FINITE-AMPLITUDE VIBRATION OF ORTHOTROPIC AXISYMMETRIC VARIABLE THICKNESS ANNULAR PLATE

by

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NOMENCLATURE

$r, \theta, z$ cylindrical coordinates used to describe the undeformed configuration of the plate.

$h(r), a$ thickness function and radius of plate.

$h_0$ thickness at center.

t time variable.

$u, w$ radial and transverse displacement of the middle plane.

$\varepsilon_r, \varepsilon_\theta$ radial and circumferential strains.

$\sigma_r, \sigma_\theta$ radial and circumferential stresses.

$a_{11}, a_{12}, a_{22}$ stress-strain relation coefficients.

$N_r, N_\theta$ middle plane forces per unit length.

$M_r, M_\theta$ bending moments per unit length.

$Q(\xi), Q^*(\xi)$ dimensionless loading distributions.

$q(r, t)$ loading intensity.

$K$ kinetic energy of the plate.

$V_s, V_b$ strain energy due to stretching of the middle plane and due to bending of the plate respectively.

$W$ work done on the plate by the external forces.

$\nu$ Poisson's ratio $= -a_{12}/a_{22}$

$c$ ratio of elastic constants $= a_{11}/a_{22}$

$D(r)$ flexural rigidity of the plate $= a_{22}h^3/12(a_{11}a_{22}-a_{12}^2)$

$\psi, \phi$ stress functions

$\xi, \tau$ dimensionless space and time variables respectively.

$X$ dimensionless transverse displacement

$g(\xi), f(\xi)$ shape functions of vibration.

$A, \alpha$ amplitude parameters.
\( \lambda \) nondimensional nonlinear eigenvalue.
\[ \omega = (\lambda)^{1/2} \]
non-dimensional angular frequency.
\( \bar{Y}, \bar{Z}, \bar{H} \) (6x1) vector functions
\( M, N \) coefficient matrices.
\( \bar{0} \) (3x1) null matrix
\( 'r, 't \) partial derivatives with respect to \( r \) & \( t \)
\( \eta(\xi) \) variable thickness function.
\( \tau \) frequency parameter.
\( \delta x \) first variation of \( \chi = \delta G \sin \omega t \)
\( \bar{\eta} \) adjustable data in the related initial value problem.
\( \{ \} \) indicates a column vector.
\( \Delta \) Del operator.
INTRODUCTION

Composite materials find large application in design of structural elements in the present age. These structures which are mainly in the form of plates, are subjected to severe operational conditions, and should thus be able to withstand large amplitudes of vibration. If the amplitude of vibration is of the same order of magnitude as the thickness of the plate, then the deformation of the mid-plane can no longer be neglected. In the development of a suitable thin plate theory, anisotropic properties and geometric non-linearities arising in the coupling of membrane and bending theories should be included. The resulting governing differential equations can be solved by approximate numerical methods due to the complexity of the problem.


The above and many other investigators (18-30) have worked on either solid, circular, variable thickness, anisotropic plates or annular orthotropic plates. This present investigation is thus concerned with harmonic, large
amplitude, free vibrations of orthotropic, axisymmetric, annular plates of variable thickness.

The essence of the approximate method is to approximate the continuous system by a discrete one having a finite number of degrees of freedom. The discrete representation is achieved through an assumed space mode. Substitution of this in the differential equations along with the requirement that some measure of the error be minimized, the assumed space mode can be eliminated. The problem thus reduces to a nonlinear ordinary differential equation with time as an independent variable. This equation is similar to a one-degree of freedom Duffings equation (5).

This present work assumes the existence of harmonic vibrations. The time variable is eliminated by the application of a Ritz-Kantorovich averaging method. The basic governing equations thus reduce to a pair of ordinary differential equations, with a reformulated set of boundary conditions. A numerical study of these equations is proposed by introducing the related initial value problem.

The cases considered are, a parabolic variable thickness annular plate, with the variable thickness function of the form
\[ \eta = 0.815 - 0.5 x^2 \]
and a convex variable thickness annular plate, with the variable thickness function of the form
\[ \eta = 1.0 - 0.5 x^{1/2} \].
Both the above plates have the same volume and the same boundary conditions. The boundary conditions are free on the outside and fixed on the inside. The corresponding curves for the frequency responses, bending stresses, and membrane stresses are presented.
CHAPTER I
DERIVATION OF THE GOVERNING EQUATIONS

Consider a thin annular orthotropic plate, the elastic properties of which are different in the radial and circumferential directions. The fundamental assumptions made as regards to the flexural deformations of the plate are:

1. The loads and deflections are symmetric with respect to the z axis which passes through the center of the annulus.
2. The normals to the middle plane in the undeformed plate remain straight and normal to the middle plane in the deformed plate.
3. It follows the Hooke's Law.
4. Transverse shearing deformations are not included.
5. The maximum thickness of the plate is small in comparison to the radius of the plate.

Keeping in mind the above assumptions, the following strain-displacement relations are written:

\[ \varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2} \left( \frac{\partial w}{\partial r} \right)^2 - \frac{z}{r} \frac{\partial^2 w}{\partial r^2} \]

or in indicial notation as:

\[ \varepsilon_r = u_{,r} + \frac{1}{2} w_{,r} - z w_{,rr} \] (1)

In the \( \theta \) direction:

\[ \varepsilon_\theta = \frac{u}{r} - \frac{z}{r} \frac{\partial w}{\partial r} \]

or in indicial notation as:

\[ \varepsilon_\theta = \frac{u}{r} - \frac{z}{r} w_{,r} \] (2)
In view of the orthotropy of the plate considered, the Hooke's Law can be written as:

\[ \varepsilon_\theta = a_{11} \sigma_\theta + a_{12} \sigma_r \]  
\[ \varepsilon_r = a_{12} \sigma_\theta + a_{22} \sigma_r \]  

From the above two equations, it follows

\[ \sigma_r = \frac{a_{11}}{a_{11} a_{22} - a_{12}^2} \left( \varepsilon_r - \frac{a_{12}}{a_{11}} \varepsilon_\theta \right) \]  
\[ \sigma_\theta = \frac{a_{22}}{a_{11} a_{22} - a_{12}^2} \left( \varepsilon_\theta - \frac{a_{12}}{a_{22}} \varepsilon_r \right) \]

where \( a_{11}, a_{22}, a_{12} \) are the elastic constants and \( \sigma_r, \sigma_\theta \) are the radial and circumferential stresses.

Resubstituting \( \varepsilon_\theta \) and \( \varepsilon_r \) from (1) and (2), we have

\[ \sigma_r = \frac{a_{11}}{a_{11} a_{22} - a_{12}^2} \left( u_r + \frac{1}{2} \omega_r^2 - \frac{a_{12}}{a_{11}} \left( \frac{u}{r} - \frac{z}{r^2} \omega_r \right) \right) \]  
\[ \sigma_\theta = \frac{a_{22}}{a_{11} a_{22} - a_{12}^2} \left( \frac{u}{r} - \frac{a_{12}}{a_{22}} \left( u_r + \frac{1}{2} \omega_r^2 - \frac{z}{r^2} \omega_r \right) - \frac{z}{r^2} \right) \]

Expressions for the radial and circumferential forces per unit length, \( N_r \) and \( N_\theta \), are obtained by integrating the respective stresses across the thickness of the plate.

\[ N_\theta = \int_{-h/2}^{h/2} \sigma_\theta \, dz = \frac{a_{22} h(r)}{a_{11} a_{22} - a_{12}^2} \left( \frac{u}{r} - \frac{a_{12}}{a_{22}} \left( \frac{3 u}{3 r} + \frac{1}{2} \left( \frac{3 w}{3 r} \right)^2 \right) \right) \]  
\[ N_r = \int_{-h/2}^{h/2} \sigma_r \, dz = \frac{a_{11} h(r)}{a_{11} a_{22} - a_{12}^2} \left( \frac{3 u}{3 r} + \frac{1}{2} \left( \frac{3 w}{3 r} \right)^2 \right) - \frac{a_{12}}{a_{11}} \left( \frac{u_r}{r} \right) \]
or,

\[ N_r = \frac{h(r)}{a_{22}(c-v^2)} (c(u_r + \frac{1}{2} w_r^2) + \frac{v u_r}{r}) \]  

(7a)

\[ N_\theta = \frac{h(r)}{a_{22}(c-v^2)} \left( \frac{u}{r} + v u_r + \frac{v}{2} (w_r)^2 \right) \]  

(7b)

where,

\[ c = \frac{a_{11}}{a_{22}} = \text{ratio of properties in radial and circumferential directions}. \]

\[ \nu = -\frac{a_{12}}{a_{22}} = \text{poisson's ratio}. \]

The radial and circumferential moments per unit length, \( M_r \) and \( M_\theta \), are obtained by integrating the moments of the forces about the middle plane across the thickness of the plate.

\[ M_r = \int_{-h/2}^{h/2} \sigma_r z dz = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2} \int_{-h/2}^{h/2} \left\{ \varepsilon_r - \frac{a_{12}}{a_{11}} \varepsilon_\theta \right\} z dz \]  

(8a)

\[ M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z dz = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2} \int_{-h/2}^{h/2} \left\{ \varepsilon_\theta - \frac{a_{12}}{a_{22}} \varepsilon_r \right\} z dz \]  

(8b)

or,

\[ M_r = -D \left( \frac{\nu}{r} w_r + c w_{rr} \right) \]  

(9a)

\[ M_\theta = -D \left( \frac{1}{r} w_r + \nu w_{rr} \right) \]  

(9b)

where,

\[ D = \frac{a_{22}h^3}{12(a_{11}a_{22} - a_{12}^2)} = \frac{E_\theta h^3}{12} = \frac{h^3}{12G} \]

\[ \beta = \frac{a_{11}a_{22} - a_{12}^2}{a_{22}} \]
The Energy Method

The extended Hamilton's Principle, which states that, within the interval of time, \( t_1 \) and \( t_2 \), the first variation of the action integral is equal to zero, is made use of here; i.e.:

\[
\delta \int_{t_1}^{t_2} L dt = 0
\]

Here the Lagrangian is \( L = K - V_s - V_b + W \)

where

\( K = \) Kinetic Energy
\( V_s = \) strain energy due to stretching of the middle plane
\( V_b = \) strain energy due to the bending of the plate.
\( W = \) work done by the time dependent external forces.

