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On the relation between the $S$–matrix and the spectrum of the interior Laplacian

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Abstract
The main results of this paper are:
1) a proof that a necessary condition for 1 to be an eigenvalue of the $S$-matrix is real analyticity of the boundary of the obstacle,
2) a short proof of the conclusion stating that if 1 is an eigenvalue of the $S$-matrix, then $k^2$ is an eigenvalue of the Laplacian of the interior problem, and that in this case there exists a solution to the interior Dirichlet problem for the Laplacian, which admits an analytic continuation to the whole space $\mathbb{R}^3$ as an entire function.

1. Introduction and Statement of the Result
We consider below the obstacle scattering problem in $\mathbb{R}^3$, but the argument and the results remain valid in $\mathbb{R}^n$, $n \geq 2$.

Let the obstacle $D \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary $S$. Denote by $D' = \mathbb{R}^3 \setminus D$ the exterior domain and by $N$, the unit normal to $S$, pointing into $D'$. Let $k > 0$ be the wave number, and $S^2$ be the unit sphere in $\mathbb{R}^3$. The scattering matrix $S = S(k) = I - \frac{k^2}{2\pi} A$ for the obstacle scattering problem is a unitary operator in $L^2(S^2)$, $I$ is the identity operator and $A$ is an integral operator in $L^2(S^2)$, whose kernel $A(\beta, \alpha, k)$ is the scattering amplitude, which is defined in formula (5) below. The operator $S$ has an eigenvalue 1 if and only if equation $Aw = 0$ has a non-trivial solution. The eigenvalues of $S$ have 1 as an accumulation point, they all have absolute values equal to 1 since $S$ is unitary.

The following conjecture, (the Doron-Smilansky (DS) conjecture) is known:

DS conjecture: A number $k^2 > 0$ is a Dirichlet eigenvalue of the Laplacian in a bounded domain $D$ if and only if the corresponding $S$-matrix for the scattering problem by the obstacle $D$ has an eigenvalue 1.

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This conjecture is discussed in [1]-[3], and in [2] a counterexample to this conjecture is mentioned.

From the definition of the \( S \)-matrix it follows that 1 is its eigenvalue if and only if 0 is an eigenvalue of \( A \), that is, equation (12) (see below) has a non-trivial solution.

We prove (see Theorem 2) that if equation (12) has a non-trivial solution, then the boundary \( S \) of \( D \) is an analytic set. Since generically \( S \) is not an analytic set, it follows that the DS conjecture is incorrect. Our result gives a necessary condition for 1 to be an eigenvalue of the \( S \)-matrix. This condition is necessary but not sufficient for 1 to be an eigenvalue of the \( S \)-matrix (and, therefore, not sufficient for the DS conjecture to hold for the domain \( D \)).

In [2] it is proved that if \( D \subset R^2 \) is a bounded domain with a sufficiently smooth boundary \( S \), and if 1 is a Dirichlet eigenvalue of \( S \), then \( k^2 \) is a Dirichlet eigenvalue of the Laplacian in \( D \). An open problem, stated in [2], is to give a proof of such a statement for \( D \subset R^n \) with \( n > 2 \). This is done in our paper by a method different from the one in [2]. Our proof is short and simple.

Let \( S^2_j, j = 1, 2 \), be arbitrary small fixed open subsets of \( S^2 \), and the boundary conditions on \( S \) be either the Dirichlet, or the Neumann, or the Robin conditions.

The following theorem is proved in [5], p.85:

**Theorem (Ramm)** The knowledge of \( A(\beta, \alpha, k), \forall \alpha \in S^2_1, \forall \beta \in S^2_2 \), and for a fixed \( k > 0 \), determines \( S \) and the boundary conditions on \( S \) uniquely.

It follows from this result that the knowledge of the \( S \)-matrix \( S(k) \) at a fixed \( k > 0 \) determines the boundary \( S \) of the obstacle and the boundary condition on \( S \) uniquely.

Therefore, the discrete spectrum of the Laplacian in \( D \), corresponding to this boundary condition, is determined uniquely by the knowledge of \( S(k) \) at a fixed \( k > 0 \).

This conclusion establishes a relation between the \( S \)-matrix and the spectrum of the Laplacian in \( D \).

Let us now formulate the obstacle scattering problem, introduce basic notions, and formulate our results.

