ON UNIMODULAR HYPERGRAPHS

by

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CHAPTER 1

Throughout this thesis we will consider only finite graphs and hypergraphs.

Let \( X \) be a set of \( n \) distinct elements where \( n \) is a positive natural number. Let \( x, y \in X \) (\( x = y \) is allowed), then define \([x,y]\) to be a nonordered pair, i.e., \([x,y] = [y,x]\). Let \( E \subseteq \{[x,y] : x, y \in X\} \) where \( U\{[x,y] : [x,y] \in E\} = X \) then the pair \((X,E) = G\) is called a graph.

An element in \( X \) is called a vertex and an element \([x,y] \in E\) is called an edge connecting \( x \) and \( y \), where \( x \) and \( y \) are endpoints of \([x,y]\).

A loop is an edge of the form \([x,x]\).

A chain of length \( q \) is a sequence \( \mu = (x_1, \ldots, x_{q+1}) \) of vertices of \( G \) such that, for \( i = 1, \ldots, q \), \([x_i,x_{i+1}] \in E\). We say \( x_1 \) is the initial vertex and \( x_{q+1} \) is the terminal vertex. Let \( \lambda(\mu) = q \).

A cycle is a chain \( \mu \) such that \( \lambda(\mu) > 1 \), the initial and terminal vertices are the same and for \( x_i, x_{i+1}, x_j \) and \( x_j+1 \) in \( \mu \), we have that if \( x_i = x_j \) then \( x_{i+1} \neq x_{j+1} \).

An elementary cycle is a cycle in which the only vertex in the cycle that is repeated, is the initial and terminal vertex, which is repeated only then.

A connected graph is a graph such that for each \( x, y \in X \), \( x \neq y \), there exists a chain connecting \( x \) and \( y \).

Let \( S \) be a subset of \( X \), then \( S \) is called a stable set if no edge joins any two distinct vertices in \( S \).
Let $G = (X,E)$ be a graph and let $(S_1,S_2)$ be a partition of $X$ where $S_1$ and $S_2$ are stable sets, then $(S_1,S_2)$ is a bicoloring for $G$.

**Lemma 1.1** Let $G$ be a connected graph with a bicoloring, $(S_1,S_2)$. Then for any $x \in S_1$, $y \in S_2$ and chain $\mu$ connecting $x$ and $y$, we have $\lambda(\mu)$ is odd.

**Proof** Since $G$ is connected, there is a chain $\mu$ connecting $x$ and $y$. Since $(S_1,S_2)$ is a bicoloring of $X$, $\mu$ is of the form $(x,y_1,x_1, \ldots , y_p, x_p, y)$ for some $p$, where each $x_i$ is in $S_1$ and each $y_1$ is in $S_2$. Hence $\lambda(\mu) = 2p + 1$ which is odd.

**Proposition 1.2** Let $G$ be a graph with an odd cycle, then $G$ has an elementary odd cycle.

**Proof** Assume $G$ has an odd cycle, $\mu$. If $\mu$ is elementary then we are finished. Assume $\mu$ is not elementary, then $\mu$ can be decomposed as $(\mu_1,\mu_2)$ where either $\mu_1$ or $\mu_2$ is odd. Assume $\mu_1$ is odd. If $\mu_1$ is elementary then we are finished. If $\mu_1$ is not elementary, then repeat the above process as many times as possible. Since $\mu$ had only finite length, there is only a finite number of times we can repeat the process. Let $\mu'$ be the final odd cycle. Then by the definition of elementary cycle, $\mu'$ is an odd elementary cycle.

**Lemma 1.3** Let $T$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, then $T$ is onto iff $T^{-1}$ exists.

**Proof** For the proof of this lemma see [Curtis, Theorem 13.10].
Since any matrix $M$ can be represented as a linear transformation $T_M$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ and any linear transformation can be represented as a matrix with respect to some basis, we have the following theorem.

**Theorem 1.4** Let $M$ be an integral $n \times n$ matrix, then $\det M = \pm 1$ iff for each integral $x \in \mathbb{R}^n$ there is an integral $x' \in \mathbb{R}^n$ such that $Mx' = x$.

**Proof** Let $\det M = \pm 1$ and $M$ be integral, then $M^{-1}$ is integral by the well known Cramer's Rule. Let $x \in \mathbb{R}^n$ be integral then $x' := M^{-1}x$ is integral such that $Mx' = M(M^{-1}x) = x$. Therefore for each integral $x \in \mathbb{R}^n$ there is an integral $x' \in \mathbb{R}^n$ such that $Mx' = x$.

Now let $M$ be integral with the property that for each integral $x \in \mathbb{R}^n$ there is an integral $x' \in \mathbb{R}^n$ such that $Mx' = x$.

Define $e_i$ to be the n-vector that has all zeros except for a 1 in the $i^{th}$ position. Then for each $i \in \{1, ..., n\}$ there is an $x_i$ such that $Mx_i = e_i$. Since $\{e_1, ..., e_n\}$ is a basis for $\mathbb{R}^n$, $M$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Hence by lemma 1.3 $M^{-1}$ exists. Therefore by the hypothesis, $M^{-1}e_i$ is integral for each $i$. Note that $M^{-1}e_i$ is the $i^{th}$ column of $M^{-1}$. Hence $M^{-1}$ is integral.

Since $\det M^{-1} = (\det M)^{-1}$ and $M^{-1}$ is integral, $\det M$ and $(\det M)^{-1}$ are integral. However, the only integers with this property are $\pm 1$.

Let $X$ be a set and $S \subseteq X$, then $S^c$ is denoted to be the complement of $S$ in $X$. 
Proposition 1.5  Let \( P(J) \) be the power set of \( J = \{1, \ldots, n\} \), then \((P(J), +)\) is a group under "+" where "+" is defined, for \( A, B \in P(J), \) by \( A + B = (A \cap B^c) \cup (B \cap A^c) = (A \cup B) \cap (A \cap B)^c. \)

Proof  Clearly for all \( A, B \in P(J), (A + B) \in P(J), \) \( A + \phi = \phi + A = A \) and \( A + A = \phi. \) Therefore \( P(J) \) is closed under "+", there is an identity element and each element has an inverse. Let \( A, B, C \in P(J), \) then

\[
(A + B) + C = \]
\[
[((A \cap B^c) \cup (A^c \cap B)) \cap C^c] \cup [[(A \cup B) \cap (A \cap B)^c] \cap C] =
]
\[
[(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)] \cup [(A \cup B \cap C) \cap (A \cup B)^c \cap C] =
]
\[
[(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)] \cup [(A \cup B \cap C) \cap (C \cap A^c \cap B^c)] =
]
\[
[(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)] \cup [(A \cup B \cap C^c) \cup (C \cap A^c \cap B^c)] =
]
\[
[(A \cap (B \cup C)^c) \cup (A \cap B \cap C^c)] \cup [(A^c \cap (B \cap C^c) \cup (B^c \cap C))] =
]
\[
[A \cap ((B \cap C) \cup (B \cup C)^c)] \cup [(A^c \cap ((B \cap C^c) \cup (B^c \cap C))] =
]
\[
[A \cap ((B \cap C) \cap (B \cap C)^c)] \cup [(A^c \cap ((B \cap C^c) \cup (B^c \cap C))] =
\]
\[ A + (B + C). \]

Hence we have associativity. Therefore \( P(J) \) is a group under "+".

Let \( Z_2 := \{0, 1\} \) be the field of order 2. Let \( A \in P(J) \) and \( \alpha \in Z_2, \) then define \( \alpha A = A \) if \( \alpha = 1 \) and \( \alpha A = \phi \) if \( \alpha = 0. \)

Proposition 1.6  \( P(J) \) is a vector space over \( Z_2 \) with the above definitions.

Proof  Let \( A, B, C \in P(J) \) and \( \alpha, \beta \in Z_2, \) then \( (A + B) + C = A + (B + C), A + B = B + A, A + \phi = A, A + A = \phi, \alpha(A + B) = \alpha A + \alpha B, \)
(α + β)A = αA + βA, (αβ)A = α(β)A and 1·A = A. Therefore P(J) is a vector space over \( \mathbb{Z}_2 \).

Let \( A \) be an \( n \times m \) integral matrix with \( J := \{1, \ldots , m\} \) indexing the columns and \( I := \{1, \ldots , n\} \) indexing the rows. Let \( K \in P(J) \), then define \( \Delta K = \{i \in I : \sum_{j \in K} a_{ij} \text{ is even}\} \) and define \( f_A(K) = (\Delta K)^C \).

**Proposition 1.7** Let \( A \) be an \( n \times m \) integral matrix with \( J \) indexing the columns and \( I \) indexing the rows and let \( f_A \) be defined as above, then \( f_A \) is a linear function from \( P(J) \) into \( P(I) \).

**Proof** Let \( S, T \in P(J) \). Then all that need be shown is

\[
f_A(S + T) = f_A(S) + f_A(T).
\]

\[
f_A(S) + f_A(T) = (f_A(S) U f_A(T)) \cup (f_A(S) \cap f_A(T))^C = (((\Delta S)^C \cup (\Delta T)^C) \cap (((\Delta S)^C \cap (\Delta T)^C))^C = ((\Delta S) \cap (\Delta T))^C \cap (((\Delta S) \cup (\Delta T))^C)^C = (\Delta S \cup \Delta T)^C = \Delta S + \Delta T.
\]

Hence all that need be shown is \( \Delta S + \Delta T = (\Delta(S + T))^C \), since \( f_A(S + T) = (\Delta(S + T))^C \). Now since \( \Delta S + \Delta T = (\Delta(S + T))^C \) iff \( \Delta(S + T) \cap (\Delta S + \Delta T) = \phi \) and \( \Delta(S + T) \cup (\Delta S + \Delta T) = I \), all that need be shown is \( \Delta(S + T) \cap (\Delta S + \Delta T) = \phi \) and \( \Delta(S + T) \cup (\Delta S + \Delta T) = I \).

