EXISTENCE, UNIQUENESS AND BLOW-UP RESULTS FOR NON-LINEAR WAVE EQUATIONS

by

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B. A., Assumption College, 1983

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1985

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To Mom and Dad

who made it all possible,

and

To Diane

who made it all worthwhile

ACKNOWLEDGMENTS

I would like to express my deepest appreciation to Dr. Jacqueline E. Barab, under whose guidance this thesis was written. Her honesty, assistance and friendship were extremely helpful in the preparation of this text. I am also grateful to Drs. Willard Parker and Wendell Curtis, members of my committee and former teachers, for all their help. A special thanks to Mrs. Jaquith, my typist, for her patience and expertise. A final note of appreciation to Mr. Edward L. Thome. Ed's friendship and encouragement will always be a fond memory of this undertaking.

NOTATION

n = space dimension.

$\int = \int dx = \int_{\mathbb{R}^n} d^n x.$

dS; dS_y = various surface measures.

∂K = boundary of set $K \subset R^n$.

Variables

$t \in \mathbb{R}$.

$X = (x_1,\ldots,x_n) \in \mathbb{R}^n$.

$U = U(t) = U(x,t)$.

Operations

$X \cdot Y = \Sigma_K X_K Y_K$ [dot product].

$|x| = \sqrt{x \cdot x}$ [norm]

$f(x) * g(x) = \int f(x-y)g(y)dy$ [convolution].

Derivatives

$\dot{y} = \frac{dy}{dt}$.

$U_t = \partial_t U = \frac{\partial U}{\partial t}$.

$U_k = \partial_K U = \frac{\partial U}{\partial X_K}$. [partial derivatives]

If u is a scalar and $U = (U^1,\ldots,U^n)$ is a vector, define

the scalars $\begin{cases} \nabla \cdot U = \sum_K \partial_K U^K & \text{[divergence]} \\ \Delta U = \sum_K \partial_K^2 U & \text{[Laplacian]} \end{cases}$

and the vector $\{\nabla U = (\partial_1 U,\ldots,\partial_n U)$ [gradient].

Spaces

$L^q = L^q(\mathbb{R}^n)$ $(1 \le q \le \infty)$

$$\|u\|_q = \begin{cases} \|u\|_{L^q} & \text{if } u \in \mathbb{R} \\ \|u\|_{\underbrace{L^q \times \cdots \times L^q}_{n \text{ times}}} & \text{if } u \in \mathbb{R}^n. \end{cases}$$

Constants

$$\omega_n = \int_{|x|=1} dS_x \quad \text{[area of unit sphere in } \mathbb{R}^n\text{]}.$$

Therefore, $$\int_{|x|\le 1} dx = \frac{\omega_n}{n} \quad \text{[volume of unit ball in } \mathbb{R}^n\text{]}.$$

C, C_k, $C(\cdot)$ are positive constants which change from line to line.

References

(8) and (Lemma 3) refer to a numbered line and a lemma, respectively, in the same chapter.

Figure 1.3.2 refers to the second figure in section three of chapter 1.

(A2) and [3] refer to entries in the appendix and bibliography, respectively.

CONTENTS

1.1 Introduction

As was the case of ordinary differential equations, mathematicians did not consciously create the subject of partial differential equations. As they continued to explore physical problems that had led them to the theory of ordinary differential equations and secured a better grasp of the physical principles associated with these problems, mathematicians formulated mathematical statements which now comprise the field of partial differential equations. Earlier the displacement of a vibrating string had been studied separately as a function of time and as a function of the distance of a point on the string from one end. During the eighteenth century the study of the displacement as a function of both variables and the attempt to comprehend all the possible motions led to a partial differential equation which will be discussed in the next section. After the study of the vibrating string came the investigation of the sounds created by the string as they propagate in air. This study introduced additional partial differential equations. After studying these sounds the mathematicians took up the sounds given off by horns, bells, drums, and other instruments.

The first real success with partial differential equations came in regards to further study on the Vibrating String Problem. In the first approaches to the Vibrating String Problem, it was regarded as a "string of beads." That is, the string was considered to contain n discrete equal and equally spaced weights joined to each other by pieces of weightless, flexible, and elastic thread. To approximate the continuous string, the number of weights was allowed to become infinite while the size and mass of each was decreased, so that the total mass of the increasing number of individual "beads" approached the mass of the continuous string. Although ignored at the time, there were mathematical difficulties in passing to this limit.

Work continued throughout the eighteenth century on the Vibrating String Problem. Much debate on the subject of the hypotheses and conclusions associated with the problem raged throughout the 1760s and 1770s among d'Alembert, Euler, Daniel Bernoulli and Lagrange. Many of the arguments each presented were grossly incorrect; and the results, in the eighteenth century, were inconclusive. One major issue, the representability of an arbitrary function by trigonometric series, was not settled until the work of Fourier and most importantly Dirichlet. D'Alembert, Euler and Lagrange were on the threshold of discovering the significance of Fourier series but did not appreciate what lay before them. Judging by the knowledge of the times, all three men and Bernoulli were correct in their main contentions.

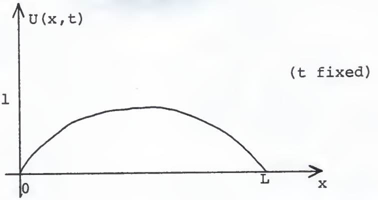
Even though the controversy over the vibrating string was still being carried on, interest in musical instruments prompted further work, not only on vibrations of physical structures but also on hydrodynamical questions which concern the propagation of sound in air. Mathematically, these involve extensions of the wave equations in different space dimensions.

During the eighteenth century efforts were directed toward solving the special equations that arose in physical problems. The theory of the solution of partial differential equations remained to be formulated and the subject as a whole was still in its infancy. It wasn't until the nineteenth century that partial differential equations were studied for their mathematical qualities rather than for some physical phenomenon associated with them.

1.2 The Vibrating String Problem

Consider the motion of a string which is fixed at its end. Set up coordinates as shown in figure 1.2.1

Figure 1.2.1



where the displacement $U(x,t)$ of the string is unknown. In order to find the equation of motion of the string, consider a short piece whose ends are at x and $x + \Delta x$ and apply Newton's second law of motion to it. The portions of the string to the right and left of our element exert forces on it which cause acceleration. There are four standing assumptions on the problem above.

Assumption 1. The string is perfectly flexible, offers no resistance to bending.

Assumption 2. A point on the string moves only in the vertical direction. Equivalently, assume that the horizontal component of the tension is constant.

Assumption 3. The string is homogeneous and its cross-section is neligible compared to its length.

Assumption 4. The displacements are relatively small.

The partial differential equation which emerges from the physical problem above is

$$(1) \quad c^2 U_{xx} = U_{tt}$$

where $c = \sqrt{T/m}$ with T the constant tension of the string and m, the constant linear mass density of the string. Equation (1), is called the one-dimensional wave equation.

In describing the motion of an object, one must specify not only the equation of motion, but also both the initial position and velocity of the object. The initial conditions for the string, then, must state the initial displacement of every particle, namely $U(x,0)$, and the initial velocity of every particle, $U_t(x,0)$.

For the vibrating string as described above, the boundary conditions are zero displacement at the ends, so the initial-boundary value problem for the string is

$$(2)\qquad \begin{cases} U_{xx} = \frac{1}{c^2}U_{tt} & 0 < x < L,\quad 0 < t \\ U(0,t) = 0 & \\ U(L,t) = 0 & 0 < t \\ U(x,0) = \phi(x) & \\ U_t(x,0) = \psi(x) & 0 < x < L \end{cases}$$

under the assumptions noted plus the assumption that gravity is negligible.

Seek a formal solution of (2) by the method of separation of variables. That is, look for solutions of the form

$$(3)\qquad U(x,t) = f(x)g(t).$$

Substitute (3) into (2) to get

$$c^2 f''(x)g(t) = f(x)g''(t)$$

or

$$(4)\qquad \frac{f''(x)}{f(x)} = \frac{1}{c^2}\frac{g''(t)}{g(t)} .$$

Since the left side is a function of x and the right side is a function of t, equation (4) can hold only if both sides are constant. Write this constant as $-\lambda^2$ and separate the above into two ordinary differential equations.

$$(5)\qquad f'' + \lambda^2 f = 0 \qquad 0 < x < L$$

$$(6)\qquad g'' + c^2\lambda^2 g = 0 \qquad t > 0.$$

The boundary conditions become

$$f(0)g(t) = 0, \quad f(L)g(t) = 0 \qquad t > 0$$

and, since $g(t) = 0$ gives a trivial solution for $U(x,t)$,

$$(7) \qquad f(0) = 0, \quad f(L) = 0.$$

The eigenvalue problem, equations (5) and (7) have eigenvalues and eigenfunctions

$$\lambda_n^2 = (\frac{n\pi}{L})^2, \quad f_n(x) = \sin(\frac{n\pi x}{L}), \quad n = 1,2,3,\ldots .$$

Equation (6) has solution

$$g_n(t) = a_n \cos \lambda_n ct + b_n \sin \lambda_n ct$$

where a_n and b_n are arbitrary.

For each $n = 1,2,3,\ldots$

$$U_n(x,t) = \sin \lambda_n x[a_n \cos \lambda_n ct + b_n \sin \lambda_n ct]$$

which, for any choice of a_n and b_n, is a solution of the one-dimensional wave equation and also satisfies the boundary conditions. Therefore linear combinations of the $U_n(x,t)$ also satisfy this wave equation with given boundary conditions. Since the a_n and b_n are arbitrary, obtain

$$(8) \quad U(x,t) = \sum_{n=1}^{\infty} \sin \lambda_n x[a_n \cos \lambda_n ct + b_n \sin \lambda_n ct].$$

The initial conditions which remain to be satisfied have the form

$$(9) \quad U(x,0) = \sum a_n \sin(\frac{n\pi x}{L}) = \phi(x) \qquad 0 < x < L$$

$$(10) \quad U_t(x,0)^* = \sum b_n \frac{n\pi}{L} c \sin(\frac{n\pi x}{L}) = \psi(x) \quad 0 < x < L.$$

Both of the initial conditions have the form of Fourier sine series.

From equation (9)

$$(11) \quad \phi(x) = a_1 \sin \frac{\pi x}{L} + a_2 \sin(\frac{2\pi x}{L}) + \cdots + a_n \sin \frac{n\pi x}{L} + \cdots .$$

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*It is assumed that the series for U may be differentiated term by term.

The problem remains to find the coefficients a_n when the function $\phi(x)$ is given. To this end, multiply (11) through by $\sin \frac{n\pi x}{L}$ and integrate the result, term by term from 0 to L. When these operations are carried out, the fact that

$$\int_0^L \sin mx \sin nx dx = 0 \quad \text{when } m \neq n \qquad \text{(A1)}$$

leaves

$$\int_0^L \phi(x) \sin \frac{n\pi x}{L} dx = a_n \int_0^L \sin^2 \frac{n\pi x}{L} dx \; ;$$

and since

$$\int_0^L \sin^2 \frac{n\pi x}{L} dx = \frac{1}{2} \int_0^L (1 - \cos \frac{2n\pi x}{L}) dx = \frac{L}{2}$$

$$a_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx \; .$$

Similarly it can be shown that

$$b_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin(\frac{n\pi x}{L}) dx \; .$$

If the functions $\phi(x)$ and $\psi(x)$ are sectionally smooth on the interval $0 < x < L$, then the initial conditions are really satisfied, except possibly at points of discontinuity of ϕ on ψ. By the nature of the problem, however, one would expect that ϕ, at least, would be continuous and would satisfy $f(0) = f(L) = 0$. Thus, one would expect the series for ϕ to converge uniformly. Questions and answers relative to the meaning and validity of the above solution and corresponding coefficients form the main theme of the theory of Fourier series.

1.3 The Initial Value Problem: d'Alembert's Solution

Consider the following initial value problem for the one-dimensional wave equation

$$(12)\quad \begin{cases} C^2U_{xx} = U_{tt} & t > 0 \\ U(x,0) = \phi(x) & -\infty < x < \infty \\ U_t(x,0) = \psi(x) & \end{cases}$$

where $\phi(x) \varepsilon C^2(\mathbb{R})$ and $\psi(x) \varepsilon C^1(\mathbb{R})$.

For the remainder of the paper assume $C^2 = 1$. If C^2 were not equal to 1, it is possible to "scale out" the C^2 through a change of variables. The above IVP (12) is also referred to as the infinite string problem due to the lack of restrictions on x.

Express the wave equation in terms of the following change of variables

$$w = x + t$$
$$z = x - t,$$

and let $U(x,t) = V(w,z)$.

By the chain rule calculate

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial w}\ \frac{\partial w}{\partial x} + \frac{\partial V}{\partial z}\ \frac{\partial z}{\partial x} = \frac{\partial V}{\partial w} + \frac{\partial V}{\partial z}, \text{ and}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial w}\left(\frac{\partial V}{\partial w} + \frac{\partial V}{\partial z}\right)\frac{\partial w}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial w} + \frac{\partial V}{\partial z}\right)\frac{\partial z}{\partial x}$$

$$= \frac{\partial^2 V}{\partial w^2} + 2\frac{\partial^2 V}{\partial w \partial z} + \frac{\partial^2 V}{\partial z^2} .$$

And similarly

$$\frac{\partial^2 U}{\partial t^2} = \left(\frac{\partial^2 V}{\partial w^2} - 2\frac{\partial^2 V}{\partial w \partial z} + \frac{\partial^2 V}{\partial z^2}\right) .$$

If $U(x,t)$ satisfies the wave equation, then in terms of the function V and the new independent variables, this equation becomes

$$\frac{\partial^2 V}{\partial w^2} + 2\frac{\partial^2 V}{\partial w \partial z} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial w^2} - 2\frac{\partial^2 V}{\partial w \partial z} + \frac{\partial^2 V}{\partial z^2} .$$

Simplify to obtain

$$\frac{\partial^2 V}{\partial w \partial z} = 0 \ ,$$

or

$$\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial w}\right) = 0 \ ,$$

which means that $\frac{\partial V}{\partial w}$ is independent of z or

$\frac{\partial V}{\partial w} = f(w)$, where f is an arbitrary function with continuous derivatives.

Integrate this equation to obtain

$$V = \int f(w)\,dw + G(z) \ .$$

Here, $G(z)$ plays the role of an integration constant. Since the integral of $f(w)$ is also a function of w, write the general solution of the partial differential equation above as

$$V(w,z) = F(w) + G(z) \quad \text{where}$$

F and G are arbitrary functions with continuous derivatives. Substitute the original variables to get

$$U(x,t) = F(x+t) + G(x-t) \ ,$$

as the general solution of the one-dimensional wave equation.

To solve the initial value problem (12) consider the following two initial value problems (12a) and (12b). After (12a) and (12b) have been solved, apply the principle of superposition to arrive at the solution for (12),

$$(12a) \quad \begin{cases} U_{xx} = U_{tt} & t > 0 \quad -\infty < x < \infty \\ U(x,0) = \phi(x) & \\ U_t(x,0) = 0 \ , & \end{cases}$$

$$(12b) \quad \begin{cases} U_{xx} = U_{tt} & t > 0 \quad -\infty < x < \infty \\ U(x,0) = 0 & \\ U_t(x,0) = \psi(x) \ , & \end{cases}$$

with the same assumptions on ϕ and ψ.

Since the general solution to (12a) is

$$U(x,t) = F(x+t) + G(x-t),$$

apply the initial conditions to obtain

$$(13)\quad U(x,0) = F(x) + G(x) = \phi(x) \text{ and}$$

$$U_t(x,0) = F'(x) - G'(x) = 0 .$$

This last equation states that $F'(x) = G'(x)$

$$\text{or}\quad (14)\quad F(x) = G(x) + C ,$$

$$\text{or}\quad (15)\quad G(x) = F(x) - C .$$

Substitute (14) into equation (13) to obtain

$$G(x) + G(x) + C = \phi(x)$$

$$\text{or}\quad G(x) = \frac{1}{2}\phi(x) - \frac{1}{2}C .$$

This is also true for $x = x - t$, therefore

$$(16)\quad G(x - t) = \frac{1}{2}\phi(x - t) - \frac{1}{2}C.$$

Substitute (15) into equation (13) to obtain

$$F(x) - C + F(x) = \phi(x)$$

$$\text{or}\quad F(x) = \frac{1}{2}\phi(x) + \frac{1}{2}C .$$

This is also true for $x + t$, therefore

$$(17)\quad F(x+t) = \frac{1}{2}\phi(x+t) + \frac{1}{2}C .$$

Combine (16) and (17) to obtain

$$(18)\quad U(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)].$$

Consider the IVP (12b) which is known to have general solution $U(x,t)$ where

$$U(x,t) = F(x+t) + G(x-t) .$$

Apply the initial conditions to obtain

$$(19)\quad U(x,0) = F(x) + G(x) = 0 ,$$

$$\text{and}\quad (20)\quad U_t(x,0) = F'(x) - G'(x) = \psi(x) .$$

Since $F(x)$ and $G(x)$ are assumed to be differentiable, equation (19) states that

$$F'(x) = -G'(x) .$$

Substitute this into equation (20) to obtain

$$G'(x) = -\frac{1}{2}\psi(x) .$$

Integrate this last equation from an arbitrary x_0 to x to obtain

$$G(x) = -\frac{1}{2}\int_{x_0}^{x} \psi(s)ds + G(x_0) .$$

Similarly

$$F(x) = \frac{1}{2}\int_{x_0}^{x} \psi(s)ds - G(x_0) .$$

Both equations are true for any x, in particular, $x-t$, and $x+t$ respectively. Therefore

$$G(x-t) = -\frac{1}{2}\int_{x_0}^{x-t} \psi(s)ds + G(x_0) = \frac{1}{2}\int_{x-t}^{x_0} \psi(s)ds + G(x_0)$$

and $$F(x+t) = \frac{1}{2}\int_{x_0}^{x+t} \psi(s)ds - G(x_0) .$$

Combine these last two equations to obtain

$$(20) \qquad U(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds .$$

Therefore by the principle of superposition, combine (18) and (20) to give the following solution to the IVP (12):

$$(21) \qquad U(x,t) = \frac{\phi(x+t) + \phi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds.$$

1.4 The Semi-Infinite String

Consider the initial value problem

$$(22)\begin{cases} U_{xx} = U_{tt} & 0 < x < \infty \qquad t > 0 \\ U(0,t) = 0 \\ U(x,0) = \phi(x) \\ U_t(x,0) = \psi(x) \end{cases}$$

where $\phi(x) \in C^2(\mathbb{R})$ and $\psi(x) \in C^1(\mathbb{R})$.

