section 2 the integral equation for the electromagnetic field scattered by many small particles embedded in a given medium is studied. The limiting equation (15) is derived for the effective field in the medium when the number of the embedded particles tends to infinity while their size tends to zero.

In Section 3 a derivation of the integral equation (10) is given, its relation to the limiting equation (15) is discussed, and a possible numerical procedure for solving equation (15) is proposed. This procedure is a version of the projection method.

We continue this Introduction with the formulation and proof of an auxiliary result concerning convergence of certain sums. Such sums appear in a study of wave scattering by many small particles. The auxiliary result, formulated below as Theorem 1, is used in Sections 2 and 3 in a study of the limiting behavior of the electromagnetic field in the limiting medium created by embedding many small particles when their characteristic size a tends to zero and their number tends to infinity.

The derivation of the integral equation for the effective field in the medium is based on the existence of the limit of the sums, similar to the following one:

$$I := \lim_{a\to 0} \varphi(a) \sum_{x_m \in D} f(x_m). \tag{1}$$

Here $\varphi(a) > 0$ is a monotone continuous strictly growing function, $\varphi(0) = 0$, x_m are some points distributed in a bounded domain $D \subset \mathbb{R}^3$ according to the following law:

$$\sum_{x_m \in \Delta} 1 := \mathcal{N}(\Delta) = \frac{1}{\varphi(a)} \int_\Delta N(x)dx[1 + o(1)], \qquad a \to 0, \tag{2}$$

where $\Delta \subset D$ is an arbitrary subdomain in D, $N(x) \geq 0$ and $f(x)$ are Riemann-integrable functions, $N \in P_0$, and $f \in P_\nu$. The inclusion $f(x) \in P_\nu$ means that the following estimate holds:

$$|f(x)| \leq \frac{c}{[\rho(x, S)]^\nu}, \qquad \nu \leq 3, \tag{3}$$

where $c > 0$ is a constant and $\rho(x, S)$ is the Euclidean distance from the point x to the set S.

For $\nu = 0$ condition (3) means that $\sup_{x\in D} |f(x)| \leq c$.

Since we assume f and N to be Riemann-integrable, their sets of discontinuities have Lebesgue measure zero in $\mathbb{R}^3$, see [12]. In applications to scattering theory, studied in this paper, the set S consists of the singular point of a Green's function. This singular point is not necessarily a fixed point. For example, if the Green's function is $g(x,y)=\frac{e^{ik|x-y|}}{4\pi|x-y|}$, then the singular point $y\in D$ can be arbitrary.

Let

$$D_\delta := \{x : x\in D, \rho(x,S)\geq\delta\}.$$

We assume that the limit

$$\lim_{\delta\to 0}\int_{D_\delta} f(x)N(x)dx := \int_D f(x)N(x)dx \tag{4}$$

exists.

If $f\in P_\nu$, $\nu<3$, and $N\in P_0$, then the existence of the limit (4) means that the integral on the right-hand side of (4) exists as an improper integral. If $\nu=3$, then the integral on the right-hand side of (4) is a singular integral which exists in the sense of the Cauchy principal value. The definition and properties of singular integrals one finds in [2].

If f is unbounded, then the sum (1) is not well-defined because $f(x_m)$ is infinite when $x_m\in S$. In this case we *define the quantity* I *in* (1) *as follows:*

$$I := \lim_{a\to 0}\varphi(a)\sum_{x_m\in D} f(x_m) := \lim_{\delta\to 0}\lim_{a\to 0}\varphi(a)\sum_{x_m\in D_\delta} f(x_m). \tag{5}$$

Theorem 1 *If* $N\in P_0$, $f\in P_\nu$, $\nu\leq 3$, *and the assumptions* (2)–(4) *hold, then*

$$I=\int_D f(x)N(x)dx. \tag{6}$$

Proof. In the set D_δ the functions $f(x)$ and $N(x)$ are bounded and can be assumed continuous because the set of discontinuities of Riemann-integrable functions is of Lebesgue measure zero. This set we include into the set S.

For such functions we prove below that

$$\lim_{\delta\to 0}\lim_{a\to 0}\varphi(a)\sum_{x_m\in D_\delta} f(x_m)=\int_{D_\delta} f(x)N(x)dx, \tag{7}$$

provided that assumption (4) holds.

If (7) is proved, then (6) follows from (7) and (5).