1. Neglecting the radial part of inertia force, \( \partial u/\partial t \ll \partial w/\partial t \), the kinetic energy is

\[
K = \pi \int_c^a \rho h(r) \left( \frac{w^2}{2} \right) rdr
\]

2. The strain energy due to stretching of the middle plane is obtained as follows:

\[
V_s = 2\pi \int_c^a \frac{N_r \varepsilon_r}{2} + \frac{N_0 \varepsilon_0}{2} rdr
\]

Substituting values of \( N_r, N_0, \varepsilon_r, \varepsilon_0 \) and rearranging,

\[
V_s = \frac{\pi}{2} \int_c^a \left\{ cu_r^2 + \frac{c}{4} w_r^4 + cu_r w_r^2 + 2u_r u_r + \nu u_r^2 \right\} hrdr
\]

3. The strain energy due to bending of the plate:
\[ V_b = -\pi \int_c^a \left\{ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right\} rdr \]

Substituting \( M_r \) and \( M_\theta \) and rearranging,

\[ V_b = \pi \int_c^a D(r) \left\{ \frac{c w^2}{r^2} + 2 \frac{v}{r} w w' + \frac{1}{2} \frac{w'^2}{r^2} \right\} rdr \tag{15} \]

4. The work done by the exciting force function \( p(r,t) \),

\[ W = -2\pi \int_c^a p(r,t) wrdr \tag{16} \]

Substituting equations (12), (14), (15) and (16) into (11) we obtain the Lagrangian as:

\[ L = \pi \int_c^a \left[ \frac{1}{2} \rho (r) w_t^2 \right] rdr \]

\[ - \frac{\pi}{\beta} \int_c^a \left\{ \frac{c u^2}{r} + \frac{c w}{r} + \frac{c}{4} w^4 + 2 \frac{u}{r} u_r + \frac{u}{r} w_r^2 + \left( \frac{u_r}{r} \right)^2 \right\} hrdr \]

\[ - \frac{\pi}{12\beta} \int_c^a \left\{ \frac{c w^2}{r^2} + 2 \frac{v}{r} w w' r + \left( \frac{w_r}{r} \right)^2 \right\} h^3 rdr \]

\[ + 2\pi \int_c^a p(r,t) wrdr \tag{17} \]

Now the integral can be written symbolically as:

\[ I = \int_{t_1}^{t_2} \int_c^a f(t,r,w,u,w_r,u_r,w_t,w_{rr}) rdrdt \]

where,

\[ f = \left\{ \rho hr w_t^2 - \frac{h}{\beta} \left[ cu^2 + \frac{c}{4} w^4 + cu rw^2 + 2 vu u_r + vu w_r^2 + \frac{u^2}{r} \right] \right\} \]

\[ - \frac{h^3}{12\beta} \left[ cw^2 + 2vw w' r + \frac{1}{r} w_r^2 \right] + 2p(r,t)rw \]
The first variation of $I$ vanishes

$$
\delta I = \int_{t_1}^{t_2} \int_c \left\{ \left( \frac{\partial f}{\partial w} \right) \delta w + \left( \frac{\partial f}{\partial u} \right) \delta u + \left( \frac{\partial f}{\partial \omega_r} \right) \delta \omega_r + \left( \frac{\partial f}{\partial \omega_t} \right) \delta \omega_t \right. \\
+ \left. \left( \frac{\partial f}{\partial \omega_{rr}} \right) \delta \omega_{rr} \right\} \, dr \, dt
$$

(18)

Now,

$$
\int_{t_1}^{t_2} \int_c \left( \frac{\partial f}{\partial w} \right) \delta w \, dr \, dt = \int_{t_1}^{t_2} \int_c (2pr) \delta w \, dr \, dt
$$

(19)

$$
\int_{t_1}^{t_2} \int_c \left( \frac{\partial f}{\partial u} \right) \delta u \, dr \, dt = \int_{t_1}^{t_2} \int_c -\frac{h}{\beta} \left( 2\nu u_r + \nu \omega_r^2 + \frac{2u_r^2}{\nu} \right) \delta u \, dr \, dt
$$

(20)

$$
\int_{t_1}^{t_2} \int_c \left( \frac{\partial f}{\partial \omega_r} \right) \delta \omega_r \, dr \, dt = \int_{t_1}^{t_2} \int_c \left( \frac{\partial f}{\partial \omega_r} \right) \delta \omega_r \, dr \, dt - \int_{t_1}^{t_2} \int_c \delta \omega \left|_c \right. \, dt
$$

(21a)

by partial integration.

Substituting values of $\frac{d}{dr} \left( \frac{\partial f}{\partial \omega_r} \right)$ and $\frac{\partial f}{\partial \omega_r}$ into equation (21a) we obtain:

$$
\int_{t_1}^{t_2} \int_c \left( \frac{\partial f}{\partial \omega_r} \right) \delta \omega_r \, dr \, dt = \int_{t_1}^{t_2} \int_c -\frac{h}{\beta} \left\{ c \omega_r^3 + 2c u_r \omega_r + 2\nu u \omega_r \right\} \\
- D(r) \left\{ 2w_{rr} + \frac{2w_r}{r} \right\} \delta w \left|_c \right. \, dt
$$

$$
- \int_{t_1}^{t_2} \int_c \left\{ -\frac{h}{\beta} \left[ 3c r \omega_r^2 \omega_{rr} + c \omega_r^3 + 2c u_r \omega_r \right] \\
+ 2\nu u \omega_{rr} + 2c u_r \omega_{rr} + 2c u \omega_r \omega_r + \right.
$$
\[ + 2v u_r w_r ] - D_r [2v w_{rr} + \frac{2w}{r}] \]

\[ - \frac{h}{\beta} [cw^3_r + 2c u_r w_r + 2v w w_r] \]

\[ - D(r)[2v w_{rr} + \frac{2w_{rr}}{r} - \frac{2w}{r^2} \}] \delta w r d r t \quad (21) \]

By similar method, substituting values of \( \frac{d}{dr} \left( \frac{\partial f}{\partial u_r} \right) \) and \( \frac{\partial f}{\partial u_r} \) in equation:

\[ \int_{t_1}^{t_2} \int_c^a \left( \frac{\partial f}{\partial u_r} \right) \delta u_r d r d t = \int_{t_1}^{t_2} \frac{\partial f}{\partial u_r} \delta u_c d t - \int_{t_1}^{t_2} \int_c^a \frac{d}{dr} \left( \frac{\partial f}{\partial u_r} \right) \delta u d r d t \]

We get,

\[ \int_{t_1}^{t_2} \int_c^a \left( \frac{\partial f}{\partial u_r} \right) \delta u_r d r d t = \int_{t_1}^{t_2} - \frac{h}{\beta} [2c u_r + c w^2_r + 2v u] \delta u_c d t \]

\[ - \int_{t_1}^{t_2} \int_c^a \left(- \frac{h}{\beta} [2c u_r + 2c u_r r + c w^2_r + 2c w_r w_{rr} \]

\[ + 2v u_r] - \frac{h}{\beta} [2c u_r + c w^2_r + 2v u] \} \delta u d r d t \quad (22) \]

Substituting values of \( \frac{\partial f}{\partial w_t} \) and \( \frac{d}{dt} \left( \frac{\partial f}{\partial w_t} \right) \) in equation:

\[ \int_{t_1}^{t_2} \int_c^a \frac{\partial f}{\partial w_t} \delta w_r d r d t = \int_{t_1}^{t_2} \frac{\partial f}{\partial w_t} \delta w_c d r - \int_{t_1}^{t_2} \int_c^a \left\{ \frac{d}{dt} \left( \frac{\partial f}{\partial w_t} \right) \right\} \delta w d r d t \]

And as, \( \delta w = 0 \) at \( t_1, t_2 \), the first integral = 0, hence the above equation reduces to:

\[ \int_{t_1}^{t_2} \int_c^a \frac{\partial f}{\partial w_t} \delta w_r d r d t = - \int_{t_1}^{t_2} \int_c^a (2 \phi h r w_{tt}) \delta w d r d t \quad (23) \]
And finally we have,

$$\int_{t_1}^{t_2} \int_c^a \frac{\partial f}{\partial w_{rr}} \delta w_{rr} \, dr \, dt = \left[ \int_{t_1}^{t_2} \frac{\partial f}{\partial w_{rr}} \delta w_r \bigg|^a_c \right] dt$$

$$- \int_{t_1}^{t_2} \frac{d}{dr} \left( \frac{\partial f}{\partial w_{rr}} \right) \delta w \bigg|^a_c + \int_{t_1}^{t_2} \int_c^a \frac{d^2}{dr^2} \left( \frac{\partial f}{\partial w_{rr}} \right) \delta w \, dr \, dt$$

after integrating by parts twice, and substituting the values of the various constituents of the equation we have,

$$\int_{t_1}^{t_2} \int_c^a \frac{\partial f}{\partial w_{rr}} \delta w_{rr} \, dr \, dt = \left[ \int_{t_1}^{t_2} - D(r)[2c_rw_{rr} + 2vw_r] \delta w_r \bigg|^a_c \right] dt$$

$$- \int_{t_1}^{t_2} \left\{ -D(r)[2c_rw_{rr} + 2crw_{rrr} + 2vw_{rrr}] \right\} \delta w \bigg|^a_c \, dt$$

$$+ \int_{t_1}^{t_2} \int_c^a \left\{ -D(r)[4c_rw_{rrr} + 2crw_{rrrr} + 2vw_{rrrr}] \right\} \delta w \, dr \, dt$$

$$- D_r[2crw_{rr} + 2vw_r] \right\} \delta w \bigg|^a_c \, dt$$

$$+ \int_{t_1}^{t_2} \left\{ -D(r)[4c_rw_{rrr} + 2crw_{rrrr} + 2vw_{rrrr}] \right\} \delta w \, dr \, dt$$

Substituting equations (19) to (24) into (18) we get:

$$\delta I = \int_{t_1}^{t_2} \int_c^a 2pr \delta w \, dr \, dt + \int_{t_1}^{t_2} \int_c^a \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right) \delta w \, dr \, dt$$

$$+ \int_{t_1}^{t_2} \left\{ - \frac{h}{\beta} (c_r^3 + 2c_rw_r + 2vw_r) - D(r)(2vw_r + \frac{2w_r}{r}) \right\} \delta w \bigg|^a_c \, dt$$

$$+ \int_{t_1}^{t_2} \left\{ - \frac{h}{\beta} (c_r^3 + 2c_rw_r + 2vw_r) - D(r)(2vw_r + \frac{2w_r}{r}) \right\} \delta w \bigg|^a_c \, dt$$
For equation (25) to hold true, the integrands in the double and single integrals should vanish separately.
The double integral yields the Euler-Lagrange equations:

\[
D(r) \left[ cw_{rrrr} + \frac{2c}{r} w_{rr} - \frac{w_{rr}}{r^2} + \frac{w_r}{r^3} \right] + D_r \left[ 2cw_{rrr} + \frac{2c}{r} w_{rr} + \frac{v}{r} w_{rr} - \frac{w_r}{r^2} \right]
\]

\[
+ D_{rr} \left[ cw_{rr} + \frac{v w_r}{r} \right] + \rho h(r) w_{tt} = p(r, t) + \frac{1}{\beta} \left\{ h \left[ c u_{w_{rr}} + u_{rr} w_r \right] + \frac{u_r w_r}{r} + \frac{3}{2} w_{rr} + \frac{w_r^3}{2r} + \frac{v}{r} \left( u_{w_{rr}} + u_r w_r \right) \right. + h_r \left( c u_r w_r + \frac{c}{2} w_r^3 + \frac{v}{r} u w_r \right) \}
\]

and

\[
h \left\{ cu_{rr} + \frac{c u_r}{r} - \frac{u}{r^2} + (c - v) \frac{w_r^2}{2r} + c w_r \frac{w_{rr}}{r} \right\}
\]

\[
+ h_{rr} \left\{ c u_r + \frac{c}{2} w_r^2 + \frac{v u_r}{r} \right\} = 0
\]

(26)

The single integrals yield, the boundary conditions,

\[
w = 0 \quad \text{or} \quad -\frac{h}{8} w_r \left( \frac{c}{2} w_r^2 + cu_r + \frac{v}{r} \right) + D(r) \left\{ -\frac{w_r}{2} + c w_{rr} + \frac{cw_{rrr}}{r} \right\}
\]

\[
+ D_r \left( cw_{rr} + \frac{v}{r} w_r \right) = 0 \quad \Rightarrow \quad \text{shear} = 0
\]

deflection = 0

\[
w_r = 0 \quad \text{or} \quad D(r) (2c w_{rr} + 2v w_r) = 0 \quad \Rightarrow \quad \text{Moment} = 0
\]

slope = 0

and

\[
u = 0 \quad \text{or} \quad rh(r) \left( cu_r + \frac{c w_r^2}{2} + \frac{u}{r} \right) = 0 \quad \Rightarrow \quad \text{Force} = 0
\]