The solution to (22) is very closely related to that of the initial value problem (12) which has a solution

$$U(x,t) = F(x+t) + G(x-t) \quad \text{where}$$

$$F(x+t) = \frac{1}{2}\phi(x+t) + \frac{1}{2}\int_{x_0}^{x+t} \psi(s)\,ds$$

and $$G(x-t) = \frac{1}{2}\phi(x-t) - \frac{1}{2}\int_{x_0}^{x-t} \psi(s)\,ds ,$$

except now the respective arguments must be positive. Write y in place of $x+t$ in the first equation and y in place of $x-t$ in the second equation to obtain

$$(23)\ F(y) = \frac{1}{2}\phi(y) + \frac{1}{2}\int_{x_0}^{y} \psi(s)\,ds$$

$$y > 0,$$

and $$(24)\ G(y) = \frac{1}{2}\phi(y) - \frac{1}{2}\int_{x_0}^{y} \psi(s)\,ds .$$

From the last section, the values of F and G are determined for positive values of their arguments.

A problem arises when $0 < x < t$. To determine the values of G for negative values of its argument, substitute the boundary condition of (22) into the general solution to obtain

$$F(t) + G(-t) = 0 \qquad t > 0, \quad \text{or set}$$

$y = -t$ and note $y < 0$ for $t > 0$

$$(25)\ F(-y) + G(y) = 0 \qquad y < 0.$$

But $y < 0$ implies $-y > 0$, so that $F(-y)$ for $y < 0$ can be found by replacing y by $-y$ in equation (23). Substitute this into (25) to obtain

$$G(y) = -\frac{1}{2}\phi(-y) - \frac{1}{2}\int_0^{-y} \psi(s)\,ds, \qquad y < 0.$$

Thus the value of G for negative values of its argument is furnished. Therefore the formal solution $U(x,t)$ of the IVP (22) is

$$(26) \quad U(x,t) = \begin{cases} \dfrac{\phi(t+x) - \phi(t-x)}{2} + \dfrac{1}{2}\displaystyle\int_{t-x}^{x+t} \psi(s)\,ds & 0 < x < t \\[2ex] \dfrac{\phi(x+t) + \phi(x-t)}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} \psi(s)\,ds & x > t. \end{cases}$$

1.5 Domain of Dependence and Range of Influence

Recall that

$$(27)\quad U(x,t) = \frac{\phi(x+t) + \phi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\,ds$$

is the explicit solution of the initial value problem for the wave equation. This equation illustrates a fundamental mathematical property of solutions of the wave equation which corresponds to a distinguishing feature of the physical phenomena described by the wave equation.

Suppose that $\phi(x)$ and $\psi(x)$ vanish outside an interval $|x| \leq a$. Interpret $\phi(x)$ and $\psi(x)$ as a disturbance from equilibrium in the interval $|x| \leq a$ of an infinite string. The question then becomes, what is the effect of this disturbance outside the interval?

Assume that either $x \geq a + t$ or $x \leq -a - t$. In the first case for $t > 0$, $x - t \geq a$ and, a fortiori, $x + t \geq a$. It follows that $\phi(x+t) = \phi(x-t) = 0$ since $\phi(s) = 0$ for $s \geq a$. Similarly, since $\psi(s) = 0$ for $s \geq a$,

$$\int_{x-t}^{x+t} \psi(s)\,ds = 0.$$

Combine these statements to obtain

$$(28)\quad U(x,t) = 0 \quad \text{outside the interval} \quad -a - t \leq x \leq a + t.$$

The ends of this interval outside of which the string remains in equilibrium travel with velocities $\pm c = 1$. Therefore, for the wave equation a disturbance is propogated with finite speed. This is the distinguishing factor of the wave equation. Where $U(x,t) \neq 0$ is referred to as the support of U. For each t

$$\operatorname{supp}_x \; U(x,t) \subset \{x\colon |x| \leq a + t\}.$$

The preceding discussion can be illustrated best with the use of diagrams in the (x,t) or space-time plane. Let (x_0,t_0) be a point in the plane. According to equation (27), the value $U(x_0,t_0)$ depends only on the values of ϕ and ψ in the interval $[x_0 - t_0, x_0 + t_0]$. This interval is called the domain of dependence of the point (x_0,t_0). In particular, if ϕ and ψ vanish there, then $U(x_0,t_0) = 0$. In figure 1.5.1 the domain of dependence is seen to be the base of the characteristic triangle in the (x,t) plane formed by the x-axis and the straight lines of slope $\pm 1/C$, in this case ± 1, through (x_0,t_0). These lines are called the characteristic lines of the wave equation through the point. In 3 space, the characteristic triangle becomes the backward characteristic cone. This will be seen in later sections.

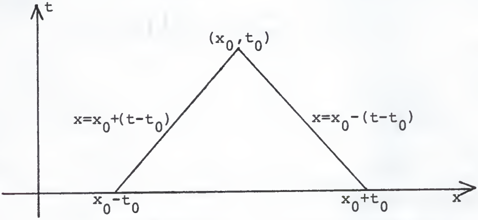


Fig. 1.5.1 Characteristic triangle for (x_0,t_0).

The inverse question can then be asked. Given a point X on the initial line, which points (x,t) are influenced by it? It is clear that this so-called Range of Influence of the initial point X_1 is the set of points (x,t) bounded by the two characteristic lines issuing from $(x_1,0)$. See figure 1.5.2.

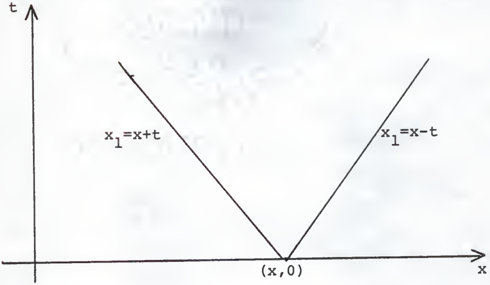


Fig. 1.5.2 Range of Influence of initial data at $(x_1,0)$.

In figure 1.5.3, note the characteristic lines (of slope ± 1) through the endpoints $\pm a$ of the interval $[-a,a]$. The region of the (x,t)-plane bounded by the characteristic lines and the interval is called the <u>region of influence</u> of the interval. Recall from the above, that the segment of the string which is disturbed at time $t = t_0$ must lie on the segment of the line $t=t_0$ included in the region of influence of the interval $[-a,a]$. To make this clear, consider the point (x_0,t_0) moving along the line $t=t_0$, and use the fact that U must vanish at all such points whose domain of dependence does not intersect the segment $[-a,a]$.

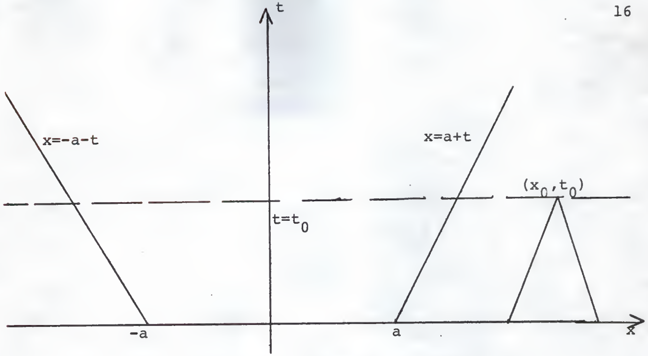


Fig. 1.5.3 Region of influence of [-a,a].

These topics will prove effective in the proof of existence of solutions to non-linear wave equations in chapter 2.

1.6 The 3-Dimensional Wave Equation

Consider the following initial value problem for the 3-dimensional wave equation,

$$(29)\quad \begin{cases} U_{tt} = \Delta U & (x \in \mathbb{R}^3, t > 0) \\ U(x,0) = 0 & \\ U_t(x,0) = \psi(x), & \end{cases}$$

where $\psi(x) \in C^2(\mathbb{R}^3)$.

Claim 1. $U(x,t)$ is a solution of the initial value problem (29), where

$$(30)\qquad U(x,t) = \frac{1}{4\pi t} \int_{|y-x|=t} \psi(y)\,ds_y.$$

Note: This solution can be derived from a method introduced by Fritz John called spherical means. A closer look at the solution $U(x,t)$ illustrates where the name of this method originated.

Proof. The following change of variables will be used throughout the proof. Let

$y = x + t\omega$, where ω is the unit vector. Therefore

$ds_y = t^2 d\omega$.

Apply the change of variables to equation (30) to obtain

$$(31)\quad U(x,t) = \frac{1}{4\pi t} \int_{|\omega|=1} \psi(x + t\omega)\,t^2 d\omega = \frac{t}{4\pi} \int_{|\omega|=1} \psi(x + t\omega)\,d\omega.$$

The change of variables will facilitate the following differentiation. To see that (31) does satisfy the 3-dimensional wave equation, differentiate (31) with respect to t to get

$$\frac{\partial U}{\partial t} = \frac{1}{4\pi} \int_{|\omega|=1} \psi(x + t\omega)\,d\omega + \frac{t}{4\pi}\frac{\partial}{\partial t}\Big(\int_{|\omega|=1} \psi(x + t\omega)\,d\omega\Big)$$

$$= \frac{1}{t}U(x,t) + \frac{t}{4\pi}\int_{|\omega|=1}\frac{\partial}{\partial t}(\psi(x + t\omega))d\omega.$$

Substitute the original variables to obtain

$$\frac{\partial U}{\partial t} = \frac{1}{t}U(x,t) + \frac{1}{4\pi t}\int_{|y-x|=t}\nabla\psi(y)\cdot\omega ds_y .$$

Differentiate this last equality to obtain

$$\frac{\partial^2 U}{\partial t^2} = -\frac{1}{t^2}U(x,t) + \frac{1}{t}\frac{\partial}{\partial t}U(x,t) - \frac{1}{4\pi t^2}\int_{|y-x|=t}\nabla\psi(y^2)\cdot\omega ds_y$$

$$+ \frac{1}{4\pi t}\frac{\partial}{\partial t}\left(\int_{|y-x|=t}\nabla\psi(y)\cdot\omega ds_y\right) .$$

Use the computation of $\frac{\partial U}{\partial t}$

$$\frac{\partial^2 U}{\partial t^2} = -\frac{1}{t^2}U(x,t) + \frac{1}{t^2}U(x,t) + \frac{1}{4\pi t^2}\int_{|y-x|=t}\nabla\psi(y)\cdot\omega ds_y$$

$$- \frac{1}{4\pi t^2}\int_{|y-x|=t}\nabla\psi(y)\cdot\omega ds_y + \frac{1}{4\pi t}\frac{\partial}{\partial t}\int_{|y-x|=t}\nabla\psi(y)\cdot\omega ds_y$$

$$= \frac{1}{4\pi t}\frac{\partial}{\partial t}\left(\int_{|y-x|=t}\nabla\psi(y)\cdot\omega ds_y\right) .$$

By the Divergence Theorem, this last equation becomes

$$\frac{\partial^2 U}{\partial t^2} = \frac{1}{4\pi t}\frac{\partial}{\partial t}\left(\int_{|y-x|=t}\nabla\cdot\nabla\psi(y)dv\right).$$

But this can be rewritten as

$$\frac{\partial^2 U}{\partial t^2} \quad \frac{1}{4\pi t}\frac{\partial}{\partial t}\left(\int_0^t\int_{|y-x|=\rho}\nabla\psi(y)ds_y d\rho\right) ,$$

$$\frac{\partial^2 U}{\partial t^2} = \frac{1}{4\pi t}\int_{|y-x|=t}\Delta\psi(y)ds_y .$$

Now from (30)

$$\Delta_x U(x,t) = \frac{1}{4\pi t}\int_{|y-x|=t}\Delta_x\psi(y)ds_y = \frac{\partial^2 U}{\partial t^2} .$$

Therefore, U(x,t) as stated in (30) satisfies the 3-dimensional wave equation.

To verify that the initial conditions are satisfied, first show that

$$\lim_{t\to 0^+} U_t(x,t) = \psi(x) .$$

To this end, note that with the change of variables

$$\lim_{t\to 0^+} U_t(x,t) = \lim_{t\to 0^+} \frac{1}{t}U(x,t) + \frac{t}{4\pi}\int_{|\omega|=1} \nabla\psi(x + t\omega)d\omega$$

$$= \lim_{t\to 0^+} \frac{1}{2}[\frac{t}{4\pi}\int_{|\omega|} \psi(x + t\omega)d\omega] + \frac{t}{4\pi}\int_{|\omega|=1} \nabla\psi(x+t\omega)\cdot\omega d\omega.$$

Take the limit to obtain

$$\lim_{t\to 0^+} U_t(x,t) = \frac{1}{4\pi}\int_{|\omega|=1} \psi(x)d\omega .$$

Switch to spherical coordinates to get

$$\lim_{t\to 0^+} U_t(x,t) = \frac{1}{4\pi}\psi(x) \int_0^{2\pi}\int_0^{\pi} \sin \phi \; d\phi d\theta = \frac{1}{4\pi} \psi(x)\int_0^{2\pi} 2d\theta = \psi(x) .$$

Therefore,

$$\lim_{t\to 0^+} U_t(x,t) = \psi(x) .$$

To see that the other initial condition is satisfied, namely

$$\lim_{t\to 0^+} U(x,t) = 0,$$

consider, with the change of variables, the following inequality

$$|U(x,t)| = |\frac{t}{4\pi}\int_{|\omega|=1} \psi(x + t\omega)d\omega| \leq \frac{t}{4\pi}\int_{|\omega|=1} |\psi(x + t\omega)|d\omega.$$

Observe that a ball B of radius 1 and center x is compact in $\mathbb{R}^3$. Since $\psi \in C^2(\mathbb{R}^3)$, there exists an $M > 0$ such that $|\psi(y)| \leq M$ $\forall y \in B$. For all $t \leq 1$, the argument $(x + tw) \in B$. Hence $|\psi(x + t\omega)| \leq M$. This shows that

$$|U(x,t)| \leq t\cdot(\frac{1}{4\pi}\int_{|\omega|=1} Md\omega) = t\frac{M}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin \phi \; d\phi d\theta = tM \text{ for } 0 < t \leq 1.$$

Take the limit as $t \to 0^+$ to obtain

$$\lim_{t\to 0^+} U(x,t) = 0.$$

From the above $U(x,t)$ is a solution to the initial problem (29).

Claim 1 is then established.

Claim 2. If $\psi \in C^2(\mathbb{R}^3)$, then the unique solution $U(x,t) \in C^2$ of the initial value problem (29) is given by equation (31).

Proof. Assume $U(x,t)$ is any classical solution to (29) and define $V(r,t)$ where

$$(32) \quad V(r,t) = \frac{r}{4\pi} \int_{|\omega|=1} U(x+\omega r,t)\,d\omega.$$

Claim 3. $V(r,t)$ satisfies the one-dimensional wave equation.

Proof. The proof is left to the reader and follows a similar pattern to that of claim 1.

Since $V(r,t)$ satisfies the 1-dimensional wave equation

$$V(r,t) = F(r+t) + G(r-t) .$$

By the definition of $V(r,t)$, equation (32)

$$\lim_{r\to 0^+} V(r,t) = \lim_{r\to 0^+} F(r+t) + G(r-t) = 0, \text{ which implies}$$

that $$F(t) + G(-t) = 0.$$

Differentiate with respect to t to obtain

$$(33) \qquad F'(t) = G'(-t).$$

Note that

$$U(x,t) = \lim_{r\to 0^+} \frac{V(r,t)}{r} = \lim_{r\to 0^+} \frac{F(r+t) + G(r-t)}{r} .$$

This last limit is in indeterminate form $(\frac{0}{0})$, therefore apply L'Hospital's Rule to get

$$U(x,t) = \lim_{r\to 0^+} F'(r+t) + G'(r-t) = F'(t) + G'(-t) .$$

But from (33) this last equation implies that (34) $U(x,t) = 2F'(t)$, $\forall t$ in particular for $t = r$.