Let us prove (7).

Consider a partition of D_δ into a union U_δ of cubes Δ_j, such that $D_\delta=\cup_{j=1}^J\Delta_j := U_\delta$, where Δ_j is a cube with a side b centered at a point $x^{(j)}$, and $b=b(a)$. The intersection $\Delta_i\cap\Delta_j$, $i\neq j$, does not contain interior points, that is Δ_i and Δ_j for $i\neq j$ may have common points of their boundaries but do not have common interior points. For a finite J such a partition may not exist for D_δ. In this case we consider the smallest U_δ containing D_δ and extend f to $U_\delta\setminus D_\delta$ by setting $f=0$ in $U_\delta\setminus D_\delta$. After this is done, one redefines the set D_δ by setting $D_\delta=U_\delta$, and our arguments remain valid.

We assume that:

$$\lim_{a\to 0}\frac{a}{b(a)}=0, \qquad \lim_{a\to 0} b(a)=0. \tag{8}$$

Let us use relation (2) with $\Delta = \Delta_j$ and write:

$$\begin{aligned} \varphi(a) \sum_{x_m \in D_\delta} f(x_m) &= \varphi(a) \sum_{j=1}^{J} \sum_{x_m \in \Delta_j} f(x_m) \\ &= \varphi(a) \sum_{j=1}^{J} f(x^{(j)})[1+o(1)] \sum_{x_m \in \Delta_j} 1 \\ &= \sum_{j=1}^{J} f(x^{(j)}) N(x^{(j)}) |\Delta_j| [1+o(1)], \end{aligned} \tag{9}$$

where $|\Delta_j|$ is the volume of Δ_j,

$$f(x_m) = f(x^{(j)})[1+o(1)], \qquad \forall x_m \in \Delta_j,$$

and $o(1) \to 0$ as $a \to 0$ because f is continuous in Δ_j and $\operatorname{diam} D_j \to 0$ as $a \to 0$.

The sum on the right in (9) is the Riemannian sum for the continuous in D_δ function $f(x)N(x)$. It is known that if the function $f(x)N(x)$ is Riemann-integrable, then the limit of this sum, as $\max_{1 \le j \le J} \operatorname{diam}(\Delta_j) \to 0$, exists and is equal to the integral $\int_{D_\delta} f(x)N(x)dx$ (see, e.g., [12], p.269). Since $\operatorname{diam}(\Delta_j) = b(a)\sqrt{3} \to 0$, formula (7) is proved. The first assumption (8) guarantees that there are many points x_m in Δ_j if $N(x) \ge 0$ in Δ_j.

Theorem 1 is proved. $\square$

Remark 1 In the usual definition of the Cauchy principal value for a singular integral $\int_D f(x)dx$, where $D \subset \mathbb{R}^3$, one assumes that $f \in P_3$, that is, (3) holds with $\nu = 3$, $S = \{x\}$ consists of one point, and $D_\delta = \{y : y \in D, |x-y| \ge \delta\}$. Necessary and sufficient conditions for the existence of singular integrals in the sense of the Cauchy principal value can be found in [2], p. 221.

Our definition deals with the case when S may consist of more than one point. However, in the applications considered in [5], [8], and in this paper, the set of the points of f at which f is unbounded consists of one point, and $\nu = 1$, so that the integral on the right-hand side of (4) exists as an improper integral.

Our arguments are valid in $\mathbb{R}^n$ with any $n \ge 1$.

2 Electromagnetic wave scattering and creating materials with a desired refraction coefficient

Let us consider many-body scattering problem for electromagnetic (EM) waves in the case of small bodies $ka \ll 1$, where a is the characteristic size of these bodies, $k = \omega\sqrt{\epsilon_0 \mu_0}$ is the wavenumber in the free space, ω is the frequency, ϵ_0, μ_0 are dielectric and magnetic parameters.

Assume that there are $M \gg 1$ small bodies D_m, $1 \le m \le M$, embedded in a bounded domain D. Each of the bodies D_m is characterized by its dielectric constant

$$\epsilon'_m = \epsilon_m + i\frac{\sigma_m}{\omega},$$

where $\epsilon_m > 0$ is the permittivity and $\sigma_m > 0$ is the conductivity of the material in D_m.

We assume that $\mu_m = \mu_0$ for all m, where μ_m is the magnetic permeability of D_m and μ_0 is a constant magnetic permeability of the free space.