The scattering solution \( u(x, \alpha, k) \) is the solution to the following scattering problem:

\[
Lu := (\nabla^2 + k^2)u = 0 \text{ in } D',
\]

\[
u \big|_{S} = 0,
\]

\[
u = u_0 + v, \quad u_0 := e^{ik\alpha \cdot x},
\]

\[
\frac{\partial v}{\partial r} - ikr = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty.
\]

Here \( \alpha \in S^2 \) is the incident direction, i.e., the direction of the incident plane wave \( u_0 \), \( v \) is the scattered field which satisfies the radiation condition (4). This condition implies that

\[
v := v(x, \alpha, k) = A(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \beta := \frac{x}{r}.
\]
The function $A := A(\beta, \alpha, k)$ is called the scattering amplitude. Let us denote by $A : L^2(S^2) \to L^2(S^2)$ the operator

$$ Aw := \int_{S^2} A(\beta, \alpha, k) w(\alpha) d\alpha. \quad (6) $$

It is well known (see [5]), that problem (1) − (4) has a unique solution $u(x, \alpha, k)$,

$$ A(\beta, \alpha, k) = -\frac{1}{4\pi} \int_{S} e^{-ik\beta \cdot s} u_N(s, \alpha, k) ds, \quad (7) $$

where $u_N(s, \alpha, k)$ is the normal derivative of the scattering solution $u(x, \alpha, k)$ on $S$, and the following relation holds:

$$ u(x, \alpha, k) = e^{ik\alpha \cdot x} - \int_{S} g(x, s, k) u_N(s, \alpha, k) ds. \quad (8) $$

Here $G$, the resolvent kernel of the Dirichlet Laplacian in the exterior domain $D'$, satisfies the following equation:

$$ G(x, y, k) = g(x, y, k) - \int_{S} g(x, s, k) G_N(s, y, k) ds, \quad (9) $$

where

$$ g(x, y, k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \quad (10) $$

The function $G$ solves the boundary value problem:

$$ LG = -\delta(x-y) \text{ in } D', \quad G|_S = 0, \quad (11) $$

and satisfies the radiation condition (4).

Let $\sigma$ denote the set of the eigenvalues of the Dirichlet Laplacian in $D$. This set is discrete.

It is proved in [5], pp.52-57, that:

a) The function $A(\beta, \alpha, k)$ admits a meromorphic continuation as a function of $k$ from the ray $(0, \infty)$ to the whole complex $k$-plane,

b) The scattering amplitude $A(\beta, \alpha, k)$ is analytic in the region $\Im k \geq 0$ (if $D \subset R^{2n}$ then $k = 0$ is a logarithmic branch point),

c) $A(\beta, \alpha, k)$ has infinitely many poles on the imaginary axis in the region $\Im k < 0$,

d) As a function of $\alpha$ and $\beta$, the scattering amplitude $A(\beta, \alpha, k)$ admits analytic continuation from $S^2 \times S^2$ to the set $M \times M$, where $M := \{ \Theta : \Theta \in C^3, \Theta \cdot \Theta = 1 \}$, $\Theta \cdot \omega := \sum_{j=1}^{3} \Theta_j \omega_j$. The set $M$ is a non-compact algebraic variety in $C^3$.

Let us now state our basic results:
Theorem 1. If $S(k)$ has an eigenvalue 1, that is, the equation

$$Aw = \int_{S^2} A(\beta, \alpha, k)w(\alpha)d\alpha = 0 \quad (12)$$

has a non-trivial solution $w$, then $k^2 \in \sigma$, and there is a solution to the problem $(\nabla^2 + k^2)W = 0$ in $D$, $W|_S = 0$, which can be extended from $D$ to $R^3$ as a bounded entire function of $x$.

Theorem 2. If equation (12) has a non-trivial solution, then the boundary $S$ is an analytic set.

An analytic set is a set of zeros of (a finite collection of) analytic functions. One can find definition and properties of analytic sets in [4], Section 1.4. If $S$ is an analytic set, then $S$ is piecewise real analytic surface. Generically, $S$ is not piecewise real analytic surface. Therefore, it follows from Theorem 2 that the DS conjecture is incorrect.