Equation (26) can be expressed as:
\[
D(cw_{rrrr} + \frac{2c}{r} w_{rrr} - \frac{1}{r^2} w_{rr} + \frac{1}{3} w_r) + D_r (2cw_{rrrr} + (2c+\nu) \frac{1}{r} w_{rr} - \frac{1}{r^2} w_r)
\]

\[+ D_r (cw_{rr} + \frac{\nu}{r} w_r) + \rho hw_{tt} = p(r,t) + \frac{1}{\beta} \frac{3}{\partial r} [hrw_r (cu_r + \frac{c}{2} w^2_r + \frac{\nu}{r} u)]\]

(28)

Proceeding further with the stress formulation, and substituting the following into equations (27) & (23)

\[\psi = rN_r \quad \text{and} \quad \frac{\partial \psi}{\partial r} = N_0\]

\[\frac{\partial u}{\partial r} + \frac{1}{r} w_r = \frac{a_{22}}{h} (N_r - \nu N_0) = \frac{a_{22}}{h} (\frac{\psi}{r} - \nu \psi)\]

\[\frac{u}{r} = \frac{a_{22}}{h} (c\psi - \frac{\nu}{r} \psi)\]

We have the following equation:

\[
D(cw_{rrrr} + \frac{2c}{r} w_{rrr} - \frac{1}{r^2} w_{rr} + \frac{1}{3} w_r) + D_r (2cw_{rrrr} + (2c+\nu) \frac{1}{r} w_{rr} - \frac{1}{r^2} w_r)
\]

\[+ D_r (cw_{rr} + \frac{\nu}{r} w_r) + \rho hw_{tt} = p(r,t) + \frac{1}{r} [w_r \psi]_r\]

(29)

\[c\psi_{rr} + \frac{c\psi}{r} - \frac{\psi}{2} + \frac{h}{2a_{22}} [-c\psi + \frac{\nu \psi}{r}] + \frac{h}{2a_{22}} w^2_r = 0\]

(30)

and the boundary conditions become:

\[w = 0 \quad \text{or} \quad D[cw_{rr} + \frac{c}{r} w_{rr} - \frac{w_r}{r^2}] + D_r [cw_{rr} + \frac{\nu}{r} w_r] - \frac{1}{r} w_r \psi = 0\]

\[w_r = 0 \quad \text{or} \quad cw_{rr} + \frac{\nu}{r} w_r = 0\]

\[\psi = 0 \quad \text{or} \quad c\psi_r - \nu \frac{\psi}{r} = 0\]

Using the substitutions,
\[ \chi = \frac{w}{a} \]
\[ \xi = \frac{r}{a} \]
\[ \tau = t \left[ \frac{D_0}{ha^4(1-v^2)} \right]^{1/2} \]
\[ \phi = \frac{a_{22}}{h_0 a} \psi \]
\[ D = \frac{h^3}{128} ; \quad \beta = a_{22}(c-v^2) = \frac{h_0^3 n^3}{12(c-v^2)a_{22}} \]
\[ D_0 = \frac{h_0^3}{12a_{22}} \]

we get the following non-dimensional form:

\[ n^3 [c \chi''' + \frac{2c}{\xi} \chi'' - \frac{1}{\xi^2} \chi' + \frac{1}{\xi^3} \chi] + (n^3) \xi \left[ 2c \chi'''' + (2c+v) \frac{1}{\xi} \chi'' \right] \]
\(- \frac{1}{\xi} \chi' \right] + (n^3) \xi \left[ c \chi'' + \frac{v}{\xi} \chi' \right] + n \left( \frac{c-v^2}{1-v^2} \right) \chi_{\tau \tau} \]
\[ = 12(c-v^2)a_{22}\left( \frac{a}{h_0} \right)^3 p + 12(c-v^2)\left( \frac{a}{h_0} \right)^2 \frac{1}{\xi} \left[ \chi' \phi \right]' \]

(31)

and,

\[ [c \phi'' + \frac{c \phi'}{\xi} - \frac{\phi}{\xi^2}] + \frac{n}{n} \left[ -c \phi' + \frac{v}{\xi} \phi \right] + \frac{n}{2\xi} \left[ \chi_\xi \right]^2 \]

(32)

The boundary conditions become,

\[ \chi' = 0 \quad \text{or} \quad c \chi'' + \frac{v}{\xi} \chi' = 0 \]  

(33a)

\[ \chi = 0 \quad \text{or} \quad [c \chi''' + \frac{c}{\xi} \chi'' - \frac{1}{\xi^2} \chi'] + \frac{3n}{n} \left[ c \chi'''' + \frac{v}{\xi} \chi' \right] \]
\[-12(c-v^2) \left( \frac{a}{h_0} \right)^2 \frac{1}{n^3} \xi \left( \chi' \phi \right) = 0 \]

(33b)

\[ \phi = 0 \quad \text{or} \quad c \phi' - \frac{v}{\xi} \phi = 0 \]

(33c)
CHAPTER II

APPROXIMATE ANALYSIS

There is, at present, no exact method known, for the solution of the differential equations (31) and (32), also the standard Fourier analysis used in linear vibration problems is not applicable, because the nonlinear character of the differential equation, causes coupling of vibration modes.

Consequently, this nonlinear coupled problem, can only be solved by some approximate numerical method. Approximate solutions of large amplitude vibrations can be achieved by separation of variables method, or implementing function space methods to eliminate the space coordinate with an assumed mode shape function. The problem is thus reduced to a non-linear ordinary differential equations with time $t$, as an independent variable. The resulting one degree of freedom Duffings equation is solved and the solutions are in terms of elliptical functions. This is called the assumed - space - mode solution. The Kantorovich Averaging method is proposed to find an assumed time-mode solution of the equations (31) and (32) and the boundary conditions equations (33).

Kantorovich Averaging Method:

A sinusoidal form of the loading intensity is assumed here:

$$ P(\xi, \tau) = Q(\xi) \sin \omega \tau $$

(34)

also, the steady state response can be closely approximated by the expressions,

$$ X(\xi, \tau) = G(\xi) \sin \omega \tau $$

(35a)

$$ \phi(\xi, \tau) = F(\xi) \sin^2 \omega \tau $$

(35b)

where $G(\xi)$ and $F(\xi)$ are the undetermined shape functions of vibrations.
Substituting equations (35) into equation (32), we have

\[
\frac{c}{\xi} \frac{d^2 F}{d\xi^2} + \frac{c}{\xi} \frac{dF}{d\xi} - \frac{F}{\xi^2} + \frac{h}{2\xi} (G')^2 + \frac{\eta}{\xi} \left[ -c \frac{dF}{d\xi} + \nu \frac{F}{\xi} \right] = 0
\]  

(36)

As the expressions (34) and (35) cannot satisfy equations (31) for all \( \tau \), the integral,

\[
I_A = \int_{-\infty}^{\infty} \left\{ n^3 \left[ c \phi'''' + \frac{2c}{\xi} \phi''' - \frac{1}{\xi^2} \phi'' + \frac{1}{\xi^3} \phi' \right] 
+ (n^3)_{\xi} \left[ 2c \phi'''' + \frac{(2c+\nu)}{\xi} \phi''' - \frac{1}{\xi^2} \phi'' \right] 
+ (n^3)_{\xi\xi} \left[ c \phi'''' + \frac{\nu}{\xi} \phi'' \right] + n \left( \frac{c-v^2}{1-\nu^2} \right) \phi' \right\} \delta\phi d\xi
\]

(37)

where

\[ P = 12a \frac{a}{h_0} \frac{(a)}{\hbar}^3 p, \]

is used to obtain a governing equation which closely resembles equation (31), within the limits of assumed form of motion and loading as given in equations (34) and (35).

Substituting expressions (34) and (35) into (37) and equating the average virtual work over a period of oscillation to zero, or explicitly:

\[
I' = \int_{0}^{2\pi/\omega} I_A d\tau = 0
\]

yields:

\[
n^3 \left[ cG'''' + \frac{2c}{\xi} G''' - \frac{1}{\xi^2} G'' + \frac{1}{\xi^3} G' \right] + (n^3)_{\xi} \left[ 2cG'''' + \frac{(2c+\nu)}{\xi} G'' \right] 
- \frac{1}{\xi^2} G' + (n^3)_{\xi\xi} \left[ cG'''' + \frac{\nu}{\xi} G'' \right] - \omega^2 n \left( \frac{c-v^2}{1-\nu^2} \right) G 
- 9 (c-v^2) \frac{a}{h_0} \frac{2}{\xi} \left[ G' F \right]' = (c-v^2) Q
\]

(38)
The problem therefore becomes governed by a pair of nonlinear coupled ordinary differential equations (36) and (38). For convenience in conducting a parametric study, let

\[
G(\xi) = A g(\xi) \quad (39a)
\]
\[
F(\xi) = A^2 f(\xi) \quad (39b)
\]

where \( A \) is amplitude parameter, and \( g(\xi) \) and \( f(\xi) \) are shape functions.

Substituting these into equations (35) and (38) we get,

\[
c \frac{d^2 f}{d\xi^2} + c(1 - \frac{n^2}{\xi}) \frac{f'}{\xi} - (1 - \frac{n}{\xi} \nu \xi) \frac{f}{\xi} + \frac{n}{2\xi} (g')^2 = 0 \quad (40a)
\]
\[
cn^3 g''' + \left( \frac{2c}{\xi} n^3 + 6cn'n^2 \right) g''' + \left( - \frac{n^3}{\xi^2} + \frac{3(2c+\nu)n^2}{\xi} \right) = 0
\]
\[
+ 3cn'n^2 + 6cn(n')^2 \right) g'''
\]
\[
+ \left( \frac{1}{\xi^3} n^3 - 3n'n^2 \frac{1}{\xi} + (3n''n^2 + 6n(n')^2) \frac{\nu}{\xi} \right) g' - n\lambda \frac{(c-\nu^2)}{(1-\nu^2)} g.
\]
\[
- 9(c-\nu^2) \alpha \frac{1}{\xi} (g'f') = (c-\nu^2) \frac{Q^*}{\sqrt{\alpha}} \quad (40b)
\]

This can be expressed as:

\[
A_1 g'''' + A_2 g''' + A_3 g'' + A_4 g' - n\lambda \frac{(c-\nu^2)}{(1-\nu^2)} g - \frac{9(c-\nu^2)}{(c-\nu^2)} \alpha (g'f')
\]
\[
= \frac{(c-\nu^2)}{(c-\nu^2)} Q^* \quad (40b)
\]

where

\[
A_1 = cn^3
\]
\[
A_2 = \frac{2c}{\xi} n^3 + 6cn'n^2
\]
\[
A_3 = - \frac{1}{\xi^2} n^3 + 3(2c+\nu) \frac{n'n^2}{\xi} + 3cn'n^2 + 6c(n')^2 n
\]
\[ A_4 = \frac{1}{\xi^3} \eta^3 - \frac{3n'n}{\xi^2} + \frac{(3n''n' + 6n(n')^2)}{\xi} \]

\[ \alpha = A\left(\frac{e}{h_0}\right)^2 \]

\[ \lambda = \omega^2 \]

\[ Q^* = \frac{Aa}{h_o} Q \]

The above equations together with the boundary conditions selected from Table I, constitute a two-point boundary problem which is solved through the solution of the related initial value problem.
<table>
<thead>
<tr>
<th>Type of Edge</th>
<th>Boundary Condition at Edge $\xi_1 = R$ or $1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clamped Immovable</td>
<td>$g = 0$                                                   $g' = 0$                                                   $cf' - \frac{v}{\xi}f = 0$</td>
</tr>
<tr>
<td>Clamped Movable</td>
<td>$g = 0$                                                   $g' = 0$                                                   $f = 0$</td>
</tr>
<tr>
<td>Hinged Immovable</td>
<td>$g = 0$ $cg'' + \frac{v}{\xi}g' = 0$                                                   $cf' - \frac{v}{\xi}f = 0$</td>
</tr>
<tr>
<td>Hinged Movable</td>
<td>$g = 0$                                                   $cg'' + \frac{v}{\xi}g' = 0$                                                   $f = 0$</td>
</tr>
<tr>
<td>Free</td>
<td>$cg'' + \frac{v}{\xi}g' = 0$ $cg''' + c\left[\frac{1}{\xi} + \frac{3n'}{n}\right]g'' + \frac{f}{\xi} = 0$ $- \frac{1}{\xi^2} - \frac{3n'}{n} \frac{v}{\xi}g' = 0$</td>
</tr>
</tbody>
</table>
CHAPTER III

NUMERICAL ANALYSIS

The solutions of nonlinear boundary values and nonlinear eigenvalue problems, are very complicated and hence these are solved by converting them to initial value problems.

