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Slow manifolds for dissipative dynamical systems

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Abstract

A class of infinite-dimensional dissipative dynamical systems is defined for which the slow invariant manifolds can be calculated. Large-time behavior of the evolution of such systems is studied.

Key words: Dissipative systems; dynamical systems; attractors; invariant manifolds; nonlinear evolution.

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1 Introduction and statement of the results

A dynamical system is described by the evolution problem:

$$\dot{u} = -F(u), \quad u(0) = u_0; \quad \dot{u} := \frac{du}{dt}, \tag{1}$$

where $u_0 \in D(F)$ is arbitrary, $D(F)$ is the domain of F. The system is called dissipative if $F : H \to H$ is a monotone, closed and hemicontinuous operator in a Hilbert space H: $(F(u) - F(v), u - v) \geq 0, \quad u, v \in D(F)$. Here (u, v) is the inner product in H, $D(F)$ is assumed to be a linear set, dense in H, and F is maximal monotone, $R(I + F) = H$, where $R(F)$ is the range of F. Under these assumptions problem (1) has a unique solution $u(t) := S(t)u_0$, defined for all $t \geq 0$, and the operator family $S(t)$ is a semigroup. A set $\mathcal{A}$ is called a global attractor for problem (1) if for any $u_0 \in H$ $\lim_{t\to\infty} d(u(t), \mathcal{A}) = 0$, where $d(u, \mathcal{A})$ is the distance between u and the set $\mathcal{A}$, and $u(t)$ solves (1). A set M is called an invariant set for problem (1) if $u_0 \in M$ implies $u(t) \in M$ for all $t > 0$. If an invariant set M is a manifold, it is called an invariant manifold for problem (1). Attractors and invariant manifolds for dissipative dynamical systems are studied in [?], [?], and [?]. In [?], Chapter 3, and in [?] a class of dissipative nonlinear systems is studied. This class consists of passive nonlinear networks.

Assume that $A = A^* \geq m > 0$ is a selfadjoint operator in H, denote by $\sigma(A)$ its spectrum, and by E_s its resolution of the identity. If $\Delta_\delta = [m, m+\delta]$, the $E(\Delta_\delta) = E_{m+\delta} - E_{m-0}$ is an orthogonal projection operator in H and $E(\Delta_\delta)H$ is an invariant subspace of A. By definition, $E_{s+0} = E_s$. If λ is an eigenvalue of A, then $[E_\lambda - E_{\lambda-0}]H$ is the corresponding eigenspace. By $\sigma(A)$ we denote the spectrum of A, $m = \inf \sigma(A)$.

One is often interested in finding "slow" invariant manifolds for problem (1). If $F = A$ is a linear operator, then its invariant manifold is called "slow", if it corresponds to the smallest (lowest) eigenvalue of A. The corresponding eigenspace of A is a linear invariant manifold for problem (1). A method for finding "slow" invariant manifolds for problem (1) is proposed in [?]. It consists of solving the problem

$$\dot{u} = -Au + b(u(t))u, \quad u(0) = u_0; \quad b(t) := b(u(t)) := \frac{(Au, u)}{(u, u)}, \tag{2}$$

and studying the limit $\lim_{t\to\infty} u(t) := v$. The existence of this limit will be established in this paper under suitable assumptions. One hopes that this limit, if it exists, is an eigenvector of A, corresponding to the smallest eigenvalue Λ of A. Our goal is to find sufficient conditions for the validity of such a conclusion.

In [?] no rigorous results have been established for the global existence of the solution to equation (2), for the existence of the limit $\lim_{t\to\infty} u(t)$, and for finding slow manifolds in an infinite-dimensional Hilbert space. Our aim is to establish such results in this paper.

Let us formulate our results. Their proofs are outlined in Section 2.

Theorem 1. *Problem (2) has a unique global solution $u(t)$ which is given by the formula*

$$u(t) = \frac{e^{-tA}u_0}{(1 - 2\int_0^t h(\tau)d\tau)^{1/2}}, \quad h(t) := (Ae^{-2tA}u_0, u_0), \tag{3}$$

and $||u(t)|| = ||u_0||$ for all $t > 0$.

Remark 1. The last statement of Theorem 1 allows one to assume without loss of generality that $||u_0|| = 1$. Everywhere below we make this assumption. The closed form solution of the nonlinear evolution problem (2) is quite useful: among other things, it yields existence and uniqueness of the global solution to problem (2) in an infinite-dimensional Hilbert space.

