SHEAVES OF DIFFERENTIAL OPERATORS AND D-MODULES OVER NON-COMMUTATIVE PROJECTIVE SPACES

by

LEE W. GOERL

B.S., Kansas State University, 2005

AN ABSTRACT OF A DISSERTATION

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Abstract

For a scheme, let D be the sheaf of differential operators, assigning to any open subscheme it's ring of differential operators. The study of D-modules advances their theory independently, but pervades many other areas of modern mathematics as well. Most notably, the theory provided a framework to solve Hilbert's 21-st problem, and to develop the Riemann-Hilbert correspondence, and eventually led to the resolution of the Kazhdan-Lustig conjecture in representation theory. For an affine patch of the scheme having dimension n, the sheaf will assign the n-th Weyl algebra. In [1], Hayashi develops the quantized Weyl algebra, a deformation of this algebra, and in [2] Lunts and Rosenberg develop versions of β and quantum differential operators for a graded non-commutative algebra. Iyer and McCune compute in [3] the ring of these quantum differential operators of Lunts and Rosenberg over the polynomial algebra in n-variables, or, over affine n-space. In [4], Bischof examines how a reconciliation of the β deformation in [2] and a 2-cocycle deformation of the graded algebra influence the category of these quantum D-modules, and considers some localizations. One naturally wonders about the category of modules for these quantum differential operators on a non-commutative space; about it's objects and it's structure. With the aim of future study in non-commutative grassmannians and flag varieties, of $U_q(\mathfrak{sl}_n)$, for example, we consider a non-commutative projective space glued together from a covering of 2-cocycle deformed polynomial rings, as proposed in [5] and [4]. We determine when there exists a deformed polynomial ring from which we can obtain this covering, and the category of quasi-coherent sheaves can be realized via the categorical Proj construction. With a guiding hand from Rosenberg's [5] we develop a general ring structure for containing these quantum differential operators on polynomial algebras. Finally, towards the goal of defining holonomic quantum D-modules, we consider the GK-dimension of the corresponding associated graded algebra

for the purpose of determining the dimension of what might be considered the singular support for a quantum D-module.

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Acknowledgments

In turns, there have been times when I did not think I was capable of completing the PhD. program I had entered into, and others in which I honestly did not want to complete it. Here I wish to thank those that offered their support regardless of how I felt about the program, but especially when I wavered.

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Dedication

This work is dedicated to the memory of my advisor Alexander Rosenberg, who passed away the semester I completed my qualification exams, just before I started my research in earnest. It is no exaggeration to say that I would not be a mathematician had it not been for Alex. I probably would not have even considered it. But, I did; he not only suggested the notion but inspired an interest, and a following from his four students.

From Alex I inherited much of the mathematical language, setting, and interests he had spent so much time developing. I can only hope that this work and anything that comes out of it, will shed some small light and can extend that vast body of work even marginally.

Chapter 1

Introduction

1.1 Overview

There are two main motivations for the work herein. The first, more general motivation is a general interest in non-commutative algebraic geometry. Typically, non-commutative algebraic geometries arise either from considering non-commutative rings over an existing space, by deformation of the rings of the structure sheaf or by considering the differential operators of those rings; they may also arise by constructing, via one of the various theories for non-commutative algebraic geometry, a space made with a non-commutative spectrum which has a structure sheaf of non-commutative rings.

1.1.1 Commutative Algebraic Geometry

It could be said that modern abstract algebraic geometry, was, at least initially, concerned with providing a setting for studying geometry over an arbitrary field which would contain the points that make up the various geometric objects of interest. This ambitious endeavour lead to numerous advances in the theory of commutative algebra. With the benefit of hindsight, we outline briefly some of the well-developed framework.

Over a topological space we can consider sheaves and presheaves; essentially, an assignment to each open set of the topological space an algebraic object. An example to keep in mind is the correspondence of a subspace to the ring of functions over the subspace.

Definition 1.1.1. Let X be a topological space with $\mathrm{Open}(X)$ the category of open subsets in X, whose morphisms are inclusions $U \hookrightarrow V$ for $U \subseteq V$. A *presheaf* of objects in a category $\mathcal{C}$ on X is a contravariant functor from $\mathcal{F} : \mathrm{Open}(X) \to \mathcal{C}$.

This categorical language obfuscates much of what a presheaf is. First of all, it is an assignment of objects in $\mathcal{C}$ to objects in $\mathrm{Open}(X)$. Being contravariant means that a presheaf reverses the direction of arrows in $\mathrm{Open}(X)$, by which we obtain, for $U \hookrightarrow V$, the restriction morphism $\mathrm{res}_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$. For triples $U \hookrightarrow V \hookrightarrow W$ we have that $\mathrm{res}_{U,W} = \mathrm{res}_{V,W} \circ \mathrm{res}_{U,V}$. Furthermore, for every U, we have $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$.

A sheaf is a special kind of presheaf, one whose global properties are determined locally. The classical example is the sheaf $\mathcal{F}$ of functions over the space X, assigning to a subspace $U \subseteq X$ the ring of functions over U. A sheaf encodes the idea that if a function $f \in \mathcal{F}(U)$ and a function $g \in \mathcal{F}(V)$ agree on the intersection $U \cap V$, then these functions must glue together to be the same function on the union $U \cup V$. Let us restrict ourselves to categories $\mathcal{C}$ whose objects have elements (sets, groups, rings, algebras, etc); we call the elements of $\mathcal{F}(U) \in \mathcal{C}$ the sections of $\mathcal{F}$ over U. The sections of $\mathcal{F}(X)$ as called the *global sections*.

Definition 1.1.2. Let $\mathcal{F}$ be a presheaf of objects in $\mathcal{C}$ on the space X, and $U = \bigcup_{i \in I} U_i \subseteq X$. $\mathcal{F}$ is called a *sheaf* if it satisfies the following

(S1) If $f, f' \in \mathcal{F}(U)$ are such that $f|_{U_i} = f'|_{U_j}$ for all $i, j \in I$, then $f = f'$,

(S2) If $f_i \in \mathcal{F}(U_i)$ are such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Let R be a commutative, unital ring.

Definition 1.1.3. We call the set of all prime ideals of the ring R the *prime spectrum*, or simply the spectrum, and denote it by $\mathbf{Spec}(R)$.

We denote $X = \mathbf{Spec}(R)$ to emphasize that it is being considered as a topological space, rather than just a collection of ideals. The topology on X is the Zariski topology, which is defined by its closed subsets. For an ideal $I \subseteq R$, we have the closed subset $V(I) = \{\mathfrak{p} \in X \mid I \subseteq \mathfrak{p}\}$. There is a basis for the Zariski topology, consisting of the *distinguished* open sets of X: for $f \in R$, denote by $X_f = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\} = X \setminus V(Rf)$.

The first quintessential example to consider is $R = k[x]$ for k an algebraically closed field of characteristic 0. We know then that each polynomial in R can be factored into linear factors of the form $(x-a)$ for $a \in k$, which comprise the totality of prime ideals of R, besides the zero ideal (0). Thus, $X = \mathbf{Spec}(R) = k$.

Setting $\mathcal{O}_X(X_f) = R_f = S_f^{-1}R$ as the localization of R by the multiplicitive subset $S_f = \{f^n \mid n = 0, 1, ...\}$ constructs a presheaf $\mathcal{O}_X$ on the distinguished base of $X = \mathrm{Spec}(R)$. To determine the restriction maps, note that $X_f \subseteq X_g$ if and only if $f^n \in Rg$ for some n, which gives $\mathrm{res}_{X_g,X_f} : R_g \to R_f = R_{gf} = (R_g)_f$ as just a subsequent localization. Verifying $\mathrm{res}_{X_g,X_h} = \mathrm{res}_{X_f,X_h} \circ \mathrm{res}_{X_g,X_f}$ for $X_h \subseteq X_f \subseteq X_g$ gives us a presheaf on X.

Theorem 1.1.4. *The presheaf $\mathcal{O}_X$ defined on $X = \mathrm{Spec}(R)$ as above is a sheaf of rings with respect to the Zariski topology.*

Proof. See [6]. □

The sheaf $\mathcal{O}_X$ is referred to as the *structure sheaf* of $X = \mathrm{Spec}(R)$, and such a pair $(X, \mathcal{O}_X)$ will be called an *affine scheme*. For an open $U \subset X$ we have that the pair $(U, \mathcal{O}_U = \mathcal{O}_X|_U)$ is an open affine subscheme of X. A sheaf $\mathcal{M}$ on X is called a sheaf of $\mathcal{O}_X$-modules, or, an $\mathcal{O}_X$-module, if $\mathcal{M}$ is a sheaf of abelian groups on X together with a map of presheaves of sets $\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$, such that, for each open $U \subset X$, the resulting map $\mathcal{O}_X(U) \times \mathcal{M}(U) \to \mathcal{M}(U)$ is an abelian group homomorphism defining on $\mathcal{M}(U)$ the structure of an $\mathcal{O}_X(U)$-module.

Definition 1.1.5. An $\mathcal{O}_X$ module $\mathcal{M}$ on $X = \mathrm{Spec}(R)$ is called *quasi-coherent* if $\mathcal{M}$ is locally presentable. That is, for all $x \in X$, there exists an open subset $x \in U \subseteq X$ such that the following sequence of $\mathcal{O}_X$-modules is exact:

$$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{M}_U \to 0,$$

where I and J are indexing sets and 0 denotes the sheaf having constant value $\{0\}$ for all open $U \subseteq X$.

Quasi-coherent sheaves on $X = \operatorname{Spec}(R)$ form a category, $\operatorname{Qcoh}(X)$, which turns out to have a very nice description. Consider the category R-mod, and an R-module M therein. Denote by $\mathcal{M}$ the presheaf of $\mathcal{O}_X$ modules formed by the assignment of distinguished opens $X_f \mapsto S_f^{-1}M = R_f \bigotimes_R M$. One can verify that $\mathcal{M}$ is indeed a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Denote the assignment by $\Delta(M) = \mathcal{M}$. Conversely, consider a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F} \in \operatorname{Qcoh}(X)$, and denote by $\Gamma(X, \mathcal{F}) = \mathcal{F}(X) \in \mathcal{O}_X(X)$-mod $= R$-mod.

Theorem 1.1.6. $\Delta : \operatorname{Qcoh}(X) \to R$*-mod* $: \Gamma$ *form an adjoint pair of functors, resulting in an equivalence of the categories* $\operatorname{Qcoh}(X)$ *and* R*-mod.*

Of course, there are more parts to the story of affine schemes, but these are sufficient for our needs. We are interested in something more general than an affine scheme.

Definition 1.1.7. A *scheme* is a pairing $(X, \mathcal{O}_X)$ of a topological space X and a structure sheaf of rings $\mathcal{O}_X$ over that space, such that for any point $x \in X$ there exists a neighborhood U of X containing x for which the pair $(U, \mathcal{O}_X|_U)$ is an affine scheme.

One common method for constructing a scheme is to glue more than one affine scheme together. As you might expect here, the word "glue" is not simply a colloquialism, but a precise term. To glue schemes $\{X_i\}_I$ together, we glue by open subschemes $X_{ij} = X_i \cap X_j \subset X_i$ which are isomorphic via morphisms $\phi_{ij} : \mathcal{O}_{X_j}|_{X_{ji}} \simeq \mathcal{O}_{X_i}|_{X_{ij}}$, wherein $\phi_{ii} = id_{X_i}$, such that the *cocycle condition* is met:

$$\phi_{ik}|_{X_{ki} \cap X_{kj}} = \phi_{ij}|_{X_{ji} \cap X_{jk}} \circ \phi_{jk}|_{X_{ki} \cap X_{kj}} \tag{1.1}$$

so that the three pairwise intersections are equal.

A classic example of a non-affine scheme which can be constructed in such a manner is that of projective n-space, $\mathbb{P}^n_k$. We will consider the case when $n = 2$. Let k be an algebraically closed ring of characteristic 0, and denote by $R = k[x_0, x_1, x_2]$ the ring of polynomials in three indeterminants with coefficients in k, which has as it's elements k-linear sums of monomials $x^a = x_0^{a_0}x_1^{a_1}x_2^{a_2}$ with $0 \leq a_i \in \mathbb{Z}$ for all i. We say that an element $f = \sum b_\alpha x^{a_\alpha} \in R$ is homogeneous of degree n if $\sum_{i=0}^{3} a_{\alpha i} = n$ for all α. In this way we see that R is $\mathbb{Z}$-graded.

Let $S_j = \{x_j^i \mid 0 \leq i \in \mathbb{N}\} \subset R$ be the multiplicitive set generated by the element x_j, and denote $R_j = (S_j^{-1}R)_0$ the homogeneous elements of degree 0 of the localization $S_j^{-1}R = k[x_0, x_1, x_2, x_j^{-1}]$. Each element $f \in R_j$ can then be written as a k-linear sum of monomials $\frac{x^a}{x_j^n}$ where $n = \sum_{i=0}^{2} a_i$. Hence, $R_j \simeq k[x_{0/j}, x_{1/j}, x_{2/j}]$, where $x_{j/j} = 1$. To construct $\mathbb{P}_k^2$ we will glue together the affine schemes $X_j = \text{Spec}(R_j)$ corresponding to the R_j by considering the open sets X_{jk} obtained by removing from X_j the locus of points wherein $x_{k/j} = 0$. Thus we have, for instance, $\mathcal{O}_{X_2}(X_{20}) = k[x_{0/2}, x_{1/2}, x_{0/2}^{-1}]$, and $\mathcal{O}_{X_2}(X_{20} \cap X_{21}) = k[x_{0/2}, x_{1/2}, x_{0/2}^{-1}, x_{1/2}^{-1}]$. Note, here, that $X_{20} \cap X_{21}$ is the locus of points in $\text{Spec}(R_2)$ where both $x_{0/2} \neq 0$, and$x_{1/2} \neq 0$.

To establish the morphisms $\phi_{ij} : \mathcal{O}_{X_j|X_{ji}} \simeq \mathcal{O}_{X_i|X_{ij}}$, we simply utilize the isomorphisms of rings of global sections $\phi_{ij} : k[x_{0/j}, x_{1/j}, x_{2/j}, x_{i/j}^{-1}] \simeq k[x_{0/i}, ..., x_{3/i}, x_{j/i}^{-1}]$ defined by $x_{k/j} \mapsto x_{k/i}x_{j/i}^{-1}$. We verify condition 1.1 for the element $x_{0/2}$; the others verify similarly. Behold,

$$(\phi_{01} \circ \phi_{12})(x_{0/2}) = \phi_{01}(x_{0/1}x_{2/1}^{-1}) = (x_{1/0})(x_{2/0}^{-1}x_{1/0}) = x_{2/0}^{-1} = \phi_{02}(x_{0/2}).$$

With the cocycle condition verified, we conclude that we have indeed glued together the scheme $\mathbb{P}_k^2$. Since $\mathcal{O}_{\mathbb{P}_k^2}$ is a quasi-coherent (indeed, coherent) sheaf, the process for gluing together the quasi-coherent sheaves in the category $\text{Qcoh}(\mathbb{P}_k^n)$ will be similar, and the process is formally explicated in the final chapter, section 6.1.2.

1.1.2 Goals

Inspired thusly, the work in this thesis addresses the following problems:

1. Developing an adaptation of the quantum differential operators of Lunts and Rosenberg [2] from which we can glue together a sheaf of differential operators for a quasi-affine non-commutative space.

2. Gluing together a suitable non-commutative quasi-affine space over which we will consider these differential operators. In this case, the space is a non-commutative version of projective space with an affine cover.

3. In the direction of holonomic D-modules, calculating the associated graded algebra for the algebra of differential operators, and its Gelfand-Kirillov dimension.

1.2 Outline

This dissertation is organized into five chapters, aimed at addressing the problems enumerated above in 1.1.2. Chapters 2, 3, and 6 are concerned with the first two of those problems, while Chapter 4 attempts to address the third. Chapter 5 offers no help in the pursuit, but provides a few very interesting research questions.

In [7] Grothendieck establishes a particularly generalizable concept of differential operators for modules of commutative rings by considering the bimodules of morphisms between them. Lunts and Rosenberg generalize this concept to non-commutative rings in [2], and, specifically, to graded-non-commutative algebras where they are able to deform the defining relation with a bicharacter γ. We recall these definitions at the beginning of Chapter 2, before outlining an adaption more suitable to our specific setting: shifted γ-differential operators. Namely, the setting in which k is an algebraically closed field of characteristic 0, an $R = \oplus_\Gamma R_a$ a commutative k-algebra graded by an abelian group Γ. For Γ we fix a bicharacter $\gamma : \Gamma \times \Gamma \to k^\times$. Over R we consider a Γ-graded R-bimodule M. For $a \in \Gamma$, we define

$$\mathfrak{Z}^0_{\gamma,a}(M,R) = \{m \in M \mid r \cdot m - \gamma(|r| + a, |m|)m \cdot r = 0 \text{ for all } r \in R\},$$

analagous to the centers used by Grothendieck and Lunts and Rosenberg, and put $D^0_\gamma(M,R) = \sum_{a\in\Gamma} \mathfrak{Z}^0_{\gamma,a}(M,R)$. We define $D_\gamma(M,R) = \cup D^i_\gamma(M,R)$, with each $D^i_\gamma(M,R)$ defined similarly to the degree 0 case. In particular, we wish to consider the differential operators of the k-algebra R, so we specify the bimodule $\mathrm{End}_k(R)$ of homogeneous k-linear endomorphisms of

R, and denote the shifted γ-differential operators on R simply as $D_\gamma(R)$. Half of the reason we choose to restrict ourselves to commutative algebras is because there exists literature detailing how to work with the differential operators of Lunts and Rosenberg over commutative rings, namely, [3] of Iyer and McCune, and half is due to the results of Bischof's [4], revealing to us a method to determining the sheaves of differential operators for a specific non-commutative algebraic variety. In [3] Iyer and McCune determine generators for Lunts and Rosenberg's quantum differential operators over the polynomial algebra $k[x_1, ..., x_n]$. From this we obtain what our shifted γ-differential operators are on the same algebra.

Chapter 3 addresses the second issue in problem 1 above: establishing the localizations necessary to glue together, via descent, the differential operators of Chapter 2 over for non-commutative projective space. Let $S \subseteq R$ be a multiplicitive subset of R; denote by $R' = S^{-1}R$ the algebra of fractions of R with demoninators in S, and $M' = M \otimes_R R'$ for M an R-bimodule. We first determine the elements of $D_\gamma(M', R')$ and how to obtain them from $D_\gamma(M', R)$. Switching back to the differential operators of Lunts and Rosenberg over a non-commutative algebra R, we present a result from the classical theory of localization of differential operators, that $D_\gamma(RS^{-1}) \simeq D_\gamma(R)S^{-1}$, which carries over to our setting. We end the chapter by applying the results to our polynomial algebra $k[x_1, ..., x_n]$, and show the shifted γ-differential operators are not affected by differeing the order of localization by two multiplicitive subsets $S_1, S_2 \subseteq R$, which is necessary when gluing together sheaves.

With all objects now collected in Chapters 2 and 3 we can go about laying the groundwork for putting them together. After reviewing notions for descent and gluing together categories, we apply the machinery to construct both our non-commutative projective space and its sheaves of differential operators in Chapter 6. We construct the non-commutative projective space from an affine covering of varieties corresponding to the algebras $k_{\beta_i}[x_1, ..., x_n]$, where $\{\beta_i \mid i = 0, ..., n\}$ is a collection of 2-cocycle deformations of the multiplication in the standard, commutative polynomial algebra. This means that $x_i x_j = \beta_i(e_i, e_j) x_j x_i$. In doing so we find when this collection can be determined as homogeneous degree 0 elements from localizations of $k_\beta[x_0, ..., x_n]$ by the Ore subsets $S_i = \{1, x_i, x_i^2, ...\}$.