In Section 2 Theorems 1 and 2 are proved. In the proofs, the following result of the author is used:

Lemma 1. ([5], p.46) One has

$$G(x, y, k) = \frac{e^{ik|x|}}{4\pi|y|} u(x, \alpha, k)[1 + o(1)], \quad |y| \to \infty, \quad \frac{y}{|y|} = -\alpha, \quad (13)$$

where $u(x, \alpha, k)$ is the scattering solution, i.e., the solution to (1) - (4).

Lemma 1 yields formula (8) as a consequence of (9), while formula (9) is obtained by Green’s formula. Formula (7) follows from (8).

2. Proofs.

Proof of Theorem 1. Let us prove that if $w \neq 0$ solves (12) then $k^2 \in \sigma$.

Assume that equation (12) has a non-trivial solution $w$. Multiply (7) by $w = w(\alpha)$ and integrate over $S^2$ with respect to $\alpha$. The result is

$$\int_S e^{-ik|s-p|}\frac{1}{4\pi|y|} u_N(s, \alpha, k)w(\alpha)d\alpha = 0 \quad (14)$$

Let us prove that $p(s) \neq 0$.

Indeed, if

$$p(s) = \int_{S^2} u_N(s, \alpha, k)w(\alpha)d\alpha = 0 \quad \forall s \in S, \quad (15)$$

then the function $w(\alpha) = 0$ because the set $\{u_N(s, \alpha, k)\}_{\alpha \in S^2}$ is total (dense) in $L^2(S)$ for any fixed $k > 0$ ([5], p.162).

Let us continue the proof of Theorem 1 and prove that $k^2 \in \sigma$ if equation (12) has a non-trivial solution.

Equation (14) and Lemma 1 imply that

$$\nu(x) := \int_{S^2} \frac{e^{ik|x-s|}}{4\pi|x-s|} p(s)ds = 0 \quad \text{in } D'.$$
Indeed, this $\nu$ solves equation (1), satisfies the radiation condition (4), and (14) implies

$$\nu(x) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty. \quad (17)$$

Relation (17) and Lemma 1 in [5], p.25, imply that

$$\nu(x) = 0 \quad \text{in } D'. \quad (18)$$

Therefore, by the jump formula for the normal derivative of the single layer potential (16) ([5], p.14), one gets

$$\frac{\partial \nu}{\partial N_+} = p(s) \neq 0, \quad (19)$$

where $\frac{\partial}{\partial N_+}$ denotes the limiting value on $S$ of the normal derivative from inside of $D$.

This implies that $k^2 \in \sigma$.

Indeed, $\nu(x)$ solves the equation

$$(\nabla^2 + k^2)\nu = 0 \quad \text{in } D', \quad (20)$$

and satisfies the boundary condition

$$\nu|_S = 0, \quad (21)$$

due to (18) and the continuity of $\nu$ across $S$. Finally, $\nu \neq 0$ in $D$ because of (19).

The last statement of Theorem 1, namely, the existence of the solution to problem (20)-(21) which can be analytically continued to the whole space $R^3$ as an entire function of $x$, is proved as follows.

The reciprocity relation $A(\beta, \alpha, k) = A(-\alpha, -\beta, k)$ (see [5], p.53) and equation (12) imply:

$$0 = \int_{S^2} A(\beta, \alpha, k)w(\alpha)d\alpha = -\frac{1}{4\pi} \int_S \left( \int_{S^2} e^{ik\alpha \cdot s}w(\alpha)d\alpha \right) u_N(s, -\beta)ds \quad \forall \beta \in S^2. \quad (22)$$

Since the set $\{u_N(s, \alpha, k)\}_{\alpha \in S^2}$ is total (dense) in $L^2(S)$ for any fixed $k > 0$ ([5], p.162), relation (22) implies

$$\int_{S^2} e^{ik\alpha \cdot s}w(\alpha)d\alpha = 0 \quad \forall s \in S. \quad (23)$$

Therefore, the function

$$W(x) := \int_{S^2} e^{ik\alpha \cdot x}w(\alpha)d\alpha, \quad x \in R^3, \quad (24)$$

satisfies all the requirements mentioned in the last statement of Theorem 1.