Let \( i_0 \in (\Delta S + \Delta T) = (\Delta S \cap (\Delta T)^C) \cup (\Delta T \cap (\Delta S)^C) \). Since \( (\Delta S \cap (\Delta T)^C) \cap (\Delta T \cap (\Delta S)^C) = \phi \) we have either \( i_0 \in (\Delta S \cap (\Delta T)^C) \) or \( i_0 \in (\Delta T \cap (\Delta S)^C) \) but not both. Assume \( i_0 \in (\Delta S \cap (\Delta T)^C) \) then \( \sum_{j \in S} a_{i_0 j} \) is even and \( \sum_{j \in T} a_{i_0 j} \) is odd. Hence

\[
(\sum_{j \in S} a_{i_0 j} + \sum_{j \in T} a_{i_0 j}) \text{ is odd. Hence}
\]
\[ i_0 \in \Delta((S \cap T) \cup (T \cap S^c)) = \Delta(S + T). \text{ Similarly, if } i_0 \in (\Delta T \cap (\Delta S)^c), \text{ then } i_0 \notin \Delta(S + T). \text{ Therefore } (\Delta S + \Delta T) \cap \Delta(S + T) = \phi. \]

Let \( i_0 \in I \) to prove \( \Delta(S + T) \cup (\Delta S + \Delta T) = I. \)

**Case 1** \( \sum_{j \in S} a_{i_0j} \) and \( \sum_{j \in T} a_{i_0j} \) are both even. Then
\[
(\sum_{j \in S} (S \cap T^c) a_{i_0j} + \sum_{j \in T} (S^c \cap T) a_{i_0j}) \text{ is even. Hence } i_0 \in \Delta(S + T).
\]

**Case 2** \( \sum_{j \in S} a_{i_0j} \) and \( \sum_{j \in T} a_{i_0j} \) are both odd. Then
\[
(\sum_{j \in S} (S \cap T^c) a_{i_0j} + \sum_{j \in T} (S^c \cap T) a_{i_0j}) \text{ is even. Hence } i_0 \in \Delta(S + T).
\]

**Case 3** \( \sum_{j \in S} a_{i_0j} \) is even and \( \sum_{j \in T} a_{i_0j} \) is odd. Then
\[
i_0 \in (\Delta S \cap (\Delta T)^c). \text{ Hence } i_0 \in (\Delta S + \Delta T).
\]

**Case 4** \( \sum_{j \in S} a_{i_0j} \) is odd and \( \sum_{j \in T} a_{i_0j} \) is even. Then
\[
i_0 \in (\Delta T \cap (\Delta S)^c). \text{ Hence } i_0 \in (\Delta S + \Delta T). \text{ Therefore }
i_0 \in [(\Delta S + \Delta T) \cup \Delta(S + T)]. \text{ Hence } (\Delta S + \Delta T) \cup \Delta(S + T) = I.
\]

Therefore \( f_A \) is a linear function from \( P(J) \) into \( P(I) \).

Note that the need for the last three propositions is to help prove Theorem 3.15.

**Lemma 1.8** Let \( A_n \) be an \( n \times n \) matrix with \( n \geq 3 \) be that
\[
A_n = \begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 0 \\
& & & & \vdots & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]
then \( \det(A_n) \) is 0 if \( n \) is even or 2 if \( n \) is odd.

**Proof** By expanding along the first column we see that
\[ \det A_n = \det \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} + (-1)^{n-1} \det \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} = 1 + (-1)^{n-1}. \]

Hence, \( \det A_n \) is 0 if \( n \) is even or 2 if \( n \) is odd.
CHAPTER 2

Let $X$ be a finite set and let $E \subseteq P(X)$ where $\emptyset \notin E$ and $UE = X$, then we call the pair $(X,E) = H$ a hypergraph.

Each element in $X$ is called a vertex and each $E \in E$ is called an edge.

A subhypergraph of $H = (X,E)$ generated by the set $A \subseteq X$ is defined to be the hypergraph $H_A = (A,E_A)$ where $E_A := \{E_A := (E \cap A) : E \in E \text{ and } (E \cap A) \neq \emptyset\}$.

In a hypergraph $H = (X,E)$ a chain of length $q$ is defined to be a sequence $\eta = (x_1,E_1,x_2,E_2,\ldots,x_q,E_{q+1})$ such that $x_1,\ldots,x_q$ are distinct vertices of $H$, $E_1,\ldots,E_q$ are distinct edges of $H$ and for each $k \in \{1,\ldots,q\}$, $x_k,x_{k+1} \in E_k$. Let $\lambda(\eta)$ denote the length of $\eta$.

A cycle of length $q$ ($q > 1$) is a chain of length $q$ with $x_1 = x_{q+1}$.

A bicoloring of a hypergraph $H = (X,E)$ is a partition $(S_1,S_2)$ of $X$ such that if $E \in E$ with $|E| > 1$ then $E \not\subseteq S_1$ and $E \not\subseteq S_2$.

An equitable bicoloring of a hypergraph $H = (X,E)$ is a partition $(S_1,S_2)$ of $X$ such that, for each $E \in E$, $|E \cap S_1| - |E \cap S_2| \leq 1$.

Proposition 2.1 An equitable bicoloring of a hypergraph $H = (X,E)$ is a bicoloring of $H$.

Proof Let $(S_1,S_2)$ be an equitable bicoloring and $E \in E$ with $|E| > 1$, then since $|E \cap S_1| - |E \cap S_2| \leq 1$ and $S_1 \cap S_2 = \emptyset$, $E \not\subseteq S_1$ and $E \not\subseteq S_2$. Therefore $(S_1,S_2)$ is a bicoloring.

The following example shows that the converse is not true.
Example 2.2 Let $X := \{1, 2, 3, 4\}$, $E := \{X\}$, $S_1 := \{1, 2, 3\}$ and $S_2 := \{4\}$, then $(S_1, S_2)$ is a bicoloring but not an equitable bicoloring.

Proof Clearly $(S_1, S_2)$ is a bicoloring and $|X \cap S_1| - |X \cap S_2| = 2$.

Let $G = (X, E)$ be a graph. Define $C_G = \{\mu : \mu \text{ is a cycle of } G\}$, $\text{Ch}_G = \{\mu : \mu \text{ is a chain of } G\}$. For each $\mu \in \text{Ch}_G$, let $E_\mu := \{x : x \text{ is a vertex of } \mu\}$.

Proposition 2.3 Let $G = (X, E)$ be a graph with a bicoloring $(S_1, S_2)$. Let $E = \{E_\mu : \mu \in C_G\}$, then $(S_1, S_2)$ is a bicoloring of the hypergraph $H = (X, E)$.

Proof Let $G$ be a graph with a bicoloring $(S_1, S_2)$. Let $E$ be any element in $C_G$. Since $(S_1, S_2)$ is a bicoloring for $G$, $E_\mu \not\in S_1$ and $E_\mu \not\in S_2$. Therefore $(S_1, S_2)$ is a bicoloring for $H$.

A hypergraph $H = (X, E)$ is defined to be unimodular if for each $S \subseteq X$, where $|S| \geq 2$, the subhypergraph $H_S$ has an equitable bicoloring.

Example 2.4 Let $X := \{1, \ldots, n\}$, $n \geq 2$, and $F := \{[i, i+1] : 1 \leq i \leq n\}$. Define $G = (X, F)$. Choose $C \in \mathcal{P}(\text{Ch}_G)$ such that $\bigcup_{\mu \in C} E_\mu = X$, then for $E := \{E_\mu : \mu \in C\}$ we have $H = (X, E)$ is a unimodular hypergraph.

Proof Let $G$ be a graph defined as in the hypothesis. Choose $C \in \mathcal{P}(\text{Ch}_G)$ such that $\bigcup_{\mu \in C} E_\mu = X$ (such a $C$ exists since $\mu = (1, \ldots, n)$ is a chain in $G$). Let $S \subseteq X$, where $|S| \geq 2$. 
Define $I_j = \{i \in S : i < j\}$, $S_1 = \{j : |I_j| \text{ is even}\}$ and $S_2 = \{j : |I_j| \text{ is odd}\}$. Then $(S_1, S_2)$ is a partition of $S$.

Let $E \subseteq E_S$. If $|E|$ is even then by the definition of $S_1$ and $S_2$ and the structure of each chain in $G$, $|E \cap S_1| - |E \cap S_2| = 0$. Also by the same reasoning, if $|E|$ is odd, then $|E \cap S_1| - |E \cap S_2| = 1$. Hence for each $E \subseteq E_S$, $|E \cap S_1| - |E \cap S_2| \leq 1$. Therefore $(S_1, S_2)$ is an equitable bicoloring for $H_S$. Therefore for each $S \subseteq X$, $H_S$ has an equitable bicoloring. Therefore $H$ is a unimodular hypergraph.

