ESTIMATION OF AN UPPER BOUND FOR EXPECTED MAINTENANCE COST OF A SYSTEM WITH PARTIALLY KNOWN, INCREASING FAILURE RATE DISTRIBUTION

by

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CHAPTER 1

INTRODUCTION

For nearly two decades, there has been a large and continuing interest in the study of maintenance models for items with stochastic failure. This interest has its roots in many military and industrial applications. Lately, however, new applications have arisen in such areas as health, ecology, and the environment. Although it is not possible to detail these many applications of maintenance models, some of them are: the maintenance of complex electronic and/or mechanical equipment, maintenance of human body, inspection and control of pollutants in the environment, and maintenance of ecological balance in populations of plants and animals. This interest has been provoked by the high cost and extraordinary demands made of modern equipment like jetliners, electronic computers, ballistic missiles, etc. Operational requirements can be achieved only by observing relatively sophisticated maintenance policies.

The practical need for subtle and delicate maintenance policies has stimulated theoretical interest and in many cases has led to the development of policies that posses theoretical novelty and practical importance. The maintenance models can be basically divided into two types i.e, the preventive and preparedness maintenance policies. The distinctive feature of preparedness maintenance is that the state of the system (e.g. operative or failed) is at all times known during its service as opposed to the preparedness maintenance for which the state of the system is only known through inspection or checking. The preventive maintenance policies are justified when the cost of unscheduled maintenance action is higher than the cost of scheduled maintenance. The name of preparedness has been given
to its policy since in some cases there are standby emergency systems which are put into operation when the main system suddenly fails. These standby systems should be checked for their preparedness; but not all of the applications of preparedness maintenance modelling lies within emergency systems. There has been already a lot of work done in both preventive maintenance [1,2,4,5,10,12,19,23,28,31] and preparedness maintenance [1,3,8,11,14,18,23,28,29,32] areas. The basic criterion is to minimize the total cost of maintenance by optimum scheduling of the maintenance actions such as repair, inspection or replacement.

The optimization objective for preparedness inspection models is to optimally balance the cost of undetected system failure over the inspection cost.

Assuming:
\[ f(t) = \text{The density function of the time to failure of the equipment;} \]
\[ I = \text{The cost of an inspection except at time } t_0 \text{ where } I_0 = 0; \]
\[ a = \text{The cost per unit time associated with an undetected failed system;} \]

and that the inspections to be performed at times \( t_1, t_2, t_3, \ldots \), until the system failure is detected (See Fig 1.), then, if failure occurs between time \( t_0 \) and \( t_1 \), say at \( X_1 \), the cost of the cycle of operation (See Fig. 1.) would be:

\[ I(1) + a(t_1 - X_1) \]  \hspace{1cm} (1.1)

and the expected value of this cost in the interval \([t_0, t_1]\) is:

\[ t_1 \]
\[ \int_{t_0}^{t_1} [I(1) + a(t_1 - x)] f(x)dx . \]  \hspace{1cm} (1.2)

If failure occurs between \( t_1 \) and \( t_2 \), say at time \( X_2 \), the cost of the
Inspection starts at
Time $t_0 = 0$

Failure occurs at $t_n-1$
Final inspection at $t_n$

One Cycle

Fig. 1. The Illustration of One Time Span or One Cycle.
cycle in the interval \([t_1, t_2]\) would be:

\[ I(1+1) + a(t_2 - x_2) \]  

and the expected value of this cost would be:

\[
\int_{t_1}^{t_2} \left[ I(1+1) + a(t_2 - x) \right] f(x) \, dx .
\]  

In a manner similar to the above, the costs and probabilities of all possible cycles can be determined to give the expected cost per cycle as:

Expected cost per cycle

\[
\begin{align*}
&= \int_0^{t_1} \left[ I(0+1) + a(t_1 - x) \right] f(x) \, dx \\
&+ \int_{t_1}^{t_2} \left[ I(1+1) + a(t_2 - x) \right] f(x) \, dx \\
&+ \int_{t_2}^{t_3} \left[ I(2+1) + a(t_3 - x) \right] f(x) \, dx \\
&+ \text{ etc.}
\end{align*}
\]  

Therefore if

\[ L(t_1, t_2, t_3, ...) = \text{Expected Cost Per Cycle} \]

then Eq. (1.5) can be written as

\[
L(t_1, t_2, t_3, ...) = \sum_{k=0}^{k=n-1} \int_{t_k}^{t_{k+1}} \left[ I(k+1) + a(t_{k+1} - x) \right] f(x) \, dx .
\]  

The objective function to be minimized is \( L(t_1, t_2, t_3, ...) \) if the failed system is replaced, renewed, repaired etc. at the end of its fixed
maximum life time. If the system is replaced or renewed or repaired when failed, then, the objective function to be minimized would be expected cost per unit time which is:

\[
\text{Expected Cost Per Unit Time} = \frac{\text{Expected Cost Per Cycle}}{\text{Expected Cycle Length}}. \quad (1.7)
\]

The optimum solution gives the optimum number of inspections \( n \), and optimum timing of inspections, i.e., \( t_1, t_2, t_3, \ldots, t_n \). Most of the literature [6, 17, 20] is about the optimum solution which minimizes the total maintenance cost per unit time when the failure distribution of the system is completely known in advance. But there are many cases where the failure distribution of the system is completely unknown or is partially known. In these cases different methods have been devised depending on the kind and amount of information which is available about the failure characteristics of the system [26].

Minimax policies have been devised to cope with the situation in which the decision maker has virtually no information about the failure distribution. This method minimizes the maximum possible loss or maintenance cost that can occur due to the failure characteristics by designating the optimum number of inspection and optimum timing of them. If \( P = \{t_1, t_2, t_3, \ldots, t_n\} \) is the space of the inspection times \( t_1, t_2, t_3, \ldots, t_n \), and if \( F \) is any failure distribution within the constraints of the problem related to the failure characteristics of the system, then, the minimax policy can be formulated as

\[
L^{**}(P^{**},F^{**}) = \min_P \max_F L(P,F), \quad (1.8)
\]

where \( L(P,F) \) is the expected maintenance cost per cycle as given by Eq. (1.6). \( L^{**}, P^{**} \) and \( F^{**} \) are the minimax expected maintenance cost, the optimum number and timing space of inspections and the failure distribution which gives the
maximum possible rise to the total maintenance cost $L$ respectively. The minimax policy is not limited to preparedness or inspection models and has been implemented in the area of preventive maintenance by Barlow [4]. But the major work in minimax policy has been done for the preparedness and inspection policies. In minimax policies applied to inspection models the time horizon $T$ is assumed to be finite; i.e., the cost accounting stops at either the first inspection to detect failure or at time $T$, whichever happens first. The reason for this is that for any possible inspection schedule there exists a distribution which would induce an arbitrarily high expected cost during an infinite time horizon. Hence, a minimax solution would not then exist. The finite horizon assumption can be found to have many implications. As an example, consider the problem of detecting the occurrence of an event (say, the arrival of an enemy missile or the presence of some grave illness such as cancer) when the time of occurrence is not known in advance. Each inspection involves a cost so that we do not wish to check too often. On the other hand, there is a penalty cost associated with the lapsed time between occurrence and its detection so that we wish to check often enough to avoid a long lapse of time between failure and its detection.

Derman [9] has found analytically the optimum number of inspections and timing, applying the minimax policy, when the failure distribution is completely unknown, to the loss function, given by Eq. (1.6), under the following assumptions.

1. The maximum life or service time of the system $T$ is limited and known, i.e., until time $T$ the system has either failed or the system
will be put out of service or repaired, renewed etc. at the end of fixed periods of length $T$ equal to the maximum life time of the system.

2. The failure can only be detected by inspection with a certain probability $P'(P' > 0)$ and the inspection does not affect the failure characteristics.

3. The inspection time is negligible.

4. Each check entails a cost $I$.

5. The time elapsed between system failure and its discovery at the next check has a cost per unit of time $a$.

In many cases there are estimable costs associated with the resumption in service (storage) of a unit which has failed. If the unit is a production system the costs are associated with the amount of defective product produced, if it is material in storage (e.g., certain kind of missile fuels) the costs are derived from considering the various implications of using, unknowingly, the unserviceable material; ... and the like.

Derman [9], proved that under the above assumptions the minimax schedule is given by

$$t_i = iP' \left( \frac{T}{nP'+1} + \frac{I}{2a} \left( \frac{n[(n+1)P'+2]}{nP'+1} - (i+1) \right) \right)$$

$$i = 0, 1, \ldots, n,$$

where $n$, the number of inspections, is the largest integer such that

$$IP' + 2n^2 + IP'(2-P') n+2(I-P'aT) < 0.$$  \hspace{1cm} (1.10)

The minimax expected cost $L^{**}$, when $P'=1$ is given by

$$L^{**} = \frac{aT}{n+1} + \frac{I}{2} \frac{n(n+3)}{n+1},$$  \hspace{1cm} (1.11)
and $t_i$ is the time of $i^{th}$ inspection. Roeloffs [30], obtained analytically the minimax schedule for the Derman's Problem [9] with the further assumption that the location $x'$, of the $100.P^{th}$ percentile of the otherwise unknown cumulative failure distribution function $F$, of the system is known. That is,

$$F(t=x') = p \quad x' \geq 0; \quad 0 < p < 1.$$ (1.12)

A surveillance or inspection schedule $t$, is a set of $m+n$ points, such that

$$0 \leq t_1 \leq \ldots \leq t_m \leq x' \leq t_{m+1} \leq \ldots \leq t_{m+n} \leq T.$$ (1.13)

The minimax inspection schedule and the minimax expected cost analytically obtained by Roeloffs [30] does not have a simple form as Derman's does and is more sophisticated and involved. The Roeloffs' solution is also under the assumption that the probability of the detection of failure $p'$, upon inspection is one. Roeloffs also showed that the added information about the location of a percentile of the failure distribution improves the minimax expected loss or maintenance cost function given by Eq. (1.6) and results in less maintenance cost compared to the solution given by Derman, i.e., Eqs. (1.9-11).

The implication of dynamic programming in the area of maintenance is well known to the researchers in this field. Hasting [13], Jardine [16, 17], Bellman [7] and many others have applied the dynamic programming to a variety of replacement-repair maintenance problems, but to the best of my knowledge, all of them except Kander [20,21], have assumed a known system failure distribution and in some cases even more specifically they have assumed a certain type of failure distribution. Kander solved several different kinds
of inspection scheduling problems by converting the minimization of the loss function, i.e., Eq. (1.6), into optimization of recurrence relationships [21].

Introduction of the implication of dynamic programming in minimax policy for the solution of the problem of optimum inspection frequency and timing has been accomplished by Kander [22], however, the idea and theory of minimax dynamic programming was well introduced and established by Bellman [7], the pioneer in this field and later repeated by Jacobs [15]. Kander implemented the dynamic programming methodology and combined it with minimax policy to obtain numerically the optimum solution to the Roeloffs' problem, discussed earlier, with a further assumption that the system failure distribution is IFR (Increasing Failure Rate). The detail of this method has been fully explained in Chapter 2 since the present work is highly based on the Kander's solution procedure; but for the time being it should be mentioned that Kander showed that his solution reduces the total maintenance cost compared to Roeloffs' solution [30].

The purpose of the present work has been directed toward obtaining an upper bound for the expected total cost of the system maintenance when optimum policy is applied to the actual failure distribution of the system and under Roeloffs' assumptions [30] with addition to:

A. The locations \( x_1 \) and \( x_2 \), of the 100.\( p_1 \) th and 100.\( p_2 \) th percentiles respectively of the otherwise unknown life or failure distribution of the system are known. That is

\[
F(t_1 = x_1) = p_1 \quad x_1 > 0; \quad 0 < p_1 < 1. \quad (1.14)
\]

\[
F(t_2 = x_2) = p_2 \quad x_2 > 0; \quad 0 < p_2 < 1. \quad (1.15)
\]
In order that an IFR distribution pass through these two points it is necessary for \( p_1, p_2, x_1 \) and \( x_2 \) to satisfy the following inequality

\[
\frac{\log (1-p_1)}{x_1} \geq \frac{\log \left( \frac{1-p_2}{1-p_1} \right)}{x_2 - x_1}
\]  

(1.16)

B. The inspection times can only occur at discrete points in time between time 0 to \( T \).

Kander did not consider the restriction on the inspection time imposed by assumption B. Assumption B, which reduces the computation time when utilized in the dynamic program, is a realistic assumption in many cases, i.e., in cases where the inspector is available for inspection only at certain times and not all the times. Assumption A, which is different from Kander's by the information about the location of two percentiles of the system failure distribution instead of one, is utilized to give the relative value of the added information and the reduction in estimation of the total inspection or maintenance cost of the system.

In Chapter 2, the properties of IFR distributions in which the basic formulation lies are presented. The loss function or Eq. (1.6), has been rearranged in an order to be suitable for dynamic programming formulation. The recurrence relationships, state and control variables, stages and logic of the problem have been formed and explained in detail.

In Chapter 3, the details of the computational procedure for the computer programming of the two models, i.e., with information about one point and two points of the system failure distribution are stated.

In Chapter 4, the convergence and accuracy of the solutions are being shown through an example and then the results are presented in the form of
tables and figures showing the sensitivity of the upper bound for optimum expected total system maintenance cost $L^{**}$, respect to one and two known locations of the percentiles of system failure distribution together with sensitivity of $L^{**}$ respect to the values of the elements of the information parameter vector which consists of elements like $a$, the cost of undetected failure, and $I$, the cost of every inspection. The value of information v.s. the location of the known percentiles of a IFR distribution has been illustrated by an example. Also an example about the application of the models is given.

Chapter 5, gives the conclusion derived from the outputs of the computer program and results of Chapter 4 plus the comparison of the upper bound expected total system maintenance cost, having information about one point of the failure distribution (Model A) and two point of the failure distribution (Model B). This will be followed by the possibilities for further research in this area.
CHAPTER 2

FORMULATION

2.1 INTRODUCTION

Maintenance policies for stochastically failing equipment have been calculated mainly for the case where the failure distribution is assumed to be known, even though in real life situations the failure characteristics of the equipment are mostly not known in advance and the only way for finding the failure characteristics and distribution is through study and tests which can be very costly. Before introduction of a new system or equipment, the management needs a precise or at least an estimate of the costs associated with the equipment or system so that it can choose the best alternative through a cost analysis. In the case where little is known about the new system an upper bound for the total cost is very helpful in decision making. It is clear that the more information about the system the lower the upper bound for costs will be. In this work, using a minimax method, dynamic programming has been implemented to find an upper bound for the optimum cost of the system.

Minimax policies were devised with the assumption of no knowledge or at most the knowledge about the location of one percentile of the system failure distribution.

Assuming the failure or the life time distribution of a system to be \( F_0 \) and given certain costs (e.g. inspection and undetected failure costs), an optimal maintenance policy \( P_0 \) can be calculated which leads to the minimum loss or total maintenance cost \( L^* \):
\[ L^* = L(P_0, F_0) = \min_P L(P, F_0) \leq L(P, F_0), \quad (2.1.1) \]

where \( P \) is any inspection policy.

The method devised for partial knowledge (as detailed above) assumes a distribution \( F^* \) while subsequently deriving an optimal policy \( P^* \) with minimal loss \( L^* \)

\[ L^* = L(P^*, F^*) = \min_P L(P, F), \quad (2.1.2) \]

In general \( L^* \) can be larger or smaller than \( L^* \), but the virtue of minimax policy is that \( L^* \) represents generally an upper bound.

For a minimax policy we have

\[ L(P, F^*) = \max_F L(P, F) \geq L(P, F), \quad (2.1.3) \]

and

\[ L^* = L(P^*, F^*) = \min_P L(P, F^*) = \min_P \max_F L(P, F). \quad (2.1.4) \]

If a saddle point exists then:

\[ L^* = \max_F L(P^*, F). \]

Now having \( L^* \) defined by Eq. (2.1.4) we can write

\[ L^* = \min_P \max_F L(P, F) \geq L(P_0, F_0) = L^* \quad (2.1.5) \]

The proof is as follows.

Proof: If (2.1.5) is not true then we should have \( L^* < L^* \). But according to (2.1.1) we have \( L^* \leq L(P_0, F_0) \) for any policy \( P \) including minimax policy \( P^* \) so that

\[ L^* < L^* \leq L(P^*, F_0). \quad (2.1.6) \]
On the other hand according to (2.1.3) we have \( L(P, F^**) \geq L(P, F) \) for any policy \( P \) and failure distribution \( F \) including \( P^* \) and \( F_0 \) respectively so that we can write \( L(P^*, F^**) \geq L(P^*, F_0) \). But \( L(P^*, F^**) \) according to 2.1.4 is the minimax cost \( L^* \) so we should have \( L^* \geq L(P^*, F_0) \). This contradicts Eq. (2.1.6) derived on the assumption that \( L^* < L^* \). So it is proved that indeed \( L^* > L^* \). That is the minimax loss or cost \( L^* \) is an upper bound for the optimum inspection policies \( P_0 \) applied to failure distributions \( F_0 \).

In this work improved upper bounds for the optimum total expected cost will be obtained by assuming the knowledge about the location of one and two percentiles of f.d. and also that failure distributions are IFR (Increasing Failure Rate), using dynamic programming methodology.

In this chapter, first the assumptions of the present study are stated, followed by the mathematical preparation, model presentation, formulation of the loss function and finally the dynamic programming formulation of the models.

2.2. THE PROBLEM STATEMENT

The objective is to find the upper bound cost for the optimum inspection policies for known failure distributions \( L^* \) under the following assumptions:

(a) The probability of the failure of the system at time \( T \) is one.
   This means that \( T \) is the maximum life time of the system and the inspection ends some time between first inspection at time \( O \) and time \( T \) upon the detection of failure.

(b) The period for renewal, replacement, replenishment etc. is fixed and is equal to \( T \).
A system failure is detected only through inspection, which costs \$x\$ dollars each. Inspection is carried also at times \(t = 0\) and \(t = T\).

The time elapsed between system failure and its detection at the next inspection costs \$a\$ dollars per unit time.

Inspection takes negligible time, the system cannot fail during an inspection and is not degraded by inspection.

The inspection can only be performed at certain discrete points in time.

The failure distribution of the system is IFR (Increasing Failure Rate).

The minimax solution gives the upper bound for optimum expected total maintenance cost per cycle \(L^{**}\) according to (2.1.5) where \(L\) is given by

\[
\text{Expected Maintenance Cost Per Cycle, } L = \sum_{i=0}^{n-1} F^*(t_i) + a \left( \sum_{i=0}^{n-1} F^*(t_i)(t_{i+1} - t_i) - \int_0^t F^*(t) \, dt \right), \tag{2.2.1}
\]

and \(n\) is the total number of inspections and \(F^*(t) = 1 - F(t)\) is the probability that the system has not failed until time \(t\).

The failure distribution \(F(t)\) is unknown except at the time \(T\) where \(F(T) = 1\) and also one of the following two additional informations depending on one of the two models A or B is given. That is

For Model A

The location \(x'\), of the \(100P\)th percentile of the otherwise unknown failure distribution of the system is known. That is

\[
F(t' = x') = p \quad x' > 0; \quad 0 < p < 1. \tag{2.2.2}
\]
For Model B

The locations $x_1^i$ and $x_2^i$, of the 100.$p_1$th and 100.$p_2$th percentiles respectively of the otherwise unknown failure distribution of the system are known. That is:

$$F(t_1^i = x_1^i) = p_1 \quad x_1^i \geq 0; \quad 0 < p_1 < 1, \quad (2.2.3)$$

and

$$F(t_2^i = x_2^i) = p_2 \quad x_2^i \geq 0; \quad 0 < p_2 < 1. \quad (2.2.4)$$

In order that IFR distributions pass through these two points, i.e., $(x_1^i, p_1)$ and $(x_2^i, p_2)$, it is necessary for $p_1, p_2, x_1^i$ and $x_2^i$ to satisfy the following inequality

$$\frac{\log (1-p_1)}{x_1^i} \leq - \frac{\log (1-p_2)}{x_2^i - x_1^i} \quad (2.2.5)$$

2.3 MATHEMATICAL PREPARATIONS

According to [6], a failure distribution $F(u), u > 0$ which is IFR, crosses the exponential distribution $1-e^{-\alpha u}, \alpha > 0$ from below at most once in addition to coincidence at $u = 0$ and $\infty$ unless they coincide identically. This can be easily seen, that is if $F$ is any IFR and the exponential distribution is $1-e^{-\alpha u}$ then the two functions agree at the origin (See Fig. 2) and equating the two functions

$$F(t) = 1-e^{-\alpha t} \quad (2.3.1)$$

and shifting $F(t)$ to the right and $e^{-\alpha t}$ to the left of the equal sign and calling $1-F(t)$ as $\bar{F}(t)$ and extracting the logarithms then:

$$G(t) = \log_e \bar{F}(t) - (-\alpha t). \quad (2.3.2)$$
Fig. 2. Exponential and Arbitrary IFR Distribution, $F(t)$

Crossing Exponential Distribution from Below
It is clear that $G(t)$ is zero at $t = 0$ since for $t = 0$, $F(t) = 1$ and $\log_e 1 = 0$, so $G(t) = 0 - 0 = 0$. According to the definition of failure rate we have

$$\text{Failure Rate} = r(u) = \frac{F(u + \Delta) - F(u)}{F(u)}$$  \hspace{1cm} (2.3.3)

where $\Delta > 0$ is an increment of time. The definition of an IFR distribution signifies that

$$r(u_2) \geq r(u_1), \quad u_2 \geq u_1.$$  \hspace{1cm} (2.3.4)

Applying Eq. (2.3.3) to exponential distribution $1 - e^{-at}$, we have

$$\text{Exponential F.R.} = r_e(t) = \frac{1 - e^{-a(t + \Delta)} - (1 - e^{-at})}{1 - (1 - e^{-at})} = 1 - e^{-a\Delta}$$

$$= \text{constant}, \hspace{1cm} (2.3.5)$$

and calling the failure rate of the arbitrary IFR distribution $F(t)$ as $r(t)$, then $F(t)$ crosses $1 - e^{-at}$ at most one more time from below (See Fig. 2.). $F(t)$ cannot cross $1 - e^{-at}$ from above since if that happens then

$$r(t^*_2) < r_e(t^*_1) = 1 - e^{-a\Delta} = \text{constant where } t^*_2 > 0 \text{ is the first crossing point after the origin.}$$

But the first crossing from above cannot happen unless $r(0) > r_e(0) = 1 - e^{-a\Delta} = \text{constant.}$ This means that $r(0) > 1 - e^{-a\Delta} > r(t^*_1)$ which violates the condition for $F(t)$ to be an IFR. As a result the first crossing of $1 - e^{-at}$ by $F(t)$ happens from below. The proof that $F(t)$ cannot cross $1 - e^{-at}$ more than once is similar to the previous one, i.e., if $F(t)$ crosses $1 - e^{-at}$ for the second time, it should be from above which means that $r_e(t^*_2) = 1 - e^{-a\Delta} = \text{const.} > r(t^*_2)$, where $t^*_2 > t^*_1 > 0$ and $t^*_1$ and $t^*_2$ are
first and second crossing points besides origin. But \( F(t) \) has already crossed the exponential distribution from below (proved previously) at \( t^* \) so we already have \( r(t^*) > r_e(t^*) = 1 - e^{-\alpha A} = \text{const.} \). This means that \( r(t^*) > 1 - e^{-\alpha A} > r(t^*_2) \) which again violates the condition for \( F(t) \) to be an IFR. So no IFR distribution can cross exponential distribution more than once besides at \( t = 0 \) and \( t = \infty \) and this crossing should take place from below. Consequently for \( F(u = x) = p \) the common point, we obtain:

\[
F(u) \leq 1 - e^{-\alpha u} \quad \text{if} \quad u \leq x \tag{2.3.6}
\]

\[
F(u) \geq 1 - e^{-\alpha u} \quad \text{if} \quad u \geq x \tag{2.3.7}
\]

\[
\alpha = -\frac{\log_e(1 - p)}{x} \tag{2.3.8}
\]

Now let us introduce a transformed exponential distribution function - t.e.d.:

\[
F(v) = \begin{cases} 
0 & \text{if} \quad u < d \\
1 - e^{-c(u-d)} & \text{if} \quad u \geq d
\end{cases} \tag{2.3.9}
\]

from Eq. (2.3.3) it follows that the transformed distribution is of constant failure rate

\[
r(v) = 1 - e^{-cA}, \quad u > a, \quad A > 0. \tag{2.3.10}
\]

The transformed exponential distribution has the following properties [22]:
PROPERTY 1: Generalizing properties of the exponential distribution (e.d.), the t.e.d. can be crossed by an IFR distribution at most twice, the first taking place from above.

REASON: Assume that F(t), which is IFR, crosses the t.e.d. F(v), with constant failure rate r(v), first from above and then from below. At the first coincidence \( r(t_1) < r(v) \) and at the second \( r(t_2) > r(v) \) so that \( F(t) \) can indeed be IFR. A further crossing from above would mean \( r(t_3) < r(v) \), contradicting the assumption of IFR distribution.

Further, we can uniquely find parameters \( c, d \) such that the t.e.d. passes through given two points \((t_i, p_i)\): \( p_i = F(t = t_i), i = 1, 2 \).

PROPERTY 2: Given three points

\[
(t_i, p_i): p_i = F(w = t_i), \quad i = 0, 1, 2 \quad t_0 < t_1 < t_2. \quad (2.3.11)
\]

(a) An IFR distribution \( F(w) \) passes through the three points only if \((t_1, p_1)\) lies on or below the t.e.d. through \((t_0, p_0), (t_2, p_2)\).