By this point, we have established a suitable theory which will provide us with modules for the sheaf of differential operators over a non-commutative, quasi-affine space, provided we carefully apply the results of [4]. Breifly, we wished to come up with a definition for holonomic D-modules over a non-commutative space. What we obtain is an approach to such a definition. Algebraically, the concept of holonomic D-modules depends on having an assocated graded sheaf of differential operators, $\Sigma = \bigoplus_{i \geq 0} D^i/D^{i-1}$, and a way to calculate its dimension so that we can calculate the dimension of the support of the associated graded module corresponding to a D-module. In Chapter 4 we determine what the associated graded algebra is for the algebra of shifted γ-differential operators of the polynomial algebra $k[x_1, ..., x_n]$, and we calculate its GK-dimension.

The last chapter contains an extension of a construction of A. Rosenberg in [5]. Rosenberg's hyperbolic algebras are a class of algebras containing generalized Weyl algebras, aimed at simplifying the calculation of classes of irreducible representations, and thus primitive ideals. As mentioned before, differential operators, and thus Weyl algebras, provide an approach to non-commutative algebraic geometry, wherein one of the pitfalls is that the notion of points includes not only maximal ideals, as in the commutative case, but also primitive ideals. Rosenberg adapts the definition to categories, creating the hyperbolic categories, from which he outlines the calculation of their spectrum. We provide no results to accompany our definition of graded hyperbolic algebras, other than that they describe the algebras of differential operators we have been working with throughout this thesis. Future work would include an adaptation to define graded hyperbolic categories, and the calculation of the spectrum of said category, which would provide an actual collection of points to define the space in which we are studying these geometries.

1.3 Questions for Future Research

First and foremost it must be emphasized again that the construction for affine non-commutative schemes presented in this thesis eschews the requirement of having an underlying topological space corresponding to the spectrum of the ring of global sections. To that end, the highest priority for future research is adapting the (left) spectrum developed by

Rosenberg in [5] for hyperbolic rings and categories to the graded hyperbolic rings defined in chapter 5 and the yet-to-be-defined graded hyperbolic categories.

Additionally, though it is not explored in this thesis, the construction employed here and the example on which we build is general enough to extend from projective spaces to grassmannians and geometric representation theory via D-modules on the quantized flag variety. Indeed, Lunts and Rosenberg developed the theory for, via the quantum differential operators we will see in Section 2.2.3, and formulated an analogue of the Beilinson-Bernstein correspondence, which Tanisaki proved in [8]. Work here could have an eye toward an analogue for the Riemann-Hilbert correspondence in this setting of Lunts and Rosenberg, and a Kazhdan-Lustig conjecture in the setting of quantized enveloping algebras.

Chapter 2

Shifted Differential Operators

Here we wish to offer an additional description to the quantum differential operators of Lunts and Rosenberg, introduced in [2], the reference we will follow until Section 2.3. The innocuous change to the general definition of differential operators by Grothendieck made in that work introduces a second grading on the ring of differential operators. In this chapter we explore this additional structure with some standard results, and see what this ring looks like in the case of our affine covers in our attempt to construct the differenial operators on a non-commutative projective space.

2.1 Differential Operators on Commutative Rings

Just to set the stage for the differential operators we will be working with, we will review the setting for the differential operators of Grothendieck.

Let k be a commutative ring, R a commutative k-algebra, and M an R-bimodule. Recall that for $r \in R$, $\mathrm{ad}_r \in \mathrm{End}_k(M)$ is defined by $\mathrm{ad}_r(m) = r \cdot m - m \cdot r$, for all $m \in M$.

Definition 2.1.1. A filtration by R-sub-bimodules $M_i \subseteq M_j$ for all $i \leq j \in \mathbb{Z}$ of an R-bimodule M, is called a *D-filtration* if

a) $M_i = 0$ for all $i < 0$

b) $\mathrm{ad}_r(M_i) \subset M_{i-1}$ for all $r \in R$

A natural choice of filtraion, and the largest with respect to inclusion, is the filtration on M by $M_i^\vee := \{m \in M | \ \mathrm{ad}_r(m) \in M_{i-1}^\vee\}$. One can show, using the commutativity of R, that each of these $M_i^\vee$ is an R-sub-bimodule of M, hence $M^\vee = \bigcup_{i \geq -1} M_i^\vee$ is as well. It is called the *differential part of* M. Note that if R is non-commutative, this is only a k-submodule.

Recall that for $M, N \in R$-mod, we have $\mathrm{Hom}_k(M, N)$ as an R-bimodule, with $(r \cdot f)(m) = r \cdot f(m)$ and $(f \cdot r)(m) = f(r \cdot m)$. We can now make the differential part of $\mathrm{Hom}_k(M, N)$, $\mathrm{Diff}(M, N) := \mathrm{Hom}_k(M, N)^\vee$ whose elements are called k-linear differential operators from M to N.

This defines for us an order for the differential operators from M to N, by $\mathrm{Diff}(M, N)_i = \mathrm{Hom}_k(M, N)_i^\vee = \{f \in \mathrm{Hom}_k(M, N) | \ \mathrm{ad}_r(f) \in \mathrm{Hom}_k(M, N)_{i-1}^\vee\}$, and this notion of degree lines up with the notion of order of differential operator, wherein an operator d is of order n precisely if $n+1$ is the minimum number of elements in R such that $[r_{n+1}, [r_n, ...[r_1, d]...]] = 0$ for all $r_1, ..., r_{n+1} \in R$, considered as operators. Furthermore, when $M = N$ and $\mathrm{End}_k(M)$ is considered as a k-algebra, then $\mathrm{End}_k(M)_i \circ \mathrm{End}_k(M)_j \subseteq \mathrm{End}_k(M)_{i+j}$. For the sake of convenience and tradition, we will denote

$$\mathrm{Diff}(M) = \mathrm{Diff}(M, M) \subset \mathrm{End}_k(M).$$

There are numerous results that can be proven from these definitions, such as establishing localization and showing that we can construct a sheaf of differential operators for an affine scheme, and that its sections can be determined on any open subscheme, but all we really need for the moment is the construction itself.

2.2 Quantum Differential Operators

When R is not necessarily commutative, there are actually two notions of differential operator algebras between the previously outlined formulation of Grothendieck's differential operators on a commutative ring and those to be oulined in this section. These are, unsurprisingly, differential operators on a non-commutative ring, and β-differential operators, for a bicharacter β of a Γ-graded ring R. We will elucidate these briefly.

2.2.1 Differential Operators on non-commutative rings

Let R be a k-algebra which is not necessarily commutative. If we tried to take Definition 2.1.1 and use it for the bimodule $\mathrm{End}_k(R)$, we would find that left multiplication by an element of R is not necessarily a differential operator, and assuredly not a degree 0 differential operator as in the commutative case. Indeed, with $\lambda_s(t) = st$, we would need $\mathrm{ad}_r(\lambda_s)(t) = r \cdot \lambda_s(t) - \lambda_s \cdot r(t) = rst - srt = 0$ for all $r, t \in R$ for λ_r to be in $\mathrm{End}_k(R)_0^\vee$, which, obviously, we do not have in the non-commutative case.

Recall that a filtration by R-sub-bimodules $M_i \subseteq M_j$ for $i \leq j \in \mathbb{Z}$ is exhaustive if $\cup M_i = M$. In particular,

Definition 2.2.1. An R-bimodule M is *differential* if it has an exhaustive filtration $0 = M_{-1} \subseteq M_0 \subseteq M_1 \subseteq ...$, such that each M_i/M_{i-1} is a quotient of $\bigoplus\limits_{i\in I} R$, for I some not-necessarily-finite indexing set.

Remark 2.2.2. Note that the second condition on this filtration, that M_i/M_{i-1} is a quotient of $\bigoplus\limits_{i\in I} R$, is not a condition imposed in the commutative case, though it is satisfied. We will not go into detail explaining how this condition is used in creating a workable theory of differential operators on non-commutative rings, suffice to say that it allows for extension of the operators to a localization of the ring R by an Ore subset.

Definition 2.2.3. The *center* of a bimodule M is defined as usual, with k-submodule $\mathfrak{Z}(M) := \mathrm{span}_k\langle m \in M \mid \mathrm{ad}_r(m) = 0 \text{ for all } r \in R\rangle$.

Starting with this center it can be shown, as before, there exists a maximal chain of R-sub-bimodules satisfying the filtration described in Definition 2.2.1:

$$\mathfrak{z}_0 M := R\mathfrak{Z}(M)R,$$

$$\mathfrak{z}_i M := R\pi_i^{-1}(\mathfrak{Z}(M/\mathfrak{z}_{i-1}M))R,$$

where $\pi_i : M \to M/\mathfrak{z}_{i-1}M$ is the canonical R-bimodule morphism. Notice that at each level of this inductive definition we have to *generate* the R-subbimodule $\mathfrak{z}_i M$. Just like in the commutative case, to talk about differential operators of the left R-module M, we consider the R-bimodule $\mathrm{End}_k(M)$, and set $\mathrm{Diff}(M) = \cup \mathfrak{z}_i \, \mathrm{End}_k(M)$.

2.2.2 β-Differential Operators

As hinted by the use of the bicharacter β in β-differential operators, the setting will be more involved. However, it will allows for the generalization to quantum differential operators we will see next.

Let k be a field and Γ an abelian group. Let $R = \bigoplus_{a \in \Gamma} R_a$ be a Γ-graded k-algebra, with $R^h = \cup R_a$ denoting the homogeneous elements of R, and $|r| = a$ for $r \in R_a$ their degree. As we are now examining a graded ring, we will require graded modules $M = \bigoplus_{a \in \Gamma} M_a$. Denote by $\mathrm{gr}_\Gamma\, R$-bimod the category of all such Γ-graded R-bimodules.

Definition 2.2.4. Let $k^\times$ the multiplicitive group of k. A function $\beta : \Gamma \times \Gamma \to k^\times$ is called a *bicharacter* if for all $a, b, c \in \Gamma$,

$$\beta(a+b, c) = \beta(a, c)\beta(b, c), \text{ and } \qquad \beta(a, b+c) = \beta(a, b)\beta(a, c).$$

A bicharacter β is called *skew-symmetric* if $\chi(a, b)^{-1} = \chi(b, a)$.

Fix a skew-symmetric bicharacter $\beta : \Gamma \times \Gamma \to k^\star$. The construction of β-differential operators is much the same as those on a general non-commutative ring, with two main differences: the use of the *β-center* rather than the usual center, and the notion of the required filtration on a module tweaked to include the *grading action* from Γ.

Definition 2.2.5. For each $M \in \mathrm{gr}_\Gamma\, R$-mod, this action, via β, define the *grading action* of Γ on M as the representation $\sigma_M : \Gamma \to \mathrm{Aut}_k(M)$ defined by

$$\sigma_M(a)|_{M_b} := \beta(a, b)\, \mathrm{id}_{M_b} \text{ for } a, b \in \Gamma.$$

This means that Γ acts on the k-algebra R as well, via the representation σ_R. We call

$$R_\Gamma := k[\Gamma]\#R = \bigoplus_{a \in \Gamma} Ra$$

the crossed product algebra resulting from this action. Explicitly, we have

$$a \cdot r = \sigma_R(a)(r)a = \beta(a,b)ra, \text{ for } r \in R_b,\ a,b \in \Gamma.$$

Definition 2.2.6. Let β be a bicharacter of Γ with R a Γ-graded ring and $M \in \mathrm{gr}_\Gamma\, R$-bimod. For homogeneous elements $r \in R^h$ and $m \in M^h$ we define the β-*commutator* as

$$[r,m]_\beta = mr - \beta(|m|,|r|)rm.$$

Definition 2.2.7. The β-*center* of a graded R-bimodule M is defined as the k-module

$$\mathfrak{Z}_\beta(M) = \mathrm{span}_k\langle m \in M^h \mid [r,m]_\beta = 0 \text{ for all } r \in R^h\rangle.$$

We say an exhaustive filtration by R-sub-bimodules $0 = M_{-1} \subseteq M_0 \subseteq ...$ is β-*differential* if M_i/M_{i+1} is a quotient of $\bigoplus_{i \in I} R_\Gamma$.

It is with this center we construct our maximal β-differential sub-bimodule in exactly the same way as in the general non-commutative case. It can be shown that $\cup \mathfrak{z}_{\beta i} M$ is maximal among β-differential R-sub-bimodules of M, where

$$\mathfrak{z}_{\beta 0} M := R\mathfrak{Z}_\beta(M)R,$$
$$\mathfrak{z}_{\beta i} M := R\pi_i^{-1}(\mathfrak{Z}_\beta(M/\mathfrak{z}_{\beta i-1}M))R,$$

and $\pi_i : M \to M/\mathfrak{z}_{\beta i-1}M$ is the canonical R-bimodule morphism. To talk about β-differential operators, we consider the R-bimodule $\mathrm{End}_k(M)$ for a left R-module M, and as before we set $\mathrm{Diff}_\beta(M) = \bigcup_i \mathfrak{z}_{\beta i}\,\mathrm{End}_k(M)$.

2.2.3 Quantum Differential Operators

Finally, we arrive at the quantum differential operators of Luntz and Rosenberg. The setting is much the same, except, as noted, there is an additional grading introduced. For $M \in \mathrm{gr}_\Gamma\, k$-mod, and $\gamma \in \Gamma$, let

$$M[\gamma] = \bigoplus_{a\in\Gamma} M[\gamma]_a, \text{ with } M[\gamma]_a = M_{\gamma+a} \text{ for any } a \in \Gamma.$$

Note that, as before, this defines the action of Γ on the objects of $\mathrm{gr}_\Gamma\, R$-bimod, with

$$\sigma_{M[\gamma]}|_{M[\gamma]_b} := \beta(\gamma + a, b)\,\mathrm{id}_{M[\gamma]_b}.$$

For R a Γ-graded k-algebra, the previously defined R_Γ is a Γ-graded R-bimodule. Denote by R_Γ^q the sum

$$R_\Gamma^q = \bigoplus_{\gamma\in\Gamma} R_\Gamma[\gamma].$$

Definition 2.2.8. A module $M \in \mathrm{gr}_\Gamma\, R$-bimod is called *q-differential* if it has an exhaustive filtration by graded R-sub-bimodules with $M_{-1} = 0$, such that each successive quotient M_i/M_{i-1} is a subquotient of a direct sum of copies of the R-bimodule R_Γ^q.

Definition 2.2.9. For $M \in \mathrm{gr}_\Gamma\, R$-bimod, the *q-center* of M, $\mathfrak{Z}_q(M)$ is defined as the k-span of homogeneous elements in $m \in M^h$, for which there exists a $b \in \Gamma$ such that

$$mr = \beta(|m| + b, |r|)rm \text{ for all } r \in R.$$

M is called *q-central* if $M = R\mathfrak{Z}_q(M)R$.

For a general bimodule M, there is a canonical, maximal q-differential sub-bimodule

$M_q = \bigcup_{i \geq -1} \mathfrak{z}_{qi} M$ where the $\mathfrak{z}_{qi} M$ are defined iteratively:

$$\mathfrak{z}_{q0} M := R\mathfrak{Z}_q(M)R,$$
$$\mathfrak{z}_{qi} M/\mathfrak{z}_{qi-1} M = R\mathfrak{Z}_q(M/\mathfrak{z}_{qi-1} M)R.$$

As in the commutative case, there are certain R-bimodules we wish to consider; namely, the (bi)modules of morphisms between graded left R-modules. Let M and N be graded left R-modules and let $\operatorname{gr}\operatorname{Hom}(M,N)$ denote the k-submodule of $\operatorname{Hom}_k(M,N) = \bigoplus_{\gamma \in \Gamma} \operatorname{Hom}_k(M,N)_\gamma$ spanned by homogeneous elments.

Definition 2.2.10. For $M,$ and $N \in \operatorname{gr}_\Gamma R$-mod, define the *q-differential operators* of order at most n from M to N to be

$$\operatorname{Diff}_q^n(M,N) = \mathfrak{z}_{qn} \operatorname{gr}\operatorname{Hom}(M,N),$$

and $\operatorname{Diff}_q(M,N) = \bigcup_{i \geq -1} \operatorname{Diff}_q^i(M,N)$.

Notice, again, that at each step of constructing the *q-differential* submodule of a graded bimodule, we must generate the submodule over R. It may seem obvious from this generating, but the problem is that the center is not closed under action from the ring. Explicitly, consider $m \in \mathfrak{Z}_q(M)$, then the claim is that $rm \notin \mathfrak{Z}_q(M)$. Indeed, $(rm)r' = \beta(|m|,|r'|)r'(rm) \neq \beta(|rm|,|r'|)r'rm$ unless $\beta(|r|,|r'|) = 1$.

2.3 Shifted Differential Operators

We will retain the notation from the setting of the quantum differential operators of [2] in section 2.2.3 above with the significant alteration that R is now a commutative, Γ-graded k-algebra. The loss of generality is, frankly, disappointing, but allows us to describe out specific setting, and, as noted in [4], will extend to some special non-commutative rings. The exact ones, in fact, used as examples in [2]. As we just saw, the introduction of quantum differential operators and this idea of shifting introduces an additional grading. Instead of combining all of those graded components into one "quantum" center and generating the bimodule at each step, we can instead make a center for each shift, whence it turns out we do not have to generate to get a module closed under the ring action. After all, with R commutative, we are now guaranteed that $\beta(|r|, |r'|) = 1$ for all $r \in R$.

Definition 2.3.1. Let M be a Γ-graded R-bimodule. For all $a \in \Gamma$ we define the *a-shifted center* of M as

$$\mathfrak{Z}^0_{\beta,a}(M,R) := \langle m \in M^h \mid mr = \beta(|m| + a, |r|)rm \text{ for all } r \in R^h \rangle_k. \tag{2.1}$$

Further, define the 0-degree *shifted-β-differential operators* as

$$D^0_\beta(M,R) := \sum_{a \in \Gamma} \mathfrak{Z}^0_{\beta,a}(M,R). \tag{2.2}$$

With a skew-symmetric β, it is immediate that we also must have $rm = \beta(|r|, |m| + a)mr$ for all $r \in R^h$ and $m \in \mathfrak{Z}^0_{\beta,a}(M,R)$, an element of the R-bimodule M.

The following theorem shows us that the action of R moves elements of M between these shifted centers, just as it shifts them between the graded components of the module itself.

Theorem 2.3.2.

1. *$\mathfrak{Z}^0_{\beta,a}(M,R)$ is a Γ-graded k-vector subspace of M.*

2. *For all $r \in R^h$,* $\begin{cases} r\mathfrak{Z}^0_{\beta,a}(M,R) \subseteq \mathfrak{Z}^0_{\beta,a-|r|}(M,R), \\ \mathfrak{Z}^0_{\beta,a}(M,R)r \subseteq \mathfrak{Z}^0_{\beta,a-|r|}(M,R). \end{cases}$

   In particular, $D^0_\beta(M,R)$ is an R-bimodule.

Proof. The first is obvious. For the second, consider $m \in \mathfrak{Z}^0_{\beta,a}(M,R)$ for some $a \in \Gamma$. Then, by definition, $mr = \beta(|m| + a, |r|)rm$ for all $r \in R$. Thus, for any $s \in R$, we have

$$\begin{aligned}(rm)s = r(ms) &= \beta(|m| + a, |s|)r(sm) \\ &= \beta(|rm| - |r| + a, |s|)rsm \\ &= \beta(|rm| - |r| + a, |s|)s(rm),\end{aligned}$$

since for a Γ-graded module M we have $|rm| = |r| + |m|$. Thus, $rm \in \mathfrak{Z}^0_{\beta,a-|r|}(M, R)$ for $m \in \mathfrak{Z}^0_{\beta,a}(M, R)$. □

We will again rely on these shifted centers, gathering them to form the collection of differential operators for each degree. To define the higher order operators we need, just as in the classical case, a bracket. Here, we need one that takes into account both the bicharacter β relating the left and right actions on our bimodules and the shifting discussed in the above theorem, 2.3.2.

