New metrics on networks arising from modulus and applications of Fulkerson duality

by

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B.Sc., University of Colombo, Sri-Lanka, 2011

M.S., Kansas State University, 2016

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2018

Abstract

This thesis contains six chapters. In the first chapter, the continuous and the discrete cases of p-modulus is introduced. We present properties of p-modulus and its connection to classical quantities. We also introduce use Arne Beurling's criterion for extremality to build insight and intuition regarding the modulus. After building an intuitive understanding of the p-modulus, we then proceed to switch perspectives to that of convex analysis. Using the theory of convex analysis, the uniqueness and existence of extremal densities is shown. We end this chapter with the introduction of the probabilistic interpretation of Modulus.

In the second chapter, we introduce the Fulkerson duality. After defining the Fulkerson dual, we will investigate the blocking duality for different families of objects that the NODE research group has been studying and has been established. An important result that connects the Fulkerson dual and modulus is given at the end of this chapter. This important theorem will be used in proving one of the main results that δ_p (introduced in Chapter 4) is a metric on graphs.

The third chapter will discuss about metrics and ultrametrics on networks. Among these metrics, effective resistance is given special attention because the proof of δ_p metric also serves as a new proof that effective resistance is a metric on graphs. We define effective resistance and give two different proves that show it is a metric, namely flows and the Laplacian.

Two new families of metrics on graphs that arises through modulus are introduced in the fourth chapter. We also show how the two families are related as the d_p metric is viewed as a snowflaked version of the δ_p metric. We end this chapter with some numerical examples that proves this connection and also serves as a set of plentiful examples of modulus calculations.

Clutters and blockers is also another topic that is very much related to families of objects. While it has different rules and conditions, the study of clutters and blockers can give more insights to both modulus and clutters. We explore these relations in chapter 5. We provide some examples of clutters and blockers and finally reveal the relationship between the blocker and Fulkerson dual.

Finally, in chapter 6, we end the thesis by presenting some of the open questions that we would like to explore and find answers in the future.

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Acknowledgments

Firstly, I would like to thank my adviser, Professor Pietro Poggi-Corradini. You always had faith in me even when I did not. Thank you for believing in me. You pushed me and encouraged me to be my best. You have guided me since my very first talk at the applied math seminar to many other domestic and international talks. Like a father, you have been kind and gentle even when I made mistakes and provide me with constructive criticism to improve myself . You have played a vital part in shaping the professional in me, and taught me skills that I will be using for the rest of my career and life. I am truly blessed to have such great guidance and a role model to look up to.

I'd also like to thank my committee members...

...Professor Nathan Albin, you have helped guide me at various occasions whether it be a brainstorming session, reviewing some material or helping me with a practice talk. You have guided me and encouraged me even before I applied to Kansas State University. If you had not replied to my e mail 5 years ago, I would not have even thought about pursuing a PhD in mathematics, so soon after the completion of my Bachelors degree.

...Professor Hrant Hakobyan, for always guiding me whenever I communicate my future plans with you and for enlightening me with important information and experience to reach my goals successfully. Thank you for all the feedback that you have given me throughout my grad life to make my work better quality.

...Professor Mike Higgins, for bouncing off ideas in our NODE meetings and for providing us with new perspectives and potential new uses of the theory of modulus.

There are many professors for whom I am grateful, that have volunteered their own time to help me grow as a mathematics professional...

...Professor Sarah Reznikoff and Professor Marianne Korten, for helping me with the less attractive side of academia. Your assistance with REUs, qualifying exam preparation, job applications and interview preparation has saved me so many headaches.

...Professor Fabio Milner from Arizona State University and Professor Charles Moore of Washington State University for further advice in job application and career guidance in general.

Finally, I would like to thank my family and friends....

...Mom and Dad, for always been there for me and investing your time, energy and money in me. I am forever grateful for all the values that you have embedded in me to make me a better teacher and a better human. I cannot count the numerous times my dad would have waited for me at the bus halt to walk me home and I cannot imagine the number of times my mom slept in the car until my tuition class was over. You are my all, I love you!

...My sister Akila, for putting up with me all these years. Staying up with me numerous nights just to keep me company while I studied for exams. For being calm and helping me to find things and ensure everything is fine, when I am anxious and lose my head during exams. Also, for advising me when I cannot think straight. I love you, Aki!

...Jose Guzman, my partner, for always encouraging me and believing in me. Every time I cried out of anxiety, you would calm me with your presence and let me know that I am not alone. You have reached out to help me even when I am mad at you and I want you to know that I have so much respect for you for not giving up on me. I love you for been there for me.

...Sheila Barros for being the big sister I never had. Thanks for being such a strong role model for me to look up to and for caring and looking after me all the time.

...Numerous friends from Sri Lanka and the USA, that I am forever in debted to. It is a huge list of you and I want to thank you and say you all have helped me to become who I am today. I love you all.

Dedication

My thesis is dedicated to my mom, dad and sister. I am nothing without you.

Chapter 1

Introduction to Modulus

1.1 Modulus in the continuum

The theory of conformal modulus was originally developed in complex analysis, see the comment on p. 81 in Ahlfors[1]. The more general theory of p-modulus grew out of the study of quasiconformal maps, which generalize the notion of conformal maps to higher dimensional real Euclidean spaces and, in fact, to abstract metric measure spaces. Intuitively, p-modulus provides a method for quantifying the richness of a family of curves, in the sense that a family with many short curves will have a larger modulus than a family with fewer and longer curves. The parameter p tends to favor the "many curves" aspect when p is close to 1 and the "short curves" aspect as p becomes large. This phenomenon was explored more precisely in [2] in the context of networks. The concept of discrete modulus on networks is not new, see for instance papers by Duffin[3] and Schramm[4]. However, recently Albin and Poggi-Corradini have started developing the theory of p-modulus as a graph-theoretic quantity, see [5,2], with the goal of finding applications, for instance to the study of epidemics [6,7].

The concept of blocking duality explored in the second chapter in this thesis is an analog of the concept of conjugate families in the continuum. As motivation for the discrete theory to follow, then, let us recall the relevant definitions from the continuum theory. For now, it is convenient to restrict attention to the 2-modulus of curves in the plane, which, as it happens, is a conformal invariant and thus has been carefully studied in the literature.

Let Ω be a domain in $\mathbb{C}$, and let E, F be two continua in $\overline{\Omega}$. Define $\Gamma = \Gamma_\Omega(E, F)$ to be the family of all rectifiable curves connecting E to F in Ω. A *density* is a Borel measurable function $\rho : \Omega \to [0, \infty)$. We say that ρ is *admissible* for Γ and write $\rho \in \mathrm{Adm}(\Gamma)$, if

$$\int_\gamma \rho \, ds \geq 1 \qquad \forall \gamma \in \Gamma. \tag{1.1.1}$$

Now, we define the modulus of Γ as

$$\mathrm{Mod}_2(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_\Omega \rho^2 dA. \tag{1.1.2}$$

Example 1.1.1 (The Rectangle)**.** Consider a rectangle

$$\Omega := \{z = x + iy \in \mathbb{C} : 0 < x < L, 0 < y < H\}$$

of height H and length L. Set $E := \{z \in \overline{\Omega} : \operatorname{Re} z = 0\}$ and $F := \{z \in \overline{\Omega} : \operatorname{Re} z = L\}$ to be the leftmost and rightmost vertical sides respectively. If $\Gamma = \Gamma_\Omega(E, F)$ then,

$$\mathrm{Mod}_2(\Gamma) = \frac{H}{L}. \tag{1.1.3}$$

To see this, assume $\rho \in \mathrm{Adm}(\Gamma)$. Then for all $0 < y < H$, $\gamma_y(t) := t + iy$ is a curve in Γ, so

$$\int_{\gamma_y} \rho ds = \int_0^L \rho(t, y) dt \geq 1.$$

Using the Cauchy-Schwarz inequality we obtain,

$$1 \leq \left[\int_0^L \rho(t,y)dt\right]^2 \leq L\int_0^L \rho^2(t,y)dt.$$

In particular, $L^{-1} \leq \int_0^L \rho^2(t,y)dt$. Integrating over y, we get

$$\frac{H}{L} \leq \int_\Omega \rho^2 dA.$$

So since ρ was an arbitrary admissible density, $\mathrm{Mod}_2(\Gamma) \geq \frac{H}{L}$.

In the other direction, define $\rho_0(z) = \frac{1}{L}\mathbb{1}_\Omega(z)$ and observe that $\int_\Omega \rho_0^2 dA = \frac{HL}{L^2} = \frac{H}{L}$. Hence, if we show that $\rho_0 \in \mathrm{Adm}(\Gamma)$, then $\mathrm{Mod}(\Gamma) \leq \frac{H}{L}$. To see this note that for any $\gamma \in \Gamma$:

$$\int_0^L \frac{1}{L}|\dot\gamma(t)|dt \geq \frac{1}{L}\int_0^L |\operatorname{Re}\dot\gamma(t)|dt \geq \frac{1}{L}\big(\operatorname{Re}\gamma(1) - \operatorname{Re}\gamma(0)\big) \geq 1.$$

This proves the formula (1.1.3).

A famous and very useful result in this context is the notion of a *conjugate family* of a connecting family. For instance, in the case of the rectangle, the conjugate family $\Gamma^* = \Gamma^*_\Omega(E,F)$ for $\Gamma_\Omega(E,F)$ consists of all curves that "block" or intercept every curve $\gamma \in \Gamma_\Omega(E,F)$. It's clear in this case that Γ^* is also a connecting family, namely it includes every curve connecting the two horizontal sides of Ω. In particular, by (1.1.3), we must have $\mathrm{Mod}_2(\Gamma^*) = L/H$. So we deduce that

$$\mathrm{Mod}_2(\Gamma_\Omega(E,F)) \cdot \mathrm{Mod}_2(\Gamma^*_\Omega(E,F)) = 1. \tag{1.1.4}$$

One reason this reciprocal relation is useful is that upper-bounds for modulus are fairly easy to obtain by choosing reasonable admissible densities and computing their energy. However, lower-bounds are typically harder to obtain. However, when an equation like (1.1.4) holds, then upper-bounds for the modulus of the conjugate family translate to lower-bounds for the given family. We will discuss more on this topic later in chapter two.

In higher dimensions, say in $\mathbb{R}^3$, the conjugate family of a connecting family of curves consists of a family of surfaces, and therefore one must consider the concept of *surface modulus*, see for instance Rajala[8] and references therein. It is also possible to generalize the concept of modulus by replacing the exponent 2 in (1.1.2) with $p \geq 1$ and by replacing dA with a different measure.

In the present work, we will be using a discrete version of the above theory and the next section will explore this idea further.

1.2 Modulus on networks

A general framework for modulus of objects on networks was developed by Albin and Poggi-Corradini in [9]. In what follows, $G = (V, E, \sigma)$ is taken to be a finite graph with vertex set V and edge set E. The edge weights $\sigma \in \mathbb{R}^E_{\geq 0}$. We also assume for simplicity that G is undirected and simple. The theory applies to any finite family of "objects" Γ for which each $\gamma \in \Gamma$ can be assigned an associated function $\mathcal{N}(\gamma, \cdot) : E \to \mathbb{R}_{\geq 0}$ that measures the *usage of edge e by γ*. Notationally, it is convenient to consider $\mathcal{N}(\gamma, \cdot)$ as a row vector in $\mathbb{R}^E_{\geq 0}$.

Some examples of objects and their associated usage functions are the following.

- To a walk $\gamma = x_0\ e_1\ x_1\ \cdots\ e_n\ x_n$ we can associate the traversal-counting function $\mathcal{N}(\gamma, e) =$ number times γ traverses e. In this case $\mathcal{N}(\gamma, \cdot) \in \mathbb{Z}^E_{\geq 0}$.

- To each subset of edges $T \subset E$ we can associate the indicator function $\mathcal{N}(T, e) = \mathbb{1}_T(e) = 1$ if $e \in T$ and 0 otherwise. Here, $\mathcal{N}(\gamma, \cdot) \in \{0, 1\}^E$.

- To each flow (see 3.3) f we can associate the volume function $\mathcal{N}(f, e) = |f(e)|$. Therefore, $\mathcal{N}(\gamma, \cdot) \in \mathbb{R}^E_{\geq 0}$.

For the purposes of the present work in this thesis, it is sufficient to restrict attention to families of walks. In fact, if Γ is the set of all walks between two distinct vertices, then it turns out that modulus can be computed by considering only simple paths (walks that do not visit any node more than once). This is discussed later in Theorem 1.2.9.

We define a *density* on G to be a nonnegative function on the edge set: $\rho : E \to [0, \infty)$. The value $\rho(e)$ can be thought of as the *cost of using edge* e. For an object $\gamma \in \Gamma$, we define

$$\ell_\rho(\gamma) := \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) = (\mathcal{N}\rho)(\gamma),$$

which represents the *total usage cost* for γ with the given edge costs ρ. A density $\rho \in \mathbb{R}^E_{\geq 0}$ is *admissible for* Γ, if

$$\ell_\rho(\Gamma) := \inf_{\gamma \in \Gamma} \ell_\rho(\gamma) \geq 1.$$

Let

$$\mathrm{Adm}(\Gamma) := \left\{\rho \in \mathbb{R}^E_{\geq 0} : \ell_\rho(\Gamma) \geq 1\right\} \tag{1.2.1}$$

be the set of admissible densities.

Given an exponent $p \in [1, \infty]$ we define the *p-energy* of a density ρ as

$$\mathcal{E}_{p,\sigma}(\rho) := \sum_{e \in E} \sigma(e)\rho(e)^p \quad \text{if } p < \infty \qquad \text{and} \qquad \mathcal{E}_\infty(\rho) := \lim_{p \to \infty} \left(\mathcal{E}_{p,\sigma}(\rho)\right)^{\frac{1}{p}} = \max_{e \in E} \sigma(e)\rho(e) \quad \text{if } p = \infty.$$

Definition 1.2.1. A *non-trivial family* of objects is a family that is non-empty and every object $\gamma \in \Gamma$ has at least one edge $e \in \gamma$ such that $\mathcal{N}(\gamma, e) > 0$.

Definition 1.2.2. Let $G = (V, E, \sigma)$ be a simple finite graph and let Γ be a finite non-trivial family of objects with usage matrix $\mathcal{N} \in \mathbb{R}^{\Gamma \times E}$. For $p \in [1, \infty]$, the *p-modulus* of Γ is

$$\mathrm{Mod}_{p,\sigma}(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \mathcal{E}_{p,\sigma}(\rho)$$

Equivalently, p-modulus corresponds to the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \mathcal{E}_{p,\sigma}(\rho) \\ \text{subject to} \quad & \rho \geq 0, \quad \mathcal{N}\rho \geq \mathbf{1} \end{aligned} \tag{1.2.2}$$

where each object $\gamma \in \Gamma$ determines one inequality constraint.

1.2.1 Properties of the p-Modulus

Here we present the basic properties of the p-modulus of families of walks and discuss suitable examples to lend intuition to how these properties are used.

Remark 1.2.1. If Γ is a family of walks on a finite graph G that contains a constant walk, i.e., a walk with zero hops, then $\operatorname{Mod}_p \Gamma = \infty$. Indeed, if $\gamma_0 \in \Gamma$ is a constant walk (a walk that does not use any edge), then $\ell_\rho(\gamma_0) = 0$ for every ρ- density. Consequently, $\operatorname{Adm}(\Gamma) = \emptyset$ and $\inf_{\rho\in\emptyset} \mathcal{E}_p(\rho) = \infty$.

For this reason, we exclude constant walks from now on.

Proposition 1.2.3. (The modulus is an outer measure). *Assume that Γ_j is a family of walks in a finite graph G for each $j \in \mathbb{N}$. Then,*

1. **(Empty Family):** *If $\Gamma_1 = \emptyset$, the empty family, then* $\operatorname{Mod}_p \Gamma_1 = 0$.
2. **(Monotonicity):** *If $\Gamma_1 \subset \Gamma_2$, then* $\operatorname{Mod}_p(\Gamma_1) \leq \operatorname{Mod}_p(\Gamma_2)$.
3. **(Countable Subadditivity):** $\operatorname{Mod}_p \left(\bigcup_{j=1}^{\infty} \Gamma_j \right) \leq \sum_{j=1}^{\infty} \operatorname{Mod}_p \Gamma_j$.

Proof. For each $j \in \mathbb{N}$ let Γ_j be a family of walks in a finite graph G.

1. If $\Gamma_1 = \emptyset$, then every ρ is admissible, including $\rho_0 \equiv 0$. Hence $0 \leq \operatorname{Mod}_p \Gamma \leq \mathcal{E}_p(\rho_0) = 0$.

2. If $\Gamma_1 \subset \Gamma_2$ then $\rho \in \mathrm{Adm}(\Gamma_2)$ implies that $\rho \in \mathrm{Adm}(\Gamma_1)$, so that $\mathrm{Adm}(\Gamma_2) \subset \mathrm{Adm}(\Gamma_1)$. Therefore,

$$\mathrm{Mod}_p \Gamma_1 = \inf_{\rho \in \mathrm{Adm}\,\Gamma_1} \mathcal{E}_p(\rho) \leq \inf_{\rho \in \mathrm{Adm}\,\Gamma_2} \mathcal{E}_p(\rho) = \mathrm{Mod}_p \Gamma_2.$$

3. Let $\Gamma := \bigcup_{j=1}^{\infty} \Gamma_j$ and fix $\epsilon > 0$. For each j, choose $\rho_j \in \mathrm{Adm}(\Gamma_j)$ such that

$$\mathcal{E}_p(\rho_j) \leq \mathrm{Mod}_p \Gamma_j + \frac{\epsilon}{2^j}.$$

Define $\rho := \left(\sum_{j=1}^{\infty} \rho_j^p\right)^{\frac{1}{p}}$. For any $\gamma \in \Gamma$, there exists $k \in \mathbb{N}$ so that $\gamma \in \Gamma_k$. Since $\rho \geq \rho_k$ we have that $\ell_\rho(\gamma) \geq 1$. Hence, $\rho \in \mathrm{Adm}(\Gamma)$. Moreover,

$$\begin{aligned}\mathrm{Mod}_p \Gamma \leq \mathcal{E}_p(\rho) = \sum_{e \in E} \rho(e)^p = \sum_{e \in E} \sum_{j=1}^{\infty} \rho_j(e)^p = \sum_{j=1}^{\infty} \sum_{e \in E} \rho_j(e)^p \\ = \sum_{j=1}^{\infty} \mathcal{E}_p(\rho_j) \leq \epsilon + \sum_{j=1}^{\infty} \mathrm{Mod}_p \Gamma_j.\end{aligned}$$

We can interchange the order of summation without concern by Tonelli's theorem. Hence, taking ϵ to zero attains the desired result. □

Remark 1.2.2. The following is another useful basic property to add to monotonicity and countable subadditivity:

4. **(Subordination):** *With the hypothesis of Proposition 1.2.3, suppose that Γ and Γ' are families of objects on G, and suppose that for every object $\gamma \in \Gamma$ there is an object $\gamma' \in \Gamma'$ such that $\mathcal{N}(\gamma', e) \leq \mathcal{N}(\gamma, e)$, for all $e \in E$ (we say Γ is subordinated to Γ'). Then,*

$$\mathrm{Mod}_{p,\sigma}(\Gamma) \leq \mathrm{Mod}_{p,\sigma}(\Gamma'). \tag{1.2.3}$$

Proof. Assume $\rho \in \mathrm{Adm}(\Gamma')$, then for every $\gamma \in \Gamma$, there is $\gamma' \in \Gamma$ such that

$$\sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) \geq \sum_{e \in E} \mathcal{N}(\gamma', e)\rho(e) \geq 1.$$

Namely, ρ is admissible for Γ as well. Hence, $\text{Adm}(\Gamma') \subset \text{Adm}(\Gamma)$. $\square$

1.2.2 Connection to classical quantities

The concept of p-modulus generalizes known several classical ways of measuring the richness of a family of walks, see [2]. Let a and b be two nodes in V be given. We define the *connecting family* $\Gamma(a,b)$ to be the family of all simple paths in G that start at a and end at b. To this family, we assign the usage function $\mathcal{N}(\gamma, e)$ to be 1 when $e \in \gamma$ and 0 otherwise.

Theorem 1.2.4 (See[2])**.** *Let $G = (V, E, \sigma)$ be a graph with edge weights σ. Let Γ be a nontrivial family of objects on G with usage matrix $\mathcal{N}$ and let $\sigma(E) := \sum_{e\in E} \sigma(e)$. Then the function $p \mapsto \text{Mod}_{p,\sigma}(\Gamma)$ is continuous for $1 \leq p < \infty$, and the following two monotonicity properties hold for $1 \leq p \leq p' < \infty$.*

$$\mathcal{N}_{min}^{p} \operatorname{Mod}_{p,\sigma}(\Gamma) \geq \mathcal{N}_{min}^{p'} \operatorname{Mod}_{p',\sigma}(\Gamma), \tag{1.2.4}$$

$$\left(\sigma(E)^{-1} \operatorname{Mod}_{p,\sigma}(\Gamma)\right)^{1/p} \leq \left(\sigma(E)^{-1} \operatorname{Mod}_{p',\sigma}(\Gamma)\right)^{1/p'}. \tag{1.2.5}$$

Moreover, let $a \neq b$ in V be given and set $\Gamma = \Gamma(a,b)$. Then,

(i) For $\mathbf{p} = \infty$:

$$\lim_{p\to\infty} \operatorname{Mod}_p(\Gamma)^{\frac{1}{p}} = \operatorname{Mod}_\infty(\Gamma) = \frac{1}{\ell(\Gamma)}.$$

(ii) For $\mathbf{p} = 1$,

$$\operatorname{Mod}_1(\Gamma) = \min\{|\partial S| : S \text{ an } ab\text{-cut}\} = MC(a,b).$$

(iii) For $\mathbf{p} = 2$,

$$\operatorname{Mod}_2(\Gamma) = C_{\text{eff}}(a,b) = R_{\text{eff}}(a,b)^{-1}.$$

Remark 1.2.3. In other words, as p varies continuously from 1 to 2 and to ∞, the quantity $\text{Mod}_p(\Gamma(a,b))$ recovers the classical notions of min cut - MC(a,b) (see 3.1.3), effective conductance - $C_{\text{eff}}(a,b)$ (see 3.1.1) or the reciprocal of effective resistance - $R_{\text{eff}}(a,b)$, and shortest path.

Here we give a proof for 1.2.4 since a complete proof of it is not presented in [2].

Proof. First note that the extremal density ρ^* satisfies $0 \leq \rho^* \leq \mathcal{N}_{\min}^{-1}$, where $\mathcal{N}_{\min}$ is defined as

$$\mathcal{N}_{\min} := \min_{\gamma\in\Gamma} \min_{e:\mathcal{N}(\gamma,e)\neq 0} \mathcal{N}(\gamma,e). \tag{1.2.6}$$

The upper bound on ρ^* follows from the fact that each row of $\mathcal{N}$ contains at least one nonzero entry, which must be at least as large as $\mathcal{N}_{\min}$. In the special case when $\mathcal{N}$ is integer valued, the upper bound can be taken to be 1 and gives rise to an inequality of the form $0 \leq \rho^* \leq 1$. Howeve, when $\mathcal{N}$ is not restricted to integer values, the bound on ρ^* should be replaced by $0 \leq \rho^* \leq \mathcal{N}_{\min}^{-1}$.

We have that $0 \leq \rho^*\mathcal{N}_{\min} \leq 1$, and since $p' > p$, this imples that

$$\sum_{e\in E} \sigma(e)\rho^*(e)^p \mathcal{N}_{\min}^p \geq \sum_{e\in E} \sigma(e)\rho^*(e)^p \mathcal{N}_{\min}^p$$

which gives,

$$\mathcal{N}_{\min}^p \operatorname{Mod}_{p,\sigma}(\Gamma) \geq \mathcal{N}_{\min}^{p'} \operatorname{Mod}_{p',\sigma}(\Gamma)$$

□

Example 1.2.1 (Basic Example)**.** Let G be a graph consisting of k simple paths in parallel, each path taking ℓ hops to connect a given vertex s to a given vertex t. Let Γ be the family consisting of the k simple paths from s to t. Then $\ell(\Gamma) = \ell$ and the size of the minimum cut is k. A straightforward computation (see 1.2.6) shows that

$$\operatorname{Mod}_p(\Gamma) = \frac{k}{\ell^{p-1}} \quad \text{for } 1 \leq p < \infty, \qquad \operatorname{Mod}_\infty(\Gamma) = \frac{1}{\ell}.$$

Intuitively, when $p \approx 1$, $\mathrm{Mod}_p(\Gamma)$ is more sensitive to the number of parallel paths, while for $p \gg 1$, $\mathrm{Mod}_p(\Gamma)$ is more sensitive to short walks.

1.2.3 Beurling's Criterion and Extremal Densities

We begin this section by looking at the modulus problem as a convex problem (1.2.2). By borrowing techniques from convex analysis we derive conditions for the existence and uniqueness of an *extremal density*.