(b) The IFR distribution \( F(w) \) which passes through the three points and possesses maximum area \( \int_{t_0}^{t_2} F(w) \, dw \), is given by the two t.e.d.'s which meet at \((t_1, p_1)\):

\[
F_i(w) = 1 - e^{-c_i(w - d_i)} \quad (2.3.12)
\]

\[
c_i = \frac{1}{t_i - t_{i-1}} \log_e \left( \frac{1 - p_{i-1}}{1 - p_i} \right) \quad (2.3.13)
\]

\[
d_i = \frac{1}{c_i} \log_e (1 - p_i) + t_i \quad (2.3.14)
\]

\[
t_{i-1} \leq w \leq t_i, \quad i = 1, 2, \quad (2.3.15)
\]
\[ c_2 \geq c_1 \]  \hspace{1cm} (2.3.16)

**REASON (a):** This property is a direct result of property l.

**REASON (b):** Let us define the following distribution functions (See Fig. 3.): t.e.d. connecting points \((t_2, p_2), (t_0, p_0)\)

\[ F_3(w) = 1 - e^{-c_3(w - d_3)} \]  \hspace{1cm} (2.3.17)

t.e.d. connecting points \((t_1, p_1), (t_0, p_0)\)

\[ F_1(w) = 1 - e^{-c_1(w - d_1)} \]  \hspace{1cm} (2.3.18)

t.e.d. connecting points \((t_2, p_2), (t_1, p_1)\)

\[ F_2(w) = 1 - e^{-c_2(w - d_2)} \]  \hspace{1cm} (2.3.19)

We assume according to (a) that

\[ p_1 \leq 1 - e^{-c_3(t_1 - d_3)} \]  \hspace{1cm} (2.3.20)

Since \(F_1(w)\) crosses t.e.d. \(F_3(w)\) from above (See Fig. 3.) it follows directly from definition (2.3.3) that:

\[ r_3(t_0) \geq r_1(t_0) \]  \hspace{1cm} (2.3.21)

where \(r_i(\cdot)\) is the failure rate for \(F_i(\cdot)\). Similarly since \(F_2(w)\) crosses \(F_3(w)\) from below (See Fig. 3.)

\[ r_3(t_2) \leq r_2(t_2) \]  \hspace{1cm} (2.3.22)

so that from (2.3.21), (2.3.22) we obtain:

\[ r_2(t_2) \geq r_1(t_0) \]  \hspace{1cm} (2.3.23)
Fig. 3. The Relative Positions of the Three Points and Distributions.
from which by (2.3.10) we have:

\[ c_2 \geq c_1. \]

The Eqs. (2.3.12 - 15) can easily be obtained since \( F_1(w) \) and \( F_2(w) \) have point \((t_1, p_1)\) in common.

Viewing now \( F_1(w), w \leq t_1 \) and \( F_2(w) \geq t_1 \) as parts of one distribution \( F(w) \), we obtain for the failure rate of the latter from (2.3.3), (2.3.10) and (2.3.12 - 16):

\[
F(w) = \begin{cases} 
1 - e^{-c_1 \Delta} & \text{if } w + \Delta < t \\
1 - \frac{e^{-c_2 (w+\Delta - d_2)}}{e^{-c_1 (w - d_1)}} = 1 - \frac{e^{-c_2 (\Delta - d_2)}}{e^{+c_1 d_1}} e^{-(c_2-c_1)w} & \text{if } w \leq t \text{ or } w + \Delta \geq t \\
1 - e^{-c_2 \Delta} & \text{if } w > t \\
\end{cases}
\]

\[ \Delta > 0, \]

which is increasing in \( w \), so that \( F(w) \) is IFR.

From Property 1 it follows that no IFR distribution can pass through points \((t_i, p_i), i = 1, 2\) and above \( F(w) \). Therefore indeed

\[
\int_{t_0}^{t_2} F(w) \, dw \text{ is maximum among all IFR distributions.}
\]

2.4 THE OBJECTIVE FUNCTION

In Chapter 1 it was shown that the expected total maintenance cost per cycle could be found from Eq. (1.6) which is stated as
This equation is valid when there is no maximum life time for the system and inspection continues until the system fails. According to the assumption \( a \) of Section 2.2 there is a maximum life time \( T \) for the system, i.e., \( F(T) = 1 \) for both model A and model B presented here. Also the definition of the problem signifies that inspection is performed at time \( t_0 = 0 \) and \( t_n = T \). With these points in mind Eq. (1.6) can be written as

\[
L(t_0 = 0, t_1, t_2, \ldots, t_n = T) = \sum_{k=0}^{k=n-2} \int_{t_k}^{t_{k+1}} [I(k+1) + a(t_{k+1}-x)] f(x) dx, \quad t_n = T
\]

\[
= \int_{t_{n-1}}^{t_n} [I(n) + a(T-x)] f(x) dx, \quad (2.4.1)
\]

where \( N \) is the maximum possible number of inspections between \( t = 0 \) to \( t = T \) including.

Equation (2.4.1) can be written in the following form

\[
L = I \sum_{k=0}^{n-1} F*(t_k) + a \left( \sum_{k=0}^{n-1} F*(t_k)(t_{k+1} - t_k) - \int_0^{t_n} F*(t) dt \right), \quad (2.4.2)
\]

where \( F*(t) = 1 - F(t) \) and \( F(t) \) is the cumulative failure distribution of the system. The algebraic manipulation which transforms Eq. (2.4.1) into Eq. (2.4.2) is presented in detail in Appendix A.
2.5 FORMULATION

It is the objective of this work to maximize objective function $L$ given by (2.4.2) for a given inspection policy $p = \{t_0 = 0, t_1, t_2, ..., t_n = T\}$ by searching all IFR distributions $F$ which pass through one or two known points depending on the model, i.e., A or B respectively.

The form of the IFR distributions $F(w)$ leading to Max. $L$ is given, according to Property 2 by transformed exponential distribution functions (t.e.d) connecting all points $(t_i, p = F(t_i)), i = 0, 1, 2, ..., n$ and the known points.

2.5.1 Model A

In this model we assume that one point of the failure distribution $(x, p)$ is known. If $m$ is the number of inspections before $x$, we obtain the time sequence $t_0, t_1, t_2, ..., t_m, x, t_{m+1}, ..., t_n$, according to which above points are arranged. We recode this sequence as

$$\{(t_i, p_i); \quad i = 0, 1, ..., n+1\}, \quad (2.5.1)$$

while

$$(t_0, p_0) = (0, 0) = (t(0), p(0))$$

$$(x, p) = (t(m+1), p(m+1)),$$  \[ c = c(m+1) \]

$$(t_{n-1}, p_{n-1}) = (t(n), p(n)) \quad c_{n-1} = c(n)$$

$$(t_n, p_n) = (T, 1) = (t(n+1), p(n+1)),$$

knowing these we can define $F(w)$ as:
We observe that \( F(w) \) passes through points \((0,0), (x,p)\) and possesses a jump from \((t_{n-1}, p_{n-1})\) to \((t^+_{n-1}, 1)\) (as allowed for an IFR distribution in [6, 22]).

The objective function \( L \), given feasible points \((t_i, p_i), i = 1, 2; \ldots, n-1\), is indeed maximized by function \( F(w) \) since
\[ t_0 = T \]
\[ \int_0^{t_0} F^*(w) \, dw \text{ becomes minimum. This can be seen easily since} \]
\[ t_0 = T \]
\[ \int_0^{t_0} F^*(w) \, dw = \int_0^{t_0} (1 - F(w)) \, dw \]
\[ = T - \int_0^{t_0} F(w) \, dw, \tag{2.5.4} \]
but \[ \int_0^{t_0} F(w) \, dw \text{ becomes maximized according to Property 2 (2.3) if F(w) is} \]
given by the set of the transformed exponential distributions defined in (2.5.2). The value of maximum life time \( T \) is fixed so Eq. (2.5.4) gives the minimum value of the expression for all IFR distributions \( F(t) \).

The feasible region of \( F(t) \) v.s. \( t \) diagram is shown in Fig. 4. Between \( t = 0 \) and \( t = x \) the region is the area confined from above by \( F(t) = 1 - e^{-\alpha t} \), from below by \( F(t) = 0 \) and from left and right by \( t = 0 \) and \( t = x \) respectively. The reason for this is given by Property (2-a-2.3) which states that any IFR distribution will pass through three points (e.g. \( [t = 0, 0], [t_i, p_i], [t = x, p] \)) only if \( (t_i, p_i) \) lies on or below the t.e.d through the first and third points (e.g. \( [t = 0, 0], [t = x, p] \)). Between \( t = x \) and \( t = T \) the region is the area confined from above by \( F(t) = 1 \), from below by \( F(t) = 1 - e^{-\alpha t} \) and from left and right by \( t = x \) and \( t = T \) respectively (See Fig. 4.). The reason that \( (t_i, p_i) \) cannot be below \( F(t) = 1 - e^{-\alpha t} \) is that any IFR distribution which crosses \( F(t) \) at point \( (t = x, p) \) should have a higher failure rate at this point than \( 1 - e^{-\Delta \alpha} \). So no point \( (t_i, p_i) \) for \( t_i > x \) can lie below \( F(t) = 1 - e^{-\alpha t} \). In addition to this the points \( (t_i, p_i) \), must satisfy the equations given in (2.5.2).
The Feasible Region for Points \((t_f, p_f)\) for Model A.

\[
F(t) = 1 - e^{-at}
\]

\[
(t = x, p)
\]

\[
\alpha = \frac{\text{Log}_e(1-p)}{x}
\]

Fig. 4. The Feasible Region for Points \((t_f, p_f)\) for Model A.
The following relationship exists between two successive points \((t_i, p_i), (t_{i-1}, p_{i-1})\). That is

\[
F^*(t_i) = F^*(t_{i-1}) e^{-c(i)(t_i - t_{i-1})}
\]

\[
i = 1, 2, \ldots, n.
\]

(2.5.5)

This can be obtained from equations (2.5.2). That is

\[
F(t_i) = 1 - e^{-c(i)(t_i - d(i))},
\]

and

\[
F(t_{i-1}) = 1 - e^{-c(i)(t_{i-1} - d(i))},
\]

since both points \((t_{i-1}, p_{i-1})\) and \((t_i, p_i)\) are the two ends of \(F(t)\).

Now shifting 1 to the left side of both equations and multiplying both sides by -1 and dividing by each other then we have

\[
\frac{1 - F(t_i)}{1 - F(t_{i-1})} = e^{c(i)(t_i - d(i) - t_{i-1} + d(i))}
\]

\[
\frac{F^*(t_i)}{F^*(t_{i-1})} = e^{-c(i)(t_i - t_{i-1})}.
\]

Maximization for given policy can now be carried out by dynamic programming methodology. The search extends over all \(p_i = F(t_i), i = 1, 2, \ldots, n\).

We define \(J_i\), as loss or maintenance cost on the interval between \((i-1)\)th and \(i\)th inspection \([t_{i-1}, t_i]\), which is also a function of the failure probability at \((i-1)\)th and \(i\)th inspection. That is
\[
J_i = J_i(t_{i-1}, p_{i-1}, t_i, p_i) = [I + a(t_i - t_{i-1})] F^*(t_{i-1})
\]

\[
= \begin{cases} 
[I + a(t_i - t_{i-1})] F^*(t_{i-1}) + a \frac{F^*(t_i) - F^*(t_{i-1})}{c_i} & \text{if } i=1,2,\ldots, m,m+2,\ldots,n-1 \\
[I + a(t_{m+1} - t_m)] F^*(t_m) + \frac{(1-p) - F(t_m)}{c_m} + \frac{F^*(t_{m+1}) - (1-p)}{c_{m+1}} & \text{if } i=m+1 \\
[1 + a(t_n - t_{n-1})] F^*(t_{n-1}) & \text{if } i=n
\end{cases}
\]

The first expressions on the right hand side of equations in (2.5.7), i.e., \([I + a(t_i - t_{i-1})] F^*(t_{i-1})\) are obviously the share of \(J_i\) from the first expression in the expected total maintenance cost \(L\) given by Eq. (2.4.2), i.e., \[I \sum_{i=0}^{n-1} F^*(t_i) + a \sum_{i=0}^{n-1} F^*(t_i)(t_{i+1} - t_i).\] The second
expression a \int_{t_{i-1}}^{t_i} F*(w) \, dw, is also the share of J from L, i.e., a \int_{0}^{t_n=T} F*(w) \, dw.

For values of i = 1, 2, ..., m, m + 2, ..., n-1, \int_{t_{i-1}}^{t_i} F*(w) \, dw can be calculated from equations (2.5.2). That is

\int_{t_{i-1}}^{t_i} F*(w) \, dw = \int_{t_{i-1}}^{t_i} e^{-c_i(w-d_i)} \, dw

= \left[ -\frac{e^{-c_i(w-d_i)}}{c_i} \right]_{t_{i-1}}^{t_i}

= \frac{-c_i(t_{i-1} - d_i)}{c_i} - \frac{c_i(t_i - d_i)}{c_i}

= e^{-c_i(t_{i-1} - d_i)} - e^{-c_i(t_i - d_i)}

= \frac{F*(t_{i-1}) - F*(t_i)}{c_i}.

(2.5.8)

For i = m+1, F(w) passes through the known point (x,p), so

\int_{t_m}^{t_{m+1}} F*(w) \, dw = \int_{t_m}^{x} F*(w) \, dw + \int_{x}^{t_{m+1}} F*(w) \, dw

= \int_{t_m}^{x} e^{-c_m(w-d_m)} \, dw + \int_{x}^{t_{m+1}} e^{-c_{m+1}(w-d_{m+1})} \, dw

= \left[ -\frac{e^{-c_m(w-d_m)}}{c_m} \right]_{t_m}^{x} + \left[ -\frac{e^{-c_{m+1}(w-d_{m+1})}}{c} \right]_{x}^{t_{m+1}}
\[
\frac{-c_m(t_m - d_m) - c_m(x - d_m)}{c_m}
\]
\[
+ \frac{-c_{m+1}(x - d_{m+1}) - c_{m+1}(w - d_{m+1})}{c_{m+1}}
\]
but \(e^{-c_m(x - d_m)} = e^{-c_{m+1}(x - d_{m+1})} = F^*(x) = (1-p)\) since all describe the same point \((x, p)\). Also \(c_m = c\) according to recodification in Section 5 of this chapter. So

\[
\int_{t_m}^{t_{m+1}} F^*(w) \, dw = \frac{F^*(t_m) - (1-p)}{c} + \frac{(1-p) - F^*(t_{m+1})}{c_{m+1}}.
\]

For \(i = n\), i.e., to find \(J_n^*\) between one to the last inspection at \(t_{n-1}\) and last inspection at \(t_n = T\), according to Eqs. (2.5.7), \(F(w) = 1\) or equivalently \(F^*(w) = 0\) so

\[
\int_{t_{n-1}}^{t_n} F^*(w) \, dw = 0.
\]

Let us define \(K_i(t_i, p_i, c_{i+1})\) as loss or expected maintenance cost during time \((0, t_i]\) with \(F(t_i) = p_i\) when the parameter of the transformed failure distribution connecting point \((t_i, p_i)\) to \((t_{i+1}, p_{i+1})\) is \(c_{i+1}\). Then we have

\[
K_i(t_i, p_i, c_2) = J_i(t_0, p_0, t_1, p_1) = J_i(0, 0, t_i, p_i)
\]
\[
K_i(t_i, p_i, c_{i+1}) = K_{i-1}(t_i-1, p_{i-1}, c_i) + J_i(t_{i-1}, p_{i-1}, t_i, p_i)
\]
\[
= \sum_{j=1}^{i} J_j(t_{i-1}, p_{i-1}, t_i, p_i).
\]
Let \( K_n(T) \) signify the loss for the entire time period \( T \) at \( F(T) = 1 \) and any t.e.d. parameter \( c_{n+1} \)

\[
K_n(T) = K_n(t_n = T, p_n = 1, c_{n+1})
\]

\[
= K_{n-1}(t_{n-1}, p_{n-1}, c_n) + J_n(t_{n-1}, p_{n-1}, t_n, p_n)
\]

\[
= L.
\]  

(2.5.12)

From this it should be clear that there are three state variables in the system, i.e., \( t_i, p_i \) and \( c_{i+1} \). Optimization can now be carried out in two phases. In the first phase \( K_{m+1}(t_{m+1}, p_{m+1}, c_{m+1}) \), i.e., the loss between \( t = 0 \) to the time of the first inspection larger or equal to \( x \), \( t_{m+1} \) will be optimized for all possible values of \( t_{m+1} \) and \( p_{m+1} \). In the second and final phase \( K_n(t_{n=m+j} = T, p_{n=m+j} = 1, c_{n+1} = m+j = \infty) = L \) will be optimized. In the second phase, the optimization will be achieved, taking the optimum \( K_{m+1} \) values for state variables at time \( t = t_{m+1} \) as the \( K \) values for the first stage of the second phase and the values of \( L \) will be calculated for all possible number of stages both in phase 1 and phase 2 and the optimum value of \( L \) is obtained by search among these values.

First optimization phase - Let the minimax optimized loss for time period \( (0, t_i] \) at \( F(t_i) = p_i \) when the parameter of the transformed failure distribution connecting \( (t_i, p_i) \) to \( (t_{i+1}, p_{i+1}) \) is \( c_{i+1} \) be

\[
K_i(t_i, p_i, c_{i+1}) = \min_{P} \max_{F} K_i(t_i, p_i, c_i),
\]  

(2.5.13)

where \( P \) is the inspection policy and \( F \) is the failure distribution given by Eqs. (2.5.2). By dynamic programming procedure we then obtain
\[ K_0(t_0 = 0, p_0 = 0, c_{0+1}) = 0. \] (2.5.14)

\[ K_1(t_1, p_1, c_2) = I(x, 0, t_1, p_1) = I + at_1 + a \frac{F(t_1) - 1}{c_1}, \quad c_1 < \alpha \]

\[ O < t_1 < x \]

\[ i = 1 \]

\[ p''(t_1) < p_1 < p'(t_1) \]

\[ K_2(t_2, p_2, c_3) = \min \left\{ \max \{ K_1(t_1, p_1, c_2) + J_2(t_1, p_1, t_2, p_2) \} \right\} \]

\[ O < t_2 < x \]

\[ 0 < t_1 < t_2 \]

\[ v(t_2, p_2, c_3, t_1) < p_1 < G(t_2, p_2, t_1) \]

\[ p''(t_2) < p_2 < p'(t_2) \]

\[ i = 2 \]

\[ K_i(t_i, p_i, c_{i+1}) \]

\[ O < t_i < x \]

\[ p''(t_i) < p_i < p'(t_i) \]

\[ = \min \left\{ \max \{ K_{i-1}(t_{i-1}, p_{i-1}, c_i) + J_1(t_{i-1}, p_{i-1}, t_i, p_i) \} \right\}, \]

\[ 0 < t_{i-1} < t_i \]

\[ v(t_i, p_i, c_{i+1}, t_{i-1}) < p_{i-1} < G(t_i, p_i, t_{i-1}) \]

\[ i = 3, 4, \ldots, m \]

\[ K_{m+1}(t_{m+1}, p_{m+1}, c_{m+1}) \]

\[ x < t_{m+1} < T \]

\[ p''(t_{m+1}) < p_{m+1} < p'_{m+1}(t_{m+1}) \]
\[ \begin{align*}
= \min \quad & \max \{K_m(t_m, p_m, c) + J_{n+1}(t_m, p_m, t_m, p_{m+1})\}, \\
0 < t_m < x \quad & \forall (t_{m+1}, p_{m+1}, t_m) < p_m < G(t_m) \\
& \text{for } m = 0, 1, 2, \ldots, N_1,
\end{align*} \]

\[ \bar{K}_{n+1}(t_m, p_m, t_m) = T, \quad p_{m+1} = p_n = 1, \quad c_{m+1} = c_n = \infty \]

\[ \begin{align*}
= \min \quad & \max \{K_m(t_m, p_m, c) + J_{n+1}(t_m, p_m, t_m, p_{m+1})\}, \\
0 < t_m < x \quad & 0 < p_m < G(t_m) \\
& \text{for } m = 0, 1, 2, \ldots, N_1.
\end{align*} \]

\[ c = - \frac{\log_e \left( \frac{1-p}{1-p_m} \right)}{x-t_m}. \]

\[ N_1 \] is the maximum possible number of inspections in the interval \((0,x)\) and we have \(c_1 \leq c_2 \leq \ldots \leq c_m \leq c_{m+1}\) since the failure distribution \(F(w)\) is IFR. Also since at point \((x,p)\), \(F(w)\) crosses the exponential distribution \(F(u)\) \((2.3.6-8)\), we should have \(a \leq c \leq c_{m+1}\), as stated already in Eqs. \((2.5.2)\).

\(p'(t_i)\) and \(p''(t_i)\) which are the upper and lower limits for the possible values of \(p_i\) in the feasible region (See Fig. 5.) are given by

\[ p'(t_i) = \begin{cases} 
1 - e^{-\alpha t_i} & \text{for } 0 \leq t_i < x \\
1 & \text{for } x < t_i < T
\end{cases} \quad (2.5.15) \]

and

\[ p''(t_i) = \begin{cases} 
0 & \text{for } 0 \leq t_i < x \\
1 - e^{-\alpha t_i} & \text{for } x < t_i < T
\end{cases} \quad (2.5.16) \]

At \(t_i = x\) and \(t_i = T\) we have
The Feasible Region for IFR to Pass Through Points \((0,0)\) and \((x,p)\) and \((t_{i-1},p_{i-1})\) and \((t_i,p_i)\) with \(c_{i+1} > c_i\).

Fig. 5. The Upper and Lower Limits for \(p_i\) and \(p_{i-1}\) for \(t_i \in (0,x)\).
\[ p'(x) = p''(x) = p \]  \hspace{1cm} (2.5.17)

\[ p''(T) = 1. \]  \hspace{1cm} (2.5.18)

In order for any IFR distribution to pass through points \((0, 0)\), \((t_{i-1}, p_{i-1})\) and \((t_i, p_i)\), it is necessary for point \((t_{i-1}, p_{i-1})\) to be located below the transformed failure distribution (t.e.d.) through points \((0, 0)\) and \((t_i, p_i)\) according to Property (2.5-2.3) for \(t_i\) in the interval \((0, x)\) (See t.e.d. No 2 in Fig. 5.). The parameter of the t.e.d. No 2 called \(c(t_i, p_i)\) can be calculated from

\[ c(t_i, p_i) = \frac{\log_e(1-p_i)}{t_i} \]  \hspace{1cm} (2.5.19)

and the maximum possible value of \(p_{i-1}\) given by \(G(t_i, p_i, t_{i-1})\) according to Eq. (2.5.5) is

\[ G(t_i, p_i, t_{i-1}) = 1 - (1-p_i)e^{c(t_i, p_i)(t_i - t_{i-1})}. \]  \hspace{1cm} (2.5.20)

\(G(t_m)\) is the particular value of \(G\) function at \(t_m\) and can be found with a similar type of reasoning as for Eqs. (2.5.19-20) to be

\[ G(t_m) = 1 - e^{-\alpha t_m}. \]  \hspace{1cm} (2.5.21)

In order for a transformed exponential distribution to connect points \((t_{i-1}, p_{i-1})\) and \((t_i, p_i)\) and have a parameter \(c_i\) so that \(c_i < c_{i+1}\), where \(c_{i+1}\) is the parameter of the transformed exponential distribution connecting points \((t_i, p_i)\) and \((t_{i+1}, p_{i+1})\) as required for an IFR distribution (See Fig. 5 t.e.d. No 1), the point \((t_{i-1}, p_{i-1})\) should be above t.e.d. No 1. The reason is obvious. Let us call the t.e.d. passing through points \((t_i, p_i)\) and
\((t_i+1, p_{i+1})\) as \(v_1(t)\) then if point \((t_{i-1}, p_{i-1})\) is located below \(v_1(t)\) we should have \(v_1(t_{i-1}) > p_{i-1}\). Now we have

\[
c_i = - \frac{\log e(1-p_{i-1})}{t_i - t_{i-1}},
\]

(2.5.22)

and assuming \(\bar{c} = \) parameter of \(v_1(t) = - \frac{\log e(1-F(t_{i-1}))}{t_i - t_{i-1}}\),

(2.5.23)

we can write

\[
c_i - \bar{c} = - \frac{1}{t_i - t_{i-1}} \left( \log e(1-p_{i-1}) - \log e(1-F(t_{i-1})) \right)
\]

(2.5.24)

\[
= - \frac{1}{t_i - t_{i-1}} \log e(1-F(t_{i-1}))
\]

but \(- \frac{1}{t_i - t_{i-1}}\) is negative since \(t_i > t_{i-1}\) and \(\log e(1-F(t_{i-1}))\) is also negative since \(v_1(t_{i-1}) > p_{i-1}\) according to the assumption and so \(1-F(t_{i-1}) < 1 - p_{i-1}\) which shows that \(\log e(A<1)\). But \(\log e(A<1)\) is always negative. This means that \(c_i - \bar{c} = B\) where \(B\) is a positive amount so \(c_i \geq \bar{c} = c_{i+1}\). But we should have \(c_i \leq c_{i+1}\) in order that the failure distribution to be IFR. As a result point \((t_{i-1}, p_{i-1})\) should be above \(v_1(t)\). The lower limit on \(p_{i-1}\) is given by \(v\) function which its value is determined by the above mentioned \(v_1(t)\) distribution or regional boundaries and has different forms at different stages and different points. \(v(t_i, p_i, c_{i+1}, t_{i-1})\) according to Eq. (2.5.5) is given by

\[
v(t_i, p_i, c_{i+1}, t_{i-1}) = \text{Max}\{1. - (1-p_i)e^{c_{i+1}(t_i-t_{i-1})} \text{ or } 0.0\}.
\]

(2.5.25)
\( v(t_{m+1}, p_{m+1}, t_m) \) is the particular value of \( v \) function at \( t_{m+1} \), i.e., the value at \((m+1)\)th stage and is given by

\[
v(t_{m+1}, p_{m+1}, t_m) = \max\{1-(1-p)e^{c_{m+1}(x-t_m)} \text{ or } 0.0\}, \tag{2.5.26}
\]

where

\[
c_{m+1} = -\frac{\log(1-p)}{e^{t-1-p}}. \tag{2.5.27}
\]

The value of \( c_i \), i.e., the parameter for the t.e.d. connecting points \((t_i, p_i)\) and \((t_{i-1}, p_{i-1})\) is given according to Eq. (2.5.5) by

\[
c_i = -\frac{\log(1-p)}{t_i-t_{i-1}}. \tag{2.5.28}
\]

We now choose for each \((t_{m+1}, p_{m+1})\) that \( m=m^* \) which renders \( K_{m+1}^* \) a minimum. That is

\[
K_{m+1}^{**}(t_{m+1}, p_{m+1}, c_{m+1}) = \min\{K_{m+1}(t_{m+1}, p_{m+1}, c_{m+1})\}
\]

\[
x_{m+1} < t_{m+1} < T
\]

\[
p''(t_{m+1}) < p_{m+1} < p'(t_{m+1}) \quad m = 0, 1, 2, \ldots, N
\]

\[
K_{n=m+1}^{**}(t_{m+1}=t_n=T, p_{m+1}=p_n=1, c_{m+1}=c_n=\infty)
\]

\[
= \min\{\bar{K}_{n=m+1}^*(T, 1, \infty)\} \quad m = 0, 1, 2, \ldots, N. \tag{2.5.30}
\]

\( \bar{K}_{n=m+1}^{**}(T, 1, \infty) \), the total expected loss when there is only one inspection in the interval \( (x, T] \) is given by the last equation in (2.5.14). \( K_{m+1}^{**} \) is
the minimax expected total maintenance cost in the time interval $(0, t_{m^*+1})$
where $x < t_{m^*+1} < T$. $m^*$ is the optimum number of inspections in the interval
$(0, x)$. From the above formulations it becomes clear that the control variables are the previous inspection time $t_{i-1}$ and previous cumulative failure probability $p_{i-1}$. The stage number $m$ represents the number of inspections.