Initial Value Method

Due to the extremely nonlinear form of these equations, after conversion to an initial value problem the shooting technique is used. An associated variational problem is developed and used in Newton-Raphson iteration scheme.

The governing equations (40a) and (40b) can be written as a system of six first order differential equations,

\[ \frac{d\bar{Y}}{d\xi} = \bar{H}(\xi, \bar{Y}, \alpha, \lambda, Q^*) , \quad R < \xi < 1 \quad (41) \]

where

\[ \bar{Y}(\xi) = \begin{bmatrix} g \\ g' \\ g'' \\ f \\ f' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} , \quad (\cdot)' = \frac{d}{d\xi} , \text{ and} \]

\[ \bar{H} \text{ is the appropriately defined (6x1) vector function:} \]
\[ \bar{H} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \\ y_6' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ \frac{-A_2}{A_1} y_4 - \frac{A_3}{A_1} y_3 - \frac{A_2}{A_1} y_2 + \frac{n\lambda}{A_1} y_1 \frac{(c-v^2)}{(1-v^2)} \\ + \frac{9(c-v^2)}{\xi A_1} \alpha (y_3 y_5 + y_2 y_6) + \frac{(c-v^2)}{\sqrt{\alpha} A_1} Q* \\ (1 - \frac{n'}{\eta} \nu \xi) \frac{y_5}{\xi} - (1 - \frac{n'}{\eta} \xi) \frac{y_6}{\xi} - \frac{n}{2\xi} (y_2')^2 \end{bmatrix} \]

The parameters \( \alpha \) and \( \lambda \) are at present not known and hence two additional restraints are imposed to evaluate these. One component of \( \bar{Y}(l) \) is normalized to fulfill the requirement.

The boundary conditions can be expressed as:

\[ M\bar{Y}(l) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (42a) \]

\[ N\bar{Y}(R) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (42b) \]

where, \( M \) and \( N \) are \((4x6)\) and \((3x6)\) coefficient matrices of rank 4 and 3 respectively. The first row of \( M \) normalizes a component of \( \bar{Y}(l) \), and the remaining rows of \( M \) and \( N \) are obtained by taking into consideration the boundary conditions at the two ends.

The corresponding initial value problem may be expressed as

\[ \frac{d\bar{Z}}{d\xi} = \bar{H}(\xi, \bar{Z}; \alpha, \lambda, Q*) \quad (43a) \]
\[
\tilde{Z}(1) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \tilde{\gamma}
\]

where \( \tilde{\gamma} \) is a \((6 \times 1)\) vector of initial values.

Substitution of these initial values \( \tilde{Z}(1) = \tilde{\gamma} \) into the equation \((42a)\) yields:

\[
M\tilde{Z}(1) = M\tilde{\gamma} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

As \( M \) is of rank four, two additional values are required, and by the implicit function theorem,

\[
\tilde{\gamma} = \tilde{\gamma}^*(\eta_1, \eta_2)
\]

is a solution of equation \((44)\), and \( \eta_1, \eta_2 \) are arbitrary initial values.

The related initial value problem can thus be written as:

\[
\frac{d\tilde{Z}}{d\xi} = \bar{H}(\xi, \tilde{Z}; \alpha, \lambda, Q^*)
\]

\[
\tilde{Z}(1) = \tilde{\gamma}^*(\eta_1, \eta_2)
\]

The above contains the initial values, which satisfy the boundary conditions \((42a)\).

Assuming a continuous function \( Q^*(\xi) \), a solution of the initial value problem \((45)\) is obtained over a closed interval \([R, 1]\), and is denoted by
From the boundary condition (42b),
\[ N\bar{Z}(R; \bar{n}, \alpha) = 0 \]  

Stating a well-known matrix theorem, "For a system of equations
\[ N\bar{Z}(\xi; \bar{n}, \alpha) = 0 \]  
a necessary and sufficient condition for a unique solution
\[ \bar{n} = \bar{n}(\alpha), \]  
is that the determinant of the Jacobian matrix,
\[ J = \frac{\partial}{\partial \bar{n}} [N\bar{Z}(R, \bar{n}, \alpha)] \]  
is not equal to zero," assuming also that \( \bar{Z} \) is continuously differentiable
with respect to \( \bar{n} \) and \( \alpha \).

Thus there exists a locally unique function at \( \xi = R \), such that,
\[ N\bar{Z}(R, n(\alpha), \alpha) = 0 \]

Or, we can put it as
\[ \bar{Y}(\xi, \alpha) = \bar{Z}(\xi, \bar{n}(\alpha), \alpha) . \]

This forms an \( \alpha \)-dependent family of solutions to (42) each of which
is a solution to the initial value problem.

For a fixed value of \( \alpha \), say \( \alpha^0 \), equations (46) reduce to three
trancendental equations,
\[ N\bar{Z}(R, \bar{n}, \alpha^0) = 0 \]  

A root \( \bar{n}^0 \) of (47) may be obtained by Newtons iteration method. Starting
with an initial guess, \( \bar{n} = \bar{n}_1 \) the iterative sequence,
\[ \bar{n}_{k+1} = \bar{n}_k + \Delta \bar{n}_k , \quad k = 1, 2, 3, ... \]  
is generated.

This can be expanded in the Taylor's Series. Retaining only the
linear terms, gives,
\[ \Delta \bar{\eta}_k = -\left[ N \frac{\partial}{\partial \bar{\eta}_k} Z(R, \eta_k, \alpha^0) \right]^{-1} \bar{N\bar{Z}}(R; \bar{\eta}_k, \alpha^0) \] (48b)

where, at the \(k\)th step, the (6x3) matrix \(J_1\) is defined as,

\[
(J_1) = \begin{pmatrix}
\frac{\partial \bar{Z}}{\partial \bar{\eta}} \\
\frac{\partial \bar{Z}}{\partial n_j} \xi = R
\end{pmatrix} = \begin{pmatrix}
\frac{\partial Z_i}{\partial \bar{\eta}} \\
\frac{\partial Z_i}{\partial n_j} \xi = R
\end{pmatrix} \quad i = 1, 2, \ldots, 6 \\
j = 1, 2, 3
\] (49)

Physically, this represents the change of final values with respect to \(\bar{\eta}\).

The expression \(N\bar{Z}(\xi, \bar{\eta}_k, \alpha^0)\) represents the \(k\)th error vector.

If the initial guess \(\bar{n}\), is in the neighborhood of \(\eta^0\), then the convergence of the sequence \(\bar{\eta}_k\) to the root \(\bar{\eta}^0\) is feasible.

In order to generate the sequence \(\bar{\eta}_k\), it is necessary to evaluate the matrix \((J_1)_k\) at each step, \(k\), of the iteration process. To do this, an associated variational problem is introduced.

Formally differentiating (45) with respect to \(\bar{\eta}\), gives

\[
\frac{d}{d\xi} \left( \frac{\partial \bar{Z}}{\partial \bar{n}} \right) = \frac{\partial \bar{H}}{\partial \bar{\eta}} + \left( \frac{\partial \bar{H}}{\partial \bar{Z}} \right) \left( \frac{\partial \bar{Z}}{\partial \bar{n}} \right)
\] (50a)

\[
\left( \frac{\partial \bar{Z}}{\partial \bar{n}} \right)_{\xi=1} = \frac{\partial \bar{y}^*}{\partial \bar{n}}
\] (50b)

which constitute eighteen first order equations, and a corresponding set of initial values.

\[
\frac{dZ_1}{d\xi} = Z_2
\]

\[
\frac{dZ_2}{d\xi} = Z_3
\]
\[
\frac{dz_3}{d\xi} = z_4
\]
\[
\frac{dz_4}{d\xi} = n^2 \frac{2}{(1-v^2)} \frac{(c-v^2)}{y_1 - A_4y_2 - A_3y_3 - A_2y_4}
\]
\[
+ \frac{9(c-v^2)}{\xi} \frac{\alpha}{A_1} (y_3y_5 + y_2y_6) + \frac{(c-v^2)}{\sqrt{\alpha} A_1} \text{ (51)}
\]
\[
\frac{dz_5}{d\xi} = z_6
\]
\[
\frac{dz_6}{d\xi} = \frac{1}{c} \left[ 1 - \frac{n1}{n} \xi v \right] \left[ \frac{1}{\xi} y_5 - \left[ 1 - \frac{n1}{n} \xi \right] \frac{1}{\xi} y_6 - \frac{n}{2\xi c} y_2^2 \right]
\]

Differentiating the above with respect to \((n_1, n_2, \lambda)\) we get the following variational equations:

\[
\frac{d}{d\xi} \left( \frac{\partial z_1}{\partial n_1} \right) = \frac{\partial z_2}{\partial n_1}
\]
\[
\frac{d}{d\xi} \left( \frac{\partial z_2}{\partial n_1} \right) = \frac{\partial z_3}{\partial n_1}
\]
\[
\frac{d}{d\xi} \left( \frac{\partial z_3}{\partial n_1} \right) = \frac{\partial z_4}{\partial n_1}
\]
\[
\frac{d}{d\xi} \left( \frac{\partial z_4}{\partial n_1} \right) = \left[ \frac{2}{n^2} \frac{2}{(1-v^2)} \frac{(c-v^2)}{\partial n_1} \right] - A_4 \frac{\partial z_2}{\partial n_1} - A_3 \frac{\partial z_3}{\partial n_1} - A_2 \frac{\partial z_4}{\partial n_1}
\]
\[
+ \frac{9(c-v^2)}{\xi} \frac{\alpha}{A_1} \left[ \frac{\partial z_3}{\partial n_1} z_5 + \frac{\partial z_5}{\partial n_1} z_3 + \frac{\partial z_6}{\partial n_1} z_2 \right]
\]
\[
+ \frac{\partial z_2}{\partial n_1} z_6 \right] / A_1 \text{ (52a)}
\]
\[
\frac{d}{d\xi} \left( \frac{\partial z_5}{\partial n_1} \right) = \frac{\partial z_6}{\partial n_1}
\]
\[
\frac{d}{d\xi} \left( \frac{\partial Z_6}{\partial n_1} \right) = [1 - \frac{n'}{n} \xi v] \frac{\partial Z_5}{\partial n_1} / c\xi^2 - [1 - \frac{n'}{n} \xi] \frac{\partial Z_6}{\partial n_1} / \xi
\]

\[
- \frac{n}{c\xi} Z_2 \frac{\partial Z_2}{\partial n_1}
\]

\[
\frac{d}{d\xi} \left( \frac{\partial Z_1}{\partial n_2} \right) = \frac{\partial Z_2}{\partial n_2}
\]

\[
\frac{d}{d\xi} \left( \frac{\partial Z_2}{\partial n_2} \right) = \frac{\partial Z_3}{\partial n_2}
\]

