OPTIMIZATION OF MANAGEMENT SYSTEMS

BY SECOND VARIATION

by 45

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# TABLE OF CONTENTS

1. INTRODUCTION 1

2. GRADIENT TECHNIQUES 3
   2.1 First Variation Method 5
   2.2 Second Variation Method 6
   2.3 Derivation of the Second Variation Method 7
   2.4 Advantages and Disadvantages of the Second Variation Method 22

3. APPLICATIONS 24
   3.1 An Inventory Model 25
      The Model 25
      Recursive Relations 26
      Numerical Results 44
   3.2 An Inventory and Advertising Model 46
      The Model 46
      Recursive Relations 48
      Numerical Results 58
   3.3 A Chemical Manufacturing Problem with Advertisement 98
      The Model 98
      Recursive Relations 101
      Discussion 118

4. CONCLUSION 119

5. ACKNOWLEDGMENT 121

6. BIBLIOGRAPHY 122

7. APPENDIX 125
   7.1 Computer Program for the Inventory Model 126
   7.2 Computer Program for the Inventory and Advertising Model 128
   7.3 Computer Program for the Chemical Manufacturing Problem with Advertisement 134
1. INTRODUCTION

Optimization techniques can be divided into two classes, single stage and multistage. In multistage optimization techniques, a certain relationship is used to isolate the interconnections between the various stages. Thus one stage is searched at a time instead of all the N stages simultaneously. In this way, an N-dimensional problem is converted into N one-dimensional problems if the problem has only one control variable. The multistage optimization techniques can be classified into classical techniques (calculus of variation) and dynamic programming.

In case of calculus of variation, the resulting equations form a two-point boundary value problem (2,15). The differential equations encountered in practical applications are generally nonlinear and cannot be solved analytically. Finding numerical answers for this nonlinear boundary value problem is very tedious especially if there is a large number of equations with a large number of initial values missing. This has limited the use of the calculus of variation.

The maximum principle is a very powerful tool for obtaining analytical solutions of linear optimization problems with inequality constraints on control variable (7). But when the problem is nonlinear and an analytical solution cannot be obtained, the maximum principle gives rise to similar boundary value difficulties.

Dynamic programming, although free from the boundary value difficulty, has a serious drawback because of its storage requirements on the computer. Instead of solving any individual process, the dynamic programming technique solves a family of related processes (20). In here, as in the other multistage techniques, the problem of an N dimensional search is reduced to N
one-dimensional search problems if the problem has only one control variable. However, in investigating one stage at a time, all possible combinations of the stage variables for the previously calculated stage must be stored in the memory of the computer.

This storage requirement, often referred to as the "curse of dimensionality," becomes too excessive to permit the use of dynamic programming for a problem in which more than three state variables are involved. Thus, if a three-dimensional problem, i.e. involving three state variables, is to be solved and if it is decided to have each state variable discretized into 50 values, then because of the interpolation required in the dynamic programming approach, $(50)^3$ values have to be stored. Thus it is frequently impossible to handle even a three-state variable problem with straightforward dynamic programming.

Thus it is seen that the dimensionality difficulty in dynamic programming and the boundary value problem in the classical methods limit the number of state variables in a problem that can be treated by these techniques. It should be noted, however, that these two difficulties are totally different from each other. The dimensionality difficulty requires more computer memory while the boundary value demands more computer time. Also, the classical boundary value problem approach represents an iterative procedure to obtain the numerical solution while dynamic programming represents an expansion of the original problem.
2. GRADIENT TECHNIQUES

The methods of gradients seem to remove the difficulties experienced in dynamic programming and the classical multistage techniques. Although there are various approaches with these methods, the basic philosophy remains the same. When use of the gradient methods is contemplated, the problem is formulated as a final value problem. In other words, the performance index or the objective function is selected as the value of some function at the end of the process. This is not a serious restriction. Thus if the performance index is

\[ J = \int_{0}^{t_f} f(x) dt \]

Then

\[ \frac{\partial J}{\partial t} = f(x) \]

Introducing an additional state variable \( x_{n+1} \)

\[ \frac{dx_{n+1}}{dt} = f(n) \]

and \( x_{n+1}(t_0) = 0 \).

The original integral performance criterion is replaced by a criterion which calls for extremizing the final value of an element of the state vector. Philosophically at least, extremization of any performance criterion should be possible by using the following approach underlying the methods of gradients.

First a sequence of values of control vector is taken. Then a computation is made of the gradient of the performance index with respect to each
control vector. Next each control vector is improved by moving it in the appropriate direction along the individual gradients. This improved sequence of control vectors then becomes the basis for the next iteration.

In the following sections, the first variation method, a technique suitable for optimizing nonlinear complex problems, is summarized. Then the second variation method, which is more sophisticated than the first variation method, is discussed. Three applications of this method in the field of production planning and control illustrate the advantages and disadvantages of this method.
2.1 The First Variation Method

Because of its computational appeal, various versions of the gradient methods have been developed for optimization calculations. A gradient technique for the numerical solution of dynamic optimization problems is generally known as the functional or serial gradient technique. This technique has been applied successfully to solve problems in aerospace, control and chemical engineering systems (5, 6, 10, 16, 17, 20, 21). The continuous version of the functional gradient technique was developed independently by Kelley (10) and by Brayson and his coworkers (5). A comprehensive treatment of this technique and of the gradient methods in general can be found in the article by Kelley (21).

In this method, the convergence is generally independent of the initial guess used in the iterative procedure, although the rate of convergence or, alternatively, the computer time, is affected by the initial guess. The number of equations to be integrated in the forward direction is \((n+1)\); i.e. these equations are integrated from \(t=0\) to \(t=t_f\). There are \((n+1)\) recursive equations. There are, however, no equations to be integrated in the backward direction from \(t=t_f\) to \(t=0\). The first variation equations are simpler than those of the second variation method.

The main drawback of the first variation method is that a very large number of iterations must be made in order to approach the optimal trajectory. More important is the fact that the trajectory approaches the optimum but does not actually reach it within a finite number of iterations. In some cases, the trajectory is far from the optimum after a large number of iterations and the rate of convergence becomes too slow to permit further iterations. This method cannot conveniently handle the problems with inequality constraints on the state variables.
2.2 Second Variation Method

The pioneer work in the area of second variation method has been carried out by Bryson and his coworkers \(^{(4,5)}\), Kelley and his coworkers \(^{(10,11)}\), Merriam \(^{(25)}\) and Jaswinski \(^{(9)}\). Mitter \(^{(26)}\) and Breakwell and Ho \(^{(8)}\) have also added to the work in this field.