Now note that

$$(35) \quad \lim_{t\to 0^+} (V_r + V_t) = 2F'(r) = U(x,r) \text{ from equation (34).}$$

Compute the $\lim_{t\to 0^+} (V_r + V_t)$ using the definition of V, equation (32), to get

$$U(x,r) = \lim_{t\to 0}\left[\frac{V(r,t)}{r} + \frac{r}{4\pi}\int_{|\omega|=1} \nabla U(x+\omega r,t)\cdot\omega d\omega\right.$$

$$\left. + \frac{r}{4\pi}\int_{|\omega|=1} U_t(x+\omega r,t)d\omega\right] .$$

From (32)

$$V(r,0) = \frac{r}{4\pi}\int U(x+\omega r,0)d\omega = 0 \quad \text{from (29).}$$

Since $U(x,0) = 0$, this implies that $\nabla U(x,0) = 0$ which implies that the second term in [...] is 0. Therefore

$$U(x,r) = \frac{r}{4\pi}\int_{|\omega|=1} \psi(x+\omega r)d\omega, \quad \forall r.$$

Let $r = t$ to obtain

$$U(x,t) = \frac{t}{4\pi}\int_{|\omega|=1} \psi(x+\omega t)d\omega .$$

Make the change of variables to show that

$$U(x,t) + \frac{1}{4\pi t}\int_{|y-x|=t} \psi(y)ds_y .$$

Claim 2 is proved.

<u>Lemma 1</u> If $U = U(r,t)$ satisfies

$$(36) \quad \begin{cases} U_{tt} - \Delta U = 0 & (x \in \mathbb{R}^3 \quad t > 0) \\ U(x,0) = 0 & \\ U_t(x,0) = \phi(x) , & \end{cases}$$

then $V(x,t) \stackrel{\text{def}}{=} \frac{\partial}{\partial t}(U(x,t))$ satisfies

$$(37) \quad \begin{cases} V_{tt} - \Delta V = 0 & (x \in \mathbb{R}^3 \quad t > 0) \\ V(x,0) = \phi(x) & \\ V_t(x,0) = 0, & \end{cases}$$

where $\phi(x) \in C^3(\mathbb{R}^3)$.

<u>Proof</u>. Differentiate $V(x,t)$ with respect to t to obtain

$$V_t(x,t) = U_{tt}(x,t) = \Delta U \quad \text{by (36).}$$

Differentiate this with respect to t to get

$$V_{tt}(x,t) = \frac{\partial}{\partial t}(\Delta U).$$

Now take the second derivative of $V(x,t)$ with respect to x to obtain

$$\Delta V = \frac{\partial}{\partial t}(\Delta U).$$

Therefore $V(x,t)$ satisfies the three-dimensional wave equation. To see that the initial conditions of (37) are satisfied, note that

$$V(x,0) = U_t(x,0) = \phi(x) \quad \text{and}$$
$$V_t(x,0) = \Delta U(x,0) = 0 \ .$$

The proof of lemma 1 is complete.

Now consider the initial value problem

$$(38) \quad \begin{cases} U_{tt} - \Delta U = 0 & (x \in \mathbb{R}^3 \quad t > 0) \\ U(x,0) = \phi(x) & \\ U_t(x,0) = 0 \ , & \end{cases}$$

where $\phi(x) \in C^3(\mathbb{R}^3)$.

From lemma 1 and Claim 1 the solution $U(x,t)$ to the IVP (38) is

$$U(x,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi t} \int_{|y-x|=t} \phi(y)ds_y\right] .$$

Using the principle of superposition the solution $U(x,t)$ to the following IVP

$$(39) \quad \begin{cases} U_{tt} - \Delta U = 0 & (x \in \mathbb{R}^3 \quad t > 0) \\ U(x,0) = \phi(x) & \\ U_t(x,0) = \psi(x) & \end{cases}$$

where $\phi(x) \in C^3(\mathbb{R}^3)$ and $\psi(x) \in C^2(\mathbb{R}^3)$, is

$$(40) \quad U(x,t) = \frac{1}{4\pi t} \int_{|y-x|=t} \psi(y)ds_y + \frac{\partial}{\partial t}\left[\frac{1}{4\pi t} \int_{|y-x|=t} \phi(y)ds_y\right] .$$

1.7 Conservation of Energy

Now turn to a topic that is crucial to the theory of partial differential equations, the conservation of energy. The following lemma and corresponding remarks will play a major role in the theorems in Chapter 2. Consider the following lemma:

Lemma 2 (Conservation of Energy) If U is a smooth solution to

$$U_{tt} - \Delta U = 0 \qquad x \in \mathbb{R}^n,$$

for $0 \le t \le T \le \infty$, then

$$\frac{d}{dt}\left[\frac{1}{2}\left(\| U_t(t) \|_2^2 + \| \nabla U(t) \|_2^2\right]\right. = 0 \quad \text{for } 0 \le t \le T.$$

Proof. Assume U satisfies

$$U_{tt} - \Delta U = 0.$$

Multiply by U_t to get

$$U_t U_{tt} - U_t \Delta U = 0.$$

Note that

$$U_t U_{tt} = \frac{1}{2}\partial_t (U_t^2) \quad , \text{ so}$$

$$\begin{aligned}\frac{1}{2}\partial_t (U_t^2) &= U_t \Delta U \\ &= \nabla \cdot (U_t \nabla U) - \nabla U_t \cdot \nabla U \\ &= \nabla \cdot [U_t \nabla U] - \partial_t (\tfrac{1}{2}|\nabla U|^2).\end{aligned}$$

Therefore

$$(41) \quad \partial_t [\tfrac{1}{2}(U_t^2 + |\nabla U|^2)] = \nabla \cdot [U_t \nabla U] \ .$$

Integration over all space and use of the Divergence Theorem on the RHS completes the proof, the details of which will be seen in Chapter 2.

Remark 1. Equation (41) is known as the energy identity, and the [...] on the LHS in equation (41) is referred to as the energy. Since the derivative with respect to t of [...] is zero, [...] $= C \ \forall t$, in particular for $t = 0$. Therefore, from the initial conditions

$$(42) \qquad \frac{1}{2}(\|\psi\|_2^2 + \|\nabla\phi\|_2^2) \equiv E ,$$

The assumption of finite energy data is that the energy at $t = 0$, E, is finite. If U is a smooth solution to the wave equation with finite energy data, then Lemma 2 implies

$$\frac{1}{2}(\|\partial_t U(t)\|_2^2 + \|\nabla U(t)\|_2^2) = E$$

for $0 \le t \le T$. Since each term in the energy is positive definite, it follows that

$$(43) \qquad \|\partial_t U(t)\|_2^2 , \|\nabla U(t)\|_2^2 \le 2E \quad \text{for } 0 \le t \le T .$$

Consider the following claim:

Claim 4. If $U(x,t)$ is a solution to the one-dimensional wave equation and satisfies the IVP (12) and ϕ,ψ have compact support with

$$\operatorname*{supp}_y U(y,t) \subset \{|y| \le k + t\}$$

then $\|U(\cdot,t)\|_\infty \le C$, where C only depends on the initial data.

Proof. Note that

$$U^2(x,t) = \int_{-\infty}^{x} \frac{\partial}{\partial y}[U^2(y,t)]dy$$

$$\le 2\int_{-\infty}^{x} U(y,t)\cdot U_y(y,t)dy .$$

Take absolute values to get

$$|U^2(x,t)| \le 2 \int_{-\infty}^{x} |U(y,t)\cdot U_y(y,t)|dy,$$

$$\le 2 \int_{-\infty}^{\infty} |U(y,t)\cdot U_y(y,t)|dy .$$

By Schwarz's inequality

$$|U^2(x,t)| \le 2\|U(x,t)\|_2 \|U_x(x,t)\|_2 .$$

From inequality (43)

$$U^2(x,t) \le C(\phi,\psi) ,$$

where C depends on the initial data. Therefore

$$|U(x,t)| \leq C .$$

Since this is true for all x, this implies

$$\| U(\cdot,t)\|_{\infty} \leq C .$$

Claim 4 is established.

Remark. This argument is essentially the proof of an elementary Sobalev inequality and will be used in the proof of global existence for a nonlinear wave equation in Chapter 2.

1.8 Uniqueness for Solutions of Linear Wave Equations

Consider the following two linear nonhomogeneous wave equations

$$(44a)\begin{cases} U_{tt} - \Delta U = h_1(x,t) & x \in \mathbb{R}^3 \\ U(x,0) = f_1(x) \\ U_t(x,0) = g_1(x) \end{cases} \qquad (44b)\begin{cases} V_{tt} - \Delta V = h_2(x,t) & x \in \mathbb{R}^3 \\ V(x,0) = f_2(x) \\ V_t(x,0) = g_2(x) \ . \end{cases}$$

Theorem 1. If U is a solution to (44a) and V is a solution to (44b) and $h_1 = h_2$ on $K(x_0,t_0)$, where $K(x_0,t_0)$ is the backward characteristic cone with apex (x_0,t_0) (see fig. 1.6.1) and $f_1 = f_2$, $g_1 = g_2$ for $|x - x_0| \leq t_0$, then $U(x,t) = V(x,t)$ for all $(x,t) \in K(x_0,t_0)$.

Proof. Define $W(x,t) = U(x,t) - V(x,t)$. Then $W(x,t)$ satisfies

$$(44c)\begin{cases} W_{tt} - \Delta W = 0 \\ W(x,0) = 0 \\ W_t(x,0) = 0 \ . \end{cases} \qquad (x,t) \in K(x_0,t_0)$$

The goal is to show that $W(x,t) \equiv 0$ $\forall x,t \in K(x_0,t_0)$. To this end take $T \in (0,t_0)$ and show $W = 0$ in the truncated cone $K_T(x_0,t_0)$. (See figure 1.6.2.) Once this has been shown since T is arbitrary and $W(x,t)$ is continuous, take the limit as $T \to t_0$ and the theorem will be proved.

Consider the energy equation (45) for the three-dimensional wave equation

$$(45) \quad \frac{\partial}{\partial t}[\frac{1}{2}(W_t^2 + |\nabla W|^2)] = \nabla\cdot(W_t \nabla W)$$

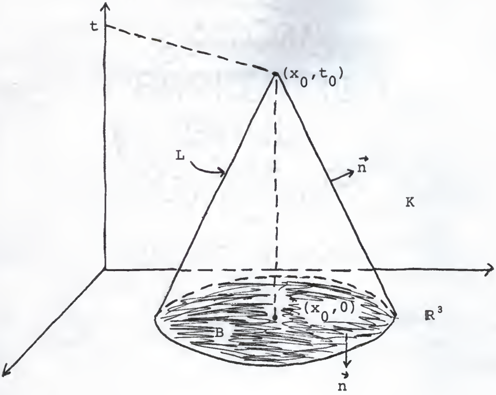
or, equivalently,

$$\frac{\partial}{\partial t}[\frac{1}{2}(W_t^2 + |\nabla W|^2)] - \nabla\cdot(W_t \nabla W) = 0 \ .$$

Rewrite as a four-dimensional divergence* and integrate over the

---

$*\nabla_4 \equiv (\nabla_3, \partial_t)$.



Each point has the form $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$. (The plane represents $\mathbb{R}^3$.)

For each $(x_0,t_0) \in \mathbb{R}^3 \times [0,\infty)$, define

the <u>backward characteristic cone</u>

$$K = K(x_0,t_0) \equiv \{(x,t) \in \mathbb{R}^3 \times [0,t_0]: |x-x_0| \leq t_0 - t\} ,$$

the <u>lateral surface of</u> K

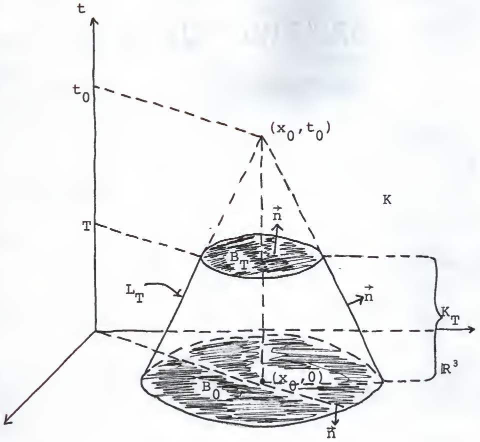
$$L = L(x_0,t_0) \equiv \{(x,t) \in \mathbb{R}^3 \times [0,t_0]: |x-x_0| = t_0 - t\} ,$$

and the <u>base of</u> K

$$B = B(x_0,t_0) \equiv \{(x,0) \in \mathbb{R}^3 \times \{0\}: |x-x_0| \leq t_0\} .$$

The outer unit normal $\vec{n} = \vec{n}(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ is given by $\vec{n} = (0,-1)$ on B and $\vec{n} = \frac{1}{\sqrt{2}}(\omega,1)$ on L, where $\omega = \dfrac{x-x_0}{|x-x_0|}$.

Figure 1.6.1



For each $(x_0,t_0) \in \mathbb{R}^3 \times [0,\infty)$ and $T \in (0,t_0)$, define

the <u>backward characteristic cone</u>

$$K = K(x_0,t_0) \equiv \{(x,t) \in \mathbb{R}^3 \times [0,t_0]: |x-x_0| \leq t_0 - t\},$$

the <u>truncated cone</u>

$$K_T = K_T(x_0,t_0) \equiv \{(x,t) \in \mathbb{R}^3 \times [0,T]: |x-x_0| \leq t_0 - t\},$$

the <u>lateral surface</u> of K_T

$$L_T = L_T(x_0,t_0) \equiv \{(x,t) \in \mathbb{R}^3 \times [0,T]: |x-x_0| = t_0 - t\}, \text{ and}$$

the <u>bases of</u> K_T

$$B_s = B_s(x_0,t_0) \equiv \{(x,t) \in \mathbb{R}^3 \times \{s\}: |x-x_0| \leq t_0 - s\} \text{ for } s=0,T.$$

The outer unit normal $\vec{n} = \vec{n}(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ is

$\vec{n}=(0,-1)$ on B_0, $\vec{n} = (0,1)$ on B_T, and $\vec{n} = \frac{1}{\sqrt{2}}(\omega,1)$ on L_T,

where $\omega = \dfrac{x-x_0}{|x-x_0|}$.

Figure 1.6.2

solid truncated cone $K_T(x_0,t_0)$ to obtain

$$\int_{K_T} \nabla_4\cdot[-W_t\nabla W, \tfrac{1}{2}(W_t^2 + |\nabla W|^2)]dxdt = 0 .$$

Now use the four-dimensional Divergence Theorem to get

$$\int_{\partial K_T} n\cdot(-W_t\nabla W, \tfrac{1}{2}(W_t^2 + |\nabla W|^2))ds = 0$$

where n is the unit outer normal.

Use the explicit vectors n (figure 1.6.2) to obtain

$$\int_{B_0} (0,-1)\cdot(-W_t\nabla W, \tfrac{1}{2}(W_t^2+|\nabla W|^2))ds + \int_L \frac{1}{\sqrt{2}}(\omega,1)\cdot(-W_t\nabla W, \tfrac{1}{2}(W_t^2+|\nabla W|^2))ds$$
$$+ \int_{B_T} (0,1)\cdot(-W_t\nabla W, \tfrac{1}{2}(W_t^2 + |\nabla W|^2))ds = 0 .$$

Compute the dot product to get

$$(46)\quad \int_{B_0} -\tfrac{1}{2}(W_t^2(x,0) + |\nabla W(x,0)|^2)ds + \frac{1}{\sqrt{2}}\int_L [\omega\cdot(-W_t\nabla W) + \tfrac{1}{2}(W_t^2+|\nabla W|^2)]ds$$
$$+ \int_{B_T} \tfrac{1}{2}(W_t^2(x,t) + |\nabla W(x,t)|^2)ds = 0 .$$

Note that the first integral in equation (46) is zero, since $W_t(x,0) = 0$ and because $W(x,0) = 0$ implies $\nabla W(x,0) = 0$ for every $x \varepsilon B_0(x_0,t_0)$. Further, in regards to the second integral, since $|\omega| = 1$ and

$$|\omega\cdot(-W_t\nabla W)| \leq |\omega||W_t\nabla W| \leq |W_t||\nabla W| \leq \tfrac{1}{2}(W_t^2 + |\nabla W|^2) ,$$

it is implied that

$$(\omega\cdot(-W_t\nabla W) + \tfrac{1}{2}(|W_t|^2 + |\nabla W|^2) \geq \tfrac{1}{2}(|W_t|^2 + |\nabla W|^2 - |\omega\cdot -W_t\nabla W|$$
$$\geq \tfrac{1}{2}(|W_t|^2 + |\nabla W|^2) - \tfrac{1}{2}(|W_t|^2 + |\nabla W|^2) = 0 .$$

Equation (46) then reduces to

$$\frac{1}{\sqrt{2}}\int_L [\omega\cdot(-W_t\nabla W) + \tfrac{1}{2}(W_t^2 + |\nabla W|^2)]ds + \int_{B_T} \tfrac{1}{2}(W_t^2(x,T) + |\nabla W(x,T)|^2)ds = 0 .$$

Since the first integrand is positive and the second is positive definite, both integrands must be identically zero. In particular

$$W_t^2(x,T) + |\nabla W(x,T)|^2 \equiv 0$$

Therefore

$$(47) \qquad W_t(x,T) = 0 \quad \text{and} \quad |\nabla W(x,T)| = 0 \ .$$

By the Mean Value Theorem, for each x there exist $\xi \in (0,T)$ such that

$$W(x,T) = W(x,0) + W_t(x,\xi)(T-0) \ .$$

From (47) and (44a) it follows that

$$W(x,T) = 0 \ .$$

Take the following limit

$$0 = \lim_{T \to t_0^-} W(x,T) = W(x,t_0)$$

to imply that

$$W(x,t) = 0 \quad \forall x,t \in K(x_0,t_0) \ .$$

The proof of Theorem 1 is complete.