\[
\frac{d}{d\xi} \left( \frac{\partial Z_3}{\partial n_2} \right) = \frac{\partial Z_4}{\partial n_2}
\]

\[
\frac{d}{d\xi} \left( \frac{\partial Z_4}{\partial n_2} \right) = \left[ \frac{2}{n} \omega \frac{(c-v^2)}{(1-v^2)} \right] \frac{\partial Z_1}{\partial n_2} - \frac{\partial Z_2}{\partial n_2} - \frac{\partial Z_3}{\partial n_2} - \frac{\partial Z_4}{\partial n_2} + g(c-v^2) \frac{\partial}{\partial \xi} \left( \frac{\partial Z_3}{\partial n_2} Z_5 + \frac{\partial Z_5}{\partial n_2} Z_3 + \frac{\partial Z_2}{\partial n_2} Z_6 + \frac{\partial Z_6}{\partial n_2} Z_2 \right) / A_1
\]

\[
\frac{d}{d\xi} \left( \frac{\partial Z_5}{\partial n_2} \right) = \frac{\partial Z_6}{\partial n_2}
\]

\[
\frac{d}{d\xi} \left( \frac{\partial Z_6}{\partial n_2} \right) = [1 - \frac{n'}{n} \xi v] \frac{\partial Z_5}{\partial n_2} / c\xi^2 - [1 - \frac{n'}{n} \xi] \frac{\partial Z_6}{\partial n_2} / \xi
\]

\[
- \frac{n}{c\xi} Z_2 \frac{\partial Z_2}{\partial n_2}
\]
For a given vector \( \vec{n} \) and \( \alpha = \alpha^0 \), this derived problem along with the initial value problem (45) may be integrated simultaneously on the interval \([R,1]\). Corresponding to a given value of \( \vec{n} \) and \( \alpha = \alpha^0 \), the calculation of the resulting solution to the variational problem at \( \xi = 1 \) provides the Jacobian \( (J_1) \). By setting \( \vec{n} = \vec{n}_1 \) and integrating equations (45) & (50) from \( \xi=1 \) to \( \xi=R \), gives the first correction vector \( \vec{n}_2 \). By repeating this procedure, the desired sequence \( \vec{n}_k \) is obtained, which converges to \( \vec{n}^0 \) within a specified error bound to the accuracy of the system.

Having obtained \( \vec{n}^0 \), corresponding to \( \alpha^0 \), the value of \( \alpha \) can now be perturbed,

\[
\alpha = \alpha^0 + \Delta \alpha^o = \alpha^1
\]
The problem is reinstated, for this value of \( a \), starting from \( \eta = \eta^0 \). If \( \Delta a^0 \) is small, then \( \eta^0 \) is contained in the new contraction domain of Newton's method, the iterations converging to \( \eta^1 \) corresponding to \( a = a^1 \). Successive completion of this operation \( j \) number of times, yields,

\[
\eta^i = \eta^{i-1} \left( a^{i-1} \right), \quad i = 0, 1, 2, \ldots, j
\]

By setting \( a = a^i + \Delta a^j = a^{i+1} \) and starting integration from \( \eta^{-j} \), one obtains \( \eta^{-j+1} \) provided \( \Delta a^j \) results in convergence.

The range of \( a \) is limited as the elastic plate cannot withstand unbounded amplitudes.
CHAPTER IV

NUMERICAL COMPUTATIONS

The above theoretical analysis suggests the use of a numerical integration technique.

Use is made of a fourth order Runge-Kutta-Gill method to integrate the initial value problems (45) and (50) over the interval \([R,1]\). The following approach is suggested.

The problem is first reduced to that of a free vibration by setting \(Q^* = 0\) and \(\alpha^0 = 0\). By subjecting this equation to a particular set of boundary conditions the linear eigenvalues and mode shape functions are determined.

This information leads to a basis for making a reasonable starting guess, \(n_1\), required by the initial value method.

For \(n = n_1\), the initial value problems (45) and (50) are integrated numerically over \([R,1]\) with a step size \(\Delta \mu = 1/40\). Successive correction is carried out till all equations in (47) satisfy the range of prescribed error; this being consistent with the order 0 \((|\Delta \mu|^{4})\), of Runge Kutta Gill method.

By gradually incrementing the value of \(\alpha\) and restarting the correction and integration procedure from the values of \((n_1, n_2, \lambda)\), obtained from the solution corresponding to the previous \(\alpha\), the resonance curve and other solutions are evaluated. This procedure is terminated at a particular value of \(\alpha^m\), because of reasons mentioned earlier.
Cases Considered:

The cases considered are:

(1) Annular circular plate with convex variable thickness and free on the outside, fixed on the inside.

(2) Annular circular plate with parabolic variable thickness and free on the outside, fixed on the inside.

The figures pertaining to the above two cases are as shown in Appendix I.

The governing equations and boundary conditions are written as:

\[
\frac{d\vec{Y}}{d\xi} = \bar{H}(\xi, \vec{\nu}; a, \lambda, Q^*) \quad ; \quad R < \xi < 1
\]  

\[
M \vec{Y}(1) = \begin{cases} 
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 
\end{cases}
\]  

\[
N \vec{Y}(R) = \begin{cases} 
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 
\end{cases}
\]

where

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \nu & 0 & 0 & 0 & 0 & 0 \\
0 & -(1 - \frac{3n}{n}) \nu & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -\frac{\nu}{R} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c
\end{bmatrix}
\]

\[
N = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\nu}{R} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c
\end{bmatrix}
\]
The related initial value problem, is defined as,

\[
\frac{d\vec{Z}}{d\xi} = H(\xi, \vec{Z}; a, \lambda, Q^*)
\]  

(54a)

\[
\vec{Z}(1) = \gamma^*(n_1, n_2) = \begin{pmatrix}
1 \\
- \frac{\nu}{c} n_1 \\
(1+\nu) c n_1 \\
0 \\
n_2
\end{pmatrix}
\]  

(54b)

and the variational problem is

\[
\frac{d}{d\xi} \left( \frac{\partial \vec{Z}}{\partial n_1} \right) = \left( \frac{\partial H}{\partial \vec{Z}} \right) \left( \frac{\partial \vec{Z}}{\partial n_1} \right)
\]

(53a)

\[
\frac{d}{d\xi} \left( \frac{\partial \vec{Z}}{\partial n_1} \right) = \begin{pmatrix}
0 \\
1 + \nu \\
0 \\
0 \\
0
\end{pmatrix}
\]

(55b)

\[
\frac{d}{d\xi} \left( \frac{\partial \vec{Z}}{\partial n_2} \right) = \left( \frac{\partial H}{\partial \vec{Z}} \right) \left( \frac{\partial \vec{Z}}{\partial n_2} \right)
\]

(55c)

\[
\left( \frac{\partial \vec{Z}}{\partial n_2} \right) = \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

(55d)
\[
\frac{d}{d\xi} \left\{ \frac{\partial Z}{\partial \lambda} \right\} = \left\{ \frac{\partial H}{\partial Z} \right\} \left\{ \frac{\partial Z}{\partial \lambda} \right\} + \left\{ \frac{\partial H}{\partial \lambda} \right\} \tag{55e}
\]

\[
\left\{ \frac{\partial Z}{\partial \lambda} \right\} = \left\{ \begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right\} \tag{55f}
\]

The unification of the above is as symbolized in equation (50) with \( \bar{n} = (n_1, n_2, \lambda) \) while the value of \( \alpha \) is held constant.

\[
\begin{align*}
\Delta n_1 & = \left[ \frac{\partial Z_1}{\partial n_1} - \frac{\partial Z_1}{\partial n_2} + \frac{\partial Z_1}{\partial \lambda} \right]^{-1} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \end{pmatrix} \\
\Delta n_2 & = \left[ \frac{\partial Z_2}{\partial n_1} - \frac{\partial Z_2}{\partial n_2} + \frac{\partial Z_2}{\partial \lambda} \right]^{-1} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \end{pmatrix} \\
\Delta n_3 & = \left[ \frac{\partial Z_3}{\partial n_1} - \frac{\partial Z_3}{\partial n_2} + \frac{\partial Z_3}{\partial \lambda} \right]^{-1} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \end{pmatrix}
\end{align*}
\]

provides the linear correction of the estimated values \((n_1, n_2, \lambda)\), where \( Z_i \) are components of \( \bar{Z} \).

For each value of \( \alpha \), a sequence which defines discrete values of \( \lambda \), successive corrections of \((n_1, n_2, \lambda)\) were performed, till the final values of \( \bar{Z}(R) \) are satisfied,

\[
\max_{1 < i < 3} \sum_{j=1}^{6} n_{ij} Z_j(R) \leq 0.1 \times 10^{-5} \tag{57}
\]

where \( n_{ij} = N \).

Perturbing the amplitude \( \alpha \), the process is started using the values of \( \bar{n} \) obtained after the first cycle is completed. At least five to six iterations were required for most values of \( \alpha \).
Stresses:

From all the discussion carried out so far, it is obvious that the amplitude influences the distribution of bending stress to a greater extent, as these are related to the derivatives of the transverse shape function \( g(\xi) \).

The expressions for the bending and membrane stresses are:

\[
\sigma_r^b = -\frac{6M_r}{h^2} = \frac{h_o}{a} \frac{n(\xi)}{2a_{22}(c-v^2)} \left[c\chi'' + \frac{v}{\xi} \chi'\right]
\]

\[
\sigma_\theta^b = -\frac{6M_\theta}{h^2} = \frac{h_o}{a} \frac{n(\xi)}{2a_{22}(c-v^2)} \left[\frac{1}{\xi} \chi' + \chi''\right]
\]

\[
\sigma_r^m = \frac{N_r}{h} = \frac{1}{a_{22}n(\xi)} \phi
\]

\[
\sigma_\theta^m = \frac{N_\theta}{h} = \frac{1}{a_{22}n(\xi)} \phi'
\]

These are the radial bending stress, circumferential bending stress, radial membrane stress and circumferential membrane stress respectively, in terms of the dimensionless deflection, \( \chi \), and stress function \( \phi \), respectively. Taking the previous assumption into consideration, i.e.,

\[
\chi(\xi, \tau) = A g(\xi) \sin \omega \tau
\]

\[
\phi(\xi, \tau) = A^2 f(\xi) \sin^2 \omega \tau
\]

and also taking into consideration the fact that when time, \( \tau \), is equal to the odd multiple of \( \pi/2\omega \), we have the maximum stresses,

\[
\frac{\sigma_r^b a^2}{h_o^2} = \pm \frac{\sqrt{a}}{2} \frac{n(\xi)}{2(c-v^2)} \left[c g_{\xi \xi} + \frac{v}{\xi} g_\xi\right] \tag{58a}
\]

\[
\frac{\sigma_\theta^b a^2}{h_o^2} = \pm \frac{\sqrt{a}}{2} \frac{n(\xi)}{2(c-v^2)} \left[v g_{\xi \xi} + \frac{1}{\xi} g_\xi\right] \tag{58b}
\]
\[
\frac{a^m r^2 a_{22}}{h_o^2} = \frac{a}{n(\xi)} \left(\frac{r}{\xi}\right) \quad (58c)
\]
\[
\frac{a^m \sigma_0^2 a_{22}}{h_o^2} = \frac{a}{n(\xi)} \left(f'\right) \quad (58d)
\]

The above equations were used in the computer program.
CHAPTER 5: CONCLUSIONS

This study is based on the supposition of harmonic oscillations. The assumed solutions (35) contradict the inseparability of modes in Von Karman's dynamic equations. Nevertheless, for moderate amplitude of vibrations, physical arguments may be made to justify such assumptions.