Definition 2.3.3. Let $M \in \mathrm{gr}_\Gamma\, R$-mod, with $m \in M^h$ and $r \in R^h$. For $a \in \Gamma$ define their *a-shifted β-commutator* as

$$[m, r]_{\beta,a} = mr - \beta(|m| + a, |r|)rm.$$

Definition 2.3.4. For a bimodule $M \in \mathrm{gr}_\Gamma\, R$-mod, we define the *$a$-shifted, $i+1$-th degree center* iteratively to be

$$\mathfrak{Z}^{i+1}_{\beta,a}(M, R) := \langle m \in M^h \mid [m, r]_{\beta,a} \in D^i_\beta(M, R) \text{ for all } r \in R^h\rangle_k.$$

We then make $D^{i+1}_\beta(M, R) := \sum_{a\in\Gamma} \mathfrak{Z}^{i+1}_{\beta,a}(M, R)$.

Proposition 2.3.5.

1. *$\mathfrak{Z}^{i+1}_{\beta,a}(M, R)$ is a Γ-graded k-vector subspace of M.*

2. For all $r \in R^h$ $\begin{cases} r\mathfrak{Z}^{i+1}_{\beta,a}(M,R) \subseteq \mathfrak{Z}^{i+1}_{\beta,a-|r|}(M,R), \\ \mathfrak{Z}^{i+1}_{\beta,a}(M,R)r \subseteq \mathfrak{Z}^{i+1}_{\beta,a-|r|}(M,R). \end{cases}$

In particular, $D^{i+1}_\beta(M,R)$ is an R-bimodule.

Proof. Again, the first is obvious. For the second, consider $m \in \mathfrak{Z}^0_{\beta,a}(M,R)$ for some $a \in \Gamma$. Then, by definition, $[m,r]_{\beta,a} \in D^i_\beta(M,R)$ for all $r \in R$. Thus, for any $s \in R$, we have

$$\begin{aligned}(rm)s = r(ms) &= r(\beta(|m|+a,|s|)sm + d) \\ &= \beta(|rm| - |r| + a, |s|)rsm + rd \\ &= \beta(|rm| - |r| + a, |s|)s(rm) + rd,\end{aligned}$$

where $[m,s]_{\beta,a} = d \in D^i_\beta(M,R)$. □

Though we have made our definitions for general bimodules, recall that we have a specific motivation here: the bimodule $\mathrm{Hom}_k(M,N)$ for left-R-modules M and N. But, really, we seek $\mathrm{End}_k(M)$ for a left-R-module M. But really, really, $\mathrm{End}_k(R)$. To this end, we denote $D_\beta(M) := \bigcup_{i \geq -1} D^i_\beta(\mathrm{End}_k(M), R)$, with $D^{-1}_\beta = 0$.

Proposition 2.3.6. *For $M \in \mathrm{gr}_\Gamma$ R-mod, $D_\beta(M)$ is a ring. In particular, we have*

$$D^i_{\beta,a}(M) \circ D^j_{\beta,b}(M) \subseteq D^{i+j}_{\beta,a+b}(M).$$

Proof. It is known that $\mathrm{End}_k(M)$ is a ring; we simply need to show $D_\beta(M) \subseteq \mathrm{End}_k(M)$, obviously containing identity elements, is closed under composition. Let $\varphi \in D^i_{\beta,a}(M)$ and $\psi \in D^j_{\beta,b}(M)$, then for $r \in R^h$ we have

$$\begin{aligned}(\varphi \circ \psi)r &= \varphi \circ \beta(|\psi| + b, |r|)r\psi \\ &= \beta(|\varphi| + a, |r|)\beta(|\psi| + b, |r|)r\varphi \circ \psi \\ &= \beta(|\varphi| + |\psi| + a + b, |r|)r(\varphi \circ \psi).\end{aligned}$$

Thus, $\varphi \circ \psi \in D^{i+j}_{\beta,a+b}(M)$. $\square$

In the next chapter we will discuss the localization of our rings of differential operators, as it is imperative we understand the construction for us to pursue our goal of examining the differential operators on non-commutative projective space and their modules. Imminently, we will apply this construction to the relevant commutative ring with we plan to work. Eventually, however, we must address the fact that our construction is for commutative rings, while we have again, just now, claimed the goal is to work with a non-commutative projective space.

2.4 Examples on Polynomial Algebras

As in [3], we will start out on $k[x]$, polynomials of one variable before moving on to the general n-variable case. Let k be an algebraically closed field of characteristic 0. In this case, our grading group $\Gamma = \mathbb{Z}$. Denote by $\beta : \mathbb{Z} \times \mathbb{Z} \to k^\times$ the bicharacter determining grading action by $\mathbb{Z}$ on $k[x]$, defined by $\beta(a,b) = q^{ab}$ for some $q \in k^\times$, then we have for $M \in \operatorname{gr}_{\mathbb{Z}} k[x]$-bimod, $\sigma_M : \mathbb{Z} \to \mathrm{GL}_k(M)$ defined as before:

$$\sigma_M(a)(m) = \beta(a, |m|)m = q^{a|m|}m \text{ for all } a \in \mathbb{Z} \text{ and } m \in M^h.$$

Be reminded, we are considering homogeneous k-linear morphisms; that is, morphisms $\varphi \in \operatorname{Hom}_k(M,N) = \bigoplus_{\alpha \in \mathbb{Z}} \operatorname{Hom}_k(M,N)_\alpha$ such that $\varphi(M_a) \subseteq N_{|\varphi|+a}$. For $f \in k[x]$ we will denote by λ_f the endomorphism of $k[x]$ defined as left multiplication by f. Explicitly: $\lambda_f(g) = fg$ for all $g \in k[x]$. We will use this convention when referring to the image of elements of $k[x]$ under the standard embedding $k[x] \hookrightarrow \operatorname{End}_k(k[x])$. i.e. $f \mapsto \lambda_f$. There is a right multiplication as well, ρ_f. Note that this means $\lambda_f \in \operatorname{End}_k(k[x])_a$ for f of homogeneous degree a. On $\mathrm{End}_k(k[x])$ we define the a-shifted, β-commutator

$$[\varphi, \psi]_{\beta,a} := \varphi\psi - \beta(|\varphi| + a, |\psi|)\psi\varphi \text{ for homogeneous } \varphi, \psi \in \mathrm{End}_k(k[x]).$$

It is important to note that $\sigma_a \in \mathrm{End}_k(k[x])_0$ for all $a \in \mathbb{Z}$, since the operators simply add an invertible scalar coefficient, and thus do not change the homogeneous degree of any $f \in k[x]$.

Now that we have the setting, the first obvious question to ask is about what elements are in $\mathfrak{Z}^0_{\beta,a}(\mathrm{End}_k(k[x]), k[x])$.

Proposition 2.4.1. *If* $\varphi \in \mathfrak{Z}^0_{\beta,a}(\mathrm{End}_k(k[x]), k[x])$, *then* $\varphi = \rho_f \sigma_a$.

Proof. Suppose $\varphi \in \mathfrak{Z}^0_{\beta,a}(\mathrm{End}_k(k[x]), k[x])$, and suppose that $\varphi(1) = f \in k[x]$. Then

$$\begin{aligned} [\varphi, \lambda_g]_{\beta,a}(1) = \varphi\lambda_g(1) - \beta(|\varphi| + a, |g|)\lambda_g\varphi(1) = 0 \\ \implies \varphi(g) = \beta(|\varphi| + a, |g|)gf \\ \implies \varphi = \rho_f\sigma_a. \end{aligned}$$

On the other hand,

$$\begin{aligned} [\rho_f\sigma_a, \lambda_g]_{\beta,|f|+a}(h) &= \rho_f\sigma_{|f|+a}(gh) - \beta(|f| + a, |\lambda_g|)\rho_f\sigma_{|f|+a}(h) \\ &= \beta(|f| + a, |gh|)ghf - \beta(|f| + a, |g|)\beta(|f| + a, |h|)ghf \\ &= 0. \end{aligned}$$

Thus $\rho_f\sigma_{|f|+a} \in \mathfrak{Z}^0_{\beta,a}(\mathrm{End}_k(k[x]), k[x])$ for any $f \in k[x]$ and $a \in \mathbb{Z}$. $\square$

Natrually, now we wish to consider $\mathfrak{Z}^1_\beta(\mathrm{End}_k(k[x]), k[x])$, and to do so we will need degree-lowering operators. Of course, we have some special ones for this specific setting; these are defined in [3].

Definition 2.4.2. For $a \in \mathbb{Z}$, the operator ∂^{β^a} is defined by the equation

$$\begin{aligned}\partial^{\beta^a}(x^n) :&= [n]_{\beta(a,1)}x^{n-1} \\ &= (1 + \beta(a,1) + \beta(a,2) + ... + \beta(a,n-1))x^{n-1}.\end{aligned}$$

For $a \in \mathbb{Z}$, we can rewrite this in terms of the operators σ_a as

$$\partial^{\beta^a} = \frac{1-\beta(1,1)}{1-\beta(1,a)}\partial^{\beta^1}[\sigma_0 + \sigma_1 + ... + \sigma_{a-1}], \text{ for } a > 0, \tag{2.3}$$

$$\partial^{\beta^a} = \frac{1-\beta(1,1)}{1-\beta(1,-a)}\partial^{\beta^{-1}}[\sigma_0 + \sigma_{-1} + ... + \sigma_{a-1}] \text{ for } a < 0. \tag{2.4}$$

It's clear that ∂^{β^a} is an element of $\mathfrak{Z}^1_{\beta,a}(\mathrm{End}_k(k[x]), k[x])$. Indeed,

$$\begin{aligned}[\partial^{\beta^a}, x]_{\beta,a+1}(x^n) &= \partial^{\beta^a}(x^{n+1}) - \beta(|\partial^{\beta^a}| + a + 1, 1)x\partial^{\beta^a}(x^n) \\ &= [n+1]_{\beta(a,1)}x^n - \beta(a,1)[n]_{\beta(a,1)}x^n \\ &= x^n.\end{aligned}$$

Thus, $[\partial^{\beta^a}, x]_{\beta,a+1} = 1 \in D^0_\beta$. While this is just an example, we get a representation in the form of $\lambda_f\sigma_a$ for any other $[\partial^{\beta^a}, x^n]_{\beta,a+1}$, for which there exists a $b \in \mathbb{Z}$ so that $[[\partial^{\beta^a}, x^n]_{\beta,a}, x^m]_{\beta,b} = 0$.

Theorem 2.4.3. *The first order differential operators $\{\partial^{\beta^a} \mid \text{for } a = 0, 1\}$ generate D^1_β as an R-module.*

Proof. Since we have defined D^1_β as the collection $\bigcup\limits_{a\in\Gamma} \mathfrak{Z}^1_{\beta,a}(\mathrm{End}_k(k[x]), k[x])$, this is really just a statement about $\mathfrak{Z}^1_{\beta,a}(\mathrm{End}_k(k[x]), k[x])$. $\square$

Chapter 3

Localization

Here, we expand the notion of shifted differential operators introduced in the previous chapter so that we may mimic the algebraic geometry construction of schemes. The main goal is to describe what happens when we make the differential operators on the localization of the ring, and to compare this to what happens to the ring of differential operators when we localize it. Specifically, in the language of shifted differential operators: what happens to the shifted centers? This may sound like a banal algebraic result, but localization is an extremely important construction in algebraic geometry in general, and specifically here for obtaining our main objective.

Let us recall the localization concept in detail. Let R be a commutative, unital ring and $1 \in S \subseteq R$ be a multiplicitive subset, i.e. a multiplicitive submonoid of R. The essential idea is that we are adding the formal multiplicitive inverses to the elements of S to the ring R. To this end, we make the cartesian product $R \times S$ and quotient by the equivalence relation

$$(r, s) \sim (r', s') \text{ if and only if there exists } s_1 \in S \text{ such that } s_1(rs' - r's) = 0.$$

This ring of fractions denoted $S^{-1}R$ consists of the equivalence classes for (r, s) denoted by

$\frac{r}{s}$. This collection is a ring with the familiar addition and multiplication:

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'} \text{ and}$$
$$\frac{r}{s}\frac{r'}{s'} = \frac{rr'}{ss'},$$

with additive identity $\frac{0}{1}$, additive inverses $-\frac{a}{b}$, and multiplicitive identity $\frac{1}{1}$. It is routine to check that these operations are well-defined.

In the non-commutative case, we have to be more careful about how we define our elements. For instance, if we wish to consider *right inverses* of our multiplicitive set $S \subseteq R$ of the non-commutative, unital ring R, we mean that for $s \in S$ there exists s_r^{-1} such that $ss_r^{-1} = 1$. Thus the ring RS^{-1} has as its elements rs^{-1} for $r \in R$ and $s \in S$. Since these s^{-1} are still elements of the ring, they can be multiplied on the left, and the necessary well-defined-ness of multiplication of these elements demands the ability to write $s^{-1}r$ as $r's'^{-1}$ for some $r' \in R$ and $s' \in S$.

3.1 Localization of Shifted Differential Operators

Definition 3.1.1. Let S be a multiplicitive subset of a ring R, then S is *right Ore* if for each $r \in R$ and $s \in S$, there exists $s' \in S$ and $r' \in R$ such that $rs' = sr'$. Alternatively, $rS \cap sR \neq \emptyset$. We will also require that s is not a zero divisor for all $s \in S$, or, a regular element.

The following lemmas make it much easier to show that a multiplicitive subset is Ore than is done simply by definition. For a longer, more detailed discussion of the localization non-commutative rings, see [9].

Lemma 3.1.2. *The multiplicatively closed subset of a ring R generated by a family of right Ore sets of R is right Ore.*

Proof. See Lemma 4.1 in [10]. □

Definition 3.1.3. Let V be a k-vector space. A k-endomorphism α of V is called *locally nilpotent* if for all $v \in V$ there exists $m \in \mathbb{N}$ such that $\alpha(v) = 0$.

Lemma 3.1.4. *Let x be a a regular element of R such that the map $\delta_x : a \mapsto ax - xa$ is locally nilpotent, then $S = \{1, x, x^2, ...\}$ is Ore.*

Proof. See Lemma 4.7 in [10]. □

Example 3.1.5. Consider $\gamma : \mathbb{Z}^n \times \mathbb{Z}^n \to k^\times$ a skew symmetric bicharacter and a 2-cocycle for the group $\mathbb{Z}^n$ which grades the ring of polynomials in n variables $k[x_1, ..., x_n]$ over the field k. Instead, we will work with the ring of *skew polynomials* $k_\gamma[x_1, ..., x_n]$, wherein $x_i x_j = \gamma(e_i, e_j)x_j x_i$, with e_i the i^{th} basis vector in $\mathbb{Z}^n$. Note here, that with the specified grading, the homogeneous elements of $k_\gamma[x_1, ..., x_n]$ are just monomials.

If we make $S_i = \{1, x_i, x_i^2, ...\}$, then S_i is right Ore. In the notation of definition 3.1.1, we set $r = x_1^{a_1} x_2^{a_2} ... x_m^{a_m}$ and $s = x_i^m$. Clearly, $s' = x_i^m$ and $r' = C x_1^{b_1} x_2^{b_2} ... x_n^{b_n}$ with $C \in k$. We have,

$$rs' = (x_1^{a_1} x_2^{a_2} ... x_n^{a_n})x_i^n = \left(\prod_{k=i+1}^{n} \gamma(a_k e_k, m e_i)\right) x_1^{a_1} ... x_i^{a_i+m} ... x_n^{a_n} \text{ and,}$$
$$sr' = x_i^m (C x_1^{b_1} x_2^{b_2} ... x_n^{b_n}) = \left(\prod_{k=1}^{i-1} \gamma(m e_i, b_k e_k)\right) x_1^{b_1} ... x_i^{b_i+m} ... x_n^{b_n}$$

We need $s = s'$, and $r' = Cr$ with $C = \frac{\prod_{k=i+1}^{n} \gamma(a_k e_k, n e_i)}{\prod_{k=1}^{i-1} \gamma(n e_i, a_k e_k)} = \prod_{\substack{k=1 \\ k \neq i}}^{n} \gamma(a_k e_k, n e_i)$. Consequently, S_i is a right Ore subset of $k_\gamma[x_1, ..., x_n]$. The ring $k_\gamma[x_1, ..., x_n]S_i^{-1}$ is the ring whose elements are $f x_i^{-m}$ for $f \in k_\gamma[x_1, ..., x_n]$; its homogeneous elements are monomials $c x_1^{a_1} ... x_n^{a_n} x_i^{-m}$ with $c \in k$. Additionally, 3.1.2 allows us to generate Ore subsets containing monomials in more than one variable by using these Ore subsets S_i. From those, we can make the localization containing the inverses of these monomials.

It's important to note the grading here as well. We have $x_i^{-1} \in k_\gamma[x_1, ..., x_n]_{-e_i}$, and thus the degree of the monomial $x_1^{a_1}...x_n^{a_n}x_i^{-m}$ is

$$\deg(x_1^{a_1}...x_n^{a_n}x_i^{-m}) = (a_1, ..., a_n) - me_i = (a_1, ..., a_i - m, ..., a_n) \in \mathbb{Z}^n.$$

Example 3.1.6. Let $D = D(k[x_1, ..., x_n])$ denote the ring of differential operators on the ring of polynomials in n variables. Since our ring is commutative, we work in the setting of section 2.1. It is known that $D \simeq A_n$, the n-dimensional Weyl algebra, the k-algebra generated by $x_1, ..., x_n$ and $\partial_1, ...\partial_n$ subject to the relations

$$[\partial_i, \partial_j] = [x_i, x_j] = 0, \text{ and, } [\partial_i, x_j] = \delta_{ij} \text{ for all } 1 \leq i, j \leq n,$$

with δ_{ij} the usual Kronecker delta function. Due to these relations, we can write any element of D as a sum of monomials of the form

$$ax_1^{a_1}...x_n^{a_n}\partial_1^{b_1}...\partial_n^{b_n}. \tag{3.1}$$

As before, consider the multiplicitive subset $S_i = \{1, x_i, x_i^2, ...\} \subset D$. It is evident that D is a infinite-dimensional k-vector space, and any finite dimensional subspace will be generated by monomials as in equation (3.1). Suppose $V \subseteq D$ is one such finite dimensional subspace and $v \in V$ is one of its finitely many generators. If $v = x_1^{a_1}...x_n^{a_n}$, then we have directly $[x_i, v] = 0$. If v contains non-zero powers of any of the partial derivatives ∂_k, then $v \in D^l$ is a differential operator of order $l \geq 0$ and thus $[v, f] = -[f, v] \in D^{l-1}$ for any $f \in k[x_1, ..., x_n]$. Simply set $f = x_i$ and proceed iteratively to see that $[x_i, \cdot]$ is nipotent for any such V. Thus, by Lemma 3.1.4, $S_i \subset D$ is Ore in D, and we can form the ring of right fractions DS_i^{-1}.