This method is a natural evolution of the first order linearizations used in the first variation method in which the equations are linearized by truncating after all linear terms. The second order and higher order terms are thus ignored. It is well-known that the use of a linear approximation in a gradient search procedure is an excellent means for arriving near the optimum point quickly and from almost and stationary starting point. Near the optimum, however, the linear approximation becomes deficient and it is necessary to turn to a second order approximation to achieve the optimum. A useful optimization procedure is to initially use the first variation to get near the optimum trajectory and then to switch to the second order method for refinement.
2.3 Derivation of the Second Variation Method

Consider a process which can be represented by

\[
\frac{dx}{dt} = f[x(t), \theta(t)]
\]  

where \( x \) is a \( n \) dimensional state vector, \( \theta \) is a \( r \) dimensional control vector and \( x(0) \) is prescribed. No terminal constraints are to be imposed on \( x(t_f) \), although the final time, \( t_f \), may be specified.

Suppose it is desired to minimize the following performance index:

\[
I[x(0), t_f] = I = \int_0^{t_f} J(x, \theta, t) dt
\]

From Equation 2, this equation results

\[
\frac{dI}{dt} = J(x, \theta, t)
\]  

Since the performance index as given by Equation 2 is subject to the system constraints of Equation 1, consider the minimization of the unconstrained performance index as

\[
I^* = I + \int_0^{t_f} z'(f - \frac{dx}{dt}) dt
\]

where \( z \) is a vector of \( n \) Lagrange multipliers. Substituting Equation 2 into Equation 3 results in

\[
I^* = \int_0^{t_f} [J(x, \theta, t) + z'(f - \frac{dx}{dt})] dt
\]

In order to minimize \( I^* \), an iteration algorithm can be constructed such that
\[ I^{*(j+1)} = \int_0^T \left[ J^{(j+1)} + \frac{\partial}{\partial x^{(j+1)}} \frac{d}{dt} \right] dt \]  

(5)

converges in a desirable way. The superscript \((j+1)\) is used to indicate the number of iteration, and it is desired to have

\[ I^{*(0)} > I^{*(1)} > \ldots > I^{*(j)} > I^{*(j+1)} > \ldots \]  

(6)

To construct the desired iterative algorithm, the values of the functions at iteration \((j+1)\) can be expressed in terms of the \(j^{th}\) iteration by means of Taylor's series expansion. Retaining only the terms up to the second order gives

\[ J^{(j+1)} = J^{(j)} + \left( \frac{\partial J^{(j)}}{\partial x^{(j)}} \right) \delta x^{(j)} + \left( \frac{\partial J^{(j)}}{\partial \theta^{(j)}} \right) \delta \theta^{(j)} \]

\[ + \frac{1}{2} \delta x^{(j)'} \left( \frac{\partial^2 J^{(j)}}{\partial x^{(j)2}} \right) \delta x^{(j)} + \delta \theta^{(j)'} \left( \frac{\partial^2 J^{(j)}}{\partial \theta (j) \cdot \partial x^{(j)}} \right) \delta x^{(j)} \]

\[ + \frac{1}{2} \delta \theta^{(j)'} \left( \frac{\partial^2 J^{(j)}}{\partial \theta^{(j)2}} \right) \delta \theta^{(j)} \]  

(7)

where,

\[ \delta x^{(j)} = x^{(j+1)} - x^{(j)} \]

\[ \delta \theta^{(j)} = \theta^{(j+1)} - \theta^{(j)} \]  

(8)
The superscript \((j)\) has been omitted in Equation (9) for clarity. Thus it is seen that

\[
\begin{align*}
\frac{\partial^2 J}{\partial x_2} &= \left[ \begin{array}{cccc}
\frac{\partial^2 J}{\partial x_1^2} & \frac{\partial^2 J}{\partial x_1 \partial x_2} & \ldots & \frac{\partial^2 J}{\partial x_1 \partial x_n} \\
\frac{\partial^2 J}{\partial x_2 \partial x_1} & \frac{\partial^2 J}{\partial x_2^2} & \ldots & \frac{\partial^2 J}{\partial x_2 \partial x_n} \\
\frac{\partial^2 J}{\partial x_n \partial x_1} & \frac{\partial^2 J}{\partial x_n \partial x_2} & \ldots & \frac{\partial^2 J}{\partial x_n \partial x_n}
\end{array} \right], \\
\frac{\partial^2 J}{\partial \theta \partial x} &= \left[ \begin{array}{cccc}
\frac{\partial^2 J}{\partial \theta_1 \partial x_1} & \frac{\partial^2 J}{\partial \theta_1 \partial x_2} & \ldots & \frac{\partial^2 J}{\partial \theta_1 \partial x_n} \\
\frac{\partial^2 J}{\partial \theta_2 \partial x_1} & \frac{\partial^2 J}{\partial \theta_2 \partial x_2} & \ldots & \frac{\partial^2 J}{\partial \theta_2 \partial x_n} \\
\frac{\partial^2 J}{\partial \theta_n \partial x_1} & \frac{\partial^2 J}{\partial \theta_n \partial x_2} & \ldots & \frac{\partial^2 J}{\partial \theta_n \partial x_n}
\end{array} \right]
\end{align*}
\]

Next, define the Hamiltonian

\[
\bar{H} = z' f
\]

and expand \(\bar{H}\) at the \((j+1)\)th iteration up to the second order terms as a function of \(\bar{H}\) at the \(j\)th iteration. Note that \(\bar{H}\) is a function of \(x, \theta,\)
and \( z \) and that \( \frac{\partial^2 H}{\partial z^2} = 0 \)

\[
\tilde{H}(j+1) = \tilde{H}(j) + \left( \frac{\partial \tilde{H}(j)}{\partial x(j)} \right) \delta x(j) + \left( \frac{\partial \tilde{H}(j)}{\partial \theta(j)} \right) \delta \theta(j)
\]

\[
+ \left( \frac{\partial \tilde{H}(j)}{\partial z(j)} \right) \frac{1}{2} \delta x(j) + 1 \frac{\partial^2 \tilde{H}(j)}{\partial x(j)^2} \delta x(j)
\]

(12)

\[
+ \delta \theta(j) \left( \frac{\partial^2 \tilde{H}(j)}{\partial \theta(j) \partial x(j)} \right) \delta x(j) + \frac{1}{2} \delta \theta(j) \left( \frac{\partial^2 \tilde{H}(j)}{\partial \theta(j)^2} \right) \delta \theta(j)
\]

\[
+ \delta \theta(j) \left( \frac{\partial^2 \tilde{H}(j)}{\partial \theta(j) \partial z(j)} \right) \delta z(j) + \delta z(j) \left( \frac{\partial^2 \tilde{H}(j)}{\partial x(j) \partial z(j)} \right) \delta x(j) \delta z(j)
\]

Now consider the nonlinear performance equations. If these equations are linearized by Taylor-series expansions and by retaining only the first order terms, the result is

\[
\delta \left( \frac{dx(j)}{dt} \right) = \left( \frac{\partial f(j)}{\partial x(j)} \right) \delta x(j) + \left( \frac{\partial f(j)}{\partial \theta(j)} \right) \delta \theta(j)
\]

(13)

with \( \delta x(0) = 0 \) since the initial conditions are constant. This last equation may be rearranged by noting that

\[
\frac{dx(j+1)}{dt} = \delta \left( \frac{dx(j)}{dt} \right) + f(j)
\]