Second optimization phase – we continue the iterative procedure, taking the $K_{m^*+1}^{**}$ values at each point $(t_{m^*+1}, p_{m^*+1})$ as the minimax expected maintenance cost up to that point or state for the first value of $K^{**}$ at the first stage in the second phase of the optimization. That is

$$K_{m^*+j}^{**}(t_{m^*+j}, p_{m^*+j}, c_{m^*+j})$$

$$x < t_{m^*+j} < T$$

$$p''(t_{m^*+j}) < p_{m^*+j} < p'(t_{m^*+j})$$

$$= \text{Min} \quad \text{Max}(K_{m^*+j-1}^{**}(t_{m^*+j-1}, p_{m^*+j-1}, c_{m^*+j-1}), \text{v}(t_{m^*+j}, p_{m^*+j}, c_{m^*+j}) < p_{m^*+j-1} < G(t_{m^*+j}, p_{m^*+j}, c_{m^*+j}), t_{m^*+j-1}$$

$$+ \sum_{j=m^*+j}^{j+1} (t_{m^*+j-1}, p_{m^*+j-1}, t_{m^*+j}, p_{m^*+j}), \text{v}(t_{m^*+j}, p_{m^*+j}, c_{m^*+j}))$$

(2.5.31)

$$j = 2, 3, 4, \ldots, N_2$$

$$K_{m^*+j}^{**}(t_{m^*+j} = t, p_{m^*+j} = p, c_{m^*+j} = c),$$

$$x < t_{m^*+j-1} < T \quad \text{v}(t_{m^*+j-1}) < p_{m^*+j-1}$$
\[ j = 2, 3, \ldots, N_2 \]

where \( N_2 \) is the maximum possible number of inspections or equivalently maximum number of stages in the second phase. Also we should have

\[ \alpha \leq c_{m+1} \leq c_{m+2} \leq \ldots \leq c_{n-1} \]

according to the definition of the failure distribution \( F(w) \) given in Eqs. (2.5.2-3). \( p'(t_{m+j}) \) and \( p''(t_{m+j}) \) are given in Eqs. (2.5.15-18). Function \( G \) with a similar reasoning as for phase one is given by

\[ G(t_{m+j}, P_{m+j}, t_{m+j-1}) = 1 - (1-p)e^{-c(t_{m+j}, P_{m+j})(t_{m+j-1}-x)}, \]

where

\[ c(t_{m+j}, P_{m+j}) = -\frac{\log(\frac{1-p_{m+j}}{1-p})}{t_{m+j-1}-x}. \]

Function \( v \) with a similar reasoning as stated in phase one for Eqs. (2.5.22-24) is given by

\[ v(t_{m+j}, P_{m+j}, c_{m+j+1}, t_{m+j}) \]

\[ = \text{Max}(1 - (1-p_{m+j})c_{m+j+1}(t_{m+j}-t_{m+j-1}) \text{ or } 1 - e^{-\alpha t_{m+j-1}}), \]

and function \( v(t_{m+j-1}) \) is a particular value of \( v \) function at \( t_{m+j-1} \).
That is
\[ v(t_{m^*+j-1}) = 1 - e^{-at_{m^*+j-1}}. \] (2.5.35)

The value of the parameter of the transformed exponential distribution which passes through \((t_{m^*+j-1}, p_{m^*+j-1})\) and \((t_{m^*+j}, p_{m^*+j})\), i.e., \(c_{m^*+j}\) is given by
\[
\frac{1-p_{m^*+j}}{\log(1-p_{m^*+j-1})} = \frac{t_{m^*+j}}{t_{m^*+j} - t_{m^*+j-1}}.
\] (2.5.36)

\(K_{n=m^*+j}^{**}(T, l, \infty)\), given by Eq. (2.5.32) is the minimax total expected cost in the interval \([0,T]\) when \(m^*\) inspections are performed in the interval \([0,x)\) and \(j\) inspection performed in the interval \([x,T]\). Now the optimal loss \(L^{**}\) at \(n^* = m^*+j^*\) is:
\[ L^{**} = \min(K_{n=m^*+j}^{**}(T, l, \infty)) \quad \text{or} \quad \hat{R}_{n=m^*+1}^{**}(T, l, \infty), \] (2.5.37)
\[ j = 2, 3, 4, ..., N_2 \]

where \(\hat{R}_{n=m^*+1}^{**}(T, l, \infty)\) is the minimax total expected cost in the interval \([0,T]\) with \(m^*\) inspections in the interval \([0,x)\) and one inspection at time \(T\) and is given by Eq. (2.5.30). \(j^*\) is the optimal number of inspections in the interval \([x,T]\).

The policy \(p^{**} = \{t_1^*, t_2^*, ..., t_n^*\}\) and the failure distribution \(F^{**}\) defined by (2.1.3-4) can also be found if needed by a backward recursive procedure from the computer printout (as explained in Chapter 3).
2.5.2 Model B

In this model we assume that the location $x_1$ and $x_2$ of the 100.$p_1$ th and 100.$p_2$ th percentiles respectively of the otherwise unknown failure distribution of the system are known. If $m_1$ is the number of checks or inspections before $x_1$ and $m_2$ is the number of inspections before $x_2$, we obtain the time sequence $t_0, \ldots, t_{m_1}, x_1, t_{m_1+1}, \ldots, t_{m_2}, x_2, t_{m_2+1}, \ldots, t_n$, according to which above points are arranged. We recode this sequence as

$$\{(t(i), p(i)); \quad i = 0, 1, \ldots, n+2\}, \quad (2.5.38)$$

while

$$(t_0, p_0) = (0, 0) = (t_0, p_0)$$

$$(x_1, p_1) = (t_{m_1+1}, p_{m_1+1}) \quad c_1^* = c_{m_1+1}$$

$$(t_{n-1}, p_{n-1}) = (t_{n'}, p_{n'}) \quad c_{n-1}^* = c_{n'}$$

$$(x_2, p_2) = (t_{m_2+2}, p_{m_2+2}) \quad c_2^* = c_{m_2+2}$$

$$(t_{n-1}, p_{n-1}) = (t_{n+1}, p_{n+1}) \quad c_{n-1} = c_{n+1}$$

$$(t_n, p_n) = (T, 1) = (t_{n+2}, p_{n+2})$$

knowing these we can define $F(w)$ with a series of transformed exponential distributions connected together as

$$F(w) = \begin{cases} 
1 - e^{-c(1) w} & \text{if } t(0) = 0 \leq w \leq t(1) \\
1 - e^{-c(i) (w-d(i))} & \text{if } t(i-1) \leq w \leq t(i), \quad 1 \leq i \leq n+1 \\
1 & \text{if } t(n+1) \leq w \leq t(n+2) = T
\end{cases}$$

(2.5.39)
\[ c(1) < \alpha_1 = -\frac{\log_e(1-p_1)}{x_1}, \]
\[ \alpha_2 = -\frac{\log_e(1-p_2)}{x_2-x_1} \geq \alpha_1 = -\frac{\log_e(1-p_1)}{x_1}. \]
\[ c(1) \leq \cdots < c(m_1) \leq c(m_1+1) \leq \cdots \leq c(m_2+1) \leq c(m_2+2) \leq \cdots \leq c(n+1) \]
\[ \alpha_1 < c(m_1+1) < \alpha_2 \quad c(m_2+2) > \alpha_2 \]
\[ c(i) = \frac{1}{t(i) - t(i-1)} \log_e(\frac{1-p(i-1)}{1-p(i)}), \quad i = 1, 2, \ldots, n \]
\[ d(i) = \frac{1}{c(i)} \log_e(1-p(i)) + t(i), \]

while we assumed

\[ c(i) = \begin{cases} 
  c_1 & \text{if } 1 \leq i \leq m_1 \\
  c_1^* & \text{if } i = m_1 + 1 \\
  c_{1-1} & \text{if } m_1+2 \leq i \leq m_2+1 \\
  c_2^* & \text{if } i = m_2+2 \\
  c_{1-2} & \text{if } m_2+3 \leq i \leq n+2
\end{cases} \quad (2.5.40) \]

We observe that \( F(w) \) passes through points \((0, 0), (x_1, p_1), (x_2, p_2)\) and possesses a jump from \((t_{n-1}, p_{n-1})\) to \((t_{n-1}^+, 1)\) (as allowed for an IFR distribution in \([6, 22]\)).
As it was stated and proved for model A, Eq. (2.5.4), objective function \( L \) (2.4.2), given feasible points \((t_i, p_i)\), \(i = 1, 2, \ldots, n-1\), is indeed maximized by function \( F(w) \) defined above by Eqs. (2.5.39-40) which is a set of the transformed exponential distributions.

The feasible region of \( F(t) \) v.s.t. diagram for the set of IFR distributions which pass through points \((0,0), (x_1, p_1), \) and \((x_2, p_2)\) is shown in Fig. 6. Between \( t = 0 \) to \( t = x_1 \) the region is the area confined from above by \( F_1(t) = 1 - e^{-\alpha_1 t} \), for the same reason stated for Model A. The region is confined from below by \( \text{Max}(F_2(t), 0) \), where \( F_2(t) \) is the t.e.d. connecting points \((x_1, p_1)\) and \((x_2, p_2)\) and according to Eq. (2.5.5) is given by

\[
F_2(t) = 1 - (1-p_1) e^{-\alpha_2(t-x_1)}, \tag{2.5.41}
\]

\[
\alpha_2 = \frac{\log \left( \frac{1-p_2}{1-p_1} \right)}{x_2 - x_1}. \tag{2.5.42}
\]

The reason for this is that any point \((t_i, p_1)\) below \( F_2(t) \) will be connected to point \((x_1, p_1)\) by an IFR distribution which crosses t.e.d. \( F_2(t) \) from below. But according to property 1 (2.3) no IFR distribution can cross a t.e.d. at two points (e.g. \((x_1, p_1), (x_2, p_2)\)) first from below. From left and right the region is confined by \( t = 0 \) and \( t = x_1 \) respectively. Between \( t > x_1 \) and \( t = x_2 \), the region is confined from above by t.e.d. \( F_2(t) \) since according to property 2-a (2.3), an IFR passes through three points (e.g. \((x_1, p_1), (t_i, p_1), (x_2, p_2)\)) only if \((t_i, p_1)\) lies on or below the t.e.d. through \((x_1, p_1)\) and \((x_2, p_2)\), i.e., \( F_2(t) \). The region is confined from below by t.e.d. \( F_1(t) \) which connects points \((0,0)\) to \((x_1, p_1)\). The
reason that \((t_i, p_i)\) cannot be below \(F_1(t) = 1 - e^{-\alpha_1 t}\) is that any IFR distribution which crosses at point \((t = x_1, p_1)\) should have a higher failure rate at this point than \(1 - e^{-\alpha_1 t}\). So no point \((t_i, p_i)\) for \(t > x_1\) can lie below \(F_1(t)\). The region is confined from left and right by \(t = x_1\) and \(t = x_2\) respectively. Between \(t > x_2\) and \(t = T\) the region is confined from above by \(F(t) = 1\) as it is obviously clear and from below by \(F_2(t)\). The reason that \((t_i, p_i)\) cannot be below \(F_2(t)\) is that any IFR distribution which crosses at point \((x_2, p_2)\) should have a higher failure rate at this point than \(1 - e^{-\alpha_2 t}\). So any point \((t_i, p_i)\) for \(t_i > x_2\) cannot lie below \(F_2(t)\). The region is confined from left and right by \(t = x_2\) and \(t = T\) respectively. In addition to the above limitations the point \((t_i, p_i)\) must satisfy the equations given in (2.5.39)

Maximization for given policy can now be carried out by dynamic programming methodology. The search extends over all \(p_i = F(t_i), i = 1, 2, \ldots, n\). As for Model A, we define \(J_i\), as loss or maintenance cost on the interval between \((i-1)th\) and \((i)th\) inspection \([t_{i-1}, t_i]\), which is also a function of the failure probability at \((i-1)th\) and \(i\)th inspection. That is

\[
J_i = J_i(t_{i-1}, p_{i-1}, t_i, p_i) = [1 + a(t_i - t_{i-1})] F^*(t_{i-1})
\]

\[
- a \int_{t_{i-1}}^{t_i} F^*(w) \, dw
\]

(2.5.43)

\(t_i < w < t_{i-1}, \ i = 1, 2, \ldots, n\).

If \(J_i^* = \text{Max} J_i\), we obtain from Eqs. (2.5.39) that

\[
J_i^* = J_i^*(t_{i-1}, p_{i-1}, t_i, p_i) = \text{Max} J_i(t_{i-1}, p_{i-1}, t_i, p_i)
\]

(2.5.44)
Fig. 6. The Feasible Region for Points \((t_1, p_1)\) for Model B.
\[
\begin{align*}
&= [I + a(t_i - t_{i-1})] F^*(t_{i-1}) + a \frac{F^*(t_i) - F^*(t_{i-1})}{c_i} \\
&\quad \text{if } i = 1, 2, \ldots, m_1, m_1+2, \ldots, m_2, m_2+2, \ldots, n-1, \\
&= [I + a(t_i - t_{i-1})] F^*(t_{i-1}) + a \left( \frac{(1-p_1) - F^*(t_{m_1})}{c_i^*} + \frac{F^*(t_{m_1+1}) - (1-p_1)}{c_{m_1+1}} \right) \\
&\quad \text{if } i = m_1+1, \\
&= [I + a(t_{m_2+1} - t_{m_1})] F^*(t_{m_1}) + a \left( \frac{(1-p_1) - F^*(t_{m_1})}{c_i^*} + \frac{p_1-p_2}{\alpha_2} + \right. \\
&\quad \left. \frac{F^*(t_{m_2+1}) - (1-p_2)}{c_{m_2+1}} \right) \\
&\quad \text{if } i = m_2+1 \text{ and } m_2 = m_1, \\
&= [I + a(t_i - t_{i-1})] F^*(t_{i-1}) + a \left( \frac{(1-p_2) - F^*(t_{m_2})}{c_i^*} + \frac{F^*(t_{m_2+1}) - (1-p_2)}{c_{m_2+1}} \right) \\
&\quad \text{if } i = m_2+1 \text{ and } m_2 \neq m_1, \\
&= [I + a(t_i - t_{i-1})] F^*(t_{i-1}) \quad \text{if } i = n.
\end{align*}
\]

As mentioned for Model A, the first expressions on the right side of equations in (2.5.44), i.e., \([I + a(t_i - t_{i-1})] F^*(t_{i-1})\) are obviously the share of \(J_i\) from the first expression in the expected total maintenance cost \(L\) given by Eq. (2.4.2), i.e.,
\[
I \sum_{i=0}^{n-1} F^*(t_i) + a \sum_{i=0}^{n-1} F^*(t_i)(t_{i+1} - t_i).
\]
The second expression is, a \(\int_{t_{i-1}}^{t_i} F^*(w) \, dw\), which is also the share of \(J_i\) from \(L\), i.e.,
\[ a \int_0^{t_n} F^*(w) \, dw. \] For values of \( i \), except for \( i = m_2+1 \) when \( m_2=0 \), the calculation of the values of \( a \int_{t_{i-1}}^{t_i} F^*(w) \, dw \) is the same as for model A. But for \( i = m_1+1 \) when \( t_i \geq x_2 \) we have

\[
\int_{t_{i-1}}^{t_i} F^*(w) \, dw = \int_{t_{i-1}}^{x_1} F^*(w) \, dw + \int_{x_1}^{x_2} F^*(w) \, dw + \int_{x_2}^{t_i} F^*(w) \, dw. \tag{2.5.45}
\]

Substituting the values of \( F^*(w) \) from Eqs. (2.5.38-40) into equation (2.5.45) we have

\[
\int_{t_{i-1}}^{t_i} F^*(w) \, dw = \int_{t_{i-1}}^{x_1} e^{-c(m_1+1)(w-d(m_1+1))} \, dw + \int_{x_1}^{x_2} (1-p_i) e^{-c(m_1+1)(w-d(m_1+1))} \, dw + \int_{x_2}^{t_i} e^{-c(m_1+3)(w-d(m_1+3))} \, dw
\]

\[
= \left[ -e^{-c(m_1+1)(w-d(m_1+1))} \right]_{x_1}^{t_{i-1}} + \left[ -\frac{e^{-c(m_1+1)(w-d(m_1+1))}}{c(m_1+1)} \right]_{t_{i-1}}^{t_i} + \left[ \frac{e^{-c(m_1+3)(w-d(m_1+3))}}{c(m_1+3)} \right]_{x_2}^{t_i}.
\]
\[
\begin{align*}
&\left(-\frac{c(m_2+3) (w - d(m_2+3))}{c(m_2+3)}\right)_{x_2}^{-1} \\
&= e^{-c(m_1+1)(t_i-1 - d(m_1+1))} - e^{-c(m_1+1)(x_1 - d(m_1+1))} \\
&\quad + \frac{(1-p_1) e^{-\alpha_2(x_1-x_1)}}{\alpha_2} - (1-p_1) e^{-\alpha_2(x_2-x_1)} \\
&\quad + \frac{e^{-c(m_2+3)(x_2 - d(m_2+3))}}{c(m_2+3)} - e^{-c(m_2+3)(t_i - d(m_2+3))} \\
&= \frac{F^*(t_{(m_1)}) - F^*(t_{(m_1+1)})}{c(m_1+1)} + \frac{(1-p_1) - (1-p_2)}{\alpha_2} + \\
&\quad + \frac{F^*(t_{(m_2+2)}) - F^*(t_{(m_2+3)})}{c(m_2+3)} ,
\end{align*}
\]

since transformed exponential distribution \(F(w)\), which connects \((p_1, x_1)\) and \((p_2, x_2)\) has already been given by \(F_2(t)\) as

\[
F_2(t) = 1 - (1 - p_1) e^{-\alpha_2(t-x_1)} ,
\]

and also according to recodification (2.5.38) and (2.5.40) we have
\[ F^*(t_{m_1}) = F^*(t_m) \]
\[ F^*(t_{m_1+1}) = F^*(x_1) = (1-p_1) \]
\[ F^*(t_{m_2+1}) = F^*(x_2) = (1-p_2) \]
\[ F^*(t_{m_2+3}) = F^*(t_{m_2+1}) \]
\[ c_{m_1+1} = c_1^* \]
\[ c_{m_2+3} = c_{m_2+1}^* \]

so that

\[
\int_{t_{i-1}}^{t_i} F^*(w) = \frac{F^*(t_{m_1}) - (1-p_1)}{c_i^*} + \frac{p_2-p_1}{a_2} + \frac{(1-p_2) - F^*(t_{m_2+1})}{c_{m_2+1}}
\]

In Model B also we define \( K_i \) \((t_i, p_i, c_{i+1})\) as minimax expected maintenance cost during time interval \([0, t_i]\) with \( F(t_i) = p_i \) when the parameter of the t.e.d. connecting \((t_i, p_i)\) to \((t_{i+1}, p_{i+1})\) is less than or equal to a certain amount \( c_{i+1} \). We can write

\[
K_i \left( t_i, p_i, c_{i+1} \right) = J_i^* \left( t_0, p_0, t_1, p_1 \right) = J_i^* \left( 0, 0, t_1, p_1 \right)
\]

\[
K_i \left( t_i, p_i, c_{i+1} \right) = K_{i-1} \left( t_{i-1}, p_{i-1}, c_i \right) + J_i^* \left( t_{i-1}, p_{i-1}, t_i, p_i \right)
\]

\[
= \sum_{j=1}^{i} J_j^* \left( t_{i-1}, p_{i-1}, t_i, p_i \right). \quad (2.5.46)
\]
Let $K_n(T)$ signify the loss for the entire time period $T$ at $F(T) = 1$ and t.e.d. with parameter $c_{n+1} = \infty$

$$K_n(T) = K_n(t_n = T, p_n = 1, c_{n+1} = \infty)$$

$$= K_{n-1}(t_{n-1}, p_{n-1}, c_n) + J^*(t_{n-1}, p_{n-1}, t_n, p_n)$$

$$= L. \quad (2.5.47)$$

In this model as for Model A, we have three state variable, i.e., $t_1$, $p_1$ and $c_{i+1}$. There are two control variable, i.e., $t_{i-1}$, $p_{i-1}$. Minimaxation will be carried out in three phases. In the first phase the minimax expected maintenance cost from time $t_0 = 0$ to the time of first inspection in the interval $[x_1, x_2]$, i.e., $K_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+2})$ and the minimax expected maintenance cost from time $t_0 = 0$ to the time of first inspection in the interval $[x_2, T]$, i.e., $K_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+2})$ will be obtained.