Lemma 1.2.5. *Given Γ and $1 \leq p \leq \infty$, there exists an extremal density ρ^*. If $1 < p < \infty$, then ρ^* is unique.*

Proof. For any $1 \leq p < \infty$, the function $\rho \to \mathcal{E}_p(\rho)^{1/p}$ is a norm on the m-dimensional space of edge densities, as is the function $\rho \to \mathcal{E}_\infty(\rho)$. Moreover, the set $A(\Gamma)$ is closed and convex. Thus, the modulus problem can be restated as the problem of finding a point in a closed convex subset of $\mathbb{R}^m$ that is closest to the origin in a particular norm. Such a point always exists, and is unique provided the norm is strictly convex, as it is in the cases $1 < p < \infty$. $\square$

As discussed earlier, since the modulus is defined as an infimum, it is easy to get upper-bounds on the value of the modulus. Typically, it is difficult to attain non-trivial (for instance the modulus is always non-negative) lower bounds. However, Arne Beurling's famous criteria for extremality is a very useful sufficient condition to test for extremality of a density.

Theorem 1.2.6. (Beurling's Extremality Criterion). *For a fixed $1 < p < \infty$, a density $\rho \in \mathrm{Adm}\,\Gamma$ is extremal if there exists $\widetilde{\Gamma} \subset \Gamma$ with $\ell_\rho(\gamma) = 1$ for all $\gamma \in \widetilde{\Gamma}$ such that*

$$\sum_{e\in E} h(e)\rho^{p-1}(e) \geq 0 \text{ whenever } h : E \to \mathbb{R} \text{ with } \ell_h(\gamma) \geq 0 \text{ for all } \gamma \in \widetilde{\Gamma}. \tag{1.2.7}$$

Proof. Let $\rho, \widetilde{\Gamma}$ be as in the hypothesis of (1.2.7) and let $\sigma \in \operatorname{Adm}\Gamma$. Define $h := \sigma - \rho$. Then $\ell_h(\gamma) \geq 0$ for all $\gamma \in \widetilde{\Gamma}$. By (1.2.7) we have

$$\sum_{e\in E} \sigma(e)\rho(e)^{p-1} - \sum_{e\in E} \rho(e)^p = \sum_{e\in E} h(e)\rho^{p-1}(e) \geq 0,$$

so that

$$0 \leq \sum_{e\in E} \rho(e)^p \leq \sum_{e\in E} \sigma(e)\rho(e)^{p-1} \leq \left(\sum_{e\in E} \sigma(e)^p\right)^{\frac{1}{p}} \left(\sum_{e\in E} \rho(e)^{(p-1)q}\right)^{\frac{1}{q}}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

The second inequality is an application of Hölder's Inequality. Since $q = \frac{p}{p-1}$, the above reads

$$\sum_{e\in E} \rho(e)^p \leq \left(\sum_{e\in E} \sigma(e)^p\right)^{\frac{1}{p}} \left(\sum_{e\in E} \rho(e)^p\right)^{1-\frac{1}{p}}.$$

Consequently,

$$\left(\sum_{e\in E} \rho(e)^p\right)^{\frac{1}{p}} \leq \left(\sum_{e\in E} \sigma(e)^p\right)^{\frac{1}{p}}.$$

Taking the pth power of both sides and recalling that σ is an arbitrary admissible density shows that ρ is indeed extremal. □

Definition 1.2.7. A subfamily $\widetilde{\Gamma}$ that satisfies the hypothesis in Beurling's Extremality Criterion is called a **Beurling subfamily**.

Theorem 1.2.8 (Converse of Beurling's Criterion.)**.** *If $\rho \in \operatorname{Adm}(\Gamma)$ is extremal, then*

$$\Gamma_0(\rho) := \{\gamma \in \Gamma : \ell_\rho(\gamma) = 1\}. \tag{1.2.8}$$

is a Buerling subfamily.

For a proof of this theorem see [5].

Theorem 1.2.9. *If $\widetilde{\Gamma}$ is a Beurling subfamily of Γ, then for each $1 < p < \infty$ it follows $\operatorname{Mod}_p \widetilde{\Gamma} = \operatorname{Mod}_p \Gamma$.*

Proof. Since $\widetilde{\Gamma}$ is a Beurling subfamily, letting $\rho_0 \in \operatorname{Adm}(\Gamma)$ be the extremal density for Γ, we see that $\widetilde{\Gamma} \subset \Gamma_0(\rho_0)$ and by hypothesis (1.2.7) holds for ρ_0 with $\widetilde{\Gamma}$.

By monotonicity, $\operatorname{Mod}_p(\widetilde{\Gamma}) \leq \operatorname{Mod}_p(\Gamma)$. On the other hand, let $\rho \in \operatorname{Adm}(\widetilde{\Gamma})$. Define $h : E \to \mathbb{R}$ by $h = \rho - \rho_0$. Then, for every $\gamma \in \widetilde{\Gamma}$ we have $\ell_h(\gamma) = \ell_\rho(\gamma) - \ell_{\rho_0}(\gamma) = 0$ and

$$\sum_{e \in E} h(e)\rho_0^{p-1}(e) \geq 0,$$

follows as a consequence of (1.2.7). Expanding out h, yields

$$\mathcal{E}_p(\rho_0) \leq \sum_{e \in E} \rho(e)\rho_0^{p-1}(e) \leq \left(\sum_{e \in E} \rho(e)^p\right)^{\frac{1}{p}} \left(\sum_{e \in E} \rho_0(e)^p\right)^{1-\frac{1}{p}}.$$

The latter inequality follows by applying Hölder. Dividing over we get,

$$\left(\sum_{e \in E} \rho_0(e)^p\right)^{\frac{1}{p}} \leq \left(\sum_{e \in E} \rho(e)^p\right)^{\frac{1}{p}} \Rightarrow \sum_{e \in E} \rho_0(e)^p \leq \sum_{e \in E} \rho(e)^p$$

which implies $\mathcal{E}(\rho) \geq \mathcal{E}(\rho_0) = \operatorname{Mod}(\Gamma)$. Minimizing over all $\rho \in \operatorname{Adm}(\widetilde{\Gamma})$ we have that $\operatorname{Mod}(\widetilde{\Gamma}) \geq \operatorname{Mod}(\Gamma)$. $\square$

Example 1.2.2. Consider the House graph G as in Figure 1.1 and the connecting family of walks $\Gamma = \Gamma_G(\{1\}, \{2\})$. If we restrict ourselves to the simple walks in Γ, denoted Γ_s, then it's straightforward to verify that Γ_s only contains the three paths $P_1 = (1, 2), P_2 = (1, 5, 2), P_3 = (1, 4, 3, 2)$. For clarity, if $e = \{x, y\}$, we write $h(e) = h(x, y)$. Letting Γ_s play the role of the Beurling subfamily, we can verify that ρ_0 defined as in the right half of Figure

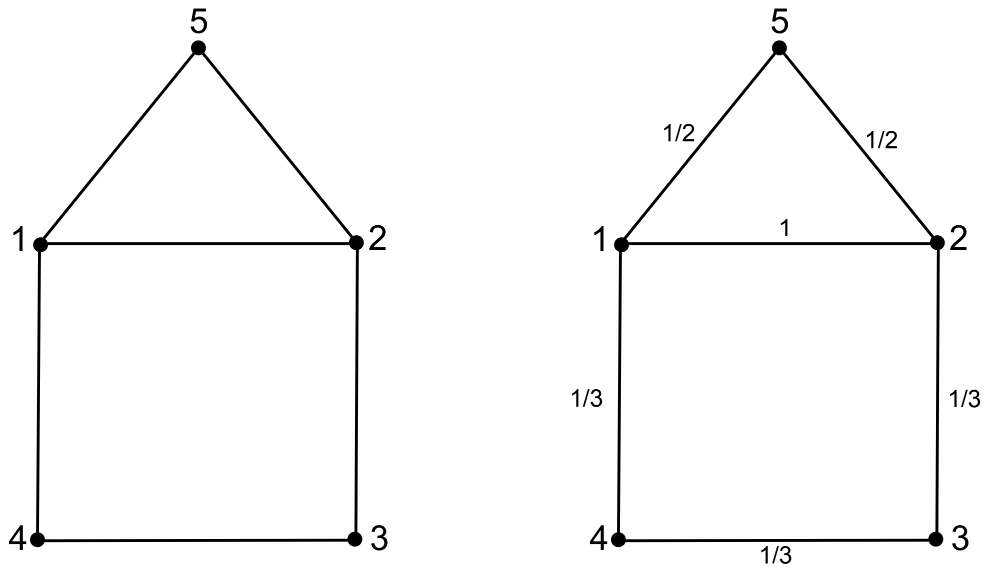

Figure 1.1: Left: House Graph. Right: House graph with the values assigned to each edge by the extremal density of the connecting family $\Gamma(\{1\},\{2\})$ for $1 < p < \infty$.

1.1 is extremal. Indeed, for $h : E \to \mathbb{R}$ suppose h satisfies

$$\ell_h(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma_s. \tag{1.2.9}$$

For each $k = 1, 2, 3$, let γ_k be the walk that traverses the vertices of P_k in the same order. Since the walks γ_1, γ_2, and γ_3 partition the edges of G and traverse each edge exactly once,

$$\sum_{e\in E} h(e)\rho_0^{p-1}(e) = \sum_{e\in\gamma_1} h(e)\rho_0^{p-1}(e) + \sum_{e\in\gamma_2} h(e)\rho_0^{p-1}(e) + \sum_{e\in\gamma_3} h(e)\rho_0^{p-1}(e).$$

Moreover, since ρ_0 is constant on each walk γ_j, the above can be reduced to

$$\begin{aligned}\sum_{e\in E} \rho_0^{p-1}(e)h(e) &= \sum_{e\in\gamma_1} h(e) + \left(\frac{1}{2}\right)^{p-1} \sum_{e\in\gamma_2} h(e) + \left(\frac{1}{3}\right)^{p-1} \sum_{e\in\gamma_3} h(e) \\ &= \ell_h(\gamma_1) + 2^{1-p}\, \ell_h(\gamma_2) + 3^{1-p}\, \ell_h(\gamma_3) \geq 0,\end{aligned}$$

due to (1.2.9). Therefore, ρ_0 and Γ_s satisfy the hypothesis of Theorem 1.2.6 for each $p > 1$.

Consequently, ρ_0 is the extremal metric for the p-modulus on the House graph G. It is a coincidence, caused by the lack of edges shared by any walks in Γ_s, that the same density ρ_0 is extremal for all p.

1.3 Lagrangian Duality and the Probabilistic Interpretation of Modulus

The optimization problem (1.2.2) is an ordinary convex program, in the sense of Rockafellar[10] Sec. 28. It involves minimizing the convex function $\mathcal{E}_p$ over the convex set $\mathrm{Adm}(\Gamma)$ defined by a set of linear inequalities. It was shown in [5] that this set of inequalities can be assumed finite.

Theorem 1.3.1 ([5])**.** *Let Γ be a given family of walks on a graph. There exists a finite subfamily $\Gamma^* \subset \Gamma$ such that $\mathrm{Adm}(\Gamma^*) = \mathrm{Adm}(\Gamma)$.*

Such a finite subfamily is called an *essential subfamily* for the modulus problem. Ensuring that $\ell_\rho(\gamma) \geq 1$ is true for all walks in the essential subfamily is equivalent to ensuring the inequality for the entire family. Thus, even when a described set of walks (e.g., the connecting family $\Gamma(a, b)$) contains infinitely many members, it can be always assumed that Γ has been replaced by a finite essential subfamily (e.g., the simple paths from a to b when $\Gamma = \Gamma(a, b)$. See section 7 of [5].

Existence of a minimizer follows from compactness, and uniqueness holds when $1 < p < \infty$ by strict convexity of the objective function. Furthermore, it can be shown that strong duality holds in the sense that a maximizer of the Lagrangian dual problem exists and has dual energy equal to the modulus. The Lagrangian dual problem was derived in detail in [2]. The Lagrangian dual was later reinterpreted in a probabilistic setting in [9].

In order to formulate the probabilistic dual, we let $\mathcal{P}(\Gamma)$ represent the set of probability mass functions (pmfs) on the set Γ. In other words, $\mathcal{P}(\Gamma)$ contains the set of vectors $\mu \in \mathbb{R}^{\Gamma}_{\geq 0}$

with the property that $\mu^T\mathbf{1} = 1$. Given such a μ, we can define a Γ-valued random variable $\underline{\gamma}$ with distribution given by μ: $\mathbb{P}_\mu\left(\underline{\gamma} = \gamma\right) = \mu(\gamma)$. Given an edge $e \in E$, the value $\mathcal{N}(\underline{\gamma}, e)$ is again a random variable, and we represent its expectation (depending on the pmf μ) as

$$\mathbb{E}_\mu\left[\mathcal{N}(\underline{\gamma}, e)\right] := \sum_{\gamma\in\Gamma} \mu(\gamma)\mathcal{N}(\gamma, e) = (\mathcal{N}^T\mu)(e).$$

The probabilistic interpretation of the Lagrangian dual can now be stated as follows.

Theorem 1.3.2. *Let $G = (V, E, \sigma)$ be a finite graph with edge weights σ, and let Γ be a non-trivial finite family of objects on G with usage matrix $\mathcal{N}$. Then, for any $1 < p < \infty$, letting $q := p/(p-1)$ be the conjugate exponent to p, we have*

$$\mathrm{Mod}_{p,\sigma}(\Gamma)^{-\frac{1}{p}} = \left(\min_{\mu\in\mathcal{P}(\Gamma)} \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \mathbb{E}_\mu\left[\mathcal{N}(\underline{\gamma}, e)\right]^q\right)^{\frac{1}{q}}. \tag{1.3.1}$$

Moreover, any optimal measure μ^*, must satisfy

$$\mathbb{E}_{\mu^*}\left[\mathcal{N}(\underline{\gamma}, e)\right] = \frac{\sigma(e)\rho^*(e)^{\frac{p}{q}}}{\mathrm{Mod}_{p,\sigma}(\Gamma)} \qquad \forall e \in E,$$

where ρ^* is the unique extremal density for $\mathrm{Mod}_{p,\sigma}(\Gamma)$.

Theorem 1.3.2 is a consequence of the theory developed by Albin and Poggi-Corradini[9]. However, since it was only remarked on in [9], we provide a detailed proof here.

Proof. The optimization problem (1.2.2) is a standard convex optimization problem. Its Lagrangian dual problem, derived in [2], is

$$\begin{aligned} \text{maximize} \quad & \sum_{\gamma\in\Gamma} \lambda(\gamma) - (p-1)\sum_{e\in E} \sigma(e)\left(\frac{1}{p\sigma(e)}\sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)\lambda(\gamma)\right)^{\frac{p}{p-1}} \\ \text{subject to} \quad & \lambda(\gamma) \geq 0 \quad \forall \gamma \in \Gamma. \end{aligned} \tag{1.3.2}$$

It can be readily verified that strong duality holds (i.e., that the minimum in (1.2.2) equals the maximum in (1.3.2)) and that both extrema are attained. Moreover, if ρ^* is the unique minimizer of the modulus problem and λ^* is any maximizer of the Lagrangian dual, then the optimality conditions imply that

$$\rho^*(e) = \left(\frac{1}{p\sigma(e)} \sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)\lambda^*(\gamma) \right)^{\frac{1}{p-1}}. \tag{1.3.3}$$

By decomposing $\lambda \in \mathbb{R}^{\Gamma}_{\geq 0}$ as $\lambda = \nu\mu$ with $\nu \geq 0$ and $\mu \in \mathcal{P}(\Gamma)$, we can rewrite (1.3.2) as

$$\max_{\nu\geq 0} \left\{ \nu - (p-1) \left(\frac{\nu}{p} \right)^q \min_{\mu\in\mathcal{P}(\Gamma)} \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \left(\sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)\mu(\gamma) \right)^q \right\}.$$

The minimum over μ can be recognized as the minimum in (1.3.1). Let α be its minimum value. Then the maximum over $\nu \geq 0$ is attained at $\nu^* := p\alpha^{-\frac{p}{q}}$, and strong duality implies that

$$\text{Mod}_{p,\sigma}(\Gamma) = \nu^* - (p-1) \left(\frac{\nu^*}{p} \right)^q \alpha = \alpha^{-\frac{p}{q}}.$$

Thus,

$$\min_{\mu\in\mathcal{P}(\Gamma)} \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \mathbb{E}_{\mu} \left[\mathcal{N}(\underline{\gamma}, e) \right]^q = \alpha = \text{Mod}_{p,\sigma}(\Gamma)^{-\frac{q}{p}},$$

proving (1.3.1). The remainder of the theorem follows from (1.3.3):

$$\rho^*(e) = \left(\frac{\nu^*}{p\sigma(e)} \sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)\mu^*(\gamma) \right)^{\frac{1}{p-1}} = \alpha^{-1}\sigma(e)^{-\frac{q}{p}} \mathbb{E}_{\mu^*} \left[\mathcal{N}(\underline{\gamma}, e) \right]^{\frac{q}{p}}$$

□

Remark 1.3.1. The probabilistic interpretation is particularly informative when $p = 2$, $\sigma \equiv 1$, and Γ is a collection of subsets of E, so that $\mathcal{N}$ is a $(0,1)$-matrix defined as $\mathcal{N}(\gamma, e) =$

$\mathbb{1}_\gamma(e)$. In this case, this duality relation can be expressed as

$$\mathrm{Mod}_2(\Gamma)^{-1} = \min_{\mu \in \mathcal{P}(\Gamma)} \mathbb{E}_\mu \left| \underline{\gamma} \cap \underline{\gamma}' \right|,$$

where $\underline{\gamma}$ and $\underline{\gamma}'$ are two independent random variables chosen according to the pmf μ, and $\left| \underline{\gamma} \cap \underline{\gamma}' \right|$ is their *overlap* (also a random variable). In other words, computing the 2-modulus in this setting is equivalent to finding a pmf that minimizes the expected overlap of two iid Γ-valued random variables.

Proof. To see the above result, first recall that $\mathbb{E}_\mu(\mathcal{N}(\underline{\gamma}, e)) = \sum_\gamma \mu(\gamma)\mathcal{N}(\gamma, e)$. Also,

$$\begin{aligned}
\sum_{e \in E} \mathbb{E}_\mu(\mathcal{N}(\underline{\gamma}, e))^2 &= \sum_{e \in E} \sum_{\gamma, \gamma'} \mu(\gamma)\mu(\gamma')\mathcal{N}(\gamma, e)\mathcal{N}(\gamma', e) \\
&= \sum_{\gamma, \gamma'} \mu(\gamma)\mu(\gamma') \sum_{e \in E} \mathcal{N}(\gamma, e)\mathcal{N}(\gamma', e) \\
&= \sum_{\gamma, \gamma'} \mu(\gamma)\mu(\gamma') \left| \gamma \cap \gamma' \right| \\
&= \mathbb{E} \left| \gamma \cap \gamma' \right|
\end{aligned}$$

□

In the present work, we are interested in a different but closely related duality called *blocking duality* which will be discussed in the next chapter.

Chapter 2

Fulkerson duality

In this chapter, we introduce blocking duality for modulus. In order to do so, we shall focus on the set of inequalities defining $\mathrm{Adm}(\Gamma)$.

2.1 Fulkerson's theorem

First, we recall some general definition. Let $\mathcal{K}$ be the set of all closed convex sets $K \subset \mathbb{R}^E_{\geq 0}$ that are *recessive*, in the sense that $K + \mathbb{R}^E_{\geq 0} = K$. To avoid trivial cases, we shall assume that $\varnothing \subsetneq K \subsetneq \mathbb{R}^E_{\geq 0}$, for $K \in \mathcal{K}$.

Definition 2.1.1. For each $K \in \mathcal{K}$ there is an associated *blocking polyhedron*, or *blocker*,

$$\mathrm{BL}(K) := \left\{ \eta \in \mathbb{R}^E_{\geq 0} : \eta^T \rho \geq 1, \ \ \forall \rho \in K \right\}.$$

Observe that every element η in the blocker would induce an admissible ρ. We can do this by assigning $\rho = 1$ for the edges in the blocker and 0 otherwise.

Definition 2.1.2. Given $K \in \mathcal{K}$ and a point $x \in K$ we say that x is an *extreme point* of K if $x = tx_1 + (1-t)x_2$ for some $x_1, x_2 \in K$ and some $t \in (0, 1)$, implies that $x_1 = x_2 = x$. Moreover, we let $\mathrm{ext}(K)$ be the set of all extreme points of K .

Definition 2.1.3. A set C is convex if the line segment between any two points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have $\theta x_1 + (1-\theta)x_2 \in C$.

We call a point of the form $\theta_1 x_1 + \cdots + \theta k x_k$, where $\theta_1 + \cdots + \theta_k = 1$ and $\theta_i \geq 0, i = 1, \ldots, k$, a convex combination of the points $x_1, \ldots, x_k$.

The convex hull of a set C, denoted co(C), is the set of all convex combinations of points in C:

$$co(C) = \{\theta_1 x_1 + \cdots + \theta_k x_k | x_i \in C, \theta_i \geq 0, i = 1, \ldots, k, \theta_1 + \cdots + \theta_k = 1\}.$$

Definition 2.1.4. The *dominant* of a set $P \subset \mathbb{R}^E_{\geq 0}$ is the recessive closed convex set

$$\mathrm{Dom}(P) = \mathrm{co}(P) + \mathbb{R}^E_{\geq 0}.$$

If Γ is a finite non-trivial family of objects on a graph G, the admissible set $\mathrm{Adm}(\Gamma)$ is determined by finitely many inequalities, in particular $\mathrm{Adm}(\Gamma)$ has finitely many faces. However, $\mathrm{Adm}(\Gamma)$ is also determined by its finitely many extreme points, or "vertices". In fact, it is well-known that $\mathrm{Adm}(\Gamma)$ is the dominant of its extreme points $\mathrm{ext}(\mathrm{Adm}(\Gamma))$, see Theorem 18.5 in [10].

Definition 2.1.5. Suppose $G = (V, E)$ is a finite graph and Γ is a finite non-trivial family of objects on G. We say that the family

$$\hat{\Gamma} := \mathrm{ext}(\mathrm{Adm}(\Gamma)) = \{\hat{\gamma}_1, \ldots, \hat{\gamma}_s\} \subset \mathbb{R}^E_{\geq 0},$$

consisting of the extreme points of $\mathrm{Adm}(\Gamma)$, is the *Fulkerson blocker* of Γ. Also, we define the matrix $\hat{\mathcal{N}} \in \mathbb{R}^{\hat{\Gamma} \times E}_{\geq 0}$ to be the matrix whose rows are the vectors $\hat{\gamma}^T$, for $\hat{\gamma} \in \hat{\Gamma}$.

Theorem 2.1.6 (Fulkerson[11])**.** *Let $G = (V, E)$ be a graph and let Γ be a non-trivial finite family of objects on G. We identify Γ with its edge-usage functions, hence we think of Γ as a finite subset of $\mathbb{R}^E_{\geq 0}$. Let $\hat{\Gamma}$ be the Fulkerson blocker of Γ. Then*

(1) $\mathrm{Adm}(\Gamma) = \mathrm{Dom}(\hat{\Gamma}) = \mathrm{BL}(\mathrm{Adm}(\hat{\Gamma}))$*;*

(2) $\mathrm{Adm}(\hat{\Gamma}) = \mathrm{Dom}(\Gamma) = \mathrm{BL}(\mathrm{Adm}(\Gamma))$*;*

(3) $\hat{\hat{\Gamma}} \subset \Gamma$.

In words, (3) says that the extreme points of $\mathrm{Adm}(\hat{\Gamma})$ are a subset of Γ. Combining (1) and (2) we get the following.

Corollary 2.1.7. *Let $G = (V, E)$ be a graph and let Γ be a nontrivial finite family of objects on G. Identify Γ with the subset of $\mathbb{R}^E_{\geq 0}$ consisting of all the edge-usage functions for objects in Γ. Then,*

$$\mathrm{BL}(\mathrm{BL}(\mathrm{Adm}(\Gamma))) = \mathrm{Adm}(\Gamma) \qquad \textit{and} \qquad \mathrm{BL}(\mathrm{BL}(\mathrm{Dom}(\Gamma))) = \mathrm{Dom}(\Gamma).$$

as well as

$$\mathrm{Adm}(\Gamma) = \mathrm{BL}\left(\mathrm{Dom}(\Gamma)\right) \qquad \textit{and} \qquad \mathrm{BL}(\mathrm{Adm}(\Gamma)) = \mathrm{Dom}(\Gamma).$$

We include a proof of Theorem 2.1.6 for the reader's convenience.

Proof. We first prove (2). Suppose $\eta \in \mathrm{BL}(\mathrm{Adm}(\Gamma))$. Then $\eta^T \rho \geq 1$, for every $\rho \in \mathrm{Adm}(\Gamma)$. In particular, since every row of $\hat{\mathcal{N}}$ is an extreme point of $\mathrm{Adm}(\Gamma)$, we have

$$\hat{\mathcal{N}}\eta \geq 1. \tag{2.1.1}$$

In other words, $\eta \in \mathrm{Adm}(\hat{\Gamma})$. Conversely, suppose $\eta \in \mathrm{Adm}(\hat{\Gamma})$, that is (2.1.1) holds. Since

$$\mathrm{Adm}(\Gamma) = \mathrm{co}(\hat{\Gamma}) + \mathbb{R}^E_{\geq 0},$$

for every $\rho \in \mathrm{Adm}(\Gamma)$, there is a probability measure $\nu \in \mathcal{P}(\hat{\Gamma})$ and a vector $z \geq 0$ such that

$$\rho = \hat{\mathcal{N}}^T \nu + z$$

And by (2.1.1),

$$\eta^T \rho = \eta^T \hat{\mathcal{N}}^T \nu + \eta^T z \geq \nu^T 1 + \eta^T z \geq 1.$$

So $\eta \in \mathrm{BL}(\mathrm{Adm}(\Gamma))$.