It is proved in Section 3 that the integral equation for electromagnetic field scattered by M small particles embedded in D and having constant refraction coefficients K_m^2 in D_m, $1 \leq m \leq M$, is a vector integral equation, which we write as the following system of scalar integral equations:

$$\begin{aligned} E_i(x) =& E_{0i}(x) + \sum_{m=1}^{M} (K_m^2 - k^2) \int_{D_m} g(x,y) E_i(y) dy \\ &+ \frac{\partial}{\partial x_i} \sum_{m=1}^{M} \frac{K_m^2 - k^2}{k^2} \int_{D_m} \frac{\partial g(x,y)}{\partial x_j} E_j(y) dy, \qquad 1 \leq i \leq 3, \end{aligned} \tag{10}$$

where $E = E_i e_i$, $\{e_i\}_{i=1}^3$ is the Euclidean orthonormal basis of $\mathbb{R}^3$, *summation is understood over the repeated indices here and below*, E_{0i} are the Cartesian components of the incident field $E_0(x)$,

$$g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \qquad K_m^2 = \omega^2 \epsilon'_m \mu_0 := K^2(x_m), \tag{11}$$

$\epsilon'(x)$ is a function in D such that

$$\epsilon'(x_m) = \epsilon'_m, \tag{12}$$

and $K^2(x)$ is the refraction coefficient, a function in D, such that (11) holds, that is, $K^2(x_m) = K_m^2$.

From the point of view of applications, it is of interest to emphasize that K_m^2 can be a tensor, and it can have an imaginary part, which describes the absorption of energy in the material of particles. In the limiting medium, which one obtains in the limit $a \to 0$, the refraction coefficient $K_1^2(x)$ (see formula (16) below) can be a tensor describing the anisotropy and absorption of the created material.

To make this paper self-contained, the derivation of equation (10) is given in Section 3, where the statement of the electromagnetic wave scattering problem by a body of an arbitrary shape is also given. This derivation follows the one in [10].

The first sum in (10) we write as

$$\sum_{m=1}^{M} (K_m^2 - k^2) \int_{D_m} g(x,y) E_i(y) dy = \sum_{m=1}^{M} (K_m^2 - k^2) g(x, x_m) E_i(x_m) |D_m| [1 + o(1)], \tag{13}$$

where $|D_m| = O(a^3)$ is the volume of D_m, $x \notin D_m$, and $o(1) \to 0$ as $a \to 0$.

The second sum in (10) we write as

$$\sum_{m=1}^{M} \frac{K_m^2 - k^2}{k^2} \frac{\partial^2 g(x, x_m)}{\partial x_j \partial x_i} E_j(x_m) |D_m| [1 + o(1)]. \tag{14}$$

Let us assume that D_m is a cube, centered at the point x_m, of side a. Then $|D_m| = a^3$.

Using Theorem 1 with $\varphi(a) = a^3$, one passes to the limit, as $a \to 0$, in the sums (13) and (14), and obtains the following integral equation for the effective electromagnetic field $E(x) = E_i(x) e_i$

in the limiting medium:

$$\begin{aligned} E_i(x) =& E_{0i}(x) + \int_D g(x,y)(K^2(y)-k^2)N(y)E_i(y)dy \\ &+ \frac{\partial}{\partial x_i}\int_D \frac{\partial g(x,y)}{\partial x_j}\frac{K^2(y)-k^2}{k^2}N(y)E_j(y)dy, \qquad 1 \le i \le 3, \end{aligned} \tag{15}$$

where $N(y)$ is the function from formula (2), describing the density of the distribution of small bodies (particles), and $K^2(x)$ is a function such that $K^2(x_m) = K_m^2$.

The quantities $E_j(x_m)$ in the sums (13) and (14) depend on a, but converge to finite limits as $a \to 0$, as follows from the convergence of the collocation method, studied in [9]. Therefore Theorem 1 is applicable for passing to the limit $a \to 0$ in the sums (13) and (14).

A continuous in D function $K^2(x)$ is uniquely determined by its values at a set of points $\{x_m\}$ dense in D. In the limit $M \to \infty$, or, which is the same, in the limit $a \to 0$, the set of points $\{x_m\}$ is dense in D, so the values K_m^2 determine $K^2(x)$ uniquely in this limit.

As $a \to 0$, one obtains the limiting medium created by embedding small particles D_m into D.