In the second phase $K_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+2})$, i.e., the minimax loss from $t_0 = 0$ to the time of the first inspection $t_{m_2+1} = x_2$ will be optimzed for all possible values of $t_{m_2+1}$, $p_{m_2+1}$ and $c_{m_2+2} = c_{m_2+1}$. In the third and final phase $K_n(t_n = m_2+j = T, p_n = m_2+j = 1, c_{n+1} = \infty) = L$ will be minimaxed. In the second phase the minimaxation will be achieved, taking the minimax $K_{m_1+1}$ values for state variables at time $t = t_{m_1+1}$ as the $K$ values for the first stage of the second phase and in third phase also in the same way the minimax $K_{m_2+1}$ values for state variables at time $t = t_{m_2+1}$ are taken as the $K$ values for the first stage of the third phase. The value of $L$ will be calculated for all possible number of stages in phase 1, phase two and phase 3 and the optimum value of $L$ will be found by a simple search among these values.
First optimization phase - Let the minimax optimized loss for time period \((0, t_i]\) at \(F(t_i) = p_i\) when the parameter of the t.e.d. connecting \((t_i, p_i)\) to \((t_i+1, p_{i+1})\) is equal to a certain amount \(c_{i+1}\), be

\[
K^*_i(t_i, p_i, c_{i+1}) = \min_F \max_{p_i} K_i(t_i, p_i, c_i)
\]

(2.5.48)

where \(P\) is the inspection policy (2.2.1) and \(F\) is failure distribution given by Eqs. (2.5.39). Now by using dynamic programming procedure we obtain

\[
K^*_0(t_0=0, p_0=0, c_{0+1}) = 0,
\]

(2.5.49)

\[
K^*_1(t_1, p_1, c_2) = J^*_1(0, 0, t_1, p_1) = I + \alpha(t_1) + a \frac{F^*(t_1) - 1}{c^*_1}, \quad c^*_1 < a_1
\]

\(0 < t_1 \leq x_1\)

\(p''(t_1) < p_1 < p'(t_1)\)

\[
K^*_2(t_2, p_2, c_3) = \min_0 \max_{t_2} \{K^*_1(t_1, p_1, c_2) + J^*_2(t_1, p_1, t_2, p_2)\}
\]

\(0 < t_2 \leq x_1\) \quad \(0 < t_1 < t_2\)

\(v(t_2, p_2, c_3, t_1) \leq p_1 \leq G(t_2, p_2, t_1)\)

\(p''(t_2) \leq p_2 < p'(t_2)\)

\[
K^*_i(t_i, p_i, c_{i+1}) = \min_0 \max_{t_i} \{K^*_i-1(t_{i-1}, p_{i-1}, c_i) + J^*_i(t_{i-1}, p_{i-1}, t_i, p_i)\}
\]

\(0 < t_i \leq x_1\) \quad \(0 < t_{i-1} < t_i\)

\(v(t_i, p_i, c_{i+1}, t_{i-1}) < p_{i-1} \leq G(t_i, p_i, t_{i-1})\)

\(p''(t_i) < p_i < p'(t_i)\) \quad i = 3, 4, \ldots, m_1\)

\[
K^*_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1})
\]

\(x_1 \leq t_{m_1+1} \leq x_2\)
\[ p''(t_{m_1} + 1) < p_{m_1} + 1 < p'(t_{m_1} + 1) \]

\[
\begin{align*}
\text{Min} & \quad \text{Max}\{K^*_m(t_{m_1}, p_{m_1}, c^*_m) + J^*_m + 1(t_{m_1}, p_{m_1}, t_{m_1} + 1, p_{m_1} + 1)\} \\
0 < t_{m_1} < x_1 & \quad v(t_{m_1} + 1, p_{m_1} + 1, t_{m_1}) \leq p_{m_1} \leq G_0(t_{m_1}) \\
m_1 = 0, 1, 2, \ldots, N_1,
\end{align*}
\]

\[ C^*_m = - \frac{\log(1 - p_{m_1})}{x_1 - t_{m_1}} \]

\[ R_{m_1 + 1}(t_{m_1} + 1, p_{m_1} + 1, c_{m_1} + 1) \]

\[ x_2 \leq t_{m_1} + 1 \leq T \]

\[ p''(t_{m_2} + 1) < p_{m_2} + 1 < p'(t_{m_2} + 1) \]

\[
\begin{align*}
\text{Min} & \quad \text{Max}\{K^*_m(t_{m_1}, p_{m_1}, c^*_m) + J^*_m + 1(t_{m_1}, p_{m_1}, t_{m_1} + 1, p_{m_1} + 1)\} \\
0 < t_{m_1} < x_1 & \quad v(t_{m_1} + 1, p_{m_1} + 1, t_{m_1}) \leq p_{m_1} \leq G_0(t_{m_1}) \\
m_1 = 0, 1, 2, \ldots, N_1,
\end{align*}
\]

where \( N_1 \) is the maximum possible number of inspections in the interval \((0, x_1)\) and \( c_1 \leq c_2 \leq \ldots \leq c^*_1 \leq c_{m_1} + 1, \ldots \leq c_{m_2} \leq c^*_2 \leq c_{m_2} + 1 \leq \ldots \leq c_{n-1} \)

since the \( F(w) \) distribution is IFR. Also because at point \((x_1, p_1), F(w)\) crosses \( F(u) = 1 - e^{-\alpha_{1}u} \) from below, we should have \( \alpha_1 < c^*_1 \leq c_{m_1} + 1, \)

\( p'(t_1) \) and \( p''(t_1) \) are defined the same way as in Model A (See Fig. 5.). But their values are given by (See Fig. 6.)
\[
p'(t_i) = \begin{cases} 
1 - e^{-\alpha_1 t_i} & 0 \leq t_i < x_1 \\
1 - p (1-p) e^{-\alpha_2(t_i-x_1)} & x_1 \leq t_i \leq x_2 \\
1 & x_2 \leq t_i \leq T 
\end{cases}
\]

\[
p''(t_i) = \begin{cases} 
\text{Max}[0, 1 - (1-p) e^{-\alpha_2(t_i-x_1)}] & 0 \leq t_i \leq x_1 \\
1 - e^{-\alpha_1 t_i} & x_1 \leq t_i \leq x_2 \\
1 - (1-p) e^{-\alpha_2(t_i-x_1)} & x_2 \leq t_i \leq T 
\end{cases}
\]

and at \( t_i = T \) we have \( p''(T) = 1 \).

The definition of \( G \) and \( v \) functions for Eqs. (2.5.49) are the same as for Model A and with a similar reasoning we have (See Fig. 5-6):

\[
G(t_i, p_i, t_{i-1}) = 1 - (1 - p_i) e^{c(t_i, p_i)(t_i - t_{i-1})},
\]

\[0 < t_i \leq x_1\]  

(2.5.52)

where

\[
c(t_i, p_i) = - \frac{\log (1-p_i)}{t_i}.
\]

(2.5.53)

\(G_0(t_{m1})\) is the particular value of \( G \) function at \( t_{i-1} = t_{m1} \) and its value is (See Fig. 5-6).
\[ G_0(t_{m_1}) = 1 - e^{-\alpha_1 t_{m_1}}, \]  
(2.5.54)

also

\[ v(t_i, p_i, c_{i+1}, t_{i-1}) = \text{Max}(1 - (1 - p_i) e^{c_{i+1}(t_i - t_{i-1})}, 0), \]
\[ 0 < t_i < x_1 \]  
(2.5.55)

and \( v(t_{m_1+1}, p_{m_1+1}, t_{m_1}) \) is the particular value of \( v \) function at \( t_{m_1} \) and its value is

\[ v(t_{m_1+1}, p_{m_1+1}, t_{m_1}) = \text{Max}(1 - (1 - p_1)e^{-c_{m_1+1}(t_{m_1} - x_1)}, 0). \]  
(2.5.57)

The value of \( c_i \), i.e., the parameter of the t.e.d. connecting point \( (t_i, p_i) \) to \( (t_{i-1}, p_{i-1}) \), according to Eq. (2.5.5) is given by

\[ c_i = -\frac{\log_e(1-p_i)}{t_i - t_{i-1}}, \]  
(2.5.58)

and the value of \( c_{m_1+1} \) is given by

\[ c_{m_1+1} = -\frac{\log_e(1-p_{m_1+1})}{t_{m_1+1} - x_1}. \]  
(2.5.59)

We now choose for each \( (t_{m_1+1}, p_{m_1+1}) \) that \( m_1 = m_1^* \) which renders \( K_{m_1+1}^* \) a minimum. That is

\[ K_{m_1+1}^{**}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1}) = \text{Min}\{K_{m_1+1}^*(t_{m_1+1}, p_{m_1+1}, c_{m_1+1}) \}. \]
\[ m_1 = 0, 1, \ldots, N_1. \]  
(2.5.60)
$K^{**+1}(t_m^{**+1}, P_m^{**+1}, c_m^{**+1})$ is the minimax expected total maintenance cost from time $t_0 = 0$ to $t_m^{**+1}$ where $t_m^{**+1}$ is larger or equal to $x_1$ with $F(t_m^{**+1}) = P_m^{**+1}$. Also $m^*$ is the optimum number of checks or inspections before time $t = x_1$.

Second optimization phase - The iterative procedure continues taking the $K^{**+1}$ values at each point $(t_m^{**+1}, P_m^{**+1})$, $x_1 < t_m^{**+1} < x_2$, as the minimax expected maintenance costs up to that point for the first stage of the second phase of the optimization. That is

$$K^{**+2}(t_m^{**+2}, P_m^{**+2}, c_m^{**+2})$$

$$x_1 < t_m^{**+2} < x_2$$

$$p''(t_m^{**+2}) < p_m^{**+2} < p'(t_m^{**+2})$$

$$= \text{Min} \quad \text{Max}(K^{**+1}(t_m^{**+1}, P_m^{**+1}, c_m^{**+1}))$$

$$x_1 \leq t_m^{**+1} < t_m^{**+2} \quad v(t_m^{**+2}, P_m^{**+2}, c_m^{**+3}, t_m^{**+1}) < P_m^{**+1} < G(t_m^{**+2}, P_m^{**+2}, t_m^{**+1})$$

$$+ J^{**+2}(t_m^{**+1}, P_m^{**+1}, t_m^{**+2}, P_m^{**+2})$$

$$c_m^{**+1} < a_2$$

$$K^{**+j}(t_m^{**+j}, P_m^{**+j}, c_m^{**+j+1})$$

$$x_1 < t_m^{**+j} < x_2$$

$$p''(t_m^{**+j}) < p_m^{**+j} < p'(t_m^{**+j})$$

$$= \text{Min} \quad \text{Max}(K^{**+j}(t_m^{**+j}, P_m^{**+j}, c_m^{**+j}))$$

$$x_1 < t_m^{**+j-1} < t_m^{**+j} \quad v(t_m^{**+j-1}, P_m^{**+j}, c_m^{**+j+1}, t_m^{**+j-1}) < P_m^{**+j-1} <$$

$$G(t_m^{**+j}, P_m^{**+j}, t_m^{**+j-1})$$
\[
+ J_m^{*} + j \left( t_m^{*} + j - 1, \ p_m^{*} + j - 1, \ t_m^{*} + j, \ p_m^{*} + j \right) \quad j = 3, 4, \ldots, \ m_2 - m_1
\]

\[
K_m^{**}(t_m^{*} + 1, \ p_m + 1, \ c_m + 1)
\]

(2.5.62)

\[x_2 \leq t_{m_2} + 1 \leq T\]

\[
= \text{Min} \quad \text{Max}(K_m^{**}(t_m^{*} + p_m, c_m), J_m^{*} + J_m^{*} + 1(t_m^{*} + p_m, t_m^{*} + 1, p_m + 1)) \quad x_1 \leq t_m \leq x_2
\]

\[v(t_m^{*} + 1, p_m + 1, c_m + 1, t_m) < p_m < G(t_m)\]

\[m_2 = 1 + m_1, 2 + m_1, \ldots, N_2 + m_1, \]

\[c_2^* = \frac{- \log \left( \frac{1 - p_2}{1 - p_m^2} \right)}{x_2 - t_{m_2}}.\]

\(N_2\) is the maximum possible number of inspections in the interval \([x_1, x_2]\) and \(c_m + 1 \leq c_m + 2, \ldots, \leq c_m + 1\) since failure distribution \(F(w)\) is IFR. \(p'(t_1)\) and \(p''(t_1)\) are given by (2.5.2.13) and (2.5.2.14) respectively. The value of \(G\) function is given by (See Fig. 5.6)

\[
G(t_m^{*} + j, p_m^{*} + j, t_m^{*} + j - 1) = 1 - (1 - p_1)e^{-c(t_m^{*} + j, p_m^{*} + j)(t_m^{*} + j - 1 - x)}
\]

where \(1 - p_m^{*} + j \leq \log \left( \frac{1 - p_m^*}{1 - p_1} \right)\)

\[c(t_m^{*} + j, p_m^{*} + j) = - \frac{1 - p_m^{*} + j}{t_m^{*} + j - x_1}.\]
\( G_0(t_{m_2}^*) \) is the particular value of \( G \) function at point \( t_{m_2}^* \) (See Fig. 5.6.)

\[
G_0(t_{m_2}) = 1 - (1 - p_2) e^{-\alpha_2(t_{m_2} - x_1)}.
\] (2.5.65)

The value of \( v \) function is given by

\[
v(t_{m_1}^* + j, p_{m_1}^* + j, c_{m_1}^* + j + 1, t_{m_1}^* + j - 1) = \text{Max}\{1 - (1 - p_{m_1}^* + j) e^{-\alpha_1(t_{m_1}^* + j - 1)}
\]

or \( 1 - e^{-\alpha_1 t_{m_1}^* + j - 1} \}
\] (2.5.66)

\( x_1 < t_{m_1}^* + j < x_2. \)

\( v(t_{m_2}^* + 1, p_{m_2}^* + 1, t_{m_2}) \) is the particular value of \( v \) function at \( t_{m_2} \)

and its value is (See Fig. 5.6.)

\[
v(t_{m_2}^* + 1, p_{m_2}^* + 1, t_{m_2}) = \text{Max}\{1 - (1 - p_2) e^{-\alpha_1 t_{m_2}} \}
\]

where

\[
c_{m_2}^* + 1 = -\frac{\log_e(\frac{1 - p_{m_2}^* + 1}{1 - p_2})}{t_{m_2}^* + 1 - x_2}.
\] (2.5.68)

The value of \( c_{m_1}^* + j \), i.e., the parameter of t.e.d. connecting point \((t_{m_1}^* + j, p_{m_1}^* + j)\) to \((t_{m_1}^* + j - 1, p_{m_1}^* + j - 1)\), according to Eq. (2.5.5) is given by
Now we choose for each \((t_{m_2+1}, p_{m_2+1})\) that \(m_2 = m_2^* = m_1^* + j^*\) which renders \(K_{m_2+1}^{**}\) a minimum. That is

\[
K_{m_2+1}^{**}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1}) = \min \{K_{m_2+1}^{**}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})
\]

\[
\text{or } R_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})
\]

\[
m_2 = m_1^* \text{ or } j^* = 0,
\]

where \(R_{m_2+1}\) is calculated in the first optimization phase for \(m_2 = m_1\) or equivalently when no inspection is performed in the interval \([x_1, x_2]\).

\[
R_{m_2+1} = n(T, 1, \infty) = \min \{K_{m_2+1}^{**}(T, 1, \infty)
\]

\[
\text{or } R_{m_2+1}(T, 1, \infty),
\]

\[
m_2 = m_1 = 0, 1, 2, \ldots, N_1
\]

Also \(m_2^* = m_1^* + j^*\) where \(m_2^*\) is the optimal total number of inspections at the end of second stage and \(j^*\) is the optimal number of inspections only in the second stage.
Third optimization phase - The iterative procedure continues taking the $K_{m^2+1}$ values at each point $(t_{m^2+1}, p_{m^2+1})$, $x_2 < t_{m^2+1} < T$ as the minimax expected total maintenance cost up to that point for the first stage of the third phase of optimization. That is

$$K_{m^2+2} (t_{m^2+2}, p_{m^2+2}, c_{m^2+3})$$

$x_2 < t_{m^2+2} < T$

$$p''(t_{m^2+2}) < p_{m^2+2} < p'(t_{m^2+2})$$

$$= \text{Min} \quad \text{Max}(K_{m^2+1}(t_{m^2+1}, p_{m^2+1}, c_{m^2+1}))$$

$$x_2 < t_{m^2+1} < t_{m^2+2} \quad v(t_{m^2+2}, p_{m^2+2}, c_{m^2+3}, t_{m^2+1}) < p_{m^2+1} < G(t_{m^2+2}, p_{m^2+2}, t_{m^2+1})$$

$$+ J_{m^2+2} (t_{m^2+1}, p_{m^2+1}, t_{m^2+2}, p_{m^2+2})$$

and

$$K_{m^2+\lambda} (t_{m^2+\lambda}, p_{m^2+\lambda}, c_{m^2+\lambda+1})$$

$x_2 < t_{m^2+\lambda} < T$

$$p''(t_{m^2+\lambda}) < p_{m^2+\lambda} < p'(t_{m^2+\lambda})$$

$$= \text{Min} \quad \text{Max}(K_{m^2+\lambda-1}(t_{m^2+\lambda-1}, p_{m^2+\lambda-1}, c_{m^2+\lambda}))$$

$$x_2 < t_{m^2+\lambda-1} < t_{m^2+\lambda} \quad v(t_{m^2+\lambda}, p_{m^2+\lambda}, c_{m^2+\lambda+1}, t_{m^2+\lambda-1}) < p_{m^2+\lambda-1} < G(t_{m^2+\lambda}, p_{m^2+\lambda}, t_{m^2+\lambda-1})$$
In particular, the term for the total optimal loss with \( n = m_2^{*+\ell} \) inspections, is

\[
K_{n=m_2^{*+\ell}}^{**}(T,1,\infty)
\]

\[
= \min \max \left( K_{m_2^{*+\ell}+1}^{**}(t_{m_2^{*+\ell}+1}, t_{m_2^{*+\ell}+1}, p_{m_2^{*+\ell}+1}, p_{m_2^{*+\ell}+1}, t_1^*, p_1^{*-1}) \right)
\]

\[\ell = 1, 3, \ldots, N_3, \quad (2.5.73)\]

where \( N_3 \) is the maximum possible number of inspections and \( G \) and \( v \) functions are given by (See Fig. 5-6.)

\[
G(t_{m_2^{*+\ell}}, p_{m_2^{*+\ell}}, t_{m_2^{*+\ell}+1}) = 1 - (1 - p_{m_2^{*+\ell}})(t_{m_2^{*+\ell}+1} - x_2)
\]

\[\ell = 1, 3, \ldots, N_3, \quad (2.5.74)\]

where

\[
c(t_{m_2^{*+\ell}}, p_{m_2^{*+\ell}}) = \frac{1 - p_{m_2^{*+\ell}}}{\log_e \left( \frac{1 - p_{m_2^{*+\ell}}}{1 - p_{m_2^{*+\ell}}} \right)}
\]

\[\ell = 1, 3, \ldots, N_3, \quad (2.5.74)\]

and \( G(t_{m_2^{*+\ell}+1}) \) is particular value of \( G \) function at \( t_{n-1} = t_{m_2^{*+\ell}+1} \) and is given by

\[
G(t_{m_2^{*+\ell}+1}) = 1
\]

\[\ell = 1, 3, \ldots, N_3, \quad (2.5.75)\]
\[ v(t_{m^*_2+\varepsilon}, p_{m^*_2+\varepsilon}, c_{m^*_2+\varepsilon}+1, t_{m^*_2+\varepsilon}-1) \]

\[ = \text{Max}\{ 1 - (1 - p_{m^*_2+\varepsilon}) e^{-\alpha_2(t_{m^*_2+\varepsilon}-1 - x_2)} \} \]

or \( 1 - (1 - p_{m^*_2+\varepsilon}) e^{-\alpha_2(t_{m^*_2+\varepsilon}-1 - x_2)} \),

and at \( t_{n-1} = t_{m^*_2+\varepsilon}-1 \) we have

\[ v(t_{m^*_2+\varepsilon}-1) = 1 - (1 - p_{m^*_2+\varepsilon}) e^{-\alpha_2(t_{m^*_2+\varepsilon}-1 - x_2)} \].

The value of \( c_{m^*_2+\varepsilon} \) can be found according to Eq. (2.5.5) for \( c_n \neq c_{m^*_2+\varepsilon} \) by

\[ c_{m^*_2+\varepsilon} = \frac{1 - p_{m^*_2+\varepsilon}}{\log_e \left( \frac{t_{m^*_2+\varepsilon} - t_{m^*_2+\varepsilon}-1}{t_{m^*_2+\varepsilon}} \right)} \]

\[ \varepsilon = 2, \ldots, N_3, \]

subject to \( \alpha_2 < c_{m^*_2+1} \leq c_{m^*_2+2} \leq \ldots \leq c_{n-1} = m^*_2+\varepsilon-1 \) since the failure distribution should be IFR and \( c_{m^*_2+1} \) is given by Eq. (2.5.68) for \( m^*_2 = m_2^* \).

The optimal loss \( L^{**} \) at \( n^* = m_2^* + \varepsilon^* = m_1^* + j^* + \varepsilon^* \) where \( n^* \) is the optimal total number of inspection in the interval \([0,T]\) and \( \varepsilon^* \) is the optimum number of inspections in the interval \([x_2,T]\) is then found from

\[ L^{**} = \text{Min}\left\{ \frac{K^{**}(T,1,\varepsilon)}{n=m_2^*+\varepsilon}, \quad \frac{R(T,1,\varepsilon)}{n=m_2^*+1} \right\}, \]

\[ \varepsilon = 2, 3, \ldots, N_3 \]

(2.5.78)

where \( R(T,1,\varepsilon) \), the total expected loss if \( \varepsilon = 1 \), is given by Eq. (2.5.71).
The P** and failure distribution F** defined by Eqs. (2.1.3.4) can also be found if needed by a backward recursive procedure from the computer print out.

2.6 DISCUSSION

In this chapter the mathematical meaning of a minimax policy was stated followed by the statement of the present problem which consists of two models A and B according to having information about one point or two points of an increasing failure rate (IFR) distribution of a system respectively. The special properties of IFR distribution utilized in the logic of the formulation of the model A and B were proved and explained by property No 1 and property No 2.a and 2.b. The form of the objective function suitable for a recursive relationship was presented and derived. Both model A and model B have been formulated by functional equations of dynamic programming. Three state variables, i.e., the last inspection time \( t_i \), the cumulative failure probability of the system \( F(t_i) = p_i \) up to time \( t_i \) and the parameter of a transformed failure distribution passing through points \( (t_i, p_i) \) and \( (t_{i+1}, p_{i+1}) \) or \( c_{i+1} \) represent the state of the system. The control variables are \( t_{i-1} \), the timing of the previous inspection and \( p_{i-1} \), the failure probability of the system at \( t_{i-1} \). The stages represent the number of inspections which ranges from 2 to maximum possible number of inspections in the interval \( [0,T] \). The minimax policy first maximizes at each stage and for a fixed state vector \( (t_i, p_i, c_{i+1}) \), the total expected loss from initial state vector \( (t_0 = 0, p_0 = 0, c_{0+1}) \) to the final state \( (t_i, p_i, c_{i+1}) \) by the choice of the control variable \( p_{i-1} \) and then minimizes the total expected loss by the choice of control variable \( t_{i-1} \). As it is clear the number of stages should be also optimized.
This has been done in model A in two phases, i.e., in phase one the optimal number of inspections or stages in the interval, \([0,x_1]\) is obtained and in the second and final phase the total optimum number of inspections and their timing is obtained. In model B optimization has been accomplished in three phases where the optimal number of inspections in the intervals \([0,x_1), [x_1,x_2), [x_2,T]\) are obtained.

In Chapter 3 the computational details and procedures will be presented.
CHAPTER 3

COMPUTATIONAL PROCEDURE

3.1 INTRODUCTION

In this chapter the numerical procedure for the computer program is presented for each of the two models, i.e., Model A and Model B. For each model first the range of the possible values of state and control variables is stated and then the computational and numerical procedures is followed step by step with reference to the computer program in Appendix B.

3.2 MODEL A

In Chapter 2, there are three state variables and two control variables in the formulation of the problem. The state variables are:

1. The inspection or checking time of the ith inspection or checking denoted by \( t_i \).
2. The cumulative failure probability of the system at ith inspection denoted by \( p_i \).
3. The parameter of the transformed exponential failure distribution connecting point \( (t_i, p_i) \) to point \( (t_{i+1}, p_{i+1}) \) on the \( F \) v.s. \( t \) diagram denoted by \( c_{i+1} \).

The control variables are:

1. The inspection time of the (i-1)th inspection denoted by \( t_{i-1} \).
2. The cumulative failure probability of the system at (i-1)th inspection denoted by \( p_{i-1} \).

\( t_i \) and \( t_{i-1} \) according to the assumption \( f \) in (2.2) are discrete values ranging from 0 to \( T \) the maximum life time of the system. \( p_i \) and \( p_{i-1} \) have continuous values whose range is given by \( p' \) and \( p'' \) functions defined
in Chapter 2. \( c_{i+1} \) is also continuous and the range of its values is given by

\[
R_1 \leq c_{i+1} \leq R_2
\]

(3.2.1)

where the values of \( R_1 \) and \( R_2 \) are given by

\[
R_1(t_i, p_i) = \begin{cases} 
\frac{\log_e(1-p_i)}{t_i} & 0 < t_i \leq x \\
\frac{\log_e(1-p_i)}{t_i-x} & x < t_i \leq T
\end{cases}
\]

(3.2.2)

and

\[
R_2(t_i, p_i) = \begin{cases} 
\frac{\log_e(1-p_i)}{x-t_i} & 0 < t_i < x \\
\frac{\log_e(1-p_i)}{t_{i+1}-t_i} & x \leq t_i < T
\end{cases}
\]

(3.2.3)

The reason for the above values for \( R_1 \) and \( R_2 \) is based on the properties of exponential and IFR failure distributions explained in (2.3) and on the same line of reasoning used in (2.5.2) for establishing the feasible region for IFR distributions passing through points \((0,0),(x_1,p_1)\) and \((x_2,p_2)\) on the F v.s. t diagram (See Fig. 6) with the difference that here the feasible region is given for the IFR distributions which pass through points \((0,0),(t_i,p_i)\) and \((x,p)\) for \( t_i \) in the time interval \((0,x)\) and through points \((x,p),(t_i,p_i)\) and \((T,1)\) for \( t_i \) in the time interval \((x,T)\).
The input vector elements are:

\( a \) = The cost of undetected failure per unit time.

\( I \) = The cost of every inspection with the exception of the inspection cost at \( t = 0 \) where \( I(0) = 0 \).

\( x \) = the time at which the cumulative failure probability of the system is known.

\( p \) = The known cumulative failure probability of the system at time \( x \).

\( T' = \{t_0 = 0, t_1, t_2, \ldots, x, \ldots t_N = T\} \)

= The set of possible inspection times.

\( PN_1 \) = Number of increments of \( p_1 \) & \( p_{1-1} \) in the feasible region at \( t_1 \) & \( t_{1-1} \) respectively

\( AL_1 \) = Number of increments of \( c_{i+1} \) in the feasible range given by Eqs. (3.1.1-3).

The computational procedure consist of the following steps;

Phase 1:

1. Divide the feasible range of \( p_i \) values, given by \( p' \) and \( p'' \) functions in Eqs. (2.5.15-18), at each time \( t_i \) where \( t_i \) is an element of the set \( T' \), into \( PN_1 \) increments.

2. Divide the feasible range of \( c_{i+1} \) values, given by \( R_1 \) and \( R_2 \) functions in Eqs. (3.2.1-3), at each point \((t_i, p_i)\) on the \( F \) v.s. \( t \) diagram, in to \( AL_1 \) increments.

3. Assume no inspection in the time interval \((0, T)\) with only one inspection at time \( T \), i.e., set the stage number \( m \) equal to 0.

4. Calculate the maximum expected total maintenance cost when \( m \) inspections are performed in the interval \((0, x)\), i.e., \( R^*_{n=m+1}(T, 1, n) \) and
\( K_{m+1}(t_{m+1}, p_{m+1}, c_{m+1}) \) for all possible values of \( t_{m+1}, p_{m+1} \) and \( c_{m+1} \) from Eq. (2.5.7) for \( i = m+1 \) and set of equations (2.5.14).

5. Set current \( m = \) old \( m+1 \) and calculate \( K_0(t_i, p_i, c_{i+1}) \) given by Eq. (2.5.14) when \( i = m \) and for all possible values of \( t_i, p_i \) and \( c_{i+1} \).