To see what the ring DS_i^{-1} looks like, consider $A_1 = D(k[t])$, and let $t = x^{-1}$. We have $\frac{d}{dt} = \frac{dx}{dt}\frac{d}{dx} = (\frac{d}{dx}t)^{-1}\frac{d}{dx} = (-x^2)\frac{d}{dx}$. Thus, we can write $\partial_{x^{-1}} = -x^2\frac{d}{dx}$. To find the commutation relations, we have in $D(k[t])$ that $\partial_t t - t\partial_t - 1 = 0$, which gives us

$$\partial_{x^{-1}} x^{-1} - x^{-1}\partial_{x^{-1}} - 1 = 0,$$
$$(-x^2\partial_x)x^{-1} + x^{-1}x^2\partial_x - 1 = 0, \text{ or}$$
$$x^{-1}\partial_x - \partial_x x^{-1} - x^{-2} = 0,$$

after multiplying by x^{-2} on both sides. This makes the ring $DS_i^{-1} = \langle x_1, ..., x_n, x_i^{-1}, \partial_1, ..., \partial_n \rangle_k$, subject to the relations

$$[\partial_i, \partial_j] = [x_i, x_j] = 0, \text{ and, } [\partial_i, x_j] = \delta_{ij} \text{ for all } 1 \leq i, j \leq n,$$
$$\text{with the addition } [\partial_j, x_i^{-1}] = 0, \text{ and } [\partial_i, x_i^{-1}] = -x_i^{-2}.$$

As we noted in the previous chapter, there is another grading on the ring of shifted differential operators and it is our goal here to see what happens to both these gradings and the filtration by which we define these operators.

Consider $R = \bigoplus_{a\in\Gamma} R_a$, a Γ-graded, commutative ring and $\beta : \Gamma \times \Gamma \to k^\times$ a bicharacter of the group Γ. Suppose $S \subseteq R$ is a multiplicitive Ore subset of Γ-homogeneous elements of R, and denote by $R' = RS^{-1}$ the ring of right fractions with denominators in S.

Proposition 3.1.7. *For $M \in \mathrm{gr}_\Gamma\, R$-bimod and $M' = R' \bigotimes_R M \bigotimes_R R'$ we have, for all $s \in S$*

1. $\frac{1}{s}\,\mathfrak{Z}^0_{\beta,a}(M, R) \subseteq \mathfrak{Z}^0_{\beta,a+|s|}(M', R)$,
2. $\mathfrak{Z}^0_{\beta,a}(M', R') = \mathfrak{Z}^0_{\beta,a}(M', R)$.

Proof. For the first, simply observe for $m \in \mathfrak{Z}^0_{\beta,a}(M, R)$ and $r \in R^h$:

$$\begin{aligned}(\frac{1}{s}m)r &= \frac{1}{s}(mr)\\ &= \beta(|m| + a, r)\frac{1}{s}rm\\ &= \beta(|\frac{1}{s}m| + a + |s|, r)r\frac{1}{s}m.\end{aligned}$$

Hence $\frac{1}{s}m \in \mathfrak{Z}^0_{\beta,a+|s|}(M', R)$, and 1) follows.

Now, consider $m' = \frac{1}{s} \otimes m \otimes \frac{1}{s'} \in \mathfrak{Z}^0_{\beta,a}(M', R)$. For $t \in S$ we have by definition,

$$\begin{aligned}\frac{1}{s} \otimes m \otimes \frac{1}{s'} &= \left(\frac{1}{s} \otimes m \otimes \frac{1}{s'}\right) t\frac{1}{t} \\ &= \beta(|m'| + a, |t|)t\left(\frac{1}{s} \otimes m \otimes \frac{1}{s'}\right)\frac{1}{t},\end{aligned}$$

thereby forcing $\left(\frac{1}{s} \otimes m \otimes \frac{1}{s'}\right)\frac{1}{t} = \beta(|m'|+a, -|t|)\frac{1}{t}\left(\frac{1}{s} \otimes m \otimes \frac{1}{s'}\right)$. This puts $m' \in \mathfrak{Z}^0_{\beta,a}(M', R')$ and shows $\mathfrak{Z}^0_{\beta,a}(M', R) \subseteq \mathfrak{Z}^0_{\beta,a}(M', R')$. For the reverse inclusion, simply note that $R \subseteq R'$ since we've specified that S contains no zero-divisors. $\square$

Recalling that we have defined $D^0_\beta(M, R)$ as the sum $\sum_{a\in\Gamma} \mathfrak{Z}^0_{\beta,a}(M, R)$, the next theorem is essentially a corollary of the previous proposition. It appears in [2] in Lemma 1.2.1.1 and Theorems 1.2.1 and 3.2.2; here, we couch it in terms of the shifted differential operators and provide an alternative proof.

Theorem 3.1.8. *For $R = \bigoplus_{a\in\Gamma} R_a$ a commutative, Γ-graded integral domain and $S \subset R$ a multiplicitive set, denote $R' = RS^{-1}$. Let $M \in \mathrm{gr}_\Gamma$ R-bimod, a bimodule and $M' = R' \otimes M$. We have*

$$D^i_\beta(M', R') = D^i_\beta(M', R) = R' \bigotimes_R D^i_\beta(M, R) \bigotimes_R R',$$

for all $i \geq 0$.

Proof. We will consider the base case $i = 0$ and proceed by induction. We have already seen that for all $m' \in \mathfrak{Z}^0_{\beta,a}(M', R)$ there exist $s', s \in S$ such that $m = sm's' \in \mathfrak{Z}^0_{\beta,a-|s|-|s'|}(M, R)$.

Hence, $m' \in \frac{1}{s}\mathfrak{Z}^0_{\beta,a-|s|-|s'|}(M, R)\frac{1}{s'}$. and we have

$$\begin{aligned}R'(\sum_{a\in\Gamma} \mathfrak{Z}^0_{\beta,a}(M, R))R' &= \sum_{a\in\Gamma} \mathfrak{Z}^0_{\beta,a}(M', R) \\ &= \sum_{a\in\Gamma} \mathfrak{Z}^0_{\beta,a}(M', R'),\end{aligned}$$

where the last equality is from the previous proposition.

Now assume that the statement is true for i. That assumption grants

$$\begin{aligned}
\mathfrak{Z}^{i+1}_{\beta,a}(M',R) &= \langle m \in M^h \mid [m,r]_{\beta,a} \in D^i_\beta(M',R) \text{ for all } r \in R^h\rangle_k \\
&= \langle m \in M^h \mid [m,r]_{\beta,a} \in D^i_\beta(M',R') \text{ for all } r \in R^h\rangle_k \\
&= \langle m \in M^h \mid [m,r]_{\beta,a} \in R'D^i_\beta(M,R)R' \text{ for all } r \in R^h\rangle_k, \text{ and} \\
D^{i+1}_\beta(M',R) &= \sum_{a\in\Gamma} \mathfrak{Z}^{i+1}_{\beta,a}(M',R),
\end{aligned}$$

where $[m,r]_{\beta,a} = mr - \beta(|m|+a,|r|)rm$. Let $s \in S$ and $m \in \mathfrak{Z}^{i+1}_{\beta,a}(M,R)$, then we have

$$\begin{aligned}
(\frac{1}{s}m)r = \frac{1}{s}(mr) &= \frac{1}{s}(\beta(|m|+a,|r|)rm + d), \text{ for some } d \in D^i_\beta(M,R) \\
&= \beta(|\frac{1}{s}m| + a + |s|, |r|)r\frac{1}{s}m + \frac{1}{s}d,
\end{aligned}$$

for any $r \in R^h$, and thus $\frac{1}{s}\mathfrak{Z}^{i+1}_{\beta,a}(M,R) \subseteq \mathfrak{Z}^{i+1}_{\beta,a+|s|}(M',R)$. On the other hand, if $m' \in \mathfrak{Z}^{i+1}_{\beta,a}(M',R)$, there exist $s,s' \in S$ such that $m = sm's' \in M$, with $m \in \mathfrak{Z}^{i+1}_{\beta,a-|s|-|s'|}(M,R)$, via the same proof as Proposition 3.1.7. Hence

$$\begin{aligned}
R' \bigotimes_R \sum_{a\in\Gamma} \mathfrak{Z}^{i+1}_{\beta,a}(M,R) \bigotimes_R R' &= \sum_{a\in\Gamma} \mathfrak{Z}^{i+1}_{\beta,a}(M',R) = D^{i+1}_\beta(M',R) \\
&= \sum_{a\in\Gamma} \mathfrak{Z}^{i+1}_{\beta,a}(M',R') = D^{i+1}_\beta(M',R').
\end{aligned}$$

□

Additionally, applying Proposition 2.3.6 from the last chapter to the bimodule $\mathrm{End}_k(M)$ for $M \in \mathrm{gr}_\Gamma R$-mod, we now know that

$$D_\beta(M') = D_\beta(\mathrm{End}_k(M)', R') = R' \bigotimes_R D_\beta(\mathrm{End}_k(M), R) \bigotimes_R R'$$

is a ring. From there we have that

$$D^{i}_{\beta,a}(M') \circ D^{j}_{\beta,b}(M') \subseteq D^{i+j}_{\beta,a+b}(M').$$

3.2 Localization of $\mathrm{Diff}_{\beta}(R)$

We are now ready to return to our desired setting: the example of $k[x_1, ..., x_n]$ and its quantum differential operators, but we are missing a big piece of the puzzle that exists in the standard case when considering differential operators, and which is essential when considering the global differential operators of a quasi-affine geometric object. This missing piece is the fact that we can localize the ring and then construct its differential operators, or we can construct the ring differential operators for the ring and then localize. In other words, for a commutative ring R, a multiplicitive subset $S_1 \subset R$, and differential operators on R denoted by $D(R)$ in the sense of section 2.1, we have the following equivalence:

$$D(RS_1^{-1}) \simeq D(R)S_1^{-1}.$$

Equally important for quasi-affine constructions is the commutativity of localization by a second multiplicitive set $S_2 \subseteq R$; it is in fact a requirement for gluing a sheaf together. In our setting this means the following diagram commutes:

$$\begin{array}{ccc} D((RS_1^{-1})S_2^{-1}) & \longrightarrow & D(R)S_1^{-1}S_2^{-1} \\ \downarrow & & \downarrow \\ D((RS_2^{-1})S_1^{-1}) & \longrightarrow & D(R)S_2^{-1}S_1^{-1}, \end{array}$$

where all arrows are isomorphisms.

Before pursuing these results let us be reminded that heretofore our ring R has indeed been commutative, which is divergent from the formulation in [2] of β and quantum differential operators for a bicharacter β on a non-commutative ring R. As noted before, this divergence is both convenient and necessary. It is convenient because, with strategic application of the results of [4], we can still study the specific case we wish to study, and necessary because of our insistence on using the shifted differential operators of the previous chapter, which allows us to simply collect all of the shifted centers together to form an R-sub-bimodule, rather than having to generate it at every step as we would otherwise be required to do. If we no longer wish to restrict ourselves to commutative rings R, we can simply collect all of the shifted centers and generate the bimodule of *ith* quantum differential operators at each step and this, by definition, coincides with the construction of [2].

We do not need to go that far, however. Let $R = \bigoplus_{a\in\Gamma} R_a$ be a unital, associative, commutative Γ-graded k-algebra, and $\gamma : \Gamma \times \Gamma \to k^\times$ a bicharacter of the group Γ. Let $D_\gamma(R)$ denote the shifted γ-differential operators of Section 2.3. Let $\lambda : R \hookrightarrow D_\gamma(R)$ be the injective mapping associating each element of R to the left multiplication by that element: $\lambda(r) := \lambda_r$ such that $\lambda_r(s) = rs$. As before, we will not distringuish between r and λ_r in $D_\gamma(R)$.

Lemma 3.2.1. *Suppose R is a finitely generated Γ-graded k-algebra and $S \subseteq R$ is a multiplicitive subset with no zero divisors. Let $T \in D_\gamma(R)$. Then there exists a unique $\bar{T} \in D_\gamma(RS^{-1})$ such that the following diagram is commutative:*

$$\begin{array}{ccc} R & \xrightarrow{T} & R \\ \downarrow{\scriptstyle q_S} & & \downarrow{\scriptstyle q_S} \\ RS^{-1} & \xrightarrow{\bar{T}} & RS^{-1}, \end{array}$$

where $q_S : r \mapsto \frac{r}{1}$ is the standard localization map.

Proof. To show uniqueness it is enough to show that for any $W \in D_\gamma(RS^{-1})$ with $W(r) = 0$ for any $g \in q_S(R)$, then $W = 0$. If the order i of the γ-differential operator W is 0 then this is immediate. Suppose $i > 0$. Then by definition $[W, t]_{\gamma,a} \in D_\gamma^{i-1}(RS^{-1})$ for all $s \in S$ and for some $a \in \Gamma$ and it annihilates $q_S(R)$. Hence by induction $[W, t]_{\gamma,a} = 0$. Let $f \in RS^{-1}$. Then there exists some $t' \in S$ such that $t'f \in q_S(R)$. Thus $t'W(f) = \gamma(|W| + a, |t'|)W(t'f) = 0$. Since $\frac{1}{t'} \in RS^{-1}$ we conclude $W(f) = 0$.

For existence see Lemma 4.1.6 in [4]. □

Proposition 3.2.2. *Suppose R is a finitely generated Γ-graded k-algebra and $S \subset R$ is a multiplicitive subset with no zero divisors. Then $q_S : R \to RS^{-1}$ induces an isomorphism*

$$p_S : D_\gamma(RS^{-1}) \xrightarrow{\sim} D_\gamma(R)S^{-1},$$

where $p_S(T) = \bar{T}$ as in 3.2.1.

Proof. S has no zero divisors and is comprised exclusively of degree zero operators, i.e. elements of RS^{-1}. Hence we have injectivity.

For surjectivity we show that for all $T \in D^i_\gamma(RS^{-1})$ there exists $s \in S$ such that $sT(R) \subseteq R$. The initial case $i = 0$ is immediate. Suppose the statement is true for arbitrary $i > 0$, and denote by $r_1, ..., r_n$ the generators of R. By definition of $D^i_\gamma(R)$, we have that $[T, r]_\gamma \in D^{i-1}_\gamma(R)$ for any $r \in R$ and by assumption there exists $s \in S$ such that $(s[T, r]_\gamma)(R) \subseteq R$. This means that for any $h \in R$ such that $sh \in q_S(R)$ we have

$$s[T, r_i]_\gamma(h) = sT(r_ih) - \gamma(|T|, |r_i|)sr_iT(h) \in R$$
$$\Rightarrow sT(r_ih) \in R.$$

Considering $r = r_ir_j \in R$, we have

$$s[T, r_ir_j]_\gamma(h) = s(Tr_ir_j)(h) - \gamma(|T|, |r_ir_j|)sr_ir_jT(h) \in R$$
$$\Rightarrow sT(r_ir_j)(h) \in R \text{ for all } h \in R.$$

Using $sT(1) \in R$ and an induction on the length of the monomials $r_1^{a_1}...r_n^{a_n}$, we see that this relations implies $sT(R) \subseteq R$. □

Corollary 3.2.3. *Let $\hat{S}_2 \subseteq RS_1^{-1}$ be a multiplicitive subset with no zero divisors and R finitely generated. Denote by $\hat{R} = RS_1^{-1}$, then by the previous proposition we have*

$$D_\gamma(\hat{R}\hat{S}_2^{-1}) \simeq D_\gamma(\hat{R})\hat{S}_2^{-1}.$$

Furthermore,

$$D_\gamma(RS_1^{-1}\hat{S}_2^{-1}) \simeq (D_\gamma(RS_1^{-1}))\hat{S}_2^{-1} \simeq (D_\gamma(R)S_1^{-1})\hat{S}_2^{-1}.$$

Finally, we wish to say these results lead us to the last result we set out to obtain, that

$$\begin{tikzcd} RS_1^{-1}\hat{S}_2^{-1} \arrow[r, "\sim"] \arrow[d, hook] & RS_2^{-1}\hat{S}_1^{-1} \arrow[d, hook] \\ D_\gamma((RS_1^{-1})\hat{S}_2^{-1}) \arrow[r, "\sim"] & D_\gamma(R)S_2^{-1}\hat{S}_1^{-1}. \end{tikzcd} \tag{3.2}$$

3.3 Examples on Polynomial Algebras

Let us now return to the quantum differential operators of Luntz and Rosenberg [2] as (partially) realized by Iyer and McCune [3]. Starting again with the quantum differential operators on $k[x_1, ..., x_n]$, we have that $D_q(k[x_1, ..., x_n])$ is the ring generated by elements $\rho_f \sigma_a$ for $f \in k[x_1, ..., x_n]$ and $a \in \Gamma$, and the first-order operators $\partial_k^{\beta^i}$ for $i = 0, 1$, with $[\partial_k^{\beta^a}, x_k]_{\beta,a} = 1$, and $[\partial_k^{\beta^a}, x_k] = \sigma_a$ Let $S_i = \{1, x_i, x_i^2, ...\}$ We find that

$$\begin{aligned} [\partial_i^{\beta^a}, x_i^{-1}](x_i^n) &= \partial_i^{\beta^a}(x_i^{n-1}) - x_i^{-1}\partial_i^{\beta^a}(x_i^n) \\ &= (1 + \beta(a, e_i) + ... + \beta(a, (n-2)e_i))x_i^{n-2} - (1 + \beta(a, e_i) + ... + \beta(a, (n-1)e_i))x_i^{n-2} \\ &= -\beta(a, (n-1)e_i)x_i^{n-2}. \end{aligned}$$

Thus $[\partial_i^{\beta^a}, x_i^{-1}] = -x_i^{-1}\sigma_a x_i^{-1}$.

From [4] this also includes the example of quantum differential operators on $k_\gamma[x_1, ..., x_n]$ for γ a 2-cocycle deformation of the multiplication in $k[x_1, ..., x_n]$. Due to the so-called twisting and untwisting oulined therein, we have no worries about how elements of R and RS^{-1} commute, ergo we obtain in these cases the result we failed to obtain in the diagram (3.2), that

$$
\begin{tikzcd}
RS_1^{-1}\hat{S}_2^{-1} \arrow[r, "\sim"] \arrow[d, hook] & RS_2^{-1}\hat{S}_1^{-1} \arrow[d, hook] \\
D_q((RS_2^{-1})\hat{S}_1^{-1}) \arrow[r] & D_q(R)S_2^{-1}\hat{S}_1^{-1},
\end{tikzcd}
$$

commutes in the category of k-algebras and thus the lower horizontal arrow is an equivalence.

Chapter 4

Gelfand-Kirillov dimensions

In the theory of holonomic D-modules, dimension plays a role of utmost importance. Indeed, it is the defining characteristic of the modules, though this is not as simple as taking the dimension of the module over the the ring of differential operators.

Let X be a smooth algebraic variety over $\mathbb{C}$ (or, an algebraically closed, zero-characteristic field k), with a structure sheaf $\mathcal{O}_X$ and sheaf of differential operators $\mathcal{D}_X$ with $\Sigma = \bigoplus_{i \geq 0} \mathcal{D}_X^i / \mathcal{D}_X^{i-1}$ the associated graded sheaf of algebras. There, a coherent $\mathcal{D}_X$-module, necessarily having a "good" filtration $M = \bigoplus_{i \geq 0} M^i \neq 0$ with $M^i \subseteq M^{i+1}$, and thus its own associated graded module $M_\Sigma = \bigoplus_{i \geq 0} M^i / M^{i-1}$ naturally has the structure of a Σ-module. What's going on here is that Σ is the sheaf of functions on the cotangent bundle $T^\star(X)$, and so we have $\mathrm{Supp}(M_\Sigma) \subset T^\star(X)$. This support is a closed subvariety and has a defining ideal $J_{M_\Sigma} = \mathrm{ann}(M_\Sigma) \subset \Sigma$, i.e. $\mathrm{Supp}(M_\Sigma) = \mathbf{Spec}(\Sigma / J_{M_\Sigma})$. Bernstein's theorem on defect, or Bernstein's Inequality, gives us a lower bound to the dimension of this support:

$$\dim(\mathrm{Supp}(M_\Sigma)) \geq \dim(X).$$

From this inequality it is obvious that there should be nonzero coherent $\mathcal{D}_X$-modules with a minimum dimension, namely $\dim(X)$, and it is with this in mind that Holonomic $\mathcal{D}_X$-modules are named to be those with this lowest possible dimension.

