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# How large is the class of operator equations solvable by a DSM Newton-type method ?

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**Abstract**

It is proved that the class of operator equations $F(y) = f$ solvable by a DSM (Dynamical Systems Method) Newton-type method

$$\dot{u} = -[F'(u) + a(t)I]^{-1}[Fu(t) + a(t)u - f], \quad u(0) = u_0, \quad (*)$$

is large. Here $F : X \to X$ is a continuously Fréchet differentiable operator in a Banach space $X$, $a(t) : [0, \infty) \to \mathbb{C}$ is a function, $\lim_{t\to\infty} |a(t)| = 0$, and there exists a $y \in X$ such that $F(y) = f$. Under weak assumptions on $F$ and $a$ it is proved that

$$\exists! u(t) \quad \forall t \geq 0; \quad \exists u(\infty); \quad F(u(\infty)) = f.$$

This justifies the DSM (*).

*MSC: 47J05, 47J07, 58C15*
*Key words:* Nonlinear operator equations; DSM (Dynamical Systems Method); Newton's method

## 1 Introduction

There is a large literature on solving nonlinear operator equations

$$F(y) = f, \tag{1}$$

where $F$ is a Fréchet differentiable operator in a Banach space $X$ ([1], [2], [14], [20] to mention a few books). We assume that the norm in $X$ is Gateaux differentiable, and equation (1) has a solution $y$, possibly non-unique. Newton-type iterative methods for solving equation (1) are widely used. In most cases it is assumed that $F \in C^2_{loc}$, i.e., $F$ is twice Fréchet differentiable in a neighborhood of the solution $y$, and the initial approximation is sufficiently close to $y$. The classical iterative Newton's method for solving equation (1) is

$$u_{n+1} = u_n - [F'(u_n)]^{-1}F(u_n), \quad u|_{n=0} = u_0, \tag{2}$$

where $u_0$ is an initial element. This method makes sense if $[F'(u_n)]^{-1}$ is a bounded linear operator. If $F'(u)$ is not boundedly invertible, then method (2) has to be modified and regularized. In [20] the DSM (Dynamical Systems Method) for solving equation (1) is developed. The DSM consists of solving the Cauchy problem

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0, \ t \geq 0; \ \dot{u} = \frac{du}{dt}, \tag{3}$$

where $\frac{du}{dt}$ is the strong derivative, $\Phi$ is chosen such that

$$\exists! u(t) \quad \forall t \geq 0; \quad \exists u(\infty); \ F(u(\infty)) = f, \tag{4}$$

i.e., problem (3) has a unique global solution, there exists $u(\infty) := \lim_{t\to\infty} u(t)$, and $u(\infty)$ solves equation (1). If

$$\Phi = -A_{a(t)}^{-1}[F(u) + a(t)u - f], \tag{5}$$

then DSM (3) is a Newton-type method,

$$A_a := A + aI, \qquad A := F'(u),$$

$I$ is the identity operator, and problem (3) takes the form

$$\dot{u}(t) = -A_{a(t)}^{-1}[F(u(t)) + a(t)u(t) - f], \quad u(0) = u_0. \tag{6}$$

The standard way to prove the local existence of the solution to (6) is based on the assumption that the operator (5) satisfies a local Lipschitz condition. However, this condition is satisfied only if $F'(u)$ satisfies a Lipschitz condition.

*We will prove the local existence of the solution to* (6) *assuming only that* $F'(u)$ *is continuous with respect to* $u$.

To do this, let us introduce some assumptions.

*Assumptions A):*

1. There exists a smooth contour $L \subset \mathbb{C}$, joining the origin and a point $z_0 \in \mathbb{C}$, $|z_0| < \epsilon_0$, where $\epsilon_0 > 0$ is an arbitrary small fixed number, such that the map $A_a : F'(u) + aI, \ a \in L$, is boundedly invertible and

$$\|A_a^{-1}\| \leq \frac{c_0}{|a|^b}, \qquad |a| > 0, \quad a \in L, \tag{7}$$

   where $b > 0$ and $c_0 > 0$ are constants,

2. The map $u \to F(u) + au$, $a \in L$, is a global homeomorphism. Thus, equation

$$F(u_a) + au_a = f \tag{8}$$

   is uniquely solvable for every $f \in X$,

3. If $u_a$ solves (8), then

$$\lim_{|a|\to 0, a\in L} u_a = y, \qquad F(y) = f. \tag{9}$$

If $a = a(t)$, where $t$ is a parameter, $t \geq 0$, and $|a(t)| := r(t)$, then we assume that

$$C|\dot{a}(t)| \leq |\dot{r}(t)|,$$

where $C \in (0,1)$ is a constant independent of $t$.