Thus, Theorem 1 is proved. \qed
Remark 1. A similar argument yields the following result:

If \( \sigma_N \) is the set of the eigenvalues of the Neumann Laplacian, and \( A_N(\beta, \alpha, k) \) is the scattering amplitude, corresponding to the plane wave scattering by the obstacle \( D \) on the boundary of which the Neumann boundary condition holds, then if equation (12), with \( A_N \) in place of \( A \), has a non-trivial solution, then \( k^2 \in \sigma_N \).

Remark 2. If \( k^2 \in \sigma \), then any non-trivial solution to (20)-(21) can be written in the form (16) with \( p(s) \) defined in (19), and the boundary condition (18) holds. Taking \( |x| \to \infty, \frac{x}{|x|} = \beta \), in (16) and using (18), one obtains

\[
\int_S e^{-ik\beta \cdot s} p(s) ds = 0 \quad \forall \beta \in S^2, \quad p(s) \neq 0.
\] (25)

Thus, if \( k^2 \in \sigma \), then equation (25) has a non-trivial solution \( p(s) \).

Proof of Theorem 2. Suppose equation (12) has a solution \( \eta \in L^2(S^2), \eta \neq 0 \). Then

\[
\int_S ds u_N(s, \alpha) \int_{S^2} e^{-ik\beta \cdot s} \eta(\beta) d\beta = 0 \quad \forall \alpha \in S^2.
\] (26)

Since the set \( \{u_N(s, \alpha)\}_{\forall \alpha \in S^2} \) is total in \( L^2(S) \), one concludes from (26) that

\[
\psi(s) := \int_{S^2} e^{-ik\beta \cdot s} \eta(\beta) d\beta = 0 \quad \forall s \in S,
\] (27)

where

\[
\psi(x) := \int_{S^2} e^{-ik\beta \cdot x} \eta(\beta) d\beta.
\]

The function \( \psi(x) \) is an entire function of \( x \), that is, an analytic function of \( x \in \mathbb{C}^3 \). It vanishes on \( S \), so \( S \) is an analytic set (see [2] for the definition and properties of analytic sets). Generically, the boundary \( S \) is not an analytic set.

Thus, Theorem 2 is proved.

Remark 3. If one uses the reciprocity relation \( A(\beta, \alpha, k) = A(-\alpha, -\beta, k) \), then one concludes that zero is an eigenvalue of \( A \) if either

\[
\int_S e^{-ik\beta \cdot s} \int_{S^2} u_N(s, \alpha, k) w(\alpha) d\alpha = 0 \quad \forall \beta \in S^2, \quad w \neq 0,
\] (28)

or

\[
\int_S \left( \int_{S^2} e^{ik\alpha \cdot s} w(\alpha) d\alpha \right) u_N(s, -\beta) ds = 0 \quad \forall \beta \in S^2, \quad w \neq 0.
\] (29)

The last relation implies equation (28) (with \( \beta = -\alpha \) and \( \eta(\beta) = w(\alpha) \)).

Let us denote \( T_k p := \int_S g(s, t, k) p(t) dt \) and \( U := U(x, k) := \int_S g(x, t, k) p(t) dt \), so \( U \mid_S = T_k p \).

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Remark 4. The operator $T_k^{-1}$ has simple poles at the points $k^2 = k_j^2$, where $k_j^2 \in \sigma$.

Remark 4 shows that the knowledge of the set of poles of the operator $T_k^{-1}$ allows one to find the spectrum of the interior Dirichlet Laplacian in $D$.

Proof of Remark 4. Consider the equation $T_k p = f$. Then

$$U(x) = \int_S g(x,t,k)p(t)dt$$

solves the problem

$$(\nabla^2 + k^2)U = 0 \text{ in } D, \quad U|_S = f. \quad (30)$$

Let

$$(\nabla^2 + k^2)\Gamma = -\delta(x - y) \text{ in } D, \quad \Gamma|_S = 0.$$

Then Green’s formula yields the following representation of the solution to problem (30):

$$U(x) = -\int_S f(t)N_1(t,x,k)dt, \quad x \in D, \quad k^2 \neq k_j^2. \quad (31)$$

Since $\Gamma(x,y,k) = \sum_{j=1}^{\infty} \frac{\phi_j(x)\overline{\phi}_j(y)}{k^2 - k_j^2}$ has a simple pole at $k^2 = k_j^2$, the claim is proved. Here $\phi_j$ are the normalized eigenfunctions of the Dirichlet Laplacian in $D$.

References