We will not be able to go so far as to follow this program to the end, and in this section the culmination is the calculation of the GK-dimension of the associated graded algebra of the ring of γ-differential operators. This would then conceivably make it possible to calculate $\dim(\Sigma/J_{M_\Sigma})$.

The idea is to use localization of our shifted differential operators, $D_\gamma = D_\gamma(k[x_1, ..., x_n])$, of section 2.3 and 2.4 to compute the gk-dim of D_γ, since, as we will see, the proper localization doesn't change the GK-dimension.

4.1 Almost Central Elements

Definition 4.1.1. Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a graded k-algebra, and V denote a graded k-subspace of R; i.e. $V = \bigoplus_{\gamma\in\Gamma}(V \cap R_\gamma) \ni 1$. We call x *almost V-central* if $xV = Vx$ implies that $(xV)^n = x^n V^n$. Additionally, x R^h is called *R-central* if there is a finite dimensional graded k-subspace $V \subset R$ such that V generates R as a k-vector space and x is almost V-central.

Proposition 4.1.2. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a graded k-algebra and let Ω be a multiplicatively closed subset of regular, almost R-central elements of R. Then*

$$\mathrm{GKdim}(R\Omega^{-1}) = \mathrm{GKdim}(R).$$

Proof. Let W be a finite dimensional subspace of $R\Omega^{-1}$ and $s \in \Omega$ a common denominator of basis elements in W, then $Ws \subset R$. Consider $V = Ws + ks + k$. As a direct consequence we have both that V is a finite dimensional k-subspace of R, and that $W \subset V^n s^{-n}$ for all $n \in \mathbb{N}$. Indeed, s is almost V-central and this follows immediately.
Thus $\dim_k(W^n) \leq \dim_k(V^n)$ $\forall$ $n \geq 0$, and so $\mathrm{GKdim}(R\Omega^{-1}) \leq \mathrm{GKdim}(R)$.
Since $\mathrm{GKdim}(R\Omega^{-1}) \geq \mathrm{GKdim}(R)$ is obvious, we have $\mathrm{GKdim}(R\Omega^{-1}) = \mathrm{GKdim}(R)$. $\square$

The rest of the chapter is a consquence of this proposition. Directly, we have:

Corollary 4.1.3. *Let S_X be the multiplicitive set generated by $x_1, ..., x_n$ in $k[x_1, ..., x_n]$, then*

$$\mathrm{GKdim}(\mathrm{gr}_\Gamma(D_\gamma(k[x_1, ..., x_n]))S_X^{-1}) = \mathrm{GKdim}(\mathrm{gr}_\Gamma(D_\gamma(k[x_1, ..., x_n]))).$$

This will allow us to calculate the GK-dimension of the localized associated-graded instead of the original, more complex one.

4.2 GK-dimension of $\mathrm{gr}_\Gamma(D_\gamma(k[x_1, ..., x_n]))$

Recall the definition of the differential operators $\partial_i^\gamma = \partial_i^{\gamma^{e_i}}$ from 2.4.2 by their action on the $x_i \in k[x_1, ..., x_n]$:

$$\begin{aligned}\partial_i^{\beta^a}(x_i^n) &:= [n]_{\beta(a,e_i)} x_i^{n-1} \\ &= (1 + \beta(a, e_i) + \beta(a, 2e_i) + ... + \beta(a, (n-1)e_i))x_i^{n-1}.\end{aligned}$$

Consider the product $\partial_i \partial_i^\gamma$; everywhere there is a problem, this is the element that causes it, and it does so because there is no "nice" way to write the commutator, or even the γ-commutator of these elements ∂_i, and ∂_i^γ. We start with reminding that $\partial_i x_i \partial_i^\gamma = \partial_i^\gamma x_i \partial_i$. On the left hand side we have

$$\begin{aligned}\partial_i x_i \partial_i^\gamma &= \partial_i(\partial_i^\gamma x_i - \sigma_{e_i}) \\ &= \partial_i \partial_i^\gamma x_i - \partial_i \sigma_{e_i},\end{aligned}$$

while on the right there is

$$\begin{aligned}\partial_i^\gamma x_i \partial_i &= \partial_i^\gamma(\partial_i x_i - 1) \\ &= \partial_i^\gamma \partial_i x_i - \partial_i^\gamma.\end{aligned}$$

Explicitly, we can write:

$$\partial_i x \partial_i^\gamma = \partial_i^\gamma x \partial_i,$$

$$\partial_i \partial_i^\gamma x_i - \partial_i \sigma_{e_i} = \partial_i^\gamma \partial_i x_i - \partial_i^\gamma,$$

$$\partial_i \partial_i^\gamma x_i - \partial_i^\gamma \partial_i x_i = \partial_i \sigma_{e_i} - \partial_i^\gamma,$$

$$(\partial_i \partial_i^\gamma - \partial_i^\gamma \partial_i) x_i = \partial_i \sigma_{e_i} - \partial_i^\gamma.$$

Thus, having now localized by $S_x = \{x_k^m \mid k = 1, ..., n, \ m \in \mathbb{N}\}$, we obtain

$$\partial_i \partial_i^\gamma - \partial_i^\gamma \partial_i = (\partial_i \sigma_{e_i} - \partial_i^\gamma) x_i^{-1}, \tag{4.1}$$

the right side of which is, finally, in the first filtered part of $D_\gamma(k[x_1, ..., x_n]))S_X^{-1}$.

This means that $\partial_i \partial_i^\gamma = \partial_i^\gamma \partial_i$ in $\mathrm{gr}_\Gamma(D_\gamma(k[x_1, ..., x_n]))S_X^{-1})$, and makes our associated graded algebra a run-of-the-mill skew-polynomial ring, when considered over $k[\Gamma]$.

Lemma 4.2.1. $\mathrm{gr}_\Gamma D_\gamma = \mathrm{gr}_\Gamma(D_\gamma(k[x_1, ..., x_n]))S_X^{-1})$ *is an iterated skew polynomial ring over* $k[\Gamma]$. *Explicitly,*

$$\mathrm{gr}_\Gamma(D_\gamma(k[x_1, ..., x_n]))S_X^{-1}) \simeq k[\Gamma][x_1, x_1^{-1}, ..., x_n, x_n^{-1}, \partial_1, \partial_1^\gamma, ..., \partial_n, \partial_n^\gamma; \theta],$$

where θ_i *is the family of automorphisms on* $k[\Gamma]$ *defined by* $\theta_i(\sigma_a) = \gamma(a, e_i)\sigma_a$.

Proof. From the example 3.3 we what elements make up $D_\gamma(k[x_1, ..., x_n]))S_X^{-1}$ and how they act on $k[x_1, ..., x_n]$, and from the discusion surrounding the relation 4.1 we have seen that

$$\partial_i \partial_i^\gamma = \partial_i^\gamma \partial_i + (\partial_i \sigma_{e_i} - \partial_i^\gamma) x^{-1},$$

where $(\partial_i \sigma_{e_i} - \partial_i^\gamma) x^{-1}$ is in the first filtered part of $\mathrm{gr}_\Gamma D_\gamma$. Thus, in $\mathrm{gr}_\Gamma D_\gamma$, we have $\partial_i \partial^\gamma =$

$\partial_i^\gamma \partial_i$. Similarly,

$$\partial_i^{\gamma^a} x_j = x_j \partial_i^{\gamma^a} + \delta_{ij}\sigma_a,$$

shows $[\partial_i^{\gamma^a}, x_j]$ is in the zeroth filtered part of $\mathrm{gr}_\Gamma D_\gamma$, and so $\partial_i^{\gamma^a} x_j = x_j \partial_i^{\gamma^a}$.

Furthermore, from the defining relations in [3] and Example 2.4 we have for $K = (k_1, ..., k_n) \in \Gamma, x^K = x_1^{k_1} ... x_n^{k_n}$,

$$\begin{aligned} x_i \sigma_a(x^K) &= \gamma(a, K) x^{K+e_i}, \\ \sigma_a x_i(x^K) &= \gamma(a, K + e_i) x^{K+e_i} \\ &= x_i \theta_i(\sigma_a)(x^K), \end{aligned}$$

and

$$\begin{aligned} \partial_i^{\gamma^b} \sigma_a(x^K) &= \gamma(a, K)[k_i]_{\gamma^b} x^{K-e_i}, \\ \sigma_a \partial_i^{\gamma^b}(x^K) &= \gamma(a, K - e_i)[k_i]_{\gamma^b} x^{K-e_i} \\ &= \gamma(a, K)\gamma(a, -e_i)[k_i]_{\gamma^b} x^{K-e_i} \\ &= \partial_i^{\gamma^b} \theta_i^{-1}(\sigma_a)(x^K). \end{aligned}$$

□

A problem remains: we have an infinite number of linear generators for $D_\gamma(k[x_1, ..., x_n])$. However, there is hope since $k[\Gamma] \subset D_\gamma(k[x_1, ..., x_n]))$, and, as we saw in chapter 2 and as noted in [3], $\partial_i^{\gamma^a}$ is generated by ∂_i^γ over $k[\Gamma]$ for all $i = 1, ..., n$ and $a \neq 0, 1 \in \Gamma$. It's important then to note that $D_\gamma(k[x_1, ..., x_n]))$ is a $k[\Gamma]$-algebra, and in fact of finite type.

Lemma 4.2.2. *for* $\Gamma = \mathbb{Z}^n, k[\Gamma] \simeq k[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$.

Proof. Elements of $k[\Gamma]$ are k-linear combinations of the automorphisms σ_a for all $a \in \Gamma$. First, note that for $a, b \in \Gamma$ we have $\sigma_a \sigma_b = \sigma_{a+b}$. This is evident from the action on

$k[x_1, ..., x_n] : (\sigma_a \circ \sigma_b)(x^K) = \sigma_a(\gamma(b, K)x^k) = \gamma(a, K)\gamma(b, K)x^K = \gamma(a + b, K)x^K$. Thus, for $a_i \geq 0 \in \mathbb{Z}$ and $(0, ..., 0, a_i, 0, ..., 0) \in \Gamma$, we have $\sigma_{a_i} = \sigma_{e_i}^{a_i}$, while if $a_i \leq 0$, we use $\sigma_{a_i} = \sigma_{-e_i}^{a_i}$. Thus, for any $a \in \Gamma, \sigma_a = \sigma_{e_1}^{b_1} \circ \sigma_{-e_1}^{c_1} \circ ... \circ \sigma_{e_1}^{b_1} \circ \sigma_{-e_n}^{c_n}$ for $b_i \geq 0, c_i \leq 0 \in \mathbb{Z}$ such that $a_1 + b_i - c_i$. Clearly, $k[\Gamma]$ is gerenated by the elements $\{\sigma_{\pm e_i} \mid i = 1, ..., n\}$.

The k-algebra morphism defined on generators by $\sigma_{\pm e_i} \mapsto t^{\pm 1}$ then gives the claimed isomorphism. □

Lemma 4.2.3. *Let A be a k-algebra with a k-derivation δ such that each finite dimensional subspace of A is contained in a δ-stable finitely generated subalgebra of A. Then* $\mathrm{GKdim}(A[x; \delta]) = \mathrm{GKdim}(A) + 1$.

Proof. see Proposition 3.5 of [10] □

Proposition 4.2.4. $\mathrm{GKdim}(\mathrm{gr}_\Gamma(D_\gamma(k[x_1, ..., x_n]))) = 3n$.

Proof. By repeated application of Lemma 4.2.3 and Proposition 4.1.2 we obtain $\mathrm{GKdim}(k[\Gamma]) = n$.

By Lemma 4.2.1 we have

$\mathrm{GKdim}(\mathrm{gr}_\Gamma(D_\gamma(k[x_1, ..., x_n]))) = \mathrm{GKdim}(k[\Gamma][x_1, x_1^{-1}, ..., x_n, x_n^{-1}, \partial_1, \partial_1^\gamma, ..., \partial_n, \partial_n^\gamma; \theta]) = n + 2n = 3n$. □

Chapter 5

Graded Hyperbolic Algebras

Hyperbolic algebras are a construction of Alexander Rosenberg, first appearing in 1989 in his preprint *Geometry of Grothendieck Categories*, and appearing later containing Vladimir Bavula's Generalized Weyl Algebras. The rings appear numerously as examples of tractable non-commutative rings, and specifically in non-commutative algebraic geometry, as we will see below. One of the greatest results obtained for hyperbolic rings is the description of their irreducible representations. While this result is not obtained here for our construction, it is in our sights as the target result of future research.

5.1 Hyperbolic Algebras

Definition 5.1.1. Let θ be an automorphism of a commutative ring R, and $\xi \in R$. Denote by $R\{\theta, \xi\}$ the R-ring generated by the indeterminants x and y subject to the relations:

$$xa = \theta(a)x \text{ and } ya = \theta^{-1}(a)y \text{ for all } a \in R,$$

$$xy = \xi, yx = \theta^{-1}(\xi)$$

The rings $R\{\theta, \xi\}$ are called hyperbolic because these defining relations can be interpreted as the equations of a non-commutative hyperbola.

Remark 5.1.2. It may not seem as though these relations encode the one between x and y, and, explicitly, they do not. The relationship cannot be seen explicitly since, in general, we do not precisely know the automorphism θ. It is hidden in the relation $xa = \theta(a)x$, since, indeed, $xy = \xi \in R$, and thus we could take $a = \xi$. In the specific example of the Weyl algebra A_1 below this will be seen readily.

Note that the ring $R\{\theta, \xi\}$ is not just an R-ring, but, more importantly, an $R[\xi]$-ring. Furthermore, that $R[\xi]$ is a commutative subalgebra of $R\{\theta, \xi\}$.

Lemma 5.1.3. *Every element of $R\{\theta, \xi\}$ can be represented as a sum $f(x) + g(y)$ where*

$$f(x) = \sum_{i \geq 0} x^i a_i \text{ and } g(y) = \sum_{i \geq 1} y^j b_j,$$

with a_i, $b_j \in R$ for all $i \geq 0, j \geq 1 \in \mathbb{Z}$.

Proof. This follows directly from the definition $xy = \xi \in R, yx = \theta^{-1}(\xi)$. □

As a direct conesquence, we have that $R[x, \theta] \oplus yR[y, \theta^{-1}] \simeq R\{\theta, \xi\}$ as R-modules, with $R[x, \theta]$ and $R[y, \theta^{-1}]$ skew polynomial rings.

5.2 Examples

5.2.1 Weyl Algebra

As we know, the first Weyl Algebra A_1 is realized as the algebra of differential operators on the polynomial ring $k[x]$. In terms of generators and relations, we have $A_1 = k[x, y]/\langle xy - yx + 1\rangle$.

To realize A_1 as a hyperbolic algebra $R\{\theta, \xi\}$, we set $\xi = xy$ and let $R = k[\xi]$. Lastly, then, we need to encode the defining relation $[x, y] = -1$ in terms of θ: since $\xi \in R$, we have, by definition,

$$x\xi = xxy = x(yx-1) = xyx - x = (\xi - 1)x = \theta(\xi)x,$$

$$y\xi = yxy = (xy+1)y = xyy + y = (\xi+1)y = \theta^{-1}(\xi)y.$$

Thus, we have $\theta(\xi) = \xi - 1$ and $\theta^{-1}(\xi) = \xi + 1$.

Alternatively, consider the ring $R = k[\xi]$ and the hyperbolic ring $R\{\theta, \xi\}$ generated by x and y over R with $xy = \xi$ and $\theta \in \mathrm{Aut}(k[\xi])$ defined on the single generator ξ by the linear equation $\theta(\xi) = \xi - 1$. Notice it is from here we obtain the defining relation of the Weyl algebra:

$$x\xi = \theta(\xi)x,$$

$$xxy = \theta(xy)x = (xy-1)x = x(yx-1),$$

which obviously gives us the relation $xy = yx - 1$, or, $[y, x] = 1$.

5.3 Graded Hyperbolic Algebras

Though hyperbolic algebras may be naturally $\mathbb{Z}$-graded (or $\mathbb{Z}^2$-graded), we wish to adapt the construction to consider the case wherein this grading comes with an action.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a commutative Γ-graded k-algebra. Let $\beta : \Gamma \times \Gamma \to k^\times$ be a 2-cocycle for the action of Γ on R. Via the map $\lambda : R \to \mathrm{End}_k(R)$ defined by $r \mapsto \lambda_r : a \mapsto ra$, for all $a \in R$, we see that the grading on R naturally extends to a grading on $\mathrm{End}_k(R) = \bigoplus_{\alpha \in \Gamma} \mathrm{End}_k(R)_\alpha$. Define the operators $\sigma_\alpha \in \mathrm{Aut}_k(R) \subseteq \mathrm{End}_k(R)$ by $\sigma_\alpha(r) = \beta(\alpha, |r|)r$, where $|\cdot|$ gives the degree of a homogeneous element in R; i.e. $|r| = \alpha$ for $r \in R_\alpha$.

Remark 5.3.1. From this, we see that Γ will also act on $\mathrm{End}_k(R)$, giving, for instance, $\hat{\sigma_\alpha}(\phi) = \beta(\alpha, |\phi|)\phi$. In the traditionally confusing manner, we will condense the notation in the case when $\phi = \lambda_r$, just using r for $r \in R$, nor will we distinguish between $\hat{\sigma_\alpha}$ and σ_α. In fact, for $\phi = \lambda_r$ we have $|\phi| = |r|$, i.e. $\hat{\sigma_\alpha}(\lambda_r) = \lambda_{\sigma_\alpha(r)}$. Furthermore,

$$\begin{aligned}\hat{\sigma_\alpha}(\phi)(x) &= \beta(\alpha, |\phi|)\phi(x) \\ &= \beta(\alpha, |r|)(rx) = \lambda_{\sigma_\alpha(r)}(x).\end{aligned}$$

Remark 5.3.2. Equivalently, we could instead define the action of Γ on the objects of $\mathrm{gr}_\Gamma R$-mod, wherein we have Γ-graded R-modules, $M = \bigoplus_{\alpha\in\Gamma} M_\alpha$. For all $M \in \mathrm{gr}_\Gamma R$-mod, there is a natural group homomorphism $\sigma_M : \Gamma \to \mathrm{Aut}_k(M)$ defined by

$$\sigma_M(\gamma)|_{M_\alpha} := \beta(\alpha, \gamma)\,\mathrm{id}_{M_\alpha}, \text{ for } \alpha, \gamma \in \Gamma.$$

5.3.1 Application to $D_\beta(k[x])$

Obviously, the goal here is to create a general construction, based on hyberbolic algebras, that contains our ring of β-differential operators as a specific example. One important aspect of this example to recall is that we could write every ∂^{β^a} as a sum

$$\partial^{\beta^a} = \frac{q-1}{q^a-1}\partial^\beta \sum_{i=0}^{a-1} \sigma_i \text{ for } a > 0 \in \Gamma, \text{ and every } \partial^{\beta^{-a}}, \tag{5.1}$$

$$\partial^{\beta^{-a}} = \frac{q-1}{q^{-a}-1}\partial^{\beta^{-1}} \sum_{i=0}^{a-1} \sigma_{-i}, \tag{5.2}$$

and that we have this relation in addition to the fact that both ∂^{β^a} and σ_a are elements of D_β for all $a \in \Gamma$. There is no way to obtain the standard partial differential operator ∂ in such a fashion; ∂ and ∂^β are not only k-linearly independent operators, but they are algebraically independent over $k[\Gamma]$.