We prove in Section 2 that $|\dot{r}(t)| \leq |\dot{a}(t)|$. Thus, our assumption implies inequality

$$|\dot{r}(t)| \leq |\dot{a}(t)| \leq C^{-1}|\dot{r}(t)|.$$

Sufficient conditions for (9) to hold are given in [19], and in [20].

Our basic result is

**Theorem 1.** *If Assumptions A) hold, then problem* (6) *has a unique global solution* $u(t)$ *and conclusions* (4) *hold.*

Theorem 1 gives a justification of the DSM for a class of operator equations in Banach space under the Assumptions A).

In Section 2 proofs are given.

In Remark 2 in Section 2 it is pointed out that the class of monotone operators in Hilbert spaces $H$ satisfies *Assumptions A)*. The contour $L$ for this class of operators is a segment $(0, \epsilon_0)$, where $\epsilon_0 > 0$, $b = c = 1$ in (7), conditions (8)-(9) hold, and $y$ is the unique minimal-norm solution of equation (1) with monotone operator $F$ in $H$. The class of monotone operators is important in many applications. In particular, the author has proved that any solvable linear equation $Au = f$ in a Hilbert space $H$ can be reduced to solving an operator equation with a monotone operator provided that $A$ is a closed, densely defined in $H$, linear operator (see [21], [22], [23]).

However, the class of operators for which *Assumptions A)* hold is much larger than the class of monotone operators. For example, it includes the operators satisfying the spectral assumption introduced in [20], p. 133.

# 2 Proofs

## 2.1 Local existence and uniqueness of the solution to problem (6)

Denote

$$v(t) := F(u(t)) + a(t)u(t) - f. \tag{10}$$

If $u \in C^1_{loc}$ then

$$\dot{v} = A_{a(t)}\dot{u} + \dot{a}u. \tag{11}$$

If (6) holds then (11) can be written as

$$\dot{v} = -v + \dot{a}G(v), \quad v(0) = F(u_0) + a(0)u_0 - f, \tag{12}$$

where $u = G(v)$ is the unique solution to (10). This solution exists and is unique by *Assumption A2)*. By the abstract inverse function theorem one concludes that $G$ is a Lipschitz map, because $F$ is Lipschitz and *Assumption A1)* holds. This $u(t)$ solves problem (6) if $v(t)$ solves problem (12). Problem (12) has a unique local solution $v(t)$ by the standard result, since the right-hand side of equation (12) satisfies a Lipschitz condition with respect to $v$. This solution $v(t)$ generates the unique $u(t) = G(v(t))$, and this $u(t)$ solves problem (6). Therefore, (6) has a unique local solution $u(t) \in C^1_{loc}$.

The above argument is new, it does not use the usual assumption that the right-hand side of equation (6) satisfies the Lipschitz condition.

## 2.2 The local solution $u(t)$ is global

The solution $v(t)$ to problem (12) exists globally if the following a priori estimate

$$\sup_{t\geq 0} \|v(t)\| \leq c < \infty \tag{13}$$

holds. Here and below $c > 0$ stands for various constants. The fact that estimate (13) is sufficient for the global existence of a solution to an evolution problem (3) with $\Phi(t, u)$ satisfying a Lipschitz condition with respect to $u$ is known (see, e.g., [20]) and is based on the following argument. If (13) holds, then the length $\ell$ of the interval of the local existence of the solution to (3) depends only on the Lipschitz constant for $\Phi$ and on the norm of $\Phi$, both of which depend only on the constant $c$ in (13). Thus, $\ell = \ell(c) > 0$. If the maximal interval of existence of the solution to (3) is $[0, T)$ and $T < \infty$, then one solves the Cauchy problem (3) with the initial data $v(T - \frac{\ell}{2})$ at $t = T - \frac{\ell}{2}$. The solution to this Cauchy problem exists on the interval $[T - \frac{\ell}{2}, T + \frac{\ell}{2})$. This contradicts the fact that $[0, T)$ is the maximal interval of existence of the solution, unless $T = \infty$. Thus, estimate (13) implies that $v(t)$, the unique solution to (12), exists for all $t \geq 0$. Therefore $u(t)$, the unique solution to (6), exists for all $t \geq 0$.