(14)
Thus Equation 13 can be rewritten as

$$\frac{dx^{(j+1)}}{dt} = f^{(j)} + \frac{\partial^2 H^{(j)}}{\partial z^{(j)} \partial x^{(j)}} \delta x^{(j)} + \frac{\partial^2 H^{(j)}}{\partial z^{(j)} \partial \theta^{(j)}} \delta \theta^{(j)}$$

(15)

Furthermore,

$$z^{(j+1)} = z^{(j)} + p^{(j)} \delta x^{(j)}$$

so that

$$\delta z^{(j)} = p^{(j)} \delta x^{(j)}$$

(16)

where the matrix $P$ is defined by

$$P = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_n}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_n} \end{pmatrix} = \left( \frac{\partial z'}{\partial x} \right)'$$

(17)

It is a symmetrical matrix,

i.e. $\frac{\partial z_i}{\partial x_j} = \frac{\partial z_j}{\partial x_i}$

Clearly $P$ is unknown explicitly at this point. For the sake of clarity, the superscript $(j)$ is omitted in the subsequent derivation.

If now the normal Hamiltonian function is defined as $H = J + z' f$, then the above expressions can be substituted into Equation 4 to yield
To further simplify Equation 18, use of the adjoint equation is made. Thus

$$\frac{dz}{dt} = -\frac{\partial J}{\partial x} - \frac{\partial f'}{\partial x} z$$

(19)

This is easily obtained by defining

$$M = \min_{\theta} [J(x, \theta, t) + z'f]$$

(20)

where the adjoint variable $z$ is defined by

$$z = \frac{\partial I^{\circ}}{\partial x}$$

(21)

But from the principle of optimality in dynamic programming, it follows that for

$$I^{\circ}(x, t) = \min_{\theta} \int_{0}^{t_f} J(x, \theta, \lambda) \, d\lambda$$

(22)

that
\[ I^0(x,t) = \min_0 \left[ \int_t^{t+\Delta t} J(x,0,\lambda) d\lambda + \int_{t+\Delta t}^{t} J(x,0,\lambda) d\lambda \right] \]

\[ = \min_0 \int_t^{t+\Delta t} J(x,0,\lambda) d\lambda + I^0 \left( x + \frac{dx}{dt} \Delta t, \ t + \Delta t \right) . \]

As \( \Delta t \) approaches zero

\[ I^0(x,t) = \min_0 \left[ J(x,0,t) \Delta t + J^0(x,t) + \left( \frac{\partial J^0}{\partial x} \right) \frac{dx}{dt} \Delta t + \frac{\partial J^0}{\partial t} \Delta t \right] . \]

i.e. \[ J^0(x,0^0,t) + \left( \frac{\partial J^0}{\partial x} \right) \frac{dx}{dt} + \frac{\partial J^0}{\partial t} = 0 \]

which may be written as

\[ M + \frac{\partial J^0}{\partial t} = 0 \] (23)

The partial differentiation of Equation 23 w.r.t. \( x \) yields

\[ \frac{\partial M}{\partial x} + \frac{\partial J^0}{\partial x} \frac{\partial z}{\partial t} = 0 \]

or

\[ \frac{\partial M}{\partial x} + \frac{\partial z}{\partial t} = 0 \] (24)

However, the total time derivative of \( z \) is

\[ \frac{dz}{dt} = \frac{\partial z}{\partial t} + \left( \frac{\partial z'}{\partial x} \right) \frac{dx}{dt} \]

\[ = - \frac{3J^0}{\partial x} \frac{\partial (z'f)}{\partial x} + \left( \frac{\partial z'}{\partial x} \right) \frac{dx}{dt} . \] (25)
Since
\[ \frac{\partial J}{\partial \theta} + \frac{\partial f}{\partial \theta} (z') = 0 \]  
(26)
due to the optimality condition. Expanding Equation 25 gives Equation 19, namely
\[ \frac{dz}{dt} = - \frac{\partial J}{\partial x} - \frac{\partial f'}{\partial x} z \]  
(27)
Similarly,
\[ \frac{dp}{dt} = - \frac{\partial^2 J}{\partial x^2} - \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial x^2} - \left\{ P \left( \frac{\partial f'}{\partial x} \right)' + \frac{\partial f'}{\partial x} P \right\} + K P \]  
(28)
where
\[ K = - \frac{\partial f'}{\partial x} \]  
(29)
and
\[ P = \frac{\partial J}{\partial \theta \cdot \partial x} + \frac{\partial f'}{\partial \theta} P + \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial \theta \partial x} \]  
(30)
To evaluate \( K \) it may be noted from Equation 17 that
\[ \frac{\partial J}{\partial \theta} + \frac{\partial f'}{\partial \theta} z = 0 \]
so that partially differentiating w.r.t. \( x \) gives
\[ \left( \frac{\partial f'}{\partial x} \right) \frac{\partial^2 J}{\partial x^2} + \frac{\partial^2 J}{\partial \theta \cdot \partial x} + \left( \frac{\partial f'}{\partial \theta} \right) \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial \theta \partial x} + \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial \theta \partial x} + \frac{\partial f'}{\partial \theta} \left( \frac{\partial f'}{\partial \theta} \right)' = 0 \]
\[ \frac{\partial J}{\partial x} = - R' \]

and \[ K = R' \left( \frac{\partial^2 J}{\partial \theta^2} + \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial \theta^2} \right) \].

Therefore it is possible to solve for \( \frac{\partial J}{\partial x} \), namely

\[ \frac{\partial J}{\partial x} = - \frac{dP}{dt} - \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial x^2} - \left\{ P \left( \frac{\partial f'}{\partial x} \right)' + \left( \frac{\partial f'}{\partial x} \right) P \right\} + Kr \]

Now, substituting Equations 27 and 33 into Equation 18 yields

\[ I^*(j+1) - I^*(j) = \int_{0}^{t_f} \left\{ \frac{1}{2} \delta \theta' \left( \frac{\partial^2 f}{\partial \theta^2} + \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial \theta^2} \right) \delta \theta \right. \\
+ \left. \left( \frac{\partial J}{\partial \theta} \right)' + \sum_{i=1}^{n} \left( \frac{\partial f_i}{\partial \theta} \right)' \right\} \delta \theta \\
+ \delta \theta' R \delta x + \frac{1}{2} \delta x' K R \delta x \right\} dt \]

In order that the performance index will converge to a minimum, the integral in Equation 34 must be less than zero, i.e.
\[
\int_{0}^{t_f} \left( \frac{1}{2} \delta \dot{\theta}' \left( \frac{\partial^2 J}{\partial \theta^2} + \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial \theta^2} \right) \delta \theta + \left( \frac{\partial J}{\partial \theta} \right)' + \sum_{i=1}^{n} z_i \left( \frac{\partial f_i}{\partial \theta} \right)' \right) \delta \theta \\
+ \delta \dot{\theta}' \cdot R \cdot \delta \mathbf{x} + \frac{1}{2} \delta \mathbf{x}' \cdot K \cdot R \cdot \delta \mathbf{x} \right) \, dt < 0
\] (35)