Note that $\eta \in \mathrm{BL}(\mathrm{Adm}(\Gamma))$ if and only if the value of the following linear program is greater or equal 1.

$$\begin{aligned} \text{minimize} \qquad & \eta^T \rho \\ \text{subject to} \qquad & \mathcal{N}\rho \geq \mathbf{1},\ \rho \geq 0, \end{aligned} \tag{2.1.2}$$

where $\mathcal{N}$ is the usage matrix for Γ. The Lagrangian for this problem is

$$\mathcal{L}(\rho, \lambda, t) := \eta^T \rho + \lambda^T (\mathbf{1} - \mathcal{N}\rho) - t^T \rho = \lambda^T \mathbf{1} + \rho^T (\eta - \mathcal{N}^T \lambda - t),$$

with $\rho \in \mathbb{R}^E$, $\lambda \in \mathbb{R}^\Gamma_{\geq 0}$ and $t \in \mathbb{R}^E_{\geq 0}$. In particular, the dual problem is

$$\begin{aligned} \text{maximize} \qquad & \lambda^T \mathbf{1} \\ \text{subject to} \qquad & \mathcal{N}^T \lambda \leq \eta,\ \lambda \geq 0. \end{aligned} \tag{2.1.3}$$

Splitting $\lambda = s\nu$, with $s \geq 0$ and $\nu \in \mathcal{P}(\Gamma)$, we can rewrite this problem as

$$\begin{aligned} \text{maximize} \qquad & s \\ \text{subject to} \qquad & s\mathcal{N}^T \nu \leq \eta,\ \nu \in \mathcal{P}(\Gamma). \end{aligned} \tag{2.1.4}$$

By strong duality, $\eta \in \mathrm{BL}(\mathrm{Adm}(\Gamma))$ if and only if there is $s \geq 1$ and $\nu \in \mathcal{P}(\Gamma)$ so that

$$\eta \geq s\mathcal{N}^T\nu.$$

Namely, $\eta \in \mathrm{BL}(\mathrm{Adm}(\Gamma))$ implies that $\eta \geq \mathcal{N}^T\nu$, so $\eta \in \mathrm{Dom}(\Gamma)$.

Conversely, if $\eta \in \mathrm{Dom}(\Gamma)$, then there is a $\nu \in \mathcal{P}(\Gamma)$ such that $\eta \geq \mathcal{N}^T\nu$. So we have proved (2). In particular, since $\hat{\hat{\Gamma}}$ is the set of extreme points of $\mathrm{Adm}(\hat{\Gamma})$ by Definition **??**, it follows from (2) that

$$\hat{\hat{\Gamma}} = \mathrm{ext}(\mathrm{Adm}(\hat{\Gamma})) = \mathrm{ext}(\mathrm{Dom}(\Gamma)).$$

Since any extreme point of $\mathrm{Dom}(\Gamma)$ must be present in Γ, we conclude that $\hat{\hat{\Gamma}} \subset \Gamma$, and hence (3) is proved as well.

To prove (1), we apply (2) to $\hat{\Gamma}$ and find that

$$\mathrm{BL}(\mathrm{Adm}(\hat{\Gamma})) = \mathrm{Adm}(\hat{\hat{\Gamma}}) \supset \mathrm{Adm}(\Gamma),$$

where the last inclusion follows from (3), since $\hat{\hat{\Gamma}} \subset \Gamma$. Also, by (3) applied to $\hat{\Gamma}$, the extreme points of $\mathrm{Adm}(\hat{\hat{\Gamma}})$ are a subset of $\hat{\Gamma}$ and therefore they are a subset of $\mathrm{ext}(\mathrm{Adm}(\Gamma))$. This implies that $\mathrm{Adm}(\hat{\hat{\Gamma}}) \subset \mathrm{Adm}(\Gamma)$. So we have $\mathrm{BL}(\mathrm{Adm}(\hat{\Gamma})) = \mathrm{Adm}(\Gamma)$.

Moreover, by (2) applied to $\hat{\Gamma}$, we get that

$$\mathrm{BL}(\mathrm{Adm}(\hat{\Gamma})) = \mathrm{Dom}(\hat{\Gamma}).$$

So (1) is proved as well. □

2.1.1 Blocking duality for p-modulus

Theorem 2.1.8. *Let $G = (V, E, \sigma)$ be a graph and let Γ be a nontrivial finite family of objects on G with Fulkerson blocker $\hat{\Gamma}$. Let the exponent $1 < p < \infty$ be given, with $q :=$*

$p/(p-1)$ its conjugate exponent. For any set of weights $\sigma \in \mathbb{R}^E_{>0}$ define the dual set of weights $\hat{\sigma}$ as $\hat{\sigma}(e) := \sigma(e)^{1-q}$, for all $e \in E$.

Then

$$\operatorname{Mod}_{p,\sigma}(\Gamma)^{\frac{1}{p}} \operatorname{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{\frac{1}{q}} = 1. \tag{2.1.5}$$

Moreover, the optimal densities $\rho^* \in \operatorname{Adm}(\Gamma)$ and $\eta^* \in \operatorname{Adm}(\hat{\Gamma})$ respectively for $\operatorname{Mod}_{p,\sigma}(\Gamma)$ and $\operatorname{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})$ are related as follows:

$$\eta^*(e) = \frac{\sigma(e)\rho^*(e)^{p-1}}{\operatorname{Mod}_{p,\sigma}(\Gamma)} \qquad \forall e \in E. \tag{2.1.6}$$

Remark 2.1.1. The case for $p = 2$ is essentially contained in Lemma 2 [12], although not stated in terms of modulus and with a different proof. However, it is worth stating it separetely. Namely,

$$\operatorname{Mod}_{2,\sigma}(\Gamma) \operatorname{Mod}_{2,\sigma^{-1}}(\hat{\Gamma}) = 1.$$

And

$$\sigma(e)\rho^*(e) = \operatorname{Mod}_{2,\sigma}(\Gamma)\eta^*(e) \qquad \forall e \in E,$$

in this case.

Proof. For all $\rho \in \operatorname{Adm}(\Gamma)$ and $\eta \in \operatorname{Adm}(\hat{\Gamma})$, Hölder's inequality implies that

$$\begin{aligned} 1 \leq \sum_{e\in E} \rho(e)\eta(e) &= \sum_{e\in E} \left(\sigma(e)^{1/p}\rho(e)\right)\left(\sigma(e)^{-1/p}\eta(e)\right) \\ &\leq \left(\sum_{e\in E} \sigma(e)\rho(e)^p\right)^{1/p} \left(\sum_{e\in E} \hat{\sigma}(e)\eta(e)^q\right)^{1/q}, \end{aligned} \tag{2.1.7}$$

so

$$\operatorname{Mod}_{p,\sigma}(\Gamma)^{1/p} \operatorname{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{1/q} \geq 1. \tag{2.1.8}$$

Now, let $\alpha := \operatorname{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{-1}$ and let $\eta^* \in \operatorname{Adm}(\hat{\Gamma})$ be the minimizer for $\operatorname{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})$.

Then (2.1.8) implies that

$$\mathrm{Mod}_{p,\sigma}(\Gamma) \geq \alpha^{\frac{p}{q}} = \alpha^{\frac{1}{q-1}}. \tag{2.1.9}$$

Define

$$\rho^*(e) := \alpha \left(\frac{\hat{\sigma}(e)}{\sigma(e)} \eta^*(e)^q \right)^{1/p} = \alpha \hat{\sigma}(e) \eta^*(e)^{q/p}. \tag{2.1.10}$$

Note that

$$\mathcal{E}_{p,\sigma}(\rho^*) = \sum_{e \in E} \sigma(e) \rho^*(e)^p = \alpha^p \sum_{e \in E} \hat{\sigma}(e) \eta^*(e)^q = \alpha^{p-1} = \alpha^{\frac{1}{q-1}}.$$

Thus, if we can show that $\rho^* \in \mathrm{Adm}(\Gamma)$, then (2.1.9) is attained and ρ^* must be extremal for $\mathrm{Mod}_{p,\sigma}(\Gamma)$. In particular, (2.1.5) would follow. Moreover, (2.1.6) is another way of writing (2.1.10).

To see that $\rho^* \in \mathrm{Adm}(\Gamma)$, we will verify that $\sum_{e \in E} \rho^*(e)\eta(e) \geq 1$ for all $\eta \in \mathrm{Adm}(\hat{\Gamma})$. First, consider $\eta = \eta^*$. In this case

$$\sum_{e \in E} \rho^*(e) \eta^*(e) = \alpha \sum_{e \in E} \hat{\sigma}(e) \eta^*(e)^q = 1.$$

Now let $\eta \in \mathrm{Adm}(\hat{\Gamma})$ be arbitrary. Since $\mathrm{Adm}(\hat{\Gamma})$ is convex, we have that $(1-\theta)\eta^* + \theta\eta \in \mathrm{Adm}(\hat{\Gamma})$ for all $\theta \in [0,1]$. So, using Taylor's theorem, we have

$$\begin{aligned}
\alpha^{-1} = \mathcal{E}_{q,\hat{\sigma}}(\eta^*) \leq \mathcal{E}_{q,\hat{\sigma}}((1-\theta)\eta^* + \theta\eta) &= \sum_{e \in E} \hat{\sigma}(e) \left[(1-\theta)\eta^*(e) + \theta\eta(e) \right]^q \\
&= \alpha^{-1} + q\theta \sum_{e \in E} \hat{\sigma}(e) \eta^*(e)^{q-1} \big(\eta(e) - \eta^*(e) \big) + O(\theta^2) \\
&= \alpha^{-1} + \alpha^{-1} q\theta \sum_{e \in E} \rho^*(e) \big(\eta(e) - \eta^*(e) \big) + O(\theta^2).
\end{aligned}$$

Since this inequality must hold for arbitrarily small $\theta > 0$, it follows that

$$\sum_{e \in E} \rho^*(e)\eta(e) \geq \sum_{e \in E} \rho^*(e)\eta^*(e) = 1,$$

and the proof is complete. ☐

2.1.2 The cases $p = 1$ and $p = \infty$

Now we turn our attention to establishing the duality relationship in the cases $p = 1$ and $p = \infty$. Recall that by Theorem 1.2.4,

$$\lim_{p\to\infty} \mathrm{Mod}_{p,\sigma}(\Gamma)^{\frac{1}{p}} = \mathrm{Mod}_{\infty,1}(\Gamma) = \frac{1}{\ell(\Gamma)},$$

where $\ell(\Gamma)$ is defined to be the smallest element of the vector $\mathcal{N}\mathbf{1}$.

In order to pass to the limit in (2.1.5), we need to establish the limits for the second term in the left-hand side product.

Lemma 2.1.9. *Under the assumptions of Theorem 2.1.8,*

$$\begin{aligned} &\lim_{q\to 1} \mathrm{Mod}_{q,\hat\sigma}(\hat\Gamma)^{\frac{1}{q}} = \mathrm{Mod}_{1,1}(\hat\Gamma) \quad \textit{and} \\ &\lim_{q\to \infty} \mathrm{Mod}_{q,\hat\sigma}(\hat\Gamma)^{\frac{1}{q}} = \mathrm{Mod}_{\infty,\sigma^{-1}}(\hat\Gamma), \end{aligned} \tag{2.1.11}$$

where $\sigma^{-1}(e) = \sigma(e)^{-1}$.

Proof. Let $\mathcal{N}, \hat{\mathcal{N}} \in \mathbb{R}^{\Gamma\times E}_{\geq 0}$ be the usage matrices for Γ and $\hat\Gamma$ respectively. Let $\sigma \in \mathbb{R}^{\mathbf{E}\times\mathbf{E}}$ be the diagonal matrix with entries $\sigma(\mathbf{e},\mathbf{e}) = \sigma(\mathbf{e})$, and define $\tilde{\mathcal{N}} = \hat{\mathcal{N}}\sigma\sigma$, with $\tilde\Gamma$ its associated family in $\mathbb{R}^E_{\geq 0}$. Note that $\eta \in \mathrm{Adm}(\hat\Gamma)$ if and only if $\sigma^{-1}\eta \in \mathrm{Adm}(\tilde\Gamma)$. Moreover, for every $\eta \in \mathrm{Adm}(\hat\Gamma)$,

$$\mathcal{E}_{q,\hat\sigma}(\eta) = \sum_{e\in E} \hat\sigma(e)\eta(e)^q = \sum_{e\in E} \sigma(e)\left(\frac{\eta(e)}{\sigma(e)}\right)^q = \mathcal{E}_{q,\sigma}(\sigma^{-1}\eta),$$

which implies that

$$\mathrm{Mod}_{q,\hat\sigma}(\hat\Gamma) = \mathrm{Mod}_{q,\sigma}(\tilde\Gamma).$$

Taking the limit as $q \to 1$ and using Theorem 1.2.4 shows that

$$\lim_{q\to 1} \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{\frac{1}{q}} = \mathrm{Mod}_{1,\sigma}(\tilde{\Gamma}) = \min_{\eta\in\mathrm{Adm}(\hat{\Gamma})} \sum_{e\in E} \sigma(e)\left(\frac{\eta(e)}{\sigma(e)}\right) = \mathrm{Mod}_{1,1}(\hat{\Gamma}).$$

Taking the limit as $q \to \infty$ and using Theorem 1.2.4 shows that

$$\lim_{q\to\infty} \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{\frac{1}{q}} = \mathrm{Mod}_{\infty,1}(\tilde{\Gamma}) = \min_{\eta\in\mathrm{Adm}(\hat{\Gamma})} \max_{e\in E} \left(\frac{\eta(e)}{\sigma(e)}\right) = \mathrm{Mod}_{\infty,\sigma^{-1}}(\hat{\Gamma}).$$

□

Taking the limit as $p \to 1$ in Theorem 2.1.8 then gives the following theorem.

Theorem 2.1.10. *Under the assumptions of Theorem 2.1.8,*

$$\mathrm{Mod}_{1,\sigma}(\Gamma)\,\mathrm{Mod}_{\infty,\sigma^{-1}}(\hat{\Gamma}) = 1. \tag{2.1.12}$$

Note that taking the limit as $p \to \infty$ simply yields the same result for the unweighted case.

2.2 Blocking Duality for Families of Objects

We begin this section with the introduction of max flow min cut theorem which will be used throughout the rest of this thesis.

2.2.1 Max-Flow Min-Cut

Modulus is also closely related to the Max-Flow Min-Cut Theorem [13]. There are a number of different ways to see this; we shall focus on the connection to Ford and Fulkerson's original work. Let $G = (V, E, \sigma)$ be a weighted, undirected graph and let $s, t \in V$ be distinct vertices. Let Γ be the family of simple paths from s to t. ($\Gamma \subset \Gamma(s, t)$ is an essential subfamily in the sense of Theorem 1.3.1.) In the terminology of Ford and Fulkerson, the positive edge function σ is called the *capacity* of the edge. A *flow* can be thought of as a function $\lambda : \Gamma \to [0, \infty)$ with the property that

$$\sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)\lambda(\gamma) \leq \sigma(e) \qquad \forall e \in E, \tag{2.2.1}$$

and the *value* of such a flow the sum of $\lambda(\gamma)$ over all $\gamma \in \Gamma$. Γ here is the family of paths between two distinct points on the graph.

A *disconnecting set* is a subset of edges $D \subseteq E$ with the property that each $\gamma \in \Gamma$ traverses at least one edge in D, and a *(minimal) cut* is a disconnecting set that contains no other disconnecting sets as proper subsets. The *value* of a disconnecting set D, denoted $v(D)$, is defined as the sum of the capacities of all edges in D. Thus a disconnecting set of minimal value is automatically a cut (we call this the min cut). The Max-Flow Min-Cut Theorem (originally known as the minimal cut theorem) is stated as follows.

Theorem 2.2.1 (Max-Flow Min-Cut)**.** *The values of the maximal flow and minimal cut on any graph are equal.*

2.2.2 Duality for 1-modulus

Suppose that $G = (V, E, \sigma)$ is a weighted graph, with weights $\sigma \in \mathbb{R}^E_{\geq 0}$, and Γ is a non-trivial, finite family of subsets of E, where $\mathcal{N}$ be the corresponding usage matrix. In this case we can equate each $\gamma \in \Gamma$ with the vector $\mathbb{1}_\gamma \in \mathbb{R}^E_{\geq 0}$, so we think of Γ as living in $\{0, 1\}^E \subset \mathbb{R}^E_{\geq 0}$. Recall that $\mathrm{Mod}_{1,\sigma}(\Gamma)$ is the value of the linear program:

$$\begin{aligned} &\text{minimize} && \sigma^T \rho \\ &\text{subject to} && \rho \geq 0, \quad \mathcal{N}\rho \geq \mathbf{1} \end{aligned} \tag{2.2.2}$$

Since it's a linear program, strong duality holds, and the dual problem is

$$\begin{aligned} \text{maximize} \quad & \lambda^T \mathbf{1} \\ \text{subject to} \quad & \lambda \geq 0, \quad \mathcal{N}^T \lambda \leq \sigma. \end{aligned} \tag{2.2.3}$$

We think of (2.2.3) as a (generalized) max-flow problem, given the weights σ. That's because the condition $\mathcal{N}^T \lambda \leq \sigma$ says that for every $e \in E$

$$\sum_{\substack{\gamma \in \Gamma \\ e \in \gamma}} \lambda(\gamma) \leq \sigma(e).$$

However, to think of (2.2.2) as a (generalized) min-cut problem, we would need to be able to restrict the densities ρ to some given subsets of E. That's exactly what the Fulkerson blocker does.

By (generalised) max-flow/ min-cut problem we mean the problems discussed in section 2.2.1 in terms of any family Γ (does not have to be the family of connecting paths necessarily).

Proposition 2.2.2. *Suppose $G = (V, E, \sigma)$ is a finite graph and Γ is a family of subsets of E with Fulkerson blocker family $\hat{\Gamma}$. Then for any set of weights $\sigma \in \mathbb{R}^E_{>0}$,*

$$\mathrm{Mod}_{1,\sigma}(\Gamma) = \min_{\hat{\gamma} \in \hat{\Gamma}} \sum_{e \in E} \hat{\mathcal{N}}(\hat{\gamma}, e) \sigma(e). \tag{2.2.4}$$

Moreover, for every $\hat{\gamma} \in \hat{\Gamma}$ there is a choice of $\sigma \in \mathbb{R}^E_{\geq 0}$ such that $\hat{\gamma}$ is the unique solution of (2.2.4).

Proof. By Theorem 2.1.6(1)

$$\mathrm{Adm}(\Gamma) = \mathrm{Dom}(\hat{\Gamma})$$

So if $\sigma \in \mathbb{R}^E_{>0}$ is a given set of weights, then, by (2.2.2), $\mathrm{Mod}_{1,\sigma}(\Gamma)$ is the value of the linear program

$$
\begin{aligned}
\text{minimize} \qquad & \sigma^T \rho \\
\text{subject to} \qquad & \rho \in \mathrm{Dom}(\hat{\Gamma}).
\end{aligned}
\tag{2.2.5}
$$

In particular, the optimal value is attained at a vertex of $\mathrm{Dom}(\hat{\Gamma})$, namely for an object $\hat{\gamma} \in \hat{\Gamma}$. Therefore, the optimization can be restricted to $\hat{\Gamma}$. $\square$

2.2.3 Connecting families

Let G be an undirected graph and let $\Gamma = \Gamma(a, b)$ be the family of all simple paths connecting two distinct nodes a and b, i.e., the ab-paths in G. Consider the family $\Gamma_{\text{cut}}(a, b)$ of all minimal ab-cuts. See 3.1.3 for the definition of min-cut.

Note that (2.2.3) in this case is exactly the Max-Flow problem. Moreover, the Max-Flow-Min-Cut Theorem implies that (2.2.2) is always attained by a minimal ab-cut, and therefore Proposition 2.2.2 shows that every element of $\hat{\Gamma}$ is a minimal ab-cut. Conversely, if $\hat{\gamma}$ is a minimal ab-cut, then we can define σ to be very small on $\hat{\gamma}$ and large otherwise, so that $\hat{\gamma}$ is the unique solution of the Min-Cut problem. Therefore, the Fulkerson blocker of $\Gamma(a, b)$ is $\hat{\Gamma}(a, b) = \Gamma_{\text{cut}}(a, b)$.

Moreover, the duality

$$\mathrm{Mod}_{p,\sigma}(\Gamma)^{\frac{1}{p}}\, \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{\frac{1}{q}} = 1$$

can be viewed as a generalization of the max-flow min-cut theorem. To see this, consider the limiting case given by Theorem 2.1.10:

$$\mathrm{Mod}_{1,\sigma}(\Gamma)\, \mathrm{Mod}_{\infty,\sigma^{-1}}(\hat{\Gamma}) = 1. \tag{2.2.6}$$

As discussed above, $\mathrm{Mod}_{1,\sigma}(\Gamma)$ takes the value of the minimum ab-cut with edge weights σ. With a little work, the second modulus in (2.2.6), can be recognized as the reciprocal of the corresponding max flow problem. Using the standard trick for ∞-norms, the modulus problem $\mathrm{Mod}_{\infty,\sigma^{-1}}(\hat{\Gamma})$ can also be transformed into a linear program taking the form

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & \sigma(e)^{-1}\eta(e) \leq t \ \forall e \in E \\ & \eta \geq 0, \quad \hat{\mathcal{N}}\eta \geq \mathbf{1} \end{aligned}$$

The minimum must occur somewhere on the boundary of $\mathrm{Adm}(\hat{\Gamma})$ and, therefore, by Theorem 2.1.6(2), must take the form

$$\eta(e) = \sum_{\gamma \in \Gamma} \lambda(\gamma) \mathbb{1}_\gamma(e) \qquad \lambda(\gamma) \geq 0, \ \sum_{\gamma \in \Gamma} \lambda(\gamma) = 1.$$

In other words, the minimum occurs at a unit *st*-flow η, and the problem can be restated as

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & \frac{1}{t}\eta(e) \leq \sigma(e) \ \forall e \in E \\ & \eta \text{ a unit } st\text{-flow} \end{aligned}$$

The minimum is attained when $\frac{1}{t}\eta$ is a maximum *st*-flow respecting edge capacities $\sigma(e)$; the value of such a flow is $1/t$, recovering the max-flow min-cut theorem.

2.2.4 Fulkerson Blocker of Spanning trees

When Γ is the set of spanning trees on an unweighted, undirected graph G with $\mathcal{N}(\gamma, \cdot) = \mathbb{1}_\gamma(\cdot)$, the Fulkerson blocker $\hat{\Gamma}$ can be interpreted as the set of (weighted) *feasible partitions*, see [[14]].

Definition 2.2.3. A *feasible partition* P of a graph $G = (V, E)$ is a partition of the vertex set V into two or more subsets, $\{V_1, \ldots, V_{k_P}\}$, such that each of the induced subgraphs

$G(V_i)$ is connected. The corresponding edge set, E_P, is defined to be the set of edges in G that connect vertices belonging to different V_i's.

The results of [14] imply the following theorem.

Theorem 2.2.4. *Let $G = (V, E)$ be a simple, connected, unweighted, undirected graph and let Γ be the family of spanning trees on G. Then the Fulkerson blocker of Γ is the set of all vectors*

$$\frac{1}{k_P - 1}\mathbb{1}_{E_P}.$$

ranging over all feasible partitions P.

This fact plays an important role in [15].

2.3 Blocking Duality and the Probabilistic Interpretation

At the end of Section 1.3 it was claimed that blocking duality was closely related to Lagrangian duality. In this section, we make this connection explicit.

Theorem 2.3.1. *Let $G = (V, E, \sigma)$ be a graph and Γ a finite family of objects on G with Fulkerson blocker $\hat{\Gamma}$. For a given $1 < p < \infty$, let μ^* be an optimal pmf for the minimization problem in (1.3.1) and let η^* be optimal for $\mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})$. Then, in the notation of Section 1.3,*

$$\eta^*(e) = \mathbb{E}_{\mu^*}\left[\mathcal{N}(\underline{\gamma}, e)\right]. \tag{2.3.1}$$

Proof. Every $\eta \in \mathrm{Adm}(\hat{\Gamma})$ can be written as the sum of a convex combination of the vertices of $\mathrm{Adm}(\hat{\Gamma})$ and a nonnegative vector. In other words, $\eta \in \mathrm{Adm}(\hat{\Gamma})$ if and only if there exists $\mu \in \mathcal{P}(\Gamma)$ and $\eta_0 \in \mathbb{R}^E_{\geq 0}$ such that $\eta = \mathcal{N}^T\mu + \eta_0$. Or, in probabilistic notation,

$$\eta(e) = \sum_{\gamma\in\Gamma}\mathcal{N}(\gamma, e)\mu(\gamma) + \eta_0(e) = \mathbb{E}_{\mu}\left[\mathcal{N}(\underline{\gamma}, e)\right] + \eta_0(e).$$

For such an η,

$$\mathcal{E}_{q,\hat{\sigma}}(\eta) = \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \eta(e)^q \geq \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \mathbb{E}_\mu \left[\mathcal{N}(\underline{\gamma}, e)\right]^q$$

with equality holding if and only if $\eta_0 = 0$. This implies that the optimal η^* must be of the form $\eta^* = \mathcal{N}^T \mu' = \mathbb{E}_{\mu'}\left[\mathcal{N}(\underline{\gamma}, \cdot)\right]$ for some $\mu' \in \mathcal{P}(\Gamma)$.