The other important aspect to consider is the relation

$$\partial x \partial^\beta = \partial^\beta x \partial. \tag{5.3}$$

Thus, inspired by the hypberbolic construction upon the ring $k[\xi]$ where $\xi = xy$, we set

$\xi_0 := x\partial$, and $\xi_1 := x\partial^{\beta}$, we make the ring $k[\xi_0, \xi_1]$. Just as in the case of hyperbolic algebras, this ring $k[\xi_0, \xi_1]$ is commutative. Indeed, by the relation 5.3, we have that

$$\xi_0\xi_1 = x\partial x\partial^{\beta} = x\partial^{\beta} x\partial = \xi_1\xi_0.$$

Furthermore, $k[\Gamma][\xi_0, \xi_1] = k[\xi_0, \xi_1][\Gamma]$ is a commutative ring as well, by the fact that

$$\xi_\gamma\sigma_\alpha = \sigma_\alpha\xi_\gamma \text{ for } \gamma = 0, 1, \text{ and } \alpha \in \Gamma,$$

since $\xi_\gamma = x\partial_\gamma$ does not change the degree of a monomial in $k[x]$, i.e. is an element of $D^{\beta}(k[x])_0$.

Definition 5.3.3. Let θ be an automorphism of a Γ-graded, commutative ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$; and let ξ_0, ξ_1 be elements of R. Denote by $R[\Gamma]\langle x, y_0, y_1\rangle$ the R-ring generated over the group algebra $R[\Gamma]$ by the indeterminants x, y_0, y_1, with the following relations:

$$xa = \theta(a)x, \;\; y_0 a = f(\theta^{-1}(a))y_0, \;\text{ and } y_1 a = f(\theta^{-1}(a))y_1 \text{ for all } a \in R[\Gamma];$$

$$xy_0 = \xi_0, xy_1 = \xi_1,$$

where $f(\theta^{-1}(a))$ is some map $f : R[\Gamma] \to R[\Gamma]$.

Of course, this is not simply an abstraction of a general construction to include the Γ action on the graded ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, it is a naturally occuring phenomenon.

Recall from section 2.2.3 that our ring $D_\beta(k[x])$ is the ring of quantum differential operators of Luntz and Rosenberg on the ring of polynomials in one variable over the algebraicly closed, 0-characteristic field k. By Iyer and McCune's [3] $D_\beta(k[x])$ is the k-algebra generated by $k[x]$ and the set $\{\partial^{\beta^{-1}}, \partial, \partial^{\beta}\}$ over $k[x]$. This description uses the relations (5.1), but ignores the facts that

1. $\sigma_a \notin k[x]$, and
2. $\partial^{\beta^{-a}} = \sigma_{-a}\partial^{\beta^{a}}$,

which is why we choose here to generate over the group algebra $R[\Gamma]$.

To show $D_\beta(k[x])$ can be realized as a graded hyperbolic ring $R[\Gamma] < x, y_0, y_1 >$, with $R = k[\xi_0, \xi_1]$, we have a few things to work through; namely, the three defining relations of the graded hyperbolic ring.

Lemma 5.3.4. *$xa = \theta(a)x$ for all $a \in k[\Gamma][\xi_0, \xi_1]$.*

Proof. Let $\beta(1,1) := q \in k^\times$. We will determine the automorphism $\theta \in \mathrm{Aut}_k(k[\Gamma])$ be its value on the elements $a = \xi_0, \xi_1,$ and σ_α for $\alpha \in \Gamma$.

1. $a = \xi_0 : x\xi_0 = x(x\partial) = x(\partial x - 1) = (x\partial - 1)x = (\xi_0 - 1)x$, where we see that $\theta(\xi_0) = \xi_0 - 1$, thus providing the relation $x\xi_0 = \theta(\xi_0)x$.

2. $a = \xi_1 : x\xi_1 = x(x\partial^\beta) = x(\partial^\beta x - \sigma_1) = x\partial^\beta x - x\sigma_1 = (x\partial^\beta - q^{-1}\sigma_1)x$, where we see that $\theta(\xi_1) = \xi_1 - q^{-1}\sigma_1$, making $x\xi_1 = \theta(\xi_1)x$.

   For the fourth equality above, recall $(x\sigma_1 - \sigma_1 x)(x^n) = q^n(1-q)x^{n+1} = (1-q)x\sigma_1 x^n$. Thus,

$$x\sigma_1 - \sigma_1 x = (1-q)x\sigma_1$$
$$x\sigma_1 - (1-q)x\sigma_1 = \sigma_1 x$$
$$qx\sigma_1 = \sigma_1 x.$$

3. $a = \sigma_\alpha$: comes directly from the relation $\sigma_\alpha x = q^\alpha x\sigma_\alpha$, providing $\theta(\sigma_\alpha) = q^{-\alpha}\sigma_\alpha$, and so $x\sigma_\alpha = \theta(\sigma_\alpha)x$.

Setting $\theta(a) = a$ for all $a \in k$, the rest is immediate. □

Lemma 5.3.5. *$\partial a = f(\theta^{-1}(a))\partial$ for all $a \in k[\Gamma][\xi_0, \xi_1]$.*

Proof.

1. $a = \xi_0 : \partial\xi_0 = \partial x\partial = (x\partial - 1)\partial = (\xi_0 + 1)\partial = \theta^{-1}(\xi_0)\partial$.

2. $a = \xi_1 : \partial\xi_1 = \partial x\partial^\beta = \partial^\beta x\partial = (x\partial^\beta + \sigma_1)\partial = (\xi_1 + \sigma_1)\partial = (\theta^{-1}(\xi_1) + (1 - q^{-1})\sigma_1)\partial$ so we have that $\partial\xi_1 = f(\theta^{-1}(\xi_1))\partial$.

3. $a = \sigma_\alpha : \partial\sigma_\alpha = q^\alpha \sigma_\alpha \partial = \theta^{-1}(\sigma_\alpha)\partial.$

$\square$

Lemma 5.3.6. *$\partial^\beta a = f(\theta^{-1}(a))\partial^\beta$ for all $a \in k[\Gamma][\xi_0, \xi_1]$.*

Proof. 1. $a = \xi_0$:

$$\begin{aligned}
\partial^\beta \xi_0 &= \partial^\beta (x\partial) \\
&= \partial x \partial^\beta \\
&= (x\partial + 1)\partial^\beta \\
&= (\xi_0 + 1)\partial^\beta \\
&= \theta^{-1}(\xi_0)\partial^\beta.
\end{aligned}$$

2. $a = \xi_1$:

$$\begin{aligned}
\partial^\beta (\xi_1) &= \partial^\beta (x\partial^\beta) \\
&= (x\partial^\beta + \sigma_1)\partial^\beta \\
&= (\xi_1 + \sigma_1)\partial^\beta \\
&= (\theta^{-1}(\xi_1) + (1 - q^{-1})\sigma_1)\partial,
\end{aligned}$$

again showing $\partial^\beta \xi_1 = f(\theta^{-1}(\xi_1))\partial^\beta$.

3. $a = \sigma_\alpha$:

$$\begin{aligned}
\partial^\beta \sigma_\alpha &= q^\alpha \sigma_\alpha \partial^\beta \\
&= \theta^{-1}(\sigma_\alpha)\partial^\beta.
\end{aligned}$$

$\square$

These lemmas show that our ring can be realized as a graded hyperbolic algebra. The next will show that our morphism θ, as defined, is indeed a k-linear endomorphism of the ring $k[\Gamma][\xi_0, \xi_1]$. Now that we know what the map does to the generators, we verify how it handles products. To summarize the above:

$$\theta(\xi_0) = \xi_0 - 1, \;\; \theta(\xi_1) = \xi_1 - q^{-1}\sigma_1, \text{ and } \theta(\sigma_\alpha) = q^{-\alpha}\sigma_\alpha.$$

Lemma 5.3.7. *As determined in Lemma 5.3.4, θ is a k-linear algebra endomorphism of $k[\Gamma][\xi_0, \xi_1]$.*

Proof. We need only verify that multipication of two of the three combinations of generators of $k[\Gamma[\xi_0, \xi_1]$ are respected by θ, since we can obtain for free that

$$\theta(\xi_0\xi_1) = \theta(\xi_1\xi_0).$$

1. ξ_0 and σ_α for $\alpha \in \Gamma$

$$\begin{aligned} \theta(\sigma_\alpha\xi_0) &= \theta(\sigma_\alpha)\theta(\xi_0) = q^{-\alpha}\sigma_\alpha(\xi_0 - 1) \\ =&\theta(\xi_0\sigma_\alpha) = \theta(\xi_0)\theta(\sigma_\alpha) = (\xi_0 - 1)q^{-\alpha}\sigma_\alpha, \end{aligned}$$

   and $q^{-\alpha}\sigma_\alpha(\xi_0 - 1) = (\xi_0 - 1)q^{-\alpha}\sigma_\alpha$ since $\sigma_\alpha\xi_a = \xi_a\sigma_\alpha$. Hence, $\theta(\sigma_\alpha\xi_0) = \theta(\xi_0\sigma_\alpha)$.

2. ξ_1 and σ_α for $\alpha \in \Gamma$

$$\begin{aligned} \theta(\sigma_\alpha\xi_1) = \theta(\sigma_\alpha)\theta(\xi_1) &= q^{-\alpha}\sigma_\alpha(\xi_1 - q^{-1}\sigma_1) \\ &= \xi_1 q^{-\alpha}\sigma_\alpha - q^{-1}\sigma_1 q^{-\alpha}\sigma_\alpha \\ &= (\xi_1 - q^{-1}\sigma_1)q^{-\alpha}\sigma_\alpha \\ &= \theta(\xi_1)\theta(\sigma_\alpha) = \theta(\xi_1\sigma_\alpha). \end{aligned}$$

   Hence $\theta(\sigma_\alpha\xi_1) = \theta(\xi_1\sigma_\alpha)$.

Thus we see that the multiplication of $k[\Gamma][\xi_0, \xi_1]$ is preserved by θ. ☐

With the inverse of θ already defined on the generators of $k[\Gamma][\xi_0, \xi_1]$, we have now seen that θ is a k-linear automorphism, and that our ring $D_\gamma(k[x])$ can be seen as a graded hyperbolic ring.

Proposition 5.3.8. *Any element of $R[\Gamma]\langle x, y_0, y_1\rangle$ can be represented as a sum of polynomials $f(x) + g(y_0, y_1)$ where*

$$f(x) = \sum_{l \geq 0} x^l r_l, \;\; g(y_0, y_1) = \sum_{n,m \geq 1} y_0^n y_1^m r_{n,m}, \;\; \textit{with } r_{n,m} \in R[\Gamma].$$

Proof. Given the defining equations

$$xy_0 = \xi_0, xy_1 = \xi_1,$$

and the fact that $\xi_0, \xi_1 \in R$, by definition, this is obvious. □

5.4 Iterated Graded Hyperbolic Rings

Definition 5.4.1. Let $\theta_1, ...\theta_n$ be a family of pairwise commuting automorphisms of a graded, commutative ring R; and $\xi_{10}, \xi_{11}, \xi_{20}, \xi_{21}, ...\xi_{n0}, \xi_{n1}$ a collection of elements in R. Denote by $R[\Gamma]\langle (x_i), (y_{i0}), (y_{i1})\rangle$ the R-ring generated over the group algebra $R[\Gamma]$ by the indeterminants x_i, y_{i0}, y_{i1} which satisfy the following relations

$$x_i a = \theta(a) x_i, \;\; y_{i0} a = f(\theta^{-1}(a)) y_{i0}, \text{ and } y_1 a = f(\theta^{-1}(a)) y_{i1};$$

$$x_i y_{i0} = \xi_{i0}, x_i y_{i1} = \xi_{i1};$$

$$x_i y_{j\alpha} = y_{j\alpha} x_i, \;\; x_i x_j = x_j x_i, \;\; y_{i\alpha} y_{j\beta} = y_{j\beta} y_{i\alpha}.$$

for every $a \in R[\Gamma]$, $1 \leq i, j \leq n$, and $\alpha, \beta \in \{0, 1\}$.

Chapter 6

Gluing and Descent

6.1 Gluing Framework

Gluing is a traditionally informal terminology referring to combining two separate topological or geometric spaces into one, wherein the original spaces are mostly preserved. There are multiple ways of doing this, but we will employ a version of descent adapted more to gluing categories together rather than gluing spaces or sheaves.

Descent allows one to construct a scheme (or quasi-coherent sheaf), by putting together pieces formed from a much coarser covering than is typically required. An example is the case of $\mathbb{P}^n_{\mathbb{C}}$ with the construction by descent on a finite collection of open affine patches, as opposed to specifying the value of the structure sheaf at arbitrary distinguished open subsets of the underlying space. It does this in a way mandated by the definition of a sheaf: by considering the intersections and triple intersections of the sets in the collection. It still must respect the cocycle condition for sheaves.

6.1.1 (Co)Monads and their (co)Algebras

Let us recall some of the categorical concepts from [11].

Definition 6.1.1. Suppose $\mathcal{A}$ and $\mathcal{B}$ are categories with functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$. We say that $\langle F, G, \eta, \varepsilon \rangle$ is an *adjunction* between $\mathcal{A}$ and $\mathcal{B}$, and that F and G are adjoint functors with F the left adjoint and G the right adjoint. We write (F, G) if there exist natrual transformations

$$\eta : Id_{\mathcal{A}} \dot{\to} GF, \qquad \varepsilon : FG \dot{\to} Id_{\mathcal{B}},$$

with η_A, ε_B universal for all objects A of $\mathcal{A}, B$ of $\mathcal{B}$.

Now consider a general endofunctor T of a category $\mathcal{A}$. Naturally, then we have that any positive power of T, $T^n = T^{n-1} \circ T$ is an endofunctor as well. T^2 is equipped with a natural transformation $\mu : T^2 \dot{\to} T$, object-wise denoted $\mu_A : T^2(A) \to T(A)$, for an object A of $\mathcal{A}$. Customarily, $T\mu : T^3 = T \circ T^2 \dot{\to} T^2$, and $\mu T : T^3 = T^2 \circ T \dot{\to} T^2$, where the former has object-wise components $(T\mu)_A = T(\mu_A) : T(T^2)(A) \to T(T(A))$, and the latter $(\mu T)_A = \mu_{T(A)}$.

Definition 6.1.2. A *monad* in a category $\mathcal{A}$ is a triple $\langle T, \eta, \mu \rangle$, with T an endofunctor of $\mathcal{A}$ while μ and η are natural transformations

$$\eta : Id_{\mathcal{A}} \dot{\to} T, \qquad \mu : T^2 \dot{\to} T, \tag{6.1}$$

such that the diagrams below commute

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T,
\end{tikzcd}
\qquad
\begin{tikzcd}
Id_{\mathcal{A}}T \arrow[r, "\eta T"] \arrow[d, equal] & T^2 \arrow[d, "\mu"] & TId_{\mathcal{A}} \arrow[l, "T\eta"'] \arrow[d, equal] \\
T \arrow[r, equal] & T \arrow[r, equal] & T.
\end{tikzcd}$$

Notice that η plays the role of the unit transformation, and so we reuse the notation from adjoint pairs above. Also of note, the diagram on the left recalls associativity of multiplication, while the right recalls the identity for multiplication.

The comonad is the dual construction in $\mathcal{A}$ to the monad, essentially with all the defining arrows in the diagrams reversed. Clearly any adjoint pair $\langle F, G, \eta, \varepsilon \rangle : \mathcal{A} \rightharpoonup \mathcal{B}$ provides a monad $\langle GF, \eta, G\varepsilon F \rangle$ on $\mathcal{A}$ and a comonad $\langle FG, \varepsilon, F\eta G \rangle$ on $\mathcal{B}$.

The next construction we need is that of *algebras* of a monad. For a monad $T = (T, \eta, \mu)$ of the category $\mathcal{A}$, the T-algebras are the sets on which the monoid T acts. It is worth noting that often, especially when the category is svelt and abelian, algebras of a monad are referred to instead as *modules*, and for good reason: the category of R-modules can be realized as the algebras of the monad $T = R \bigotimes -$ on the category of abelian groups.

Definition 6.1.3. If $T = \langle T, \eta, \mu \rangle$ is a monad in the category $\mathcal{A}$, a T-*algebra* is a pair $\langle A, h \rangle$, where A is an object of $\mathcal{A}$, and $h : T(A) \to A$ is an arrow of $\mathcal{A}$ satisfying the following equations for associatvity and unit laws: $h \circ Th = h \circ \mu_A$ and $h \circ \eta_A = \mathrm{id}_A$. The former of which can be depicted as the exact diagram

$$T^2(A) \underset{\mu_A}{\overset{Th}{\rightrightarrows}} T(A) \xrightarrow{h} A.$$

A morphism $f : \langle A, h \rangle \to \langle A', h' \rangle$ of T-algebras is an arrow $f : A \to A'$ of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd}
T(A) \arrow[r, "h"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\
T(A') \arrow[r, "h'"'] & A'
\end{tikzcd}$$

commutes. $\mathcal{A}^T$ will denote the category of T-algebras.

Example 6.1.4. For R a ring, let T_R be the endofunctor $R \otimes_{\mathbb{Z}} -$ on the category of abelian groups $\mathbb{Z}$-mod. From T_R we can make the monad (T_R, η, μ), where

$$\begin{aligned}
T_R(M) = R \otimes_{\mathbb{Z}} M, \qquad \eta_M :& M \to R \otimes M, \qquad \mu_M : T_R^2(M) = R \otimes (R \otimes M) \to R \otimes M, \\
& x \mapsto r \otimes x, \qquad\qquad\qquad\qquad r_1 \otimes (r_2 \otimes m) \mapsto r_1 r_2 \otimes m.
\end{aligned}$$

For $M \in \mathbb{Z}$-mod we have the pair (M, h) where $h : R \otimes M \to M$ by $r \otimes m \mapsto r \cdot m$. The category T_R-algebras is the category of R-modules.

Example 6.1.5. Let k be a field and consider the category k-mod of k-vector spaces. Then the category of $k[x]$-modules is the category of algebras of the monad $T = k[x] \otimes_k -$..

Dually, we can define the coalgebras of a comonad in $\mathcal{A}$ and we denote by $\mathcal{A}_T$ the category of all T-coalgebras in the category $\mathcal{A}$. The following lemma from [12] will be very useful for us.

Lemma 6.1.6. *Let $\mathcal{A}$ be an abelian category, and $\langle T, \eta, \mu \rangle$ a monad or (comonad) on $\mathcal{A}$. Assume that T is additive and right (left) exact, then the categories $\mathcal{A}^T$ of T-algebras and $\mathcal{A}_T$ of T-coalgebras are abelian, and their forgetful functors to A are exact.*

Let $\langle F, G, \eta, \varepsilon \rangle : \mathcal{A} \rightharpoonup \mathcal{B}$ be an adjunction between abelian categories $\mathcal{A}$, and $\mathcal{B}$. Let $\langle \Phi, \varepsilon, \mu \rangle = \langle FG, \varepsilon, F\eta G \rangle$ be the associated comonad on $\mathcal{B}$ and $\mathcal{B}_\Phi$ the category of coalgebras of Φ. The following theorem is a version of the venerable Barr-Beck tailored to comparing the category of coalgebras of a comonad from an adjunction to the category through which the adjunction filters. For a more general statement and conditions, see §IV.7 Theorem 1 in [11].

Theorem 6.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $\langle \Phi, \varepsilon, \mu \rangle = \langle FG, \varepsilon, F\eta G \rangle$ the comonad on $\mathcal{B}$ associated to the adjunction $\langle F, G, \eta, \varepsilon \rangle : \mathcal{A} \rightharpoonup \mathcal{B}$ wherein F is additive, exact, and faithful. Then $\mathcal{A} \simeq \mathcal{B}_\Phi$.*

Proof. See Theorem 2.6 in [12]. □

6.1.2 Descent in Categories

The entire reason for the consideration of monads and their algebras in the previous section was to develop the language for descent. Here, we again consider an adjunction between two categories, $\mathcal{A}$ and $\mathcal{B}$, but this time the category $\mathcal{B}$ has only objects which are divided into parts from other categories, some $\mathcal{B}_i$. That is, $\mathcal{B} = \oplus_J \mathcal{B}_i$.