6. Repeat step 4 and compare the \( R^*_n(T, 1, \infty) \) and \( K_{m+1}(t_{m+1}, p_{m+1}, c_{m+1}) \) values for the old \( m \) values with their values for current \( m \) respectively and choose that value of \( R^*_n \) and \( K_{m+1} \) for each point \((t_m, p_m)\) which is smaller and discard the larger values.

7. Compare the maximum value of \( m \), i.e., \( N_1 \) with the current \( m \). If \( N_1 > m+1 \), go back to step 5 and if \( N_1 < m+1 \) go to step 8.

8. Take that value of \( R^*_n(T, 1, \infty) \) and \( K_{m+1}(t_{m+1}, p_{m+1}, c_{m+1}) \) at each point \((t_{m+1}, p_{m+1})\) which is minimum for all values of \( m = 0, 1, \ldots, N_1 \) and denote them as \( R^{**}_{n=m+1}(T, 1, \infty) \) and \( K^{**}_{m+1}(t_{m*+j}, p_{m*+j}, c_{m*+j}) \) respectively. \( m^* \) is the optimum number of inspections in the time interval \((0, x)\).

**Phase 2**

9. Consider the \( K^{**}_{m+1}(t_{m*+j}, p_{m*+j}, c_{m*+j}) \) values as the minimax expected maintenance cost upto the time of the first inspection in the time interval \([x, T]\) or the first stage \( j = 1 \) of the second phase.

10. Calculate \( K^{**}_{n=m*+j}(T, 1, \infty) \) from Eq. (2.5.32) for \( j = 1 \).

11. Set current \( j = \) old \( j+1 \). Compare the maximum possible number of inspections in the interval \([x, T]\), i.e., \( N_2 \) with the current \( j \). If \( N_2 > j \) go to step 12 and if \( N_2 < j \) go to step 14.

12. Calculate \( K^{**}_{m*+j}(t_{m*+j}, p_{m*+j}, c_{m*+j+1}) \) from Eq. (2.5.31) for all feasible values of \( t_{m*+j}, p_{m*+j} \) and \( c_{m*+j+1} \).

13. Calculate \( K^{**}_{n=m*+j}(T, 1, \infty) \) from Eq. (2.5.32). Compare \( K^{**}_{n=m*+j} \) values for old \( j \) values with its value for current \( j \) and keep the smaller value and discard the larger values. Then go back to step 11.
14. Find $L^*$, the upper bound for the optimum expected total maintenance cost with known system failure distribution from Eq. (2.5.37).

At each stage $i$ and each state $(t_i, p_i, c_{i+1})$ the values of the $t_{i-1}, p_{i-1}$, i.e., the control variables of stage $i$ which minimaxes $K_i(t_i, p_i, c_{i+1})$ are recorded and printed out. The value of $c_i$ is taken as the first larger value of $R_i + \Delta r$ where $r$ is the number of increments and $\Delta$ is the size of the increment obtained from

$$\Delta = \frac{R^*_2 - R_1}{AL_1},$$

(3.2.4)

respect to $cc$ given according to Eq. (2.5.5) by

$$cc = \frac{\log_e \left( \frac{1-p_i}{1-p_{i-1}} \right)}{t_i - t_{i-1}}.$$  

(3.2.5)

The $m^*$ values for the first stage of the second phase are being recorded and printed out at the beginning of the second phase, i.e., when $j = 1$. Finally the optimum total number of stages or $n^* = m^* + j^*$ is recorded together with $L^*$. The policy $p^*$ and failure distribution $F^*$ defined by Eqs. (2.1.3-4) can then be found easily if needed by searching backward from stage $n^* = m^* + j^*$ to $n^*-1, n^*-2, \ldots$, until $m^*+1$. Then the search continues from stage $m^*$ to $m^*-1, m^*-2, \ldots$, until the first stage.

3.3 MODEL B

In this model also there are three state variables, i.e., $t_i, p_i, c_{i+1}$ and two control variables, i.e., $t_{i-1}, p_{i-1}$ which are defined in the same way as for Model A. Here also $t_i$ and $t_{i-1}$ are discrete values according to assumption $f$ in (2.2) and range from 0 to $T$ the maximum life time of
the system. $p_i$ and $p_{i-1}$ have continuous values whose range is given by $p'$ and $p''$ functions defined in Chapter 2. $c_{i+1}$ is also continuous and its range is given by

$$R_1 \leq c_{i+1} \leq R_2,$$  \hspace{1cm} (3.3.1)

where the values of $R_1$ and $R_2$ are given by

$$R_1(t_i, p_i) = \begin{cases} \frac{\log_e(1-p_i)}{t_i} & 0 < t_i \leq x_1 \\ \frac{\log_e(1-p_i)}{t_i - x_1} & x_1 < t_i \leq x_2 \\ \frac{\log_e(1-p_i)}{t_i - x_2} & x_2 < t_i \leq T \end{cases}$$  \hspace{1cm} (3.3.2)

and

$$R_2(t_i, p_i) = \begin{cases} \frac{\log_e(1-p_i)}{x_1 - t_i} & 0 < t_i < x_1 \\ \frac{\log_e(1-p_i)}{x_2 - t_i} & x_1 \leq t_i < x_2 \\ \frac{\log_e(e)}{t_{i+1} - t_i} & x_2 \leq t_i \leq T \end{cases}$$  \hspace{1cm} (3.3.3)
The reason for the above values for \( R_1 \) and \( R_2 \) is based on the properties of exponential and IFR failure distributions and on the same line of reasoning utilized in (2.5.2) for establishing the feasible region for IFR distributions passing through points \((0, 0)\) \((x_1, p_1)\) and \((x_2, p_2)\) on the F v.s. t diagram (See Fig. 6.) with the difference that here the feasible region is given for IFR distributions which pass through points \((0, 0)\), \((t_i, p_i)\) and \((x_1, p_1)\) for \(t_i\) in the interval \((0, x_1)\), through points \((x_1, p_1)\), \((t_i, p_i)\) and \((x_2, p_2)\) for \(t_i\) in the interval \((x_1, x_2)\) and through points \((x_2, p_2)\) \((t_i, p_i)\) and \((T, 1)\) for \(t_i\) in the interval \((x_2, T)\). The input vector elements consists of \( a, I, PN_1 \) and \( AL_1 \) as defined in (3.2) plus

\[ x_1 \text{ and } x_2 = \text{The times at which the cumulative failure distribution of the system are known where } x_1 < x_2. \]

\[ p_1 \text{ and } p_2 = \text{The known cumulative failure probabilities of the system at times } x_1 \text{ and } x_2 \text{ respectively where } p_1 < p_2. \]

\[ T' = \{t_0 = 0, t_1, \ldots, x_1, \ldots, x_2, \ldots, T_N = T\} \]

\[ = \text{The set of possible inspection times.} \]

The computational procedure consists of the following steps;

Phase 1:
1. Divide the feasible range of \( p_i \) values given by \( p' \) and \( p'' \) functions in Eqs. (2.5.50-51), at each time \( t_i \) where \( t_i \) is an element of the set \( T' \), into \( PN_1 \) increments.
2. Divide the feasible range of \( c_{i+1} \) values, given by \( R_1 \) and \( R_2 \) functions in Eqs. (3.3.1-3), at each point \((t_i, p_i)\) on the F v.s. t diagram, into \( AL_1 \) increments.
3. Assume no inspection in the time interval \((0, T)\) with only one inspection at time \(T\), i.e., set the stage numbers \(m_1 = m_2 = 0\).

4. Calculate the minimax cost \(K^*_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1})\) for all possible values of \(t_{m_1+1}, p_{m_1+1}\) and \(c_{m_1+1}\) from Eqs. (2.5.44) for \(i = m_1+1\), and the set of equations (2.5.49). Also calculate the minimax cost \(\bar{K}^*_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})\) for all possible values of \(t_{m_2+1}, p_{m_2+1}\) and \(c_{m_2+1}\) from Eq. (2.5.44) for \(i = m_2+1\), \(m_1 = m_2 = m_1\) and the set of equations (2.5.49).

5. Set current \(m_1 = \text{old } m_1+1\) and calculate \(K^*_i(t_i, p_i, c_{i+1})\) given by Eq. (2.5.49) when \(i = m_1\) and for all possible values of \(t_i, p_i\) and \(c_{i+1}\).

6. Repeat step 4 and compare the \(K^*_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1})\) and \(\bar{K}^*_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})\) values for old \(m_1\) values with their values for current \(m_1\) respectively and choose that value of \(K^*_{m_1+1}\) and \(\bar{K}^*_{m_2+1}\) for each point \((t_{m_1+1}, p_{m_1+1})\) and \((t_{m_2+1}, p_{m_2+1})\) which is smaller and discard the larger values.

7. Compare the maximum value of \(m_1\), i.e., \(N_1\) with the current value of \(m_1\).

   If \(N_1 \geq m_1+1\), go back to step 5 and if \(N_1 < m_1 + 1\) go to Step 8.

8. Take that value of \(K^*_{m_1+1}\) at each point \((t_{m_1+1}, p_{m_1+1})\) which is minimum for all values of \(m_1 = 0, 1, ..., N_1\) and denote it as \(K^{**}_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1})\) where \(m_1^*\) is the optimum number of inspections in the time interval \((0, x_1)\).

   Phase 2

9. Consider the \(K^{**}_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1})\) values as the minimax expected maintenance cost up to the time of the first inspection in the time interval \([x_1, x_2]\) or the first stage \(j = 1\) of the second phase.

10. Calculate \(K^{**}_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})\) values for all feasible points \((t_{m_2+1}, p_{m_2+1})\) on the \(F\) v.s. \(t\) diagram from Eq. (2.5.44) setting \(i = m_2+1\) and \(m_2 = m_1^*+1\) and from Eq. (2.5.62).
11. Set current \( m_2 = \text{old } m_2 + 1 \). Compare the maximum possible number of inspections in the interval \([x_1, x_2]\), i.e., \( N_2 + m^*_i \) with the current value of \( m_2 \). If \( N_2 + m^*_i > m_2 \) go to step 12. If \( N_2 < m_2 \) go to step 15.

12. Calculate \( K^{**}(t_{m^*_i + j}, p_{m^*_i + j}, c_{m^*_i + j}) \) for all feasible values of \( t_{m^*_i + j}, p_{m^*_i + j} \) and \( c_{m^*_i + j} \) from Eq. (2.5.44) for \( i = m^*_i + j = m_2 \) and from Eq. (2.5.61).

13. Calculate \( K^{**}_{m_2 + 1}(t_{m_2 + 1}, p_{m_2 + 1}, c_{m_2 + 1}) \) values for all feasible points \((t_{m_2 + 1}, p_{m_2 + 1})\) from Eq. (2.5.44) setting \( i = m_2 + 1 \) and from Eq. (2.5.62).

14. Compare \( K^{**}_{m_2 + 1} \) values for old \( m_2 \) values with \( K^{**}_{m_2 + 1} \) values for current \( m_2 \) and for all feasible points \((t_{m_2 + 1}, p_{m_2 + 1})\) and keep the smaller values and discard the larger ones. Go back to step 11.

15. Take that value of \( K^{**}_{m_2 + 1}(t_{m_2 + 1}, p_{m_2 + 1}, c_{m_2 + 1}) \) at each point \((t_{m_2 + 1}, p_{m_2 + 1})\) except point \((T,1)\) which is minimum for all values of \( m_2 = m^*_i + 1, m^*_i + 2, \ldots, m^*_i + N_2 \) as found in step 14 and compare it with the minimum value of \( K^{**}_{m_2 + 1}(t_{m_2 + 1}, p_{m_2 + 1}, c_{m_2 + 1}) \) as found in step 6. Denote the smaller value of them at each point \((t_{m_2 + 1}, p_{m_2 + 1})\) except point \((T,1)\) as \( K^{***}_{m_2 + 1}(t_{m^*_2 + 1}, p_{m^*_2 + 1}, c_{m^*_2 + 1}) \) where \( m^*_2 \) is the optimum number of inspection in the interval \((0, x_2)\).

Phase 3

16. Consider \( K^{***}_{m_2 + 1}(t_{m^*_2 + 1}, p_{m^*_2 + 1}, c_{m^*_2 + 1}) \) values as the minimax expected maintenance cost upto the time of the first inspection in the interval \([x_2, T)\) or the first stage \( \varepsilon = 1 \) of the third phase.

17. Calculate \( K^{**}(T,1,\omega) \) from Eq. (2.5.73) for \( \varepsilon = 1 \)

18. Set current \( \varepsilon = \text{old } \varepsilon + 1 \). Compare the maximum possible number of inspections in the interval \([x_2, T)\), i.e., \( N_3 \) with the current \( \varepsilon \). If
N_3 \geq i \text{ go to step 19 and if } N_3 < i \text{ go to step 21.}

19. Calculate \( K^{**}_{m_i^2+i} (t_{m_i^2+i}, p_{m_i^2+i}, c_{m_i^2+i+1}) \) from Eq. (2.5.72) and Eq. (2.5.44) for \( i = m_2^* + i \) and \( m_2^* \neq m_1^* \) for all feasible values of \( t_{m_2^* + i}, p_{m_2^* + i} \) and \( c_{m_2^* + i} \) and \( c_{m_2^* + i}^2 \).

20. Calculate \( K^{**}_{n=m_2^*+1} (T,1,\infty) \) from Eq. (2.5.73) and from Eq. (2.5.44) for \( i = n \). Compare \( K^{**}_{n=m_2^*+i} \) values for old \( i \) values with its value for current \( i \) and keep the smaller value and discard the larger values. Then go back to step 18.

21. Find L** the upper bound for the optimum expected total maintenance cost with known system failure distribution passing through points \((0,0),(x_1,p_1),(x_2,p_2)\) and \((T,1)\) from Eq. (2.5.78) where \( L_{n=m_2^*+1}^*(T,1,\infty) \) is given by Eq. (2.5.71). The value of \( i^* \), i.e., the optimum number of inspections in the interval \([x_2,T)\) is then obtained.

As for Model A, at each stage \( i \) and each state \((t_i, p_i, c_{i+1})\), the values of the \( t_{i-1}, p_{i-1} \), i.e., the control variables of stage \( i \) which minimaxes \( K_i(t_i, p_i, c_{i+1}) \) are recorded and printed out. The value of \( c_i \) is taken as the first larger value of \( R_1 + \Delta r \) where \( r \) is the number of increments and \( \Delta \) is the size of the increment obtained from

\[
\Delta = \frac{R_2 - R_1}{AL_1} \quad (3.3.4)
\]

respect to \( cc \) given according to Eq. (2.5.5) by

\[
cc = \frac{1 - p_i}{\log_e \left( \frac{1 - p_i}{1 - p_{i-1}} \right)} + \frac{t_i - t_{i-1}}{\ln(1 - p_i)} \quad (3.3.5)
\]

The \( m^*_i \) values for the first stage of the second phase is being recorded and printed out at the beginning of the second phase, i.e., when
\( j = 1 \). The \( m^*_1 \) values for the first stage of the third phase are also being recorded and printed out at the beginning of the third phase, i.e., when \( \varepsilon = 1 \). Finally the optimum total number of stages or \( n^* = m^*_2 + \varepsilon^* = m^*_1 + j^* + \varepsilon^* \) is recorded together with value of \( c^* \). The policy \( p^* \) and failure distribution \( F^* \) defined by Eqs. (2.1.3-4) can then be found easily if needed by searching the printout backwardly from stage \( n^* = m^*_1 + j^* + \varepsilon^* \) to \( n^*-1, n^*-2, \ldots, \) until \( m^*_1 + j^* + 1 \). Then the search continues from stage \( m^*_1 + j^* \) to \( m^*_1 + j^*-1, m^*_1 + j^*-2, \ldots, \) until \( m^*_1 + 1 \). Now the search continues from stage \( m^*_1 \) to \( m^*_1 - 1, m^*_1 - 2, \ldots, \) until the first stage.

In appendix B the computer program together with its discription is presented. In Chapter 4 the result for both Model A and Model B are presented and comparison is made between the two models on the basis of \( L^* \) values obtained by each model.
CHAPTER 4

THE RESULTS & APPLICATION

4.1 INTRODUCTION

In this chapter the results obtained from computer programming for several input data has been presented and the application of the results has been discussed through an example problem utilizing the information obtained from it. The results are presented according to the following format.

1. The convergence and accuracy of the results.
2. Evaluation of the value of information about the failure distribution of a system through comparison of the maximum expected total maintenance cost $L^{**}$, applying Model A (location of one percentile known) and Model B (location of the two percentiles known).
3. Comparison of the actual optimum expected total maintenance cost for several IFR failure distributions with the maximum of the optimum total expected maintenance cost $L^{**}$ for both Model A and Model B.
4. Evaluation of the value of information about a system failure distribution with regard to the relative location of the known points.
5. Variation of $L^{**}$ with the inspection Cost $I$ for Model A and Model B.
6. Variation of $L^{**}$ with the cost per unit time of undetected failure $a$ for Model A and Model B.
7. Example problem.

4.2 CONVERGENCE AND ACCURACY

FOR MODEL A

The following input data
a = 35. $/unit time,
I = 1.4 $/inspection,
\( x = 0.4 \) unit time,
\( p = 0.180 \)

\( T' = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.1, 1.2, 1.3, 1.4\} \),

plus each of the following sets of values:

\[
S_1 = \begin{cases} 
  \text{PN}_1 = 10 \\
  \text{AL}_1 = 5 
\end{cases} 
\quad ; \quad 
S_2 = \begin{cases} 
  \text{PN}_1 = 20 \\
  \text{AL}_1 = 9 
\end{cases} 
\quad ; \quad 
S_3 = \begin{cases} 
  \text{PN}_2 = 25 \\
  \text{AL}_1 = 10 
\end{cases}
\]

were given for the computer program in Appendix B. The values of \( F^* \), \( P^* \) and maximum expected total maintenance cost for different numbers of inspections in each of the subsets of inspection times

\( T'_1 = \{0, 0.1, 0.2, 0.3\} \) and \( T'_2 = \{0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.1, 1.2, 1.3\} \)

are given in Table 1 where \( SS_1 \), \( SS_2 \) and \( SS_3 \) refer to the values of \( F^* \), \( P^* \) and maximum expected total cost v.s. number of inspections when the number of increments of state variables, \( c \) and \( p \) are given by sets \( S_1 \), \( S_2 \) and \( S_3 \) respectively. Table 1 also shows that the minimum value with respect to number of inspections of the maximum with respect to system failure distributions of IFR type passing through point \((x = 0.4, p = 0.180)\), i.e., \( L^* \) is equal to 10.1 for \( M_2 = 5 \). \( P^* \) shows a good convergence and accuracy since shifting from set \( S_2 \) to \( S_3 \) with higher number of state increments has not changed the solution for \( P^* \). However for \( F^* \) and expected total cost the table shows that although shifting from \( S_1 \) to \( S_2 \) and from \( S_2 \) to \( S_3 \) increases the convergence and accuracy but not to more than one digit of
Table 1. The Accuracy v.s. the Number of Increments for the State Variables $c$ and $p$ for Model A.

<table>
<thead>
<tr>
<th>$F^* = {p_1, p_2, \ldots, p_N}$</th>
<th>$P^* = {t_1, t_2, \ldots, t_n}$</th>
<th>Max. Expected Total Cost v.s. $M_i$, the Total No. of Inspections in Subset $T_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$SS_1$</td>
<td>$SS_2$</td>
</tr>
<tr>
<td>$p_1$</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$p_2$</td>
<td>0.048</td>
<td>0.048</td>
</tr>
<tr>
<td>$p_3$</td>
<td>0.094</td>
<td>0.094</td>
</tr>
<tr>
<td>$p_4$</td>
<td>0.138</td>
<td>0.138</td>
</tr>
<tr>
<td>$p_5$</td>
<td>0.180</td>
<td>0.180</td>
</tr>
<tr>
<td>$p_6$</td>
<td>0.306</td>
<td>0.233</td>
</tr>
<tr>
<td>$p_7$</td>
<td>0.422</td>
<td>0.283</td>
</tr>
<tr>
<td>$p_8$</td>
<td>0.529</td>
<td>0.330</td>
</tr>
<tr>
<td>$p_9$</td>
<td>0.626</td>
<td>0.380</td>
</tr>
<tr>
<td>$p_{10}$</td>
<td>0.716</td>
<td>0.427</td>
</tr>
<tr>
<td>$p_{11}$</td>
<td>0.797</td>
<td>0.471</td>
</tr>
<tr>
<td>$p_{12}$</td>
<td>0.871</td>
<td>0.512</td>
</tr>
<tr>
<td>$p_{13}$</td>
<td>0.939</td>
<td>0.551</td>
</tr>
<tr>
<td>$p_{14}$</td>
<td>1.00</td>
<td>0.585</td>
</tr>
<tr>
<td>$p_{15}$</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
accuracy has been obtained. Surely, increasing the number of increments of state variables increases the accuracy but the computational time also increases very rapidly since there are two continuous state variables, i.e., c and p. But since this method will give an upper bound for the total expected cost, in many cases the level of accuracy on this upper bound of the order obtained for this problem (i.e., $\frac{0.1}{10.1} \times 100 = 99\%$) can be sufficient.

FOR MODEL B

The following input data

\[ a = 35. \quad $/Unit Time, \]
\[ I = 1.4 \quad $/Inspection, \]
\[ x_1 = 0.4 \quad Unit Time, \]
\[ x_2 = 0.9 \quad Unit Time, \]
\[ p_1 = 0.180, \]
\[ p_2 = 0.560, \]
\[ T' = \{0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0,1.1,1.2,1.3,1.4\}, \]

plus each of the following sets of values:

\[ S_1 = \begin{cases} PN_1 = 10 \\ AL_1 = 5 \end{cases} \quad ; \quad S_2 = \begin{cases} PN_1 = 20 \\ AL_1 = 9 \end{cases} \quad ; \quad S_3 = \begin{cases} PN_1 = 25 \\ SL_1 = 15 \end{cases} \]

were given for the computer program in Appendix B. The values of $F^*$, $P^*$ and maximum total expected maintenance cost for different numbers of inspections in each of the subsets of inspections $T'_1 = \{0,0.1,0.2,0.3\}$,
$T_2^I = \{0.4, 0.5, 0.6, 0.7, 0.8\}$ and $T_3^I = \{0.9, 1.0, 1.1, 1.2, 1.3\}$ are given in Table 2 where $SS_1$, $SS_2$ and $SS_3$ are defined in the same way as for Model A. Table 2 also shows that the minimum value with respect to number of inspections of the maximum with respect to system failure distributions of IFR type passing through points $(x_1 = 0.4, p_1 = 0.180)$ and $(x_2 = 0.9, p_2 = 0.560)$, i.e., $L^{**}$ is equal to 9.61 for $M_3 = 3$. Again here $P^{**}$ shows a good convergence and accuracy since shifting from input set $S_2$ to $S_3$ with higher number of state increments has not changed the solution for $P^{**}$, $F^{**}$ and maximum expected cost. Values in Table 2 show that an accuracy of up to one decimal point can be obtained by shifting from $S_2$ to $S_3$.

4.3 COMPARISON OF MODEL A WITH MODEL B

Figure 7 shows the variation of maximum expected cost with the number of inspections. It can easily be seen that the inspections which are performed after the time for which cumulative probability of failure is known are more important and their number determines the upper bound on the expected total optimum cost of the maintenance. Table 3 shows the value of information for IFR distributions which pass through points $(0.4, 0.180)$ and $(0.9, 0.560)$. If both informations are utilized then Model B gives the values for $L^{**}$ which is equal to 9.61. If only point $(0.4, 0.180)$ is known then Model A gives $L_0^{**}$ equal to 10.10. Similarly if only point $(0.9, 0.560)$ is known then Model A gives $L^{**}$ equal to 9.64. Table 3 also shows that the additional knowledge about point $(0.9, 0.560)$ improves the upper bound cost $L^{**}$ for about 4.85%. Another important conclusion is that even if only information about one point is available the closer this point to the final point $(T, 1)$ the more improved upper bound value, $L^{**}$ can be obtained. This
Table 2. The Accuracy v.s. the Number of Increments for the State Variables $c$ and $p$ for Model B.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$SS_1$</th>
<th>$SS_2$</th>
<th>$SS_3$</th>
<th>$t$</th>
<th>$SS_1$</th>
<th>$SS_2$</th>
<th>$SS_3$</th>
<th>Max. Expected Total Cost v.s. $M_i$, the Total No. of Inspections in Subset $T_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>$t_1$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>$SS_1$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>0.048</td>
<td>0.048</td>
<td>0.048</td>
<td>$t_2$</td>
<td>0.4</td>
<td>0.5</td>
<td>0.5</td>
<td>$SS_2$</td>
</tr>
<tr>
<td>$p_3$</td>
<td>0.094</td>
<td>0.094</td>
<td>0.094</td>
<td>$t_3$</td>
<td>0.7</td>
<td>0.8</td>
<td>0.8</td>
<td>$SS_3$</td>
</tr>
<tr>
<td>$p_4$</td>
<td>0.138</td>
<td>0.138</td>
<td>0.138</td>
<td>$t_4$</td>
<td>0.9</td>
<td>1.0</td>
<td>1.0</td>
<td>$M_1$ = 1</td>
</tr>
<tr>
<td>$p_5$</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
<td>$t_5$</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>$M_2$ = 1</td>
</tr>
<tr>
<td>$p_6$</td>
<td>0.263</td>
<td>0.223</td>
<td>0.222</td>
<td>$t_6$</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>$M_2$ = 2</td>
</tr>
<tr>
<td>$p_7$</td>
<td>0.337</td>
<td>0.318</td>
<td>0.322</td>
<td>$t_7$</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>$M_2$ = 3</td>
</tr>
<tr>
<td>$p_8$</td>
<td>0.404</td>
<td>0.401</td>
<td>0.408</td>
<td>$t_8$</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>$M_2$ = 4</td>
</tr>
<tr>
<td>$p_9$</td>
<td>0.494</td>
<td>0.474</td>
<td>0.487</td>
<td>$t_9$</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>$M_2$ = 5</td>
</tr>
<tr>
<td>$p_{10}$</td>
<td>0.560</td>
<td>0.560</td>
<td>0.560</td>
<td>$t_{10}$</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>$M_3$ = 1</td>
</tr>
<tr>
<td>$p_{11}$</td>
<td>0.643</td>
<td>0.632</td>
<td>0.628</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$M_3$ = 2</td>
</tr>
<tr>
<td>$p_{12}$</td>
<td>0.710</td>
<td>0.923</td>
<td>0.930</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$M_3$ = 3</td>
</tr>
<tr>
<td>$p_{13}$</td>
<td>0.764</td>
<td>0.984</td>
<td>0.987</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$M_3$ = 4</td>
</tr>
<tr>
<td>$p_{14}$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$M_3$ = 5</td>
</tr>
<tr>
<td>$p_{15}$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fig. 7. Variation of Upper Bound for the Expected Total Cost with $M_i$ Number of Inspections.
Table 3. The Value of Information about IFR Distributions Passing Through Points (0.4, 0.180) and (0.9, 0.560).