## 2.3 Existence of $u(\infty)$

Denote by $w(t)$ the unique solution to (8) with $a \in L$, $a = a(t) \in C^1([0, \infty))$, $\lim_{t\to\infty} a(t) = 0$. Differentiating (8) with respect to $t$, one gets

$$A_{a(t)}\dot{w} + \dot{a}(t)w(t) = 0, \tag{14}$$

so by (7) and (9) one obtains

$$\|\dot{w}\| \leq \frac{c_0|\dot{a}(t)|}{|a(t)|^b}\|w(t)\| \leq \frac{c_1|\dot{a}(t)|}{|a(t)|^b}, \tag{15}$$

where $c_1 > 0$ is a constant, $c_1 = c_0 \sup_{t\geq 0} \|w(t)\|$. The quantity $\sup_{t\geq 0} \|w(t)\| < \infty$ because of the assumption (9).

Let

$$r(t) := |a(t)|. \tag{16}$$

One has

$$|\dot{r}(t)| \leq |\dot{a}(t)|. \tag{17}$$

Indeed, if $a(t) = p(t) + iq(t)$, $p = \text{Re } a(t)$, $q = \text{Im } a(t)$, then $|\dot{a}(t)| = \sqrt{\dot{p}^2 + \dot{q}^2}$, $r(t) = \sqrt{p^2 + q^2}$,

$$|\dot{r}(t)| = \frac{|p\dot{p} + q\dot{q}|}{\sqrt{p^2 + q^2}} \leq \sqrt{\dot{p}^2 + \dot{q}^2} = |\dot{a}(t)|. \tag{18}$$

We have assumed that there exists a constant $C \in (0,1)$ such that $C|\dot{a}(t)| \leq |\dot{r}(t)|$. Therefore,

$$|\dot{r}(t)| \leq |\dot{a}(t)| \leq C^{-1}|\dot{r}(t)| \tag{19}$$

To prove the existence of $u(\infty)$ we use differential inequality (28), see below. Let us derive this inequality. Let

$$z(t) := u(t) - w(t), \quad h(t) := \|z(t)\|. \tag{20}$$

From (6) one derives

$$\dot{z} = -\dot{w} - A^{-1}_{a(t)}(u(t))[F(u(t)) - F(w(t)) + a(t)z(t)]. \tag{21}$$

One has

$$F(u) - F(w) = F'(u)z + \eta, \quad \|\eta\| = o(\|z\|). \tag{22}$$

From (21) and (22) it follows that

$$\dot{z} = -\dot{w} - z - A^{-1}_{a(t)}\eta. \tag{23}$$

Let us derive from equation (23) the inequality

$$\dot{h}(t) \leq -h(t) + \frac{c_0\epsilon(h(t))}{|a(t)|^b} + \frac{c_1|\dot{a}(t)|}{|a(t)|^b}, \qquad h(t) := \|z(t)\|, \quad \epsilon(h(t)) := \|\eta\|. \tag{24}$$

To derive (24), let $z(t) := e^{-t}p(t)$. Then (23) yields:

$$e^{-t}\dot{p} = -\dot{w} - A^{-1}_{a(t)}\eta.$$

Taking the norm of both sides of this equation yields

$$e^{-t}\frac{d(e^t h(t))}{dt} \leq e^{-t}\|\dot{p}\| \leq \frac{c_0\epsilon(h(t))}{|a(t)|^b} + \frac{c_1|\dot{a}(t)|}{|a(t)|^b}. \tag{25}$$

Here we have used inequality (15) and the following inequality

$$\left|\frac{d\|p\|}{dt}\right| \leq \|\dot{p}\|. \tag{26}$$

To derive (26), use the triangle inequality $\|p(t+s)\| - \|p(t)\| \leq \|p(t+s) - p(t)\|$, divide it by $s > 0$, take $s \to 0$, and get $\frac{d\|p(t)\|}{dt} \leq \|\dot{p}(t)\|$. Similarly, one gets (26). From inequality (25) one obtains inequality (24).

Let us assume that

$$|\epsilon(h)| \le c_2 h^{1+\nu}, \tag{27}$$

where $\nu = const > 0$.

From (24), (27) and (19) one gets

$$\dot h(t) \le -h(t) + \frac{c_0 \epsilon(h)}{r^b(t)} + \frac{C_1 |\dot r(t)|}{r^b(t)}, \tag{28}$$

where $C_1 = c_1 C^{-1}$ is a positive constant. Let us use the following lemma ( see papers [24],[25], cf [7]).