**Proposition 2.5** If $H$ is a unimodular hypergraph, then each subhypergraph of $H$ is also unimodular.

**Proof** Let $H$ be a unimodular hypergraph and $S \subseteq X$, $|S| \geq 2$, then for any $T \subseteq S$, $|T| \geq 2$, we have $(H_S)_T = H_T$, which has an equitable bicoloring. Hence $H_S$ is unimodular. Therefore each subhypergraph is unimodular.

**Theorem 2.6** Let $G = (X, E)$ be a graph with no odd cycles, then $G$ has a bicoloring.

**Proof** Case 1 $G$ is connected.

Let $x' \in X$, then define $S_1 = \{y \in X : \text{there is a chain of odd length connecting } x' \text{ and } y\}$ and $S_2 = \{y \in X : \text{there is a chain of even length connecting } x' \text{ and } y\} \cup \{x'\}$.

**Claim** $(S_1, S_2)$ is a bicoloring of $X$.

**Proof of claim** Clearly if $x' \in S_1$, $G$ would have an odd cycle.

If $x' \neq y \in (S_1 \cap S_2)$ then there exist chains $\mu_1$ and $\mu_2$ of minimal length such that $\mu_1$ connects $x'$ to $y$, $\mu_2$ connects $y$ to $x'$, $\lambda(\mu_1)$ is odd and $\lambda(\mu_2)$ is even. Then $\mu = (\mu_1, \mu_2)$ is a
cycle with $\lambda(\mu) = \lambda(\mu_1) + \lambda(\mu_2)$ which is odd. But $G$ has no odd cycles, so $S_1 \cap S_2 = \phi$. Hence $(S_1, S_2)$ is a partition of $X$.

Let $y_1, y_2 \in S_1$ (the argument for $S_2$ is similar) and assume $[y_1, y_2] \in E$. Then there exists chains $\mu_1$ and $\mu_2$ of minimal length such that $\mu_1$ connects $x'$ to $y_1$, $\mu_2$ connects $y_2$ to $x'$ and both $\lambda(\mu_1)$ and $\lambda(\mu_2)$ are odd. Then $\mu = (\mu_1, y_1, y_2, \mu_2)$ is a cycle such that $\lambda(\mu) = \lambda(\mu_1) + 1 + \lambda(\mu_2)$ which is odd. But $G$ has no odd cycles, so $[y_1, y_2] \notin E$. Therefore $(S_1, S_2)$ is a bicoloring of $G$.

**Case 2** $G$ is not connected.

Define $S_1 = U(S_{i1})$ and $S_2 = U(S_{i2})$ where $(S_{i1}, S_{i2})$ is a bicoloring for the $i$th connected component of $G$ which exist since each connected component is a connected graph with no odd cycles. Then $(S_1, S_2)$ is a bicoloring for $G$.

The following theorem gives a sufficient condition for a hypergraph to be unimodular.

**Theorem 2.7** If $H = (X, E)$ is a hypergraph without odd cycles, then $H$ is unimodular.

**Proof** Let $H$ have no odd cycles, then each subhypergraph of $H$ has no odd cycles. For, if $H_S$ has an odd cycle, $(a_1^1 E_1^* \cdots a_{2n+1}^1 E_{2n+1}^* a_1^1)$ where \( \{a_1^1, \ldots, a_{2n+1}^1\} \subseteq S \subseteq X \) and for each $i$, $E_i^* = (S \cap E_i) \in E_S$ for some $E_i \in E$, then $(a_1^1 E_1^* \cdots a_{2n+1}^1 E_{2n+1}^* a_1^1)$ would be an odd cycle of $H$. Hence, by the definition of a unimodular hypergraph, we need only show that a hypergraph $H$ without odd cycles has an equitable bicoloring.
Let \( f \) be a 1-1 and onto map from \( I := \{1, \ldots, |E|\} \) to \( E \). For each \( i \in I \), denote \( E'_i = f(i) \). For each \( i \in I \), let \( f_i \) be a 1-1 and onto map from \( \{1, \ldots, r_i\} \) to \( E'_i \), where \( |E'_i| = r_i \). If \( r_i \) is even define \( F_i = \{[f_i(1), f_i(2)], [f_i(3), f_i(4)], \ldots, [f_i(r_i-1), f_i(r_i)]\} \).

If \( r_i \) is odd then define \( F_i = \{[f_i(1), f_i(2)], [f_i(3), f_i(4)], \ldots, [f_i(r_i-2), f_i(r_i-1)], [f_i(r_i), f_i(r_i)]\} \).

Define \( \hat{E} = \cup \{F_i : i \in I\} \), then consider the graph \( G = (X, \hat{E}) \).

Suppose that \( G \) contains an elementary odd cycle, \( \mu = (a_1, \ldots, a_{2p+1}, a_1) \), for some positive integer \( p \). We may assume that \( \mu \) contains two disjoint edges from the same class, \( F_i \). If not, then there would exist a cycle \( (a_1E_1 \cdots a_{2p+1}E_{2p+1}a_1) \) which is odd and hence contradicting the fact that \( H \) has no odd cycles.

Without loss of generality, let these two edges be \([a_s, a_{s+1}]\) and \([a_t, a_{t+1}]\). Transform \( F_i \) by changing the above two edges into \([a_s, a_{s+1}]\) and \([a_t, a_{s+1}]\). Replace \( \mu \) by either the sequence \( \mu' = (a_1, \ldots, a_s, a_{t+1}, \ldots, a_{2p+1}, a_1) \) or \( \mu'' = (a_{s+1}, a_{s+2}, \ldots, a_t, a_{s+1}) \) each of which are elementary cycles of odd length.

Repeat this process as many times as possible. Since \( \mu \) had only finite length, this process will terminate. At the final step, the cycle will be elementary and odd. As stated above, this odd cycle will determine an odd cycle in \( H \), which is a contradiction. Therefore \( G \) contains no elementary odd cycles. Proposition 1.2 implies that if \( G \) has odd cycles, then \( G \) has elementary odd cycles. Hence \( G \) contains no odd cycles.

Define \( S_1 \) and \( S_2 \) as in the proof of Theorem 2.6 then there
is a bicoloring \((S_1, S_2)\) of the vertices of \(G\). Since any vertex in \(S_1\) is adjacent only to itself and from the way in which \(E\) was constructed, we have \(|E \cap S_1| - |E \cap S_2| \leq 1\) for all \(E \in E\). Hence \((S_1, S_2)\) is an equitable bicoloring of \(H\).

The following example shows that a unimodular hypergraph can have odd cycles.

**Example 2.8** Define \(X = \{1, 2, 3, 4\}, E_1 = \{1, 2\}, E_2 = \{2, 3\}, E_3 = \{3, 4\}, E_4 = \{4, 1\}, E_5 = X\) and \(E = \{E_1, E_2, E_3, E_4, E_5\}\). Then \(H = (X, E)\) is unimodular and has odd cycles.

**Proof** Let \(S \subseteq X\) with \(|S| = 2\). Let \((S_1, S_2)\) be a partition of \(S\), then for all \(i\), \(||E_i \cap S_1| - |E_i \cap S_2|| \leq 1\).

Suppose \(|S| = 3\), then partition \(S\) into \((S_1, S_2)\) where \(S_1 = \{1, 2\}\) or \(S_2 = \{2, 4\}\). Clearly for \(i \neq 5\), \(||E_i \cap S_1| - |E_i \cap S_2|| = 0\).

Now for \(i = 5\) we have \(|E_5 \cap S_1| - |E_5 \cap S_2| = 2 - 1 = 1\).

Finally, suppose \(S = X\), then partition \(S\) into \((S_1, S_2)\) where \(S_1 := \{1, 3\}\) and \(S_2 := \{2, 4\}\). Then for each \(i\), \(|E_i \cap S_1| = |E_i \cap S_2|\).

Hence for each \(S \subseteq X\), where \(|S| \geq 2\), \(H_S\) has an equitable bicoloring. Therefore \(H\) is unimodular with a cycle \((1E_12E_23E_51)\) which is odd.
CHAPTER 3

A submatrix of a matrix $A$ is a matrix formed by deleting rows and columns of $A$.

A totally unimodular matrix is a matrix such that every square submatrix has determinant 0,1 or -1.

Proposition 3.1 If $A$ is totally unimodular, then each $a_{ij}$ is either 0,1 or -1.

Proof $(a_{ij})$ is a $1 \times 1$ square submatrix of $A$.

Let $A_i$ and $C_j$ denote the $i^{th}$ row and $j^{th}$ column, respectively, of the matrix $A$.

For the remainder of the thesis, the same letter will be used to denote a set of rows or columns and the matrix formed by these rows or columns. The context in which each will be used should cause no confusion.

Let $n \leq m$ and let $S$ be an $n \times m$ integral matrix then for $\{d : d$ is a determinant for some $n \times n$ submatrix of $S\} := D$ define

$$
\text{GCD } S = \begin{cases} 
0 & \text{if } D = \{0\} \\
greatest \text{ common divisor of } & \text{otherwise.} \\
\text{all the elements in } D 
\end{cases}
$$

Lemma 3.2 Let $m$ and $n$ be relatively prime integers then there exists integers $a$ and $b$ such that $an + bm = 1$. 
Proof The proof can be found in [Hernstein, Lemma 1.3.1].