Consider a collection of functors $\{\mathcal{B}_i \xrightarrow{u_{i*}} \mathcal{A} \mid i \in J\}$ wherein the u_{i*} have left adjoints which we will denote by u_i^*. Let $\mathcal{B}$ be the product category $\prod_{j \in J} \mathcal{B}_j$. If $\mathcal{A}$ has products of $|J|$ objects, then we can make $\mathfrak{u}_* : \mathcal{B} \to \mathcal{A}$ defined by $(L_i) \mapsto \prod_{j \in J} u_{j*}(L_j)$, with which we have right adjoint $\mathfrak{u}_* : \mathcal{A} \to \prod_{j \in J} \mathcal{B}_j$, with $M \mapsto (u_i^*(M) \mid i \in J)$. The adjunction morphisms are given by

$$\eta_{\mathfrak{u},M} : M \to \prod_{i\in J} u_{i*}u_i^*(M) \text{ for } M \in \mathcal{A},$$

$$\varepsilon_{\mathfrak{u},L_i} : u_i^*(\prod_{j\in J} u_{j*}(L_j)) \to u_i^*u_{i*}(L_i) \to L_i \text{ for } L = (L_i) \in \mathcal{B}.$$

As an adjoint pair $(\mathfrak{u}^*, \mathfrak{u}_*)$, we have an associated comonad $\mathcal{G}_\mathfrak{u} = \mathfrak{u}^*\mathfrak{u}_*\ \mathcal{B}$; denote this by $(\mathcal{G}_\mathfrak{u}, \delta_\mathfrak{u}) = (\mathfrak{u}^*\mathfrak{u}_*, u^*\eta_\mathfrak{u}u_*)$. Denote by $\mathcal{B}_{\mathcal{G}_\mathfrak{u}}$ the category of coalgebras of the comonad $\mathcal{G}_\mathfrak{u}$ whose objects are $((L_i), \zeta)$. Beck's theorem says that if $\mathcal{A}$, and $\mathcal{B}$ are abelian, and if $\mathfrak{u}^*$ is additive, exact, and faithful, then $\mathcal{A} \simeq \mathcal{B}_{\mathcal{G}_\mathfrak{u}}$. In terms of the local data from each $\mathcal{B}_i$, a $\mathcal{G}_\mathfrak{u}$-coalgebra (L, ζ) is a collection (L_i, ζ_i) with $(L_i) = L$ and

$$\zeta_i : L_i \to u_i^*\mathfrak{u}_*(L) = u_i^*(\prod_{j\in J} u_{j*}(L_j)),$$

satisfying the following equalization of arrows:

$$u_i^*\mathfrak{u}_*(L) = u_i^*(\prod_{j\in J} u_{j*}(L_j)) \underset{u_i^*(u_{j*}\zeta_j)}{\overset{u_i^*\eta_{\mathfrak{u},\mathfrak{u}_*(L)}\mathfrak{u}_*(L)}{\rightrightarrows}} u_i^*(\prod_{m\in J} u_{m*}u_m^*(\prod_{j\in J} u_{j*}(L_j))) = u_i^*\mathfrak{u}_*\mathfrak{u}^*\mathfrak{u}_*(L),$$

and $\varepsilon_{\mathfrak{u},L_i}(L)\zeta_i = id_{L_i}$. Note, these are simply the requirements for being a coalgebra of a comonad made from an adjoint pair $(\mathfrak{u}^*, \mathfrak{u}_*)$. In terms of exactness, we need the exactness of the diagram below for all $i \in J$

$$L_i \longrightarrow u_i^*(\prod_{j\in J} u_{j*}(L_j)) \rightrightarrows u_i^*(\prod_{m\in J} u_{m*}u_m^*(\prod_{j\in J} u_{j*}(L_j))), \text{ or}$$

$$L \xrightarrow{\zeta} \mathcal{G}_\mathfrak{u}(L) \underset{\mathcal{G}_\mathfrak{u}(\zeta)}{\overset{\eta_\mathfrak{u}(L)}{\rightrightarrows}} \mathcal{G}_\mathfrak{u}^2(L).$$

If the functors u_k^* preserve products of $|J|$ objects, then this can be simplified slightly as the exact diagram

$$L_i \longrightarrow u_i^*(\prod_{j,m\in J} u_{j*}(L_j)) \rightrightarrows \prod_{j,m\in J} u_i^*u_{m*}u_m^*u_{j*}(L_j)\ .$$

Remark 6.1.8. This should look familiar. Indeed, as in remark $I.4.3$ of [13], if B is a scheme and $B_i \subseteq B$ are subschemes and we take $\mathcal{B}_i = \mathrm{Qcoh}(B_i)$ for all $i \in J$, then $u_i^*u_{j*}(L_j)$ is in

Qcoh($B_j \cap B_i$) over the intersection of B_i and B_j, and $u_i^* u_{m*} u_m^* u_{j*}(L_j)$ the restriction sheaf on $B_i \cap B_j \cap B_m$.

The previous discussion can be summarized nicely with the following framework. Again, we are considering abelian categories $\mathcal{A}$, and $\mathcal{B} = \prod\limits_{w \in I} \mathcal{B}_w$. An endofunctor $\mathcal{G} : \mathcal{B} \to \mathcal{B}$ is a collection of components $\mathcal{G} = (\mathcal{G}_{w_1,w_2})$, where $\mathcal{G}_{w_1,w_2} : \mathcal{B}_{w_2} \to \mathcal{B}_{w_1}$.

Definition 6.1.9. A *gluing data* for $(\mathcal{B}_w)_{w \in I}$ is a comonad $(\mathcal{G}, \varepsilon, \mu)$ on $\mathcal{B}$ such that

1. $\mathcal{G}$ is additive and left exact;

2. for each $w \in I$ the morphism $\varepsilon_w \mathcal{G}_{w,w} \to Id_{\mathcal{B}_w}$ induced by ε is an isomorphism.

The category $\mathcal{B}_{\mathcal{G}}$ of coalgebras of the comonad $\mathcal{G}$ is regered to as the category glued from the $\mathcal{B}_w$ along the endofunctor $\mathcal{G}$.

Definition 6.1.10. A *localization data* for $(\mathcal{B}_w)_{w \in I}$ is an abelian category $\mathcal{A}$ and a collection of exact functors $u_w^* : \mathcal{A} \to \mathcal{B}_w$, such that u_w^* has a right adjoint u_{w*}, and the adjunction $u_w^* \circ u_{w*} \to Id_{\mathcal{B}_w}$ is an isomorphism.

Each localization data defines an adjoint pair $(\mathfrak{u}^*, \mathfrak{u}_*)$ betwee $\mathcal{A}$ and $\mathcal{B} = \prod\limits_{w \in I} B_w$, and hence a comonad $\mathcal{G}$. By Theorem 6.1.7 we have the following:

Corollary 6.1.11. *If $(\mathcal{A}, \mathcal{B}_w, u_w^*)$ is a localization data, then $\mathcal{G} = (\mathcal{G}_{w_1,w_2})$, where $\mathcal{G}_{w_1,w_2} = u_w^* u_{w*}$, is a gluing data. The exact functor $\mathcal{A} \to \mathcal{B}_{\mathcal{G}}$ of Theorem 6.1.7 is an equivalence if and only if the functor $\oplus u_w^* : \mathcal{A} \to \oplus \mathcal{B}_w$ is faithful.*

Remark 6.1.12. To make this construction a little more accessible, let us consider something which may be more familiar. Recall that a functor between abelian categories $u^* : \mathcal{A} \to \mathcal{B}$ is called a *Serre localization*, or *Serre quotient* if $\mathcal{B} \simeq \mathcal{A}/\mathcal{C}$, with $\mathcal{C}$ a *Serre subcategory* of $\mathcal{A}$. This is obtained if we have u^* exact such that there exists a right adjoint u_* such that $u^* \circ u_* \simeq Id_{\mathcal{B}}$. Suppose $\{u_w^* : \mathcal{A} \to \mathcal{B}_w\}$ is a finite set of such localizations and we set $\mathcal{G}_{w_1,w_2} = u_{w_1}^* \circ u_{w_2 *} : \mathcal{B}_{w_2} \to \mathcal{B}_{w_1}$, then their collection $\mathcal{G} = (\mathcal{G}_{w_1,w_2})$ form a comonad on the category $\mathcal{B} = \prod\limits_{w \in I} \mathcal{B}_w$. The functor $u^* : \mathcal{A} \to \mathcal{B}$ is faithful if and only if u_w^* is faithful for all w, which is obtained if and only if $\cap \mathcal{C}_w$ is the zero category, for all the corresponding Serre subcategories $\mathcal{C}_w \subset \mathcal{A}$. In this case the category $\mathcal{B}_{\mathcal{G}}$ of coalgebras of the comand $\mathcal{G}$ is equivalent to the category A, by Theorem 6.1.7. Beware, this is not the case in our construction.

6.2 β-Projective n-Space

Projective n-space over the field of complex numbers, $\mathbb{P}^n_{\mathbb{C}}$, typically has two presentations: the first employing isomorphisms of the coordinate rings on the standard open affine cover and showing that the cocycle condition on the intersections of elements of the cover is satisfied. This second presentation takes the categorical approach outlined in the previous section to construct the category of quasi-coherent sheaves over $\mathbb{P}^n_{\mathbb{C}}$ as glued together from the categories of quasi-coherent sheaves on the cover of open affines, exploiting the framework outlined in the previous section. Yet another is the projective spectrum of graded polynomial algebras. In this section we will construct the category which should be viewed as quasi-coherent sheaves on the β-projective n-space.

6.2.1 Cocycle Gluing

To glue schemes $\{X_i\}_I$ together, we glue along open subschemes $X_{ij} \subset X_i$, which are isomorphic via morphisms $\phi_{ij} : \mathcal{O}_{X_j}|_{X_{ji}} \simeq \mathcal{O}_{X_i}|_{X_{ij}}$, wherein $\phi_{ii} = id_{X_i}$.

Definition 6.2.1. A collection of isomorphic subschemes (X_{ij}, ϕ_{ij}) of schemes $\{X_i\}_I$ are said to satisfy the *cocycle condition* if:

$$\phi_{ik}|_{X_{ki} \cap X_{kj}} = \phi_{ij}|_{X_{ji} \cap X_{jk}} \circ \phi_{jk}|_{X_{ki} \cap X_{kj}}, \tag{6.2}$$

all restricted to $X_{ij} \cap X_{kj} \cap X_{ik}$.

Let k be an algebraically closed field of characteristic 0, and consider the polynomial algebra $k[x_0, ..., x_n]$, where we consider it as $\Gamma = \mathbb{Z}^{n+1}$-graded. Recall that for a 2-cocycle

$\beta : \Gamma \times \Gamma \to k^\times$ for Γ which is generated by elements $e_k = (0, ..., 1, ..., 0) \in \Gamma$, the algebra $k_\beta[x_1, ..., x_n]$ is the algebra of non-commutative polynomials in n variables $x_1, ..., x_n$ for which $x_k x_l = \beta(e_k, e_l) x_l x_k$. Notice this demands that β is a skew symmetric 2-cocycle, i.e. $\beta(a, b)^{-1} = \beta(b, a)$, which indeed exists in $k^\times$. Additionally, because $x_i x_k x_i^{-1} = \beta(e_k, -e_i) x_i x_i^{-1} x_k = \beta(e_i, e_k) x_k x_i x_i^{-1}$, we must have that $\beta(a, b) = \beta(b, -a) = \beta(b, a)^{-1}$ if β is a bicharacter (and thus a 2-cocycle). Also note that $\beta(0, b) = \beta(b, 0) = \beta(0, 0) = 1$ for all $b \in \Gamma$. Finally, associativity of multiplication demands

$$\beta(a, b)\beta(a + b, c) = \beta(b, c)\beta(a, b + c). \tag{6.3}$$

Lemma 6.2.2. *The multiplicitive subset $S_i = \{1, x, x^2, ...\} \subset k_\beta[x_0, ..., x_n]$ is an Ore subset of $k_\beta[x_0, ..., x_n]$.*

Proof. See Example 3.1.5. □

Lemma 6.2.3. *For each $i \in \{0, ..., n\}$ there exists a 2-cocycle $\beta_i : \Gamma \times \Gamma \to k^\times$ such that*

$$\rho_i : (k_\beta[x_1, ..., x_n] S_i^{-1})_0 \simeq k_{\beta_i}[x_{1/i}, ..., x_{n/i}]/(x_{i/i} - 1),$$
$$x_k x_i^{-1} \mapsto x_{k/i},$$

with $x_{k/i} = x_k x_i^{-1}$.

Proof. The proof from lemma 4.3.3 [4] is elementary, but we include it here to familiarize us with our setting. Observe, using the fact that $\beta(a, b) = \beta(b, -a)$

$$\begin{aligned}
x_{k/i} x_{l/i} &= x_k x_i^{-1} x_l x_i^{-1} \\
&= \beta(e_k, -e_i) x_i^{-1} x_k x_l x_i^{-1} \\
&= \beta(e_k, -e_i)\beta(e_k, e_l) x_i^{-1} x_l x_k x_i^{-1} \\
&= \beta(e_k, -e_i)\beta(e_k, e_l)\beta(-e_i, e_l) x_l x_i^{-1} x_k x_i^{-1} \\
&= \beta(e_k, -e_i)\beta(e_k, e_l)\beta(-e_i, e_l) \tilde{x}_l \tilde{x}_k.
\end{aligned}$$

Thus, setting $\beta_i(e_k, e_l) := \beta(e_k, -e_i)\beta(e_k, e_l)\beta(-e_i, e_l)$, the result is obtained after verifying that β_i is indeed a 2-cocycle. $\square$

In light of the lemma, we will refer to $(k_\beta[x_1, ..., x_n]S_i^{-1})_0$ as simply $k_{\beta_i}[x_{1/i}, ..., \hat{x_{i/i}}, ..., x_{n/i}]$, as it can be assumed $x_{i/i} = 1$ is intuitive and to compact notation. These induce 2-cocycles $\beta_i : \Gamma_i = \Gamma/\mathbb{Z}e_i \simeq \mathbb{Z}^{n-1} \to k^\times$. The natural question is then if, given a finite collection of such cocycles, $\beta_i : \Gamma_i \times \Gamma_i \to k^\times$, there exists another, $\beta : \Gamma \times \Gamma \to k^\times$ such that $\beta|_{\Gamma_i} = \beta_i$.

Proposition 6.2.4. *For the collection of $n+1$ algebras $k_{\beta_i}[x_{1/i}, ..., x_{n/i}] \simeq (k_\beta[x_0, ..., x_n]S_i^{-1})_0$ of homogeneous elements of degree zero in the localizations of $k_\beta[x_0, ..., x_n]$ and isomorphisms between them $\phi_{ij} : k_{\beta_j}[x_{0/j}, ..., x_{n/j}] \to k_{\beta_i}[x_{0/i}, ..., x_{n/i}]$, with $\phi_{ij} : x_{k/j} \mapsto x_{k/i}x_{j/i}^{-1}$, the cocycle condition holds. That is,*

$$\phi_{ik}|_{X_{ki} \cap X_{kj}} = \phi_{ij}|_{X_{ji} \cap X_{jk}} \circ \phi_{jk}|_{X_{ki} \cap X_{kj}}.$$

Proof. The following diagram proves extremely helpful:

$$\begin{tikzcd}
k_{\beta_i}[x_{0/i}, ..., x_{n/i}, x_{j/i}^{-1}, x_{k/i}^{-1}] & & k_{\beta_j}[x_{0/j}, ..., x_{n/j}, x_{i/j}^{-1}, x_{k/j}^{-1}] \arrow[ll, "\phi_{ij}"'] \\
& k_{\beta_k}[x_{0/k}, ..., x_{n/k}, x_{j/k}^{-1}, x_{i/k}^{-1}], \arrow[ul, "\phi_{ik}"] \arrow[ur, "\phi_{jk}"'] &
\end{tikzcd}$$

Indeed, for $x_{l/k}x_{m/k} \in k_{\beta_k}[x_{0/k}, ..., x_{n/k}, x_{i/k}^{-1}, x_{j/k}^{-1}]$ we have

$$x_{l/k}x_{m/k} = \beta_k(e_l, e_m)x_{m/k}x_{l/k} = \beta(e_i, e_l)\beta(e_l, e_m)\beta(m, i)x_{m/k}x_{l/k}.$$

Under the composition we have:

$$\begin{aligned}
(\phi_{ij} \circ \phi_{jk})(x_{l/k}x_{m/k}) &= \phi_{ij}(x_{l/j}x_{k/j}^{-1}x_{m/j}x_{k/j}^{-1}) \\
&= x_{l/i}x_{j/i}^{-1}x_{j/i}x_{k/i}^{-1}x_{m/i}x_{j/i}^{-1}x_{j/i}x_{k/i}^{-1} \\
&= x_{l/i}x_{k/i}^{-1}x_{m/i}x_{k/i}^{-1} \\
&= \beta_i(e_m, e_k)\beta_i(e_l, e_m)\beta_i(e_k, e_l)x_{m/i}x_{k/i}^{-1}x_{l/i}x_{k/i}^{-1}.
\end{aligned}$$

Expanding out the β_i we obtain:

$=\cancel{\beta(e_i,e_m)}\beta(e_m,e_k)\cancel{\beta(e_k,e_i)}\cancel{\beta(e_i,e_l)}\beta(e_l,e_m)\cancel{\beta(e_m,e_i)}\cancel{\beta(e_i,e_k)}\beta(e_k,e_l)\cancel{\beta(e_l,e_i)}x_{m/i}x_{k/i}^{-1}x_{l/i}x_{k/i}^{-1}$

$= \beta(e_m,e_k)\beta(e_l,e_m)\beta(e_l,e_m)x_{m/i}x_{k/i}^{-1}x_{l/i}x_{k/i}^{-1}$

$= \beta_k(e_l,e_m)x_{m/i}x_{k/i}^{-1}x_{l/i}x_{k/i}^{-1}$

$= \beta_k(e_l,e_m)\phi_{ik}(x_{m/k}x_{l/k}).$ □

6.2.2 A Global Cocycle

In this section we answer the question of when we can consider a collection of $n+1$ 2-cocycle deformed polynomial algebras $k_{\beta_i}[x_{0/i},...,\hat{x}_{i/i},...,x_{n/i}]$, with $\beta_i\Gamma\times\Gamma\to k^\times$ a 2-cocycle and $\beta(e_i,a)=1$ for all $a\in\Gamma$, to all be algebras of homogeneous elements of degree zero from localizations from the same deformed polynomial algebra. This is done by constructing the 2-cocycle deformation that gives rise to each deformation of the algebras in the collection.