<table>
<thead>
<tr>
<th>Utilizing The Information(s)</th>
<th>$L^{**}$</th>
<th>( \frac{L^{<strong>} - L^{0</strong>}}{L^{0**}} \times 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>That</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_1 = 0.180; \quad x_1 = 0.4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_2 = 0.560; \quad x_2 = 0.9$</td>
<td>9.61</td>
<td>4.85</td>
</tr>
<tr>
<td>(Using Model B)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p = 0.180; \quad x = 0.4$</td>
<td>10.10</td>
<td>0.00</td>
</tr>
<tr>
<td>(Using Model A)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p = 0.560; \quad x = 0.9$</td>
<td>9.64</td>
<td>4.55</td>
</tr>
<tr>
<td>(Using Model A)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For:

- $a = 35.\ $/Unit Time,
- $I = 1.4\ $/Inspection,
- $T' = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3\}$,
- and $AL_1$ & $PN_1$ given by set $S_3$.  


is clear again from Table 3 since having the knowledge about point (0.9, 0.560) is 4.55% more valuable than the knowledge about point (0.4, 0.180) which is closer to point (0, 0). In section 4.5 more results are presented regarding the location of the known point or points.

4.4 ACTUAL OPTIMAL COST V.S. L**

The optimal policy \( \bar{P}_i = \{t_1, t_2, \ldots, t_n\} \) and the optimal cost \( R_i \) has been found numerically by using computer program of Appendix C for several different IFR failure distributions \( F_i \) which consist of points \((\bar{t}_i, p_i)\) connected by transformed exponential distributions. Tables 4 and 5 give the policies \( \bar{P}_i \) and optimal costs \( R_i \) and also the ratio of \( R_i \) to \( L** \). It can be seen that in both tables this ratio is less than one as could have been expected since by definition \( L** \) is an upper bound for the optimal expected maintenance cost for known distributions.

4.5 VALUE OF INFORMATION V.S. RELATIVE LOCATION OF THE KNOWN POINT(S).

A certain IFR failure distribution \( F_0 \) has been selected for which:

\[
\begin{align*}
    p(t = 0.0) &= 0.000; &
    p(t = 0.2) &= 0.150; &
    p(t = 0.3) &= 0.216; \\
    p(t = 0.4) &= 0.278; &
    p(t = 0.5) &= 0.535; &
    p(t = 0.6) &= 0.700.
\end{align*}
\]

The set of possible inspection times \( T' \) is given by

\[
    T' = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}.
\]

Now for the above failure distribution the values of \( L** \) are plotted in Fig. 8 and Fig. 9 assuming that only one point \((x, p_x)\) or two points \((x_1, p_{x_1})\) and \((x_2, p_{x_2})\) of the distribution \( F_0 \) is known. Figure 8 shows that the value of \( L** \) improves as the known location \( x \) of the 100.\( p_x \) th percentile of the failure distribution moves toward the maximum life time
Table 4. Optimal Policy $P_i$ and Cost $R_i$ for Some IFR Distributions $F_i$ which Pass Through Points $(0.4, 0.180)$ and $(0.9, 0.560)$ and Their Comparison with $L^{**}$

<table>
<thead>
<tr>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
<th>$F_5$</th>
<th>$F^{**}=F_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$P_5$</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
</tr>
<tr>
<td>$P_7$</td>
<td>0.305</td>
<td>0.322</td>
<td>0.270</td>
<td>0.270</td>
<td>0.356</td>
</tr>
<tr>
<td>$P_9$</td>
<td>0.444</td>
<td>0.480</td>
<td>0.364</td>
<td>0.444</td>
<td>0.494</td>
</tr>
<tr>
<td>$P_{10}$</td>
<td>0.560</td>
<td>0.560</td>
<td>0.560</td>
<td>0.560</td>
<td>0.560</td>
</tr>
<tr>
<td>$P_{11}$</td>
<td>0.822</td>
<td>0.644</td>
<td>0.822</td>
<td>0.692</td>
<td>0.966</td>
</tr>
<tr>
<td>$P_{13}$</td>
<td>0.975</td>
<td>0.848</td>
<td>1.000</td>
<td>0.874</td>
<td>1.000</td>
</tr>
<tr>
<td>$P_{14}$</td>
<td>1.000</td>
<td>0.978</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>$P_{15}$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>$t_1$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$t_2$</td>
<td>0.4</td>
<td>0.4</td>
<td>0.5</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>$t_3$</td>
<td>0.7</td>
<td>0.7</td>
<td>0.9</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>$t_4$</td>
<td>0.9</td>
<td>0.9</td>
<td>1.0</td>
<td>0.9</td>
<td>1.0</td>
</tr>
<tr>
<td>$t_5$</td>
<td>1.0</td>
<td>1.1</td>
<td>1.1</td>
<td>1.0</td>
<td>1.1</td>
</tr>
<tr>
<td>$t_6$</td>
<td>1.1</td>
<td>1.2</td>
<td>1.4</td>
<td>1.1</td>
<td>1.4</td>
</tr>
<tr>
<td>$t_7$</td>
<td>1.2</td>
<td>1.3</td>
<td>1.2</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>$t_8$</td>
<td>1.3</td>
<td>1.4</td>
<td>1.3</td>
<td>1.4</td>
<td></td>
</tr>
<tr>
<td>$t_9$</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_{10}$</td>
<td>8.28</td>
<td>8.81</td>
<td>8.16</td>
<td>8.77</td>
<td>7.99</td>
</tr>
<tr>
<td>$R_i$</td>
<td>8.32</td>
<td>8.32</td>
<td>8.32</td>
<td>8.32</td>
<td>8.32</td>
</tr>
<tr>
<td>$R_i/L^{**}$</td>
<td>0.861</td>
<td>0.916</td>
<td>0.849</td>
<td>0.912</td>
<td>0.831</td>
</tr>
</tbody>
</table>

$L^{**} = 9.61$ $F_i = (F(\xi_i)) = p_i$ for $i = 1, 5, 7, 9, 10, 11, 13, 14, 15$

$a = 35$. $$/Unit Time

$I = 1.4$ $$/Inspection \quad \xi_1 = 0.0$ Unit Time; $\xi_5 = 0.4$ Unit Time
Table 4. (continued)

\[ T' = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3\} \]

\( AL_1 \) & \( AL_2 \) given by Set \( S_3 \)

\( \xi_7 = 0.6 \) Unit Time; \( \xi_9 = 0.8 \) Unit Time

\( \xi_{10} = 0.9 \) Unit Time; \( \xi_{11} = 1.0 \) Unit Time

\( \xi_{13} = 1.2 \) Unit Time; \( \xi_{14} = 1.3 \) Unit Time

\( \xi_{15} = 1.4 \) Unit Time;
Table 5. Optimal Policy $P_i$ and Cost $R_i$ for Some IFR Distributions $F_i$ which Pass Through Point (0.4, 0.180) and Their Comparison with $L^{**}$

<table>
<thead>
<tr>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
<th>$F_5$</th>
<th>$F^{**}=F_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$p_5$</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
</tr>
<tr>
<td>$p_8$</td>
<td>0.382</td>
<td>0.441</td>
<td>1.000</td>
<td>0.412</td>
<td>0.553</td>
</tr>
<tr>
<td>$p_{10}$</td>
<td>0.680</td>
<td>0.760</td>
<td>1.000</td>
<td>0.653</td>
<td>0.813</td>
</tr>
<tr>
<td>$p_{12}$</td>
<td>0.903</td>
<td>0.976</td>
<td>1.000</td>
<td>0.827</td>
<td>0.956</td>
</tr>
<tr>
<td>$p_{14}$</td>
<td>0.978</td>
<td>1.000</td>
<td>1.000</td>
<td>0.918</td>
<td>1.000</td>
</tr>
<tr>
<td>$p_{15}$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.563</td>
</tr>
<tr>
<td>$t_1$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$t_2$</td>
<td>0.5</td>
<td>0.3</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$t_3$</td>
<td>0.8</td>
<td>0.6</td>
<td>1.4</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>$t_4$</td>
<td>1.0</td>
<td>0.8</td>
<td>1.0</td>
<td>0.8</td>
<td>1.1</td>
</tr>
<tr>
<td>$t_5$</td>
<td>1.1</td>
<td>0.9</td>
<td>1.1</td>
<td>0.9</td>
<td>1.4</td>
</tr>
<tr>
<td>$t_6$</td>
<td>1.2</td>
<td>1.0</td>
<td>1.2</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>$t_7$</td>
<td>1.3</td>
<td>1.1</td>
<td>1.3</td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>$t_8$</td>
<td>1.4</td>
<td>1.2</td>
<td>1.4</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>$t_9$</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td></td>
</tr>
<tr>
<td>$t_{10}$</td>
<td>1.3</td>
<td>1.1</td>
<td>1.3</td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>$R_1$</td>
<td>8.22</td>
<td>7.83</td>
<td>6.20</td>
<td>8.70</td>
<td>7.80</td>
</tr>
<tr>
<td>$R_1/L^{**}$</td>
<td>0.814</td>
<td>0.775</td>
<td>0.614</td>
<td>0.861</td>
<td>0.772</td>
</tr>
</tbody>
</table>

$L^{**} = 10.10 \$ \quad F_i = \{F(\xi_i) = p_i \text{ for } i = 1, 5, 8, 10, 12, 14, 15\}$

$a = 35. \text{ } \$ \text{Unit Time}$

$I = 1.4 \text{ } \$/\text{Inspection} \quad \xi_1 = 0.0 \text{ Unit Time}; \quad \xi_5 = 0.4 \text{ Unit Time}$

$T' = \text{Defined in Table 4.} \quad \xi_8 = 0.7 \text{ Unit Time}; \quad \xi_{10} = 0.9 \text{ Unit Time}$

$AL_1 \text{ and } AL_2 \text{ given by} \quad \xi_{12} = 1.1 \text{ Unit Time}; \quad \xi_{14} = 1.3 \text{ Unit Time}$

$\text{Set } S_3 \quad \xi_{15} = 1.4 \text{ Unit Time}$
The Time where $p_x = F(x)$ is known in units of time.

$a = 20. \$/Unit Time
$I = 0.8 \$/Inspection
$AL_1 = 10$
$PN_1 = 20$

Fig. 8. Variation of $L^*$ with the Time $x$ at which $p_x = F(x)$ is known.
of the system. The same trend can be seen in Fig. 9 for constant value of $x_2$. In addition to this, Fig. 9 shows that $L^{**}$ improves as the location of $x_2$ moves toward the maximum life time of the system.

4.6 VARIATION OF $L^{**}$ WITH $I$

Figure 10 shows the variation of $L^{**}$, the upper bound total expected cost with $I$ the inspection cost for each inspection. As it can be seen the rate of increase of $L^{**}$ with the increase of $I$ is higher for lower values of $I$ for both cases, i.e., when the information about two points of distribution are known or the information about only one point is known. Table 6 gives the values of $L^{**}$ and $I$ which are plotted in Fig. 10 where $L_1^{**}$ and $L_2^{**}$ stand for the values of $L^{**}$ when two points and one point of the distribution are known respectively. The third column of Table 6 shows the percent improvement in $L^{**}$ values when additional point of distribution is known for different values of $I$.

4.7 VARIATION OF $L^{**}$ WITH $a$

Figure 11 shows the variation of $L^{**}$ with $a$ the cost per unit time of undetected failure. It can be seen that the rate of increase of $L^{**}$ with the increase of $a$ is higher for lower values of $a$ for both cases, i.e., when the information about two points of the distribution are known or the information about only one point is known. Table 7 gives the values of $L^{**}$ and $a$ which are plotted in Fig. 11 where $L_1^{**}$ and $L_2^{**}$ stand for the values of $L^{**}$ when two points and one point of the distribution are known respectively. The third column of Table 7 shows the percent improvement in $L^{**}$ values when additional point of distribution is known for different values of $a$. The value of $L^{**}$ at $a = 0$ is equal to 0.8 or the cost of one inspection which
Fig. 9. Variation of L** with the Times $x_1$ and $x_2$ at which $p_{x_1}$ and $p_{x_2}$ are Known.

$x$, For $x_2 = 0.6$ where $p_{x_2} = F(x_2) = 0.700.$

$\Theta$, For $x_2 = 0.5$ where $p_{x_2} = F(x_2) = 0.535.$

$a = 20. \$/Unit Time

$I = 8. \$/Inspection

$AL_1 = 10$

$AL_2 = 20$
\[ a = 20 \$/\text{Unit Time} \]
\[ p = 0.216; \ x = 0.3 \]
\[ AL_1 = 10; \ PN_1 = 20 \]

\[ a = 20 \$/\text{Unit Time} \]
\[ p_1 = 0.216; \ p_2 = 0.700 \]
\[ x_1 = 0.3; \ x_2 = 0.6 \]
\[ AL_1 = 10; \ PN_1 = 20 \]

Fig. 10. Variation of \( L^{**} \) with Cost of an Inspection \( I \).
Table 6. Percent Improvement in $L^{**}$ for Various Values of I

<table>
<thead>
<tr>
<th>$I$</th>
<th>$L^{**}_1$</th>
<th>$L^{**}_2$</th>
<th>$\frac{L^{<strong>}_2 - L^{</strong>}_1}{L^{**}_1} \times 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.34</td>
<td>1.51</td>
<td>11.3</td>
</tr>
<tr>
<td>0.2</td>
<td>2.34</td>
<td>2.58</td>
<td>9.3</td>
</tr>
<tr>
<td>0.4</td>
<td>3.10</td>
<td>3.48</td>
<td>10.9</td>
</tr>
<tr>
<td>0.8</td>
<td>4.24</td>
<td>4.60</td>
<td>7.8</td>
</tr>
<tr>
<td>1.6</td>
<td>6.07</td>
<td>6.93</td>
<td>12.4</td>
</tr>
<tr>
<td>3.2</td>
<td>8.98</td>
<td>10.22</td>
<td>12.1</td>
</tr>
</tbody>
</table>

$a = 20$
Fig. 11. Variation of $L^{**}$ with Cost per Unit Time of Undetected Failure $a$. 

$I = 0.8 \, \$/$\text{Inspection}$

$p_1 = 0.216; \quad x_1 = 0.3$

$AL_1 = 10; \quad PN_1 = 20$

$p_2 = 0.700; \quad x_2 = 0.6$

$AL_2 = 10; \quad PN_2 = 20$
Table 7. Percent Improvement in L** for Various Values of a

<table>
<thead>
<tr>
<th>a</th>
<th>L**₁</th>
<th>L**₂</th>
<th>( \frac{L<strong>₂ - L</strong>₁}{L**₂} \times 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.80</td>
<td>0.80</td>
<td>0.0</td>
</tr>
<tr>
<td>5.0</td>
<td>2.25</td>
<td>2.55</td>
<td>11.7</td>
</tr>
<tr>
<td>10.0</td>
<td>3.03</td>
<td>3.47</td>
<td>12.7</td>
</tr>
<tr>
<td>20.0</td>
<td>4.24</td>
<td>4.60</td>
<td>7.8</td>
</tr>
<tr>
<td>40.0</td>
<td>6.20</td>
<td>6.96</td>
<td>10.9</td>
</tr>
<tr>
<td>60.0</td>
<td>7.89</td>
<td>8.73</td>
<td>9.6</td>
</tr>
</tbody>
</table>

I = 0.8
at least is needed to be performed in order to detect the failure of the system.

4.8 EXAMPLE PROBLEM

Problem: The management of an organization wants to introduce a new deteriorating item into the system. There are three deteriorating items that have the same function but have different failure and cost characteristics. Because of the shortage of investing capital the company wants to choose the alternative which has the lowest possible expected cost. The characteristics of the items A, B and C are as follows:

<table>
<thead>
<tr>
<th>Item</th>
<th>Item A</th>
<th>Item B</th>
<th>Item C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Average inventory cost, $/day</td>
<td>140</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>2. Cost per inspection, $</td>
<td>56</td>
<td>20</td>
<td>80</td>
</tr>
<tr>
<td>3. Probability of failure before time $x$</td>
<td>0.180</td>
<td>0.216</td>
<td>0.535</td>
</tr>
<tr>
<td>4. $x$, is equal to</td>
<td>4th day</td>
<td>3rd day</td>
<td>10th day</td>
</tr>
<tr>
<td>5. Maximum life time</td>
<td>14 days</td>
<td>3 days</td>
<td>16 days</td>
</tr>
</tbody>
</table>

Besides these the inspection can only be performed at certain time and only once a day for example at 3 p.m. every day. The replenishment of the item will occur at the end of periods equal to the maximum life time of the item. Which alternative should be chosen with the above limited information?

Solution: Since the failure characteristics of the system are not completely known then the best estimate of the costs would be the maximum possible expected cost $L^{**}$. In many cases the systems show an IFR failure characteristic so it is assumed that failure characteristic of the items are IFR. Average inventory cost becomes equivalent to the cost of undetected
failure a since a failed item will remain in the stock and assumes cost until its failure is detected through inspection. We assume that the inspection is perfect and it does not take time and does not degrade the item. It can be shown from Eq. (2.4.2) that if $L$ is multiplied by a scale factor $u$, and $t$, the time, multiplied by another scale factor $v$ then new $L$ called $L' = (u)L$ is the expected cost of the system provided that

$$I' = (u)I,$$
$$a' = (u)\frac{a}{v} = (\frac{u}{v}) a,$$

and

$$t_{\text{max}}' = (v) t_{\text{max}}$$

where $I'$ and $a'$ are new scaled inspection and undetected failure costs. $t_{\text{max}}'$ and $t_{\text{max}}$ are the scaled and not scaled values of the maximum life time of the system.

For Item A we have:

$$I' = (40)(1.4) = 56$$
$$a' = (\frac{40}{10}) (35) = 140$$
$$x' = (10)(.4) = 4$$
$$t_{\text{max}}' = (10) (1.4) = 14.$$

So for $u = 40$ and $v = 10$, we have $I = 1.4$, $a = 35$, $x = 0.4$, $p = 0.130$ and $t_{\text{max}} = 1.4$. From Table 3 the value of $L^{**}$ for above values of input parameters is $L^{**} = 10.10$. So
L**1 = (40)(10.10) = 404 $/cycle or 404/14 = 28.86 $/day.

For Item B we have:

I' = (100)(0.2) = 20
a' = (100/10)(20) = 200
x' = (10)(0.3) = 3
t'\text{max} = (10)(0.8) = 8.

So for u = 100 and v = 10, we have I = 0.2, a = 20, x = 0.3, p = 0.216 and t'\text{max} = 0.8. From Table 6 the value of L** for above values of input parameters is L** = 2.58. So

L**' = (100)(2.58) = 258 $/cycle or 258/8 = 32.25 $/day.

For Item C we have

I' = (100)(0.8) = 80
a' = (100/20)(20) = 100
x' = (20)(0.5) = 10
t'\text{max} = (20)(0.8) = 16.

So for u = 100 and v = 20, we have I = 0.8, a = 20, x = 0.5, p = 0.535 and t'\text{max} = 0.8. From Fig. 8 the value of L** for above values of input parameters is L** = 4.45. So
L**' = (100)(4.45) = 445 $/cycle or 445/16 = 27.81 $/day.

Comparison between the cost per day for the three items shows that introduction of item C is more economical in a long run than the two other items.
CHAPTER 5

CONCLUSION

Minimax policy has been devised to cope with the problem of partial knowledge about the failure distribution of systems. In this work this policy has been adopted together with dynamic programming methodology to find the upper bound on the optimal total expected maintenance cost of a system subject to deterioration with an increasing failure rate distribution. The major difference with the previous work in this area is that here a procedure and a computer program has been devised which finds the upper bound cost not only when one point \((x, p)\) of the failure distribution is known but also when two points \((x_1, p_1)\) and \((x_2, p_2)\) of the failure distribution are known.

The basic findings can be summarized as:

1. The convergence and accuracy of the upperbound on the expected optimal total maintenance cost depends on the number of increments into which the feasible range of the state variables \(c\) and \(p\) are divided and for reasonable amount of computer time a fairly accurate upper bound value can be obtained.

2. Additional knowledge about a second point \((x_2, p_2)\) improves (decreases) the upper bound in general. But it improves the upper bound cost especially if the second point is closer to the end point \((T_{\text{max}}, 1)\) where \(T_{\text{max}}\) is the maximum life time of the system.

3. The closer the two known points \((x_1, p_1)\) and \((x_2, p_2)\) to the end point \((T_{\text{max}}, 1)\) the more improved upper bound cost can be obtained.
4. The upper bound cost seems to increase faster with the increase in the
cost per inspection, I, at lower values of I.
5. The upper bound cost seems to increase faster with the increase in the
cost per unit time of undetected failure, a, at lower values of a.
6. Several optimal expected total costs which have been found for several
different IFR distributions showed that all were lower than upper bound
cost as it could have been expected.