The time coordinate function is assumed and eliminated by a time averaging method. By elimination of the time variable, an infinite number of degrees of freedom in the space coordinate function is achieved. By the numerical integration technique used the solution of the continuous system is obtained at a number of discrete points. This reduces the number of degrees of freedom to the number of points considered.

The two cases studied are an annular plate with parabolic variable thickness and of convex variable thickness. Both are of the same volume and have the same boundary conditions of free on the outside and fixed on the inside.

The responses of the plates exhibit a behavior similar to that of a hard spring.

The parabolic variable thickness plate is stiffer than the convex variable thickness one, as is evident from the frequency responses obtained.

The bending stresses of the first plate are slightly higher than those of the second plate and the membrane stresses are just the reverse.

The membrane stresses have significant magnitudes even at relatively low amplitudes. This is due to a stress concentration factor at the edge of the hole, and is called the boundary layer.

The results obtained were compatible with those obtained by Sandman [2]. In this study the higher modes and stability of vibration have not been considered. So also, various other boundary conditions are possible, these are thus left open for future investigation.
REFERENCES


7. Sandman, B. E., Harmonic Oscillations of Circular and Annular plates at finite amplitudes, Ph.D. Dissertation, Kansas State University, Manhattan, Kansas, 1970.


Appendix A

Figures
Fig. 1. Circular Plate and the Polar Coordinate System.
Fig. 2. Frequency responses of the two plates.
Fig. 3. Normalized frequency responses.
Fig. 4. Shape function for annular, convex-variable thickness plate.
Fig. 5. Shape function for annular, parabolic, variable thickness plate.

\[ n = 0.815 - 0.5x^2 \]
Fig. 6. Radial bending stresses

\[ \frac{\sigma_{ba}}{h^2a_{22}} \]

Equation:

\[ h = 0.815 - 0.5x^2 \]

\[ h = 1.0 - 0.5x^{1/2} \]
Fig. 7. Circumferential bending stresses.
Fig. 8. Radial membrane stresses.
Fig. 9. Circumferential membrane stresses.
APPENDIX B

Computer program for annular orthotropic convex variable thickness plate --
Backward shooting
$JOB
C*******************************************************************************
C INITIAL VALUE METHOD — FREE VIBRATION OF AN ANNUAL
C ORTHOTROPIC, CONVEX VARIABLE THICKNESS PLATE, WITH
C BOUNDARY CONDITIONS AS FIXED ON THE INSIDE AND
C FREE ON THE OUTSIDE
C*******************************************************************************
C*******************************************************************************
C S=RATION OF ELASTIC CONSTANTS
C E=POISSON'S RATIO
C QL=UNIFORM LOADING INTENSITY
C A=AMPLITUDE
C R=RATIO OF INNER TO OUTER RADIUS
C P=EIGENVALUE
C H=STEP SIZE
C ETA=THICKNESS FUNCTION
C DETA=FIRST DERIVATIVE OF ETA
C DDET=SECOND DERIVATIVE OF ETA
C*******************************************************************************
1 IMPLICIT REAL*8(/-H,C-Z), INTEGER(I-N)
2 DIMENSION ETA(41), XX(41), Y(24), C(24), TP(3,3), D(6,41)
3 DIMENSION C(3), LK(3), MW(3), ER(3)
4 DIMENSION RBS(45), CBS(45), RMS(45), CMS(45)
5 112 FORMAT(5X, 'AMF=', F22.14, 3X, 'FREQ=', D22.14, 3X, 'FRER=', D22.14)
6 113 FORMAT(9X, 'W', 19X, 'Dh', 18X, 'DDW', 17X, 'DDDW')
7 114 FORMAT(4C22.14)
8 115 FORMAT(//9X, 'F', 19X, 'DF')
9 117 FORMAT(1H)
10 120 FORMAT(5X, 'S=', F10.3, 5X, 'E=', F10.3, 5X, 'QL=', F10.3)
11 121 FORMAT(//9X, 'STA', 19X, 'PRCF')
12 122 FORMAT(6X, 'RBS', 18X, 'CBS', 18X, 'RMS', 18X, 'CMS')
13 123 FORMAT(5X, 'ITER=', I2)
14 S=0.5
15 H=1./40.
16 LL=41
17 KK=9
18 JK=LL+1-KK
19 IK=1
20 QL=0.0
21 A=0.0
22 E=1./3.
23 R=0.25-0
24 DA=0.25
25 P=0.35*2
26 ETA1=1.4492
27 ETA2=-0.2426
28 BE=0.5
C*******************************************************************************
C CONSTRUCT INITIAL VALUES
C*******************************************************************************
29 560 IT=1
30 90 9 I=1,24
31 9 Y(I)=0.00-0
32 Y(1)=1.00-0
33 V(2)=ETA1
34 Y(3)=-(E*Y(2))/S
35 Y(4)=1.0*Y(2)+Y(2)/S
36 Y(6)=ETA2
37 Y(8)=1.0C-0
51

38 \( Y(9) = \frac{-E}{S} \)
39 \( Y(10) = \frac{(1.0+E)}{S} \)
40 \( Y(11) = 1. \)

C*****************************************************************************
C X=INDEPENDENT VARIABLE
C INTEGRATION BY BACKWARD SHOOTING
C*****************************************************************************
41 600 X=1.0D-0
42 623 Q(1)=0.0D-0
43 620 DO 1=1,6
44 620 D(I,LL)=Y(I)
46 HR=-H
47 624 DO 1,J=2,JK
48 624 M=LL+1-J
49 CALL RKG(X,HR,Y,C,P,L,S,E,QL,BE)
50 615 DO 1,L=1,6
51 615 D(L,M)=Y(L)
52 624 CCNTINUE

C*****************************************************************************
C ER(I)=ERROR VECTOR FOR BOUNDARY CONDITION
C*****************************************************************************
53 53 ER(1)=C(1,KK)
54 54 ER(2)=D(2,KK)
55 55 ER(3)=S*D(6,KK)-(E*D(5,KK))/R

C*****************************************************************************
C CONSTRAINT ERROR NORM
C*****************************************************************************
56 56 DO 26 I=1,4
57 26 DET=0.0D-0
58 26 IF(DER.GT.0.10-05) GO TO 28
59 26 CCNTINUE
60 26 GO TO 50
61 26 CCNTINUE

C*****************************************************************************
C NEWTON'S METHOD (ERROR NORM=(8)Y=0
C TP(I,J)= T-E JACOBIAN OF THE INITIAL VALUES
C C(I)=CORRECTION VECTOR
C*****************************************************************************
62 62 TP(1,1)=Y(7)
63 63 TP(2,1)=Y(8)
64 64 TP(3,1)=S*Y(12)-(E*Y(11))/R
65 65 TP(1,2)=Y(13)
66 66 TP(2,2)=Y(14)
67 67 TP(3,2)=S*Y(18)-(E*Y(17))/R
68 68 TP(1,3)=Y(19)
69 69 TP(2,3)=Y(20)
70 70 TP(3,3)=S*Y(24)-(E*Y(23))/R
71 71 DET=0.0D-0
72 72 CALL CMINV(TP,3,DET,LW,MW)
73 73 DO 75 I=1,3
74 74 C(I)=0.0
75 75 DO 75 J=1,3
76 76 C(I)=C(I)-TP(I,J)*ER(J)

C*****************************************************************************
C CORRECTED VALUES
C*****************************************************************************
77 77 DO 76 I=1,6
78 76 Y(I)=D(I,LL)
79 76 Y(2)=Y(2)+C(1)
50  \[ Y(6) = Y(6) + C(2) \]
51  \[ P = P + C(3) \]
52  \[ Y(3) = -(E \cdot Y(2))/S \]
53  \[ Y(4) = ((1+E) \cdot Y(2))/S \]
54  DO 77 I = 7, 24
55 77  \[ Y(I) = 0.00 - 0 \]
56  \[ Y(8) = 1.00 - 0 \]
57  \[ Y(9) = E/S \]
58  \[ Y(10) = (1.0+E)/S \]
59  \[ Y(18) = 1.0 \]
60  \[ IT = IT + 1 \]
61  IF (IT.GT.10) GO TO 550
62  GO TO 600

C************************************************************
C FINAL RESULTS
C************************************************************
63  950  SRA = DSQRT(A)
64     SP = DSQRT(P)
65     DO 975 J = KK, LL, 4
66 975  \[ DJ = J - 1 \]
67     \[ XX(J) = OJ \times H \]
68  \[ ETA(J) = 1.0 - B \times (XX(J)**(0.5)) \]
69  \[ IF (XX(J).GT.0.0) GO TO 905 \]
70  \[ RBS(J) = SRA \times C(3, J)/2.0 *(1.0 - E) \]
71  \[ CBS(J) = RBS(J) \]
72  \[ RKS(J) = A \times D(6, J) \]
73  \[ CMS(J) = RKS(J) \]
74  \[ GO TO 795 \]
75  \[ RBS(J) = SRA \times ETA(J) \times (S + D(3, J) + E \times D(2, J)/XX(J))/2.0 *(S - E)**2 \]
76  \[ CBS(J) = SRA \times ETA(J) \times \{C(2, J)/XX(J) + E \times D(3, J))/2.0 *(S - E)**2 \}
77  \[ RKS(J) = A \times D(5, J)/ETA(J) \times XX(J) \]
78  \[ CMS(J) = A \times D(6, J)/ETA(J) \]
79  \[ CCNTIME \]

C*****************************************
C FOR FREQUENCY RATIO
C*****************************************
80  \[ IF (A.GT.0.0) GO TO 906 \]
81  \[ SPC = SP \]
82  \[ SPR = SP/SPO \]
83  \[ WRITE(6, 117) \]
84  \[ WRITE(6, 120) S, E, CL \]
85  \[ WRITE(6, 117) \]
86  \[ WRITE(6, 122) \]
87  \[ WRITE(6, 117) \]
88  \[ WRITE(6, 113) \]
89  \[ DO 901 J = KK, LL, 4 \]
90 901  \[ WRITE(6, 114) (D(I, J), I = 1, 4) \]
91  \[ WRITE(6, 115) \]
92  \[ WRITE(6, 122) \]
93  \[ WRITE(6, 114) (D(L, J), L = 5, 6) \]
94  \[ WRITE(6, 117) \]
95  \[ WRITE(6, 124) \]
96  \[ WRITE(6, 117) \]
97  \[ WRITE(6, 125) \]
98  \[ WRITE(6, 114) \]
99  \[ WRITE(6, 126) \]
100  \[ WRITE(6, 117) \]
101  \[ WRITE(6, 127) \]
102  \[ WRITE(6, 114) \]
103  \[ WRITE(6, 128) \]
104  \[ WRITE(6, 129) \]
105  \[ WRITE(6, 121) \]
106  \[ WRITE(6, 120) \]
107  \[ D0 924 J = KK, LL, 4 \]
108  \[ WRITE(6, 114) XX(J), ETA(J) \]
109  \[ WRITE(6, 117) \]
SUBROUTINE PKGTATIGN(H,T,Y,P,AP,S,E,QL,BE)

IMPLICIT REAL*8(A-H,C-I),INTEGER(I-N)

DIMENSION Y(24),C(24),DY(24),A(2)

A(1)=0.292893198135
A(2)=1.7671567011665
H2=0.5*h

CALL DERIV(X,H,Y,DY,P,AP,S,E,QL,BE)

DO 13 I=1,24
R=A(J)*(H*DY(I)-C(J))
Y(I)=Y(I)+R
Q(I)=Q(I)-3.0*R-A(J)*H*DY(I)

CONTINUE

SUBROUTINE DERIV(X,H,Y,DY,P,AP,S,E,QL,BE)

IMPLICIT REAL*8(A-H,C-I),INTEGER(I-N)

DIMENSION Y(24),C(24)