In addition, the convergence ideally should be as fast as possible, so the minimization of the integral is considered:

\[
V(\delta \mathbf{x}, \mathbf{t}) = \int_{0}^{t_f} \left( \frac{1}{2} \delta \dot{\theta}' \left( \frac{\partial^2 J}{\partial \theta^2} + \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial \theta^2} \right) \delta \theta \\
+ \left( \frac{\partial J}{\partial \theta} \right)' + \sum_{i=1}^{n} z_i \left( \frac{\partial f_i}{\partial \theta} \right)' \right) \delta \theta \\
+ \delta \dot{\theta}' \cdot R \cdot \delta \mathbf{x} + \frac{1}{2} \delta \mathbf{x}' \cdot K \cdot R \cdot \delta \mathbf{x} \right) \, dt
\] (36)

Through the proper choice of \( \delta \theta \) and denoting the minimum by \( V^0(\delta \mathbf{x}, \mathbf{t}) \), since \( V(\delta \mathbf{x}, \mathbf{t}) \) as given by Equation 36 is quadratic in \( \delta \mathbf{x} \), the minimum of \( V(\delta \mathbf{x}, \mathbf{t}) \) may be written as a quadratic expression, as

\[
V^0(\delta \mathbf{x}, \mathbf{t}) = q(t) + (q(t))' \delta \mathbf{x} + \delta \mathbf{x}' Q(t) \delta \mathbf{x}
\] (37)

where \( q(t) \) = scalar function of \( t \)

\[
q(t) = (n \times l) \text{ vector function of } t
\]

\[
Q(t) = (n \times m) \text{ matrix function of } t \text{ (symmetric)}
\]

and \( q(t_f) = 0 \quad q(t_f) = 0 \quad Q(t_f) = 0 \).
From Equation 37
\[
\frac{\partial V^0(\delta x, t)}{\partial t} = \frac{dq(t)}{dt} + \left( \frac{dq(t)}{dt} \right)' \delta x + \delta x' \frac{dQ(t)}{dt} \cdot \delta x
\]  
(38)

and
\[
\frac{\partial V^0(\delta x, t)}{\partial x} = q(t) + 2Q(t) \delta x
\]  
(39)

Minimization of \( V(\delta x, t) \) as given by Equation 36 gives
\[
\frac{1}{2} \delta^0 \left[ \sum_{i=1}^{n} z_{i} \frac{\partial^2 f_{i}}{\partial \theta^2} + \frac{R}{z} \sum_{i=1}^{n} z_{i} \frac{\partial^2 f_{i}}{\partial \theta^2} \right] \delta^0 + \left( \frac{\partial^0}{\partial \theta} \right)' + \sum_{i=1}^{n} z_{i} \frac{\partial f_{i}}{\partial \theta} \right] \delta^0
\]
\[
+ \frac{1}{2} \delta x' R \cdot \delta x + \frac{1}{2} \delta x'' K R \delta x
\]
\[
+ \left( q'(t) + 2\delta x'Q(t) \right) \delta \left( \frac{dx}{dt} \right) + \frac{dq(t)}{dt}
\]
\[
+ \left( \frac{dq(t)}{dx} \right)' \delta x' + \delta x' \frac{dQ(t)}{dt} \delta x = 0
\]  
(40)

where
\[
\delta^0 = - \left( \frac{\partial^2 f}{\partial \theta^2} + \sum_{i=1}^{n} z_{i} \frac{\partial^2 f_{i}}{\partial \theta^2} \right)^{-1} \left( \frac{\partial f}{\partial \theta} + \sum_{i=1}^{n} z_{i} \frac{\partial f_{i}}{\partial \theta} \right)
\]
\[
+ R \delta x + \left( \frac{\partial f'}{\partial \theta} \right) (q(t) + 2Q(t) \delta x)
\]  
(41)

and
\[
\frac{\delta dx}{dt} = \left( \frac{\partial f'}{\partial x} \right) \delta x + \left( \frac{\partial f'}{\partial \theta} \right) \delta \theta .
\] (42)

When the optimal control as given by Equation 41 is substituted into Equation 40 and the coefficients of \( \delta x \) and \( \delta x' \cdot \delta x \) along with the terms not containing \( \delta x \) are all put equal to zero (to satisfy the identify for any \( \delta x \)) the following results:

\[
\frac{dq}{dt} = \frac{1}{2} \left( S' T^{-1} S + S' T^{-1} \left( \frac{\partial f'}{\partial \theta} \right) q + q' \left( \frac{\partial f'}{\partial \theta} \right)' T^{-1} S \right)
+ q' \left( \frac{\partial f'}{\partial \theta} \right)' T^{-1} \left( \frac{\partial f'}{\partial \theta} \right) q \right) (43)
\]

\[
\frac{dq}{dt} = R' T^{-1} S + R' T^{-1} \left( \frac{\partial f'}{\partial \theta} \right) q - \frac{\partial f'}{\partial x} q
+ 2Q \left( \frac{\partial f'}{\partial \theta} \right)' T^{-1} S + \frac{\partial f'}{\partial \theta} T^{-1} \left( \frac{\partial f'}{\partial x} \right) q \right) (44)
\]

and

\[
\frac{dQ}{dt} = 2 \left\{ Q \left( \frac{\partial f'}{\partial \theta} \right)' T^{-1} R + Q \left( \frac{\partial f'}{\partial \theta} \right)' T^{-1} \frac{\partial f'}{\partial \theta} Q - Q \left( \frac{\partial f'}{\partial x} \right)' \right\} (45)
\]

where \( S \) and \( T \) are introduced to condense the notation and are given by

\[
S = \frac{\partial J}{\partial \theta} + \sum_{i=1}^{n} z_i \frac{\partial f_i}{\partial \theta} (46)
\]

\[
T = \frac{\partial^2 J}{\partial \theta^2} + \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial \theta^2} (47)
\]
At this point it is noted that the matrix $Q$ contributes only insignificantly to the control. Furthermore $Q$ appears as a second order term itself. Therefore, to facilitate programming on the digital computer these $\frac{n(n+1)}{2}$ equations shall be discarded and $Q$ shall be put equal to zero. Equation 43 is not required for the evaluation of control; therefore Equation 44 will take the form

$$\frac{dq}{dt} = R' T^{-1} S + R' T^{-1} \left( \frac{\partial f'}{\partial \theta} \right) q - \left( \frac{\partial f'}{\partial x} \right) q$$

and the change in the control simplifies to

$$\delta \theta = -T^{-1} \left( S + R \delta x + \frac{\partial f'}{\partial \theta} q \right).$$

To prevent overstepping in control adjustment, Mehmam [23] has suggested the introduction of a constant $\varepsilon$ where $0 < \varepsilon < 1$ in Equation 49 to give

$$\delta \theta = -\varepsilon T^{-1} (S + R \delta x + \frac{\partial f'}{\partial \theta} q) - T^{-1} R \delta x.$$  

Thus it is now worthwhile to detail the application of the second variation method equations developed above.