The basic disadvantage of the present computational procedure is due to
the existence of three state variables. In this work the state variable,
time, has been assumed to be discrete in order to increase the computational
feasibility. This assumption although limits the scope of the applicability
of the procedure but nonetheless it can be a practical assumption since in
many cases the maintenance and inspection actions can only be done at dis-
crete points in time. The other disadvantage is that it only provides for
the replenishment to take place at the end of its maximum life time. In
general this work and the previous works in this area are worthwhile to be
continued in a wider sense since in many cases when a new system or deterior-
ating item is introduced into the existing system the complete knowledge on
the failure characteristic of it is rare and it takes time to build up
knowledge about the system failure characteristics and the management needs
to have some estimate of the cost even before introducing a new system in
order to compare different options or alternatives.
REFERENCES


APPENDIX A

MATHEMATICAL TRANSFORMATION
OF THE EXPECTED TOTAL MAINTENANCE COST
APPENDIX A

MATHEMATICAL TRANSFORMATION
OF THE EXPECTED TOTAL MAINTENANCE COST

As stated in (2.4), Eq. (2.4.1), i.e.,

\[
L(t_0 = 0, t_1, t_2, \ldots, t_n = T) = \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} \left[ I(k+1) + a(t_{k+1} - x) \right] f(x) \, dx \\
+ \int_{t_{n-1}}^{t_n} \left[ I(n) + a(T - x) \right] f(x) \, dx
\]

\( n = 1, 2, \ldots, N, \) \( (1) \)

can be transformed into Eq. (2.4.2), i.e.,

\[
L = I \sum_{k=0}^{n-1} F^*(t_k) + a \left( \sum_{k=0}^{n-1} F^*(t_k)(t_{k+1} - t_k) - \int_0^{t_n} F^*(t) \, dt \right)
\]

\( n = 1, 2, \ldots, N. \) \( (2) \)

PROOF:

\[
\sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} \left[ I(k+1) + a(t_{k+1} - x) \right] f(x) \, dx = \sum_{k=0}^{n-2} \left( \int_{t_k}^{t_{k+1}} I(k+1) f(x) \, dx \right) + a \int_{t_k}^{t_{k+1}} t_{k+1} f(x) dx - a \int_{t_k}^{t_{k+1}} x f(x) \, dx
\]

\( (3) \)

but

\[
\int_{t_k}^{t_{k+1}} I(k+1) f(x) \, dx = I(k+1)(F(t_{k+1}) - F(t_k)), \]

\( (4) \)
and

\[ \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} I(k+1)f(x)dx = \sum_{k=0}^{n-2} I(k+1)(F(t_{k+1}) - F(t_k)) \]

\[ = I(0+1)(F(t_1) - F(t_0)) + I(1+1)(F(t_2) - F(t_1)) + I(2+1)(F(t_3) - F(t_2)) + \ldots + I(n-2+1)(F(t_{n-1}) - F(t_{n-2})) \]

\[ = -IF(t_0) - IF(t_1) - IF(t_2) \ldots - IF(t_{n-2}) + I(n-1)F(t_{n-1}) \]

\[ = -I \sum_{k=0}^{n-2} F(t_k) + I(n-1)F(t_{n-1}), \quad (5) \]

also

\[ \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} f(x)dx = \sum_{k=0}^{n-2} t_{k+1}(F(t_{k+1}) - F(t_k)). \quad (6) \]

We know that according to the integration by part formula

\[ \int_a^b vu'du = (vu)_a^b - \int_a^b v'u \ du, \quad (7) \]

where v' and u' are the derivatives of functions v and u. Assuming u'du = f(x)dx and v = x and using Eq. (7) we can write

\[ \int_{t_k}^{t_{k+1}} xf(x)dx = (xF(x))_{t_k}^{t_{k+1}} - \int_{t_k}^{t_{k+1}} F(x)dx \]
\[ \begin{align*}
&= t_{k+1} F(t_{k+1}) - t_k F(t_k) - \int_{t_k}^{t_{k+1}} F(x) \, dx, \\
\text{and} \\
&\sum_{k=0}^{n-2} \frac{t_{k+1}}{t_k} - a \int_{t_k}^{t_{k+1}} xf(x) \, dx = -a \sum_{k=0}^{n-2} \left( t_{k+1} F(t_{k+1}) - t_k F(t_k) \right) \\
&\quad + a \int_{0}^{t_{n-1}} F(x) \, dx. \\
\end{align*} \]

Now substituting the equivalent values from Eqs. (5), (6) and (9) in Eq. (3) we have

\[ \begin{align*}
&\sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} [I(k+1) + a(t_{k+1} - x)] f(x) \, dx = - I \sum_{k=0}^{n-2} F(t_k) + \\
&\quad I(n-1) F(t_{n-1}) + \\
&\quad a \sum_{k=0}^{n-2} t_{k+1} (F(t_{k+1}) - F(t_k)) \\
&\quad - a \sum_{k=0}^{n-2} (t_{k+1} F(t_{k+1}) - t_k F(t_k)) \\
&\quad + a \int_{0}^{t_{n-1}} F(x) \, dx \\
&\quad = - I \sum_{k=0}^{n-2} F(t_k) + I(n-1) F(t_{n-1}) \\
&\quad + a \left[ - \sum_{k=0}^{n-2} F(t_k) (t_{k+1} - t_k) + \\
&\quad \int_{0}^{t_{n-1}} F(x) \, dx \right]. \\
\end{align*} \]
The second portion of the right hand side of Eq. (1) can be written in the same way as

\[ \int_{t_{n-1}}^{t_n} [I(n) + a(T-x)]f(x)dx = I(n)(F(T) - F(t_{n-1})) + \]
\[ + a \int_{t_{n-1}}^{t_n} F(x) dx - a(TF(T) - t_{n-1}F(t_{n-1})) \]
\[ = I(n)(1 - F(t_{n-1})) + \]
\[ a \left( - F(t_{n-1})(T - t_{n-1}) + \right. \]
\[ \left. \int_{t_{n-1}}^{t_n} F(x) dx \right) , \quad (11) \]

now substituting Eq. (10) and Eq. (11) in Eq. (1) and naming \( t = x \) we have

\[ L = - I \sum_{k=0}^{n-2} F(t_k) + I(n-1)F(t_{n-1}) + a \left( - \sum_{k=0}^{n-2} F(t_k)(t_{k+1} - t_k) \right. \]
\[ + \int_{0}^{t_{n-1}} F(t)dt + I(n)(1-F(t_{n-1})) + a \left( -F(t_{n-1})(T - t_{n-1}) + \right. \]
\[ \left. \int_{t_{n-1}}^{t_n} F(t)dt \right) \]
\[ = I \sum_{k=0}^{n-1} (1 - F(t_k)) + a \left( - \sum_{k=0}^{n-1} F(t_k)(t_{k+1} - t_k) + \right. \]
\[ \left. \int_{0}^{t_{n-1}} F(t)dt \right) , \quad (12) \]
denoting $F^*(t)$ as $F^*(t) = 1 - F(t)$ and substituting it instead of $1 - F(t)$ in Eq. (12) we have

$$
L = \sum_{k=0}^{n-1} F*(t_k) + a \left\{- \sum_{k=0}^{n-1} (1 - F^*(t_k))(t_{k+1} - t_k) + \int_{0}^{t_n = T} (1 - F^*(t))dt \right\}
$$

$$
= \sum_{k=0}^{n-1} F^*(t_k) + a \left\{ \sum_{k=0}^{n-1} (t_{k+1} - t_k) + \sum_{k=0}^{n-1} F^*(t_k)(t_{k+1} - t_k) + \int_{0}^{t_n = T} F^*(t)dt \right\}
$$

$$
= \sum_{k=0}^{n-1} F^*(t_k) + a \left\{ \sum_{k=0}^{n-1} F^*(t_k)(t_{k+1} - t_k) - \int_{0}^{t_n = T} F^*(t)dt \right\}.
$$
APPENDIX B

COMPUTER PROGRAM FOR CALCULATING AN UPPER BOUND
FOR THE EXPECTED TOTAL COST FOR BOTH MODELS A AND B
CALCULATION OF THE UPPER BOUND FOR THE OPTIMAL EXPECTED TOTAL MAINTENANCE COST OF A SYSTEM WITH PARTIALLY KNOWN FAILURE DISTRIBUTION

NUTATIONS
NB=1—CALCULATES UPPER BOUND COST WHEN ONLY ONE POINT OF THE DISTRIBUTION IS KNOWN—MODEL A
NB=2—CALCULATES UPPER BOUND COST WHEN TWO POINTS OF THE DISTRIBUTION ARE KNOWN—MODEL B
P1—THE KNOWN CUMULATIVE FAILURE PROBABILITY AT TIME 01
P2—THE KNOWN CUMULATIVE FAILURE PROBABILITY AT TIME 02
AI—INSPECTION COST PER INSPECTION
AA—COST OF UNDETECTED FAILURE PER UNIT TIME
NP—NUMBER OF INCREMENTS INTO WHICH THE FEASIBLE RANGE OF FAILURE PROBABILITY P IS DIVIDED MINUS ONE
LA—NUMBER OF INCREMENTS INTO WHICH THE FEASIBLE RANGE OF C PARAMETER OF TRANSFORMED EXPONENTIAL DISTRIBUTION IS DIVIDED MINUS ONE
EI—MAXIMUM POSSIBLE NUMBER OF INSPECTIONS
RKKK(I,J,K),RKK(I,J,K)—THE MAXIMUM EXPECTED MAINTENANCE COST UP TO TIME T(I) WITH P(J),C(K) AND AT STAGE N-1 AND N RESPECTIVELY

THIS PROGRAM WAS WRITTEN BY NURUSH-KARAMPISHRI, DEPARTMENT OF INDUSTRIAL ENGINEERING KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS, MAY, 1979

INTEGER D1,D2,E1,D11,001,002,003,004,005,006
DIMENSION RKKK(10,20,10),RKK(10,20,10),M(10,20,10),A(10,20,10),DD(10,20,10),P(10,20) ,PK2(10),1A1(10),MM2(10),N(10)
DATA 01=2,02=2
821 READ(5,822)P1,A1,AA,NP,E1,D1,LA
822 FORMAT(3F10.5,4I5)
02=D1
P2=P1
GO TO 823
823 READ(5,1)P1,P2,AA,NP,E1,D1,J2,LA
1 FORMAT(4F10.5,3I5)
823 PAC=1.E-25
0C 2 LL=1,E1
READ(5,600)I(T I)
600 FORMAT(F13.6)
2 CONTINUE
CIC1=-ALOG(1.-P1)/T(01)
IF(NB-1)100,824,825
STEP 1 FOR BOTH MODELS A&B
824 CIC2=CIC1
GU TO 831
825 CIC2=-ALOG(1.-P2)/T(D2)-T(01)
631  P(1,1)=0.
       NP1=NP-1
       PNI=NP-1
       O2=O2-1
       D01=O1+1
       D02=02-1
        
       828  PP2=1.-(1.-P1)*EXP(CIC2*(T(U1)-T(L2)))
            IF(PP2>3,4,4
            IF((PP2)3,4,4
            3  DP=PP1/PN1
                DB TO 6
                4  DP=(PP1-PP2)/PN1
                6  P(L2,1)=PP1
                    DC / L3=2,NP1
                7  P(L2,L3)=P(L2,L3-1)-DP
                    IF(NB-1)100,3,828
                886  IF((PP2)3,9,9
                9  P(L2,NP)=0.0
                    GO TO 10
                10 CONTINUE
                    P(D1,1)=P1
                    IF(NB-1)100,829,830
            30  DD 12  L4=DD01,DD2
                PP1=1.-(1.-P1)*EXP(-CIC1*T(L2))
                PP2=1.-(CIC2*(T(L4)-T(U1)))
                    JP=(PP1-PP2)/PN1
                    P(L4,1)=PP1
                        DD 11  L5=2,NP1
                11  P(L4,L5)=P(L4,L5-1)-DP
                    P(L4,NP)=PP2
                12 CONTINUE
                    P(D2,1)=P2
                829  DD 14  LB=DD02,E1
                    PP1=1.-(PAC
                    PP2=1.-(1.-P2)*EXP(-CIC2*(T(LB)-T(U2)))
                    JP=(PP1-PP2)/PN1
                    P(L6,1)=1.-PAC
                        DD 13  L7=2,NP1
                13  P(L6,L7)=P(L6,L7-1)-DP
                    P(L6,NP)=PP2
                14 CONTINUE

STEP 2 FOR BOTH MODELS A & B

DC=CIC1/AL1
C(1,1,1)=0.
DD 27 K=2,LAI
27  C(1,1,K)=C(1,1,K-1)+LC
     C(1,1,LAI)=CIC1
        DD 17  L9=2,D01
        DD 17  L10=2,NP1
L1=(-AL05(1.-P(L9,L11))/T(L)9
L2=(-AL06((1.-P1)/(1.-P(L9,L11)))/T(U1)-T(L9))
DC = (C2 - C1) / AL1
C(L9, L10, 1) = C1
DO 16 K = 2, LAL1
16 C(L9, L10, K) = C(L9, L10, K-1) + DC
C(L9, L1D, LA) = C2
17 CONTINUE
IF(NE-1)100, 332, 833
833 DC = (CIC2 - CIC1) / AL1
C(D1, L, I) = CIC1
DO 23 K = 2, LAL1
23 C(D1, I, K) = C(D1, I, K-1) + DC
C(D1, I, LA) = CIC2
DO 20 L5 = DDUI, U02
20 C(L10) = 2, NPL
G1 = ALUG((1 - P(L9, L1)) / (1 - P1)) / (T(L9) - T(J1))
G2 = ALUG((1 - P2) / (1 - P(L9, L10))) / (T(D2) - T(L9))
DC = (C2 - C1) / AL1
C(L9, L10, 1) = G1
DO 19 K = 2, LAL1
19 C(L9, L10, K) = C(L9, L10, K-1) + DC
C(L9, L1D, LA) = C2
20 CONTINUE
832 C(D2, L, I) = CIC2
PPL = 1 - (1 - P2) * EXP(-CIC2 * (T(D02) - T(D2)))
DP = (1 - PAC - PP1) / AL1
DO 21 K = 2, LAL1
AK = K
21 C(D2, I, K) = ALUG((1 - P1 - (AK - 1.1) * DP) / (1 - P2)) / (T(D02) - T(D2))
C(D2, I, LA) = ALUG(PAC / (1 - P2)) / (T(D02) - T(D2))
DO 24 L11 = D02, E1
24 L12 = L11, NPL
IF(L12 - 1) GO TO 707, 708
707 C1 = ALUG(PAC / (1 - P2)) / (T(L11) - T(D2))
GC TO 702
708 C1 = ALUG((1 - P(L11, L12)) / (1 - P2)) / (T(L11) - T(J2))
709 C(L11, L12, 1) = C1
IF(L11 - EL) GO TO 512, 24, 24
512 CONTINUE
24 CONTINUE
DO 610 LL3 = D02, E1
610 LL4 = 2, NPL
PP1 = 1 - (1 - P2) * EXP(-C(L13, L14, 1) * (T(L13 + 1) - T(J2)))
DP = (1 - PAC - PP1) / AL1
DO 700 K = 2, LAL1
AK = K
700 C(L13, L14, K) = ALUG((1 - PP1 - (AK - 1.1) * DP) / (1 - P(L13, L14))) / (T(L13 + 1) - T(L13))
610 C(L13, L14, LA) = ALUG(PAC / (1 - P(L13, L14))) / (T(L13 + 1) - T(L13))

STEPS 3&4 FOR BOTH MODELS AG3

M=1
I=D1
RKK(D1, 1, 2) = A1 * AA * T(D1) - AA * P(D1, 1) / CIC1
JO 25 K = 2, LA
RKK(D1, 1, K) = RKK(D1, 1, 2)
MF(D1, 1, K) = 1
MT(D1, 1, K) = 1
25 MK(D1, 1, K) = 1
IF(NE-1)100, 335, 36
836 DO 26 I=0,0,0
26 J=2,0,P1
KKK(I,J,1)=A1+AA*T(I)-AA*P(I,J,1)/C(I,J,1)
MIF(I,J,1)=1
MT(I,J,1)=1
26 MK(I,J,1)=1
I=D2
KKK(D2,1,J)=A1+AA*T(D2)-AA*P(D2,1)/C(D2,1)
MIF(D2,1,J)=1
MT(D2,1,J)=1
DO 601 K=2,LA
KKK(D2,1,K)=KKK(D2,1,2)
MIF(D2,1,K)=1
MT(D2,1,K)=1
601 MK(D2,1,K)=1
835 DO 29 I=0,0,0
29 J=2,0,P1
KKK(I,J,1)=A1+AA*T(I)-AA*P(I,J,1)/C(I,J,1)
MIF(I,J,1)=1
MT(I,J,1)=1
MK(I,J,1)=1
IF(I-E1)=29,570,100
570 CONTINUE
604 FORMAT(4X,'\$(E1,1,1)'

STEP 5 FOR BOTH MODELS A & B WHEN M=2

M=2
WRITE(6,32)M
32 FORMAT(2X,'(M,1,15)
30 DO 30 I=2,0,01
30 J=2,0,P1
KKK(I,J,LA)=A1+AA*T(I)-AA*P(I,J)/C(I,J,1)
MIFM2=1
MTM2=1
WRITE(6,002)KKK(I,J,LA),P(MTM2,MIFM2),AM1,I,P(I,J),C(I,J,1)
002 FORMAT(2X,'(M,1,15)
30 CONTINUE

STEP 6 FOR BOTH MODELS A & B

160 DO 31 I=M,0,D1
31 JR(I)=3000
31 JR(I)=2
47 K=2,LA
LL=1
42 IF(I-JR(I)-NP1)=40,40,41
40 IF=I-JR(I)
PPP1=L+(-P(I,J,1))+C*(T(I,J,1)-T(I,I))
33 IF=PPP1-P(I,J,1),34,34,35
34 IF=K(I,JR,LA)+PPP1,35,36,37
35 C=K(I,JR,LA)+PPP1,36,37,38
36 IF=K(I,JR,LA),PPP1,38,34,39
39 JR=JR+1
IF(JR-NP1)=33,33,35
38 PK2(II)=PPK1
MMF1(II)=JR
JR=JR+1
IF(JR-NP1)=33,33,35
35 JR(II)=JR
41 IF(LL-1)=100,515,43
515 LL=2
PK3=PK2(II)
IF(PK2(II)=7000)=145,146,140
145 MT2=II
MF2=MMF1(II)
146 IF(II-DD1)=42,42,44
43 IF(PK2(II)=PK3)=45,40,40
46 II=II+1
IF(II-DD1)=42,42,44
45 PK3=PK2(II)
IF(PK2(II)=7000)=147,148,148
147 MT2=II
MF2=MMF1(II)
148 II=II+1
IF(II-DD1)=42,42,44
44 IF(RKK(D1,1,K)=PK3)=47,47,48
48 RKK(D1,1,K)=PK3
MF(D1,1,K)=MF2
MT(D1,1,K)=MT2
MK(D1,1,K)=M
47 CONTINUE
IF(NB=1)=100,860,380
880 MN=0J2
JVR=2
GO TO 361
860 MN=1
JVR=1
361 DC 66 I=0DD1,MN
DO 60 J=JVR,NP1
II=M
LL=1
56 PPK2=3000
JR=2
L=1
FP=C(I,J,1)*(T(JL)-T(II))
IF(FP=10)=138,869,869
869 PPK1=0.0
GO TO 49
868 PPK1=-1,-P(JL,11)*EXP(C(I,J,1)*(T(JL)-T(II))
45 IF(PPK1-P(II,JR))=50,50,51
50 IF(RKK(I,JR,K)=40)=361,53,53
361 PPK1=11+AA*(T(I)-T(II))*1,-P(I,JR))+AA*((P(I,JR)-P(JL,11))/
1C(I,JR,K)+P(D1,1)-P(JL,1))/C(I,J,11)+RKK(I,JR,K)
IF(L-1)=100,63,52
86 L=2
PPK2=PPK1
MMF1=JR
JR=JR+1
IF(JR-NP1)=49,49,51
52 IF(PPK2=PPK1)=54,53,53
53 JR=JR+1
IF(JR-NP1)=49,49,51
54 PPK2=PPK1
   MFM1=JR
   JR=JR+1
   IF(JR-NP1)J72,72,75
51 IF(LL-1)100,55,56
55 LL=2
   PPK3=PPK2
   IF(PPK2=7000.)J61,62,62
61 MFM2=MFM1
   MTM1=II
62 II=II+1
   IF(II-UD1)56,56,57
58 IF(PPK2-PPK3)59,60,60
60 II=II+1
   IF(II-UD1)56,56,57
59 PPK3=PPK2
   IF(PPK2-7000.)J64,65,65
64 MFM2=MFM1
   MTM1=II
65 II=II+1
   IF(II-UD1)56,56,57
57 IF(RKK(I,J,1)-PPK3)J70,670,67
67 RKK(I,J,1)=PPK3
   MFI(I,J,1)=MFM2
   MT(I,J,1)=MTM1
   MK(I,J,1)=M
670 IF(NO-1)1100,60,60
862 IF(I-E1)66,530,100
66 CONTINUE
I=02
II=M
LL=1
75 PPK2=8000.
   JR=2
   L=1
72 IF(RKK(I,J,1)-LA)-4000.)J382,74,74
382 PPK1=(AI+AA*(T(C2)-T(I1)))*((1-P(I1,JR))+R*(P(I1,JR)-P(D1,1))
   L(T(I,1,1)+P(D1,1)-P(D2,1))/CIC2)+RKK(I,J,1)
   IF(LL-1)100,70,71
70 L=2
   PPK2=PPK1
   MFM1=JR
   JR=JR+1
   IF(JR-NP1)J72,72,75
71 IF(PPK2-PPK1)J73,74,74
74 JR=JR+1
   IF(JR-NP1)J72,72,75
73 PPK2=PPK1
   MFM1=JR
   JR=JR+1
   IF(JR-NP1)72,72,75
75 IF(LL-1)100,76,77
76 LL=2
   PPK3=PPK2
   IF(PPK2-7000.)J385,386,386
385 MFM2=MFM1
   MTM1=II
386 II=II+1
   IF(II-UD1)79,79,80
77 IF(PPK3-PPK2)82,82,82
62 END
11=11+1
IF(I1=J01)79,79,80
83 PPK3=PPK2
IF(PPK2=7000.)367,383,388
387 MFM2=MFM1
MTM1=11
3d8 11=11+1
IF(I1=J01)79,79,80
86 IF(PPK(I2,1,2)-PPK3)85,85,86
86 RKK(I2,1,2)=PPK3
MF(I2,1,2)=MFM2
MT(I2,1,2)=MTM1
M(I2,1,2)=M
DO 87 K=2,LA
RKK(I2,1,K)=RKK(I2,1,2)
MF(I2,1,K)=MF(I2,1,2)
MT(I2,1,K)=MT(I2,1,2)
87 M(I2,1,K)=M
85 DO 105 J=1,NPL
105 IF(L=1)105,92,94
90 L=1
92 IF(PPK(I1,J1,LA)=4000.)369,94,94
399 PPK=(A1+AA*(I*(1-T(I))+(I-P(I,J,R))+(P(I1,J1)-P(I1,J1))/
LC(I1,J1,LA)+P(I1,J1)-P(I1,J1))/LC2+(P(I2,1)-P(I,J)))/C(I,J,1))
2*PPK(I1,J1,LA)
IF(L-1)100,90,91
90 L=2
PPK2=PPK1
MFM1=JR
JR=JR+1
IF(JR=NPL)92,92,95
91 IF(PPK2=PPK1)93,54,94
94 JR=JR+1
IF(JR=NPL)92,92,95
93 PPK2=PPK1
MFM1=JR
JR=JR+1
IF(JR=NPL)92,92,95
95 IF(LL=L)100,90,97
96 LL=2
PPK3=PPK2
IF(PPK2=7000.)390,391,391
390 4Fm2=MFM1
MTM1=II
391 II=II+1
IF(II=J01)79,79,101
97 IF(PPK3=PPK2)102,102,103
102 II=II+1
IF(II=J01)99,99,101
103 PPK3=PPK2
IF(PPK2=7000.)392,393,393
392 MFM2=MFM1
MTM1=II
393 II=II+1
IF(II=J01)59,59,161
101 IF(PPK(I1,J1,1)-PPK3)535,535,136
RKK(I,J,1)=PPK3
MF(I,J,1)=MFM2
MT(I,J,1)=MTM1
MK(I,J,1)=M

IF(I-E1)105,530,100
105 CONTINUE
530 WRITE(6,805)PPK3,M(TM1,MFM2),MTM1,M
805 FORMAT(4X,'M=',10,6X,'KRITE(6,605JPPK,P(MTM1,MFM2),MTM1,M

STEP 7 FOR BOTH MODELS A&B

IF(M-DU1)180,345,100
845 IF(NB-1)100,353,218

STEP 5 FOR BOTH MODELS A&B WHEN M>2

100 M=M+1
   WRITE(6,125)M
125 FORMAT(2X,'M=',I5)
   DO 129 I=M,DU1
   DO 129 J=2,NP1
   II=M-1
   I2=I-1
   DO 111 II=II,12
   PK2(I)=800C.
   PPI=1-(1.-P(I,J))*EXP(C(I,J,L)*(T(I)-T(II)))
   DO 109 JR=2,NP1
   IF(PPI-P(II,JR))105,110,110
109 CONTINUE
   JJR(I)=NP
   GO TO 111
110 JJR(I)=JR
   111 CONTINUE
   DO 129 K=2,LA
   IF(I-DU1)523,524,100
524 IF(K-LA)129,523,100
523 LL=1
   II=MI
   I2=I-1
   JC113:II=II,12
   IF(JJR(I)-NP)112,547,100
112 PPP2=1-(1.-P(I,J))*EXP(C(I,J,K)*(T(I)-T(II)))
   JR=JJR(I)
   IF(PPI-P(I,JR))114,114,113
113 CL=-ALOG((1.-P(I,J))/((1.-P(I,JR))/T(I)-T(II))
   IF(M-3)100,171,172
172 DO 173 KP=2,LA
   IF(CL(I,J,K)-CL)173,174,174
173 CONTINUE
   KKP=LA
   GO TO 520
174 KKP=KP
520 IF(RKK(I,JR,KKP)-400.)394,110,110
394 PPKI=AL+AA*(T(I)-T(III))*((1.-P(I,JR))+AA*(P(I,JR)-P(I,J))/CL
   I+RKK(I,JR,KKP)
   GO TO 521
171 IF(RKK(I,JR,LA)-4000.)395,110,110
395 PPKI=AL+AA*(T(I)-T(III))*((1.-P(I,JR))+AA*(P(I,JR)-P(I,J))/CL
   I+RKK(I,JR,LA)
521 IF(PK2(II)-7000.)190,115,115
150 IF(PK2(II)=PPK1)115,116,110
116 JR=JR+1
115 IF(JR=NPl)117,117,113
115 PK2(II)=PPK1
115 MMF1(II)=JR
115 JR=JR+1
115 IF(JR=NPl)117,117,113
113 JJR(II)=JR
547 IF(LL-1)100,119,120
119 LL=2
119 PPK3=PK2(II)
119 IF(PK2(II)=7000.)121,118,118
121 MMF2=MMF1(II)
121 MTF1=11
121 GO TO 118
120 IF((PPK3=PK2(II))118,118,123
123 PPK3=PK2(II)
123 IF(PK2(II)=7000.)124,118,118
124 MMF2=MMF1(II)
124 MTF1=11
118 CONTINUE
118 RBB(I,J,K)=PPK3
118 IF(PPK3=7000.)127,126,128
127 WRITE(6,120)RBB(I,J,K),P(MT1,MT2),MTM1,I,P(I,J),C(I,J,K)
126 FORMAT(2X,'R=',F12.6,2X,'I=',I5,2X,'J=',J5)
126 CONTINUE
129 CONTINUE
129 JU=730 I=M,DJ1
129 JU=730 J=2,NP1
129 JU=730 K=2,LA
129 IF(I=DI1)731,732,100
732 IF(K=LA)730,731,100
731 RKK(I,J,K)=RBB(I,J,K)
730 CONTINUE
100 GO TO 100

STEPS & FCB MODEL 6

216 M=1
219 FORMAT(2X,'N=',I5)
219 I=01
219 I=01
220 DO 220 K=2,LA
220 MV1=4F(01,1,K)
220 MV2=MT(01,1,K)
220 WRITE(6,221)RKKU(01,1,K),P(MV2,MV1),MTU(01,1,K),PKU(01,1,K),I,
220 IP(I,J),C(I,J,K)
221 FORMAT(2X,'R=',F12.6,2X,'I=',I5,2X,'J=',J5)
221 CONTINUE
220 DO 223 I=0001,302
223 DO 223 J=2,NP1
223 RKKU(I,J,LA)=RKKU(I,J,1)
223 MV1=4F(I,J,1)
223 MV2=MT(I,J,1)
223 WRITE(6,224)RKKU(I,J,1),P(MV2,MV1),MT(I,J,1),PKU(I,J,1),I,P(I,
STEP 12 FOR MODEL B

CONTINUE

GO TO 300

WRITE(6,225)M

FORMAT(2X,'M='),15)

I=JO+1+M

DO 226 I=1,1002

DO 226 J=2,NP1

I3=M+JO-1

I2=I-1

JO 227 II=I3,I2

PK2(II)=8000.