ETA=1.-BE*(X**(-0.5))
DETA=-0.5*BE*(X**(0.5))
DDETA=-0.5*BE*(X**(1.5))

DO 10 I=1,3
DY(I)=Y(I+1)

DO 12 I=1,24
R=(H*DY(I)-2.0*Q(I))/6.0
Y(I)=Y(I)+R
Q(I)=Q(I)-3.0*R-H2*DY(I)

RETURN

END

SUBROUTINE CMINV(A,N,D,L,M)
DIMENSION A(N),L(L,M)
DOUBLE PRECISION A,D,BIGA,HCLD,CABS
D=1.0
AK=-N
DO 80 K=1,N
N*K=NK+N
L(K)=K
M(K)=K
KK=AK+K
BIGA=A(KK)

185
DY(4)*3.*P*Y(11)/8.**(ETA**2)+27.**(1.-E**2)*AP*Y(3)*Y(6)
1/4.*{ETA**3}
150
DY(4)=DY(4)*9.*(1.+E)*DDET*Y(3)/8.*ETA
151
IF(AP) 18,19,19
152
19 DY(4)=DY(4)+3.*Y(8)/(ETA**2)*DSQRT(AP)
153
18
DY(4)=DY(4)
154
DY(10)=3.*P*Y(11)/8.**(ETA**2)+27.**(1.-E**2)*AP*Y(9)*
1Y(6)*Y(3)*Y(12)/4.**(ETA**2)
155
DY(10)=DY(10)-9.*(1.+E)*DDET*Y(3)/8.*ETA
156
DY(16)=2.*P*Y(13)/8.**(ETA**2)+27.**(1.-E**2)*AP*Y(15)*
1Y(6)*Y(3)*Y(18)/4.**(ETA**3)
157
DY(16)=DY(16)-9.*(1.+E)*DDET*Y(15)/8.*ETA
158
DY(22)=3.*P*Y(19)/8.**(ETA**2)+3.*Y(1)/8.*{ETA**3}+27.*
{1.-E**2)*AP*Y(21)*Y(6)+Y(3)+Y(24)}*4.**(ETA**3)
159
DY(22)=DY(22)-9.*(1.+E)*DDET*Y(21)/8.*ETA
200
DY(6)=0.0
201
DY(12)=0.0
202
DY(18)=0.0
203
DY(24)=0.0
204
GO TO 70
205
17 B0=(9.*(S-(E**2))/(S*(X**2))-(S-E**2)/<S*(X**2))*0SaRT(AP)
206
B1=(S-E**2)/(S*(X**2))+(2./X)
207
B2=((6.*CETA)/ETA)+(2./X)
208
B3=[(-1./{S*(X**2))]+(3.*DETA*(2.*S+E))/{S*X*ETA}+(3.*
1DDET/ETA*{S*(X**2)}+(S*X*ETA)+(3.*E*DDET)/
1EETA*{S*X}+6.*{ ETA*{ ETA**2})*CETA**2})+

209
B4=(S*(X**2))-(3.*DETA/(S*X*ETA)+(3.*E*DDET)/
1EETA*{S*X}+6.*{ ETA*{ ETA**2})*CETA**2})

210
DY(4)=8C*AP*(Y(3)+Y(5)+Y(7)+Y(9)+Y(11)+Y(13)+Y(15)+Y(17))
1-3*Y(3)-82*Y(4)
211
IF(AP) 100,100,101
212
101
DY(4)=DY(4)*(S-E**2)+QL/(S*(ETA**3)*DSQRT(AP))
213
100
DY(4)=DY(4)
214
DY(6)=1.0-E*X*DETA/ETA)*Y(3)/(S*(X**2))-1.0-E*X*DETA/ETA
1*Y(6)/X-(ETA*Y(2)+2)/{2.*S}
215
DY(10)=00*AP*Y(6)*Y(8)+Y(5)*Y(5)+Y(12)*Y(3)*Y(11)
1.01*P*Y(7)-B2*Y(10)-B3*Y(9)-B4*Y(8)
216
DY(12)=(1.-E*X*DETA/ETA)*Y(11)/(S*(X**2))-1.0-E*X*DETA/
1EETA*Y(12)-X-EETA*Y(2)*Y(9)/(S*X)
217
DY(16)=E0*AP*Y(15)*Y(3)*Y(17)+Y(14)*Y(6)*Y(2)*
1Y(18)+E1*P*Y(13)-B2*Y(16)-B3*Y(15)-B4*Y(14)
218
DY(18)=1.0-E*X*DETA/ETA)*Y(17)/(S*(X**2))-1.0-E*X*DETA/
1EETA*Y(18)-X-EETA*Y(2)*Y(14)/(S*X)
219
DY(22)=00*AP*Y(21)*Y(5)+Y(3)*Y(23)+Y(20)*Y(6)+Y(21)*
1Y(24)+E1*P*Y(19)+B1*Y(1)-82*Y(22)-B3*Y(21)-B4*Y(20)
220
DY(24)=(1.-E*X*DETA/ETA)*Y(23)/(S*(X**2))-1.0-E*X*DETA/
1EETA*Y(24)/X-EETA*Y(2)*Y(20)/(S*X)
221
70 RETURN
222
END
DO 20 J=K,N
IZ=NZ(J-J-1)
DO 20 I=K,N
IJ=IZ+1
10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20
15 BIGA=A(IJ)
L(K)=I
M(K)=J
20 CONTINUE
J=L(K)
IF(J<1)35,35,25
KI=Ki-N
DO 30 I=1,N
JI=KI-N
30 A(JI)=HCLD
35 I=M(K)
38 JP=N*(I-1)
DO 40 J=1,N
40 A(JI)=HCLD
45 IF(BIGA) 48,46,48
46 D=O.0
RETURN
48 DC 55 I=1,N
49 IF(I=K) 50,55,50
50 IK=IK+1
55 CONTINUE
DO 65 I=1,N
65 A(IK)=A(IK)/(BIGA)
66 IK=NK+I
65 CONTINUE
DO 65 J=1,N
67 I=IJ+1
60 IF(I<K) 62,65,62
61 IF(J=K) 62,65,62
62 KJ=IJ+1
64 A(IJ)=HCLD*A(KJ)+A(IJ)
65 CONTINUE
70 KJ=K-N
75 CONTINUE
DO 80 J=1,N
79 CONTINUE
DO 75 J=1,N
80 A(KK)=1.0/BIGA
70 A(KJ)=A(KJ)/BIGA
75 CONTINUE
D=-8*BIGA
76 A(KJ)=A(KJ)/BIGA
77 CONTINUE
K=N
100 K=(K-1)
105 I=L(K)
120 IF(1-K) 120,120,120
Computer program for annular orthotropic convex variable thickness plate -- Forward shooting.
$JOB$

C*******************************************************************************
C INITIAL VALUE METHOD - FREE VIBRATION OF AN ANNUAL
C ORTHOTROPIC, CONVEX VARIABLE THICKNESS PLATE, WITH
C BOUNDARY CONDITIONS AS FIXED ON THE INSIDE AND
C FREE ON THE OUTSIDE
C*******************************************************************************

C S=RATIO OF ELASTIC CONSTANTS
C E=POISSON RATIO
C QL=UNIFORM LOADING INTENSITY
C A=AMPLITUDE
C R=RATIO OF INNER TO OUTER RADIUS
C P=EIGENVALUE
C H=STEP SIZE
C ETA=THICKNESS FUNCTION
C DETA=FIRST DERIVATIVE OF ETA
C DDETA=SECOND DERIVATIVE OF ETA
C*******************************************************************************

IMPLICIT REAL*3 (A-H,O-Z), INTEGER (I-N)

DIMENSION ETA(41), XX(41), Y(30), C(30), TP(4,4), C(8,41)

DIMENSION RMS(45), COS(45), CMS(45), RBS(45), COS(45)

1 FORMAT(5X,'AMP=', 022.14, 3X,'FREQ=', 022.14, 3X,'TBE=', 022.14)

2 FORMAT(9X,'W', 19X,'CW', 18X,'DCW', 17X,'DCCW')

3 FORMAT(4D22.14)

4 FORMAT(5X,'F', 19X,'OF')

5 FORMAT(6X,'R8S', 18X,'CnS', 18X,'RMS', 18X,'CMS')

6 FORMAT(5X,'ITER=', 12)

7 S=0.5

8 H=1./40.

9 L=41

10 KK=9

11 JK=LL+1-KK

12 IK=1

13 QL=0.0

14 A=0.0

15 E=1./3.

16 R=0.2C-0

17 DA=0.25

18 P=4.0**2

19 ETA1=8.00

20 ETA2=-58.0

21 ETA3=0.06

22 BE=0.5

C*******************************************************************************
C.construct initial values
C*******************************************************************************

560 IT=1

20 D0 9 I=1,30

9 Y(I)=0.00-0

23 Y(1)=1.00-0

34 Y(3)=ETA1

35 Y(4)=ETA2

36 Y(5)=ETA3

37 Y(6)=E+ETA3/(R*S)
DO 623 I=1,30
623 Q(I)=0.6C-0,
DO 620 I=1,6
620 C(I,KK)=Y(I)
KJ=KK+1
DO 624 J=KJ,LL
CALL RK4(XH,YQ,AP,A,S,E,QL,BE)
DO 615 L=1,6
615 D(L,J)=Y(L)
CONTINUE
C ER(I)=ERROR VECTOR FOR BOUNDARY CONDITIONS

C CONSTRUCT ERROR NCRM
C NEVTONS METHOD (ERRCR NCRM=8) Y=0
C TP(I,J)= THE JACOBIAN OF THE INITIAL VALUES
C C(I)=CORRECTION VECTOR

TP(1,1)=Y(7)
TP(2,1)=E*Y(8)+S*Y(9)
TP(3,1)=-(1.+1.5*E)*Y(8)-S*0.5*Y(9)+S*Y(10)
TP(4,1)=Y(11)
TP(5,1)=Y(13)
TP(6,2)=E*Y(14)+S*Y(15)
TP(7,2)=-(1.+1.5*E)*Y(14)-S*0.5*Y(15)+S*Y(16)
TP(8,2)=Y(17)
TP(9,3)=Y(19)
TP(10,3)=E*Y(20)+S*Y(21)
TP(11,3)=-(1.+1.5*E)*Y(20)-S*0.5*Y(21)+S*Y(22)
TP(12,3)=Y(23)
TP(13,4)=Y(25)
TP(14,4)=E*Y(26)+S*Y(27)
TP(15,4)=Y(29)
TP(16,4)=-(1.+1.5*E)*Y(26)-S*0.5*Y(27)+S*Y(28)
TP(17,4)=Y(30)
DET=0.0C=0
CALL DIINV(TP,4,CET,LL,MW)
DO 75 I=1,4
C(I)=0.0
CUFFIXCCorrected Values
C******************************************************************************
C
DO 76 I=1,6
76 Y(I)=C(I,*KK)
Y(3)=Y(3)+C(1)
Y(4)=Y(4)+C(2)
Y(5)=Y(5)+C(3)
P=P+C(4)
Y(6)=E*Y(5)/(R*S)
DO 77 I=7,30
77 Y(I)=0.0D-0
Y(9)=1.0
Y(16)=1.0
Y(23)=1.0
Y(24)=E/(R*S)
IT=IT+1
IF(IT.GT.10) GO TO 550
GO TO 600
C******************************************************************************
C FINAL RESULTS
C******************************************************************************