Corollary 1. There exists at most one solution to (44a).

Proof. This follows immediately from Theorem 1.

The arguments above work for all n. $n = 3$ was used to provide background for the second chapter.

1.9 Huygens' Principle

Recall the result from section 1.5 concerning the domain of dependence of the solution of the initial value problem at the point (x,t): $U(x,t)$ depends on the values of the initial data on the part of the initial surface cut off by the backward characteristic cone with apex at (x,t). This part of the initial surface is the closed ball $B(x,t)$ in the x-space $\mathbb{R}^n$ with center at x and radius t. An examination of the formulas (21) and (40) shows that indeed for $n = 1,3$, the value of $U(x,t)$ depends on the values of the initial data in $B(x,t)$. However, a peculiarity occurs in the case $n=3$: $U(x,t)$ depends only on the data (and their derivatives) over the boundary $S(x,t)$ of the ball $B(x,t)$. This phenomenon was first discovered by Huygens and is known as the Strong Huygens' principle. While formula (40) shows that the Strong Huygens' principle holds for $n=3$, formula (21) shows that the Strong Huygens' principle does not hold for $n=1$. In the case $n=1$, $U(x,t)$ depends on the values of the data over the whole ball $B(x,t)$. This is sometimes referred to as the Weak Huygens' principle. In general, it can be shown that the Strong Huygens' principle holds for every odd $n \geq 3$ and the Weak Huygens' principle holds for all $n \geq 1$.

To better understand the implications of the Strong Huygens' principle, consider the initial value problem for the n-dimensional wave equation with the initial data vanishing everywhere except in a small ball $B(x_0,k)$ with center at the point x_0 and radius k. Let x be a fixed point in $\mathbb{R}^n$ outside of $B(x_0,k)$ and study the values of $U(x,t)$ for $t \geq 0$ (see Fig. 1.7.1). If $n = 3$ (or n is odd and ≥ 3), the Strong Huygens' principle holds and $U(x,t)$

is given in terms of integrals of the data and their derivatives over $S(x,t)$. Therefore $U(x,t) = 0$ for all t for which $S(x,t)$ does not intersect $B(x_0,k)$. If T is the distance between x and x_0, $S(x,t)$ intersects $B(x_0,k)$ only when t is in the interval $T-k \le t \le T+k$. Consequently $U(x,t) = 0$ for $t < T-k$ and for $t > T+k$. If $n=1$ (or n is even), the Strong Huygens' principle does not hold and $U(x,t)$ is given in terms of the values of the data over the whole ball $B(x,t)$. Since $B(x,t)$ intersects $B(x_0,k)$ for all $t \ge T-k$, it follows that $U(x,t)=0$ for $t < T-k$, while $U(x,t)$ may be nonzero for all $t \ge T-k$. Figure 1.9.2 illustrates the above. Note that the history of $U(x,t)$ at the point x is described along the line passing through x and parallel to the t axis. Figures 1.9.3(a) and (b) show the regions where $U(x,t)$ may be nonzero if the data are not zero only in the ball $B(x_0,k)$.

The Strong Huygens' principle will play a very important part in Theorem 3 of Chapter 2.

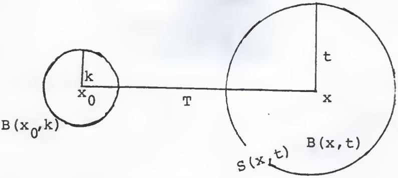


Fig. 1.9.1

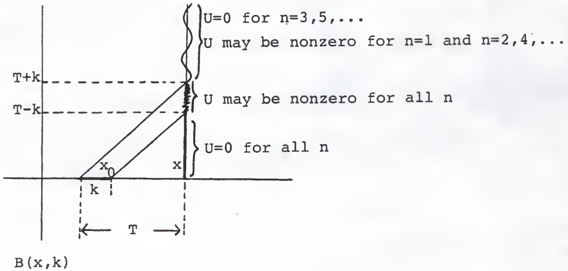


Fig. 1.9.2

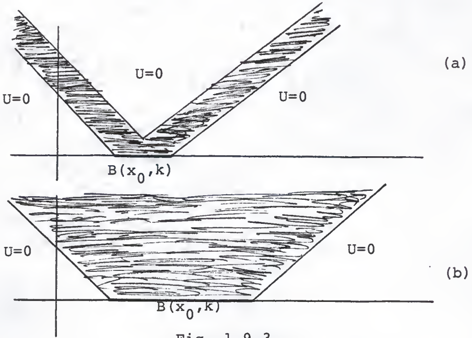


Fig. 1.9.3

2.1 Introduction

Numerous types of differential equations were created by mathematicians during the eighteenth and nineteenth centuries, but methods to solve many of these equations were not available. Due to the failure to find explicit solutions to these differential equations, mathematicians turned to the proof of the existence of solutions. These proofs serve several useful purposes, but either they do not exhibit a solution or they do not exhibit it in a useful form. In almost all cases these mathematical equations were formulations of physical phenomena with no guarantees that these equations could be solved. Hence, the proof of the existence of a solution would at least insure that a search for a solution would not be attempting the impossible. The proof of existence would also answer: What must be known about a given physical situation, in other words, what initial and boundary conditions insure a solution, particularly a unique one? From the investigation of the theory of existence proofs pose some other objectives. Does the solution change continuously with the initial conditions, or does some totally new phenomenon enter when the initial or boundary conditions are varied slightly?

Cauchy spent much time on the work of existence theorems. He emphasized that existence can often be established where an explicit solution is not available. Cauchy noted that certain partial differential equations of order greater than one can be reduced to a system of first order partial differential equations, and then proceeded to show the local existence of a solution for the system. The method he used is known today as the method of majorant functions.

Independently, Sophie Kowalewsky (1850-91), a pupil of Weierstrass, was completing work on systems similar to that of Cauchy only in somewhat of an improved form. In 1888, Kowalewsky won a prize awarded by the French Academy for a work in the integration of the equation of motion for a solid body rotating around a fixed point. Goursat later improved the proofs of Cauchy and Kowalewsky.

Many other mathematicians worked on the existence of solutions during the nineteenth century such as Poincaré, Dirichlet, Hilbert and DuBois-Reymond, to name a few. At the end of the nineteenth century the systematic theory of boundary and initial-value problems for partial differential equations was still in its infancy. The work in this area expanded rapidly in the twentieth century compared to the nineteenth century due to the work of John and Segal in relation to nonlinear wave equations.

The following is an exposition of some of the modern results in the theory of non-linear wave equations. It will examine existence, uniqueness, and blow-up of solutions to non-linear wave equations in one- and three-space dimensions. Since much of partial differential equation theory has been motivated in the past by ordinary differential equations, a similar route follows.

2.2 An Example

Consider the following initial value problem:

$$(1)\begin{cases} \ddot{U} = -U^3 & (t > 0) \\ U(0) = \phi & \phi,\psi \in \mathbb{R} \\ \dot{U}(0) = \psi . & \end{cases}$$

Integration of the ODE twice with respect to t yields

$$(2)\quad U(t) = \psi t + \phi + \int_0^t -(t-\tau)(U^3(\tau))d\tau .$$

Note. This integral equation says that

$$U(t) = W(t) + \int_0^t V(t-\tau,\tau)d\tau, \quad \text{where}$$

(i) W satisfies the linear equation with the same data

$$\ddot{W} = 0$$
$$W(0) = \phi$$
$$\dot{W}(0) = \psi$$

and

(ii) the family $V(t,\tau)$ parametrized by τ formally satisfies

$$\ddot{V} = 0$$
$$V(0,\tau) = 0$$
$$\dot{V}(0,\tau) = -U^3(\tau) .$$

The analogous statement of this note in the partial differential equation case will be crucial to the main results of the chapter.

To see the equivalence of (1) and (2), first assume U is a continuous solution of the integral equation (2) and show U is a solution to (1).

Differentiate (2) to obtain

$$\dot{U} = \psi - \int_0^t U^3(\tau)d\tau,$$

and again to conclude that

$$\ddot{U} = -U^3(t).$$

Now note that the right-hand side of the integral equation is continuous and differentiable. Therefore $U(t)$ is differentiable. Through the same reasoning $U(t)$ is also twice differentiable.

Further

$$U(0) = \phi + \int_0^0 (0 - \tau)(U^3(\tau))d\tau = \phi$$

$$\dot{U}(0) = \psi + \int_0^0 -(0 - \tau)(U^3(\tau))d\tau = \psi .$$

The equivalence between (1) and (2) is established.

The goal is to prove local and global existence for the initial value problem (1), in a way which motivates the corresponding proof for the nonlinear wave equation. Observe that if an operator $\mathcal{J}$ is defined by

$$\mathcal{J}[f] = \psi t + \phi + \int_0^t -(t - \tau)(f^3(\tau))d\tau,$$

then f is a solution of the integral equation (2) iff f is a fixed point of $\mathcal{J}$, i.e., iff $\mathcal{J}[f] = f$.

<u>Definition</u>. If X is a Banach space, then an operator $\mathcal{J} : X \to X$ is called a <u>contraction operator</u> on X if there is a constant α satisfying $0 \leq \alpha < 1$ such that for every pair of functions f and g in X

$$\| \mathcal{J}[f] - \mathcal{J}[g] \|_X \leq \alpha \| f - g \|_X .$$

<u>Contraction Mapping Theorem</u>. Let $\mathcal{J}: X \to X$ be a contraction operator on the Banach Space X. Then there exists a unique f in X such that

$$\mathcal{J}[f] = f .$$

<u>Theorem 1</u>. There exists a time $T > 0$ and a unique $U \in C([0,T],\mathbb{R})$ such that U is a solution of the integral equation (2) on $(0,T)$.

<u>Proof</u>. Define for each $B > 0$ and $T > 0$ the Banach Space

$$X(B,T) = \{f \in C([0,T],\mathbb{R}): \| f \|_X \leq B\}$$

with $\| f \|_X = \| f \|_{X(B,T)} = \sup_{0 \leq t \leq T} |f(t)|$ and the positive number B is yet to be chosen.

Define the operator $\mathcal{J}: X \to X$ by the equation

$$\mathcal{J}[f] = \psi t + \phi + \int_0^t -(t-\tau)(f^3(\tau))d\tau \ .$$

By the Second Fundamental Theorem of Calculus, $\mathcal{J}[f] \in C([0,T],\mathbb{R})$ whenever $f \in C([0,T],\mathbb{R})$. To see that $\|\mathcal{J}[f]\|_X \leq B$ whenever $\|f\|_X \leq B$. Note that

$$|\mathcal{J}[f]| = |\psi t + \phi + \int_0^t -(t-\tau)(f^3(\tau))d\tau|$$

$$\leq |\psi t| + |\phi| + \int_0^t |(t-\tau)||f^3(\tau)|d\tau \ .$$

Take the supremum over t $[0,T]$ to obtain

$$\sup_{0 \leq t \leq T} |\mathcal{J}[f]| \leq |\psi|T + |\phi| + C(T)B^3 \ .$$

*Note that $C(T)$ can be made arbitrarily small by choosing T arbitrarily small. Further $C(t)$ increases with t.

If $\phi = 0$, then choose an arbitrary B and then T small enough to ensure

$$|\psi|T + C(T)B^3 \leq B \ .$$

Otherwise let $B = 3|\phi|$ and note the following

(i) $|\phi| \leq |\phi|$;

(ii) $|\psi|T \leq |\phi|$ whenever T is chosen such that $T \leq \frac{|\phi|}{|\psi|}$. If $|\psi| = 0$, then the inequality $|\psi|T \leq |\phi|$ is obtained immediately;

and (iii) $C(T)B^3 \leq |\phi|$ whenever $C(T) \leq \frac{|\phi|}{(3|\phi|)^3}$.

Therefore $|\psi|T + |\phi| + C(T)B^3 \leq 3|\phi| = B$. Thus $\mathcal{J}[f] \in X$ whenever $f \in X$, provided B,T are chosen to satisfy this last discussion.

For all f and g in X consider the difference

$$\mathcal{T}[f(t)] - \mathcal{T}[g(t)] = \int_0^t -(t-\tau)[f^3(\tau)]d\tau + \int_0^t (t-\tau)[g^3(\tau)]d\tau$$

$$= \int_0^t (t-\tau)[g^3(\tau) - f^3(\tau)]d\tau \ .$$

Take absolute values and factor to obtain

$$|\mathcal{T}[f(t)] - \mathcal{T}[g(t)]|$$

$$\leq \int_0^t (t-\tau)\,|(g(\tau)-f(\tau))(g^2(\tau)+g(\tau)f(\tau) + f^2(\tau))|d\tau$$

$$\leq \int_0^t (t-\tau)\,|f(\tau) - g(\tau)|\,|g^2(\tau) + g(\tau)f(\tau) + f^2(\tau)|d\tau$$

$$\leq \int_0^t (t-\tau)\,|f(\tau) - g(\tau)|\,(|g(\tau)|^2 + |g(\tau)|\,|f(\tau)| + |f(\tau)|^2)d\tau \ .$$

Now f and g in X implies $|f(\tau)|,|g(\tau)| \leq B \ \forall\tau \ \varepsilon \ [0,T]$, so

$$|\mathcal{T}[f] - \mathcal{T}[g]| \leq \int_0^t |f(\tau) - g(\tau)|C(\tau)d\tau$$

where $C(T) = T(3B^2)$.

Therefore,

$$\sup_{0\leq t\leq T} |\mathcal{T}[f] - \mathcal{T}[g]| \leq \sup_{0\leq t\leq T} C(T) \int_0^t |f(\tau) - g(\tau)|d\tau$$

$$= \sup_{0\leq t\leq T} |f-g| \ C(T)$$

$$= C_1(T) \sup_{0\leq t\leq T} |f-g| \ .$$

Now choose T such that $\alpha = 0 \leq C_1(T) < 1$, thus $\mathcal{T}$ is a contraction operator, implying with the use of the contraction mapping theorem that $\mathcal{T}$ has a fixed point U in X. The proof is complete.

Due to the equivalence of the integral equation and the initial value problem apply Theorem 1 and the following is proved:

Theorem 2. There exists a time $T > 0$ and a unique $U \ \varepsilon \ C^2([0,T],\mathbb{R})$, such that U is the solution of the initial value problem (1) on $[0,T]$.

Theorem 3 (Global Existence). The initial value problem (1) has a global solution.

Proof. Let U be the solution to (1) on $0 \leq t \leq t_0$. Multiply the ODE by $\dot{U}$ to get

$$\ddot{U}\dot{U} + \dot{U}U^3 = 0,$$

and rewrite as

$$\frac{d}{dt}[\frac{1}{2}\dot{U}^2 + \frac{1}{4}U^4] = 0 .$$

Hence, the [...], the "energy" is independent of t and must equal its value at $t = 0$. Therefore

$$\frac{1}{2}[\dot{U}^2(t) + \frac{1}{2}U^4(t)] = \frac{1}{2}[\psi^2 + \frac{1}{2}\phi^4] .$$

Note that each term in the energy is positive definite. It follows that

$$U^2(t), U^4(t) \leq C(\phi,\psi) \quad \text{whenever } 0 \leq t < t_0,\ t_0 \text{ arbitrary.}$$

Therefore $\qquad |U(t)| \leq C$.

Invoke the boundedness implies existence theorem (A2) and the proof is complete.

Now consider the possibility of a solution to an ODE being restricted to a finite interval. Again techniques used in the following example will serve as motivation in the analogous nonlinear partial differential equations. Note the nonlinear term in the initial value problem (1) which has a global solution and the nonlinear term in the following initial value problem (3) which will be shown not to have a global solution.

Consider the following IVP

$$(3) \qquad \begin{cases} \ddot{U} = U^3 & \\ U(0) = \phi & 0 < \phi < \sqrt[4]{2\psi^2} \\ \dot{U}(0) = \psi & \psi > 0 . \end{cases}$$

Claim 1. If a solution U to (3) exists on $[0,T]$, then $T < \infty$.

Proof. Suppose U is a smooth solution to (3) for $0 \le t < T$. Multiply both sides of the ODE by $\dot{U}$

$$\ddot{U}\dot{U} = U^3\dot{U}$$

and integrate from 0 to t

$$\int_0^t \ddot{U}\dot{U} = \int_0^t U^3\dot{U}$$

to obtain

$$\frac{\dot{U}^2}{2}\Big|_0^t = \frac{U^4}{4}\Big|_0^t ,$$

which is equivalent to

$$\frac{\dot{U}^2(t)}{2} - \frac{\dot{U}^2(0)}{2} = \frac{U^4(t)}{4} - \frac{U^4(0)}{4} .$$

Substitute the initial condition into the equation above to obtain

$$\dot{U}^2(t) = \frac{U^4(t)}{2} - \frac{\phi^4}{2} + \psi^2 .$$

But

$$\dot{U}^2(t) = \frac{U^4(t)}{2} + C \ge \frac{U^4(t)}{2} ,$$

where $C = \psi^2 - \frac{\phi^4}{2} \ge 0$.

In order to solve for $\dot{U}(t)$, $\dot{U}$ must be greater than or equal to zero $\forall t \in [0,T]$, hence the following:

Claim 2. $\dot{U} \ge 0 \quad \forall\, t \in [0,T)$.