IF(II-D01)100,227,313

PP1=1.-(1.-P(I,J)) EXP(C(I,J,1)*(T(I)-T(II))

DO 229 JR=2,NP1

IF(PP1-P(II,JR))229,210,210

CONTINUE

J JR(II)=NP

GO TO 227

210 JR(II)=J

CONTINUE

DO 226 K=2,LA

IF(I-DD2)550,551,100

551 IF(K-LA)226,550,100

550 LL=1

I3=M+DD1-1

I2=I-1

DO 228 I=I3,I2

IF(II-D01)100,230,231

231 IF(JR(II)-NP)232,545,100

232 PP2=1.-(1.-P(I,J)) EXP(C(I,J,K)*(T(I)-T(II))

JK=JR(II)

249 IF(PP2-P(II,JK))236,236,233

236 CL=ALOG((1.-P(I,J))/(1.-P(II,JK)))/(T(I)-T(II))

IF(M-2)100,237,237

238 DU 239 KP=2,LA

IF(C(I,J,KP)-C(I)239,249,249

239 CONTINUE

KP=LA

240 IF(RKK(I,J,KP)-4000.)32,2+3,2+8

402 PKK1=(AI+T(I)-T(II))*AA*(1.-P(II,JK)) AA*(P(II,JK)-P(I,J))/CL

1+RKK(I,JF,KP)

GO TO 241

237 IF(RKK(I,JR,1)-4000.)403,243,243

403 PKK1=(AI+AA*(T(I)-T(II)))*(1.-P(I,JK)) AA*(P(II,JK)-P(I,J))/CL

1+RKK(I,JR,1)

GO TO 241

230 DU 2+2 KP=2,LA

IF(C(I,J,1,KP)-C(I,J,1))242,2+3,2+3

242 CONTINUE

KP=LA

2+3 PKK3=(AI+AA*(T(I)-T(II)))*(1.-P(J1,1))+AA*(P(J1,1)-P(I,J))/CL(I,J

1,1)+RKK(J1,1,KP)

MEM2=1

MTM1=J1
LL=2
GO TO 226
241 IF(PK2(II) - 700) = 1316, 247, 247
316 IF(PK2(II) = PKK1) = 247, 248, 249
248 JR = JR + 1
IF(PK2(II) = 249, 247, 233
247 PK2(II) = PKK1
3M11(II) = JR
JR = JR + 1
IF(PK2(II) = 249, 247, 233
233 JR(II) = JR
548 IF(LL - 1100) = 252, 253
252 LL = 2
PKK3 = PK2(II)
IF(PKK3 = 254, 226, 228
228 MFM2 =MMF1(II)
MTM1 = II
GO TO 228
253 IF(PKK3 = PK2(II)) = 226, 228, 255
255 PKK3 = PK2(II)
IF(PKK3 = 254, 226, 228, 228
256 MFM2 = MMF1(II)
MTM1 = II
228 CONTINUE
259 WRITE(6, 258) RKB(I, J, K), P(I, MTM1, MFF2), MTM1, 1, P(I, J, J), LL(I, J, K)
GO TO 226
259 WRITE(6, 258) RKB(I, J, K), I, J
260 FCKMAT(2X, 'R= ', F12.6, 2X, I=I5, 2X, J=I5)
226 CONTINUE
I1 = DD1 + M
DO 735 I = 11, 001
DO 735 J = 2, NP1
DO 735 K = 2, LA
IF(J = 735, 730, 100)
737 IF(K = 735, 730, 100)
736 RKB(I, J, K) = RKB(I, J, K)
735 CONTINUE

STEPS 15, 1361 + FOR MODEL B

300 I = 02
I1 = M + 001
DO 301 I1 = 11, 002
IF(I1 = 301, 303, 303
303 PK2(II) = 3000.
JR(II) = 2
301 CONTINUE
DO 304 K = 2, LA
LL = 1
I1 = M + 001
DO 305 I1 = 11, 002
IF(I1 = 301, 303, 307
307 IF(JR(II) = NP1) 308, 308, 309
308 JR = JR(II)
FP = C(D2, I, K) * (T(D2) = T(II))
IF(FP = 100) 311, 512, 312
812 PP1 = 0.0
GO TO 321
811 PP1 = 1.- (1.-P(O2,1)) * EXP(C(D2,1,K) * (T(D2) - T(I1))
321 IF(PP1 = PIII, JR) 310, 310, 311
310 IF(RKK(I1, JR, LA) = 4000.) 407, 320, 320
407 PP1 = (AI + AA * (T(D2) - T(I1)) * (1.-P(I1, JR)) + AA * (P(I1, JR) - P(O2, I1))
/ LC(I1, JR, LA) + RKK(I1, JR, LA)
GO TO 332
332 PP1 = (AI + (T(D2) - T(I1)) * AA) * (1.-P(O1, I1)) + AA * (P(O1, I1) - P(D2, I1))
/ LC(I1, JR, LA)
GO TO 305
GO TO 321
318 IF(PK2(I1) = 1.000, 320, 320
320 JR = JR + 1
319 PK2(I1) = PP1
321 IF(JR = 100, 321, 321, 321
311 JR = JR + 1
305 IF(LL = 0, 400, 305
322 LL = 2
300 PP3 = PK2(I1)
311 IF(PK2(I1) = 1.000, 322, 322, 322
324 MFM2 = MFM1(I1)
323 IF(JR = 100, 324, 324, 324
325 IF(PK2(I1) = 1.000, 325, 325, 325
326 MTM2 = MFM1(I1)
305 CONTINUE
327 RKK(D2,1,K) = PP3
328 IF(D2,1,K) = MF + 1
329 IF(D2,1,K) = MT + 1
304 CONTINUE
330 PP2 = 8000.
332 JR = 2
333 PL = C(I,J,L) * (T(O2) - T(I1))
334 IF(P1 = 100, 333, 333, 333
605 PP1 = 0.0
GO TO 338
606 PP1 = 1.- (1.-P(O2,1)) * EXP(C(I,J,L) * (T(O2) - T(I1))
332 IF(P1 = PII, JR) 334, 334, 334
334 IF(RKK(I1, JR, LA) = 4000, 403, 341, 341
403 PP1 = (AI + AA * (T(I1) - T(I1)) * (1.-P(I1, JR)) + AA * (P(I1, JR) - P(O2, I1))
/ LC(I1, JR, LA) + (P(O2, I1) - P(I1, JR)) * C(I,J,L) + RKK(I1, JR, LA)
IF(L=1) 100, 336, 337

330 L=2
PPK2=PPK1
MFM1=JR
JR=JR+1
IF(JR=NP1) 336, 336, 336
337 IF(PPK2=PPK1) 340, 341, 341
341 JR=JR+1
IF(JR=NP1) 336, 336, 336
340 PPK2=PPK1
MFM1=JR
JR=JR+1
IF(JR=NP1) 336, 336, 336
335 IF(LL=1) 100, 343, 344
343 LL=2
PPK3=PPK2
IF(PPK2=7000) 345, 331, 331
345 MFM2=MFM1
MTM1=II
GO TO 331
344 IF(PPK2=PPK3) 346, 331, 331
346 PPK3=PPK2
IF(PPK2=7000) 347, 331, 331
347 MFM2=MFM1
MTM1=II
GO TO 331
332 PPK3=(AI+AA*(T(I)-T(J,1)))*((1-P(D1,1))-P(D2,1))/CIC
12+(P(D2,1)-P(I,J))/C(I,J,1)+KKK(D1,1,K)
LL=2
MFM2=1
MTM1=II
GO TO 331
331 CONTINUE
IF(KKK(I,J,1)=PPK3) 532, 532, 100
303 KKK(I,J,1)=PPK3
MFM(I,J,1)=MFM2
MTM(I,J,1)=MTM1
MK(I,J,1)=M
532 IF(I=E1) 330, 332, 100
330 CONTINUE
352 WRITE(6,361) PPK3, MTM1, MFM2, MTM1, M
001 FORMAT(4X, 'RG=', 'F12.6, 2X, 'J=', 'F12.6, 2X, 'T=', 'F12.6, 2X, 'M=', 'I5, 15)

STEP 11 FOR MODEL B

M=M+1
IF(M=02+01) 350, 350, 353

STEPS 889 FOR MODEL A, STEPS 15210 FOR MODEL B

353 M=1
WRITE(6,358) M
358 FORMAT(2X, 'V=', 'I5
1=C2
J=1
DO 35 K=2, LA
MV1=MFM(D2,1,K)
MV2=MTM2(C2,1,K)
WRITE(6,360) KKK(D2,1,K), MV2, MV1, MT(D2,1,K), MK(D2,1,K), I, IP(I,J,1)
350 FORMAT(2X, 'R=', 'F12.6, 2X, 'F=', 'F12.6, 2X, 'T=', 'F12.6, 2X, 'M=', 'I5, 2X,

1. I = 'I, I5, 2X, 'J = 'F12, 2X, 'C = 'F14, 0

359 CONTINUE
DO 362 I=0, D2, E1
DO 362 J=I, NP1
KKK(I, J, LA) = KKK(I, J, 1)
MV1 = MV1 - 1
MV2 = MV2 - 1
WRITE(6, 363) KKK(I, J, 1), P(MV2, MV1), MT(I, J, 1), MK(I, J, 1), P(I, J), C(I, J, 1)

363 FORMAT(4X, 'R', 'F12, 6, 2X, 'F', 'F12, 6, 2X, 'T', 'F12, 6, 2X, 'T', 'F12, 6, 2X, 'T', 'F12, 6, 2X, 'C', 'F14, 6')
IF (I-E1) 362, 433, 100

362 CONTINUE

STEP 10 FOR MODEL A, STEP 17 FOR MODEL B

430 II = M+DD2
DO 372 II = I1, E1
LL = 1
PKK2 = D0000.
IF (II = D2) 100, 370, 371

370 PKK3 = (A1 + AA*(T(E1)-T(D2)))*L - P(D2, 1) + KKK(D2, 1, LA)
MFM2 = 1
MTM1 = D2
LL = 2
GO TO 372

371 DO 414 JR = 1, NP1
IF (KKK(I1, JR, LA) = 4000.) 410, 415, 416
IF (II = D2) 100, 370, 371

410 PKK1 = (A1 + AA*(T(E1)-T(I1)))*L - P(I1, JR) + KKK(I1, JR, LA)
IF (PKK2 = 7000.) 412, 413, 414

412 PKK2 = PKK1
MFM1 = JR
GO TO 414

415 PKK2 = PKK1
MFM1 = JR
GO TO 414

416 CONTINUE
IF (LL = I1) 410, 415, 416

416 PKK3 = PKK2
IF (PKK2 = 7000.) 418, 372, 372

418 MFM2 = MFM1
MTM1 = II
GO TO 372

417 IF (PKK3 = PKK2) 372, 372, 419

419 PKK3 = PKK2
IF (PKK2 = 7000.) 420, 372, 372

420 MFM2 = MFM1
MTM1 = II
GO TO 372

372 CONTINUE
WRITE(6, 380) PKK3, P(MT, A1, MFM2), MTM1, M

000 FORMAT(4X, 'R', 'F12, 6, 2X, 'J', 'F12, 6, 2X, 'T', 'F12, 6, 2X, 'M', 'F12, 6, 2X, 'A', 'F12, 6, 2X, 'C', 'F14, 6')
IF (PKK3 = RKK(E1, 1, LA)) 422, 423, 423

422 RKK(E1, 1, LA) = PKK3
MFM1 = MFM2
MTM1 = MTM1
MK(E1, 1, 1) = M

423 CONTINUE

STEP 11 FOR MODEL A, STEP 18 FOR MODEL B
M=M+1
IF(M-E1+O2)431,431,432

STEP 12 FOR MODEL A, STEP 19 FOR MODEL B

431 WRITE(6,470)M
470 FORMAT(2A,5X,I9)
    1=I1+O2
    J=1
    LL=1
    I2=M+O2-1
    I3=I-1
    DD 435 11=I2;13
    PKK2=OJO.
    IF(I1-02)100,436,437
436 LL=2
    DD 501 K=2;LA
    IF(C(I2,1,K)-C(I,J,1))501,503,503
501 CONTINUE
    K=LA
    PKK3=(AI*AA*(T(I)-T(O2)))*(1.-P(O2,1))+AA*(P(O2,1)-P(I,J)) /
    IC(I,J,1)+RKK(O2,1,K)
    MMF2=1
    MTM1=O2
    GU TC 435
437 PPPl=1.-(1.-P(I,1))*EXP(C(I,J,1)*(T(I)-T(I)))
    JR=2
441 IF(PPPl-P(I,J))440,439,439
440 JR=JR+1
    IF(JR-NP1)441,441,459
439 JJV=JR
450 IF(M-2)100,690,620
620 L1=-ALG(PAC/(1.-P(I,JV)))/(T(I)-T(I))
    DD 621 KP=2;LA
    IF(C(I,J,JV,KP)-CL)621,622,622
621 CONTINUE
    KP=LA
622 IF(RKK(I,J,JV,KP)-4000,1423,445,445
623 PKK1=(AI*AA*(T(I)-T(I)))*(1.-P(I,JV))+/AA*(P(I,JV)-P(I,J)) /
    IC(J1)+RKK(I,J,JV,KP)
    GU TO 623
690 IF(RKK(I,J,JV,LA)-4000,1443,445,445
443 L1=-ALG(PAC/(1.-P(I,JV)))/(T(I)-T(I))
    PKK1=(AI*AA*(T(I)-T(I)))*(1.-P(I,JV))+/AA*(P(I,JV)-P(I,1)
    J1)/CL+RKK(I,J,JV,LA)
625 IF(PKK2-7000,1447,446,446
448 PKK2=PKK1
    MMF1=JLV
445 JJV=JLV+1
    IF(JJV-NP1)450,450,450
447 IF(PKK2-PKK1)52,450,456
452 PKK2=PKK1
    MMF1=JLV
456 JJV=JLV+1
    IF(JJV-NP1)450,450,459
459 IF(LL-1)100,460,401
40 C LL=2
    PKK3=PKK2
    IF(PKK3=7000,1462,435,435
462 MFM2=MFM1
    MTM1=11
    GO TO 433
464 IF(PKK3-PKK2)435,435,464
465 MFM2=MFM1
    MTM1=11

435 CONTINUE

RKK(I,J,LA)=PKK3
    IF(PKK3=7000.) 472,473,474
472 WRITE(6,474)RKK(I,J,LA),P(MT41,MFM2),MT41,I,P(I,J),C(I,J,L)
474 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'I=',I5,2X,'J=',I5)
    GO TO 705
473 WRITE(6,475)RKK(I,J,LA),I,J
475 FORMAT(2X,'R=',F12.6,2X,'I=',I5,2X,'J=',I5)
705 DU 434 J=2,NP1
    DU 030 I=I2,I3
    PK2(I1)=8000.
    IF(I1=32)100,630,631
631 PPP1=L-(1.-P(I,J))*EXP(C(I,J,1)*(T(I)-T(I))/)
    DU 032 JR=2,NP1
    IF(PPP1-P(I1,JK))632,633,633
632 CONTINUE
    JJR(I1)=NP
    GO TO 630
633 JJK(I1)=JK
634 CONTINUE
    DU 434 K=2,LA
    IF(I1=EE1)637,638,100
638 IF(K=LA)434,637,100
637 LL=1
    DU 039 N=I2,I3
    IF(I1=02)100,040,641
641 IF(JJR(I1)=NP)642,643,100
642 FP=C(I,J,K)*(T(I)-T(I))/)
    IF(FP=1.0)712,713,710
710 PPP2=0.
    GO TO 711
712 PPP2=L-(1.-P(I,J))*/EXP(C(I,J,K)*/T(I)-T(I))/)
711 JR=JJR(I1)
663 IF(PPP2-P(I1,JK))644,644,645
644 C1=-ALOG((1.-P(I,J))/(1.-P(I1,JK)))/(T(I)-T(I))/)
    IF(M=2)100,047,646
646 DU 047 KP=2,LA
    IF(C(I1,JK,KP)-C1)649,650,650
649 CONTINUE
    KP=LA
650 IF(PKK(I1,JR,KP)=4000.)651,652,652
651 PKK1=(A1*(T(I)-T(I))/2)*/(1.-P(I1,JK))+AA*(P(I1,JK)-P(I,J))/C1
    1+RKK(I1,JK,KP)
    GO TO 653
657 IF(RKK(I1,JR,KP)=4000.)654,662,662
654 PKK1=(A1+AA*(T(I)-T(I)))/(1.-P(I1,JK))+AA*(P(I1,JK)-P(I1,J))/C1
    1+RKK(I1,JK,KP)
    GO TO 663
660 UG 055 KP=2,LA
    IF(C(I2,1,KP)-C(I1,J,1))655,656,656
655 CONTINUE
KP=L A
656 PKK3=(AI+AA*(T(J)-T(J2)))*(1-P(J2,1))+AA*(P(J2,1)-P(J,1))/(C(I,J)
L1,1)+KK(K(J2,1,KP))
MFM2=1
MTM1=D2
LL=2
GU TO 633
053 IF(PK2(J-700),663,661,661
660 IF(PK2(J)-PKK1),661,662,662
662 JR=JK+1
IF(JR-NP1),663,663,645
661 PK2(J2)=PKK1
MMFL(J2)=JK
JR=JR+1
IF(JR-NP1),663,663,645
045 JR(J2)=JR
643 IF(LL-1),IOO,667,668
607 LL=2
PKK3=PK2(J2)
IF(PK2(J-700),669,639,639
605 MFM2=MMFL(J2)
MTM1=I1
GU TO 639
668 IF(PKK3-PK2(I),639,639,670
670 PKK3=PK2(I)
IF(PK2(J)-700),671,639,639
671 MMFM2=MMFL(I)
MTM1=II
635 CONTINUE
NLB(I,J,K)=PKK3
IF(PKK3-700),1672,673,673
672 WRITE(0,674)KBB(I,J,K),P(MTAL,MF42),MTAL1,P(I,J),C(I,J,K)
674 FORMAT2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',15,2X,'I=',15,2X,'J='
1,F12.6,2X,'C=',F14.6)
GU TO 434
673 WRITE(1,675)I2(1),J2,1,0
675 FORMAT2X,'R=',F12.6,2,1,'I=',15,2X,'J=',15
434 CONTINUE
I1=M+2
DO 740 I=I1,IE1
DO 740 J=2,NP1
DO 740 K=2,LA
IF(I-EE1),741,742,100
742 IF(K-LA),740,741,100
741 KKK(I,J,K)=R0B(I,J,K)
740 CONTINUE
STEP 13 FOR MODEL A, STEP 20 FOR MODEL B
GO TO 430
STEP 14 FOR MODEL A, STEP 21 FOR MODEL B---PRINTS THE UPPER
BOUND EXPECTED TOTAL COST RKK(E1,1,LA)
432 MVI=MF(E1,1,1)
MV=MT(E1,1,1)
WRITE(6,500)KKK(E1,1,LA),PIV2,MVI,MT(E1,1,1),PK(E1,1,1)
500 FORMAT2X,'R0P=',F12.6,2X,'F=',F12.0,2X,'T=',15,2X,'ANN=',15)
100 STOP
END
APPENDIX C

COMPUTER PROGRAM FOR CALCULATING THE OPTIMAL EXPECTED TOTAL COST WITH COMPLETE INFORMATION ABOUT FAILURE DISTRIBUTION OF THE SYSTEM.
CALCULATION OF THE OPTIMAL EXPECTED TOTAL COST WITH KNOWN IFR FAILURE DISTRIBUTION FO

NOTATIONS
AI --- INSPECTION COST PER INSPECTION
AU --- COST OF UNDETECTED FAILURE PER UNIT TIME
T(TF) --- MAXIMUM LIFE TIME OF THE SYSTEM
RK(I), RU(I) --- OPTIMUM EXPECTED MAINTENANCE COST UP TO THE
1 TIME T(I) AND AT STAGE M-1 AND M RESPECTIVELY

THIS PROGRAM WAS WRITTEN BY KUROSH-KARAPISHEM, DEPARTMENT
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KANSAS, MAY, 1979

**INT**EGER TF, TF1, TMT
**DIMEN**SION RK(15), RU(15), C(15), P(15), T(15)
READ(5, 1) AA, AI, TF
1 FORMAT(2F10.6, 15)

**IN**PUT **DATA** ABOUT **IFR FAILURE DISTRIBUTION FO**

DO 2 I=1, TF
READ(5, 5) P(I), C(I), T(I)
40 FORMAT(3F10.6)
2 CONTINUE

CALCULATES RK(IF) FOR M=1

M=1
TF1=TF-1
V=3.0
DO 4 L=2, TF1
4 V=V+(P(L-1)-P(L))/C(L)
RK(TF)=AI+AA*T(TF)+AA*V
MT=1
MM=1
WRITE(0, 7) RK(TF)
7 FORMAT(2X, 'RU=', 'F12.6)

CALCULATES RK(I) FOR M=2 AND I<TF

M=2
WRITE(0, 12) M
12 FORMAT(2X, 'M=', 'I5)
DO 10 L1=2, TF1
V=0.0
50 L2=2, L1
5 V=V+(P(L2-1)-P(L2))/C(L2)
RK(L1)=AI+AA*T(L1)+AA*V
TMT=1
WRITE(0, 11) RK(L1), L1, TMT
11 FORMAT(2X, 'K=', 'F12.6, 2X, 'I=', 'I5, 2X, 'T=', 'I5)
Calculates R_k(\text{TF}) for M \geq 2

30 DO 13 II=M,TF1
   V=0.0
   DO 31 LL=II,TF1
   V=V+(P(LL)-P(LL+1))/C(LL+1)
   PK=(AI+AA*(T(TF)-T(II)))*(1.0-P(II))+V*AA+R_k(II)

Compares R_k(\text{TF}) for older and current values of M and chooses the smaller value and discards the larger values

IF(R_k(\text{TF})-PK)>0,13,14
14 R_k(\text{TF})=PK
   MT=II
   MM=MI
13 CONTINUE

Sets new M = old M + 1 and compares it with M(MAX)

   M=M+1
   IF(TF1-M)0,15,16
   IF M<M(MAX) + 1, calculates R_k(I) for TF>M \geq 2

15 WRITE(0,10)M
16 FORMAT(2X,'M=',I5)
   DO 18 LV=M,TF1
   LV=LV-1
   M1=M-1
   LN=1
   DO 19 LP=M1,LV1
   V=0.0
   DO 20 LE=LP,LV1
   V=V+(P(LE)-P(LE+1))/C(LE+1)
   PK=(AI+AA*(T(LV1)-T(LP)))*(1.0-P(LP))+AA*V+R_k(LP)
   IF(LP=1)100,23,24
23 LN=2
   R_B(LV)=PF
   TMT=4-1
24 IF(R_B(LV)=-PF)19,19,25
25 R_B(LV)=PF
   TMT=LP
19 CONTINUE
   WRITE(0,27)R_B(LV),LV,TMT
27 FORMAT(2X,'R=',F12.6,2X,'T=',I5,2X,'TT=',I5)
16 CONTINUE
   DO 28 LC=M,TF1
28 R_k(LC)=R_k(LC)
GO TO 30

Prints the optimal expected total cost R_k(I) and optimal number of inspections 4

90 WRITE16,71)R_k(\text{TF}),MT,MM
91 FORMAT(2X,'ROPT=',F12.6,2X,'TT=',I5,2X,'TT=',I5)
100 STOP
END
ESTIMATION OF AN UPPER Bound FOR EXPECTED MAINTENANCE COST OF A SYSTEM WITH PARTIALLY KNOWN, INCREASING FAILURE RATE DISTRIBUTION

by

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AN ABSTRACT OF A MASTER'S THESIS

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MASTER OF SCIENCE

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A considerable work has been done in the field of maintenance, but except for a few almost all of this work has been based on the complete knowledge about the characteristics of the deteriorating system or equipment especially about the failure characteristics of it. But in real situations the complete information about failure distribution of a system is rarely available especially when a new one is introduced into the existing system. Minimax policy has been devised already to cope with this problem. Basically it gives the best maintenance or inspection policy or timing when the information about the system failure distribution is incomplete. It utilizes the information which is available, to minimize the total cost and at the same time maximizes the total cost with respect to all possible and feasible values of the unknown portion of the failure characteristics. The present work utilizes the information about one and two points of increasing failure rate distributions to find an upper bound for the optimal expected total maintenance cost. The basic difference of this work with previous work in this area is that a procedure and computer program has been devised which utilizes the available information not only about one point of the failure distribution but also searches for improved upper bound for the total cost when information about two points is available.

The comparison has been made on the basis of the available knowledge about a failure distribution between the upper bound total costs and the value of information has been discussed through an example. The variation of the upper bound cost with changes in the cost per inspection and the cost per unit time of undetected failure has been discussed through an example. It has been found basically that having information about two
points of the distribution improves (decreases) the upper bound total cost. Also it has been found that the closer the known time of failure probability to the maximum life time, the higher the value of information and the lower the upper bound would be. It has been found that the upperbound total cost is more sensitive to changes of the inspection and undetected failure costs at lower values of these costs. Computer program has been written and used for several different increasing failure rate distributions to find the optimal total costs and policies. These optimal costs were compared with the upper bound total costs and all were lower than the upper bound cost as it could have been expected. Finally an example problem has been worked out to illustrate the application of the results of this work in industry. There is still a lot of study needed in the field of maintenance with partial knowledge to find efficient and computationally feasible methods to optimally utilize the available partial knowledge about the system failure characteristics for finding improved upper bounds.