1. Assume a set of initial value for $\theta$.
2. Equations 1 and 2A are integrated forward from $t = 0$ to $t = t_f$; i.e. $(n+1)$ equations are integrated forward in time, namely

$$\frac{dx}{dt} = f(x, \theta)$$

$$\frac{dt}{dt} = J(x, \theta, t).$$
While the integration is carried out, the values of \( x \) are retained in the computer memory at small time intervals to approximate the continuous system.

The adjoint Equation 19 plus the additional Equations 28 and 48 are integrated backwards, i.e., \( 2n + \frac{n(n+1)}{2} \) equations are integrated backwards in time from \( t_f \) to 0, namely

\[
\frac{dz}{dt} = - \frac{\partial J}{\partial x} - \left( \frac{\partial f'}{\partial x} \right) z
\]

\[
\frac{d\varphi}{dt} = - \frac{\partial^2 J}{\partial x^2} - \sum_{i=1}^{n} z_i \frac{\partial^2 f_i}{\partial x^2} - \left( p \left( \frac{\partial f'}{\partial x} \right) + \left( \frac{\partial f'}{\partial x} \right) p \right) + R' T R
\]

\[
\frac{dq}{dt} = R' T^{-1} S + R' T^{-1} \left( \frac{\partial f'}{\partial q} \right) q - \left( \frac{\partial f'}{\partial x} \right) a.
\]

During the backward integration, the values of \( T, S, q \) and \( R \) are stored in computer memory.

The new value of control is calculated from Equation 50, i.e.

\[
\theta^{(j+1)} = \theta^{(j)} - \left( \varepsilon T^{-1} S + \frac{\partial f'}{\partial \theta} q \right) \left( \frac{\partial x^{(j+1)}}{\partial \theta} - \frac{\partial x^{(j)}}{\partial \theta} \right)
\]

and steps 2-6 are carried out again.

This iteration is continued until no further change in \( \theta \) is noticed or until the performance index does not change. The former is more sensitive [12].

If the performance index increases during some iteration, the parameter \( \varepsilon \) is halved and the iteration is continued.
For maximization problems, the derivation can be followed on the same lines and it will be seen that the resulting equations are the same.
2.4 Advantages and Disadvantages of the Second Variation Method.

The foremost advantage of the second variation method lies in its rapid convergence. Also, unlike the first variation, the optimum can be reached with reasonably high accuracy.

The theoretical attractiveness of this method, however, is more than offset by its disadvantages. First, and most important, the initially assumed trajectory of the control variable must be sufficiently close to the optimal trajectory for convergence to be obtained. Second, the number of equations to be integrated is considerably greater than required for the first variation method. In the second variation method, \((n+1)\) equations are integrated in the forward direction, i.e. from \(t=0\) to \(t=t_f\), and \((2n+n(n+1)/2)\) equations are integrated backwards where \(n\) is the number of state variables in the problem under consideration. The first variation method requires only \((n+1)\) equations to be integrated in the forward direction and there are \((n+1)\) recursive equations in the backward direction. Not only is the number of equations involved in the second variation method high but the equations themselves are more complicated. The main reason for this is that the calculations of all derivatives, both first and second order, becomes more and more tedious with the increasing number of state and control variables. All the multiplications are in terms of matrices. Again, the inverse of \(T\) has to be computed at each integration step in the backward integration. Hence the programming of the iteration scheme with the required equations can be quite complicated. Instability can arise from bad starting values, i.e. from an insufficiently good guess for the starting trajectory of the control variable. The values for the parameter \(\epsilon\) have to be established by trial and error for the particular
The higher the value the faster the convergence. Finally, this technique cannot handle problems involving inequality constraints.
3. APPLICATIONS

To illustrate the use of the second variation method, three numerical problems in the field of production planning and control are solved in the following sections.
3.1 An Inventory Model

The Model

The following is a simple problem in the field of production scheduling and inventory control. Assume that the rate of sales $Q(t)$ is known with certainty and that the rate of change of the inventory level $I(t)$ is given by

$$\frac{dI(t)}{dt} = P(t) - Q(t)$$

(51)

where $P(t)$ is the production rate at time $t$. The problem is to minimize the cost function.

$$C_T = \int_0^T \{C_I (I_M - I(t))^2 + C_P \exp (P_M - P(t))^2 \} \, dt$$

(52)

where $C_T$ is the total cost of inventory and production and $C_P$ is the minimum production cost which occurs when the production rate equals $P_M$. The quantity $P_M$ can be considered as the capacity of the manufacturing plant. Since the plant is designed for a capacity $P_M$, an increase in capacity may require additional equipment and manpower which, due to contract agreements cannot be reduced easily. $I_M$ can be considered as the capacity for the storage of inventory and $C_I$ is the inventory carrying cost. In many practical situations, the minimum storage cost is obtained when the storage capacity is completely filled. Furthermore, the cost function, Equation (2), has the smoothing capability which is frequently desirable for many manufacturing processes. In this case, $I_M$ and $P_M$ can be considered as the desirable inventory and production levels. It is further assumed that the sales forecast is known and is given by the linear relation
Q(t) = a + bt \tag{53}

and the initial inventory is

I(0) = c \tag{54}

Recursive Relations

This optimum production planning problem can be rewritten into the form required for the second variation method as

Let

\[ x_1(t) = I(t) \]
\[ \theta(t) = P(t) . \]

Equations (51) and (54) become

\[ \frac{dx_1(t)}{dt} = \theta(t) - a - b(t) \tag{55} \]
and

\[ x_1(0) = c \tag{56} \]

Let

\[ x_2(t) = \int_0^t C_I(I_M - I(t))^2 + C_P \exp (P_M - \theta(t))^2 \] \tag{57}

Then

\[ x_2(t) = C_T \tag{58} \]

\[ \frac{dx_2(t)}{dt} = C_I(I_M - x_1(t))^2 + C_P \exp (P_M - \theta(t))^2 \tag{59} \]
\[ x_2(0) = 0 \tag{60} \]

Thus, in this problem there is one state variable, namely inventory \( x_1 \).