Proof. Since U is continuous on $[0,T)$, and $U(0), \dot{U}(0) > 0$, there exists an interval of the form $[0,T_1)$ on which $U, \dot{U} > 0$. This proof establishes that T_1 can be chosen as T.

Assume there exists $\hat{T} < T$ such that $U(t), \dot{U}(t) > 0$ for $[0,\hat{T})$ but

$$U(\hat{T}) = 0 \quad \text{or} \quad \dot{U}(\hat{T}) = 0.$$

Case 1. $(U(\hat{T}) = 0)$ By the Mean Value Theorem $\exists\, \xi \in (0,\hat{T})$ such that

$$U(\hat{T}) - U(0) = \dot{U}(\xi)\cdot(\hat{T} - 0).$$

But this says a negative is equal to a positive, therefore a contradiction.

Case 2. $(\dot{U}(\hat{T}) = 0)$ By the Mean Value Theorem (applied to $\dot{U}$) $\xi \varepsilon (0,\hat{T})$ such that $\dot{U}(T) - \dot{U}(0) = \hat{U}(\xi)(\hat{T} - 0)$, which leads to the same contradiction.

Therefore there does not exist such a $\hat{T}$ implying $U, \dot{U} > 0$ for $[0,\infty)$.

Now solve for $\dot{U}$ to obtain

$$\dot{U} \geq \sqrt{\tfrac{1}{2}}U^2(t)$$

$$\frac{\dot{U}}{U^2(t)} \geq \sqrt{\frac{1}{2}} .$$

Integrate both sides of the inequality above

$$\int_0^t U^{-2}(s)\dot{U}ds \geq \int_0^t \sqrt{\frac{1}{2}}\, ds$$

to obtain

$$-U^{-1}(t) + U^{-1}(0) \geq \sqrt{\tfrac{1}{2}}\, t .$$

Multiply the inequality by -1 to get

$$\frac{1}{U(t)} - \frac{1}{U(0)} \leq -\sqrt{\frac{1}{2}}\, t .$$

Then substitute in the initial condition and solve for $U(t)$

$$0 < \frac{1}{U(t)} \leq \frac{1}{\phi} - \frac{t}{\sqrt{2}}$$

$$\frac{1}{U(t)} \leq \frac{\sqrt{2} - t\phi}{\sqrt{2}\,\phi}$$

$$U(t) \geq \frac{\phi\sqrt{2}}{\sqrt{2} - \phi t} .$$

If $T > \frac{\sqrt{2}}{\phi}$, then $\lim_{t \to \frac{\sqrt{2}}{\phi}^-} U(t) = +\infty$.

Therefore, $T < \frac{\sqrt{2}}{\phi} < \infty$. Thus Claim 1 is established.

Although many of the techniques above will be used in the corresponding partial differential equations, the difficulty will increase drastically even in the case $n = 1$ but more particularly with $n = 3$.

2.3 Local and Global Existence for $n = 1$

Consider the Cauchy problem

$$(4)\quad \begin{cases} U_{tt} - U_{xx} = -u^3 & x \in \mathbb{R}^1, \quad (0 < t < T) \\ U(x,0) = \phi(x) & \\ U_t(x,0) = \psi(x) & \end{cases}$$

where $\phi, \psi \in C^2(\mathbb{R})$ and $\operatorname{supp} \phi, \psi \subset \{x: |x| \le k\}$.

Claim 3. If $U \in C^2(\mathbb{R} \times [0,T], R)$ and U satisfies the integral equation

$$(5)\quad U(x,t) = W(x,t) + \int_0^t V(x,t-\tau,\tau)\,d\tau,\text{*} \text{ where}$$

(i) W is the solution to the linear equation with the same data

$$W_{tt} - W_{xx} = 0$$
$$W(x,0) = \phi(x)$$
$$W_t(x,0) = \psi(x)$$

and

(ii) the family $V(x,t,\tau)$ parametrized by τ, $0 \le \tau \le t$, formally satisfies

$$V_{tt} - V_{xx} = 0$$
$$V(x,0,\tau) = 0$$
$$V_t(x,0,\tau) = -u^3(x,\tau),$$

then $U(x,t)$ is a solution to (4).

Proof. Differentiate (5) with respect to t to get

$$U_t = W_t + V(x,0,t) + \int_0^t V_t(x,t-\tau,\tau)\,d\tau.$$

Use the initial conditions from (ii) to obtain

$$(6)\quad U_t = W_t + \int_0^t V_t(x,t-\tau,\tau)\,d\tau .$$

Differentiate with respect to t again to get

$$U_{tt} = W_{tt} + V_t(x,0,t) + \int_0^t V_{tt}(x,t-\tau,\tau)\,d\tau .$$

---

*This integral is referred to as Duhamel's integral.

Again, apply the initial conditions from (ii) to obtain

$$U_{tt} = W_{tt} - U^3(x,t) + \int_0^t V_{tt}(x,t-\tau,\tau)\,d\tau \ .$$

Now, differentiate (5) with respect to x twice to get

$$U_{xx} = W_{xx} + \int_0^t V_{xx}(x,t-\tau,\tau)\,d\tau \ .$$

Since W and V satisfy the linear wave equation, from the above

$$U_{tt} - U_{xx} = -U^3(x,t) \ .$$

From (5) and (6)

$$U(x,0) = W(x,0) = \phi(x) \quad \text{and}$$

$$U_t(x,0) = W_t(x,0) = \psi(x) \ .$$

Therefore Claim 3 is established.

Similar to that of the ordinary differential equation case, an equivalence exists between the initial value problem (4) and the integral equation (5). This is shown from the claim above and the fact that the right-hand side of the integral equation is twice differentiable with respect to both x and t. Again, this equivalence will play a crucial role in the proof of local existence for the initial value problem (4).

Define the operator $\mathcal{T}$ by

$$(7) \quad \mathcal{T}[f] = W(x,t) + \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} - f^3(s,\tau)\,ds\,d\tau \ ,$$

where $W(x,t)$ is the free solution and $V(x,t) = \frac{1}{2}\int_{x-t}^{x+t} - f^3(s,\tau)\,ds$ is the solution to (ii) as shown in Chapter 1.

As in the case with the ordinary differential equation, f is a solution of the integral equation (7) iff f is a fixed point of $\mathcal{T}$.

Theorem 4. There exists a time $T > 0$ and a unique continuous function U from $\mathbb{R} \times [0,T]$ into $\mathbb{R}$, such that U is a solution of the integral equation (5) on $\mathbb{R} \times [0,T]$ with $\|\phi(\cdot)\|_\infty, \|\psi(\cdot)\|_\infty < \infty$.

Proof. Define for each $B > 0$ and $T > 0$ the Banach Space

$$X(B,T) = \{f \in C(\mathbb{R} \times [0,T], \mathbb{R}) : \operatorname{supp}_x f(x,t) \subset \{x : |x| \le k + t \text{ for each } t \in [0,T]\} \quad \|f\|_X \le B\}$$

where $\|f\|_X = \|f\|_{X(B,T)} = \sup_{0 \le t \le T} \sup_{0 \le x < \infty} |f(x,t)|$ and the positive constant B is yet to be chosen. The $\sup_{0 \le x < \infty} |f(x,t)|$ will be denoted by $\|f(\cdot,t)\|_\infty$.

Define the operator $\mathcal{T}: X \to X$ by the equation (7). Note that if $f \in X$, then the continuity and compact support of $\mathcal{T}[f]$ are guaranteed by the definition of $\mathcal{T}[f]$ and the assumptions on f. Seek a B to ensure that $\|\mathcal{T}[f]\|_X \le B$ whenever $\|f\|_X \le B$. Note that

$$|\mathcal{T}[f]| = |W(x,t) + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} -|f^3(s,\tau)dsd\tau|,$$

$$\le |W(x,t)| + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f^3(s,\tau)|dsd\tau.$$

Take the supremum over $t \in [0,T]$ and $x \in [0,\infty)$ to obtain

$$(8) \quad \sup_{0 \le t \le T} \|\mathcal{T}[f]\|_\infty \le \|\phi(\cdot)\|_\infty + 2T\|\psi(\cdot)\|_\infty + C(T,k)B^3$$

where $C(T,k) = T(k + T)$.

If $\|\phi(\cdot)\|_\infty = 0$, then choose an arbitrary B and then choose T small enough to ensure inequality (8) is less than or equal to B. Otherwise choose $B = 3\|\phi(\cdot)\|_\infty$ and note the following

(i) $\|\phi(\cdot)\|_\infty \le \|\phi(\cdot)\|_\infty$

(ii) $2T\,\|\psi(\cdot)\|_\infty \le \|\phi(\cdot)\|_\infty$ whenever T is chosen such that

$$T \le \frac{\|\phi(\cdot)\|_\infty}{2T\|\psi(\cdot)\|_\infty}.$$ If $\|\psi(\cdot)\|_\infty = 0$, the inequality

$2T\,\|\psi(\cdot)\|_\infty \le \|\phi(\cdot)\|_\infty$ is obtained immediately,

and (iii) $C(T,k)B^3 \le \|\phi(\cdot)\|_\infty$ whenever T is chosen such that

$$C(T,k) \le \frac{\|\phi(\cdot)\|_\infty}{27\,\|\phi(\cdot)\|_\infty}.$$

Therefore $\|\phi(\cdot)\|_\infty + 2T\,\|\psi(\cdot)\|_\infty + C(T,k)B^3 \le 3\,\|\phi(\cdot)\|_\infty = B$.

Thus $\mathcal{T}[f] \in X$ whenever $f \in X$.

Proceed as in the ODE case to prove that $\mathcal{T}$ is a contraction and consider the difference

$$\mathcal{T}[f](x,t) - \mathcal{T}[g](x,t)$$
$$= \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} -f^3(s,\tau)\,ds\,d\tau + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} g^3(s,\tau)\,ds\,d\tau$$
$$= \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} [g^3(s,\tau) - f^3(s,\tau)]\,ds\,d\tau .$$

Factor the right-hand side to obtain

$$= \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} ((g(s,\tau)-f(s,\tau))(g^2(s,\tau)+g(s,\tau)f(s,\tau) + f^2(s,\tau)))\,ds\,d\tau .$$

Take absolute values to get

$$|\mathcal{T}[f](x,t) - \mathcal{T}[g](x,t)|$$
$$\le \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} |(g(s,\tau)-f(s,\tau))(g^2(s,\tau)+g(s,\tau)f(s,\tau)+f^2(s,\tau))|\,ds\,d\tau .$$

Since f and g have compact support $\subset \{x: |x| \le k + t\}$

$$|\mathcal{T}[f](x-t) - \mathcal{T}[g](x,t)|$$
$$\le \frac{1}{2}\int_0^t \int_{-k-t}^{k+t} |(g(s,\tau)-f(s,\tau))(g^2(s,\tau)+g(s,\tau)f(s,\tau)+f^2(s,\tau))|\,ds\,d\tau .$$

Use the fact that $\|f\cdot g\|_\infty \le \|f\|_\infty \|g\|_\infty$ and Minkowski's inequality to obtain

$$\|\mathcal{T}[f](\cdot,t) - \mathcal{T}[g](\cdot,t)\|_\infty$$

$$\le \frac{1}{2}\int_0^t \int_{-k-t}^{k+t} \|f(s,\tau)-g(s,\tau)\|_\infty \, \|g^2(s,\tau)+g(s,\tau)f(s,\tau)+f^2(s,\tau)\|_\infty \, ds\,d\tau$$

$$\le \frac{1}{2}\int_0^t \int_{-k-t}^{k+t} \|f(s,\tau)-g(s,\tau)\|_\infty (\|g(s,\tau)\|_\infty^2 + \|g(s,\tau)\|_\infty \|f(s,\tau)\|_\infty + \|f(s,\tau)\|_\infty^2) ds\,d\tau.$$

From the boundedness of f and g

$$\|\mathcal{T}[f](\cdot,t) - \mathcal{T}[g](\cdot,t)\|_\infty \le \frac{1}{2}\int_0^t \int_{-k-t}^{k+t} \|f-g\|_\infty \; 3B^2 ds\,d\tau \; ,$$

$$\le t(k+t)\|f-g\| \; 3B^2 \; .$$

Take the supremum over all $0 \le t \le T$ to get

$$\sup_{0\le t\le T} \|\mathcal{T}[f] - \mathcal{T}[g]\|_\infty \le \sup_{0\le t\le T} t(k+t)\|f-g\| \; 3B^2 = C(T,k) \sup_{0\le t\le T} \|f-g\|_\infty \; ,$$

$$\text{where } \; C(T,k) = T(k+T)3B^2 \; .$$

Similar to that of the ODE case, choose T such that $0 \le C(T,k) < 1$, thus $\mathcal{T}$ is a contraction operator, implying with the use of the contraction mapping theorem that $\mathcal{T}$ has a fixed point U in X. The proof is complete.

Since the initial value problem (4) and the integral equation (5) are equivalent, the following has been proved:

Theorem 5. There exists a time $T > 0$ and a unique function $U \in C^2(\mathbb{R}\times[0,T],\mathbb{R})$, such that U is a solution of the IVP (1) on $\mathbb{R}\times[0,T]$.

Theorem 6. The initial value problem (1) has a global solution.

Proof. From the *a priori* energy estimates (Claim 4, Chapter 1)

$$\|U(\cdot,t)\|_\infty \le C(\phi,\psi) \text{ for all } t > 0.$$

The proof is complete.

As in the case with the ODE, in proving $\mathcal{T}$ a contraction operator the fact that

(9) $|f \cdot g| \leq |f||g|$ played a crucial role.

If $|\cdot|$ is replaced by $\|\cdot\|_q$ in (9), then the inequality

$$\|f \cdot g\|_q \leq \|f\|_q \, \|g\|_q$$

holds only for $q = \infty$, which was used in the above. Since it is not to be expected that $\|U(\cdot,t)\|_\infty \leq C$ for $0 \leq t \leq T$ and $x \in \mathbb{R}^3$, the proof for local existence in the case $n = 3$ should be much more difficult.

2.4 An Example* When Global Existence Fails

Consider the Cauchy problem

$$(10)\quad \begin{cases} U_{tt} - U_{xx} = U^p & x \in \mathbb{R} \quad p \geq 2 \quad t \geq 0 \\ U(x,0) = \phi(x) \\ U_t(x,0) = \psi(x) \end{cases}$$

where $\phi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$.

The following will show that if ϕ and ψ are chosen correctly, then

$$F(t) = \int_R U^2(x,t)\,dx$$

goes to infinity in finite time.

Theorem 7. If $T > 0$ and $p \geq 2$ and $U(x,t)$ is a smooth solution to (10) on $\mathbb{R} \times (0,T)$, then $T < \infty$ with $\phi,\psi \in C_0^\infty(\mathbb{R})$.

Proof. Assume an $\alpha > 0$ and initial data ϕ and ψ can be chosen such that

$$\text{(i)} \quad (F(t)^{-\alpha})'' \leq 0 \quad \text{for all } t \geq 0$$

$$\text{(ii)} \quad (F(t)^{-\alpha})' < 0 \quad \text{at } t = 0$$

Then $F(t)^{-\alpha}$ will go to zero in finite time. See Figure (2.4.1). This is equivalent to saying $F(t)$ goes to $+\infty$ in finite time.

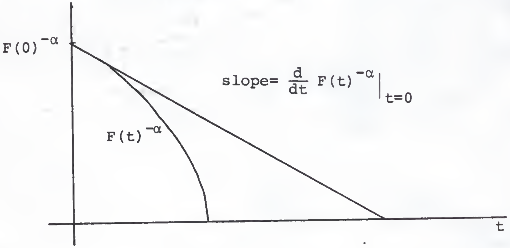


Figure 2.4.1

*This example is due to H. Levine.

Condition (ii) is automatically satisfied by choosing ϕ and ψ to have the same sign on $(-\infty,\infty)$ since

$$(F(0)^{-\alpha})' = -\alpha(F(0)^{-1-\alpha})F'(0) = -2\alpha F(0)^{-1-\alpha} \int \phi\psi dx .$$

It now remains for (i) to hold. Since $F(t) \geq 0$ is the same as showing that $Q(t) \geq 0$ where

$$Q(t) \equiv (-\alpha)^{-1}F^{\alpha+2}(F^{-\alpha})'' = F''F - (\alpha+1)(F')^2 .$$

But,

$$F'(t) = 2 \int UU_t dx, \quad \text{and}$$

$$F''(t) = 2 \int (UU_{tt} + U_t^2)dx$$

$$= 4(\alpha+1) \int U_t^2 dx + 2 \int (UU_{tt} - (2\alpha+1)U_t^2)dx .$$

Therefore,

$$Q(t) = 4(\alpha+1)\{(\int U^2 dx)(\int U_t^2 dx) - (\int UU_t dx)^2\}$$

$$+ 2F(t)\{\int UU_t dx - \int (2\alpha+1)U_t^2 dx\} .$$

Since the first term to the right of the equal sign is positive by the Schwarz inequality, it suffices to arrange for $H(t) \geq 0$ where

$$H(t) \equiv \int UU_{tt}dx - (2\alpha + 1)\int U_t^2 dx .$$

$U_{tt} = U^p + U_{xx}$ by (10), therefore

$$H(t) = \int U^{p+1}dx + \int UU_{xx}dx - (2\alpha + 1) \int U_t^2 dx .$$

Use integration by parts and the compact support of U to obtain

$$H(t) = \int U^{p+1}dx - \int U_x^2 dx - (2\alpha +1) \int U_t^2 dx .$$

Note that the conserved energy for (10) is

$$E(t) = \frac{1}{2} \int (U_x^2 + U_t^2)dx - \frac{1}{p+1} \int U^{p+1}dx ,$$

which is independent of t. Thus, choose α so that $2(2\alpha+1) = p+1$ to obtain

$$H(t) = -(p+1)E(t) + 2\alpha \int U_x^2 d_x$$

$$= -(p+1)E(0) + 2\alpha \int U_x^2 d_x \; .$$

Therefore if $E(0) < 0$, the H is always strictly positive since $\alpha = \frac{1}{4}(p-1) \geq 0$. Now choose $\phi > 0$ and $\psi > 0$ so that (ii) is satisfied and since $p + 1 > 2$, multiply ϕ by a positive constant and eventually $E(0) < 0$ implying $H(t) \geq 0$. For any such data, $F(t)$ goes to infinity in finite time, therefore $T < \infty$.