Now, let μ^* be any optimal pmf for (1.3.1) and let $\eta' = \mathcal{N}^T \mu^*$. As a convex combination of the rows of $\mathcal{N}$, $\eta' \in \mathrm{Adm}(\hat{\Gamma})$. Moreover, by optimality of μ^*,

$$\mathcal{E}_{q,\hat{\sigma}}(\eta') = \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \mathbb{E}_{\mu^*} \left[\mathcal{N}(\underline{\gamma}, e)\right]^q \leq \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \mathbb{E}_{\mu'} \left[\mathcal{N}(\underline{\gamma}, e)\right]^q = \mathcal{E}_{q,\hat{\sigma}}(\eta^*).$$

But, since $1 < q < \infty$, the minimizer for $\mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})$ is unique and, therefore, $\eta' = \eta^*$. So $\eta^* = \mathcal{N}^T \mu^* = \mathbb{E}_{\mu^*}\left[\mathcal{N}(\underline{\gamma}, \cdot)\right]$ as claimed. $\square$

Chapter 3

Distance metrics on networks

In this chapter we explore some of the known distance metrics on networks. We specifically look at the effective resistance metric and give two different proofs that it is a metric. In Chapter 4, we will present a new proof that shows effective resistance is a metric using modulus and Fulkerson duality.

First let's recall the definition of a metric.

3.1 Metrics and Ultrametrics

Definition 3.1.1. Let X be a set and $d : X \times X \to \mathbb{R}$. The function d is called a *metric* on X if it satisfies the following four properties.

(i) **Non-negativity:** $d(a, b) \geq 0$ for all $a, b \in X$.

(ii) **Non-degeneracy:** $d(a, b) = 0$ if and only if $a = b$.

(iii) **Symmetry:** $d(a, b) = d(b, a)$ for all $a, b \in X$.

(iv) **Triangle inequality:** For every $a, b, c \in X$:

$$d(a, b) \leq d(a, c) + d(c, b).$$

Definition 3.1.2. If, instead of (iv) in the previous definition, d satisfies

$$\text{(iv)'} \quad d(a,b) \leq \max\{d(a,c), d(c,b)\}, \qquad \text{for every } a,b,c \in X, \tag{3.1.1}$$

then d is called an *ultrametric.* Since (iv)' implies (iv), every ultrametric is a metric.

A **simple graph** or **simple network** is an ordered pair $G = (V, E)$ where V, the **vertex-set**, is a collection of objects treated as nodes and E, the **edge-set**, is a collection of pairs of elements in V. An element of the edge-set, denoted $e = \{v_1, v_2\} \in E$ for $v_1, v_2 \in V$ means that there is an undirected link between nodes v_1 and v_2. Moreover, in this case, we say that v_1 and v_2 are **neighbors**, and write $v_1 \sim v_2$. Further, a simple graph has no **self-loops**, i.e., $\{v, v\} \notin E$ for all $v \in V$, and edges between pairs of nodes all have multiplicity one. The above is a complete characterization of simple graphs.

A graph is called a **finite graph** if the vertex and edge sets are both finite. For a simple graph, it is sufficient that the vertex set is finite, since if N denotes the number of nodes, then there are at most $\binom{N}{2}$ edges.

When d is a metric on V (the set of nodes of a network), d is often referred to as a **graph metric** or **network metric**. Three known network metrics are shortest path, effective resistance and the (reciprocal of) minimum cut. The shortest path metric between two nodes a and b, as its name suggests, simply refers to the length of the shortest path from a to b. The proof that this quantity is a network metric is straightforward.

3.1.1 Effective Resistance

The **effective resistance** metric arises from viewing the graph G as an electrical circuit with unit resistances on each edge. The effective resistance $R_{\text{eff}}(a, b)$ is the voltage drop necessary to pass 1 amp of current between a and b through the network G (see, e.g.,[[16]]). Effective resistance also turns out to be a metric on V, see Corollary 10.8 [[17]].

3.1.2 Effective conductance

An undirected graph $G = (V, E, \sigma)$ is a model for a resistor network with edges representing resistors with conductances σ, connected at junctions represented by the vertices. Let s and t be distinct vertices in V. The effective conductance $C_{\text{eff}}(s,t)$ (the reciprocal of the effective resistance) between s and t can be found by minimizing the total power

$$\text{Power} = \sum_{(x,y)\in E} \sigma(x,y)\left(\phi(x) - \phi(y)\right)^2$$

over all voltage potentials $\phi : V \to \mathbb{R}$ satisfying $\phi(s) = 0$ and $\phi(t) = 1$. The following theorem shows that, for $1 < p < \infty$, the extremal density ρ^* for the modulus of a connecting family of walks can be related to a generalized voltage potential. Such a result, in the language of extremal distance and in the special case $p = 2$, first appeared in the work of Duffin[3] (see also Proposition 6.2[[18]]). For a version in metric spaces, see Theorem 7.31 of [[19]].

Theorem 3.1.3. *Let $G = (V, E, \sigma)$ be an undirected graph, let $\Gamma = \Gamma(s,t)$ be the connecting family of walks from s to t, two distinct vertices in V, and let $1 < p < \infty$. Let ρ^* be the extremal density. Then there exists a vertex potential $\phi^* : V \to \mathbb{R}$ such that $\phi^*(s) = 0$, $\phi^*(t) = 1$, and*

$$\rho^*(x,y) = |\phi^*(x) - \phi^*(y)| \qquad \forall (x,y) \in E. \tag{3.1.2}$$

Moreover, this ϕ^* solves the optimization problem

$$\begin{aligned} &\textit{minimize} && \sum_{(x,y)\in E} \sigma(x,y)|\phi(x) - \phi(y)|^p \\ &\textit{subject to} && \phi(s) = 0, \quad \phi(t) = 1. \end{aligned}$$

When $p = 2$, it follows that $\mathrm{Mod}_2(\Gamma(s,t)) = C_{\text{eff}}(s,t)$.

Proof. Let $\rho^* \in A(\Gamma)$ be the extremal density. Note that $\ell_{\rho^*}(\Gamma) = 1$, for, if not, the density

$\rho' = \rho^*/\ell_{\rho^*}(\Gamma)$ is also admissible and has lower p-energy than ρ^*. Define $\phi^*(s) = 0$ and for $x \in V \setminus \{s\}$

$$\phi^*(x) = \min_{\gamma \in \Gamma(s,x)} \ell_{\rho^*}(\gamma).$$

Since $\Gamma = \Gamma(s,t)$, $\phi^*(t) = 1$. To see that Equation (3.1.2) holds, define $\rho'(x,y) = |\phi^*(x) - \phi^*(y)|$ for $(x,y) \in E$. For any $\gamma = sv_2v_3 \dots v_rt \in \Gamma$,

$$\ell_{\rho'}(\gamma) = |\phi^*(s) - \phi^*(v_2)| + |\phi^*(v_2) - \phi^*(v_3)| + \cdots + |\phi^*(v_r) - \phi^*(t)| \geq |\phi^*(t) - \phi^*(s)| = 1,$$

so $\rho' \in A(\Gamma)$. Let $(x,y) \in E$ be an edge. Without loss of generality, $\phi^*(x) \leq \phi^*(y)$ and $y \neq s$. We claim that

$$0 \leq \rho'(x,y) = \phi^*(y) - \phi^*(x) \leq \rho^*(x,y). \tag{3.1.3}$$

The case $x = s$ is trivial. Suppose that $x \neq s$ and let $\gamma \in \Gamma(s,x)$ be a walk such that $\ell_{\rho^*}(\gamma) = \phi^*(x)$. Letting $\gamma' \in \Gamma(s,y)$ be the walk obtained by first traversing γ and then traversing the edge (x,y), we have

$$\phi^*(y) \leq \ell_{\rho^*}(\gamma') = \ell_{\rho^*}(\gamma) + \rho^*(x,y),$$

which implies Equation (3.1.3).

Equation (3.1.3) implies that $\mathcal{E}_p(\rho') \leq \mathcal{E}_p(\rho^*)$ and, by uniqueness of the extremal density, $\rho^* = \rho'$. To see that ϕ^* solves the optimization problem, let $\phi' : V \to \mathbb{R}$ with $\phi'(s) = 0$ and $\phi'(t) = 1$ and define $\rho'(x,y) = |\phi'(x) - \phi'(y)|$. As before, $\rho' \in A(\Gamma)$ and so

$$\sum_{(x,y)\in E} \sigma(x,y)|\phi^*(x) - \phi^*(y)|^p = \mathcal{E}_p(\rho^*) \leq \mathcal{E}_p(\rho') = \sum_{(x,y)\in E} \sigma(x,y)|\phi'(x) - \phi'(y)|^p.$$

□

3.1.3 Min-cut

In order to define the minimum cut metric, we recall that a subset $S \subset V$ is called an ab**-cut** if $a \in S$ and $b \notin S$. The size of a cut is measured by $|\partial S|$, where $\partial S = \{e = \{x, y\} \in E : x \in S, y \notin S\}$ is the *edge-boundary* of S. In this paper, we shall use the notation

$$\mathrm{MC}(a,b) = \min\big\{|\partial S| : S \text{ is an } ab\text{-cut}\big\} \quad \text{and} \quad d_{\mathrm{MC}}(a,b) = \begin{cases} 0 & \text{if } a = b, \\ \mathrm{MC}(a,b)^{-1} & \text{if } a \neq b. \end{cases}$$

That d_{MC} is a graph metric (indeed, an ultrametric) can be seen from the following argument. Suppose a, b and c are distinct vertices and let S be a minimum cut for $\mathrm{MC}(a,b)$, so that $a \in S$ and $b \notin S$. Then, either $c \in S$ or $c \notin S$. If $c \in S$, then S is a cb-cut and $\mathrm{MC}(c,b) \leq \mathrm{MC}(a,b)$, hence $\mathrm{MC}(c,b)^{-1} \geq \mathrm{MC}(a,b)^{-1}$. If $c \notin S$, then S is an ac-cut and $\mathrm{MC}(a,c) \leq \mathrm{MC}(a,b)$, so that $\mathrm{MC}(a,c)^{-1} \geq \mathrm{MC}(a,b)^{-1}$. Since one of the two inequalities must hold, it follows that

$$\mathrm{MC}(a,b)^{-1} \leq \max\{\mathrm{MC}(a,c)^{-1}, \mathrm{MC}(c,b)^{-1}\}, \tag{3.1.4}$$

showing that d_{MC} is an ultrametric.

3.1.4 Epidemic Hitting Time

A number of other interesting metrics exist on networks. For example, [7] presents a metric related to the spreading of epidemics in a contact network. There, the standard SI model of infection is applied to a network with the spreading time from infected to susceptible nodes modeled by independent exponential random variables. In this system, the time required for an infection originating at node a to reach node b is a random variable. Its expected value is called the Epidemic Hitting Time $\mathrm{EHT}(a,b)$ and was shown to be a network metric.

3.2 The effective resistance metric

In this section, we will discuss more on the effective resistance metric. We will explore this metric in two contexts.

- Using flows
- Using the Laplacian

3.3 Proving effective resistance is a metric using flows

Definition 3.3.1. $D = (V, \vec{E})$ is an orientation of $G = (V, E)$ if D is obtained from G by orienting each edge $x, y \in E$ as (x, y) or (y, x).

Let G be a finite simple graph and let $s \neq t \in V$ be vertices so that s is the *source* and t is the *sink*. A *flow* from s to t is a function $f : \vec{E} \longrightarrow \mathbb{R}$,where $\vec{E}$ is the set of oriented edges.

(a) f is *antisymmetric*: $f(x, y) = -f(y, x)$.

(b) f satisfies the *node law*:

$$\mathrm{Div}_f(x) := \sum_{y \sim x} f(x, y) = 0 \qquad \forall x \in V \setminus \{s, t\},$$

where the operator $\mathrm{Div}_f(x) := \sum_{y \sim x} f(x, y)$ is called the *divergence* of the flow f at x.

3.3.1 Value of a flow

Define the *value* of a flow from s to t to be

$$\mathrm{Val}(f) := \mathrm{Div}_f(s).$$

Note that the node law and antisymmetry lead to

$$\begin{aligned}\mathrm{Val}(f) + \mathrm{Div}_f(t) = \mathrm{Div}_f(s) + \mathrm{Div}_f(t) &= \sum_{x \in V} \mathrm{Div}_f(x) \\ &= \sum_{x \in V} \sum_{y \in V} A(x,y) f(x,y) = 0.\end{aligned}$$

So $\mathrm{Div}_f(t) = -\mathrm{Val}(f)$.

3.3.2 Energy of a flow

Given a flow from s to t and some *edge-resistances* $r(e) > 0$ for $e \in E$, let the *energy* of the flow be

$$\mathcal{E}(f) = \sum_{e \in E} r(e) |f(e)|^2.$$

We also define $c(e) := 1/r(e)$ to be the *conductance* of the edge e.

The goal is to minimize the energy among all flows from s to t of value 1:

$$\mathcal{E}^* := \min_{f : \mathrm{Val}(f) = 1} \mathcal{E}(f).$$

Existence and uniqueness of a *minimizer* f^* such that $\mathcal{E}(f^*) = \mathcal{E}^*$ can be shown using the theory of convex optimization with a similar argument that was presented in Chapter 1, proof of Lemma 1.2.5. In this context, the unique minimizer is called the *unit electrical current* flow from s to t.

3.3.3 Cycle law

Lemma 3.3.2. *If f^* is a minimizer and k is an arbitrary flow from s to t with* $\mathrm{Val}(k) = 0$*, then*

$$\sum_{e \in E} r(e) f^*(e) k(e) = 0.$$

The proof of this lemma is deferred to the next section in lemma 3.3.3.

Define a *cycle* to be a walk $W = x_0\ e_1\ x_1\ \cdots\ e_n\ x_n$ with $x_0 = x_n$ and x_j distinct . Let $k_W(x_{i-1}, x_i) = 1$ for $i = 1, \ldots, n$. Then k_W is a flow from s to t with value zero. Therefore, it implies that f^* satisfies the following *cycle law*:

$$\sum_{i=1}^{n} r(e_i) f^*(x_{i-1}, x_i) = 0 \qquad \text{for all cycles}$$

3.3.4 Potential and Ohm's law

If a flow f from s to t satisfies the cycle law, then it must admit a potential, namely, a function $h : V \to \mathbb{R}$ such that for every edge $e = \{x, y\}$:

$$r(e) f(x, y) = h(y) - h(x).$$

The existence of a potential satisfying the above equation is referred to as *Ohm's law*.

Lemma 3.3.3 (Improved-flow-stationarity)**.** *If f is a flow from s to t that admits a potential h and k is an arbitrary flow from s to t, then*

$$\sum_{e \in E} r(e) f(e) k(e) = \mathrm{Val}(k)(h(t) - h(s)).$$

Now we give the proof of this lemma.

Proof. Let f be a unit flow from s to t that admits a potential h that satisfies Ohm's law, and let k be an arbitrary flow from s to t. Then

$$\begin{aligned}
&\sum_{e\in E} r(e)f(e)k(e) \\
&= \frac{1}{2}\sum_{x,y} r(x,y)f(x,y)k(x,y) = \frac{1}{2}\sum_{x,y}(h(y)-h(x))k(x,y) \\
&= \frac{1}{2}\sum_{y} h(y)\left(\sum_{x} k(x,y)\right) - \frac{1}{2}\sum_{x} h(x)\left(\sum_{y} k(x,y)\right) \\
&= -\frac{1}{2}\sum_{y} h(y)\,\mathrm{Div}_k(y) - \frac{1}{2}\sum_{x} h(x)\,\mathrm{Div}_k(x) \\
&= -h(s)\,\mathrm{Div}_k(s) - h(t)\,\mathrm{Div}_k(t) = (h(t)-h(s))\,\mathrm{Val}(k)
\end{aligned}$$

□

We summarize this section in the following theorem.

Theorem 3.3.4. *Let $G=(V,E)$ be a finite connected simple graph with edge-resistances $r(e)\in(0,\infty)$ for every $e\in E$. Fix a source $s\in V$ and a target $t\neq s$. Suppose f is a unit flow from s to t. Then the following are equivalent:*

1. *f is an energy-minimizer (3.3.2).*

2. *f satisfies the cycle law (3.3.3).*

3. *f admits a potential that satisfies Ohm's law (3.3.4).*

3.3.5 Harmonic functions and uniqueness of current flow

The node law implies that the potential h is harmonic on $V\setminus\{s,t\}$, since for $x\neq s,t$:

$$\mathrm{Div}_f(x) = \sum_{y\sim x} f(x,y) = \sum_{y\sim x} c(x,y)(h(y)-h(x)) = 0.$$

Hence, writing $C(x) := \sum_{y\sim x} c(x,y)$, we get that the value of h at x is a weighted average of the values of h on the neighbors of x:

$$h(x) = \frac{1}{C(x)} \sum_{y\sim x} c(x,y)h(y). \tag{3.3.1}$$

This is known as the *Mean Value Property.*

Theorem 3.3.5 (Maximum Principle)**.** *Let $G = (V,E)$ be a finite connected simple graph with edge-conductances $c(e) \in (0,\infty)$ for every $e \in E$. Fix a non-empty subset $B \subset V$, e.g., $B = \{s,t\}$. Suppose that $h : V \to \mathbb{R}$ satisfies the mean value property at every $x \in V \setminus B$. Then h admits its global maximum on B.*

Proof. Suppose $x \notin B$ and $h(x) = \max_{y\in V} h(y)$. The mean value property implies what is sometime referred to as the "Lake Wobegon Effect", namely, that $h(y) = h(x)$ for every neighbor $y \sim x$. In fact, assume that there is one $y' \sim x$ where $h(y') < h(x)$. Since for every other neighbor $y'' \sim x$ we have $h(y'') \leq h(x)$ when we average we get

$$C(x)h(x) = \sum_{y\sim x} c(x,y)h(x) > \sum_{y\sim x} c(x,y)h(y) = C(x)h(x)$$

which is a contradiction. Therefore, if the maximum is attained at x it must also be attained at its neighbors. By connectedness there is a walk from x to any other vertex, therefore h must be constant. $\square$

The maximum principle can be used to prove uniqueness of the current flow. Suppose that f_1 and f_2 are two unit flow from s to t that minimize the energy in (3.3.2). By Theorem (3.3.4), let h_1 and h_2 be the corresponding potentials normalized as $h_1(s) = h_2(s) = 0$. By Lemma 3.3.2 applied to the same flow, for $j = 1,2$, we get

$$\mathcal{E}^* = \sum_{e\in E} r(e)|f_j(e)|^2 = \mathrm{Val}(f_j)(h_j(t) - h_j(s)) = h_j(t)$$

Form $u = h_1 - h_2$. Then u is harmonic on $V \setminus \{s, t\}$ and $u(s) = u(t) = 0$. By the Maximum Principle, $u(x) \leq 0$ for $x \in V$, and likewise $-u(x) \leq 0$, so $u(x) \equiv 0$ for all $x \in V$. Therefore, $h_1 = h_2$ and $f_1 = f_2$.

3.3.6 Effective resistance is a metric: Proof

We are now ready to define the effective resistance using flows.

Definition 3.3.6. If f is a unit current flow from s to t with potential h, then by applying the previous lemma 3.3.3 to itself we get

$$\mathcal{E}^* = \sum_{e \in E} r(e)|f(e)|^2 = h(t) - h(s)$$

If we replace the whole network by a single edge $\hat{e}$ with resistance $\mathcal{R}$ between s and t, but we keep the unit flow and the potential difference as before, then $\mathcal{R}$ would have to satisfy:

$$\mathcal{R}(\hat{e})|f(\hat{e})|^2 = \mathcal{R}(\hat{e}) = h(t) - h(s).$$

We call $\mathcal{R}$ the *effective resistance* of the network from s to t and write $R_{\text{eff}}(s, t)$.

We will now give the proof that effective reistance is a metric.

Proof.

- We first prove the non-degeneracy of the effective resistance.If $a = b$, then there is no flow and the potential is constant. So $R_{\text{eff}}(a, a) = 0$.

  Conversely, if $R_{\text{eff}}(a, b) = 0$, and $a \neq b$, then the electric potential satisfies $h(b) - h(a) = 0$. So by the Maximum Principle, $h(x) \equiv h(a)$ for all $x \in V$. However, unit current flow satisfies

  $$1 = \text{Val}(f) = \text{Div}_f(a) = \sum_{y \sim a} f(a, y).$$

  In particular, by connectedness, there exists $y \sim a$ such that $f(a, y) \neq 0$. But by

Ohm's law,

$$f(a,y) = r(a,y)^{-1}(h(y) - h(a)) = 0.$$

And this is a contradiction, so $a = b$ $\square$

- Now we prove that it is symmetric. Suppose f is unit current flow from a to b with potential h such that $h(a) = 0$. Then $R_{\text{eff}}(a,b) = h(b)$. Now define $g = -f$ and $u = h(b) - h$. Then

$$r(x,y)g(x,y) = h(x) - h(y) = u(y) - u(x).$$

So u is a potential for g and $u(b) = 0$. So $R_{\text{eff}}(b,a) = u(a) = h(b) = R_{\text{eff}}(a,b)$ $\square$

- Finally, we prove the triangle inequality. Without loss of generality $a \neq b \neq c \neq a$. Write $f_{s,t}$ for the unit current flow from s to t and write $h_{s,t}$ for the associated potential normalized to be 0 at s. Then $h_{s,t}(t) = R_{\text{eff}}(s,t)$ and by the maximum principle $0 \leq h_{s,t}(x) \leq R_{\text{eff}}(s,t)$ for every $x \in V \setminus \{s,t\}$.

Define $g_{a,b} := f_{a,c} + f_{c,b}$. By linearity

$$\text{Div}_{g_{a,b}}(x) = \text{Div}_{f_{a,c}}(x) + \text{Div}_{f_{c,b}}(x).$$

So for $x \neq a,b,c$, $\text{Div}_{g_{a,b}}(x) = 0$ and for $x = c$, $\text{Div}_{g_{a,b}}(c) = 1-1 = 0$. In particular, $g_{a,b}$ is a unit flow from a to b. By the minimizing property (sometimes called Thomson's

Principle),

$$
\begin{aligned}
R_{\text{eff}}(a,b) = \mathcal{E}^* \le \mathcal{E}(g_{a,b}) \\
&= \sum_{e\in E} r(e)\left(f_{a,c}(e) + f_{c,b}(e)\right)^2 \\
&= \mathcal{E}(f_{a,c}) + \mathcal{E}(f_{c,b}) + 2\sum_{e\in E} r(e) f_{a,c}(e) f_{c,b}(e) \\
&= R_{\text{eff}}(a,c) + R_{\text{eff}}(c,b) + 2\sum_{e\in E} r(e) f_{a,c}(e) f_{c,b}(e)
\end{aligned}
$$

The triangle inequality follows if we can show that $\sum_{e\in E} r(e) f_{a,c}(e) f_{c,b}(e) \le 0$. By mimicking the proof of Lemma 3.3.3, we see that

$$
\begin{aligned}
\sum_{e\in E} r(e) f_{a,c}(e) f_{c,b}(e) &= -\sum_{x\in V} h_{a,c} \operatorname{Div}_{f_{c,b}}(x) \\
&= -h_{a,c}(c) + h_{a,c}(b) \le 0
\end{aligned}
$$

where the last inequality follows from the Maximum Principle for $h_{a,c}$ and the fact that $h_{a,c}(a) = 0$.

□

3.4 Proving effective resistance is a metric using the Laplacian

3.4.1 The Laplacian matrix

For a finite simple graph $G = (V, E)$ with edge-conductances $c : E \to (0, \infty)$ the *Laplacian* is $L : V \times V \to \mathbb{R}$:

$$L(x, y) = \begin{cases} C(x) = \sum_{z \sim x} c(x, z) & \text{if } x = y \\ -c(x, y) & \text{if } x \neq y \end{cases}$$

Writing $D_c = \mathrm{diag}(C(\cdot))$ for the diagonal matrix with the generalized degrees $C(x)$ for $x \in V$, and letting the generalized adjacency matrix $A_c(x, y) = c(x, y)\mathbb{1}_{x \sim y}$, then

$$L = D_c - A_c$$

We now present the Maximum Principle that was discussed earlier in lemma 3.3.5 in the language of Laplacian.

3.4.2 Maximum principle

Thinking of a function $h : V \to \mathbb{R}$ as a vector in $\mathbb{R}^N$ we get that

$$(Lh)(x) = \sum_{y \in V} L(x, y)h(y) = C(x)h(x) - \sum_{y \sim x} c(x, y)h(y).$$

In particular, $Lh = 0$ if and only if h is harmonic at every $x \in V$.

Using Maximum Principle Theorem, we see that if h is harmonic at every $x \in V$, then h must be constant on each connected component of G. In particular, if G is connected then the only solution to the equation $Lh = 0$ are the constant functions.

3.4.3 Gradient matrix

Assume that an orientation $\vec{E}$ as described in 3.3.1 has been chosen.