**Lemma 3.3** Let $S$ be an $n \times m$ integral matrix then $S = UDV$ where $U, D$ and $V$ are integral, $U$ is $n \times n$, $D$ is $n \times m$, $V$ is $m \times m$, $|\det U| = |\det V| = 1$ and $D$ is diagonal.

Proof The proof will be by induction on $n + m$. Since the smallest $n + m$ can be is 2 we will start the induction at 2.

Let $n + m = 2$, then $n = 1$ and $m = 1$. Then $U := I_1, D := S$ and $V := I_1$ have the desired properties.

Assume that for each $(n + m) \in \{2, \ldots, p\}$ we have shown that there exists $U, D$ and $V$ with the desired properties.

Now assume $n + m = p + 1$.

Case I $m > 1$

Define $S^*$ to be the first $p$ columns of $S$, then by the induction hypothesis we can write $S^* = U^*D^*V^*$, where $U^*, D^*$ and $V^*$ have the desired properties corresponding to $S^*$. Let

$$V' := \begin{pmatrix} V^* & 0 \\ 0 & I_1 \end{pmatrix}$$

and $C'_m := (U^*)^{-1}C_m$ where $C_m$ is the $m^{th}$ column of $S$. Then $C'_m$ is integral and $S = U^*(D^*, C'_m)V'$. Let $J' := \{j : d^*_{jj} \neq 0 \text{ and } c^*_j \neq 0\}$. For each $j \in J'$, let $k_j := \gcd(d^*_{jj}, c^*_m)$ then by Lemma 3.2 there exists integers $a_j$ and $b_j$ such that $a_jd^*_{jj} + b_jc^*_m = 1$. For each $j \in J'$ let
then $V_j$ is integral and $\det V_j = \pm 1$. Let $I := \{i : c_{im}^i$ is the only nonzero element in the $i^{th}$ row\} and for each $r \in \{1, \ldots, |I|\} := I'$, let $k_r^i := \gcd(c_{0m}^i, \ldots, c_{rm}^i)$ where $i_0 < i_1 < \ldots < i_r$ and for each $i \in I \setminus \{i_0, \ldots, i_r\}$, $i_r < i$. By Lemma 3.2 we have for each $r \in I'$ there exists integers $a_r^i$ and $b_r^i$ such that $a_r^i k_{i_0 m}^i / k_r^i + b_r^i c_{i m}^i / k_r^i = 1$, where $k_0^i := c_{0m}^i$. For each $r \in I'$ let

$$U_r := \begin{pmatrix} I_{i_0-1} & 0 & 0 & 0 \\ 0 & a_r^i & 0 & b_r^i \\ 0 & 0 & I_{n-i_0-1} & 0 \\ 0 & -c_{i m}^i / k_r^i & 0 & k_{i_0 m}^i / k_r^i \end{pmatrix},$$

then $U_r$ is integral and $\det U_r = \pm 1$.

If $i_0 \leq \min\{m, n\}$ then let $P$ be the permutation matrix permuting column $i_0$ with column $m$. Otherwise let $P$ be the permutation matrix permuting row $i_0$ with row $m$.

For the following, the product over the empty set is the identity matrix. If $i_0 \leq \min\{m, n\}$ then $U := U^* (\prod_{r \in I'} U_r)^{-1}$,

$$V := (\prod_{j \in J} V_j)^{-1} p^t v^i$$

and $D := (\prod_{r \in I'} U_r)((D^*, C_m^i))(\prod_{j \in J} V_j)P$

have the desired properties. If $i_0 > \min\{m, n\}$ then
U := U^t(\prod_{r \in I^\prime} U_r)^{-1}, V := (\prod_{j \in J^\prime} V_j)^{-1}V, and
D := P(\prod_{r \in I^\prime} U_r)((D^*, C_m)(\prod_{j \in J^\prime} V_j) have the desired properties.

Case II  \( n > 1 \)

By Case I there exist \( U, D \) and \( V \), with the desired properties, such that \( S^* = UDV \). Hence \( U' := V^t, D' := D^t \) and \( V' := U^t \) have the desired properties for \( S \). Hence by induction, \( S \) has the desired factorization.

Lemma 3.4  Let \( S \) be an \( n \times m \) integral matrix with \( n \leq m \) and let \( B \) be an \( n \times n \) integral matrix such that \( \det B = \pm 1 \) then \( \gcd S = \gcd BS \).

Proof  Let \( S' := BS \) and for \( 1 \leq k \leq n \) let \( S_{i_k} \) and \( S'_{i_k} \) be the \( i_k^{th} \) columns of \( S \) and \( S' \), respectively. Now since
\[
S_{i_k}' = B S_{i_k}, (S_{i_1}', ..., S_{i_n}') = B(S_{i_1}, ..., S_{i_n}) \quad \text{and} \quad |\det (S_{i_1}', ..., S_{i_n}')| = |\det (S_{i_1}, ..., S_{i_n})|. \]

Therefore by the definition of \( \gcd \), \( \gcd S = \gcd BS \).

Lemma 3.5  Let \( S \) be an integral \( n \times m \) matrix with \( \gcd S = 1 \) and let \( A \) be an integral \( m \times m \) matrix with \( \det A = \pm 1 \) and \( SA \) diagonal, then \( \gcd SA = 1 \).
Proof Let \( M_n := \{S_{i_1}, \ldots, S_{i_n}\} \leq S : S_{i_j} \) is the \( i_j \)th column of \( S \) and \( \{S_{i_1}, \ldots, S_{i_n}\} \) is a set of \( n \) distinct elements, then \( \text{GCD} S = \) greatest common divisor of \( \{\det M : M \in M_n\} \).

Let \( d_1, \ldots, d_n \) be the diagonal elements of \( SA \), then since \( \text{GCD} S \not= 0 \) (which implies \( \text{GCD} SA \not= 0 \)), we have that for each \( i \in \{1, \ldots, n\}, d_i \not= 0 \). Hence one sees that the first \( n \) rows of \( A^{-1} \) must be

\[
\begin{pmatrix}
\frac{a_{11}}{d_1} & \ldots & \frac{a_{1m}}{d_1} \\
\vdots \\
\frac{a_{n1}}{d_n} & \ldots & \frac{a_{nm}}{d_n}
\end{pmatrix}
\]

Since \( \det A = \pm 1 \) we have by the well known Cramer’s Rule that \( A^{-1} \) is integral. Hence for each \( i, d_i \) divides every element in the \( i \)th row of \( S \).

Let \( M \in M_n \) then \( \det M = \) \( (\prod d_i) \det \left( \begin{pmatrix} m_{11}/d_1 & \ldots & m_{1n}/d_1 \\ \vdots \\ m_{n1}/d_n & \ldots & m_{nn}/d_n \end{pmatrix} \right) \).

Since for each \( i, d_i \) divides every element in the \( i \)th row of \( S \),

\[
\det \left( \begin{pmatrix} m_{11}/d_1 & \ldots & m_{1n}/d_1 \\ \vdots \\ m_{n1}/d_n & \ldots & m_{nn}/d_n \end{pmatrix} \right)
\]

is integral. Therefore \( \prod d_i \) divides \( \det M \) for all \( M \in M_n \).

Hence \( \prod d_i \) divides \( \text{GCD} S \). But \( \text{GCD} S = 1 \), so \( \prod d_i = \pm 1 \). Since for each \( i, d_i \) is integral, we have \( d_i = \pm 1 \). Therefore \( \text{GCD} SA = 1 \).

Corollary 3.6 Let \( S \) be an \( n \times m \) integral matrix with \( n \leq m \). Let \( UDV \) be a factorization of \( S \) by Lemma 3.3, and let \( \text{GCD} S = 1 \), then \( \text{GCD} D = 1 \).
Proof Since $U^{-1}$ is integral with $\det U^{-1} = \pm 1$ we have by Lemma 3.4, $\text{GCD } U^{-1} S = \text{GCD } S$. Therefore, since $V^{-1}$ is integral with $\det V^{-1} = \pm 1$, $U^{-1} SV^{-1} = D$ is diagonal and $\text{GCD } U^{-1} S = \text{GCD } S = 1$, we have by Lemma 3.5 $\text{GCD } D = 1$.

Let $x, y \in \mathbb{R}^n$ then define $x \succeq y$ if for each $i \in \{1, \ldots, n\}$, $x_i \geq y_i$, where $x_i$ and $y_i$ are the $i$th component of $x$ and $y$ respectively.

Let $A$ be any $m \times n$ matrix and $b \in \mathbb{R}^m$, then define $P(b) := \{x \in \mathbb{R}^n : Ax \succeq b\}$. Notice that $P(b)$ is not necessarily bounded. For example, let $A$ be the $1 \times 1$ matrix (i) then $P(2) = \{x \in \mathbb{R} : (1)x \succeq 2\}$ is clearly unbounded.

Let $A$ be an $m \times n$ matrix, $b \in \mathbb{R}^m$ and $S$ a subset of the rows of $A$, then define $F_S = F_S(b) = \{x \in \mathbb{R}^n : Ax \succeq b \text{ and } A_i x = b_i \text{ if } A_i \in S\}$, $L_S(b) = \{x \in \mathbb{R}^n : A_i x = b_i \text{ for all } A_i \in S\}$ and $G_S = \text{the subspace of } \mathbb{R}^n \text{ spanned by the rows of } S$.