The control variable is the production rate \( \theta(t) \). The numerical values
used for this problem are:

\[ a = 2 \quad b = 1 \quad c = 5 \]

\[ C_I = 0.1 \quad I_M = 10 \quad C_P = 0.001 \]

\[ P_M = 5 \quad T = 1 \]

The various derivatives required for obtaining the second variational equations are:

\[ \frac{3J}{3x} = \frac{3J}{3x_1} = -2C_I (I_M - xl(t)) \]

\[ \frac{3^2J}{3x^2} = 2C_I \]

\[ \frac{3J}{3\theta} = -2C_P \exp (P_M - \theta(t))^2 \cdot (P_M - \theta(t)) \]

\[ \frac{3^2J}{3\theta^2} = 2C_P \exp (P_M - \theta(t))^2 \left( 1 + 2(P_M - \theta(t))^2 \right) \]

\[ \frac{3^2J}{3\theta \partial x} = 0 \]

\[ \frac{3f_1}{3x} = 0 \quad \frac{3^2f_1}{3x3x} = 0 \]

\[ \frac{3^2f_1}{3x^2} = 0 \quad \frac{3f'}{3x} = 0 \quad \frac{3f'}{3\theta} = 1 \]
The expressions for the terms $R$, $s$, $T$ are:

\[
R = P = P \quad \text{being 1 dimensional}
\]

\[
s = -2 \, C_p \exp \left( P_M - \theta(t) \right)^2 \left( P_M - \theta(t) \right) + z_1
\]

\[
T = 2 \, C_p \exp \left( P_M - \theta(t) \right)^2 \cdot \{1 + 2 \left( P_M - \theta(t) \right)^2}\]

The second variational equations (19, 28, 48) become

\[
\frac{dz}{dt} = \frac{dz}{dt} = 2 \, C_1 \left[ I_M - x_1(t) \right] \quad \text{(61)}
\]

\[
\frac{dP}{dt} = \frac{dP}{dt} = -2 \, C_1 + P^2 \left[ 2 \, C_p \exp \left( P_M - \theta(t) \right)^2 \{1 + 2 \left( P_M - \theta(t) \right)^2}\right] \quad \text{(62)}
\]

\[
\frac{dQF}{dt} = \frac{dQF}{dt} = \frac{P\{ -2C_p \exp(P_M - \theta(t))^2 (P_M - \theta(t)) + z + QF \}}{2C_p \cdot \exp[P_M - \theta(t)]^2 \{1 + 2[P_M - \theta(t)]^2\}} \quad \text{(63)}
\]

and

\[
\theta^{(j+1)} = \theta^{(j)} - \left[ \epsilon(s + QF) \right]^{(j)} - \left[ P(j)(x_1^{(j+1)} - x_1^{(j)}) \right].
\]

Thus Equations (61), (62) and (63) are the second variational equations and Equation (64) is the equation for finding the new value of the control.
Table 1

Effect of $\varepsilon$ on the Rate of Convergence, of Inventory, $0(t) = 7$, $x_{1}(t) = 5$, $0 \leq t \leq t_f$.

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Fig. 1 Convergence rate of Inventory.
Fig. 2. Convergence rate of Inventory.
Fig. 3 Convergence rate of Inventory
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</tr>
<tr>
<td>0.94</td>
<td>4.385418000</td>
<td>8.814865000</td>
</tr>
<tr>
<td>0.95</td>
<td>4.323664000</td>
<td>8.829268000</td>
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<tr>
<td>0.96</td>
<td>4.262953000</td>
<td>8.842953000</td>
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<tr>
<td>0.97</td>
<td>4.203338000</td>
<td>8.855934000</td>
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<tr>
<td>0.98</td>
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<td>8.868217000</td>
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<tr>
<td>0.99</td>
<td>4.087779000</td>
<td>8.879816000</td>
</tr>
<tr>
<td>1.00</td>
<td>4.087779000</td>
<td>8.890743000</td>
</tr>
</tbody>
</table>
Table 2

Effect of $\epsilon$ on the Rate of Convergence
of Cost Function $x_2$, $\theta(t) = 7$, $x_1(t) = 5$, $0 \leq t \leq t_f$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$\epsilon = 0.1$</th>
<th>$\epsilon = 0.3$</th>
<th>$\epsilon = 0.5$</th>
<th>$\epsilon = 0.7$</th>
<th>$\epsilon = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.95957</td>
<td>0.95957</td>
<td>0.95956</td>
<td>0.95956</td>
<td>0.95957</td>
</tr>
<tr>
<td>5</td>
<td>0.95232</td>
<td>0.94536</td>
<td>0.94392</td>
<td>0.94356</td>
<td>0.94342</td>
</tr>
<tr>
<td>10</td>
<td>0.94694</td>
<td>0.94360</td>
<td>0.94337</td>
<td>0.94335</td>
<td>0.94335</td>
</tr>
<tr>
<td>15</td>
<td>0.94498</td>
<td>0.94339</td>
<td>0.94335</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>20</td>
<td>0.94415</td>
<td>0.94336</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>25</td>
<td>0.94376</td>
<td>0.94335</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>30</td>
<td>0.94356</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>35</td>
<td>0.94347</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>40</td>
<td>0.94342</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>45</td>
<td>0.94339</td>
<td>&quot;</td>
<td>&quot;</td>
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<td>&quot;</td>
</tr>
<tr>
<td>50</td>
<td>0.94339</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>55</td>
<td>0.94336</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>
Fig. 4 Convergence rate of Cost Function.
Fig. 5 Convergence rate of Cost Function.
Fig. 6 Convergence rate of Cost Function.