Define the $m \times N$ *gradient matrix*

$$B(\vec{e}, x) = \begin{cases} 1 & \text{if } \vec{e} = (y, x) \text{ for some } y. \\ -1 & \text{if } \vec{e} = (x, y) \text{ for some } y. \\ 0 & \text{else} \end{cases}$$

Lemma 3.4.1. *Let S be a diagonal $m \times m$ matrix listing all the edge-conductance $S(e, e) = c(e)$. Then*

$$L = B^T S B.$$

Proof. For simplicity, let's assume that S is the identity matrix, namely that all the edge-conductances are equal to 1. Then

$$(B^T B)(x, y) = \sum_{\vec{e} \in \vec{E}} B^T(x, \vec{e}) B(\vec{e}, y) = \sum_{\vec{e} \in \vec{E}} B(\vec{e}, x) B(\vec{e}, y)$$

Two cases arise. If $x = y$, then the sum is over edges that are incident to x:

$$(B^T B)(x, x) = \sum_{\vec{e} \in \vec{E}} B(\vec{e}, x)^2 = \deg(x).$$

If $x \neq y$, then the sum is over a single edge incident to both x and y.

$$(B^T B)(x, x) = B(\vec{e}, x) B(\vec{e}, y) = -1.$$

So $(B^T B)(x, y) = D(x, y) - A(x, y)$. $\square$

3.4.4 Current flow

If f is an s/t-flow, then it is a vector $f \in \mathbb{R}^m$, and for $x \in V$:

$$\begin{aligned} B^T f(x) &= \sum_{e \in E} B^T(x,e) f(e) = \sum_{e \in E} B(e,x) f(e) \\ &= \sum_{y \sim x, (y,x) \in \vec{E}} f(y,x) - \sum_{y \sim x, (x,y) \in \vec{E}} f(x,y) \\ &= -\operatorname{Div}_f(x). \end{aligned}$$

Let δ_a be the $N \times 1$ column-vector such that $\delta_a(a) = 1$ and $\delta_a(x) = 0$ otherwise.

We can then impose that f be a unit s/t-flow, by requiring that

$$B^T f = \delta_t - \delta_s.$$

Define an $m \times m$ diagonal matrix $R = \operatorname{diag}(r(e))$ with the edge-resistances on the diagonal and its inverse $S = R^{-1}$ with the conductances instead. Note that the energy of a flow can be rewritten as

$$\mathcal{E}_r(f) = \sum_{e \in E} r(e)(f(e))^2 = f^T R f = \|R^{1/2} f\|^2.$$

The problem of finding the unit current flow from s to t is

$$\text{minimize } \|R^{1/2} f\|^2 \qquad \text{given } B^T f = \delta_t - \delta_s.$$

Existence and uniqueness can then be deduced from standard results in convex optimization.

If h is the potential function of the flow f, then Ohm's law can be rewritten as:

$$f = SBh.$$

Also, by lemma 3.4.1, the Laplacian can be written as in the continuous case as the "divergence of the gradient": $L = B^T SB$. In terms of the potential we then have

$$Lh = B^T SBh = B^T f = \delta_t - \delta_s.$$

With this additional knowledge of the Laplacian, we can present a new way to prove that effective resistance satisfies the triangle inequality.

Proof. Write $f_{s,t}$ for the unit current flow from s to t and write $h_{s,t}$ for the associated potential normalized to be 0 at s. Then $h_{s,t}(t) = R_{\text{eff}}(s,t)$.By the maximum principle $0 \leq h_{s,t}(x) \leq R_{\text{eff}}(s,t)$ for every $x \in V \setminus \{s,t\}$.

Define $f := f_{a,c} + f_{c,b}$. As before f is a unit flow from a to b. Also let $h = h_{a,c} + h_{c,b}$. Then,

$$Lh = Lh_{a,c} + Lh_{c,b} = (\delta_c - \delta_a) + (\delta_b - \delta_c) = \delta_b - \delta_a.$$

However, $L(h - h_{a,b}) = 0$ and by the maximum principle we have, $h - h_{a,b} = C$ for C, some constant. $h = h_{a,c} + h_{c,b} = h_{a,b} + C = h_{a,b} + h_{c,b}(a)$. Plugging b in we get

$$\begin{aligned} h_{a,b}(b) \leq h_{a,b}(b) + h_{c,b}(a) = h(b) &= h_{a,c}(b) + h_{c,b}(b) \\ &\leq h_{a,c}(c) + h_{c,b}(b). \end{aligned}$$

We have the result since $h_{s,t}(t) = R_{\text{eff}}(s,t)$ □

Chapter 4

A new family of metrics arising from p-modulus

This chapter will introduce a new family of metrics on networks arising from p-modulus. First, we will introduce the d_p metric and later the δ_p metric. We will show how these two metrics are related through concept of "antisnowflaking". Not only would we like to present δ_p as a metric, but we would like to offer it as a new proof that effective resistance is a metric on networks.

This chapter will also present some numerical examples of calculating the modulus and thus the d_p and δ_p metrics on various families of graphs.

4.1 The d_p metric

Reinterpreting Theorem 1.2.4 that was presented in Chapter 1, we see that $\text{Mod}_p(\Gamma(a,b))^{-1}$ is a metric for $p = 1, 2, \infty$. One might naturally wonder if this fact generalizes to all $p \in [1, \infty]$. The answer turns out to be "no." However, we'll see shortly that introducing a pth root does in fact lead to a metric for all p.

Definition 4.1.1. For $1 \leq p \leq \infty$, let

$$d_p(a,b) := \mathrm{Mod}_p(\Gamma(a,b))^{-1/p} \quad \text{if } p < \infty \qquad \text{and} \qquad d_\infty(a,b) = \mathrm{Mod}_\infty(\Gamma(a,b))^{-1}.$$

Theorem 1.2.4(i) implies that $d_p(a,b) \to d_\infty(a,b)$ as $p \to \infty$. Moreover, the continuity in p and (ii) imply that $d_p(a,b) \to d_1(a,b) = \mathrm{MC}(a,b)^{-1}$ as $p \to 1$.

Remark 4.1.1. For $p = 2$, $d_2(a,b) = \sqrt{R_{\text{eff}}(a,b)}$. This is a known metric which appears for instance in the context of the discrete Gaussian Free Field, see Ding et al[20]. It also has the following alternative representation: if $\mathcal{G}$ is the pseudoinverse of the Laplacian matrix L, then

$$d_2(a,b) = \sqrt{R_{\text{eff}}(a,b)} = \|\mathcal{G}^{1/2}\mathbb{1}_a - \mathcal{G}^{1/2}\mathbb{1}_b\|_2,$$

where $\mathbb{1}_x$ is the vector with 1 at x and 0 everywhere else. This gives two different ways of verifying that d_2 is a metric. First, given a metric d and an exponent $\epsilon \in (0,1)$, then taking the fractional root d^ϵ is always a metric as well. This is called *snowflaking* the metric (see Section 4.2). Therefore since, it is known that effective resistance is a metric, it is immediate that its square-root is a metric as well. The formulation in terms of $\mathcal{G}$ shows that d_2 is the pull-back of the Euclidean norm $\|\cdot\|_2$ restricted to the set of indicator functions $\{\mathbb{1}_x\}_{x\in V}$ under the linear map given by $\mathcal{G}^{1/2}$, again showing that d_2 is a metric. Here we take an alternate approach based on the theory of modulus.

The main idea in the proof of Theorem 4.1.3 below is to compare the connecting families $\Gamma(a,c)$, $\Gamma(c,b)$, $\Gamma(a,b)$ and the *via family* $\Gamma(a,b \mid c)$—the family of all walks beginning at a, ending at b and passing through c along the way. A key lemma is the following.

Lemma 4.1.2. *Given a density $\rho : E \to [0,\infty)$, we have*

$$\ell_\rho(\Gamma(a,b \mid c)) = \ell_\rho(\Gamma(a,c)) + \ell_\rho(\Gamma(c,b)).$$

Proof. First, pick ρ-shortest walks γ_1 for $\Gamma(a,c)$ and γ_2 for $\Gamma(c,b)$. Then, the concatenation

$\gamma_0 = \gamma_1\gamma_2$, of γ_1 followed by γ_2, is a walk in $\Gamma(a, b \mid c)$. So

$$\ell_\rho(\Gamma(a, b \mid c)) \leq \ell_\rho(\gamma_0) = \ell_\rho(\gamma_1) + \ell_\rho(\gamma_2) = \ell_\rho(\Gamma(a, c)) + \ell_\rho(\Gamma(c, b)). \tag{4.1.1}$$

Conversely, let γ be a walk from a to b via c. Write γ as $\gamma' \in \Gamma(a, c)$ followed by $\gamma'' \in \Gamma(c, b)$. Then

$$\ell_\rho(\gamma) = \ell_\rho(\gamma') + \ell_\rho(\gamma'') \geq \ell_\rho(\Gamma(a, c)) + \ell_\rho(\Gamma(c, b)).$$

Taking the infimum over $\gamma \in \Gamma(a, b \mid c)$ we get that $\ell_\rho(\Gamma(a, b \mid c) \geq \ell_\rho(\Gamma(a, c)) + \ell_\rho(\Gamma(c, b))$. □

Theorem 4.1.3. *Let $G = (V, E)$ be a simple connected graph, and let $p \in [1, \infty]$. Then, d_p is a metric on V. Moreover, d_1 is an ultrametric.*

Proof. That d_1 is an ultrametric is a consequence of Theorem 1.2.4(ii) and the fact that the reciprocal of minimum cut is an ultrametric, while the fact that d_∞ is a metric is a consequence of Theorem 1.2.4(i).

For $1 < p < \infty$, we begin by verifying properties (i)-(iii) in Definition 3.1.1. Since modulus is the infimum of a non-negative energy, non-negativity holds. If $a = b$ the connecting family $\Gamma(a, a)$ contains the constant walk, and then no density can be admissible, so the p-modulus of $\Gamma(a, a)$ is infinity and $d_p(a, a) = 0$. Conversely, if $a \neq b$, consider the constant density $\rho_0 \equiv 1$. Then $\ell_0 := \ell_{\rho_0}(\Gamma(a, b))$ is the shortest-path distance from a to b, hence $\ell_0 < \infty$ since G is connected. This implies that the density $\rho_1 := \rho_0/\ell_0$ is admissible for $\Gamma(a, b)$. Therefore, for $p \in [1, \infty)$,

$$\mathrm{Mod}_p(\Gamma(a, b)) \leq \mathcal{E}_p(\rho_1) = \frac{|E|}{\ell_0^p} < \infty,$$

and $\mathrm{Mod}_\infty(\Gamma(a, b)) \leq \ell_0^{-1}$, showing that $d_p(a, b) > 0$. Finally, since every path from a to b can be reversed to a path from b to a, it follows that $\mathrm{Adm}(\Gamma(a, b)) = \mathrm{Adm}(\Gamma(b, a))$, so symmetry holds as well.

It remains to prove the triangle inequality. Without loss of generality we can assume that $a, b, c \in V$ are distinct. Let $\Gamma(a,b), \Gamma(a,c), \Gamma(c,b)$ be the corresponding families of connecting walks and let $\Gamma(a,b \mid c)$ be the family of walks from a to b via c. Let $\rho^* \in \mathrm{Adm}(\Gamma(a,b \mid c))$ be extremal for $\mathrm{Mod}_p(\Gamma(a,b \mid c))$. Then by Lemma 4.1.2 and extremality:

$$1 = \ell_{\rho^*}(\Gamma(a,b \mid c)) = \ell_{\rho^*}(\Gamma(a,c)) + \ell_{\rho^*}(\Gamma(c,b)). \tag{4.1.2}$$

We now consider two possibilities. First, suppose that $\ell_{\rho^*}(\Gamma(a,c)) > 0$ and $\ell_{\rho^*}(\Gamma(c,b)) > 0$, and define

$$\rho_1 := \frac{\rho^*}{\ell_{\rho^*}(\Gamma(a,c))} \qquad \text{and} \qquad \rho_2 := \frac{\rho^*}{\ell_{\rho^*}(\Gamma(c,b))}.$$

Then $\rho_1 \in \mathrm{Adm}(\Gamma(a,c))$ and $\rho_2 \in \mathrm{Adm}(\Gamma(c,b))$. Writing $\Gamma_1 := \Gamma(a,c)$, $\Gamma_2 := \Gamma(c,b)$, and $\Gamma := \Gamma(a,b \mid c)$, in order to simplify notation, we get

$$\begin{aligned} d_p(a,c) + d_p(c,b) &= \mathrm{Mod}_p(\Gamma_1)^{-1/p} + \mathrm{Mod}_p(\Gamma_2)^{-1/p} \\ &\geq \mathcal{E}_p(\rho_1)^{-1/p} + \mathcal{E}_p(\rho_2)^{-1/p} \\ &= \mathcal{E}_p(\rho^*)^{-1/p}\left(\ell_{\rho^*}(\Gamma_1) + \ell_{\rho^*}(\Gamma_2)\right) \\ &= \mathcal{E}_p(\rho^*)^{-1/p} = \mathrm{Mod}_p(\Gamma)^{-1/p} \end{aligned}$$

where the second to last equality follows from (4.1.2).

On the other hand, suppose that, say, $\ell_{\rho^*}(\Gamma(a,c)) = 0$. Then (4.1.2) implies that $\ell_{\rho^*}(\Gamma(c,b)) = 1$, which implies that $\rho^* \in \mathrm{Adm}(\Gamma(c,b))$. Thus,

$$d_p(a,c) + d_p(c,b) \geq d_p(c,b) = \mathrm{Mod}_p(\Gamma_2)^{-1/p} \geq \mathcal{E}_p(\rho^*)^{-1/p} = \mathrm{Mod}_p(\Gamma)^{-1/p},$$

and similarly if $\ell_{\rho^*}(\Gamma(c,b)) = 0$ and $\ell_{\rho^*}(\Gamma(a,c)) = 1$.

Now we use the Γ-monotonicity of modulus. Note that $\Gamma = \Gamma(a,b \mid c) \subset \Gamma_0 := \Gamma(a,b)$.

Thus, $\text{Mod}_p(\Gamma) \leq \text{Mod}_p(\Gamma_0)$, hence $\text{Mod}_p(\Gamma)^{-1/p} \geq \text{Mod}_p(\Gamma_0)^{-1/p} = d_p(a,b)$ and the triangle inequality holds. $\square$

4.2 Snowflaking and Antisnowflaking

As we saw in Remark 4.1.1, squaring the metric d_2 yields effective resistance, which is known to be a metric on any connected graph. Therefore, we now study the question of finding the largest exponents one can raise each d_p metric to, while maintaining the property of being a metric on arbitrary connected graphs.

Given an arbitrary metric, *snowflaking* provides an interesting way to generate new metrics on the same set. This procedure is described by the following known fact.

Fact 4.2.1. *Let d be a metric on X and let $0 < \epsilon < 1$, then taking the ϵ fractional root of d, d^ϵ is also a metric on X.*

In other words, raising a metric to a positive fractional power always results in another metric. This immediately leads one to ask the following question. Given some metric d on X, is d the snowflaked version of some other metric? In other words, does there exist a $t > 1$ such that d^t is also a metric? When such a t exists, we shall call the resulting metric d^t an *antisnowflaking* of d.

For finite X, the characterization of metrics that can be antisnowflaked is straightforward. Suppose a, b and c are distinct points in X. If $d(a,b) < d(a,c) + d(c,b)$, then the inequality also holds with d replaced by d^t for sufficiently small $t > 1$. We call such a triple of points (a,b,c) a *proper triangle*. On the other hand, if $d(a,b) = d(a,c) + d(c,b)$, then it can be seen that d^t violates the triangle inequality for arbitrarily small $t > 1$. We refer to such a triple as a *flat triangle*. Since a finite set X contains a finite number of triangles, the following theorem is evident.

Example 4.2.1. Consider the path graph on (j-i) nodes, where $j > i$. Now we calculate the d_p metric for the connecting family $\Gamma(i,j)$ for this graph.

$$d_p = (\operatorname{Mod}\Gamma(i,j))^{-1/p} = \left(\frac{1}{|i-j|^{p-1}}\right)^{-1/p} = (|i-j|)^{\frac{p-1}{p}}$$

We observe that by antisnowflaking this metric to the power of $(p/p-1)$ we can recover the Euclidean metric on this graph $|i-j|$.

Theorem 4.2.2. *Let d be a metric on a finite set X. There exists a $t>1$ such that d^t is a metric on X if and only if (X,d) contains no flat triangles.*

With this in mind, we make the following definition.

Definition 4.2.3. The *antisnowflaking exponent* of a metric d is defined as

$$\text{ASFE}(d) := \sup\{t \geq 1 : \ d^t \text{ is a metric}\}$$

For instance, it is clear that when d is an ultrametric, then $\text{ASFE}(d) = \infty$. While the antisnowflaking exponent of a particular metric on a particular graph may be interesting in certain contexts, here we will focus on the best antisnowflaking exponent for an entire family of connected graphs. Writing $d_p = d_{p,G}$ to show the dependence on the graph G, we define

$$s(p) := \inf_{G\in\mathcal{G}} ASFE(d_{p,G}) \tag{4.2.1}$$

where the infimum is taken over all simple connected graphs $\mathcal{G}$.

Note that if we find a connected graph G, an exponent $t \geq 1$, and three nodes a, b, c, such that the triangle inequality for d_p^t fails for this triple, then we are guaranteed that $s(p) \leq t$. In particular, by looking at the path graph P_3 on three nodes (Figure 4.1) we can establish the following bound.

Proposition 4.2.4. *For $1 < p < \infty$:*

$$s(p) \le \frac{p}{p-1} =: q \tag{4.2.2}$$

where q is the Hölder exponent associated with p.

Moreover, the bound is attained for $p = 1, 2, \infty$:

$$s(1) = \infty, \qquad s(2) = 2, \qquad s(\infty) = 1.$$

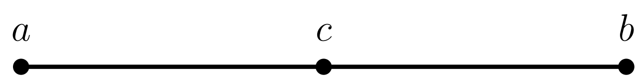

Figure 4.1: The path graph P_3 on three nodes.

Proof. Consider the path graph P_3 with nodes a, c, b and fix $p \in (1, \infty)$. It is clear that $d_p(a, c) = 1$, because to be admissible a density ρ must satisfy $\rho(a, c) = 1$, and then in order to minimize the energy, we must also have $\rho(c, b) = 0$. Likewise, $d_p(c, b) = 1$. For $d_p(a, b)$, the energy is minimized when $\rho(a, c) = \rho(c, b) = 1/2$. Thus,

$$\mathrm{Mod}_p(a, b) = (1/2)^p + (1/2)^p = 2^{1-p}$$

Hence, $d_p(a, b) = 2^{(p-1)/p} = 2^{(1-1/p)}$. The triangle inequality will fail for $t \ge 1$ such that

$$d_p(a, b)^t > d_p(a, c)^t + d_p(c, b)^t \tag{4.2.3}$$

that is,

$$2^{t(1-1/p)} > 1 + 1 = 2$$

This happens whenever $t > 1/(1 - 1/p)$. So $s(p) \le p/(p - 1)$.

The bound is attained for the case $p = 1$ because, as shown in Theorem 4.1.3, d_1 is an ultrametric on any connected graph, so $s(1) = \infty$. When $p = 2$, the metric d_2^2 is effective resistance R_{eff}, which is also a metric on connected graphs. Therefore, $s(2) \geq 2$, attaining the upper bound. For the case $p = \infty$, $d_\infty(a, c) = d_\infty(c, b) = 1$, while $d_\infty(a, b) = 2$, yielding a flat triangle. Thus, $s(\infty) = 1$. □

In fact, the numerical evidence presented in Section 4.3 will further strengthen the above proposition. We will later prove that $s(p)$ in fact attains the bound $p/(p-1)$.

4.3 Examples and numerical Results

4.3.1 Erdős-Rényi graphs

As an attempt to numerically test Proposition 4.2.4, we produced 50 Erdős-Rényi graphs on 10 nodes, with expected average degree 6 (discarding any disconnected graphs that were generated). For each graph G_i, $i = 1, 2, \ldots, 50$, we computed $d_{p,G_i}(1, 2), d_{p,G_i}(2, 3)$, and $d_{p,G_i}(1, 3)$ for a range of p values and determined the value $t_{p,i}$ such that

$$d_{p,G_i}^{t_{p,i}}(1, 2) = d_{p,G_i}^{t_{p,i}}(1, 3) + d_{p,G_i}^{t_{p,i}}(2, 3).$$

We then estimated $s(p)$ as

$$s(p) \leq t(p) = \min_{i=1,2,\ldots,50} t_{p,i}.$$

The resulting bound is shown in Figure 4.2 in blue. The red line in the same figure is the antisnowflaking exponent $s(p) = \frac{p}{p-1}$.

Observe that the blue line never goes below the red line. In other words, the worst case scenario seems to be $s(p) = \frac{p}{p-1}$.

In the following we compute some specific examples. When calculating modulus, we will often just write down the extremal metric. For simple examples, verifying that a metric ρ is extremal for p-modulus can be done using Beurling's criterion that was discussed in

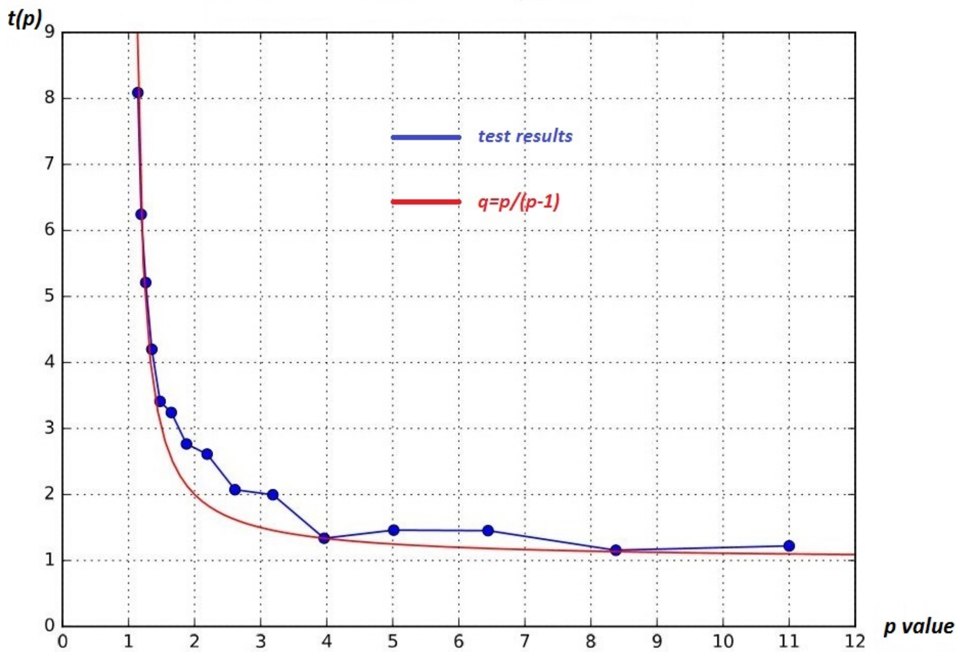

Figure 4.2: Antisnowflaking exponent for different p values.

Theorem 1.2.5 in Chapter 1.

4.3.2 Biconnected graphs

Next, we explore the anti-snowflaking exponent for more restrictive families of graphs. If $\mathcal{G}$ is a family of connected graphs, define $s_{\mathcal{G}}(p) := \inf_{G\in\mathcal{G}} ASFE(d_{p,G})$. Clearly $s(p) \le s_{\mathcal{G}}(p)$. Recall that $s(p)$ is an infimum over all connected graphs. What happens if we restrict the infimum to biconnected graphs?

Definition 4.3.1. A *biconnected graph* is a graph that remains connected after removing any node.

Let $\mathcal{B}$ be the family of all biconnected simple graphs. Define $s_{\mathcal{B}}(p) := \inf_{G\in\mathcal{B}} ASFE(d_{p,G})$.

Claim 4.3.2.

$$s_{\mathcal{B}}(p) \le p/(p-1).$$

From Proposition 4.2.4 equation (4.2.2), we see that the family of path graphs $\mathcal{P}$ satisfies $s(p) \le s_{\mathcal{P}}(p)$ for all $p's$. However, path graphs are not biconnected. So it is natural to wonder if $s_{\mathcal{B}}(p) < s_{\mathcal{P}}(p)$. We explore this question by looking at the simplest example of a biconnected graph, namely the cycle graph C_N. For distinct nodes a, b and c, as in Figure 4.3, consider the connecting families $\Gamma(a,b), \Gamma(a,c), \Gamma(c,b)$.