100 IF(A.GT.0.0) GO TO 906
SPC=SP
SPR=SP/SPO
WRITE(6,117)
WRITE(6,120) S,E,CL
WRITE(6,117)
WRITE(6,112) SRA,SP,SPR
WRITE(6,117)
WRITE(6,113)
DO 901 J=KK,LL,4
901 WRITE(6,114) (D(I,J),I=1,4)
WRITE(6,115)
DC 902 J=KK,LL,4
9C2 WRITE(6,114) (D(L,J),L=5,6)
WRITE(6,117)
WRITE(6,122)
DO 903 J=KK,LL,4
135 903 WRITE(6,114) RBS(J),CBS(J),RMS(J),CMS(J)
136 WRITE(6,117)
137 WRITE(6,123) IT
138 WRITE(6,121)
139 DO 924 J=KK,LL,4
140 924 WRITE(6,114) XX(J),ETA(J)
141 WRITE(6,117)

C*****************************************************************************
C PERTURBATION OF AMPLITUDE
C*****************************************************************************

142 A=A+CA
143 IK=IK+1
144 IF(IK.GT.26) GO TO 550
145 ETA1=C(3,KK)
146 ETA2=C(4,KK)
147 ETA3=C(5,KK)
148 P=(SP-0.3)**2
149 GO TO 500
150 550 STOP
151 END

SUBROUTINE RKG(X,H,Y,Q,P,AP,S,ET,OL,BE)

IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
DIMENSION Y(30),C(30),DY(30),A(Z)
A(1)=1.0
A(2)=1.0
A(3)=1.0
A(4)=1.0
A(5)=1.0
A(6)=1.0
A(7)=1.0
A(8)=1.0
A(9)=1.0
A(10)=1.0
A(11)=1.0
A(12)=1.0
A(13)=1.0
A(14)=1.0
A(15)=1.0
A(16)=1.0
A(17)=1.0
A(18)=1.0
A(19)=1.0
A(20)=1.0
A(21)=1.0
A(22)=1.0
A(23)=1.0
A(24)=1.0
A(25)=1.0
A(26)=1.0
A(27)=1.0
A(28)=1.0
A(29)=1.0
A(30)=1.0

DO 13 I=1,30
13 Q(I)=Q(I)+3.0*Q(I)-H2*DY(I)
14 X=X+H2
15 DO 60 J=1,2
16 CALL DERIV(X,H,Y,DY,P,AP,S,ET,BL,OL,BE)
17 DO 60 I=1,30
18 R=A(J)*(H*DY(I)-C(I))
19 Y(I)=Y(I)+R
20 DO 60 I=1,30
21 R=(H*DY(I)-2.0*Q(I))/6.0
22 Y(I)=Y(I)+R
23 Q(I)=Q(I)+3.0*Q(I)-A(J)*H*CY(I)
24 RETURN
25 CONTINUE
26 CONTINUE
27 RETURN
28 END

SUBROUTINE CERIV(X,H,Y,Q,P,AP,S,ET,BL,OL, BE)

IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
DIMENSION Y(30), CY(30)
ETA=1.0E-6*(X**(-1.5))
DET=-C.5*BE*(X**(-0.5))
ODET=0.25*BE*(X**(-1.5))
DO 10 I=1,3
10 Q(I)=Q(I)+1
11 Q(I)=Q(I)+1
12 DO 10 I=1,3
13 R=(H*DY(I)-2.0*Q(I))/6.0
14 Y(I)=Y(I)+R
15 Q(I)=Q(I)+3.0*Q(I)-A(J)*H*CY(I)
16 RETURN
17 END
DY(1) = Y(1 + 1)
DY(11) = Y(12)
123
DO 15 I = 13, 15

153
DY(1) = Y(1 + 1)
DY(I7) = Y(I8)
144
DO 16 I = 19, 21

145
DY(1) = Y(1 + 1)
DY(23) = Y(24)
197
DO 20 I = 25, 27

208
DY(1) = Y(1 + 1)
DY(29) = Y(30)
260  
IF(X*C - C1 - 0.1 - 0.2) GO TO 17
201  
DY(4) = 3.*p*Y(1)/8.*(ETA**2) + 27.*(1. - E**2) + AP*Y(3) + Y(6)
1/4.*ETA**3)
202  
DY(4) = CY(4) - 9.*(1. + E) + ODET*Y(3)/8.*ETA
203  
IF(AP) = 18, 18, 19
204  
DY(4) = DY(4) + (1. - E**2) + QL/8.* ETA**3) * SQRT(AP)
205
DY(10) = 3.*p*Y(7)/E.*ETA**3 + 27.*(1. - E**2) + AP*Y(9)*
1/4.*Y(3)/Y(12))/4.*ETA**3)
207
DY(10) = CY(10) - 9.*(1. + E) + ODET*Y(9)/8.* ETA
208
DY(16) = 3.*p*Y(13)/B.*ETA**2 + 27.*(1. - E**2) + AP*Y(15)*
1/4.*Y(18)/Y(3)/4.*ETA**3)
209
DY(16) = CY(16) - 9.*(1. + E) + ODET*Y(15)/8.* ETA
210
DY(22) = 3.*p*Y(19)/8.* ETA**2 + 3.*Y(1)/8.* ETA**3 + 27.*
1*(1. - E**2) + AP*Y(21)/8.* Y(6)/Y(3)/Y(24))/4.* ETA**3)
211
DY(22) = DY(22) - 9.*(1. + E) + ODET*Y(21)/8.* ETA
212
DY(6) = 0.0
213
DY(12) = 0.0
214
DY(18) = 0.0
215
DY(24) = 0.0
216
DY(30) = 0.0
217
GO TO 70
218
17
BO = (9.*S - E**2)/S + ETA**3)
219
B1 = (S - E**2)/S + ETA**2)/S + ETA**2)
220
B2 = (6.* ETA)/E + (2./X)
221
B3 = (1. - ETA)/ETA + S + X*ETA**2)/S + ETA**2)
222
B4 = (1. - ETA)/ETA + (6.* ETA**2)/S + ETA**2)
223
DY(4) = B6*AP*Y(3)/ETA + S + X*ETA**2)/S + ETA**2)
224
IF(AP) = 14, 10, 101
225
101
DY(4) = CY(4) + (S - E**2)/S + ETA**2)/S + ETA**2)
226
100
DY(4) = DY(4)
227
DY(6) = (1. - ETA)/ETA + Y(5)/S + X*ETA**2)/S + ETA**2)
1/Y(6)/X - ETA**2)/S + ETA**2)/S + ETA**2)
228
DY(I0) = B2*AP*Y(6)/Y(8) + Y(5)/Y(3)*Y(11)/Y(12)/Y(9)/Y(3)/Y(11)
101 + B4*Y(7) - B2*Y(10) - B3*Y(5)/B4*Y(8)
229
DY(12) = (1. - ETA)/ETA + S + X*ETA**2)/S + ETA**2)
230
DY(16) = B6*AP*Y(5)/S + X*ETA**2)/S + ETA**2)
231
DY(18) = (1. - ETA)/ETA + S + X*ETA**2)/S + ETA**2)
232
DY(22) = B6*AP*Y(21)/Y(5)/Y(3)/Y(23)/Y(20)/Y(6)/Y(21)/Y(8)/Y(2)
233
DY(24) = (1. - ETA)/ETA + S + X*ETA**2)/S + ETA**2)
234
DY(28) = B6*AP*Y(27)/Y(5)/Y(3)/Y(25)/Y(6)/Y(26)/Y(21)/Y(8)/Y(2)
62

\[ Y(30) + 81 + P \cdot Y(25) - 82 \cdot Y(20) - 83 \cdot Y(29) - 84 \cdot Y(26) - 81 \cdot Y(1) \]

\[ D(30) = (1 - E \cdot X \cdot \text{DETA}/\text{ETA}) \cdot Y(29)/(S \cdot X \cdot X) \]

\[ I - (1 - X \cdot \text{DETA}/\text{ETA}) \cdot Y(30)/(X - \text{DETA} \cdot Y(26) \cdot Y(2))/(S \cdot X) \]

236 70 RETURN

237 END

238 SUBROUTINE OMINV(A, N, D, L, M)

239 DIMENSION A(16), L(4), M(4)

240 DOUBLE PRECISION A, D, BIGA, HCLD, CABS

241 D = 1.0

242 N = -N

243 DO 20 K = 1, N

244 NK = NK + N

245 M(K) = K

246 K = NK + K

247 BIGA = A(KK)

248 DO 20 J = K, N

249 NQ = N + (J - 1)

250 DO 20 I = K, N

251 IQ = I + 1

252 IF(CABS(BIGA) - CABS(A(IJ))) 15, 2G, 20

253 15 BIGA = A(IJ)

254 L(K) = I

255 M(K) = J

256 CONTINUE

257 J = L(K)

258 IF(J = K) 35, 35, 25

259 25 KI = K - K

260 DO 30 I = 1, N

261 KI = KI + N

262 HOLD = -A(KI)

263 JI = KI - K + J

264 A(KI) = A(JI)

265 30 A(JI) = HCLD

266 35 I = M(K)

267 IF(I = K) 45, 45, 38

268 38 JP = N * (I - 1)

269 DO 40 J = K, N

270 JK = NK + J

271 JI = JP + J

272 HCLD = -A(JK)

273 A(JK) = A(JI)

274 40 A(JI) = HOLD

275 45 IF(BIGA) 43, 46, 48

276 46 D = 0.0

277 RETURN

278 DO 55 I = 1, N

279 IF(I = K) 50, 55, 50

280 50 IK = NK + I

281 A(IK) = A(IK)/(-BIGA)

282 CONTINUE

283 DC 65 I = 1, N

284 IK = NK + I

285 HOLD = A(IK)

286 IJ = I - K

287 DO 65 J = 1, N

288 IJ = IJ + K

289 65 IF(I = K) 60, 65, 60

290 60 IF(J = K) 62, 65, 62
62  KJ=IJ-I*K
253  A(IJ)=HCLD*A(KJ)+A(IJ)
254  CONTINUE
255  KJ=K-1
256  DO 75 J=1,N
257  KJ=KJ+1
258  IF(J-K) 70,75,70
259  70  A(KJ)=A(KJ)/BIGA
260  CONTINUE
261  D=C*BIGA
262  A(KK)=1.0/BIGA
263  CONTINUE
264  K=N
265  CONTINUE
266  IF(K) 150,150,105
267  I=L(K)
268  CONTINUE
269  IF(I-K) 120,120,108
270  108  JO=N*(K-1)
271  JR=N*(I-1)
272  DO 110 J=1,N
273  JK=JO+J
274  HOLD=A(JK)
275  JI=JR+J
276  A(JK)=-A(JI)
277  110  A(JI)=HCLD
278  CONTINUE
279  J=P(K)
280  IF(J-K) 100,100,125
281  125  KI=K-N
282  DO 130 I=1,N
283  KI=KI+N
284  HCLD=A(KI)
285  JI=KI-K+J
286  A(KI)=-A(JI)
287  CONTINUE
288  130  A(JI)=HCLD
289  CONTINUE
290  CONTINUE
291  RETURN
292  END
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FINITE-AMPLITUDE VIBRATION OF ORTHOTROPIC AXISYMMETRIC VARIABLE THICKNESS ANNULAR PLATE

by

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MASTER OF SCIENCE

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ABSTRACT

The problem of finite amplitude, axisymmetric free vibration of variable thickness orthotropic annular plates is formulated in terms of the Von Karman's dynamic equations. A Kantorovich averaging technique is applied to convert the nonlinear boundary value problem into a corresponding eigenvalue problem by elimination of the time variable. A numerical study is proposed by introducing the related initial value problem. By making successive corrections and perturbations of the parameters in a numerical solution to the initial value problem, approximate solutions to the boundary value problem are obtained. The cases investigated are free outside and fixed inside, parabolic and convex variable thickness orthotropic annular plates.

The hard spring behavior is evident, and it is found that the mode shape, bending stresses and membrane stresses are nonlinear functions of the amplitude of vibration.