2.5 Local Existence for $n = 3$

Consider the Cauchy problem

$$
(11) \quad \begin{cases} U_{tt} - \Delta U = -|U|^{p-1}U & x \in \mathbb{R} \qquad 0 < t < T \\ U(x,0) = \phi(x) \\ U_t(x,0) = \psi(x) \end{cases}
$$

where $\phi(x) \in C^3(\mathbb{R}^3)$, $\psi(x) \in C^2(\mathbb{R}^3)$.

As in the previous examples, the Cauchy problem (11) is equivalent to the following integral equation

$$
(12) \quad U(x,t) = W(x,t) + \int_0^t V(x,t-\tau,\tau)\,d\tau,
$$

where, as expected,

(i) W is the solution to the linear equation with the same data

(ii) the family $V(x,t,\tau)$ parametrized by τ, $0 \leq T \leq t$, formally satisfies

$$
\begin{aligned} & W_{tt} - \Delta W = 0 && \text{and} \qquad V_{tt} - \Delta V = 0 \\ & W(x,0) = \phi(x) && \qquad\qquad V(x,0,\tau) = 0 \\ & W_t(x,0) = \psi(x) && \qquad\qquad V_t(x,0,\tau) = -|U(x,\tau)|^{p-1}U(x,\tau)\ . \end{aligned}
$$

Digress for a moment and consider the following homogeneous IVP

$$
(13) \quad \begin{cases} h_{tt} - \Delta h = 0 & x \in \mathbb{R}^3 \qquad t \geq 0 \\ h(x,0) = 0 \\ h_t(x,0) = g(x)\ . \end{cases}
$$

The solution to this problem, as seen in chapter 1, is

$$
h(x,t) = \frac{1}{4\pi t} \int_{|y-x|=t} g(y)\,dS_y \ ,
$$

which when written in terms of the Riemann function

$$
h(x,t) = R(x,t) *_x g(x), \text{ where } R(x,t) = \frac{1}{4\pi t}\delta(|x|-t) \ ,
$$

where the distribution $\delta(|x|-t)$ is defined by

$$
\delta(|x|-t)g(x) \equiv \int_{|x|=t} g(x)\,dS_x \ .
$$

Also, because of the lack of an infinity bound in the case $n = 3$ some other norm must be used in order to apply the Contraction Mapping Theorem. Define the energy norm as follows:

$$\|H(\cdot,t)\|_e^2 = \|H_t(\cdot,t)\|_2^2 + \|\nabla H(\cdot,t)\|_2^2 .$$

Apply this to (13) and since from the energy estimates

$$\|h_t(\cdot,t)\|_2^2 + \|\nabla h(\cdot,t)\|_2^2 = C \quad \forall t.$$

This is true in particular for $t = 0$, therefore

$$\|h(\cdot,t)\|_e^2 = \|h_t(\cdot,0)\|_2^2 + \|\nabla h(\cdot,0)\|_2^2 .$$

Substitute the initial conditions from (13) to obtain

$$\|h(\cdot,t)\|_e^2 = \|g(\cdot)\|_2^2 \qquad \forall\ t ,$$

but since $h(x,t) = R(x,t) *_x g(x)$ then

$$\|R(\cdot,t) *_x g(x)\|_e^2 = \|g(\cdot)\|_2^2 .$$

With the above in mind, return now to the question of local existence for $n = 3$, and rewrite (12) as

$$U(x,t) = W(x,t) + \int_0^t R(\cdot,t-\tau) *_x -|U(\cdot,\tau)|^{p-1}U(\cdot,\tau)d\tau .$$

In order to produce bounds for the following proof, the Sobolev inequality $\|U\|_6 \leq C\|\nabla U\|_2$ (A3) which is bounded from the energy along with a technique called interpolation (A4) to produce bounds other than the one just mentioned will play a crucial role.

Define the operator $\mathcal{T}$ by

$$(14) \quad \mathcal{T}[f](x,t) = W(x,t) + \int_0^t R(x,t-\tau) *_x -|f(x,\tau)|^{p-1}f(x,\tau)d\tau,$$

where W is defined as in (12).

As in the case with the previous two sections, f is a solution of the integral equation (12) iff f is a fixed point of $\mathcal{T}$.

<u>Theorem 8</u>. There exists a time $T > 0$ and a unique continuous function U from $\mathbb{R}^3 \times [0,T]$ into $\mathbb{R}$, such that U is a solution of the integral equation (12) on $\mathbb{R}^3 \times [0,T]$ given $1 < p \leq 3$ and

$\|\psi\|_\infty, \|\phi\|_\infty, \|\nabla\phi\|, < \infty$.

Proof. Define for each $B > 0$ and $T > 0$ the Banach Space

$X(B,T) = \{f \in C(\mathbb{R}^3 \times [0,T], \mathbb{R})$:

$\operatorname{supp}_x f(x,t) \subset \{x: |x| \leq k+t\}$ for each $t \in [0,T]$ and $\|f\|_X \leq B\}$,

where $\|f\|_X = \|f\|_{X(B,T)} = \sup_{0\leq t\leq T} \|f(\cdot,t)\|_e$ as defined above.

Note. The compact support of f will play a slightly different role in this proof than in the case $n = 1$. The compact support will be used to infer bounds on other norms than mentioned above.

Define the operator $\mathcal{T}: X \to X$ by equation (14). Note that if $f \in X$, then the continuity and compact support of $\mathcal{T}[f]$ are guaranteed by the definition of $\mathcal{T}[f]$ and the properties of f. Now choose B to show that $\|\mathcal{T}[f]\|_X \leq B$ whenever $\|f\|_X \leq B$.

Note that

$$|\mathcal{T}[f]| = |W(x,t) + \int_0^t R(x,t-\tau) *_x - |f|^{p-1} f d\tau|$$

$$\leq |W(x,t)| + \int_0^t |R(x,t-\tau) *_x - |f|^{p-1} f| d\tau .$$

As it will be shown in the next section (Lemma 3)

$$|W(x,t)| \leq t\|\psi\|_\infty + \|\phi\|_\infty + t\|\nabla\phi\|_1 .$$

Take the supremum over all $0 \leq t \leq T$ to obtain

$$\sup_{0\leq t\leq T} |W(x,t)| \leq T\|\psi\|_\infty + \|\phi\|_\infty + T\|\nabla\phi\|_1 .$$

Also, $\int_0^t \|R(x,t-\tau) *_x - |f|^{p-1} f\|_e^{d\tau}$

$$\leq \int_0^t \|f\|^p_{2p} d\tau \qquad 2 < 2p \leq 6 .$$

Take the supremum over all $0 \leq t \leq T$ and use the boundedness of f to obtain

$$\leq TB^p .$$

As in the last two sections, if $\|\phi(\cdot)\|_\infty = 0$, the choice of B

is trivial. Otherwise choose $B = 4\|\phi(\cdot)\|_\infty$, and continue in the same manner as before to conclude that $\|\mathcal{T}[f]\|_X \le B$ whenever $\|f\|_X \le B$, therefore $\mathcal{T}[f] \in X$ whenever $f \in X$. For all f and g in X consider the difference

$$\mathcal{T}[f] - \mathcal{T}[g]$$
$$= \int_0^t (R(x,t-\tau) *_x - |f|^{p-1}f)\,d\tau - \int_0^t (R(x,t-\tau) *_x - |g|^{p-1}g)\,d\tau \;.$$

Take the energy norms to obtain

$$\|\mathcal{T}[f] - \mathcal{T}[g]\|_e \le \int_0^t \|R(x,t-\tau) *_x (g^p - f^p)\|_e\,d\tau \;.$$

From the digression above, rewrite the last inequality as

$$\|\mathcal{T}[f] - \mathcal{T}[g]\|_e \le \int_0^t \|g^p - f^p\|_2\,d\tau \;.$$

But $|g^p - f^p| \le C|g-f|\,|g^{p-1} + f^{p-1}|$ so

$$\|\mathcal{T}[f] - \mathcal{T}[g]\|_e \le \int_0^t \|\,|g-f|\,|g^{p-1} + f^{p-1}|\,\|_2\,d\tau \;.$$

Apply Hölder's and Minkowski's inequalities to obtain

$$(15)\quad \|\mathcal{T}[f] - \mathcal{T}[g]\|_e \le \int_0^t \|g-f\|_r (\|g^{p-1}\|_s + \|f^{p-1}\|_s)\,d\tau \;,$$

where $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$ and $r,s > 1$.

Note that $\|g^{p-1}\|_s = (\int (g^{p-1})^s)^{1/s} = [(\int g^{s(p-1)})^{1/s(p-1)}]^{p-1}$
$$= \|g\|_{s(p-1)}^{p-1},$$

therefore $\|g^{p-1}\|_s$ is bounded, and similarly $\|f^{p-1}\|_s$ is bounded by the Sobolev lemma and interpolation provided

$$2 \le s(p-1) \le 6 \;.$$

Use this information to simplify (15) to

$$\|\mathcal{T}[f] - \mathcal{T}[g[\|_e \le C(T) \int_0^t \|g-f\|_r\,d\tau = C(T)\int_0^t \|f-g\|_r\,d\tau \;.$$

It is necessary to obtain the energy norm on the right-hand side of the inequality above. To this end, since by the Sobolev lemma $\|f\|_6 \le C\|\nabla f\|_2$ (likewise for g), utilize the compact support features of f and g and interpolate to obtain

$$\|\mathcal{T}[f] - \mathcal{T}[g]\|_e \leq C(T)\int_0^t \|\nabla(f-g)\|_2 d\tau \text{ , provided}$$

$$2 \leq r \leq 6 .$$

By increasing the right-hand side, the inequality is maintained, therefore

$$\|\mathcal{T}[f] - \mathcal{T}[g]\|_e \leq C(T) \int_0^t (\|\nabla(f-g)\|_2 + \|(f-g)_t\|_2) d\tau .$$

But this is the definition of the energy norm, so

$$\|\mathcal{T}[f] - \mathcal{T}[g]\|_e \leq C(T) \int_0^t \|f-g\|_e d\tau = C(T)\, t\|f-g\|_e .$$

Take the supremum over all $0 \leq t \leq T$ to obtain

$$\sup_{0\leq t\leq T} \|\mathcal{T}[f] - \mathcal{T}[g]\| \leq C(T) \sup_{0\leq t\leq T} \|f-g\|_e .$$

As in the last two cases choose T such that $0 \leq C(T) < 1$.

It remains only to show that

$$(16) \quad \left\{ \begin{array}{l} \frac{1}{r} + \frac{1}{s} = \frac{1}{2} \\ r,s \geq 1 \\ 2 \leq s(p-1) \leq 6 \\ 2 \leq r \leq 6 . \end{array} \right.$$

If $r = 2$, this would imply that $s = \infty$ which would be a contradiction, so proceed with $r > 2$.

From (16) $s = \frac{2r}{r-2}$, therefore $2 \leq \frac{2r}{r-2}(p-1) \leq 6.$

Divide both sides by $\frac{2r}{r-2}$ to obtain

$$\frac{r-2}{2r} \cdot 2 \leq p-1 \leq 6\left(\frac{r-2}{2r}\right) .$$

So p must be restricted to the interval

$$\frac{r-2}{r} + 1 \leq p \leq 3\left(\frac{r-2}{r}\right) + 1 .$$

But r can only range between 2 and including 6. Therefore, r and s can be chosen to satisfy (16) provided

$$1 < p \leq 3 ,$$

the hypothesis on p. By the contraction mapping theorem, the proof is complete.

The question of global existence for the Cauchy problem (11) will be handled in a slightly different manner than in the previous examples. Several results in the modern theory of non-linear wave equations will be used. The first of which, Jörgen's cone estimate, will play a crucial role. For this reason consider the following lemma.

Lemma 1 (Cone Estimate).

If $U \in C^2(\mathbb{R} \times [0,T), \mathbb{R})$ and is a solution to (11) on $[0,T]$, then for each backward characteristic cone $K(x_0,t_0)$ with $0 < t_0 < T$ (see Figure 2.5.1)

$$\int_L \left(\frac{1}{p+1}|U|^{p+1}\right) ds \leq \sqrt{2}\, E .$$

Here $p > 1$, $0 \leq T < \infty$, and E, the conserved energy, depends only on the data and L stands for the lateral surface of K.

Proof. Define the energy density as follows:

$$e = e(x,t) \equiv \frac{1}{2}[|U_t|^2 + |\nabla U|^2 + \frac{2}{p+1}|U|^{p+1}] .$$

Thus the energy

$$E(t) = \int_{\mathbb{R}^3} e(x,t)\,dx \quad \forall t.$$

Note that the energy identity takes the form

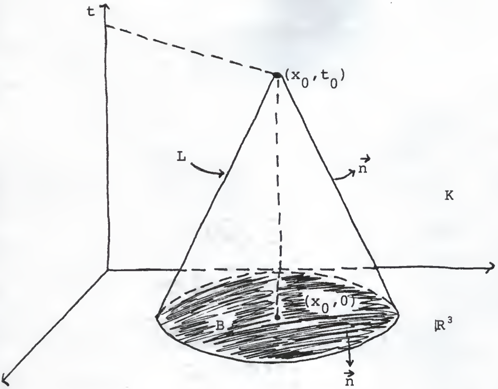
$$\partial_t e = \nabla \cdot (U_t \nabla U) .$$

Equate the above to zero to obtain

$$\partial_t e - \nabla \cdot (U_t \nabla U) = 0.$$

Remark this as a four-dimensional divergence.*

*$\nabla_4 \equiv (\nabla, \partial_t)$.



Each point has the form $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$. (The plane represents $\mathbb{R}^3$.)
For each $(x_0,t_0) \in \mathbb{R}^3 \times [0,\infty)$, define

the <u>backward characteristic cone</u>

$$K = K(x_0,t_0) \equiv \{(x,t) \in \mathbb{R}^3 \times [0,t_0]: |x - x_0| \le t_0 - t\} ,$$

the <u>lateral surface of</u> K

$$L = L(x_0,t_0) \equiv \{(x,t) \in \mathbb{R}^3 \times [0,t_0]: |x - x_0| = t_0 - t\} ,$$

and the <u>base of</u> K

$$B = B(x_0,t_0) \equiv \{(x,0) \in \mathbb{R}^3 \times \{0\}: |x - x_0| \le t_0\} .$$

The outer unit normal $\vec{n} = \vec{n}(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ is given by $\vec{n} = (0,-1)$ on B and $\vec{n} = \frac{1}{\sqrt{2}}(\omega,1)$ on L, where $\omega = \dfrac{x - x_0}{|x - x_0|}$.

Figure 2.5.1

$$\nabla_4 \cdot (-U_t \nabla U, e) = 0.$$

Integrate over an arbitrary characteristic cone K and use the (four-dimensional) Divergence Theorem to obtain

$$\int_K \nabla_4 \cdot (-U_t \nabla U, e)\, dx dt = 0 \ ,$$

$$\int_{\partial K} N \ (-U_t \nabla U, e)\, ds = 0$$

where N is the outer unit normal.

Use the explicit vectors as shown in figure 2.5.1 to get

$$(17) \quad \int_B -e(x,0)\,dx + \frac{1}{\sqrt{2}} \int_L (\omega \cdot -U_t \nabla U)\, dS + \frac{1}{\sqrt{2}} \int_L e dS = 0 \ .$$

Seek a lower bound for the integrand in the second term. Since $|\omega| = 1$, use the Schwarz inequality to get

$$|\omega \cdot - U_t \nabla U| \le |\omega| \ |U_t \nabla U| \le |U_t| \ |\nabla U| \le \tfrac{1}{2}(|U_t|^2 + |\nabla U|^2) \ .$$

This implies

$$(\omega \cdot -U_t \nabla U) + e \ge e - |\omega \cdot \ -U_t \nabla U|$$

$$\ge e - \tfrac{1}{2}(|U_t|^2 + |\nabla U|^2) \ .$$

Use the definition of e to simplify this last inequality to

$$(\omega \cdot -U_t \nabla U) + e \ge \frac{2}{p+1}|U|^{p+1} \ .$$

This together with (17) implies

$$\frac{1}{\sqrt{2}} \int_L \frac{2}{p+1}|U|^{p+1} dS \le \int_B e(x,0)\,dx \le \int_{\mathbb{R}^3} e(x,0)\,dx = \int_{\mathbb{R}^3} e(x,t)\,dx = E(t) \quad \forall\, t.$$

It then follows for every backward characteristic cone K with lateral surface L

$$\int_L \frac{1}{p+1}|U|^{p+1} dS \le \sqrt{2}\, E \ ,$$

the proof of the lemma is complete.