One may interpret this result from the physical point of view as follows:

As $a \to 0$, the limiting medium, obtained by embedding small particles D_m according to the distribution law (2) *with $\varphi(a) = a^3$ has the following refraction coefficient:*

$$K_1^2(x) = K^2(x)N(x). \tag{16}$$

By choosing $N(x)$ and $K^2(x)$, which are at the disposal of an experimentalist, one can create a desirable refraction coefficient $K_1^2(x)$, including *tensorial* ones. From the point of view of a physicist, the cofficient $K_1^2(x)$ is an analog of the coefficient $\mathcal{K}^2$ in formula (21) in Section 3.

3 Derivation of equation (10)

Consider the following scattering problem. An incident electromagnetic field (E_0, H_0) is scattered by a bounded region D, filled with a material with parameters $(\epsilon, \sigma, \mu_0)$. The exterior region D' is a homogeneous region with parameters $(\epsilon_0, \sigma = 0, \mu_0)$. Consider for simplicity the case when $\epsilon = const$ and $\sigma = const \ge 0$ in D. Let $\epsilon' = \epsilon + i\frac{\sigma}{\omega}$. The governing equations in $\mathbb{R}^3$ are

$$\nabla \times E = i\omega\mu_0 H, \qquad \nabla \times H = -i\omega\epsilon' E. \tag{17}$$

At the boundary S of D one has boundary conditions

$$[N, E^+] = [N, E^-], \tag{18}$$

and

$$N \cdot \epsilon' E^+ = N \cdot \epsilon_0 E^-, \tag{19}$$

where N is the unit normal to S, pointing into D', $E^+(E^-)$ is the limiting value of E on S from inside (outside) S, $[N, E]$ is the cross product, and $E \cdot N$ is the dot product of two vectors. The fields E and H satisfy the radiation condition at infinity.

Let

$$k^2 = \omega^2\epsilon_0\mu_0, \quad K^2 = \omega^2\epsilon'\mu_0, \quad \mathcal{K}^2 = \begin{cases} k^2, & \text{in } D', \\ K^2, & \text{in } D. \end{cases} \tag{20}$$

Equations (17) imply

$$\nabla \times \nabla \times E - \mathcal{K}^2 E = 0, \quad H = \frac{\nabla \times E}{i\omega\mu_0} \quad in\ \mathbb{R}^3. \tag{21}$$

Therefore, in order to solve problem (17)-(19) it is sufficient to find E satisfying the first equation (21), boundary conditions (18), (19), and the radiation condition

$$E = E_0 + V; \quad V_r - ikV = o\left(\frac{1}{r}\right), \qquad r := |x| \to \infty. \tag{22}$$

Equation (21) for E can be written as

$$LE := \nabla \times \nabla \times E - k^2 E = pE; \quad p := p(x) = \mathcal{K}^2 - k^2 = \begin{cases} 0, & \text{in } D', \\ K^2 - k^2, & \text{in } D. \end{cases} \tag{23}$$

The incident field E_0 solves equation (23) with $p = 0$.

Let $\delta(x)$ denote the delta-function and $\delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

Let $G = G_{ij}(x)$ be the Green's function solving the problem:

$$LG = \delta(x)\delta_{ij}, \quad G_r - ikG = o\left(\frac{1}{r}\right), \quad r \to \infty. \tag{24}$$

Then the solution to (23)-(22) solves the integral equation

$$E = E_0 + \int_{\mathbb{R}^3} G(x-y)p(y)E(y)dy. \tag{25}$$

The Green's function $G(x) = G(|x|)$ is symmetric, $G_{ij} = G_{ji}$, see formula (31) below.

Let us state and prove three Lemmas.

Lemma 1.1 *There is at most one solution to equation* (25) *satisfying conditions* (18) *and* (19).

Proof. If there are two solutions to equation (25), then their difference E solves the homogeneous equation (25), satisfies boundary conditions (18) and (19), and the radiation condition. Thus, E solves (23), (22), (18) and (19). Therefore, E and $H = \frac{\nabla \times E}{i\omega\mu_0}$ solve equations (17) and satisfy condition (18), (19) and (22). It is known (see, e.g., [4]) that this implies $E = H = 0$.
Lemma 1.1 is proved. □

Lemma 1.2 *There is at most one solution to equation* (25).