If $F_S$ is not empty, then it is called a face of $P(b)$. If $F_{S_1}$ and $F_{S_2}$ are such that $F_{S_1} \subseteq F_{S_2}$, then $F_{S_1}$ is called a subface of $F_{S_2}$.

Lemma 3.7 Let $A$ be an $m \times n$ matrix, $b \in \mathbb{R}^m$ and let $S, S^* \subseteq A$ be subsets of the rows of $A$ such that $S \subseteq S^*$; moreover let $F_S^*$ and $F_S$ be faces, then:

i) $F_S \subseteq F_S^*$

ii) $F_S$ is a minimal face iff $G_A = G_S$, i.e., iff $S$ has the same rank as $A$.

Proof of i) Let $S \subseteq S^*$ and $x \in F_S^*$ then for all $A_i \in S$, ...
$A_i x = b_i$. Hence $x \in F_S$. Therefore $F_S^* \subseteq F_S$.

**Proof of ii)** Let $S^*$ be all rows of $A$ in which $G_S = G_{S^*}$, then $A_i$ is a linear combination of the $A_i \in S$ iff $A_i \in S^*$. Therefore what must be proved is that $G_S = G_A$ iff $S^* = A$.

The first implication is proven by proving the contrapositive. If $S^* \neq A$, there is at least one row $A_k \in A \setminus S^*$. Then there is a vector $y$ such that $A_i y = 0$ for all $A_i \in S$ and $A_k y < 0$.

Let $x' \in F_S$. As $A_k x' \geq b_k$, there is a number $\lambda_k > 0$, for which $A_k (x' + \lambda_k y) = b_k$. For every $A_j \in A \setminus S^*$ such that $A_j y < 0$, the equation $A_j (x' + \lambda y) = b_j$ has a nonnegative solution. Let $\lambda_j$ be that solution. Define $\lambda = \min \{\lambda_j\}$ and let $j'$ be a value such that $\lambda = \lambda_{j'}$. Since $\lambda_k$ exists, there is at least one $\lambda_j$, so $\lambda$ exists. By the definition of $\lambda$, we have:

$$A(x' + \lambda y) \geq b,$$

$$A_i (x' + \lambda y) = b_i \text{ for all } A_i \in S \text{ and }$$

$$A_{j'} (x' + \lambda y) = b_{j'}.$$

Thus $F_S (S \cup A_{j'})$ is not empty, and is therefore a proper subface of $F_S$, since $A_{j'} x' > b_{j'}$. Hence $F_S$ is not minimal.

The second implication is also proven by proving the contrapositive.

Let $F_S$ not be minimal, then it has some proper subface $F_S (S \cup A_{j'})$. Hence there exists $x_1, x_2 \in F_S$ such that $A_j x_1 = b_j$ and $A_j x_2 > b_j$. Therefore $A_j x$ varies as $x$ ranges over $F_S$. But for $A_i \in S$, $A_i x = b_i$ is constant as $x$ varies over $F_S$. Hence $A_j$ cannot be a linear combination of the $A_i \in S$, so $A_j \in A \setminus S^*$. Hence $S^* \neq A$. Therefore $S^* = A$ iff $G_S = G_A$.

If $b$ is an $m$-tuple and $S$ is a set of $r$ linearly independ-
ent rows of $A$, then let $b_S$ be the "subvector" consisting of the $r$ components of $b$ which correspond to the rows of $S$. Let $\hat{b}$ always represent an $r$-tuple. The components of $\hat{b}$ and $b_S$ will be indexed by the indices used for the rows of $S$, not by the integers 1 to $r$.

**Lemma 3.8** Suppose $S$ is a set of $r$ linearly independent rows of $A$, where $r = \text{rank } A$, then for any $\hat{b}$ there is a $b$ such that:

1. $b_S = \hat{b}$ and,
2. $F_S$ is a minimal face of $P(b)$.

**Proof** As $S$ is a set of linearly independent rows, the equation $Sx = \hat{b}$ has at least one solution, call it $y$. Define $b$ as follows:

$$b = \begin{cases} b_i & \text{if } A_i \in S \\ [A_i y] & \text{if } A_i \notin S. \end{cases}$$

Hence $b_S = \hat{b}$ and $b$ is integral. Also $y \in F_S$, so $F_S \neq \phi$. By Lemma 3.7, $F_S$ is a minimal face, so E2 is satisfied.

**Lemma 3.9** Suppose $S^*$ is a set of rows of a matrix $A$ that has the same rank as $A$, and $S \subseteq S^*$ is a set of $r := \text{rank } A$ linearly independent rows of $A$. Then for any $b$ such that $F_{S^*}$ is a face, $F_{S^*} = L_S$.

**Proof** Let $y$ be a fixed element in $F_{S^*}$ and let $x$ be any element of $L_S$. Since for all $A_i \in S^*$ we have $A_i y = b_i$, $F_{S^*} \subseteq L_S$.

Since $S$ had rank $r$, any row $A_k \in A$ can be expressed as a linear combination of rows $A_i \in S$, i.e., $A_k = \sum_{k_i}^\alpha A_i$.

Since $A_i x = A_i y = b$ for all $A_i \in S$, $A_k x = \sum_{k_i}^\alpha A_i y = A_k y$. Hence $L_S \subseteq F_{S^*}$. Therefore $F_{S^*} = L_S$. 

Lemma 3.10 Let $A$ be an $m \times n$ matrix and $b \in \mathbb{R}^m$, then any minimal face of $P(b)$ can be expressed in the form $F_S$ where $S$ is a set of $r := \text{rank } A$ linearly independent rows of $A$.

Proof Assume the face is $F_S^*$. By Lemma 3.7, $S^*$ must have rank $r$. Let $S$ be a set of $r$ linearly independent rows of $S^*$. By Lemma 3.8 $F_S^* = L_S = F_S$.

Lemma 3.11 Let $A$ be an $m \times n$ integral matrix. Let $S$ be a set of linearly independent rows of $A$ and let $S$ have the same rank as $A$, then the following are equivalent:

i) $L_S(b)$ contains an integral point for every integral $b$

ii) $\text{GCD } S = 1$

Proof By Lemma 3.3 there exists integral matrices $U, D$ and $V$ such that $S = UV$, $|\det U| = |\det V| = 1$ and $D$ is diagonal. By Lemma 3.6 $\text{GCD } S = \text{GCD } D$. Clearly $\text{GCD } D = |d_{11} \cdots d_{rr}|$ where $r := \text{rank } A$. Hence ii) is equivalent to the condition that for $i \in \{1, \ldots, r\}$, $d_{ii} = \pm 1$.

Suppose that $d_{jj} = k > 1$ for some $j$. Without loss of generality we may assume that $j = 1$. Let $\vec{e}$ be the $r$-tuple $(1, 0, \ldots, 0)$, let $\vec{b} := U \vec{e}$ and let $x \in L_S(\vec{b})$. Then $Sx = UVx = \vec{b} = U \vec{e}$. Hence $DVx = \vec{e}$. Clearly the first component of $y = Vx$ is $1/k$. Hence $y$ is not integral. Therefore $x$ cannot be integral. Therefore $F_S(\vec{b})$ contains no integral point. Hence if $\text{GCD } S \neq 1$ then i) cannot hold.

Assume $d_{ii} = \pm 1$ for $i \in \{1, \ldots, r\}$. Let $x \in L_S(\vec{b})$ and set $Vx = (y_1, \ldots, y_r, y_{r+1}, \ldots, y_n)$. Then $U^{-1} \vec{b} = DVx = \ldots$.\ldots
(d_{11}y_1, \ldots , d_{rr}y_r)$, and so $y_1, \ldots , y_r$ are integral. Let $y = (y_1, \ldots , y_r, 0, \ldots , 0)$, then $V^{-1}y$ is integral. Since $Dy = DVx$, $S(V^{-1}y) = UDy = UDVx = \tilde{b}$. Thus $V^{-1}y \in L_S(\tilde{b})$. Therefore ii) implies i).

**Theorem 3.12** If $A$ is totally unimodular, $b$ is integral and $P(b)$ is nonempty, then $P(b)$ contains an integral point.

**Proof** Let $F_S^*$ be some minimal face of $P(b)$. By Lemma 3.10 this face can be expressed as $F_S$ where $S$ consists of $r := \text{rank } A$ linearly independent rows of $A$. By Lemma 3.9, $F_S(b) = L_S(b_S)$. Since $A$ is totally unimodular and $S$ is of full rank we have $\text{GCD } S = 1$. By Lemma 3.11, $L_S(b)$ must contain an integral point. Therefore $P(b)$ contains an integral point.

Let $A$ be an $m \times n$ matrix, $b, b' \in \mathbb{R}^m$ and $c, c' \in \mathbb{R}^n$ where $b \leq b'$ and $c \leq c'$ then define $P(b, b'; c, c') = \{x \in \mathbb{R}^n : b \leq Ax \leq b'$ and $c \leq x \leq c'\}$.

**Theorem 3.13** $A$ is totally unimodular iff for every integral $b, b', c$ and $c'$ such that $P(b, b'; c, c')$ is nonempty, then $P(b, b'; c, c')$ contains an integral point.

**Proof** Assume $A$ is totally unimodular and $P(b, b'; c, c')$ is nonempty. Consider $A_* := \begin{pmatrix} A & -A \\ -I & I \end{pmatrix}$. Clearly $A_*$ is totally unimodular.