$\epsilon = 0.7$

$1^{st}$, $5^{th}$ & OPT.
Table 3

Effect of $\varepsilon$ on Rate of Convergence of the Production Rate $\theta$, $\theta(t) = 7$, $x_1(t) = 5$, $0 < t < t_f$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$\varepsilon = 0.1$</th>
<th>$\varepsilon = 0.3$</th>
<th>$\varepsilon = 0.5$</th>
<th>$\varepsilon = 0.7$</th>
<th>$\varepsilon = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta(t_0)$</td>
<td>7.03101</td>
<td>7.09304</td>
<td>7.15507</td>
<td>7.21709</td>
<td>7.31013</td>
</tr>
<tr>
<td>$\theta(t_f)$</td>
<td>6.97778</td>
<td>6.93333</td>
<td>6.88889</td>
<td>6.84444</td>
<td>6.77778</td>
</tr>
<tr>
<td>5</td>
<td>$\theta(t_0)$</td>
<td>7.10105</td>
<td>7.17094</td>
<td>7.18717</td>
<td>7.18952</td>
</tr>
<tr>
<td></td>
<td>$\theta(t_f)$</td>
<td>6.88690</td>
<td>6.64770</td>
<td>6.38816</td>
<td>6.10440</td>
</tr>
<tr>
<td>10</td>
<td>$\theta(t_0)$</td>
<td>7.14225</td>
<td>7.18636</td>
<td>7.18927</td>
<td>7.18933</td>
</tr>
<tr>
<td></td>
<td>$\theta(t_f)$</td>
<td>6.76850</td>
<td>6.23612</td>
<td>5.58561</td>
<td>5.06248</td>
</tr>
<tr>
<td>15</td>
<td>$\theta(t_0)$</td>
<td>7.16293</td>
<td>7.18844</td>
<td>7.18933</td>
<td>7.18933</td>
</tr>
<tr>
<td></td>
<td>$\theta(t_f)$</td>
<td>6.64410</td>
<td>5.74570</td>
<td>5.03792</td>
<td>5.00015</td>
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<tr>
<td>20</td>
<td>$\theta(t_0)$</td>
<td>7.17416</td>
<td>7.18925</td>
<td>7.18933</td>
<td>7.18933</td>
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<td>$\theta(t_f)$</td>
<td>6.51294</td>
<td>5.25530</td>
<td>5.00119</td>
<td>5.00000</td>
</tr>
<tr>
<td>25</td>
<td>$\theta(t_0)$</td>
<td>7.18051</td>
<td>7.18932</td>
<td>7.18933</td>
<td>7.18933</td>
</tr>
<tr>
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<td>$\theta(t_f)$</td>
<td>6.37410</td>
<td>5.04754</td>
<td>5.00004</td>
<td>5.00000</td>
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<tr>
<td>30</td>
<td>$\theta(t_0)$</td>
<td>7.18416</td>
<td>7.18933</td>
<td>7.18933</td>
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</tr>
<tr>
<td></td>
<td>$\theta(t_f)$</td>
<td>6.22667</td>
<td>5.00802</td>
<td>5.00000</td>
<td>5.00000</td>
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<tr>
<td>35</td>
<td>$\theta(t_0)$</td>
<td>7.18629</td>
<td>7.18933</td>
<td>7.18932</td>
<td>7.18933</td>
</tr>
<tr>
<td></td>
<td>$\theta(t_f)$</td>
<td>6.06984</td>
<td>5.00135</td>
<td>5.00000</td>
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<td>40</td>
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<td>$\theta(t_f)$</td>
<td>5.90348</td>
<td>5.00023</td>
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<tr>
<td>45</td>
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<td>7.18828</td>
<td>7.18932</td>
<td>7.18933</td>
<td>7.18933</td>
</tr>
<tr>
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<td>$\theta(t_f)$</td>
<td>5.72933</td>
<td>5.00003</td>
<td>5.00000</td>
<td>5.00000</td>
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<tr>
<td>50</td>
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<td>7.18871</td>
<td>7.18932</td>
<td>7.18933</td>
<td>7.18933</td>
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<tr>
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<td>$\theta(t_f)$</td>
<td>5.55353</td>
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<td>5.38931</td>
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</table>
Fig. 7 Convergence rate of Production Rate.
Fig. 8 Convergence rate of Production Rate.
Fig. 9  Convergence rate of Production rate.
This problem was solved by two approaches. In the first, the second variation method was used in combination with the first variation method. The same problem is solved by Lee and Shaikh (20). The values of the state and control variables (at all grid points) were taken from the results of the first variation. In particular, the values of $X_1(t)$ and $\theta(t)$ were taken from the 21st iteration of the first variation and fed as good starting values for the second variation. These values are listed in Table 1A. In the second approach, the second variation was tried directly by itself. For this, a guess was made for the starting values of the state variable and the control variables.

An interesting parameter in the computation is the step size $\epsilon$ which determines the magnitude of the step taken in each iteration. In the solution of this problem, a series of values of $\epsilon$ were selected and the computation was carried out for each. Tables 1, 2 and 3 show the convergence rate of inventory $x_1$, cost function $x_2$ and the production rate $\theta(t)$, respectively. These tables are for a constant starting value of the control variable, namely $\theta(t) = 7, 0 \leq t \leq t_f$ and a constant starting value of the state variable, namely $x_1(t) = 5, 0 \leq t \leq t_f$. It is seen that for $\epsilon = 1$, the fastest convergence rate is obtained while the convergence rate slows down when its value is decreased. Figures 1 through 9 show the rate of convergence of the inventory, production rate and the cost function for different values of $\epsilon$.

Regarding the starting trajectory of the control variable, it was found that only the constant trajectories between $\theta(t) = 7$ and $\theta(t) = 8$ would lead to convergence. For all other control variable values, the
the problem would not converge. These other values were:

\[ 0(t) = 1, 2, 3, 4, 5, 6 \quad \text{and} \quad 0(t) = 9, 10, 11, \quad 0 \leq t \leq t_f \]

Also, the combination of the first and the second variation required about 50 iterations to reach the optimal with \( \epsilon = 0.3 \). A higher value of \( \epsilon \) could not be used as it led to overstepping in this situation.
3.2 An Inventory and Advertising Model

The Model

This model is an extension of the one formulated by Teichroew (27). Consider a marketing situation where only a certain number of possible customers possess certain information about a firm's product. Suppose that the total number of such possible customers remains constant and that the diffusion of information occurs only through personal contact.

The number of contacts made by an informed person in a unit time is known as contact coefficient. In a contact, the contactee receives information if he does not already have it; if he already has it, the contact is wasted insofar as increasing the number of informed persons is concerned.

Let \( K(0) = K_0 \) = number of informed persons at time \( t_0 \)

\[ N = \text{total number of persons} \]

\[ c = \text{contact coefficient, the number of contacts made by} \]

\[ \text{one informed person per unit time} \]

\[ K(t) = \text{number of informed persons at time } t. \]

Then \( K(t)/N \) = proportion of informed persons at time \( t \)

\[ 1 - K(t)/N = \text{proportion of uninformed persons at time } t \]

\[ c.K(t).dt = \text{contacts made during a time interval } dt. \]

Clearly \( dK(t) = c.K(t).dt.(1-K(t)/N) \)

Thus the equation governing the process is

\[
\frac{dK(t)}{dt} = c.K(t).(1-K(t)/N) \tag{65}
\]

Suppose next that the firm can influence the number of contacts by spending money on advertising. In particular it can increase the number
of contacts made by the informed persons (above the ones included in c) by an additional number A per unit time.

Equation (65) now becomes

\[
\frac{dK(t)}{dt} = K(t)(c+A(t))(1-K(t)/N)
\]  

(66)

If each successful contact results in the sale of n units of the firm's product and if Q(t) represents the sale at time t, then

\[Q(t) = nK(t)\]

Letting n=1 and substituting Q(t) for K(t) in Equation (66), then

\[
\frac{dQ(t)}{dt} = Q(t)(c+A(t))(1-Q(t)/N)
\]

(67)

The rate of change of the firm's inventory is given by

\[
\frac{dX(t)}{dt} = P(t) - Q(t)
\]

(68)

where P(t) = production rate at time t.

The production rate is assumed to be a linear function of time

\[P(t) = a + bt\]

(69)

where a and b are constants.

This assumption is made to simplify the model by avoiding a second control variable.

The firm's management wishes to maximize the profit

\[S_T = \int_0^T [F \cdot Q(t) - C_I (P_I - x(t))^2 - C_A A^2(t) Q(t)] dt\]

(70)

where \(S_T\) is the total net profit.
F is the revenue from the sale of one unit of the product. \( C_I \) is the inventory carrying cost and has the same significance as in the model described in Section 3.1. \( P_I \) can be considered as the capacity for the storage of inventory. \( C_A \) is the cost of advertising.