In the proof of Proposition 6.2.4 we followed the element $x_{l/k}x_{m/k}$ through the equation (6.2.1) defining the cocycle condition, seeing that for the cocycle condition to hold on a collection $\{\phi_{ij} : k_{\beta_j}[x_{0/j},...,\hat{x}_{j/j},...,x_{n/j}] \to k_{\beta_i}[x_{0/i},...,\hat{x}_{i/i},...,x_{n/i}] \mid i,j\in\{0,...,n\}\}$, we must have that

$$\beta_i(e_m,e_k)\beta_i(e_l,e_m)\beta_i(e_k,e_l)=\beta_k(e_l,e_m), \tag{6.4}$$

for all $i,k,m,l\in\{0,...,\}$. If we instead follow the element $x_{i/k}x_{j/k}$, we see

$$\begin{aligned}(\phi_{ij}\circ\phi_{jk)}(x_{i/k}x_{j/k}) &= \phi_{ij}(x_{i/j}x_{k/j}^{-1}\cancel{x_{j/j}}x_{k/j}^{-1})\\ &= \cancel{x_{i/i}x_{j/i}^{-1}x_{j/i}}x_{k/i}^{-1}x_{j/i}x_{k/i}^{-1}\\ &= \beta_i(e_j,e_k)x_{j/i}x_{k/i}^{-1}x_{k/i}^{-1}\\ &= \phi_{ik}(\beta_k(e_i,e_j)x_{j/k}x_{i/k}).\end{aligned}$$

Permuting through orders of i,j,k and choice of variables, we require, in addition to (6.4), that $\beta_i(e_j,e_k)=\beta_k(e_i,e_j)=\beta_j(e_k,e_i)$. Note, this is also evident from the proof of

Lemma 6.2.3 wherein we find $\beta_i(e_j, e_k) = \beta(e_i, e_j)\beta(e_j, e_k)\beta(e_k, e_i)$. This discussion and the previous section establishes the following result.

Proposition 6.2.5. *Suppose* $\{\beta_i : \mathbb{Z}^n \times \mathbb{Z}^n \to k^\times \mid i = 0, ..., n\}$ *is a collection of skew-symmetric 2-cocycles satisfying* $\beta_i(e_i, a) = 1$ *sufficient to produce skew polynomial rings* $k_{\beta_i}[x_{0/i}, ..., \hat{x}_{i/i}, ..., x_{n/i}]$*. Suppose also that for all* i, j, k *we have* $\beta_i(e_j, e_k) = \beta_k(e_i, e_j) = \beta_j(e_k, e_i)$*. Define* $\beta : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to k^\times$ *on the generators of* $\mathbb{Z}^{n+1}$ *by a choice of factorization in* $k^\times$ *of* $\beta_i(e_j, e_k) = \beta(e_i, e_j)\beta(e_j, e_k)\beta(e_k, e_i)$*, with* $\beta(e_i, e_i) = 1$*. Then*

1. β *is a skew-symmetric 2-cocycle of* $\mathbb{Z}^{n+1}$ *acting on* $k[x_0, ..., x_n]$ *and produces a deformation* $k_\beta[x_0, ..., x_n]$ *such that* $(k_\beta[x_0, ..., x_n]S_i^{-1})_0 \simeq k_{\beta_i}[x_{0/i}, ..., \hat{x}_{i/i}, ..., x_{n/i}]$ *for all* $i = 0, ..., n$ *and* $S_i = \{1, x_i, x_i^2, ...\}$*.*

2. *The morphisms* $(\phi_{ij} : (k_\beta[x_0, ..., x_n]S_j^{-1})_0\tilde{S}_i^{-1})_0 \to ((k_\beta[x_0, ..., x_n]S_i^{-1})_0\tilde{S}_j^{-1})_0$ *satisfy the cocycle condition of sheaves in 6.2.1*

.

6.2.3 Categorical Proj Construction

Consider first the 1-dimensional, commutative case, as in example 6.1.5. Let $\tilde{x}$ be an autoequivalence of the category $\mathrm{gr}_{\mathbb{Z}}\, k$-vect. Then we can consider the category $\mathrm{gr}_{\mathbb{Z}}\, k[x]$-mod as the category of modules of the monad constructed with this autoequivalence: $\mathcal{F}_x := \langle \tilde{x}, \eta, \mu \rangle$. As per Defintion 6.1.3, the algebras of this monad are pairs (M, u) with $M \in \mathrm{gr}_{\mathbb{Z}}\, k$-vect and u a collection of arrows $\{u_n \mid u_n : \tilde{x}(M_n) \to M_{n+1}$ for all $n \in \mathbb{N}\}$ satisfying multiplicitive and associative diagrams. Morphisms of these modules are k-linear homomorphisms $f : M \to M'$ in $\mathrm{gr}_{\mathbb{Z}}\, k$-vect such that the diagram

$$\begin{tikzcd} \tilde{x}(M_n) \arrow[r, "u_n"] \arrow[d, "\tilde{x}f"] & M_{n+1} \arrow[d, "f"] \\ \tilde{x}(M'_n) \arrow[r, "u'_n"] & M'_{n+1} \end{tikzcd}$$

is commutative. Now, let $\mathbb{T}_+$ be the full subcategory of k-vect annihilated by $\tilde{x}$, and $\mathbb{T}_+^-$ its Serre closure. The localization of the category $\mathrm{gr}_{\mathbb{Z}}\, k[x]$-mod by $\mathbb{T}_+^-$ is constructed as the category with the same objects of $\mathrm{gr}_{\mathbb{Z}}\, k[x]$-mod, but with certain morphisms turned into isomorphisms. Specifically, the structure morphisms $u_n : \tilde{x}(M_n) \to M_{n+1}$ for all $n \in \mathbb{Z}$.

We wish to compare $\mathrm{gr}_{\mathbb{Z}}\, k[x]\text{-mod}/\mathbb{T}_+^-$ to the category $(S_x^{-1}k[x])_0$-mod. First consider the natural inverse of the autoequivalence $\tilde{x}$, denoted $\tilde{x}^{-1}$, with $\gamma : \tilde{x}\tilde{x}^{-1} \to Id$, the adjunction isomorphism. Then, our objects in $\mathrm{gr}_{\mathbb{Z}}\, k[x]\text{-mod}/\mathbb{T}_+^-$ are pairs (V, v) with $v = \gamma_V : \tilde{x}\tilde{x}^{-1}V \to V$. This sets up the following,

Proposition 6.2.6. *$k[x]$-mod$/\mathbb{T}_+^- \simeq (S_x^{-1}k[x])_0$-mod $\simeq$ k-vect, as category equivalences, where $S_x = \{1, x, x^2, ...\}$.*

Proof. Let $\Phi : k[x]\text{-mod}/\mathbb{T}_+^- \to (S_x^{-1}k[x])_0\text{-mod}$ by $(M, m) \mapsto (M_0, m_0)$, and define $\Psi : (S_x^{-1}k[x])_0\text{-mod} \to gr_\Gamma k[x]\text{-mod}$ by sending (V, v) to $(\oplus \tilde{x}^n(V), v_n)$, where $v_n = id\ \forall\ n$. Thus, this is the free$\dashv$forgetful adjunction, and since $v_n = id$, Ψ sends objects to $k[x]\text{-mod}/\mathbb{T}_+^-$.

As Φ is forgetful, it is fully faithful, and all that needs to be shown is that each object $(V, v) \in (S_x^{-1}k[x])_0$-mod is equivalent to $\Phi((M, m))$ for some $(M, m) \in k[x]\text{-mod}/\mathbb{T}_+^-$. Since $\mathrm{gr}_\Gamma\, k[x]$-mod is by definition the category of modules of the monad $\mathcal{F}_x$ on k-vect, and $M_0 \subset M$ by definition must be a $k[x]_0 = k$-module, this is obvious. $\square$

Now we turn to the n-dimensional, non-commutative case. The work is much the same. Consider $J = \{1, .., n\}$ and $\{\tilde{x}_i\}_{i \in J}$ a collection of automorphisms of the category k-vect. Corresponding to our 2-cocycle β, let $\beta_{ij} : \tilde{x}_i\tilde{x}_j \simeq \tilde{x}_j\tilde{x}_i$ be a family of natural isomorphisms, $\tilde{x}_i^{-1}$ the right (and left) adjoint to $\tilde{x}_i$, and $\gamma_i : \tilde{x}_i\tilde{x}_i^{-1} \simeq Id$ their adjunction morphism. To accomodate β, we'll be using a $\Gamma := \mathbb{Z}_+^n$-grading, and it will be convenient to think of $\tilde{x}_i = \Theta(e_i)$ for some $\Theta : \mathbb{Z}_+^n \to \mathrm{Aut}(k\text{-vect})$ and $e_i = (0, ..., 0, 1, 0, ...0) \in \Gamma$.

Lemma 6.2.7. $\Theta^\star = \bigoplus_{z \in \mathbb{Z}_+^n} \Theta(z)$ *is a monad on the category k-vect.*

Proof. Obviously, $\Theta^\star$ is an endofunctor of k-vect. To make θ^* a monad, we need to constructs multiplication and unit transformations. Since Γ is a monoid generated by $\{e_i \mid i = 0, ..., n\}$, we only need consider the generators. Define $\mu : \Theta^\star \circ \Theta^\star \simeq \Theta^\star$ element-wise by the family of natural isomorphisms $\Theta(e_i) \circ \Theta(e_j) \simeq \Theta(e_i + e_j)$, and $\eta : Id \to \Theta^\star$ by the mappings

$$\eta_M : M \to \theta^*(M) = \bigoplus_{z \in \mathbb{Z}} \theta(z)(M)$$
$$m \mapsto m \in \theta(0)(M) \in \theta^*(M).$$

The diagram

$$\begin{tikzcd}
\Theta^\star \circ \Theta^\star \circ \Theta^\star \arrow[r, "\Theta^\star \mu"] \arrow[d, "\mu \Theta^\star"] & \Theta^\star \circ \Theta^\star \arrow[d, "\mu"] \\
\Theta^\star \circ \Theta^\star \arrow[r, "\mu"] & \Theta^\star
\end{tikzcd}$$

commutes via the natural isomorphism $\beta : \Theta^\star \simeq \Theta^\star$, where β is the collection of β_{ji} : $\Theta(e_j) \circ \Theta(e_i) \simeq \Theta(e_i) \circ \Theta(e_j) \simeq \Theta(e_i + e_j)$. □

In the following we will consider the concrete example of the categorical projective n-space. Let $\Gamma = \mathbb{Z}^{n+1}$. As in the 1-dimensional, commutative case, associated to the monad Θ^*, the category of Γ-graded algebras $\mathrm{gr}_\Gamma\, \Theta^\star$-mod is just the category $\mathrm{gr}_\Gamma\, k_\beta[x_0, ...x_n]$-mod. In terms of the monad $\Theta^\star$, objects of $\mathrm{gr}_\Gamma\, \Theta^\star$-mod are pairs (M, m) with $M \in k$-vect and the structure morphism a family $m_{i,n} : \tilde{x}_i(M_n) \to M_{n+e_i}$ for $n \in \Gamma$.

We will call $R = k_\beta[x_0, ...x_n]$ and R-mod= $k_\beta[x_0, ...x_n]$-mod. Now consider the localizations by the Ore sets $S_i = \{x_i, x_i^2, ...\}$, which we denote by RS_i^{-1} and are graded by $\mathbb{Z}_+ \times ... \times \mathbb{Z}_+ \times \mathbb{Z} \times \mathbb{Z}_+ \times ... \times \mathbb{Z}_+$, with $\mathbb{Z}$ in the *ith* position. If we define $\hat{e}_k := e_k - e_i$, and $\Gamma_i := \bigoplus_{k \in J} \langle \hat{e}_k \rangle \simeq \mathbb{Z}^n$, then Γ_i grade the so-called zero degree components of the localizations. Denote $R_i := k_{\beta_i}[x_{0/i}, ..., x_{n/i}]$ with $x_{i/i} = 1$, then the Lemma 6.2.3 establishes an equivalence between $(RS_i^{-1})_0 - mod$ and $R_i - mod$.

Similar to the commutative, 1-dimensional case we let $\mathbb{T}_i$ be the full subcategory of $gr_\Gamma k_\beta[x_0, ..., x_n]$-mod generated by modules M annihilated by the subfunctor $\tilde{x}_i$ of $\Theta^\star$ and $\mathbb{T}_i^-$ be its Serre closure.

Lemma 6.2.8. *$gr_\Gamma k_\beta[x_0, ..., x_n]$-mod$/\mathbb{T}_i^- \simeq (k_\beta[x_0, ..., x_n]S_i^{-1})_0$-mod*

Proof. This statement is equivalent to that of Lemma VII.2.4.3.4 of [5]. □

Denote by Q_i the canonical functor $\mathrm{gr}_\Gamma\, k_\beta[x_0,...,x_n] \to \mathrm{gr}_\Gamma\, k_\beta[x_0,...,x_n]\text{-mod}/\mathbb{T}_i^-$. This equivalence is very fortuitous for us, since $\mathbb{T}_+^-$ is a Serre subcategory of a category of modules over a unital, associative ring, and moreover, is a Grothendieck category. From here, we get that $\mathbb{T}_+^-$ is thick, and thus it is localizing. The latter means that the canonical localization functor has a (necessarily fully faithful) right adjoint.

Now, denote by Φ_i the composition of Q_i with the equivalence $gr_\Gamma k_\beta[x_0,...,x_n]\text{-mod}/\mathbb{T}_i^- \simeq (k_\beta[x_0,...,x_n]S_i^{-1})_0\text{-mod}$. Then $\{\Phi_i\}$ are a family of exact localizations with right adjoint functors, and thus define a *localization data* on the category k-vect.

Finally, define $\mathbb{T}_+ := \bigcap_{i\in J} \mathbb{T}_i \subset k_\beta[x_0,...,x_n]$-mod. By Lemma VII.2.4.3.3 of [5], since J is finite, we get that $\mathbb{T}_+^- = \bigcap_{i\in J} \mathbb{T}_i^-$, i.e. the Serre closure conicides with the (finite) intersection of the Serre closures. Hence we have the result

$$gr_\Gamma k_\beta[x_0,...,x_n]\text{-mod}/\mathbb{T}_+^- \simeq \Phi\text{-mod}, \tag{6.5}$$

akin to the classical commutative definition $\mathbf{Proj}(R) = gr_\Gamma R\text{-mod}/R_+$ for R a Γ-graded commutative ring and R_+ its irrelevant ideal.

Proposition 6.2.9. *Given a collection of 2-cocycles $\{\beta_i : \mathbb{Z}^n \times \mathbb{Z}^n \to k^\times \mid i = 0,...,n\}$ satisfying the conditions of Proposition 6.2.5, there exists a deformed polynomial algebra $k_\beta[x_0,...,x_n]$ such that $k_\beta[x_0,...,x_n]/\mathbb{T}_+^-$ gives the category of quasi-coherent sheaves on the non-commutative projective space obtained by gluing together the corresponding $k_{\beta_i}[x_1,...,x_n]$.*

Proof. This is a direct result of Proposition 6.2.5 and Lemma 6.2.8 above. □

6.3 Modules of Shifted Differential Operators on the Non-Commutative $\mathbb{P}^n_\beta$

As we have just shown in propositions 6.2.9 and 6.2.5, there is no need to consider an arbitrary finite collection of 2-cocycle deformations and the non-commutative projective space obtained by gluing together the associated rings and categories of modules, we can simply assume that they are localizations of a suitable $k_\beta[x_0, ..., x_n]$ and its own category of modules. Furthermore, we showed in example 3.3 that for a bicharacter γ, $D_q((RS_i^{-1})\hat{S}_j^{-1}) \simeq D_q(R)S_i^{-1}\hat{S}_j^{-1}$. Here we elect to proceed as in section [12], §2, primarily because we would like to consider the category of modules for $\mathbb{P}^n_\beta$ in the full generality that we can manage, which means considering a collection of (distinct) bicharacters with which to make the quantum differential operators on our affine patches. However, for the sake of completeness, in [14] it is shown that the action of $k[\Gamma]$ on the ring $k_\beta[x_0, ..., x_n]$ is *differential*, meaning, among other things, that it is compatible with taking the homogeneous elements of degree 0 in the localizations by Ore sets such as $S_i = \{1, x_i, x_i^2, ...\}$, and thus the construction could be made as in 6.2.9 above.

The aim is to construct the category of Lunts and Rosenberg's quantum D-modules on our non-commutative projective space $\mathbb{P}^n_\beta$, which itself is glued together from affine patches via $k_{\beta_i}[x_1, ..., x_n]$, where the collection of 2-cocycles β_i satisfy the cocycle condition 6.2.1. To avoid confusion, let us refer to $k_{\beta_i}[x_1, ..., x_n]$ as $k_{\beta_i}[x_{0/i}, ..., x_{n/i}]$, since, from proposition 6.2.5 we know they can be thought of as the homogeneous elements of degree 0 in localizations of an appropriate $k_\beta[x_0, ..., x_n]$ anyway. Now, from proposition 3.2.2 we know that

$$\begin{aligned}(D_q(k_\beta[x_0, ..., x_n])S_i^{-1})_0 &\simeq (D_q(k_\beta[x_0, ..., x_n]S_i^{-1}))_0 \\ &\simeq D_q(k_{\beta_i}[x_{0/i}, ..., x_{n/i}])\end{aligned}$$

We have to be careful with this statement: though our operators $\partial_i^{\beta^a}$ have a gradation, we do not consider them when taking our homogeneous degree 0 localization. From here, we employ Theorem 3.2.18 of [4] to obtain

$$D_q(k_{\beta_i}[x_{0/i}, ..., x_{n/i}])\text{-mod} \simeq D_{q'}(k[x_{0/i}, ..., x_{n/i}])\text{-mod},$$

where on the right hand side our quantum differential operators are made with the bicharacter $\gamma^{\beta_i}(a, b) := \gamma(a, b)\beta^{-1}(a, b)\beta(b, a)$. Finally, let $\mathbb{T}_i$ be the full subcategory of $D_q(k_\beta[x_0, ..., x_n])$-mod generated by modules M with $x_i \in \text{ann}(M)$, and $\mathbb{T}_i^-$ its Serre closure. Exactly as in

Lemma 6.2.8, we have that

$$(D_q(k_\beta[x_0,...,x_n])S_i^{-1})_0\text{-mod} \simeq D_q(k_\beta[x_0,...,x_n])\text{-mod}/\mathbb{T}_i^-.$$

Thus, we have a family of exact localizations $\{u_i^* : D_q(k_\beta[x_0,...,x_n])\text{-mod} \to D_{q'}(k[x_{0/i},...,x_{n/i}])\text{-mod}\}$, with right adjoints, say, u_{i*}. Following 6.1.2, name $\mathfrak{u}^* : D_q(k_\beta[x_0,...,x_n])\text{-mod} \to \prod D_{q'}(k[x_{0/i},...,x_{n/i}])\text{-mod}$ and $\mathfrak{u}_*$ its right adjoint. Then $(\mathcal{G}_\mathfrak{u}, \delta_\mathfrak{u}) = (\mathfrak{u}^*\mathfrak{u}_*, \mathfrak{u}^*\eta_\mathfrak{u}\mathfrak{u}_*)$ is a comonad on the category $\prod D_{q'}(k[x_{0/i},...,x_{n/i}])$-mod.

Definition 6.3.1. A *D_q-module* on the β-projective space $\mathbb{P}^n_\beta$ is a coalgebra of the comonad $\mathcal{G}_\mathfrak{u}$. That is, it is a collection $(L,\xi) = (L_i,\xi_i)$, with modules $L_i \in D_q(k_\beta[x_0,...,x_n]\text{-mod}/\mathbb{T}_i^-$ and

$$\xi_i : L_i \to u_i^*\mathfrak{u}_*(L) = u_i^*(\prod u_{j*}(L_j)),$$

satisfying

$$L_i \longrightarrow u_i^*(\prod_{j,m\in J} u_{j*}(L_j)) \rightrightarrows \prod_{j,m\in J} u_i^* u_{m*} u_m^* u_{j*}(L_j).$$

We will denote the category $\mathcal{G}_\mathfrak{u}$-coalg of these objects by $\mathcal{D}^\gamma_{\mathbb{P}^n_\beta}$-mod.

Following the discussion in remark 6.1.8, an object here is a collection of γ-quantum D-modules on each affine patch which have been glued together on the triple intersections.

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