**Lemma 1.** *Assume that $h(t) \ge 0$, $t \in \mathbb{R}_+ = [0, \infty)$,*

$$\dot h(t) \le -\gamma(t) h + \alpha(t, h) + \beta(t), \tag{29}$$

*where $\gamma(t)$ and $\beta(t)$ are continuous functions on $\mathbb{R}_+$ and $\alpha(t,h) \ge 0$ is continuous with respect to $t, h \in \mathbb{R}_+$, and nondecreasing with respect to $h$. Suppose that there exists a $\mu(t) \in C^1(\mathbb{R}_+)$, $u(t) > 0$, such that*

$$\alpha\left(t, \frac{1}{\mu(t)}\right) + \beta(t) \le \frac{1}{\mu(t)}\left[\gamma(t) - \frac{\dot\mu(t)}{\mu(t)}\right] \tag{30}$$

*and*

$$\mu(0) h(0) < 1. \tag{31}$$

*Then $h(t)$ exists on $\mathbb{R}_+$, and*

$$0 \le h(t) < \frac{1}{\mu(t)} \quad \forall t \in \mathbb{R}_+. \tag{32}$$

We apply Lemma 1 to inequality (28). Choose

$$\mu(t) = \frac{\lambda}{r^s(t)}, \quad s = const > 0, \quad \lambda = const > 0. \tag{33}$$

Then condition (31) is satisfied if

$$\frac{h(0)\lambda}{r^s(0)} < 1. \tag{34}$$

Let us choose $r(t)$ such that $\dot r(t) < 0$, see a possible choice of $r(t)$ in (39) below. In (28) one has

$$\gamma(t) = 1, \quad \alpha(t,h) = \frac{c_0 \epsilon(h)}{r^b(t)}, \quad \beta(t) = \frac{C_1 |\dot r(t)|}{r^b(t)}. \tag{35}$$

Condition (30) holds if

$$\frac{c_0 \epsilon\left(\frac{r^s(t)}{\lambda}\right)}{r^b(t)} + \frac{C_1 |\dot r(t)|}{r^b(t)} \le \frac{r^s(t)}{\lambda}\left(1 - \frac{s |\dot r(t)|}{r(t)}\right). \tag{36}$$

Inequality (36) holds if

$$\frac{c_0\lambda\epsilon\left(\frac{r^s(t)}{\lambda}\right)}{r^{b+s}(t)} + \frac{C_1\lambda|\dot{r}(t)|}{r^{b+s}(t)} + \frac{s|\dot{r}(t)|}{r(t)} \leq 1. \tag{37}$$

Due to (27), inequality (37) holds if

$$c_0c_2\lambda^{-\nu}r^{s\nu-b}(t) + C_1\lambda r^{-b-s}(t)|\dot{r}(t)| + sr^{-1}(t)|\dot{r}(t)| \leq 1. \tag{38}$$

Choose

$$r(t) = r_0(t+r_1)^{-k}, \quad k, r_0, r_1 = const > 0. \tag{39}$$

Then (38) holds if

$$\frac{c_0c_2\lambda^{-\nu}r_0^{s\nu-b}}{(t+r_1)^{k(s\nu-b)}} + \frac{C_1\lambda k r_0^{1-s-b}}{(t+r_1)^{k+1-k(s+b)}} + \frac{sk}{t+r_1} \leq 1, \quad \forall t \geq 0. \tag{40}$$

Inequality (40) holds if

$$s\nu > b, \quad k(s+b) < k+1, \quad \frac{c_0c_2\lambda^{-\nu}r_0^{s\nu-b}}{r_1^{k(s\nu-b)}} + \frac{C_1\lambda k r_0^{1-s-b}}{r_1^{k+1-k(s+b)}} + \frac{sk}{r_1} \leq 1. \tag{41}$$

Inequality (34) holds if

$$\frac{h(0)\lambda r_1^{ks}}{r_0^s} < 1. \tag{42}$$

One can always find positive constants $s, k, \lambda, r_0, r_1$ to satisfy inequalities (41) and (42) for any fixed $h(0) \geq 0$. For example, let $c_0, C_1, c_2, b > 0$ be arbitrary fixed positive numbers, and $0 < \nu \leq 1$. Choose

$$\lambda = \frac{r_0^s}{2h(0)r_1^{ks}}, \qquad h(0) > 0. \tag{43}$$

Then (42) holds, and (41) takes the form

$$s\nu > b, \quad ks + kb < 1 + k, \tag{44}$$

$$\frac{c_0c_2r_0^{s\nu-b}2^\nu h^\nu(0)r_1^{ks\nu}}{r_1^{k(s\nu-b)}r_0^{s\nu}} + \frac{C_1kr_0^s}{r_0^sr_1^{1+k-k(s+b)}2h(0)r_1^{ks}} + \frac{sk}{r_1} \leq 1. \tag{45}$$