We prove that if E solves equation (25), then it satisfies (22), (18), (19) and (23). Therefore by Lemma 1.1 (25) has at most one solution.

Proof. Applying operator L to (25) one obtains equation (23). The integral in (25) is the term V in formula (22). It satisfies the radiation condition because G does. Equation (23) is equivalent to (21). Equation (21) together with the formula $H = \frac{\nabla \times E}{i\omega\mu_0}$ yield both equations (17). Conditions (18) and (19) are consequences of equations (17). Therefore, every solution to (25) is in one-to-one correspondence with the solution to equations (17). This correspondence is given by the formulas $E = E$, $H := \frac{\nabla \times E}{i\omega\mu_0}$. By Lemma 1.1 equation (25) has at most one solution satisfying (18) and (19). We have proved that every solution to (25) satisfies (18) and (19). Therefore, (25) has at most one solution.
Lemma 1.2 is proved. □

Lemma 1.3 *Equation* (25) *has a unique solution.*

Proof. Uniqueness of the solution to (25) is proved in Lemma 1.2. Existence of the solution to (25) follows from the existence of the solution to the scattering problem and the fact, established in the proof of Lemma 1.2, that a solution to equation (25) solves equation (21), satisfies the radiation condition (22), and boundary conditions (18), (19).
Lemma 1.3 is proved. $\square$

From Lemmas 1.1-1.3 one obtains the following result:

Theorem 2 *Equation* (25) *has a unique solution E. This solution E generates the solution to the scattering problem by the formula $E = E$, $H := \frac{\nabla \times E}{i\omega\mu_0}$.*

Let us construct the Green's function G analytically, in closed form.
Let us look for G of the form

$$G(x) = \int_{\mathbb{R}^3} e^{i\xi\cdot x}\tilde{G}(\xi)d\xi, \quad \tilde{G}(\xi) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i\xi\cdot x}G(x)dx. \tag{26}$$

Take the Fourier transform of (24) and get

$$-[\xi,[\xi,\tilde{G}]] - k^2\tilde{G} = \frac{1}{(2\pi)^3}I, \quad I_{ij} = \delta_{ij}, \tag{27}$$

where I is the identity matrix, $[a, b]$ is the cross product of two vectors, and $a \cdot b$ is their dot product. Equation (27) implies

$$-\xi\xi\cdot\tilde{G} + (\xi^2 - k^2)\tilde{G} = \frac{1}{(2\pi)^3}I. \tag{28}$$

Multiplying (28) by ξ, one finds

$$\xi\cdot\tilde{G} = -\frac{\xi}{(2\pi)^3k^2}. \tag{29}$$

From (28) and (29) it follows that

$$\tilde{G}_{ij} = \frac{\delta_{ij}}{(2\pi)^3(\xi^2 - k^2)} - \frac{\xi_i\xi_j}{(2\pi)^3k^2(\xi^2 - k^2)}. \tag{30}$$

Taking the inverse Fourier transform of (30) and using the radiation condition (22), one gets

$$G_{ij}(x) = g(x)\delta_{ij} + \frac{1}{k^2}\partial_{ij}g(x); \quad g(x) = \frac{e^{ik|x|}}{4\pi|x|}, \quad \partial_i := \frac{\partial}{\partial x_i}. \tag{31}$$

From (31) and (25) one obtains:

$$\begin{aligned} E_i(x) = E_{0i}(x) &+ (K^2 - k^2)\int_D g(x,y)E_i(y)dy \\ &+ \frac{K^2 - k^2}{k^2}\frac{\partial}{\partial x_i}\int_D \frac{\partial g(x,y)}{\partial x_j}E_j(y)dy, \quad 1 \le i \le 3, \end{aligned} \tag{32}$$

where summation over the repeated indices is understood. So far we assumed that K^2 does not depend on y in D.

However, equation (32) yields equation (10) if one takes the region D in (32) to be the union of the small regions D_m and sets $K^2 = K_m^2$ in D_m.

From equation (10) one obtains equation (15) using Theorem 1.