For $x \in P(b, b'; c, c')$, $b \leq Ax \leq b'$ and $c \leq x \leq c'$ iff $Ax \leq b$, $-Ax \leq -b'$, $x \leq c$ and $-x \leq -c'$. Hence $P(b, b'; c, c') =$
\[ P_*(b_1, \ldots, b_m, -b'_1, \ldots, -b'_m, c_1, \ldots, c_n, -c'_1, \ldots, -c'_n)^t. \]

Since \( P_*(\cdot) \) is nonempty, we have by Theorem 3.12 \( P(b, b'; c, c') \) contains an integral point.

Suppose \( A \) is integral but not totally unimodular. Since \( A \) is not totally unimodular there is a \( p \times p \) submatrix \( A^* \) of \( A \) such that \( \text{det} A^* \neq 0 \) or 1. Hence by Theorem 1.4, there is a non-integral \( x \in \mathbb{R}^p \) such that \( A^*x \) is integral. Let \( I, K \) be the sets of indices that indexes the rows and columns, respectively, of \( A^* \). Define

\[
\begin{align*}
  x' &= \begin{cases} 
  x_i = x_i & i \in K \\
  b_i = A_i x' & i \in I \\
  0 & i \notin K
  \end{cases}, \\
  b' &= \begin{cases} 
  b_i' = b_i & i \in I \\
  [A_i x'] & i \notin I
  \end{cases}, \\
  c &= \begin{cases} 
  c_i = [x_i] & i \in K \\
  0 & i \notin K
  \end{cases}, \\
  c' &= \begin{cases} 
  c_i' = c_i + 1 & i \in K \\
  0 & i \notin K
  \end{cases}.
\end{align*}
\]

Then \( b, b', c \) and \( c' \) are integral and \( P(b, b'; c, c') \) is well defined and nonempty.

Define \( A^* = \{ A_i \in A : i \in I \} \). Since \( \text{det} A^* \neq 0 \) and \( A^* x = b_{A^*} = A' x' \) and by the definition of \( c \) and \( c' \), we have \( \{ x' \} = P(b, b'; c, c') \). Hence \( P(b, b'; c, c') \) contains no integral points.

Therefore there are integral \( b, b', c \) and \( c' \) such that \( P(b, b'; c, c') \) is nonempty and does not contain any integral points.

**Lemma 3.14** Let \( A \) be an \( m \times n \) totally unimodular matrix then for all \( x \in \{0,1,-1\}^n \) there exists a vector \( y \in \{0,1,-1\}^n \) such that \( y \equiv x \pmod{2} \) and for each \( A_i \in A \),

\[
A_i y = \begin{cases} 
  0 & \text{if } A_i x \equiv 0 \pmod{2} \\
  \pm 1 & \text{otherwise}
\end{cases}
\]

**Proof** Let \( x \in \{0,1,-1\}^n \) and \( Ax = a \). Define \( d^y \) for \( v = 1, 2 \)
to be
\[ d_i^v = \begin{cases} 
0 & \text{if } x_i \equiv 0 \mod 2 \\
1/2(x_i - 1) & \text{if } x_i \equiv 1 \mod 2 \text{ and } v = 1 \\
1/2(x_i + 1) & \text{if } x_i \equiv 1 \mod 2 \text{ and } v = 2.
\end{cases} \]

Define \( b^v \) for \( v = 1, 2 \) to be
\[ b_i^v = \begin{cases} 
1/2(a_i) & \text{if } a_i \equiv 0 \mod 2 \\
1/2(a_i - 1) & \text{if } a_i \equiv 1 \mod 2 \text{ and } v = 1 \\
1/2(a_i + 1) & \text{if } a_i \equiv 1 \mod 2 \text{ and } v = 2.
\end{cases} \]

Note that \( b_1^1 \leq 1/2(a) \leq b_2^1 \) and \( d_1^1 \leq 1/2(x) \leq d_2^2 \). Hence
\[ 1/2(x) \in P(b_1^1, b_2^2; d_1^1, d_2^2). \]
Therefore by Theorem 3.13 there exists \( x' \in \{0, 1, -1\}^n \cap P(b_1^1, b_2^2; d_1^1, d_2^2). \) Set \( y = x - 2x' \), then
\[ y \equiv x \mod 2 \text{ and } A_i y = A_i(x - 2x') = a_i - 2(1/2(a_i)) = 0 \text{ if } a_i \equiv 0 \mod 2, \]
\[ A_i y = a_i - 2(1/2(a_i - 1)) = 1 \text{ if } a_i = b_1^1 \text{ and } \]
\[ A_i y = a_i - 2(1/2(a_i + 1)) = -1 \text{ if } a_i = b_2^2. \]

**Theorem 3.15** Let \( A \) be a 0,1 \( m \times n \) matrix with I indexing the rows and \( K \) indexing the columns, then \( A \) is totally unimodular iff for each \( J \subseteq K \) there exists a partition \((J_1, J_2)\) of \( J \) such that for each \( i \in I, |\sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij}| \leq 1. \)

**Proof** Assume \( A \) is totally unimodular, then let \( A^* \) be any submatrix of \( A \), then clearly \( A^* \) is totally unimodular. So all that is needed to be shown is that for \( A_i \) there are \( K_1, K_2 \subseteq K \) such that for each \( i \in I, |\sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij}| \leq 1. \)

The proof will be by induction on \( p = |K| \).

Let \( p = 2 \) then clearly there exists \( x \in \{1, -1\}^p \) such that \( Ax \in \{0, 1, -1\}^m. \)

Suppose that for any \( m \times p \) totally unimodular matrix, where
2 \leq p \leq n$, there exists $x \in \{1, -1\}^p$ such that $Ax \in \{0,1,-1\}^m$.

Assume that $A$ is an $m \times n$ totally unimodular matrix. Then for $(C_1, \ldots, C_{n-1})$ there exists an $x \in \{1, -1\}^{n-1}$ such that $(C_1, \ldots, C_{n-1})x \in \{0,1,-1\}^m$. Define
\[
x^i = \begin{cases} x_i = x^i & \text{for } i = 1, \ldots, n-1 \\
x_n = 1 & \text{otherwise}
\end{cases}
\]
If $Ax \in \{0,1,-1\}^m$ then we are finished. Suppose not, then by Lemma 3.14 there exists a $y \in \{1, -1\}^n$ such that $Ay \in \{0,1,-1\}^m$.

Therefore by induction any $m \times p$ totally unimodular matrix $A$ is such that there exists $x \in \{1, -1\}^p$ such that $Ax \in \{0,1,-1\}^m$.

Let $A$ be an $m \times n$ totally unimodular matrix and $x \in \{1, -1\}^n$ such that $Ax \in \{0,1,-1\}^m$. Define $J_1 = \{j : x_j = 1\}$ and $J_2 = \{j : x_j = -1\}$. If either $J_1$ or $J_2$ is empty then clearly there is another choice for $x$. Assume neither $J_1$ nor $J_2$ are empty, then $(J_1, J_2)$ is a partition of $J$ and for each $i \in I$
\[
|\Sigma_{j \in J_1} a_{ij} - \Sigma_{j \in J_2} a_{ij}| \leq 1.
\]
Assume that for each $J \subseteq K$ there are $J_1, J_2 \subset J$ such that $(J_1, J_2)$ is a partition of $J$ and for each $i \in I$
\[
|\Sigma_{j \in J_1} a_{ij} - \Sigma_{j \in J_2} a_{ij}| \leq 1.
\]
Suppose $A$ contains a square submatrix, $A^*$, such that the determinant of any square submatrix of $A^*$, but not equal to $A^*$, is 0,1 or -1 and $\det A^* \neq 0,1$ or -1. Let $I_0$ index the rows of $A^*$ and $K_0$ index the columns of $A^*$.

Let $f_{A^*} : P(K_0) \rightarrow P(I_0)$ as in Chapter 1.

Suppose $f_{A^*}$ is not 1-1, then there exists $J \subseteq K_0$ such that $f_{A^*}(J) = \emptyset$. Hence $\Delta J = I_0$. Since there exists $J_1, J_2 \subseteq J$ such that
$J_1, J_2$ partition $J$ and for each $i \in I_0, \mid \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \mid \leq 1$

and $A I_0 = I_0$, we have for each $i \in I_0, \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} = 0$.

Define

$$x = \begin{cases} x_j = 1 & j \in J_1 \\ x_j = -1 & j \in J_2 \\ x_j = 0 & \text{otherwise,} \end{cases}$$

then $A^* x = 0$. But since $x \neq 0$, this contradicts the fact that $\det A^* \neq 0$.

Suppose $f_A^*$ is 1-1, then for some $i_0 \in I_0$, there exists a unique $J \subseteq K_0$ such that $f_A^*(J) = \{i_0\}$. Since there exists a partition $(J_1, J_2)$ of $J$ such that for each $i \in I_0$,

$$\sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} = 0.$$

Without loss of generality we may assume

$$\sum_{j \in J_1} a_{i_0 j} - \sum_{j \in J_2} a_{i_0 j} = 1.$$  

Let $k := |K_0|$ and define

$$(C_1, ..., C_k) = A^*.$$  

Without loss of generality we may assume $1 \in J$ and $i_0 = 1$. Then for

$$C_i^* := \sum_{j \in J_1} C_j - \sum_{j \in J_2} C_j, C_{11} = 1$$

and $C_{i_0}^* = 0$ otherwise. Hence $\det A^* = \det (C_1, ..., C_k) = \det (C_1^*, ..., C_k)$ where $A^{**}$ is formed by deleting the first row and column of $A^*$. But since $A^{**}$ is a square submatrix of $A^*$ and is not equal to $A^*$, $\det A^{**} = 0, 1$ or $-1$. But this contradicts the fact that $\det A^* \neq 0, 1$, or $-1$.