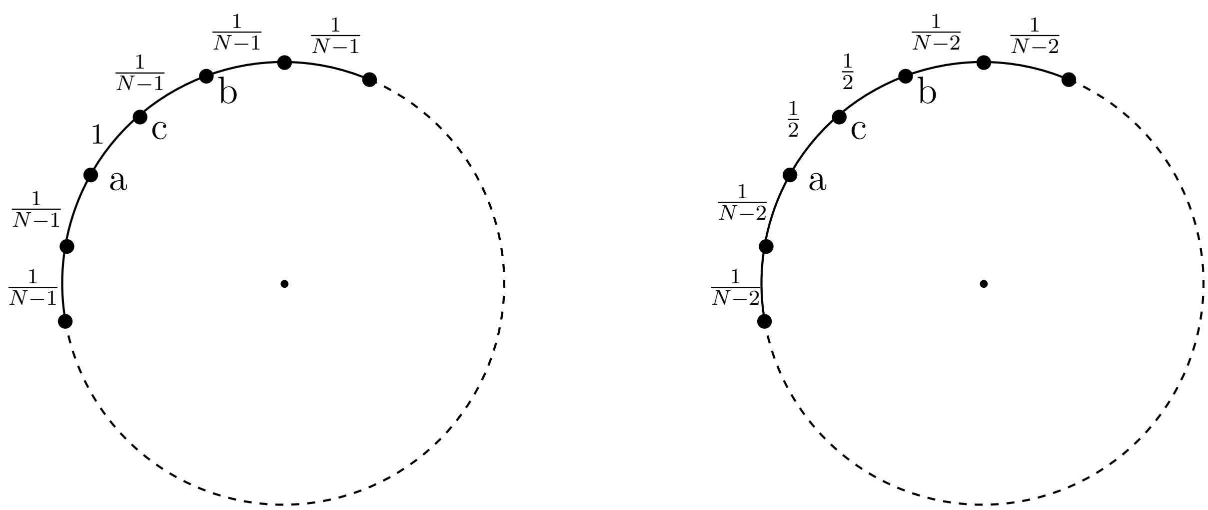

Figure 4.3: The cycle graph C_N and the extremal density ρ^* for $\Gamma(a,c)$ and $\Gamma(a,b)$.

The first diagram in Figure 4.3 shows the extremal density ρ^* for the family $\Gamma(a,c)$. The subfamily $\tilde{\Gamma}$ that verifies Beurling's criterion in this case has two simple paths, namely $a\ c$ and the path from a to c that traverses the cycle in the other direction (the long way). This gives

$$\operatorname{Mod}_p(\Gamma(a,c)) = 1 + \left(\frac{1}{N-1}\right)^p (N-1) = 1 + (N-1)^{1-p}. \tag{4.3.1}$$

By symmetry, $\operatorname{Mod}_p(\Gamma(c,b)) = 1 + (N-1)^{1-p}$ as well. We conclude that

$$d_p(a,c) = d_p(c,b) = (1 + (N-1)^{1-p})^{-1/p}.$$

The second diagram in Figure 4.3 shows the extremal density ρ^* for the family $\Gamma(a,b)$. The subfamily $\tilde{\Gamma}$ that verifies Beurling's criterion in this case has two simple paths, namely $a\ c\ b$ and the longer path from a to b in the other direction. As a consequence,

$$\operatorname{Mod}_p(\Gamma(a,b)) = 2\left(\frac{1}{2}\right)^p + (N-2)\left(\frac{1}{N-2}\right)^p = 2^{1-p} + (N-2)^{1-p}. \tag{4.3.2}$$

Therefore, $d_p(a,b) = (2^{1-p} + (N-2)^{1-p})^{-1/p}$. We will calculate the infimum of all the exponents $t \geq 1$ for which the following triangle inequality fails:

$$d_p(a,b)^t > d_p(a,c)^t + d_p(c,b)^t.$$

We get

$$\left(2^{1-p} + (N-2)^{1-p}\right)^{-t/p} > 2(1 + (N-1)^{1-p})^{-t/p},$$

hence

$$2^{p/t} < \frac{1 + (N-1)^{1-p}}{2^{1-p} + (N-2)^{1-p}}.$$

So the infimal exponent is

$$t_0 := p \left[\log_2 \left(\frac{1 + (N-1)^{1-p}}{2^{1-p} + (N-2)^{1-p}} \right) \right]^{-1}.$$

We see that as $N \to \infty$, $t_0 \to \frac{p}{-(1-p)} = \frac{p}{(p-1)} = q$. Therefore, we see that $s_{\mathcal{B}}(p) \leq p/(p-1)$.

4.3.3 Complete graphs

The *complete graph* K_N is a simple graph on N nodes, where every node is connected to each other, see Figure 4.4.

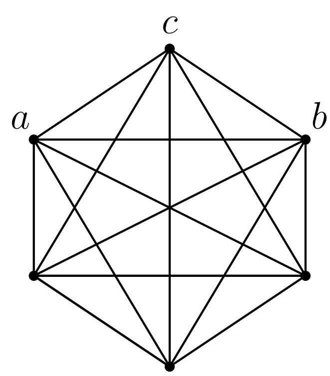

Figure 4.4: K_6- Complete graph on 6 nodes.

Observe that, by symmetry, $\mathrm{Mod}_p(\Gamma(a,c)) = \mathrm{Mod}_p(\Gamma(c,b) = \mathrm{Mod}_p(\Gamma(a,b))$, hence $d_p(a,c) = d_p(c,b) = d_p(a,b)$. Therefore, d_p is an ultrametric on complete graphs. In particular, if $\mathcal{K}$ is the family of complete graphs, then $s_{\mathcal{K}}(p) = \infty$ for all p.

It's still interesting to compute $d_p(a,b)$ for an arbitrary pair of nodes. Figure 5.6 depicts the extremal density ρ^* for $\Gamma(a,b)$ in K_N.

In formulas, $\rho^*(a,x) = 1/2 = \rho^*(b,x)$ for every $x \neq a, b$, and $\rho^*(a,b) = 1$, otherwise ρ^* is zero. To verify Beurling's criterion, consider the subfamily $\tilde{\Gamma}$ of simple paths consisting of

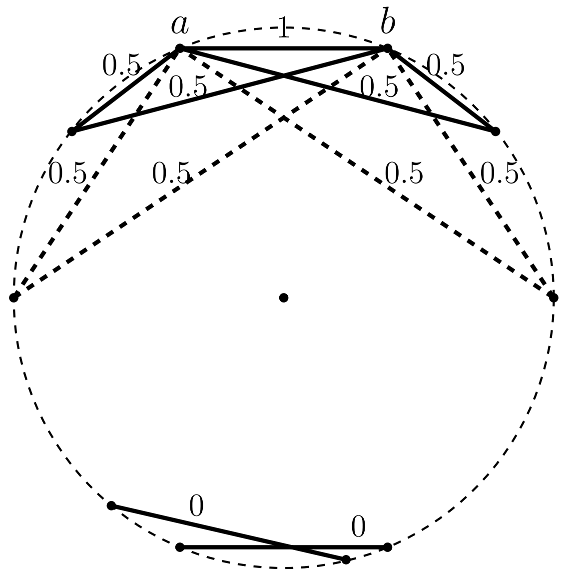

Figure 4.5: The complete graph K_N and the extremal density ρ^* for $\Gamma(a,b)$.

$a\ b$ and $a\ x\ b$ for any $x \neq a, b$. We get that

$$\mathrm{Mod}_p(\Gamma(a,b)) = 1 + 2(N-2)\frac{1}{2^p} \qquad \text{and} \qquad d_p(a,b) = \left(1 + \frac{N-2}{2^{p-1}}\right)^{-1/p}.$$

Since $s(p) \leq s_{\mathcal{B}}(p) \leq p/(p-1)$ while $s_{\mathcal{K}}(p) = \infty$, what are some natural families of graphs for which $p/(p-1) < s_{\mathcal{G}}(p) < \infty$? For instance, what happens for the family of all hypercubes? Recall that for an integer $N \geq 2$, the *hypercube* H_N is the graph whose nodes are strings of 0 and 1 of length N and two such strings are connected by an edge if they differ in exactly one position. This a question that we would like to answer in the future.

4.3.4 Graph visualization

Metrics on networks play a vital role in applications as well as in the study of intrinsic network characteristics. For instance, there are infinitely many ways to draw a network in two- or three-dimensional space. However, some choices of node layout are clearly better than others for providing a meaningful visualization of the network. Take a cycle graph on 5 nodes, for example. Drawing a regular pentagon provides a much better representation of this graph than does placing the 5 nodes randomly in the plane. To relate this to metrics, one need only observe that any time we draw a graph in the plane, its node set inherits the Euclidean metric of the plane. In this sense, different drawings of the same graph G represent different choices of metric on the vertices V and it thus seems natural that the choice of layout should be closely related to the network structure. For a beautiful example of deriving a network's layout from its intrinsic structure, see Section 2.2 of [21]. Here, we briefly discuss the relationship between graph visualization and the d_p metric.

A mapping $f : (X, d_X) \to (Y, d_Y)$ of one metric space into another is called an *isometric embedding* or *isometry* if $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. Two metric spaces are isometric if there exists a bijective isometry between them.

Theorem 4.3.3 (Shoenberg, 1935)**.** *Given a finite metric space $X = \{x_0, ..., x_m\}$ and an integer $n \in \mathbb{N}$, X embeds isometrically into $\mathbb{R}^n$ if and only if the matrix $M \in \mathbb{R}^{m\times m}$ whose entries are*

$$d(x_i, x_0)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2$$

is positive semi-definite and of rank less than or equal to n.

For convenience, we will call the matrix M the "Shoenberg matrix". As an example, consider the square in Figure 5.6. We fix the node a and derive the Shoenberg matrix so we can analyze the embeddings of this graph when the nodes are endowed with modulus metrics of the form d_p^t for some $t > 0$. By symmetry, these metrics have the property that the distance between two neighboring nodes of the square is a constant $\alpha > 0$ and the distance between diagonally opposite nodes is some other constant $\beta > 0$.

Let the columns and rows of the matrix M represent nodes b, c, d respectively. The entry M_{11} represents (b, b) and is calculated as follows:

$$d(\mathbf{a}, b)^2 + d(\mathbf{a}, b)^2 - d(b, b)^2 = 2\alpha^2.$$

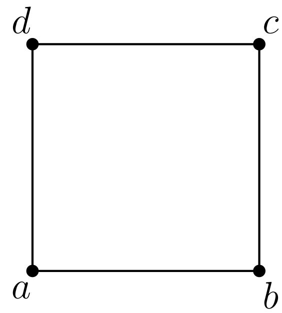

Figure 4.6: The square graph

Note that this will be the case for (d, d) as well. On the other hand, for (c, c):

$$d(\mathbf{a}, c)^2 + d(\mathbf{a}, c)^2 - d(c, c)^2 = 2\beta^2$$

The entry M_{12} represents (b, c) and is calculated as follows:

$$d(\mathbf{a}, b)^2 + d(\mathbf{a}, c)^2 - d(b, c)^2 = \beta^2$$

Likewise for (c, d) we get β^2 again. For (b, d) we have

$$d(\mathbf{a}, b)^2 + d(\mathbf{a}, d)^2 - d(b, d)^2 = 2\alpha^2 - \beta^2$$

Putting the above information together, we can derive a Shoenberg matrix M for these type of metrics on the square.

$$M = \begin{bmatrix} 2\alpha^2 & \beta^2 & 2\alpha^2 - \beta^2 \\ \beta^2 & 2\beta^2 & \beta^2 \\ 2\alpha^2 - \beta^2 & \beta^2 & 2\alpha^2 \end{bmatrix}$$

Note that for the triangle inequality to hold, we also want the condition,

$$\beta \leq 2\alpha.$$

Without loss of generality, we can normalize the edge distance to be 1, setting $\alpha = 1$, and then plot how the eigenvalues of M change with β. In Figure 4.7, we have plotted the eigenvalues of M as the normalized parameter which we still call β varies from 0 to 2. We

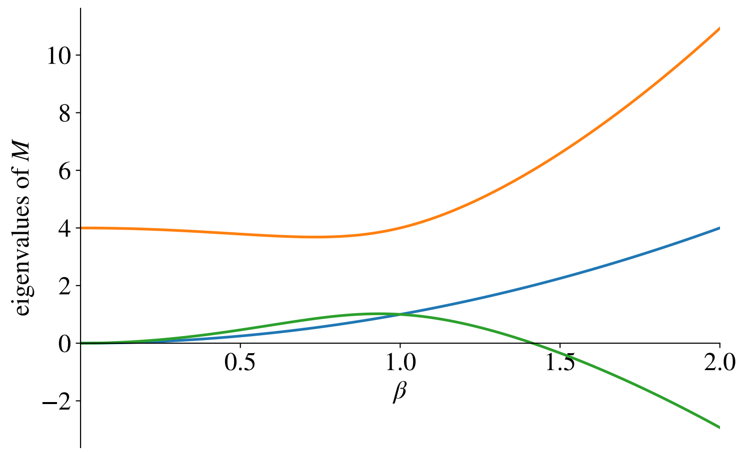

Figure 4.7: Eigenvalues of M as β varies, given $\alpha = 1$.

observe that when $\beta > \sqrt{2} \approx 1.4$, the matrix M starts having negative eigenvalues and thus fails to be positive semi-definite. For the β in $(0, \sqrt{2})$ the square is embeddable in $\mathbb{R}^3$ and for $\beta = \sqrt{2}$ it is embeddable in $\mathbb{R}^2$. We describe these embeddings by fixing one edge of the square at $(1/2, 0, 0)$ and $(-1/2, 0, 0)$, while the opposite edge is horizontal at some height $h > 0$ and by symmetry is also centered on the vertical axis. Since all four edges have unit length, the top horizontal edge must twist about the vertical axis by an an angle $\theta \geq 0$. The relationship between h and θ is governed by the parameter β, the distance between diagonally opposite nodes. A simple calculation shows that

$$h = \cos\frac{\theta}{2} \qquad \text{and} \qquad \beta = \sqrt{1 + \cos\theta}.$$

When $\theta = 0$, $\beta = \sqrt{2}$ and the square is in the xz-plane. But for $\theta \in (0, \pi)$, the embedding is three-dimensional and h tends to 0 as θ tends to π. When $\theta = \pi$, $\beta = 0$ and d stops being a metric.

To connect this to our modulus metrics, note that the square is a special case of the cycle graphs we discussed in Section 4.3.2, when $N = 4$. So by (4.3.1) and (4.3.2) applied to the edge (a, b) and the diagonal (a, c) in this case, we see that

$$d_p(a, b) = (1 + 3^{1-p})^{-1/p} \qquad \text{and} \qquad d_p(a, c) = 2^{1-2/p}.$$

Therefore, the ratio β/α for the case of the d_p metrics is

$$f(p) := 2^{1-2/p}(1 + 3^{1-p})^{1/p}.$$

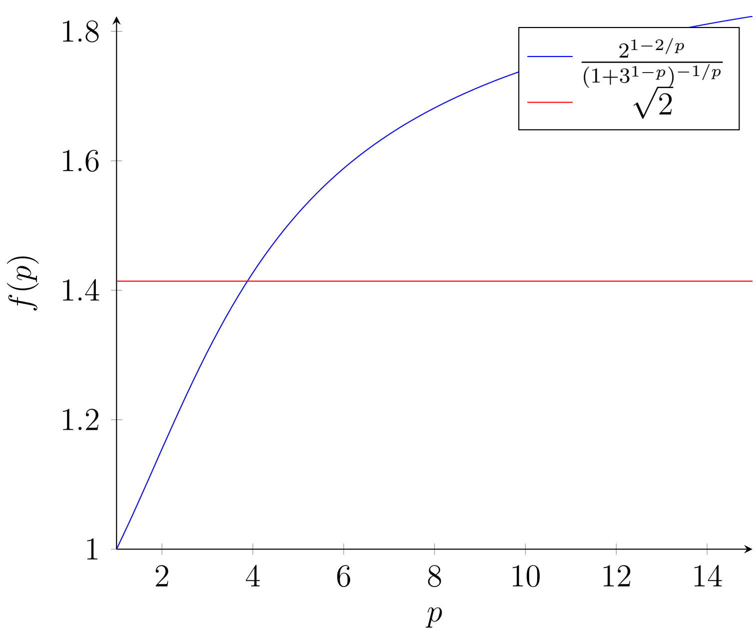

Figure 4.8: Graph for the ratio β/α against p

As seen in Figure 4.8 f is increasing, and as p decreases to 1 the ratio β/α decreases to $f(1) = 1$. Hence, the metric d_p on the square graph is embeddable in $\mathbb{R}^3$ for $1 \le p \le p_0$, for some value $p_0 \approx 3.88$.

4.4 The δ_p metric

We end this Chapter with our main theorem that shows the d_p metric is a snowflaked version of the δ_p metric defined below.

Definition 4.4.1. Let $G = (V, E, \sigma)$ be a weighted, connected, simple graph. Given $a, b \in V$, let $\Gamma(a, b)$ be the connecting family of all paths between a and b. Fix $1 < p < \infty$ and let $q := p/(p-1)$ be the Hölder conjugate exponent. Then we define

$$\delta_p(a,b) := \begin{cases} 0 & \text{if } a = b, \\ \mathrm{Mod}_{p,\sigma}(\Gamma(a,b))^{-q/p} & \text{if } a \neq b. \end{cases}$$

Theorem 4.4.2. *Suppose $G = (V, E, \sigma)$ is a weighted, connected, simple graph. Then δ_p is a metric on V. Moreover,*

(a) $\lim_{p\uparrow\infty} \delta_p = d_{\mathrm{SP}}$*;*

(b) $\delta_2 = d_{\mathrm{ER}}$*;*

(c) For $1 < p < 2$, $\mathrm{Mod}_{p,\sigma}(\Gamma(a,b))^{-1}$ is a metric and it tends to $d_{\mathrm{MC}}(a,b)$ as $p \to 1$.

Finally, for every $\epsilon > 0$ and every $p \in [1, \infty]$ there is a connected graph for which $\delta_p^{1+\epsilon}$ is not a metric.

Remark 4.4.1. Note that, in light of Theorem 1.2.4, when $p = 2$, the proof of Theorem 4.4.2 gives an alternative modulus-based proof that effective resistance is a metric.

Remark 4.4.2. It is straightforward to show that an arbitrary positive power of an ultrametric is also an ultrametric, so $(d_{\mathrm{MC}})^t$ is a metric for any $t > 0$. Using (1.2.4) and (1.2.5) it can be shown that as $p \downarrow 1$, δ_p converges to the limit

$$\lim_{t\to\infty}(d_{\mathrm{MC}}(a,b))^t = \begin{cases} 0 & \text{if } d_{\mathrm{MC}}(a,b) > 1, \\ 1 & \text{if } d_{\mathrm{MC}}(a,b) = 1, \\ \infty & \text{if } d_{\mathrm{MC}}(a,b) < 1. \end{cases}$$

For unweighted graphs, this limit essentially decomposes the graph into its 2-edge-connected components. All nodes in the same component are distance zero from one another while nodes in different components are at distance one.

Proof. Assuming the claim that δ_p is a metric, the 'Moreover' parts (a) and (b) follow from Theorem 1.2.4. For (c), recall that a metric d can always be raised to an exponent $0 < \epsilon < 1$ and still remain a metric. Since for $1 < p < 2$, we have $p/q < 1$, it follows that $\mathrm{Mod}_{p,\sigma}(\Gamma(a,b))^{-1} = \delta_p^{p/q}$ is a metric, and the claim follows from continuity in p.

Finally, the fact that the exponent 1 is sharp for the metrics δ_p can be shown as follows. Consider the (unweighted) path graph P_3 with edges $\{a,c\},\{c,b\}$ and fix $p \in (1,\infty)$. First $\mathrm{Mod}_p(\Gamma(a,c)) = 1$, because any admissible density ρ must satisfy $\rho(a,c) = 1$, furthermore, to minimize the energy, we also set $\rho(c,b) = 0$. Likewise, $\mathrm{Mod}_p(\Gamma(c,b)) = 1$. For $\mathrm{Mod}_p(\Gamma(a,b))$, the energy is minimized when $\rho(a,c) = \rho(c,b) = 1/2$. Thus,

$$\mathrm{Mod}_p(\Gamma(a,b)) = (1/2)^p + (1/2)^p = 2^{1-p}$$

Hence, $\delta_p(a,b) = 2^{q(p-1)/p} = 2 = 1+1 = \delta_p(a,c) + \delta_p(c,b)$. In particular, the triangle inequality will fail for δ_p^t as soon as $t > 1$.

The proof of the main claim hinges on the dual formulation in terms of Fulkerson blocker duality. Fix $p \in (1,\infty)$. Recall from Section 2.2.3, that the Fulkerson blocker family for

$\Gamma(a,b)$ is the family of all minimal ab-cuts $\hat{\Gamma}(a,b)$. By Theorem 2.1.8,

$$\mathrm{Mod}_{p,\sigma}(\Gamma(a,b))^{-q/p} = \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(a,b)),$$

where $q := p/(p-1)$ is the Hölder conjugate exponent of p and $\hat{\sigma} = \sigma^{-q/p}$.

An important observation at this point is that the family $\Gamma_{\mathrm{cut}}(a,b)$ of all the ab-cuts is subordinated to $\hat{\Gamma}(a,b)$, since every ab-cut contains a minimal ab-cut, see Remark 1.2.2.

Now suppose $a,b,c \in V$ are distinct. Then, for every ab-cut $S \in \hat{\Gamma}(a,b)$, we have the following mutually exclusive cases: either $c \in S$ or $c \notin S$. Therefore,

$$\hat{\Gamma}(a,b) \subset \Gamma_{\mathrm{cut}}(a,c) \cup \Gamma_{\mathrm{cut}}(c,b). \tag{4.4.1}$$

The triangle inequality then follows from monotonicity and subadditivity of modulus that was discussed in Chapter 1, Preposition 1.2.3.

$$\begin{aligned}
\delta_p(a,b) &= \mathrm{Mod}_{p,\sigma}(\Gamma(a,b))^{-q/p} && \text{(Definition)}\\
&= \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(a,b)) && \text{(Fulkerson duality)}\\
&\leq \mathrm{Mod}_{q,\hat{\sigma}}(\Gamma_{\mathrm{cut}}(a,c) \cup \Gamma_{\mathrm{cut}}(c,b)) && \text{(by (4.4.1) and Monotonicity)}\\
&\leq \mathrm{Mod}_{q,\hat{\sigma}}(\Gamma_{\mathrm{cut}}(a,c)) + \mathrm{Mod}_{q,\hat{\sigma}}(\Gamma_{\mathrm{cut}}(c,b)) && \text{(Subadditivity)}\\
&\leq \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(a,c)) + \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(c,b)) && \text{(Subordination)}\\
&= \delta_p(a,c) + \delta_p(c,b). && \text{(Fulkerson duality)}
\end{aligned}$$

$\square$

Chapter 5

Clutters, blockers and the Fulkerson dual

In this section we translate some of the terms encountered when studying clutters to our language of modulus and Fulkerson duality. We mainly refer to the article by Gérard, Cornuéjols [22].

5.1 Clutters

A **clutter** $\mathcal{C}$ is a pair $(\Omega, \mathcal{P})$ where the **ground set** Ω is finite, $\mathcal{P} \subset 2^{\Omega}$ and $P_i \not\subset P_j$ for $i \neq j$. In a clutter, a **matching** is a set of pairwise disjoint $\{P_1, \ldots, P_k\} \subset \mathcal{P}$. Note that this is not the same as perfect matchings. A **transversal** is a set of points in Ω that intersects all $P \in \mathcal{P}$.

Definition 5.1.1. A clutter is said to **pack** if the maximum cardinality of a matching equals the minimum cardinality of a transversal.

Example 5.1.1. let s and t be distinct nodes of a graph G. Menger's theorem states that the maximum number of pairwise edge-disjoint st-paths in G equals the minimum number of edges in an st-cut. Let $\mathcal{C}$ be the clutter with $\Omega = E(G)$, the edge-set of G; and $\mathcal{P} = \Gamma_{\text{simple}}(s,t)$. Some texts call this, the **clutter of st-paths** of G. The transversals of $\mathcal{C}$ are all the st-cuts. Therefore, Menger's theorem states that the clutter of st-paths packs. Similarly, we can define the clutter of st-cuts $\hat{\mathcal{C}}$. Its transversals are all connecting st-subgraphs, and $\hat{\mathcal{C}}$ packs, namely, the maximum number of pairwise disjoint st-cuts equals the length of the shortest st-path.

We are interested in the special case when the ground set $\Omega = E(G)$ is the edge-set of a graph G. Therefore, we will use Γ to denote a clutter. In this case, the objects are subsets of edges and they are determined by their 0/1-**usage** vector:

$$\mathcal{N}(\gamma, e) = \mathbb{1}_{\gamma}(e) \tag{5.1.1}$$

5.2 1-modulus

Recall that given a weight $\sigma : E \to (0, \infty)$ on the edges of G, 1-modulus, $\text{Mod}_{1,\sigma}(\Gamma)$, is the value of the linear program:

$$\begin{aligned} &\text{minimize} && \sigma^T \rho \\ &\text{subject to} && \rho \geq 0, \quad \mathcal{N}\rho \geq \mathbf{1} \end{aligned} \tag{5.2.1}$$

Since this is a feasible linear program, strong duality holds, and the dual problem is

$$\begin{aligned} &\text{maximize} && \lambda^T \mathbf{1} \\ &\text{subject to} && \lambda \geq 0, \quad \mathcal{N}^T \lambda \leq \sigma. \end{aligned} \tag{5.2.2}$$

Recall that we think of (5.2.2) as a (generalized) max-flow problem, given the weights (capacities) σ. That's because the condition $\mathcal{N}^T\lambda \leq \sigma$ says that for every $e \in E$

$$\sum_{\gamma\in\Gamma} \lambda(\gamma)\mathcal{N}(\gamma, e) = \sum_{\substack{\gamma\in\Gamma \\ e\in\gamma}} \lambda(\gamma) \leq \sigma(e). \tag{5.2.3}$$

However, to think of (5.2.1) as a (generalized) min-cut problem, we would need to be able to restrict the densities ρ to some given subsets of E. This is not always possible.