Inequality (45) can be written as

$$\frac{c_0c_2 2^\nu h^\nu(0)r_1^{kb}}{r_0^b} + \frac{C_1k}{2h(0)r_1^{1+k-kb}} + \frac{sk}{r_1} \leq 1. \tag{46}$$

If

$$r_1 \geq 2\left(\frac{C_1k}{2h(0)} + sk\right), \tag{47}$$

then

$$\frac{C_1 k}{2h(0)r_1} + \frac{sk}{r_1} \le \frac{1}{2}. \tag{48}$$

Fix $r_1$ satisfying (47), and then fix $r_0$ such that

$$r_0^b \ge 2c_0c_2 2^\nu h^\nu(0) r_1^{kb}. \tag{49}$$

Then (46) holds. Using Lemma 1, one obtains the following theorem.

**Theorem 2.** *If* (39)*,* (44)*,* (47) *and* (49) *hold, then the solution* $h(t)$ *to inequality* (28) *satisfies the estimate*

$$0 \le h(t) < \frac{r_0(t+r_1)^{-k}}{\lambda}, \quad k > 0. \tag{50}$$

Recall that $h(t) = \|u(t) - w(t)\|$, see (20). Since $\lim_{t\to\infty} a(t) = 0$, *Assumption A3)* yields

$$\lim_{t\to\infty} w(t) = y. \tag{51}$$

From (50) and (51) one concludes that

$$\lim_{t\to\infty} u(t) = y. \tag{52}$$

Therefore the following result holds.

**Theorem 3.** *If the assumptions of Theorem 2 hold, then conclusions* (4) *hold for the solution to problem* (6)*.*

**Remark 1.** *It follows from Theorem 3 that the DSM* (6) *converges to a solution* $y$ *of equation* (1) *for any choice of the initial approximation* $u_0$*, i.e., globally.*

**Remark 2.** *Let us give a simple example of a class of operators for which Assumptions A) hold. This is the class of monotone operators* $F$ *in a Hilbert space* $H$*, i.e., operators such that*

$$(F(u) - F(v), u - v) \ge 0 \quad \forall u, v \in H. \tag{53}$$

*If* $F \in C^1_{loc}$*, then the contour* $L$ *is the segment* $(0, \epsilon_0)$*, estimate* (7) *holds with* $c_0 = 1$ *and* $b = 1$*, equation* (8) *is uniquely solvable for any* $f \in H$*, and relation* (9) *holds with* $y$ *being the unique minimal-norm solution to equation* (1)*, which is assumed solvable. If* $F$ *is monotone and continuous, and equation* (1) *has a solution in* $H$*, then the set of all solutions to*(1) *is convex and closed. Such sets in a Hilbert space have a unique element with minimal norm, so the minimal-norm solution* $y$ *of* (1) *is well defined if* $F : H \to H$ *is monotone. All the statements in Remark 2 are proved in the monograph [20].*

*Various concrete choices of the function* $a(t)$ *are given in [20] and in the papers [9],[11]. For instance, the choice* $a(t) = d(c+t)^{-b}$*, where* $d, c, b > 0$ *are some constants, was used in [9], the choice* $a(t) > 0$*, monotonically decaying,* $\lim_{t\to\infty} a(t) = 0$, $\lim_{t\to\infty} \frac{|\dot{a}(t)|}{a(t)} = 0$*, was used in [11], and a piecewise-constant* $a(t) > 0$, $\lim_{t\to\infty} a(t) = 0$*, with an adaptive choice of the step size, was used in [12], [13].*

**Remark 3.** *In [3]-[8] stable methods for solving equation* (1) *given noisy data* $f_\delta$, $\|f_\delta - f\| \le \delta$, *are developed for monotone operators in Hilbert space, papers [9] and [11] are review papers in which the results of papers [3]-[11] are summarized and some new results are obtained. The methods for solving equation* (1) *given noisy data* $f_\delta$ *are based on choosing a stopping rule* $t_\delta$ *such that* $\lim_{\delta\to 0} u_\delta(t_\delta) = y$, *where* $u_\delta(t)$ *is the solution to* (6) *with* $f_\delta$ *in place of* $f$.

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