Hence $f_A^*$ is both 1-1 and not 1-1, which is absurd. Therefore every submatrix of $A$ has determinant 0, 1 or $-1$. Therefore $A$ is totally unimodular.

Note that the totally unimodular matrix $A$ in Theorem 3.14 cannot be a 0,1,-1 matrix, since
is totally unimodular and \( |a_{12} - a_{11}| = |a_{11} - a_{12}| = 2 \). The reader should notice that Berge incorrectly states this theorem in his book for this reason.

**Corollary 3.16** The incidence matrix of a unimodular hypergraph is totally unimodular.

**Proof** Let \( H = (X,E) \) be a unimodular hypergraph then for each \( S \subseteq X, |S| \geq 2 \), there exists \( S_1, S_2 \subseteq S \) such that \( S_1, S_2 \) partition \( S \) and for each \( E_i \in E, \frac{|E_i \cap S_1| - |E_i \cap S_2|}{|E_i|} \leq 1 \). Let \( A \) be the incidence matrix of \( H \). Let \( I \) index the rows of \( A \) and \( K \) index the columns of \( A \). Then there exists \( J \subseteq K \) such that each \( j \in J \) corresponds to an \( x \in S \). Hence there exists \( J_1, J_2 \subseteq J \) such that \( J_1, J_2 \) partition \( J \) and for each \( i \in I \) we have \( |E_i \cap S_1| = \Sigma_{j \in J_1} a_{ij} \) and \( |E_i \cap S_2| = \Sigma_{j \in J_2} a_{ij} \). Hence for each \( i \in I \), there is an \( E_i \in E \) such that \( 1 \geq \frac{|E_i \cap S_1| - |E_i \cap S_2|}{|E_i|} = \Sigma_{j \in J_1} a_{ij} - \Sigma_{j \in J_2} a_{ij} \). Since \( S \) was any subset of \( X \), we have for every \( J \subseteq K \) there exists \( J_1, J_2 \subseteq J \) such that \( J_1, J_2 \) partitions \( J \) and for each \( i \in I \), \( \Sigma_{j \in J_1} a_{ij} - \Sigma_{j \in J_2} a_{ij} \leq 1 \). Therefore, by Theorem 3.15, \( A \) is totally unimodular.

Define \( 1_n, 0_n \in \mathbb{R}^n \) to be such that every component of \( 1_n \) is 1 and every component of \( 0_n \) is 0.

**Corollary 3.17** Let \( H \) be unimodular and \( A \) its incidence matrix, then the extreme points of \( P(1_m, 1_m; 0_n, 1_n) \) are integral.
Proof This follows directly from Theorem 3.13.

Proposition 3.18 Let $H = (X,E)$ be a hypergraph and $A$ its incidence matrix, then every point in $P(l_1^m, l_1^m; 0_1^n, 1_1^n)$ is a convex combination of the extreme points of $P(l_1^m, l_1^m; 0_1^n, 1_1^n)$.

Proof Let $\{y_1, \ldots, y_k\}$ be the extreme points of $P(\cdot)$ and $x \in P(\cdot)$. Clearly $P(\cdot)$ is contained in the linear span of its extreme points. Hence $x = \alpha_1 y_1 + \ldots + \alpha_k y_k$. Since $\{x, y_1, \ldots, y_k\} \subseteq P(\cdot)$, $\sum_1^k \alpha_i = 1$. Therefore every point in $P(\cdot)$ is a convex combination of the extreme points of $P(\cdot)$.

Using Corollary 3.16 we have the following theory.

Let $H = (X,E)$ be a hypergraph and $\mu := (y_1 E_1 y_2 \ldots y_k E_k y_1)$ be a cycle of $H$, then $\mu$ is said to be a $T$-cycle of $H$ if for $Y := \{y_1, \ldots, y_k\}$, $|Y \cap E_i| = 2$ for $i = 1, \ldots, k$.

Theorem 3.19 Let $H = (X,E)$ be a unimodular hypergraph, then $H$ has no odd $T$-cycles.

Proof Suppose $H$ has an odd $T$-cycle, $(y_1 E_1 y_2 \ldots y_k E_k y_1)$. Let $A$ be the incidence matrix for $H$ and $A'$ be the incidence matrix of $H' = (Y, \{E_1 \cap Y, \ldots, E_k \cap Y\})$. Clearly $A'$ is a submatrix of $A$ and by Lemma 1.8 $|\det A'| = 2$. But this contradicts the fact that $A$ is totally unimodular.
CHAPTER 4

Let \( \omega \) be a function mapping \( X \) into \([0,1]\) and \( H = (X,E) \) be a hypergraph, then \( \omega \) is called a state on \( H \) if for all \( E \in E \), \( \Sigma_{x \in E} \omega(x) = 1 \). Define \( X_\omega = \{ x \in X : \omega(x) > 0 \} \), then \( X_\omega \) is called the support of \( \omega \). Two states \( \omega_1 \) and \( \omega_2 \) are said to be equal if for all \( x \in X \), \( \omega_1(x) = \omega_2(x) \). Define \( \Omega(H) = \{ \omega : \omega \) is a state on \( H \} \).

**Proposition 4.1** Let \( H = (X,E) \) be a hypergraph and \( A \) be its incidence matrix, then \( \Omega(H) \) is isomorphic to \( P(l_m^1, l_m^1; 0^n_1, 1^n_1) \).

**Proof** Define \( \phi : \Omega(H) \to P(\ast) \) by \( \phi(\omega) = (\omega(x_1), \ldots, \omega(x_n))^t \).

Let \( \omega_1, \omega_2 \in \Omega(H) \) where \( \omega_1 \neq \omega_2 \) then by the definition of two states not being equal we have \( \phi(\omega_1) = (\omega_1(x_1), \ldots, \omega_1(x_n))^t \neq (\omega_2(x_1), \ldots, \omega_2(x_n))^t = \phi(\omega_2) \). Hence \( \phi \) is 1-1. Let \( y \in P(\ast) \) and define \( \omega_y(x_i) = y_i \). By the definition of \( y \), \( \omega_y \) is a state on \( H \). Hence \( \phi \) is onto.

Let \( a, b \in \mathbb{R} \) be such that \( a > 0, b > 0 \) and \( a + b = 1 \). Define \( \theta_{a,b} \) to be such that for all \( \omega_1, \omega_2 \in \Omega(H) \), \( \omega_1 \theta_{a,b} \omega_2 = a\omega_1 + b\omega_2 \). Hence \( \phi(\omega_1 \theta_{a,b} \omega_2) = \phi(a\omega_1 + b\omega_2) = ((a\omega_1 + b\omega_2)(x_1), \ldots, (a\omega_1 + b\omega_2)(x_n))^t = a(\omega_1(x_1), \ldots, \omega_1(x_n))^t + b(\omega_2(x_1), \ldots, \omega_2(x_n))^t = a\phi(\omega_1) + b\phi(\omega_2) = \phi(\omega_1) \theta_{a,b} \phi(\omega_2) \). Hence \( \phi \) is an isomorphism. Therefore \( \Omega(H) \) is isomorphic to \( P(\ast) \).

Let \( H = (X,E) \) be a hypergraph, then a set \( T \) is called a transversal of \( H \) if for all \( E \in E, T \cap E \neq \emptyset \).

A transversal \( T \) of \( H \) is said to be thin
if for all \( E \in E, |T \cap E| = 1 \). A state, \( \psi \), on \( H \) is said to be dispersion free if for all \( x \in X, \psi(x) \in \{0,1\} \).

**Lemma 4.2** Let \( H = (X,E) \) be a hypergraph, then each dispersion free state corresponds to some thin transversal and each thin transversal corresponds to some dispersion free state.

**Proof** Let \( H \) have a thin transversal \( T \), then define

\[
\psi(x) = \begin{cases} 
1 & \text{if } x \in T \\
0 & \text{if } x \notin T.
\end{cases}
\]

Since \( T \) is thin, we have that for all \( E \in E, \sum_{x \in E} \psi(x) = 1 \).

Suppose there exists a dispersion free state, \( \psi \), then define

\( T = \{ x \in X : \psi(x) = 1 \} \). Since \( \psi \) is dispersion free, we have that for all \( E \in E, |T \cap E| = 1 \).

**Lemma 4.3** Let \( H = (X,E) \) be unimodular and \( A \) be the incidence matrix, then every extreme point in \( P(1,1;0,1) \) corresponds to a dispersion free state of \( H \).

**Proof** This follows from Proposition 4.1, Corollary 3.17 and Lemma 4.2.

**Corollary 4.4** Let \( H = (X,E) \) be unimodular and \( \Omega(H) \neq \emptyset \), then for \( \omega \in \Omega(H), \omega = \sum_{i}^{k} p_i \psi_i \) where \( \sum_{i}^{k} p_i = 1 \), for each \( i, p_i > 0 \) and \( \psi_i \) is dispersion free.