Continue now to the main result of the section.

Theorem 9 (Global Existence). The initial value problem (11) with $1 < p \leq 3$ has a global solution.

Proof. Let $U(x,t)$ be a local solution to the nonlinear wave equation (11). The equivalent integral representation of (11) is

$$U(x,t) = W(x,t) + \int_0^t \frac{1}{4\pi(t-\tau)} \int_{|x-y|=t-\tau} [-|U(y,\tau)|^{p-1}U(y,\tau)]\,dS_y\,d\tau \; .$$

Take the absolute values of both sides to get

$$|U(x,t)| \leq |W(x,t)| + \int_0^t \frac{1}{4\pi(t-\tau)} \int_{|x-y|=t-\tau} |U(y,\tau)|^p \, dS_y\,d\tau \; ,$$

$$\equiv |W(x,t)| + I_1 \; .$$

Now verify that both $|W(x,t)|$ and I_1 are bounded.

To see that $W(x,t)$ is bounded consider the following lemma.

Lemma 2.* Let $F(x) \in C^1(\mathbb{R})$ with $x \in \mathbb{R}^3$. Then for all $t \geq 0$,

$$\left| \int_{|y-x|=t} F(y)\,dS_y \right| \leq C\,\|\nabla F\|_1 \; ,$$ where the constant C is

independent of x,t.

Proof. Let n denote the unit outer normal to the ball $|y - x| = t$. Because n is the unit vector

$$\int_{|y-x|=t} F(y)\,dS_y = \int_{|y-x|=t} n \cdot (nF(y))\,dS_y \; .$$

Then by the divergence theorem

$$\int_{|y-x|=t} n \cdot (nF(y))\,dS_y = \int_{|y-x|\leq t} \operatorname{div}(nF(y))\,dy$$

$$= \int_{|y-x|\leq t} [n \cdot \nabla F + F \operatorname{div} n]\,dy \; .$$

Since $\operatorname{div} n = \frac{3}{t}$

---

*This lemma is due to Glassey [10].

$$(18) \quad \left| \int_{|y-x|=t} F(y)\,dS_y \right| \le \int_{|y-x|\le t} |\nabla F(y)|\,dy + \frac{3}{t} \int_{|y-x|\le t} |F(y)|\,dy \; .$$

Apply Hölder's inequality to the last integral

$$\int_{|y-x|\le t} |F(y)|\,dy \le \Big(\int_{|y-x|\le t} |F|^{3/2} dy \Big)^{2/3} \Big(\int_{|y-x|\le t} 1^3 dy \Big)^{1/3} \le Ct \, \|F\|_{\frac{3}{2}} \; .$$

By Sobolev's inequality, $\|F\|_{3/2} \le C \|\nabla F\|_1$

Thus

$$\int_{|y-x|\le t} |F(y)|\,dy \le Ct \, \|\nabla F\|_1 \; .$$

Insert this bound into (18) to obtain

$$\left| \int_{|y-x|=t} F(y)\,dS_y \right| \le C \, \|\nabla F\|_1 \; .$$

The proof of lemma 2 is complete.

<u>Lemma 3</u>. If $\|\psi\|_\infty$, $\|\phi\|_\infty$, $\|\nabla\phi\|_1$, $\|\Delta\phi\|_1$, and $\|\nabla\psi\|_1$ are finite,

then $|W(x,t)|$ is bounded.

<u>Proof</u>. As seen in chapter 1,

$$W(x,t) = \frac{1}{4\pi t} \int_{|y-x|=t} \psi(y)\,dS_y + \frac{\partial}{\partial t}\Big[\frac{1}{4\pi t} \int_{|y-x|=t} \phi(y)\,dS_y\Big] \; .$$

Differentiate the second term to obtain

$$W(x,t) = \frac{1}{4\pi t} \int_{|y-x|=t} \psi(y)\,dS_y + \frac{1}{4\pi t^2} \int_{|y-x|=t} \phi(y)\,dS_y + \frac{1}{4\pi t} \int_{|y-x|=t} \nabla\phi(y)\cdot\omega\,dS_y$$

Take absolute values and simplify to get

$$(19) \quad |W(x,t)| \le \left|\frac{1}{4\pi t}\right| \left| \int_{|y-x|=t} \psi(y)\,dS_y \right|$$

$$+ \left|\frac{1}{4\pi t^2}\right| \left| \int_{|y-x|=t} \phi(y)\,dS_y \right| + \left|\frac{1}{4\pi t}\right| \left| \int_{|y-x|=t} \nabla\phi(y)\cdot\omega\,dS_y \right| \; .$$

Consider two cases i) $t \in (0,1)$ and ii) $t \in [1,\infty)$.

<u>Case i</u> ($t \in (0,1)$). With the following change of variables

$$Y = x + t\omega \qquad \omega = \text{unit vector}$$

$$dS_y = t^2 d\omega$$

Rewrite inequality (19) as

$$|W(x,t)| \le |\tfrac{t}{4\pi}|\,|\int_{|\omega|=1} \psi(x+t\omega)\,d\omega + |\tfrac{1}{4\pi}|\,|\int_{|\omega|=1} \phi(x+t\omega)\,d\omega|$$

$$+ |\tfrac{t}{4\pi}|\,|\int_{|\omega|=1} \nabla\phi(x+t\omega)\cdot\omega\,d\omega|$$

$$\le |\tfrac{t}{4\pi}|\ \|\psi\|_\infty \int_{|\omega|=1} 1\,d\omega + |\tfrac{1}{4\pi}|\ \|\phi\|_\infty \int_{|\omega|=1} 1\,d\omega + |\tfrac{t}{4\pi}|\ \|\nabla\phi\|_1 \int_{|\omega|=1} 1\,d\omega$$

$$= t\,\|\psi\|_\infty + \|\phi\|_\infty + t\,\|\nabla\phi\|_1$$

which since $t \varepsilon (0,1)$

$$\le \|\psi\|_\infty + \|\phi\|_\infty + \|\nabla\phi\|_1 \ .$$

<u>Case ii</u> ($t.\varepsilon\ [1,\infty)$). Apply lemma 2 to the first two terms of inequality (19) and the divergence theorem to the last term to obtain

$$|W(x,t)| \le \tfrac{1}{t} C\|\nabla\psi\|_1 + \tfrac{1}{t}C\,\|\nabla\phi\|_1 + \tfrac{1}{t}\,\|\nabla\cdot\nabla\phi\|_1$$

$$\le \tfrac{1}{2}C(\ \|\nabla\psi\|_1 + \|\nabla\phi\|_1 + \|\Delta\phi\|_1)\ .$$

The proof of lemma 3 is complete, therefore $|W(x,t)|$ is bounded for all $t \varepsilon (0,\infty)$, and define this bound $B(T)$.

To see that I_1 is bounded rewrite the inner integral as follows:

$$\int_{|x-y|=t-\tau} |U|^p = \int_{|x-y|=t-\tau} |U|^{p-1}|U| \le \|U(\tau)\|_\infty \int_{|x-y|=t-\tau} |U|^{p-1} \ .$$

Use Hölder's inequality on this last integral to obtain

$$\int_{|x-y|=t-\tau} |U|^p \le \|U(\tau)\|_\infty \int_{|x-y|=t-\tau} 1\cdot|U|^{p-1}$$

$$\le \|U(\tau)\|_\infty \Big(\int_{|x-y|=t-\tau} 1^{\frac{p+1}{2}}\Big)^{\frac{2}{p+1}} \Big(\int_{|x-y|=t-\tau} |U|^{p-1\cdot\frac{p+1}{p-1}}\Big)^{\frac{p-1}{p+1}} \ .$$

Therefore after integrating the first integral and multiplying through by $(\frac{1}{t-\tau})$

$$I_1 \le C\int_0^t (t-\tau)^{2\cdot\frac{2}{p+1}-1}\,\|U(\tau)\|_\infty \Big(\int_{|x-y|=t-\tau} |U(\tau)|^{p+1} dS_y\Big)^{\frac{p-1}{p+1}}\ d\tau \ .$$

By the generalized Hölder's inequality (A5)

$$I_1 \le C\Big(\int_0^t (t-\tau)^{\frac{(3-p)}{p+1}\theta}\, d\tau\Big)^{\frac{1}{\theta}} \Big(\int_0^t \|U(\tau)\|_\infty^r\, d\tau\Big)^{\frac{1}{r}} \Big(\int_0^t \int_{|x-y|=t-\tau} |U(\tau)|^{p+1}\, dS_y\, d\tau\Big)^{\frac{p-1}{p+1}},$$

where $\frac{1}{\theta} + \frac{1}{r} + \frac{p-1}{p+1} = 1$, $\theta, r > 1$.

Simplify and choose θ and r such that

$$(20)\quad \frac{1}{\theta} + \frac{1}{r} = \frac{2}{p+1},$$

which is possible, provided

$$\theta > \frac{p+1}{2}.$$

For $p \le 3$, choose any $\theta > \frac{p+1}{2}$. After θ has been chosen define r so that (20) holds. Note that $r > 1$ and $\theta < \infty$ from the above.

Recall the last upper bound on I_1.

$$I_1 \le C\Big(\int_0^t (t-\tau)^{\frac{(3-p)}{p+1}\theta}\, d\tau\Big)^{\frac{1}{\theta}} \Big(\int_0^t \|U(\tau)\|_\infty^r\, d\tau\Big)^{\frac{1}{r}} \Big(\int_0^t \int_{|x-y|=t-\tau} |U(\tau)|^{p+1} dS_y\, d\tau\Big)^{\frac{p-1}{p+1}}.$$

Integrate the first integral and note that for $1 < p \le 3$ this power of t is positive and use the cone estimate (lemma 1) on the last integral to obtain

$$I_1 \le Ct^{(\frac{3-p}{p+1}\theta+1)} \Big(\int_0^t \|U(\tau)\|_\infty^r\, d\tau\Big)^{\frac{1}{r}}.$$

Recall that $|U(x,t)| \le |W(x,t)| + I_1$. Take the supremum over x and use the estimates above to conclude that for $0 \le t < T$,

$$\|U(t)\|_\infty \le B(T) + C(T)\Big(\int_0^t \|U(\tau)\|_\infty^r\, d\tau\Big)^{\frac{1}{r}},$$

where $B(T)$ is the bound found in lemma 3.

By the generalization of Gronwall's Lemma (A6), the above implies

$$\|U(t)\|_\infty \le C(T).$$

Hence $\sup_{0<t\le T} \|U(t)\|_\infty \le C(T)$. This is true for all $0 \le T < \infty$.

Global existence is then proved.

2.6 Nonexistence of Global Solutions in $\mathbb{R}^3$

Consider the following Cauchy problem

$$(21)\quad \begin{cases} U_{tt} - \Delta U = |U|^p & x \in \mathbb{R}^3 \quad t > 0 \\ U(x,0) = \phi(x) & \\ U_t(x,0) = \psi(x) & \operatorname{supp}_x \phi,\psi \subset \{x\colon |x| < k\}, \end{cases}$$

$\phi(x) \leq C^3(\mathbb{R}^3)$, $\psi(x) \in C^2(\mathbb{R}^3)$.

Theorem 10. If $T > 0$ and $1 < p < 1 + \sqrt{2}$ and $U(x,t)$ is a smooth solution of (21) on $\mathbb{R}^3 \times (0,T)$, then $T < \infty$, given $\phi(x) \in C^3(\mathbb{R}^3)$ and $\psi(x) \in C^2(\mathbb{R}^3)$.

Note: The following proof combines similar proofs by Kato, Glassey and Sideris [17].

Proof. Suppose $U(x,t)$ is a smooth solution to (21) on $\mathbb{R}^3 \times (0,T)$. Integrate the PDE above with respect to the spatial variables to obtain

$$(22)\quad \frac{d^2}{dt^2}\int_{\mathbb{R}^3} U(x,t)\,dx - \int_{\mathbb{R}^3} \Delta U(x,t)\,dx = \int_{\mathbb{R}^3} |U(x,t)|^p\,dx .$$

Due to the compact support features of $U(x,t)$

$$\int_{\mathbb{R}^3} \Delta U(x,t)\,dx = \int_{|x|\leq k+t} \Delta U(x,t)\,dx .$$

By the divergence theorem this last integral

$$\int_{|x|\leq k+t} \Delta U(x,t)\,dx = \int_{|x|=k+t} \nabla U(x,t)\cdot n\, \partial S_x ,$$

where n is the unit outer normal.

The compact support features of $U(x,t)$ implies that $\nabla U(x,t)$ is zero on $\{|x| = k+t\}$; therefore

$$\int_{\mathbb{R}^3} \Delta U(x,t)\,dx = 0 .$$

Equation (22) simplifies to

(23) $$\frac{d^2}{dt^2} \int_{\mathbb{R}^3} U(x,t)\,dx = \int_{\mathbb{R}^3} |U(x,t)|^p dx \ .$$

In order to see that $\int_{\mathbb{R}^3} U(x,t)\,dx$ becomes unbounded in finite time, define

$$F(t) = \int_{\mathbb{R}^3} U(x,t)\,dx \ .$$

Then (23) says

$$\ddot{F}(t) = \int_{\mathbb{R}^3} |U(x,t)|^p dx \ .$$

Proceed, with the use of ordinary differential inequalities to obtain a lower bound, Kato's Lower Bound, for $\ddot{F}(t)$.

Note that

(24) $$|F(t)|^p = \left| \int_{\mathbb{R}^3} U \ dx \right|^p \ .$$

Use the compact support of U to write (24) as

$$|F(t)|^p = \left| \int_{|x| \le k+t} U(x,t)\,dx \right|^p \ .$$

Now apply Hölder's inequality to obtain

$$|F(t)|^p \le \left| \left(\int_{|x| \le k+t} 1^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{|x| \le k+t} U^p \right)^{\frac{1}{p}} \right|^p$$

$$= \left(\int_{|x| \le k+t} 1 \right)^{\frac{p-1}{1} \cdot p} \left(\int_{|x| < k+t} |U|^p \right) \ .$$

The first integral is the volume of a ball in $\mathbb{R}^3$ with radius $k + t$ which implies that

$$|F(t)|^p \le \tfrac{4}{3}\pi (k + t)^{3(p-1)} \left(\int_{|x| < k+t} |U|^p \right) \ .$$

But $\int_{|x| \le k+t} |U|^p = \int_{\mathbb{R}^3} |U|^p = \ddot{F}(t)$. Substitute this into the last inequality to obtain

$$|F(t)|^p \le C(k + t)^{3(p-1)} \ddot{F}(t) \ .$$

Solve for $\ddot{F}(t)$ to obtain

$$(25) \qquad \ddot{F}(t) \geq C(k+t)^{-3(p-1)} |F(t)|^p \ .$$

Now continue with a better lower bound on $\ddot{F}(t)$ due to Glassey. As seen previously the IVP (21) satisfies the following integral equation

$$(26) \quad U(x,t) = W(x,t) + \int_0^t \frac{1}{4\pi(t-\tau)} \int_{|x-y|=t-\tau} |U(s,\tau)|^p dS_y d\tau \ ,$$

where $W(x,t)$ is the solution to the homogeneous wave equation in 3 space. Notice that the integral in (26) is positive, implying

$$U(x,t) \geq W(x,t) \ .$$

Further, since $W_{tt} - \Delta W = 0$, integrate with respect to the spatial variable and use the divergence test to obtain

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^3} W(x,t)\,dx = 0 \ .$$

Integration with respect to t yields

$$(27) \quad \int_{\mathbb{R}^3} \frac{d}{dt} W(x,t)\,dx = C \ .$$

Use the initial conditions to obtain

$$\int_{\mathbb{R}^3} \frac{d}{dt}(W(x,0))\,dx = \int_{\mathbb{R}^3} \psi(x)\,dx \equiv C_\psi \ .$$

Equation (27) now becomes

$$\int_{\mathbb{R}^3} \frac{d}{dt}(W(x,t))\,dx = C_\psi \ .$$

Integrate this with respect to t to get

$$(28) \quad \int_{\mathbb{R}^3} W(x,t)\,dx = C_\psi t + C \ .$$

Again use the initial conditions

$$\int_{\mathbb{R}^3} W(x,0)\,dx = \int_{\mathbb{R}^3} \phi(x)\,dx \equiv C_\phi \ .$$

Equation (28) can be written as

$$\int_{\mathbb{R}^3} W(x,t)\,dx = C_\psi t + C_\phi \ .$$

Since $W(x,t)$ is the solution to the homogeneous wave equation in

3 space, the strong Huygens' principle can be applied here, namely

$$\operatorname{supp}_x W(x,t) \subset \{t-k < |x| < t+k\},\ t > k .$$

Thus the last equation can be simplified to

$$(29)\quad C_\psi t + C_\phi = \int_{\mathbb{R}^3} W(x,t)\,dx = \int_{\{t-k<|x|<t+k\}} W(x,t)\,dx .$$

The goal continues to find an improved lower bound for $\ddot{F}(t) = \int_{\mathbb{R}^3} |U(x,t)|^p dx$. Recall that $U(x,t) \geq W(x,t)$, therefore from (29)

$$C_\psi t + C_\phi \leq \int_{\{t-k<|x|<t+k\}} U(x,t)\,dx = \int_{\{t-k<|x|<t+k\}} 1 \cdot U(x,t)\,dx .$$