Theorem 2. *Assume that A has a discrete spectrum. Let $\Lambda = m = \inf \sigma(A)$ be the smallest eigenvalue of A. Assume that m is an isolated point of spectrum, P_m is the orthoprojector in H onto the corresponding eigenspace, and $P_m u_0 \neq 0$.*

Under these assumptions there exists strong limit $\lim_{t\to\infty} u(t) = v$, $v \in H_m$, *and* $||v|| = ||u_0|| = 1$.

Remark 2. Suppose that the spectrum of A in the interval $[m, m+\epsilon)$ is arbitrary, containing, possibly, countably many eigenvalues λ_j, which possibly form a set dense in the interval $[m, m+\epsilon)$, so that the spectrum of A in $[m, m+\epsilon)$ contains a singular component. Here $\epsilon > 0$ is a small fixed number. Assume that m is an eigenvalue of A, possibly of infinite multiplicity, H_m is the corresponding eigenspace, P_m is the orthogonal projector onto H_m, and $P_m u_0 \neq 0$. Let $H_1 := H_m$ and $H_2 := H_1^{\perp}$. Then the conclusion of Theorem 2 remains valid.

The idea of the proof is the same as in the proof of Theorem 2. We leave the details of the proof to the reader.

Theorem 3. *If the spectrum* $\sigma(A)$ *is absolutely continuous on the interval* $[m, m+\delta] := \Delta_\delta$, $\delta > 0$, *and* $E(\Delta_\delta)u_0 \neq 0$, *then there does not exist strong limit* $\lim_{t\to\infty} u(t) = v$, $v \in H$.

Theorem 4. *If* $E(\Delta_\delta)u_0 \neq 0$, $m = \inf \sigma(A)$ *is an eigenvalue of* A, *and* m *is an isolated eigenvalue embedded into absolutely continuous spectrum of* A, *then there exists strong limit* $\lim_{t\to\infty} u(t) = v$, $v \in H_m$, *and* $||v|| = ||u_0|| = 1$.

2. **Proofs**

Proof of Theorem 1. If a solution to (2) exists, then $||u(t)|| = ||u_0||$. Indeed, multiply (2) by $u(t)$ and get $\frac{d||u(t)||^2}{dt} = 0$. This implies the desired conclusion. Therefore, without loss of generality we will assume below that $||u(t)|| = ||u_0|| = 1$.

Denote $z(t) := \int_0^t (Au(\tau), u(\tau))d\tau$, so $\dot{z} = (Au(t), u(t))$. From (2) one gets

$$u(t) = e^{z(t)} e^{-tA} u_0, \qquad z(t) := \int_0^t (Au(\tau), u(\tau))d\tau. \tag{4}$$

Apply the operator A to (4) and then multiply by u to get

$$\dot{z} = e^{2z(t)} h(t), \qquad h(t) = (Ae^{-2tA}u_0, u_0). \tag{5}$$

From (5) one gets $e^z = (1 - 2\int_0^t h(\tau)d\tau)^{-\frac{1}{2}}$. This and (4) yield (3). Theorem 1 is proved. $\square$

Corollary 1. Formula (3) for $||u_0|| = 1$ can be rewritten as

$$u(t) = \frac{\int_m^\infty e^{-st} dE_s u_0}{(\int_m^\infty e^{-2st} d\rho)^{\frac{1}{2}}}, \quad d\rho := d(E_s u_0, u_0). \tag{6}$$

To derive (6) one uses formula (3) and the spectral theorem, in particular, the relation $\int_m^\infty d\rho = ||u_0||^2 = 1$.

Proof of Theorem 2. Assume for simplicity that $\Lambda = m$ is an isolated eigenvalue of A and H_m is the corresponding eigenspace. Decompose H into an orthogonal sum of two subspaces, invariant with respect to A, one of which is H_m. Then the solution to (2) can be written as $u = u_1 + u_2$, where $u_1 \in H_m$ and u_2 is orthogonal to u_1. One has $||u(t)||^2 = ||u_1(t)||^2 + ||u_2(t)||^2 = 1$, and $||u_2(t)|| = o(||u_1(t)||)$ as $t \to \infty$. Therefore, $\lim_{t\to\infty} ||u_1(t)|| = \lim_{t\to\infty} ||u(t)|| = 1$. If $\dim H_m = 1$, and the corresponding eigenvector is ϕ, $||\phi|| = 1$, then there exists strong limit $\lim_{t\to\infty} u(t) = \phi$. In the general case, equation (**??**) is equivalent to the system of equations:

$$\dot{u}_1 = -Au_1 + bu_1, \quad \dot{u}_2 = -Au_2 + bu_2, \quad b := b(u(t)), \tag{7}$$

$$u_1(0) = u_{01} = E(\Delta_\delta)u_0, \qquad u_2 \perp u_1. \tag{8}$$

One has

$$u_j(t) = e^{z(t)-tA}u_{0j}, \quad j = 1, 2, \tag{9}$$

where $z(t)$ is defined in (4). Therefore, $\lim_{t\to\infty} \frac{\|u_2(t)\|^2}{\|u_1(t)\|^2} = 0$, and

$$\lim_{t\to\infty} e^{-mt+z(t)} = \frac{\|u_0\|}{\|u_{01}\|}. \tag{10}$$

Consequently, there exists the strong limit:

$$v := \lim_{t\to\infty} u(t) = u_{01}\frac{\|u_0\|}{\|u_{01}\|}, \quad v \in H_m, \tag{11}$$

and $||v|| = ||u_0||$. Theorem 2 is proved. □

Proof of Theorem 3. Suppose to the contrary that there exists strong limit $\lim_{t\to\infty} u(t) = v$. Clearly, $v \neq 0$, because $||u(t)|| = 1$ for all $t \geq 0$, so $||v|| = 1$. Without loss of generality assume that $u(t) \in E(\Delta_\delta)H$ and A is bounded, because the part A_1 of A in the invariant subspace $E(\Delta_\delta)H$ is bounded. Then the limit

$$\lim_{t\to\infty} (Au(t), u(t)) = (Av, v) := \lambda$$

exists, and

$$\lim_{t\to\infty} Au(t) = Av.$$

Therefore $\lim_{t\to\infty} \dot{u}(t) := w$ exists, and

$$w = -Av + \lambda v.$$

We claim that $w = 0$.

Indeed, if $w \neq 0$, then

$$u(t+h) - u(t) = \int_t^{t+h} \dot{u}d\tau = wh[1 + o(1)], \quad t \to \infty.$$

This contradicts the Cauchy criterion for the existence of the limit $\lim_{t\to\infty} u(t) = v$, unless $w = 0$. Thus, $w = 0$ and $Av = \lambda v$, $||v|| = 1$. Therefore, $\lambda \in \Delta_\delta$ is an eigenvalue of A, contrary to our assumption. Theorem 3 is proved. □

Remark 3. If the interval $\Delta = [m, m+\delta)$ consists of the points of absolutely continuous spectrum of A, and the projection of the initial data u_0 onto the invariant subspace $E(\Delta)H$ of A is non-zero, then there does not exist strong limit of the solution $u(t)$ to problem (2) as $t \to \infty$; the trajectory of the solution $u(t)$ does not stay in any fixed finite-dimensional subspace of H, and does not stay in any fixed compact subset of H. It stays on an infinite-dimensional sphere $||u(t)|| = ||u_0||$ in H. *In this sense the trajectory of the solution $u(t)$ is chaotic.*

Proof of Theorem 4. This proof is briefly sketched. If the spectrum of A in the interval $(m, m+\delta)$ is absolutely continuous, then the solution $u(t)$ to (2) can be written as $u = u_m + u'$, where $u_m \in H_m$ and u' is orthogonal to H_m, and $||u'(t)|| = o(||u_m(t)||)$ as $t \to \infty$. If the spectrum of A is absolutely continuous on $(m, m+\delta)$, then the function $d\rho = \mu(s)ds$, where $\mu \in L^1(\Delta_\delta)$, and the following estimate holds: $\int_m^{m+\delta} e^{-ts}\mu(s)ds = o(e^{-mt})$ as $t \to \infty$. On the other hand, the part of the solution, which lies in H_m is $e^{-mt}\psi$, where $\psi \in H_m$, $\psi \neq 0$. This part is the main part of the solution as $t \to \infty$. Dividing this solution by the normalizing factor as in formula (6), one gets in the limit $t \to \infty$ a normalized element v of H_m. The outline of the proof of Theorem 4 is completed. $\Box$

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