**Proof** This follows from Proposition 3.18 and Lemma 4.3.

Let \( A \) be an \( n \times n \) matrix, then \( A \) is called bistochastic if for each \( i \) we have that \( \sum_{j=1}^{n} a_{ij} = 1 \) and for each \( j \) we have
that \( \sum_{i=1}^{n} a_{ij} = 1 \) and for each \( i \) and \( j, a_{ij} \geq 0 \).

**Corollary 4.5** Let \( A \) be a bistochastic matrix, then

\[
A = \sum_{1}^{k} p_{i} P_{i}
\]

where, for each \( i, 0 < p_{i} \leq 1 \) and \( P_{i} \) is a permutation matrix.

**Proof** Let \( X = \{x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn}\} \) and

\[
E = \{E_{i}, F_{i} : E_{i} = \{x_{i1}, \ldots, x_{in}\}, F_{i} = \{x_{i1}, \ldots, x_{ni}\}\},
\]

then \( H = (X, E) \) is a hypergraph. Hence for \( i \neq j, E_{i} \cap E_{j} = \emptyset \) and

\( F_{i} \cap F_{j} = \emptyset \). Also for each \( i \) and \( j, E_{i} \cap F_{j} = \{x_{ij}\} \). Therefore \( H \) contains no odd cycles. By Theorem 2.7, \( H \) is unimodular.

Let \( A \) be bistochastic, then the function \( \omega(x) \), where

\( \omega(x_{ij}) := a_{ij} \) is a state on \( H \). By Corollary 4.4 \( \omega(x) = \sum_{1}^{k} p_{i} \psi_{i}(x) \)

where for each \( i \), we have \( 0 < p_{i} \leq 1 \) and \( \psi_{i} \) is a dispersion free state and \( \sum_{1}^{k} p_{i} = 1 \). Note that for \( \psi_{q} \) to be a state, we have that if \( 1 \leq i_{0} \leq n \) is fixed, then for some \( 1 \leq j_{0} \leq n, \psi_{q}(x_{i_{0}j_{0}}) = 1 \)

and for \( j \neq j_{0}, \psi_{q}(x_{i_{0}j}) = 0 \). Similarly if \( 1 \leq j_{*} \leq n \) is fixed

for some \( 1 \leq i_{*} \leq n, \psi_{q}(x_{ij_{*}}) = 1 \) and for \( i \neq i_{*}, \psi_{q}(x_{ij_{*}}) = 0 \).

Therefore the matrix formed by \( (\psi_{q}(x_{ij}))_{ij} \) is a permutation matrix.

Clearly the matrix formed by \( (\omega(x_{ij}))_{ij} \) is \( A \). By the above

\[
A = (\omega(x_{ij}))_{ij} = (\sum_{q=1}^{k} p_{q} \psi_{q}(x_{ij}))_{ij} = \sum_{q=1}^{k} p_{q} (\psi_{q}(x_{ij}))_{ij} = \sum_{q=1}^{k} p_{q} P_{q}
\]

where for each \( q, P_{q} \) is the permutation matrix given by

\( (\psi_{q}(x_{ij}))_{ij} \).

Note that this corollary is a result of P. Hall quoted in G. Berkhoff's paper.
Let $H = (X, E)$ be a hypergraph and $x, y \in X$ with $x \neq y$, then define $x \perp y$ if for some $E \in E$, $(x, y) \subseteq E$. Let $\Omega(H) \neq \emptyset$ and $S \subseteq \Omega(H)$ then $S$ is said to be full if, for any $x, y \in X$ where $x \neq y$, $x \perp y$ iff $\omega(x) + \omega(y) > 1$ for some $\omega \in S$. Let $\nabla$ denote the set of all dispersion free states of $\Omega(H)$.

**Example 4.6** Let $H = (X, E)$ be the hypergraph represented by Figure 1 where each line is an edge of $H$. Then $\Omega(H)$ is full and $\nabla$ is not full.

![Figure 1](image_url)

**Proof** Suppose $\Psi$ is a dispersion free state on $H$ where $\Psi(a) = \Psi(b) = 1$. Then $\Psi(c) = \Psi(d) = 1$. However $\sum_{x \in E^*} \Psi(x) = 2$, which is a contradiction. Hence $\nabla$ is not full.

Using Chart 1 below by picking any 2 vertices, $x$ and $y$, such that $x \not\perp y$, you will find a row which represents a state on $H$ such that $\omega(x) + \omega(y) > 1$. Hence $\Omega(H)$ is full.
Proposition 4.7 Let \( H = (X, E) \) be a hypergraph with \( S \subseteq \Omega(H) \) full, then \( \Omega(H) \) is full.

**Proof** Let \( x, y \in X \) where \( x \neq y \) and \( x \perp y \), then since \( S \) is full there exists some \( \omega \in S \subseteq \Omega(H) \) such that \( \omega(x) + \omega(y) > 1 \).

Theorem 4.8 Let \( H = (X, E) \) be unimodular then \( \nabla \) is full iff \( x, y \in X \), where \( x \perp y \) and \( x \neq y \), implies there is some thin transversal containing \( x \) and \( y \).

**Proof** This is immediate from Lemma 4.2.

Example 4.9 Let \( H \) be the hypergraph represented by Figure 2, where each line in Figure 2 is an edge of \( H \). Then \( \nabla \) is full and \( H \) is not unimodular.

![Figure 2](image)

**Proof** Use Table 2 as in Example 4.6 to see that \( \nabla \) is full. Consider the incidence matrix of the subhypergraph \( H_{\{1,3,5\}} \):

\[
\begin{pmatrix}
 1 & 1 & 0 \\
 0 & 1 & 1 \\
 1 & 0 & 1
\end{pmatrix}
\]

By Lemma 1.8 this matrix has determinant 2. Hence \( H \) is not uni-
modular.

Hence unimodularity does not imply a full set of dispersion free states. Note that the quest for finding what conditions need to be on a hypergraph in order that it have a full set of dispersion free states still continues.

Proposition 4.10 Let \( H \) be a hypergraph with \( \nabla \) full, then there is a state \( \omega \in \Omega(H) \) such that \( \omega(x) > 0 \) for all \( x \in X \).

Proof Let \( \nabla = \{ \psi_1, \ldots, \psi_k \} \) then clearly \( \omega = (1/k) \sum_{i=1}^{k} \psi_i \) is a state on \( H \) and for all \( x \in X \), \( \omega(x) > 0 \).

The next example shows the converse is not true.

Example 4.11 Let \( H \) be the hypergraph represented by Figure 3, where each line is an edge of \( H \), then \( \nabla \) is not full and there exists a state \( \omega \in \Omega(H) \) such that \( \omega(x) > 0 \) for all \( x \in X \).

Proof Suppose \( \Psi \) is a state such that \( \Psi(6) = \Psi(2) = 1 \), then \( \Psi(3) + \Psi(8) = 0 \), which contradicts the fact that \( \Psi \) is a state. Hence \( \nabla \) is not full. Define \( \omega \) by
\(\omega(4) = \omega(5) = \omega(3) = \omega(8) = 1/2\) and \(\omega(1) = \omega(2) = \omega(6) = \omega(7) = 1/4\), then \(\omega\) is a state on \(H\) with the desired properties.

**Theorem 4.12** Let \(H=(X,E)\) be a hypergraph such that for each \(E \in E\) there exists an \(x^* \in E\) such that for all \(E^* \in E\ \{E\}\), \(x^* \notin E^*\), then \(\nabla\) is full.

**Proof** Let \(x,y \in X, x \neq y\) and \(x \not\perp y\). Let \(\psi\) be a 0,1 function where \(\psi(x) = \psi(y) = 1\). For each \(E \in E\) such that \(x \notin E\) or \(y \notin E\) choose an \(x^* \in E\) and let \(\psi(x^*) := 1\). Then clearly \(\psi\) is a dispersion free state. Therefore \(\nabla\) is full.
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Table 2


Tamir, A., "On Totally Unimodular Matrices." Networks, 6, p. 373 - 382.


ON UNIMODULAR HYPERGRAPHS

by

GREGORY D. GOECKEL

B. S., Marymount College of Kansas, 1982

AN ABSTRACT OF A MASTER'S THESIS

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requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1985
ABSTRACT

Chapter 1 covers the essential theory of graphs and linear algebra that is necessary for the theory of unimodular hypergraphs.

Chapter 2 introduces the hypergraph and unimodular hypergraph. Two of the most important results of this chapter are:

i) each subhypergraph of a unimodular hypergraph is unimodular;

ii) if a hypergraph has no odd cycles, then it is unimodular.

Chapter 3 contains the three main theorems of this thesis. The first one is a theorem by Hoffman and Kruskal that characterizes an $m \times n$ totally unimodular matrix by a polyhedron in $\mathbb{R}^n$. The second is a theorem by Ghouila-Houri that characterizes a 0,1 totally unimodular matrix by being able to partition the columns into two sets in such a way that for any row, if you sum over one set of columns and subtract that from the sum over the other set of columns then that difference is less than or equal to 1 in absolute value. The last theorem is by Berge and is a corollary to the first two theorems that allows us to tap into the theory of totally unimodular matrices and use it to better characterize unimodular hypergraphs.

Chapter 4 deals with the applications to the theory developed in Chapter 3. The main theorem in this chapter says that any state on a unimodular hypergraph can be written as a convex combination of dispersion free states. A corollary to this theorem is that any bistochastic matrix can be written as a convex combination of permutation matrices.