Proposition 5.2.1. *Let Γ be a clutter, namely, a family of subsets of E such that $\gamma_i \not\subset \gamma_j$ for $i \neq j$. Then Γ is a clutter that "packs" if and only if there are optimal ρ^* and λ^* for the unweighted, $\sigma \equiv 1$, problems (5.2.1) and (5.2.2) respectively, such that*

$$\rho^* \in \mathbb{Z}^E_{\geq 0} \quad \textit{and} \quad \lambda^* \in \mathbb{Z}^\Gamma_{\geq 0}. \tag{5.2.4}$$

Proof of Proposition 5.2.1. Assuming that Γ is packing as in Definition 5.1.1, let $\tilde{\Gamma} := \{\gamma_1, \ldots, \gamma_k\}$ be a maximal set of pairwise disjoint objects, and let $S = \{e_1, \ldots, e_j\} \subset E$ be a minimum cardinality set such that for every $\gamma \in \Gamma$,

$$|\gamma \cap S| \geq 1.$$

Let $\rho := \mathbb{1}_S \in \mathbb{R}^E_{\geq 0}$ and let $\lambda = \mathbb{1}_{\tilde{\Gamma}} \in \mathbb{R}^\Gamma_{\geq 0}$. Then:

- λ is a feasible flow for the max-flow problem (5.2.2), since

$$\begin{aligned}
(\mathcal{N}^T\lambda)(e) &= \sum_{i=1}^{k} \mathcal{N}(\gamma_i, e)\lambda(\gamma_i) \\
&= \mathcal{N}(\gamma_j, e)\lambda(\gamma_j) && \text{(for some } j\text{, since } \gamma_i\text{'s are mutually disjoint)} \\
&\leq 1 = \sigma(e) && \text{(by (5.1.1), and because } \lambda(\gamma_j) = 1 \text{ and } \sigma \equiv 1\text{)}
\end{aligned}$$

- ρ is an admissible density for the 1-modulus problem (5.2.1), since for every $\gamma \in \Gamma$:

$$\begin{aligned}
\ell_\rho(\gamma) &= \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) \\
&= \sum_{e \in E} \mathcal{N}(\gamma, e)\mathbb{1}_S(e) \\
&= |\gamma \cap S| \geq 1.
\end{aligned}$$

Evaluating the primal and dual objectives in this case we get:

$$\sigma^T \rho = |S| \qquad \text{and} \qquad \mathbf{1}^T \lambda = |\tilde{\Gamma}|$$

Since Γ is packing, we have $|S| = |\tilde{\Gamma}|$, and by convex duality, λ and ρ must be optimal. Therefore, in this case we get that

$$\rho \in \{0, 1\}^E \quad \text{and} \quad \lambda \in \{0, 1\}^\Gamma.$$

Conversely, assume (5.2.4) holds. Then, since $\sigma \equiv 1$, feasibility says that, for every $e \in E$:

$$\sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e)\lambda^*(\gamma) \leq 1.$$

Then, by integrality, this sum is either 0 or 1, and in the latter case,

$$1 = \sum_{\gamma : \lambda^*(\gamma) > 0} \mathcal{N}(\gamma, e)\lambda^*(\gamma) \geq \sum_{\gamma : \lambda^*(\gamma) \geq 1} \mathcal{N}(\gamma, e) = |\{\gamma \in \Gamma : \lambda^*(\gamma) \geq 1, e \in \gamma\}| \tag{5.2.5}$$

So at most one γ in the support of λ^* may contain a given edge e, which implies that the two sums above range over exactly one object γ:

$$1 = \mathcal{N}(\gamma, e)\lambda^*(\gamma) \geq \mathcal{N}(\gamma, e) = 1,$$

and therefore, when this occurs, $\lambda^*(\gamma) = 1$.

Let $\tilde{\Gamma} = \{\gamma_1, \ldots, \gamma_m\} \subset \Gamma$, be the set of objects with $\lambda^*(\gamma) = 1$. Then by (5.2.5), the objects in $\tilde{\Gamma}$ must be mutually disjoint. So $\tilde{\Gamma}$ is a matching.

Moreover, since ρ^* is an admissible density, for every $i = 1, \ldots, m$,

$$\sum_{e \in E} \mathcal{N}(\gamma_i, e)\rho^*(e) \geq 1.$$

This implies that there is at least one edge $e_i \in E$ such that

$$\mathcal{N}(\gamma_i, e_i)\rho^*(e_i) > 0,$$

meaning that $e_i \in \gamma_i$ and $\rho^*(e_i) \geq 1$, since $\rho^* \in \mathbb{Z}^E_{\geq 0}$.

By strong duality and the fact that the objects in $\tilde{\Gamma}$ are pairwise disjoint,

$$|\tilde{\Gamma}| = \sum_{i=1}^{m} \lambda^*(\gamma_i) = \sum_{e \in E} \rho^*(e) \geq \sum_{i=1}^{m} \rho^*(e_i) \geq |\tilde{\Gamma}|.$$

Hence, the last inequality must be an equality and

$$\rho^*(e_i) = 1 \qquad \forall i = 1, \ldots, m.$$

Also the first inequality must be an equality, and this forces

$$\rho^*(e) = 0 \qquad \forall e \in E \setminus \{e_i\}_{i=1}^m.$$

Define $S = \{e_i\}_{i=1}^m$. Since ρ^* is admissible for Γ, for every $\gamma \in \Gamma$:

$$\sum_{e \in E} \mathcal{N}(\gamma, e)\rho^*(e) = \sum_{i=1}^{m} \mathcal{N}(\gamma, e_i) \geq 1.$$

So there must be at least an edge $e_j \in S$, for which $\mathcal{N}(\gamma, e_j) = 1$. This implies that S is a transversal.

In conclusion, since $|\tilde{\Gamma}| = |S| = m$, it must be the case that $\tilde{\Gamma}$ is a maximal matching and S is a minimal transversal. To see why, note that the cardinality of a transversal is always greater than the cardinality of a matching. Indeed, suppose $\mathcal{M} = (P_1, \ldots, P_k) \subset \mathcal{P} \subset 2^{\Omega}$ is a matching and suppose S is a transversal. By definition, S intersects every $P \in \mathcal{P}$. Therefore, $|S \cap P_j| \neq \emptyset$, for all $j = 1, \ldots, k$. In particular, there exist $e_j \in S \cap P_j$ for each $j = 1, \ldots, k$. Since the P_j are mutually disjoint, the edges e_j are also mutually disjoint. Thus $|S| \geq k = |\mathcal{M}|$. □

Corollary 5.2.2. *If the clutter Γ packs and ρ^* is unique, then for the unweighted, $\sigma \equiv 1$ case we have that ρ^* is 0,1 valued.*

Proof. The fact that the edge usage is integral follows from the Proposition 5.2.1. So it is left to show that this is 0, 1 valued.

Let's assume otherwise. That the unique ρ^* is integer valued but not just 0,1. Now define a new density ρ as follows.

$$\rho(e) = \begin{cases} 0 & \text{if } \rho^*(e) > 0, \\ 1 & \text{if } \rho^*(e) = 0, \end{cases}$$

Now observe that ρ is admissible for Γ since $\mathcal{N}$, the usage matrix is 0,1 valued. Also comparing the energies of the two densities we have that,

$$\sum_{e \in E} \rho^p < \sum_{e \in E} \rho^{*p}$$

which is a contradiction to the fact that ρ^* is the extremal point. □

Definition 5.2.3. Let Γ be a clutter, namely, a family of subsets of E such that $\gamma_i \not\subset \gamma_j$ for $i \neq j$. We say:

- Γ has the **packing property** if and only if, for all weights $\sigma \in \{0, 1, \infty\}^E$, there are optimal $\rho^* \in \mathbb{Z}_{\geq 0}^E$ and $\lambda^* \in \mathbb{Z}_{\geq 0}^\Gamma$ for the problems (5.2.1) and (5.2.2) respectively.

- Γ has the **Max Flow Min Cut** property (MFMC) if and only if, for all weights $\sigma \in \mathbb{Z}_{\geq 0}^E$, there are optimal $\rho^* \in \mathbb{Z}_{\geq 0}^E$ and $\lambda^* \in \mathbb{Z}_{\geq 0}^\Gamma$ for the problems (5.2.1) and (5.2.2) respectively.

- Γ is **ideal** if for every $\sigma \in \mathbb{R}_{\geq 0}^E$, there is an optimal $\rho^* \in \mathbb{Z}_{\geq 0}^E$ for (5.2.1).

Definition 5.2.4 (semi-ideal and semi-MFMC property). Let k be a positive integer. Clutter Γ has the **1/k -MFMC property** if it is ideal and, for all nonnegative integral vectors w, the linear program 5.2.2 has an optimal solution vector y such that ky is integral. If the clutter has the 1/k -MFMC property for some k, we say that it has semi-MFMC property.

When $k = 1$, this definition reduces to the MFMC property. If Γ has the 1/k-MFMC property, then it also has the **1/q-MFMC property** for every integer q that is a multiple of k.

The definition of the semi-idealness follows the proposition 5.2.5. Let k be a positive integer. Clutter Γ has the **1/k -ideal property** if and only if, $k\hat{\Gamma}$ is integral. If the clutter is 1/k -ideal for some k, we say that it has semi-ideal property.

Proposition 5.2.5. *Let Γ be a family of subsets of E such that $\gamma_i \not\subset \gamma_j$ for $i \neq j$. Then, Γ is ideal, if and only if, $\hat{\Gamma} \subset \mathbb{Z}_{\geq 0}^E$. In words, if and only if, the Fulkerson dual has 0/1-valued edge-usages.*

Proof. Assume first that Γ is ideal. Each $\hat{\gamma} \in \hat{\Gamma}$ is an extreme point of $\mathrm{Adm}(\Gamma)$. Hence, there is a set of weights $\sigma \in \mathbb{R}_{\geq 0}^E$, such that $\rho^*(\cdot) := \mathcal{N}(\hat{\gamma}, \cdot)$ is the unique optimal solution to (5.2.1). Therefore, $\hat{\gamma}$ must have integer-valued edge-usages.

Conversely, for any $\sigma \in \mathbb{R}_{\geq 0}^E$, the set of optimal solutions for (5.2.1) always includes at least one extreme point of $\mathrm{Adm}(\Gamma)$, which are now assumed to be integer-valued. So Γ is ideal.

The fact that these integer values are 0/1 follows from the fact that the extremal points are 0/1 in order to minimize the energy. In other words, it does not makes sense for the integer values to take anything greater than 1. See proof of Corollary 5.2.2 for an explanation. $\square$

Remark 5.2.1. We have the following implications.

- MFMC implies the packing property, but the converse is (still?) open.
- Also MFMC implies ideal, because if $\mathrm{Mod}_{1,\sigma}(\Gamma)$ admits an extremal density $\rho^* \in \mathbb{Z}^E_{\geq 0}$ whenever $\sigma \in \mathbb{Z}^E_{\geq 0}$, then by taking common denominators in the energy, the same holds whenever $\sigma \in \mathbb{Q}^E_{\geq 0}$. Then by continuity of $\mathrm{Mod}_{1,\sigma}(\Gamma)$ and ρ^*_σ in σ, it follows that the same holds for $\sigma \in \mathbb{R}^E_{\geq 0}$.
- By a result of Lehman, the packing property implies idealness.

Figure 5.1 below summarizes the above remark.

$$\begin{tikzcd}
 & \text{MFMC} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Ideal} & \text{Packing Property} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & \text{Clutters that pack}
\end{tikzcd}$$

Figure 5.1: Summary of the implications

5.3 All-star modulus

In this section, we introduce a new family of objects called, "all-stars" along with its conjectured Fulkerson dual. We will then look at the family of all-stars in different graphs and identify different characteristics of this family in the respective graphs.

5.3.1 Connection with the unoriented Laplacian

Let $G = (V, E)$ be a finite simple graph.

Definition 5.3.1. For $x \in V$, let the **star** at x be the set

$$E_x := \big\{\{x, y\} \in E : y \sim x\big\}.$$

We consider the family $\Gamma = \{E_x\}_{x \in V}$ of all the possible stars in G.

5.3.2 Stars in unweighted graphs

The usage matrix $\mathcal{N}$ is known in another context as the unoriented gradient matrix B of the graph G. Namely, for $x \in V$ and $e \in E$:

$$\mathcal{N}(E_x, e) = B(x, e) := \mathbb{1}_{\{x \in e\}}.$$

Then, the overlap matrix $C = \mathcal{N}\mathcal{N}^T$ is

$$C = D + A,$$

which is known as the **unoriented Laplacian**. Here, D and A are the degree and adjacency matrices respectively.

To see this, we will note the following two cases,

$$\begin{aligned}
x \neq y \qquad \mathcal{N}\mathcal{N}^T(x, y) &= \sum_{e \in E} B(x, e) B(y, e) \\
&= \mathbb{1}_{\{x \sim y\}} \\
x = y \qquad \mathcal{N}\mathcal{N}^T(x, y) &= \sum_{e \in E} B(x, e)^2 \\
&= deg(x).
\end{aligned}$$

In particular, if Γ is minimal, then

$$\mathrm{Mod}_{2,\sigma}(\Gamma) = \mathbf{1}^T C^{-1}\mathbf{1},$$

and

$$\rho^* = \mathcal{N}^T C^{-1}\mathbf{1}.$$

See section 4 of [9] for a proof of the above results.

Write

$$C = D(I + D^{-1}A) = 2D\tilde{P}$$

where $\tilde{P}$ is the transition matrix for the lazy random walk. It's known that the eigenvalues of $\tilde{P}$ are contained in $[0, 1]$ and that 0 is an eigenvalue if and only if G is bipartite. Therefore, since $\mathrm{Ker}\,\tilde{P} = \mathrm{Ker}\,C$, we have that C is always invertible for non-bipartite graphs.

5.3.3 Stars in weighted graphs

Let the usage matrix $\mathcal{N}$ be defined as above. Consider the Fulkerson family $\hat{\Gamma}$ with the weights $\hat{\sigma} = \sigma^{1-q}$, where q is the Hölder conjugate exponent of p. For $p = 2, q = 2$ we have, $\hat{\sigma} = \sigma^{-1}$. For this special case, the overlap matrix C is,

$$C = \mathcal{N}\Sigma^{-1}\mathcal{N}^T$$

where, Σ^{-1} is the $|E| \times |E|$ diagonal matrix with the diagonal elements $\sigma(e)^{-1}$. In matrix form, we can write this as,

$$C(\gamma_1, \gamma_2) = \sum_{e\in E} \frac{1}{\sigma(e)}\mathcal{N}(\gamma_1, e)\mathcal{N}(\gamma_2, e).$$

If Γ is minimal, then

$$\mathrm{Mod}_{2,\sigma^{-1}}(\hat{\Gamma}) = \mathbf{1}^T C^{-1}\mathbf{1},$$

and

$$\rho^*_{\sigma^{-1}} = \mathcal{N}^T C^{-1}\mathbf{1}.$$

Next, we introduce the Fulkerson dual of all-family.

5.3.4 Starcycles as Fulkerson dual

Let Γ be the family of all stars in a graph $G = (V, E)$. A **partial star** is a set of edges that share a given node $x \in V$. We are interested in studying the Fulkerson dual family $\hat{\Gamma}$ of Γ.

A **starcycle** S is an edge-cover (i.e., every vertex is incident to at least one edge in S) whose components are either partial stars or cycles of odd length. Furthermore, the usage vector corresponding to a starcycle is defined as follows: the usage of edges in a partial star component is 1, while the usage for edges in an odd cycle is 1/2.

Proposition 5.3.2. *The Fulkerson dual family $\hat{\Gamma}$ for the family Γ of all stars contains the family of all starcycles with edge usage defined as above.*

Proof. Suppose $\hat{\gamma}$ is the usage vector of a starcycle S. Then, we want to show that $\hat{\gamma}$ is an extreme point of $\operatorname{Adm}\Gamma$. Note first that $\rho := \hat{\gamma}$ is an admissible density for Γ. Indeed, assume γ is a star corresponding to node $x \in V$. Then, x is incident to at least one edge in S. If the component of S containing this edge is an odd cycle, then exactly two edges in γ must belong to S, so $\gamma^T\hat{\gamma} = (1/2) + (1/2) = 1$. If the component is a partial star, then it must have at least one edge in common with γ and so $\gamma^T\hat{\gamma} \geq 1$.

Now think of perturbing $\hat{\gamma}$ linearly. Let $\zeta \in \mathbb{R}^E$ and $t \in [-\epsilon, \epsilon]$ for some $\epsilon > 0$. We consider

$$u_t := \hat{\gamma} + t\zeta.$$

Assume that there is $\epsilon > 0$ so that u_t is admissible for Γ for every $t \in [-\epsilon, \epsilon]$, we want to show that ζ must be zero. First note that if $u_0(e) = \hat{\gamma}(e) = 0$, then $\zeta(e) = 0$, because admissible densities cannot be negative. Now assume $e = \{x, y\}$ is an edge in a component of S that happens to be a partial star $s(x)$ with vertex x. Note that $u_0(e) = \hat{\gamma}(e) = 1$. Without loss of generality assume also that $\zeta(e) > 0$. Then for $t < 0$, $u_t(e) < 1$. Now let γ be the star at y. Observe that, since $s(x)$ is a connected component of S,

$$\gamma \cap S = \gamma \cap s(x) = e.$$

Indeed, every other edge e' incident at y is not incident at x and thus cannot belong to $s(x)$. Moreover, e' also cannot belong to S, because otherwise it would belong to $s(x)$, since $s(x)$ is a component. This shows that for $t < 0$, $u_t^T \gamma < 1$ and therefore u_t is not admissible for Γ.

Finally, assume S is an odd cycle $x_0\ e_1\ x_1\ e_2\ x_2\ \cdots\ e_{2n+1}\ x_{2n+1}$, so that $x_{2n+1} = x_0$. Without loss of generality, consider the edge $e_1 = \{x_0, x_1\}$. Assume that $\zeta(x_0, x_1) > 0$, and let γ be a star with vertex at x_1. Then admissibility of u_t for every $t \in [-\epsilon, \epsilon]$ for some $\epsilon > 0$ requires that

$$\begin{aligned}
\gamma^T u_t &= u_t(x_0, x_1) + u_t(x_1, x_2) \\
&= \big(\hat{\gamma}(x_0, x_1) + t\zeta(x_0, x_1)\big) + \big(\hat{\gamma}(x_1, x_2) + t\zeta(x_1, x_2)\big) \\
&= \left(\frac{1}{2} + t\zeta(x_0, x_1)\right) + \left(\frac{1}{2} + t\zeta(x_1, x_2)\right) \\
&= 1 + t(\zeta(x_0, x_1) + \zeta(x_1, x_2)) \geq 1.
\end{aligned}$$

This implies that

$$\zeta(x_0, x_1) = -\zeta(x_1, x_2).$$

By induction, using the maximal stars at x_j for $2 = 1, \ldots, k$, we get that

$$\zeta(x_k, x_{k+1}) = (-1)^k \zeta(x_0, x_1).$$

In particular,

$$\zeta(x_{2n}, x_0) = \zeta(x_{2n}, x_{2n+1}) = \zeta(x_0, x_1).$$

However, repeating the argument above with the maximal star at x_0 we also have that

$$\zeta(x_{2n}, x_0) + \zeta(x_0, x_1) = 0$$

Therefore ζ is identically zero on the cycle S. This shows that $\zeta = 0$, and the proof is complete. $\square$

5.3.5 Examples of star families in different clutters

In this section, we will explore clutters of star families and their characteristics. We will prove the following.

- The family Γ of all stars of K_4 does not pack, but is ideal.
- The family Γ of all stars of K_3 does not pack and is not ideal.
- The star family Γ in the square packs, is ideal and has the max flow min cut property.

5.3.6 Example 1 : All-star modulus for the complete graph K_4

Let $G = K_4$ and Γ be the family of all stars $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, see Figure 5.2. The edge-usage matrix for Γ is

$$\mathcal{N} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \tag{5.3.1}$$

The Fulkerson dual family $\hat{\Gamma}$ (also the minimal transversals) of all starcycles in K_4 consists of each star γ_j in Figure 5.2, together with the three perfect matchings given by pairs of opposite edges of the tetrahedron and the four stars itself:

$$\mathcal{T} = \{T_1 = \{e_1, e_6\}, T_2 = \{e_2, e_5\}, T_3 = \{e_3, e_4\}, T_4 = \{e_1, e_2, e_3\},$$
$$T_5 = \{e_1, e_5, e_4\} T_6 = \{e_4, e_2, e_6\}, T_7 = \{e_3, e_5, e_6\}\}.$$

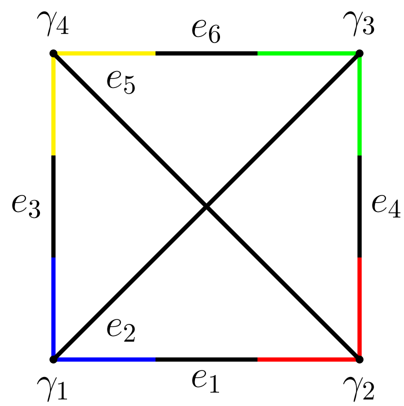

Figure 5.2: All stars for K_4

Proposition 5.3.3. *The family Γ of all stars of K_4 does not pack, but is ideal.*

Proof. We first show that the all-star family in K_4 does not pack. Notice that, as shown in Figure 5.2, the maximum number of disjoint objects is 1, since any two stars have non-empty intersection. On the other hand, the minimal transversal sets are the three vertex covers of size 2. Thus the cardinality of any minimal transversal set is at least 2. Since these two cardinalities are not equal, this clutter does not pack.

Let M be the block matrix (call this the "constraint matrix") which consists of the usage matrix $\mathcal{N}$ and the identity matrix of dimension $|E|$ (see 5.3.5). It is known that the extremal points of 5.2.1 for different σ is given uniquely by the solutions of $A\rho = b$, where A is a submatrix of M of dimension $|E|$ by $|E|$ and b is the vector of corresponding constraint values.

Below, we have formulated the system that provides all the solutions for extremal points with constraints given by 5.3.2 .

$$\mathcal{N}\rho \geq \mathbf{1} \qquad \rho \geq 0 \tag{5.3.2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.3.3}$$

Now we consider the different cases of A, submatrices of M.

Case 1: We need at least one row from $\mathcal{N}$ in order to have a non-trivial solution. Thus assume that we drop one row from the identity matrix and take one row from $\mathcal{N}$. This implies that $\rho \equiv 0$ on five edges. Which in turn means that there exists a star with zero weight. Thus this ρ is not admissible and this case cannot occur.

Case 2: Assume that we take 2 rows from $\mathcal{N}$. Figure 5.3 below depicts the two different cases of how this selection can be made. In both the cases we need to put 1 on either of the edges to make all the stars to weigh 1. Also notice that in this case, they are minimal transversals.

Figure 5.3: case 2

Case 3: Assume that we take 3 rows from $\mathcal{N}$. The three different cases that we can have are depicted in Figure 5.4 below. In the first diagram, it is clear that we need to assign 1 to all three edges that we picked to make every star weigh1. However if we select the second diagram, then we only need to assign 1 on the opposite edges to make ρ admissible. The thrid option, where the selcted edges creates a circuit will not work since this will result in one of the stars been weight 0.

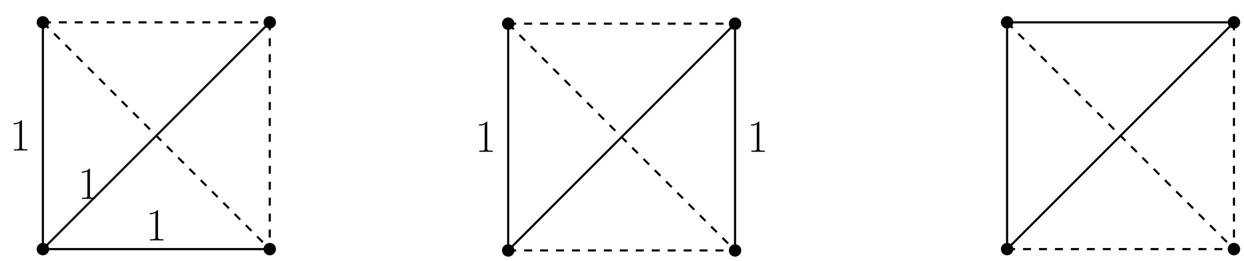

Figure 5.4: case 3

Case 4: Assume that we take all 4 rows from $\mathcal{N}$. Now solving the system,

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

we get the solution,

$$\rho = s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (1-s-t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

s and t are parameters where $\rho_1 = \rho_6 = t$ and $\rho_2 = \rho_5 = s$ and $\rho_3 = \rho_4 = 1 - s - t$. Observe that ρ is a convex combination of the three transversals. So we get that the extreme ponits are the three transversals. Thus the solutions are to have s, t or $(1 - s - t) = 1$ at a given time. □

5.3.7 Example 2 : All-star modulus for the complete graph K_3

Let Γ be the clutter consisting of family of stars in K_3

The edge-usage matrix for Γ is

$$\mathcal{N} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tag{5.3.4}$$

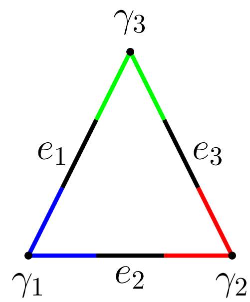

Figure 5.5: All stars for K_3

Proposition 5.3.4. *The family Γ of all stars of K_3 does not pack and is not ideal.*

Proof. Notice that, as shown in Figure 5.5, the maximum number of disjoint objects is 1, since any two stars have non-empty intersection. On the other hand, the minimal transversal sets are vertex covers with exactly 2 edges (any two edges is a transversal). Thus the cardinality of any minimal transversal set is at least 2. Since the two cardinalities are not equal, this clutter does not pack.