If one applies to equation (10) the collocation method, discussed in [9], then one obtains a linear algebraic system for the unknowns $E_i(x_m)$:

$$E_i(x_q) = E_{0i}(x_q) + \sum_{m=1, m\neq q}^{M} (K_m^2 - k^2) g(x_q, x_m) E_i(x_m)|D_m| \\ + \sum_{m=1, m\neq q}^{M} \frac{K_m^2 - k^2}{k^2} \frac{\partial^2 g(x_q, x_m)}{\partial(x_q)_j \partial(x_q)_i} E_j(x_m)|D_m|, \qquad 1 \leq q \leq M, \tag{33}$$

where $\frac{\partial^2 g(x_q,x_m)}{\partial(x_q)_j\partial(x_q)_i}$ denotes partial derivative of $g(x, x_m)$ with respect to the $j-$th and $i-$th component of the vector x, calculated at the point $x = x_q$, and *there is no summation over the index* q. The sums in (13)-(14) are the same as in (33).

It is proved in [9] that if assumption (2) holds for the distribution of the points x_m in D, then the collocation method converges, as $a \to 0$, to the solution of the limiting integral equation (15), where $N(x)$ is the function defined in (2).

In equation (15) the operator

$$TE = \int_D g(x, y)(K^2(y) - k^2)N(y)E(y)dy$$

is compact in $L^2(D)$. Let

$$\gamma(y) := \frac{K^2(y) - k^2}{k^2}, \qquad QE := \nabla \int_D \nabla_x g(x, y)\gamma(y)N(y)E(y)dy. \tag{34}$$

Then equation (15) can be written as

$$E = E_0 + TE + QE. \tag{35}$$

Numerically one can solve equation (15) by a projection method. For example, let $\{\phi_j(x)\}$ be a basis of $L^2(D)$ and $\phi_j \in H_0^1(D)$, where $H_0^1(D)$ is the closure of $C_0^\infty(D)$ functions in the norm of the Sobolev space $H^1(D)$. Multiply equation (15) by $\overline{\phi}_m$ (the bar stands for the complex conjugate), integrate over D and then the third term by parts, to get:

$$E_{im} = E_{0im} + \int_D dx \overline{\phi}_m(x) \int_D dy g(x, y)(K^2(y) - k^2)N(y) \sum_{m'=1}^{M} E_{im'}\phi_{m'}(y) \\ - \int_D dx \frac{\partial \overline{\phi}_m(x)}{\partial x_i} \int_D dy \frac{\partial g(x, y)}{\partial x_j} \gamma(y)N(y) \sum_{m'=1}^{M} E_{jm'}\phi_{m'}(y), \quad 1 \leq m \leq M, \ 1 \leq i \leq 3. \tag{36}$$

This is a linear algebraic system for finding the coefficients:

$$E_{im}^{(M)} := E_{im} := \int_D E_i(x)\overline{\phi}_m(x)dx. \tag{37}$$

The number M determines the accuracy of the appproximate solution $E(x)$. One has

$$\lim_{M\to\infty} \|E^{(M)}(x) - E(x)\|_{L^2(D)} = 0. \tag{38}$$

References

[1] V. Marchenko, E. Khruslov, *Homogenization of partial differential equations*, Birkhauser, Basel, 2006.

[2] S. Mikhlin, S. Prössdorf, *Singular integral operators*, Springer-Verlag, Berlin, 1986.

[3] G. Milton, *The theory of composites*, Cambridge University Press, Cambridge, 2002.

[4] C. Müller, *Foundations of the mathematical theory of electromagnetic waves*, Springer-Verlag, Berlin, 1969.

[5] A.G.Ramm, Many-body wave scattering by small bodies and applications, J. Math. Phys., 48, N10, (2007), 103511.

[6] A.G.Ramm, Distribution of particles which produces a "smart" material, J.Stat. Phys., 127, N5, (2007), 915-934.

[7] A.G.Ramm, Electromagnetic wave scattering by small bodies, Phys. lett.A, 372/23, (2008), 4298-4306.

[8] A.G.Ramm, Wave scattering by many small particles embedded in a medium, Phys. Lett. A, 372/17, (2008), 3064-3070.

[9] A.G.Ramm, A collocation method for solving integral equations, International Journ. of Comput. Sci. and Math. (IJCSM), 3, N2, (2009), 222-228.

[10] A.G.Ramm, A singular integral equation for electromagnetic wave scattering, Internat. Journ. Pure and Appl. Math., 55, N4, (2009), 7-11.

[11] A.G.Ramm, Creating desired potentials by embedding small inhomogeneities, J. Math. Phys., 50, N12, (2009), 123525.

[12] K.Stromberg, *An introduction to classical real analysis*, Wadsworth Int. Group, Belmont California, 1981.