Apply Hölder's inequality to obtain

$$C_\psi t + C_\phi \leq \Big(\int_{\{t-k<|x|<t+k\}} 1\Big)^{\frac{p-1}{p}} \Big(\int_{\{t-k<|x|<t+k\}} |U(x,t)|^p\Big)^{1/p}$$

$$(30)\quad \leq \Big(\int_{\{t-k<|x|<t+k\}} 1\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^3} |U(x,t)|^p\Big)^{1/p} .$$

The above inequality contains

$$\Big(\int_{\mathbb{R}^3} |U(x,t)|^p\,dx\Big)^{1/p} = (\ddot{F}(t))^{1/p},$$

also $\Big(\int_{\{t-k<|x|<t+k\}} 1\Big)^{\frac{p-1}{p}} \leq C(t+k)^{\frac{2(p-1)}{p}}$. Inequality (20) becomes

$$C_\psi t + C_\phi \leq C(t+k)^{\frac{2(p-1)}{p}} (\ddot{F}(t))^{1/p} .$$

Continue and solve for $\ddot{F}(t)$ to obtain

$$(C_\psi t + C_\phi)^p \leq C(t+k)^{2(p-1)} \ddot{F}(t),$$

$$\frac{(C_\psi t + C_\phi)^p}{C(t+k)^{2(p-1)}} \leq \ddot{F}(t) .$$

Since $C_\phi > 0$ and only large values of t are of interest, conclude that

$$\ddot{F}(t) \geq Ct^{2-p} \qquad C > 0, \text{ for all large } t.$$

Integrate twice to obtain

$$F(t) \geq Ct^{4-p} \qquad C > 0 .$$

Rewrite to look like Kato's Lower Bound (25)

$$F(t) \geq Ct^{4-p} = [C^{\frac{1}{4-p}} t]^{4-p}$$
$$= [C^{\frac{1}{4-p}}(\tfrac{1}{2}t + \tfrac{1}{2}t)]^{4-p} .$$

For all $t \geq k$ this last expression

$$\geq [C^{\frac{1}{4-p}}(\tfrac{1}{2}k + \tfrac{1}{2}t)]^{4-p} = [\tfrac{1}{2}C^{\frac{1}{4-p}}(k+t)]^{4-p} .$$

Therefore

$$F(t) \geq C_0(k+t)^{4-p} \quad \text{where} \quad C_0 \leq (\tfrac{1}{2}C^{\frac{1}{4-p}})^{4-p} .$$

Now use these two inequalities found by Kato and Glassey, respectively.

$$(31) \quad \ddot{F}(t) \geq C(k+t)^{-3(p-1)}|F(t)|^p \quad \text{and}$$

$$(32) \quad F(t) \geq C_0(k+t)^{4-p} \qquad k \leq t,$$

to prove that $T < \infty$ given $1 < p < 1 + \sqrt{2}$.

Combine (31) and (32) to get

$$(33) \quad \ddot{F}(t) \geq C_2(k+t)^{-3(p-1)+(4-p)p} .$$

Integrate (33) over $[a,t]$ to obtain

$$\dot{F}(t) \geq \dot{F}(a) + C_2 \int_a^t (k+S)^{-p^2+p+3} dS .$$

Note that if $-p^2 + p + 3 > -1$, then $\dot{F}(t) \geq \dot{F}(a) + C_2\int_a^t (k+s)^{-1} dS$ $= \dot{F}(a) + C_2[\ln(k+t) - \ln(k+a)]$. As $t \to \infty$ the RHS of the inequality gets arbitrarily large implying $\exists\, a_0$ such that $a \leq a_0 < T$ and $\dot{F}(a_0) > 0$. If no such a_0 exists, then T must be finite, thus a contradiction to the existence of a_0 is sought. Therefore restrict $-p^2 + p + 3 > -1$, which is true for

$1 < p < \frac{1+\sqrt{17}}{2}$. Note that (33) implies that $\ddot{F}(t) \geq 0$ on $[0,T]$, therefore $\dot{F}(t)$ must be a non-decreasing function, in particular

(34) $\dot{F}(t) \geq \dot{F}(a_0) > 0 \quad \forall\, t \geq a_0$.

Let $\theta \in (0,1)$ such that

(35) $\frac{1}{p} < \theta < 1 - \frac{3(p-1)-2}{p(4-p)}$,

which is true for $1 < p < 1 + \sqrt{2}$.

Use this information and interpolate between (21) and (22) to obtain

(36) $\ddot{F}(t) \geq C(k+t)^{(1-\theta)(-p^2+p+3)}F(t)^{\theta p}$.

To facilitate the following operations let $\alpha = \theta p$ and $\beta = (\theta - 1)(-p^2+p+3)$. By the inequality (35) $\alpha > 1$ and since $p \leq 1+\sqrt{2}$, $\beta < 2$. The remainder of the proof holds for both positive and negative values of $\beta < 2$. To simplify matters assume $0 \leq \beta < 2$.

Since $\dot{F}(t) > 0$ ((30)), multiply (36) by $\dot{F}(t)$ to obtain

$$\dot{F}(t)\ddot{F}(t) \geq C(k+t)^{-\beta}F(t)^{\alpha}\cdot\dot{F}(t) .$$

Integrate both sides from a_0 to t

$$\int_{a_0}^{t} \dot{F}(s)\ddot{F}(s)ds \geq C \int_{a_0}^{t} (k+s)^{-\beta}F(s)^{\alpha}\cdot\dot{F}(s)ds .$$

This is equivalent to

$$\frac{1}{2}[\dot{F}^2(t) - \dot{F}^2(a_0)] \geq C \int_{a_0}^{t} (k+s)^{-\beta}F(s)^{\alpha}\cdot\dot{F}(s)ds ,$$

$$\geq C(k+t)^{-\beta} \int_{a_0}^{t} F(s)^{\alpha}\cdot\dot{F}(s)ds .$$

Now integrate the right-hand side

(37) $\frac{1}{2}[\dot{F}^2(t) - \dot{F}^2(a_0)] \geq \frac{C}{1+\alpha}(k+t)^{-\beta}[F(t)^{1+\alpha} - F(a_0)^{1+\alpha}]$.

The proof will continue by solving for $\dot{F}(t)$ to obtain a contradiction on its positivity, hence a contradiction on the existence of an a_0. Choose C small enough but still positive so that

(38) $\frac{1}{2}\dot{F}^2(a_0) \geq C(k+a_0)^{-\beta}F(a_0)^{1+\alpha}$ in order to ensure positivity.

Solve for $\dot{F}^2(t)$ from inequality (37) to obtain

$$\dot{F}^2(t) \geq C(k+t)^{-\beta}F(t)^{1+\alpha} + C^1(t),$$

where $C^1(t) = -C(k+t)^{-\beta}F(a_0)^{1+\alpha} + \frac{1}{2}\dot{F}^2(a_0)$ which is greater than zero by (38).

Therefore

$$\dot{F}^2(t) \geq C(k+t)^{-\beta}F(t)^{1+\alpha} .$$

Take the square root of both sides to yield

$$\dot{F}(t) \geq C(k+t)^{-p/2}F(t)^{1+\alpha/2} \text{ for } t \in [a_0,T).$$

Integrate a final time from a_0 to t

$$\int_{a_0}^{t} \dot{F}(s)F(s)^{-(\frac{1+\alpha}{2})}ds \geq C\int_{a_0}^{t}(k+s)^{-\beta/2}ds$$

$$\frac{2}{1-a}[F(t)^{\frac{1-\alpha}{2}} - F(a_0)^{\frac{1-\alpha}{2}}] \geq C[(k+t)^{1-\frac{\beta}{2}} - (k+a_0)^{1-\frac{\beta}{2}}] .$$

Recall that $\alpha > 1$, therefore $\frac{2}{1-\alpha} < 0$, call it $-C_1$, so

$$-C_1[F(t)^{\frac{1-\alpha}{2}} - F(a_2)^{\frac{1-\alpha}{2}}] \geq C[(k+t)^{1-\beta/2} - (k+a_0)^{1-\beta/2}] .$$

Group the constants and distribute the -1 to obtain

$$F(a_0)^{\frac{1-\alpha}{2}} - F(t)^{\frac{1-\alpha}{2}} \geq C[(k+t)^{1-\beta/2} - (k+a_0)^{1-\beta/2}$$

Since $\beta < 2$, this implies $1 - \frac{\beta}{2} > 0$ and the RHS of the inequality is positive, but since $t \in [a_0,T)$ the LHS is less than or equal to zero. This is the desired contradiction. Therefore an a_0 cannot exist, implying $T < \infty$, for $1 < p < 1+\sqrt{2}$.

2.7 Conclusion

Local existence, global existence and global non-existence (blow-up) for non-linear wave equations has been presented. Although only the cases $n = 1$ and $n = 3$ have been addressed, similar results exist for the case $n = 2$. The bounds associated with the corresponding theorems are much more tedious to produce, hence the proofs are somewhat more difficult. This should not dissuade the reader from considering these theorems.

Although much progress has been made in the theory of non-linear partial differential equations in the twentieth century, there is still much work to be done. Other topics being considered today are decay theorems and scattering theory. Decay theorems are concerned with the value of some spacial norm placed on the solution of a non-linear wave equation as t approaches plus infinity, while scattering theory compares the global solution of a non-linear wave equation to solutions of a linear wave equation with certain data for large positive and negative times. There are many unsolved problems in the theory of non-linear wave equations which will undoubtedly provide rich areas of research in the twenty-first century.

APPENDIX

Orthogonality

(A1) $\int_0^l \text{SIN } mx \text{ SIN } nx dx = 0$ when $m \neq n$ and SIN $ml = 0$ and SIN $nl = 0$.

Proof

The eigenfunctions $U_m(x) = \text{SIN } mx$ and $U_n(x) = \text{SIN } nx$, satisfy the equations

$$U''_m = -m^2 U_m \quad \text{and} \quad U''_n = -n^2 U_n.$$

If the first equation is multiplied by U_n and the second by U_m, then the difference of the resulting equations is

$$U_n U''_m - U_m U''_n = (n^2 - m^2) U_m U_n$$

or

$$(U_n U'_m - U_m U'_n)' = (n^2 - m^2) U_m U_n.$$

Integrate both sides of this last equation from 0 to l and use the fact that $U_m(x)$ and $U_n(x)$ both vanish at 0 and l, to obtain

$$(n^2 - m^2) \int_0^l U_m(x) U_n(x) dx = [U_n(x) U'_m(x) - U_m(x) U'_n(x)]_0^l = 0.$$

Therefore $\int_0^l \text{SIN } mx \text{ SIN } nx dx = 0$ when $m \neq n$.

(A2) (Boundedness $\Longrightarrow$ Existence) [3] Theorem

Suppose that f and $\frac{\partial f}{\partial y_J}$ $(J=1,\ldots,n)$ are continuous in a given region D and suppose f is bounded on D. Let (t_0, η) be a given point of D. Then the unique solution ϕ of the system $Y' = f(t,y)$ passing through the point (t_0, η) can be extended until its graph meets the boundary of D.

Corollary. If D is the entire (t,y) space and if f and $\frac{\partial f}{\partial y_J}$ $(J=1,\ldots,n)$ are continuous on D, then the solution ϕ of $Y' = f(t,y)$ can be continued uniquely in both directions for as long as $|\phi(t)|$ remains finite.

Sobolev Inequalities

(A3) If $n = 2$ and $2 < q < \infty$, then $\|f\|_q \leq C\,\|\nabla f\|_2$.

If $n \geq 3$, then $\|f\|_{\frac{2n}{n-2}} \leq C\,\|\nabla f\|_2$.

Interpolation (Follows from Hölder's inequality)

(A4) If $1 \leq \alpha \leq \xi \leq \beta \leq \infty$, then

$$\|f\|_\xi \leq \|f\|_\alpha^\theta\,\|f\|_\beta^{1-\theta} \quad \text{where } \frac{1}{\xi} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}.$$

Generalized Hölder Inequality

(A5) If $1 < \alpha_k < \infty$ for $k = 1,2,\ldots,K$, then

$$\int f_1 f_2 \cdots f_k \leq \left(\int f_1^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\left(\int f_2^{\alpha_2}\right)^{\frac{1}{\alpha_2}}\cdots\left(\int f_K^{\alpha_K}\right)^{\frac{1}{\alpha_K}}$$

provided $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_K} = 1$.

Generalization of Gronwall's Lemma

(A6) If $\alpha \in R$, $\beta(t) \geq 0$, and both $\beta(t)$, $F(t)$ are continuous real functions on $t_0 \leq t \leq t_1$, which satisfy

$$F(t) \leq \alpha + \left\{\int_{t_0}^{t} \beta(\tau)[F(\tau)]^\gamma d\tau\right\}^{\frac{1}{\gamma}} \text{ for } t_0 \leq t \leq t_1,$$

then

$$F(t) \leq 2\alpha \exp\left[\frac{2^\gamma}{\gamma}\int_{t_0}^{t} \beta(\tau)d\tau\right].$$

BIBLIOGRAPHY

[1] Apostol, T.M. Calculus II. Blaisdell, Waltham, 1960.

[2] Barab, J. "Three Problems in the Theory of Nonlinear Wave Equations." Ph.D. thesis, Indiana University, 1982.

[3] Berg, P. and J. McGregor. Elementary Partial Differential Equations. Holden-Day, San Francisco, 1966.

[4] Bers, L. et al. Partial Differential Equations. Interscience Publishers, New York, 1964.

[5] Boyce, W. and R. DiPrima. Elementary Differential Equations and Boundary Value Problems. John Wiley & Sons, New York, 1977.

[6] Brauer, F. and J. Nobel. The Qualitative Theory of Ordinary Differential Equations An Introduction. W.A. Benjamin, Inc., New York, 1969.

[7] Braun, M. Differential Equations and Their Applications. Springer-Verlag, New York, 1975.

[8] Courant, R. and D. Hilbert. Methods of Mathematical Physics, II. Interscience, New York, 1962.

[9] Friedman, A. Partial Differential Equations. Krieger, Florida, 1969.

[10] Glassey, R.T. "An Elementary Proof of Time Decay for Solutions to Semi-linear Wave Equations in Low Space Dimension," LCDS Report #82-24, 1982.

[11] John, F. Partial Differential Equations. Springer-Verlag, New York, 1975.

[12] Kline, M. Mathematical Thought from Ancient to Modern Times. Oxford University Press, New York, 1972.

[13] Liusternik, L. and V. Sobolev. Elements of Functional Analysis. Ungar Publishing Company, New York, 1961.

[14] Powers, D. Boundary Value Problems. Academic Press, New York, 1979.

[15] Reed, M. "Abstract Non-Linear Wave Equations," Lecture Notes in Mathematics, 507. Springer-Verlag, New York, 1976.

[16] Royden, H.L. Real Analysis. Macmillan, New York, 1968.

[17] Sideris, T. "Global Behavior of Solutions to Nonlinear Wave Equations." Ph.D. thesis, Indiana University, 1981.

[18] Simmons, G. Differential Equations with Applications and Historical Notes. McGraw-Hill, New York, 1972.

[19] Stromberg, K. An Introduction to Classical Real Analysis. Wadsworth, California, 1981.

[20] Williamson, R. et al. Calculus of Vector Functions. Prentice-Hall, New Jersey, 1972.

[21] Young, E. Partial Differential Equations An Introduction. Allyn and Bacon, Boston, 1972.

[22] Zachmanoglou, E.C. and D. Thoe. Introduction to Partial Differential Equations. Williams & Wilkins, Baltimore, 1976.

EXISTENCE, UNIQUENESS AND BLOW-UP RESULTS
FOR NON-LINEAR WAVE EQUATIONS

by

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B. A., Assumption College, 1983

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1985

ABSTRACT

During the twentieth century, there has been a great deal of interest in non-linear wave equations primarily in applied physics and quantum field theory. Like most non-linear problems, these equations must to some extent be treated individually because each equation has its own special properties. As will be shown in chapter two, proofs of existence and properties of solutions often seem to depend on special properties of the particular equation studied. With using standard tools of linear functional analysis and the contraction mapping principle, a unified approach to these non-linear equations can be provided.

Chapter one introduces classical solutions to linear wave equations in one and three space dimensions. These solutions will play an important part in the theorems of chapter two. Section 1.5 presents a fundamental mathematical property of solutions of the wave equation which corresponds to a distinguishing feature of the physical phenomena described by the wave equation. Section 1.7 presents a topic crucial to the theory of partial differential equations, the conservation of energy. This will play a major role in the theorems in chapter two as well as play a key role in the proof of uniqueness for solutions of linear wave equations (1.8). Chapter one concludes with a discussion of Huygens' principle which will play an important part in the proof of Theorem 10 of section 2.6.

Chapter two presents results dealing with the local existence, global existence and global non-existence (blow-up) of solutions to non-linear wave equations in one and three space dimensions. To prove local existence of solutions the contraction mapping theorem will be used in both section 2.3 and 2.5. The example of

non-existence of global solutions in $\mathbb{R}^1$(2.4) is due to H. Levine. Jörgen's cone estimate will play a crucial role in the proof global existence of solutions in $\mathbb{R}^3$(2.5). Work by Kato and Glassey is included in the proof of non-existence of global solutions in $\mathbb{R}^3$.

Although much progress has been made in the theory of non-linear partial differential equations in the twentieth century, there is still much work to be done. Other topics being considered today, other than the question of existence/non-existence, are decay theorems and scattering theory. There are many unsolved problems in the theory of non-linear wave equations which will undoubtedly provide rich areas of research in the twenty-first century.