In order to show that this clutter is not ideal, we consider the case where we take all three rows of $\mathcal{N}$. This matrix has full rank and solving the linear system of equations,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

gives us the solution $\rho \equiv 0.5$. Since this is not integer valued we can conclude that the clutter of stars in the triangle is not ideal. □

5.3.8 Example 3 : All-star modulus for the square

As a last example for the section we present the cultter of family of stars on the square. The edge-usage matrix for Γ is

$$\mathcal{N} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{5.3.5}$$

Proposition 5.3.5. Γ*, the clutter os stars on the square packs, is ideal and has the max flow min cut property.*

Proof. Notice that, as shown in Figure 5.6, the maximum number of disjoint objects is 2, namely the stars that are not on adjacent vertices. On the other hand, the minimal

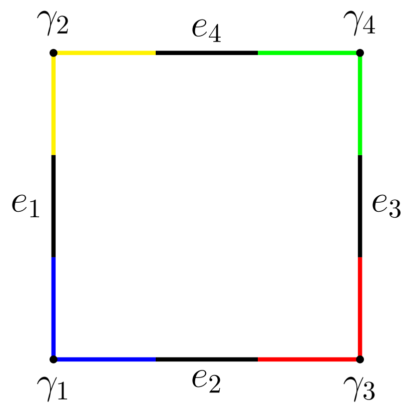

Figure 5.6: All stars for the square

transversal sets are vertex covers with exactly 2 edges. Thus the cardinality of any minimal transversal set is at least 2. Since the two cardinalities are equal, this clutter packs.

To show that it is ideal we show for all $\sigma \in \mathbb{R}^E_{\geq 0}$, the optimal $\rho^* \in \mathbb{Z}^E_{\geq 0}$ for (5.2.1) by the same method that we used in the K_4 example.

Case 1: We first consider the case where we pick one row from $\mathcal{N}$. It is clear that there exists a star that has weight zero in this case and thus this ρ is not admissible.

Case 2: Let's now consider the case where we will pick two rows from $\mathcal{N}$. We are forced to assign 1 to the two edges that are non zero. These two edges are also minimum transversals of the clutter.

Case 3: Consider the case where we will pick three rows from $\mathcal{N}$. In this case, one edge will have $\rho = 0$. Both of the edges adjacent to it will be forced to have a ρ value of 1 in order to have star weight exactly 1. This on the other hand will force the edge on the opposite of the $\rho = 0$ edge to be zero. Thus this case reduces to the case we discussed above.

Case 4: Finally, consider the case where we take all of the four rows from our usage matrix.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This matrix does not have full rank. Thus it does not define an extremal point.

Since all of the feasible submatrices with full rank produces extremal points that are intergral, we have that this clutter is ideal.

In order to show that the clutter has MFMC property, we only need to show for all $\sigma \in \mathbb{Z}^E_{\geq 0}$, the optimal $\lambda^* \in \mathbb{Z}^\Gamma_{\geq 0}$ for (5.2.2). Notice that the requirement of $\rho^* \in \mathbb{Z}^E_{\geq 0}$ has already been satisfied through the above proof of idealness.

To show this, we will look at Karush- Kuhn-Tucker (KKT) conditions for the primal and the dual problem defined by (5.2.1) and (5.2.2). These conditions will give us the following inequalities to be satified by the optimal ρ and λ for the primal and the dual problems respectively.

1. $\lambda^T(\mathbf{1} - \mathcal{N}\rho) = 0$
2. $\rho^T(\sigma - \mathcal{N}^T\lambda) = 0$
3. $\rho \geq 0$
4. $\lambda \geq 0$
5. $\mathcal{N}\rho \geq \mathbf{1}$
6. $\mathcal{N}^T\lambda \leq \sigma$

Without loss of generality we will assume the following hold for the weights of the graph.

$$\sigma_1 + \sigma_3 \leq \sigma_2 + \sigma_4$$

$$\sigma_1 \leq \sigma_3$$

$$\sigma_2 \leq \sigma_4$$

We will guess the optmial solutions for ρ as $\rho_1 = \rho_3 = 1, \rho_2 = \rho_4 = 0$. For λ we will guess $\lambda_1 = \min\{\sigma_1, \sigma_2\}, \lambda_2 = \sigma_1 - \lambda_1, \lambda_3 = \min\{\sigma_2 - \lambda_1, \sigma_3\}, \lambda_4 = \sigma_3 - \lambda_3$. Notice that if this guess works we have that $\lambda \in \mathbb{Z}^{\Gamma}_{\geq 0}$. Now we will continue to show that KKT conditions are met for this guess.

3 We will begin by checking the third condition. This is clearly met by our optimal ρ.

4 First we recongnize that $\lambda_1 \geq 0$ since both $\sigma_1, \sigma_2 \geq 0$. From our guess we also have that $\lambda_1 \leq \sigma_1$ which implies that our guess $\lambda_2 \geq 0$. Similarly $\sigma_1 \leq \sigma_2$ implies $\sigma_2 - \lambda_1 \geq 0$, which in turn implies $\lambda_3 \geq 0$ since $\sigma_3, \sigma_2 - \lambda_1 \geq 0$.

5 We have that with ρ defined as above $\mathcal{N}\rho = 1$.

1 Following from [5] above we have that $\lambda^T(1 - \mathcal{N}\rho) = 0$.

2 For this condition, we only need to check for ρ_1 and ρ_2 since when $\rho = 0$ this condition is trivially satisfied. We have that

$$(\mathcal{N}^T\lambda)_1 = \lambda_1 + \lambda_2 = \lambda_1 + \sigma_1 - \lambda_1 = \sigma_1$$

$$(\mathcal{N}^T\lambda)_3 = \lambda_3 + \lambda_4 = \lambda_3 + \sigma_3 - \lambda_3 = \sigma_3$$

6 We have already covered two cases above. Now we will consider the other two cases.

$$(\mathcal{N}^T\lambda)_2 = \lambda_1 + \lambda_3 \leq \lambda_1 + \sigma_2 - \lambda_1 = \sigma_2$$

Now for the last case,

$$(\mathcal{N}^T\lambda)_4 = \lambda_2 + \lambda_4 = \sigma_1 - \lambda_1 + \sigma_3 - \lambda_3 = (\sigma_1 + \sigma_3) - (\lambda_1 + \lambda_3)$$

We need to verify that the last expresion in the above equation is less than equal to σ_4. Equivalently we want to show that

$$\sigma_4 - (\sigma_1 + \sigma_3) + (\lambda_1 + \lambda_3) \geq (\lambda_1 + \lambda_3) - \sigma_2$$

We need the last part of the above inequality to be greater than or equal to zero. Now if $\lambda_1 = \sigma_2$ we are done since we get $\lambda_3 \geq 0$.

Otherwise$\lambda_1 = \sigma_1$. This also impliws that $\sigma_1 \leq \sigma_2$ since λ_1 is the minimum of these two quantities. Now, if $\lambda_3 = \sigma_2 - \sigma_1$ we are done since it gives us a zero. Else we have $\lambda_1 + \lambda_3 = \sigma_1 + \sigma_3$, which in case implies that$\lambda_2, \lambda_4 = 0$. Thus we have,

$$(\mathcal{N}^T \lambda)_4 = \lambda_2 + \lambda_4 = 0$$

Thus our guess seem to satisfy all the KKT conditions and we are done.

$\square$

5.4 Blockers

Definition 5.4.1. The **blocker** $b(\Gamma)$ of a clutter Γ is the clutter $(E, b(\Gamma))$, with the same edge set E and Γ whose objects are the minimal transversals of Γ. That is, $b(\Gamma)$ consists of the minimal members of

$$\{B \subseteq E : |B \cap A| \geq 1, \quad \text{for all } A \in \Gamma\}.$$

In other words, the rows of $\mathcal{N}(b(\Gamma))$ are the minimal 0,1 vectors x^T such that x belongs to the polyhedron $\operatorname{Adm}\Gamma = \{x \geq 0 : \mathcal{N}(\Gamma)x \geq \mathbf{1}\}$.

Example 5.4.1. Let G be a graph and s, t be distinct nodes of G. If Γ is the clutter of st-paths, then $b(\Gamma)$ is the clutter of minimal st-cuts.

Example 5.4.2. In a connected bipartite graph if Γ is the clutter of family of stars then the blocker will be perfect matchings.

Remark 5.4.1. We know that the Fulkerson dual family $\mathcal{N}(\hat{\Gamma})$ of st-paths(Γ) is st-cuts. So we observe here that, $\mathcal{N}(\hat{\Gamma}) = \mathcal{N}(b(\Gamma)) \Rightarrow \hat{\Gamma} = b(\Gamma)$ in this case.

Proposition 5.4.2. *Let H and K be two clutters defined on the same ground set. Call $\mathcal{P}$, "objects" of $\mathcal{C}$ for convenience. If,*

(i) every object of H contains an object of K and

(ii) every object of K contains an object of H,

then $H = K$.

Proof. If (i) and (ii) hold, then for all $h_i \in H$ there exists $k_l \in K$ s.t $k_l \subseteq h_i$ and for every k_l there exists h_j s.t $h_j \subseteq k_l$. Thus we have $h_j \subseteq k_l \subseteq h_i$. Since both h_i and h_l are minimal transversals this implies that $h_i = h_j = k_l$. Therefore for every $h_i \in H$ we have that $h_i \in K$. Which implies $H \subset K$. We can argue similarly to show that $K \subset H$. Thus we have that $H = K$. $\square$

Theorem 5.4.3. *[Edmonds and Fulkerson] If Γ is a clutter, then $b(b(\Gamma)) = \Gamma$.*

Proof. Let A be an object of Γ. The definition of $b(\Gamma)$ implies that $|A \cap B| \geq 1$, for all objects $B \in b(\Gamma)$. Thus A is a transversal of $b(\Gamma)$. This implies that A contains an object of $b(b(\Gamma))$ (we cannot claim it is an object of $b(b(\Gamma))$ since minimality of A is not guaranteed).

Now let A be an object of $b(b(\Gamma))$. We claim that A contains an object of Γ. Suppose otherwise (i.e. A does not contain any objects of Γ). Then $E \setminus A$ intersects all the objects of Γ. Thus $E \setminus A$ is a transversal of Γ, implying that it contains an object B in $b(\Gamma)$ with the property that $B \cap A = \emptyset$. However, since A is in $b(b(\Gamma))$, we also have $|B \cap A| \geq 1$, hence we have reached a contradiction. Therefore, A contains an object of Γ. By Proposition 5.4.2, we have that $b(b(\Gamma)) = \Gamma$. $\square$

Definition 5.4.4. Two 0,1 matrices of the form $\mathcal{N}(\Gamma)$ and $\mathcal{N}(b(\Gamma))$ are said to form a blocking pair.

Theorem 5.4.5 (Lehman)**.** *For a blocking pair $\mathcal{N}_1, \mathcal{N}_2$ of 0,1 matrices, the polyhedron $P := Adm(\Gamma)$ defined by*

$$\begin{aligned} \mathcal{N}_1\rho &\geq \mathbf{1} \\ \rho &\geq 0 \end{aligned} \tag{5.4.1}$$

is integral if and only if the polyhedron Q defined by

$$\begin{aligned} \mathcal{N}_2\eta &\geq \mathbf{1} \\ \eta &\geq 0 \end{aligned} \tag{5.4.2}$$

is integral.

We have state this theorem in a different way in 5.4.7. The proof of the theorem uses the preposition below.

Proposition 5.4.6. *(i) The rows of $\mathcal{N}_2$ are exactly the 0,1 extreme points of P.*

(ii) If an extreme point x of P satisfies $x^T \geq \lambda^T \mathcal{N}_2$ where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then x is a 0,1 extreme point of P.

Proof. (i) follows from the fact that the rows of $\mathcal{N}_2$ are the minimal 0,1 vectors in P.

(ii) Let x be an extreme point of P with the above characteristics. Note that x is then also an extreme point of $P_I = \{X : X^T \geq \lambda^T \mathcal{N}_2$ where $\lambda_i \geq 0$ and $\sum \lambda_i = 1\} = \mathrm{Dom}(b(\Gamma))$ for otherwise x would be a convex combination of distinct $x_1, x_2 \in P_I$ and, since $P_I \subseteq P$, this would contradict the assumption that x is an extreme point of P. Now (ii) follows by observing that the extreme points of P_I are exactly the rows of $\mathcal{N}_2$.

□

Theorem 5.4.7. *[Lehman] A clutter is ideal if and only if its blocker is.*

Proof. Proof: By Theorem 5.4.3, it suffices to show that if P defined by (5.4.1) is integral, then Q defined by (5.4.2) is also integral. Let $\tilde{\gamma}$ be an arbitrary extreme point of Q. Then $\mathcal{N}_2\tilde{\gamma} \geq 1$, i.e. $\tilde{\gamma}^T\hat{\gamma} \geq 1$ is satisfied by every extreme point of P. Also we have that, $\tilde{\gamma} \geq 0$. This fact combined with the minimality of the extreme points give us that $\tilde{\gamma}^T x \geq 1$ for all points $x \in P$.

Furthermore, it can be shown that $\tilde{\gamma}^T x_0 \geq 1$ for some $x_0 \in P$ (see proof of theorem 5.5.2). Now, by linear programming duality, we have,

$$1 = \min\{\tilde{\gamma}^T x : x \in P\} = \max\{\lambda^T 1 : \lambda^T \mathcal{N}_1 < \tilde{\gamma}^T, \lambda \geq 0\}.$$

Therefore, by Proposition 5.4.6 (ii) applied to Q, $\tilde{\gamma}$ is a 0,1 extreme point of Q. □

5.5 st-cuts and st-paths

In this section, we will investigate more on the st-cuts and st-paths.

Consider a digraph (V, E) with $s, t \in V$. Let C be the clutter of the family of st-paths.

Theorem 5.5.1 (Ford-Fulkerson). *The clutter C of st-paths has the MFMC property.*

Proof. For any edge capacities $w \in Z^E_{\geq 0}$, the Ford-Fulkerson theorem [23] states that (5.2.1) and (5.2.2) both have optimal solutions that are integral: (5.2.1) is the min cut problem and (5.2.2) is the max flow problem. Using the terminology introduced in Denition 5.2.3, the Ford-Fulkerson theorem states that the clutter C of st-paths has the MFMC property. □

It could also be shown that the clutter of st-cuts also has the MFMC property.

In a network, the product of the minimum number of edges in an st-path by the minimum number of edges in an st-cut is at most equal to the total number of edges in the network. This width-length inequality can be generalized to any non negative edge lengths l_e and widths w_e: the minimum length of an st-path times the minimum width of an st-cut is at most equal to the scalar product $l^T w$. This width-length inequality was observed by Moore and Shannon and Duffin. A length and a width can be defined for any clutter and its blocker. Lehman showed that the width-length inequality can be used as a characterization of idealness.

Theorem 5.5.2 (Width-length inequality, Lehman)**.** *For a clutter Γ and its blocker $b(\Gamma)$, the following statements are equivalent.*

(i) *Γ and $b(\Gamma)$ are ideal, i.e., $\hat{\Gamma}$ and $\widehat{b(\Gamma)}$ are $0/1$.*

(ii) *For all $w \in \mathrm{Adm}(\Gamma)$, and $z \in \mathrm{Adm}(b(\Gamma))$:*

$$\ell_w(\Gamma)\ell_z(b(\Gamma)) \leq w^T z,$$

where $\ell_\rho(\Gamma) := \min_{\gamma\in\Gamma} \sum_{e\in\gamma} \rho(e)$.

Proof. Let $\mathcal{N} = M(\Gamma)$ and $(b\mathcal{N}) = M(b(\Gamma))$ be the blocking pair of 0,1 matrices associated with Γ and $b(\Gamma)$ respectively. First assume that Γ and $b(\Gamma)$ are ideal.

It is known that $\mathrm{Adm}(\Gamma) = \mathrm{Dom}(\hat{\Gamma})$. Thus $w \in \mathrm{Adm}(\Gamma) \Rightarrow w \in \mathrm{Dom}(\hat{\Gamma}) = \mathrm{co}(\hat{\Gamma}) + \mathbb{R}^E_{\geq 0}$ and we have that there exists a measure $\nu \in \mathcal{P}(\hat{\Gamma})$ such that $w \geq (b\mathcal{N})^T \nu$ (See [Corollary 2.1.7 in Chapter 2).

Since the clutter is ideal, from proposition 5.2.5 we have that $\hat{\Gamma}$ is integer valued. More precisely 0,1 valued and we have $b(\Gamma) = \hat{\Gamma}$.

We also have that $\mathrm{Adm}\,\hat{\Gamma} = \mathrm{Dom}(\Gamma)$. Thus $z \in \mathrm{Adm}\,\hat{\Gamma} \Rightarrow z \in \mathrm{Dom}(\Gamma)$ and there exists a measure $\mu \in \mathcal{P}(\Gamma)$ such that $z \geq \mathcal{N}^T \mu$ (See [Corollary 2.1.7 in Chapter 2). Let J denote the matrix of ones. Notice that $(b\mathcal{N})\mathcal{N}^T \geq J$ since $(b\mathcal{N})\mathcal{N}^T$ have at least one common edge (recall that rows of $(b\mathcal{N})$ consists of minimal transversals of objects in $\mathcal{N}$). Thus we have,

$$w^T z \geq \nu^T (b\mathcal{N})\mathcal{N}^T \mu \geq \nu^T J \mu = 1 = \ell_w(\Gamma)\ell_z(b(\Gamma))$$

Now we prove the converse. Let $w \in \hat{\Gamma}$, be an extreme point for the admissible set of Γ. In particular, we have that $\ell_w(\Gamma) \geq 1$ since w is admissible for Γ. Also for any point $z \in \mathrm{Adm}(b(\Gamma))$ we have that $\ell_z(b(\Gamma)) \geq 1$. By hypothesis it follows that, $w^T z \geq \ell_w(\Gamma)\ell_z(b(\Gamma)) \geq 1$ for all $z \in \mathrm{Adm}\, b(\Gamma)$. In particular it can be shown that there exists at least one $z_0 \in \mathrm{Adm}\, b(\Gamma)$ such that $w^T z_0 = 1$ (this will be shown later in the proof). By the fact that $w^T z \geq 1$ and that there exists a z_0 such that $w^T z_0 = 1$ we have,

$$1 = \min\{w^T z : z \in \mathrm{Adm}\, b(\Gamma)\}$$

Now by linear programming duality we have,

$$1 = \min\{w^T z : z \in \mathrm{Adm}\, b(\Gamma)\} = \max\{\mu^T 1 : \mu^T (b\mathcal{N}) \leq w^T, \mu \geq 0\}.$$

It follows from Proposition 5.4.6 (ii) that w is a 0,1 extreme point of $\mathrm{Adm}(\Gamma)$. Therefore by Proposition 5.2.5, Γ is ideal. By Theorem 5.4.7 , $b(\Gamma)$ is also ideal.

Showing the existence of $z_0 \in \mathrm{Adm}\, b(\Gamma)$ such that $w^T z_0 = 1$ and $w \in \hat{\Gamma}$.

Since $w \in \hat{\Gamma}$, there exists a weight set $\sigma \in R^E_{\geq 0}$ such that w is the unique extremal density. In other words, it is the solution for the optimization problem,

$$\begin{array}{ll} \text{minimize} & \sigma^T \rho \\ \text{subject to} & \rho \geq 0, \quad \mathcal{N}\rho \geq \mathbf{1} \end{array}$$

and $\mathrm{Mod}_{1,\sigma}(\Gamma) = \sigma^T w$. Thus, for all $\rho \in \mathrm{Adm}(\Gamma)$ which are different from w, we have that $\sigma^T \rho > \sigma^T w$. Now consider the weight set $z_0 = \left(\frac{\sigma}{\mathrm{Mod}_{1,\sigma(\Gamma)}}\right)$. Notice that $w^T z_0 = 1$. Also observe that $z_0 \in \operatorname{Adm}\hat{\Gamma}(\hat{\Gamma} = b(\Gamma)$ in this context since our clutter is ideal). To see this, assume that $\hat{\gamma} \in \operatorname{Adm}(\Gamma) \setminus w$. Then we have that $\sigma^T \hat{\gamma} \geq \sigma^T w = \operatorname{Mod}_{1,\sigma} \Rightarrow \left(\frac{\sigma}{\operatorname{Mod}_{1,\sigma(\Gamma)}}\right)^T \hat{\gamma} \geq 1$. $\square$

We end this Chapter with the main theorem given below.

5.6 The relationship between blocker and the Fulkerson dual of a clutter

Theorem 5.6.1. *Let $b(\Gamma)$ be the blocker of a clutter Γ. Then, $b(\Gamma) \subseteq \hat{\Gamma}$, where $\hat{\Gamma}$ is the Fulkerson blocker of the clutter Γ.*

Proof. Let $T \in b(\Gamma)$ be a minimal transversal (minimal 0-1 vector). Since $|T \cap \gamma| \geq 1$ for all $\gamma \in \Gamma$, the density $\rho := \mathbb{1}_T$ is admissible for Γ. We want to show that ρ is not a strict convex combination of two densities $\rho_0 \neq \rho_1$ that are in the admissible set of Γ. This will imply that ρ is an extremal point of $\operatorname{Adm}\Gamma$ and therefore that ρ is in $\hat{\Gamma}$.

Assume, by contradiction, that $\rho = t\rho_1 + (1-t)\rho_0$, with $t \in (0,1)$ and $\rho_0 \neq \rho_1 \in \operatorname{Adm}(\Gamma)$. Since admissible densities are non-negative, this implies that $\rho_0(e), \rho_1(e) = 0$ for every $e \in E \setminus T$. Notice that if $\rho_0(e), \rho_1(e) \geq 1$ for all $e \in T$, then $t = 0$ or 1, which is a contradiction since $t \in (0,1)$. Thus, we can assume, without loss of generality, that $0 < \rho_1(e') < 1$, for some $e' \in T$. This means that all $\gamma \in \Gamma$ such that $e' \in \gamma$ have the property $|T \cap \gamma| \geq 2$, otherwise $T \cap \gamma = \{e'\}$ and $\rho_1^T \gamma = \rho_1(e') < 1$, which contradicts admissibility of ρ_1. Thus, e' can be removed from T without affecting the transversality property of T with respect to Γ. This contradicts the fact that T is a minimal transversal. $\square$

Chapter 6

Future work

In this chapter, we will present some of the avenues related to our work that we would like to explore in the future.

6.1 Relationship between EHT and δ_p metric

It was shown in [7] that

$$\delta_2(a,b) = R_{\text{eff}}(a,b) \leq \text{EHT}(a,b) \leq \lim_{p\uparrow\infty} \delta_p = d_{\text{SP}}$$

where $R_{\text{eff}}(a,b)$ is the effective resistance (see 3.1.1), $\text{EHT}(a,b)$ is epidemic hitting time (see 3.1.4)and d_{SP} is the shortest path metric. We would like to answer the question, "is there a relationship between the epidemic hitting time metric and δ_p for some p?"

In other words, "can the gap be shrunk?"

6.2 Computing the anti-snowflaking exponent for different families

The example of the square graphs in Section 4.3.4 raises the following question: Is it true that for an arbitrary simple graph $G = (V, E)$, there is always a value p_0 so that (V, d_{p_0}) embeds isometrically in some $\mathbb{R}^n$, and if so, does this imply the same is true for all $p < p_0$? Similar questions can be asked for other powers of d_p.

In the example of the square graph in Section 4.3.4, the d_p metric can be raised to an exponent that is strictly larger than q. In fact, to find the largest possible exponent $t_{\max}$ in this case it's enough to set

$$d_p(a, c)^t = 2d_p(a, b)^t,$$

and solve for t. When this happen we say that the graph contains a "flat triangle". A calculation shows that

$$t_{\max} = \frac{1}{1 - \frac{1}{p}\left(2 - \log_2(1 + 3^{1-p})\right)} > \frac{1}{1 - \frac{1}{p}} = q.$$

More generally, we intend to do the same computation for the cube in $\mathbb{R}^3$ and study whether the anti-snowflaking exponent $s_{\mathcal{H}}(p)$ can be computed for the family of hypercubes $\mathcal{H}$. These are graphs that arise in the theory of expander graphs and are considered to be very well connected, hence, their triangles should be far from being flat.

6.3 Properties of the δ_p metric

We are interested in studying the properties of the δ_p metric. Among these properties, we are particularly interested in the embeddings of the metric in different spaces and also the monotonicity properties of the metric.

6.4 Continue translation from clutters to modulus

We have tried to translate some of the terms encountered when studying clutters to our language of modulus and Fulkerson duality. This was done in Chapter 5. However, there is so much more that could be translated into our language which can strengthen the understanding of both clutters and modulus. It is also expected to unravel the connections between the two concepts through these translations/analysis.

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