Equations (67) through (70) represent the system under consideration. The system has two state variables, inventory \( X(t) \) and sales \( Q(t) \), and there is one control variable, advertising \( A(t) \).

The initial conditions and the numerical values used are:

\[
\begin{align*}
a &= 0.7 & b &= 1.0 & c &= 2.0 & N &= 1.5 & F &= 10.0 \\
C_I &= 0.15 & P_I &= 1.0 & C_A &= 1.0 & X(0) &= 0.2 & Q(0) &= 0.2
\end{align*}
\]

Recursive Relation

The necessary relations for the second variation can be obtained in the following manner. Note that in these derivations \( x(t) \) denotes the state variable vector while \( x(t) \) denotes the inventory. From Equation 70, then,

\[
J = Q \cdot F - C_I(P_I - x(t))^2 - C_AQA^2(t).
\]

The various derivatives required for obtaining the second variation equations are:

\[
\frac{\partial J}{\partial x} = \begin{bmatrix} \frac{\partial J}{\partial x} \\ \frac{\partial J}{\partial Q} \end{bmatrix} = \begin{bmatrix} 2C_I(P_I - x(t)) \\ F - C_A^2(t) \end{bmatrix}
\]
\[
\frac{\partial^2 J}{\partial x^2} = \begin{bmatrix}
\frac{\partial}{\partial x} \left( \frac{\partial J}{\partial x} \right) & \frac{\partial}{\partial x} \left( \frac{\partial J}{\partial Q} \right)
\end{bmatrix} = \begin{bmatrix}
-2C_A & 0
\end{bmatrix}
\]

\[
\frac{\partial J}{\partial \theta} = \frac{\partial J}{\partial A(t)} = -2C_A Q(t) A(t)
\]

\[
\frac{\partial^2 J}{\partial \theta^2} = \frac{\partial^2 J}{\partial A^2(t)} = -2C_A Q(t)
\]

\[
\frac{\partial^2 J}{\partial \theta \partial x} = \frac{\partial}{\partial A(t)} \begin{bmatrix}
\frac{\partial J}{\partial x} \\
\frac{\partial J}{\partial Q}
\end{bmatrix} = \begin{bmatrix}
0 & -2C_A A(t)
\end{bmatrix}
\]

\[
\frac{\partial f_1}{\partial x} = \begin{bmatrix}
0 \\
-1
\end{bmatrix}, \quad \frac{\partial^2 f_1}{\partial x \partial \theta} = \begin{bmatrix}
0, 0
\end{bmatrix}
\]

\[
\frac{\partial f_2}{\partial x} = \begin{bmatrix}
0 \\
\left( C + A(t) \right) \left( 1 - \frac{2Q(t)}{N} \right)
\end{bmatrix}, \quad \frac{\partial^2 f_2}{\partial x \partial \theta} = \begin{bmatrix}
0, \left( 1 - \frac{2Q(t)}{N} \right)
\end{bmatrix}
\]
\[ \frac{\partial f_1}{\partial \theta} = 0 \quad \frac{\partial^2 f_1}{\partial \theta^2} = 0 \]

\[ \frac{\partial f_2}{\partial \theta} = Q(t) \left( 1 - \frac{Q(t)}{N} \right) \quad \frac{\partial^2 f_2}{\partial \theta^2} = 0 \]

\[ \frac{\partial f'}{\partial \dot{x}} = \begin{bmatrix} \frac{\partial}{\partial x} [f_1] & \frac{\partial}{\partial x} [f_2] \\ \frac{\partial}{\partial Q} [f_1] & \frac{\partial}{\partial Q} [f_2] \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & [C + A(t)] \left( 1 - \frac{2Q(t)}{N} \right) \end{bmatrix} \]

\[ \frac{\partial^2 f_1}{\partial x^2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ \frac{\partial^2 f_2}{\partial x^2} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2}{N} (C + A(t)) \end{bmatrix} \]

\[ \frac{\partial f'}{\partial \theta} = \frac{\partial}{\partial A(t)} [f_1 \quad f_2] = \begin{bmatrix} 0 & Q(t) \left( 1 - \frac{Q(t)}{N} \right) \end{bmatrix} \]
Expressions for the terms \( R, s, T \) from Equation 30 result in

\[
R = \begin{bmatrix} 0 & -2C_A A(t) \end{bmatrix} + \begin{bmatrix} 0 & Q(t) \left(1 - \frac{Q(t)}{N}\right) \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}
\]

\[
+ \begin{bmatrix} 0 & z_2 \left(1 - \frac{2Q(t)}{N}\right) \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & -2C_A A(t) \end{bmatrix} + \begin{bmatrix} P_{12} Q(t) \left(1 - \frac{Q(t)}{N}\right), & P_{22} Q(t) \left(1 - \frac{Q(t)}{N}\right) \end{bmatrix}
\]

\[
+ \begin{bmatrix} 0 & z_2 \left(1 - \frac{2Q(t)}{N}\right) \end{bmatrix}
\]

Let \( R = [R_1, \ R_2] \)

where

\[
R_1 = P_{12} Q(t) \left(1 - \frac{Q(t)}{N}\right)
\]

\[
R_2 = -2C_A A(t) + P_{22} Q(t) \left(1 - \frac{Q(t)}{N}\right)
\]

\[
+ z_2 \left(1 - \frac{2Q(t)}{N}\right)
\]

Equation 46 gives

\[
S = -2C_A Q(t) A(t) + z_2 Q(t) \left(1 - \frac{Q(t)}{N}\right)
\]

and Equation 47 gives,
\[ T = -2C_A Q(t). \]

It is now possible to determine the \( 2n + \frac{n(n+1)}{2} \) i.e. \((2+2+3)\) or seven equations to be integrated backwards. Equation (19) becomes

\[
\frac{dz}{dt} = \begin{pmatrix}
-2C_I (P_I - x(t)) \\
-F + C_A A^2(t)
\end{pmatrix} - \begin{pmatrix}
0 & 0 \\
-1 & -1
\end{pmatrix} \begin{pmatrix}
[C+A(t)] \left(1 - \frac{2Q(t)}{N}\right) \\
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
-2C_I (P_I - x(t)) \\
-F + C_A A^2(t)
\end{pmatrix} - \begin{pmatrix}
0 & 0 \\
-1 & -1
\end{pmatrix} \begin{pmatrix}
-z_1 + z_2 [C + A(t)] \left(1 - \frac{2Q(t)}{N}\right) \\
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
\]

\[
\frac{dz_1}{dt} = -2C_I [P_I - x(t)]
\] (71)

\[
\frac{dz_2}{dt} = -F + C_A A^2(t) + z_1 - z_2 [C + A(t)] \left(1 - \frac{2Q(t)}{N}\right)
\] (72)

Thus Equations (71) and (72) correspond to Equation (19). Equation (28) in this case becomes
\[
\frac{\mathbf{dP}}{\mathbf{dt}} = \begin{pmatrix}
2C_1 & 0 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & \frac{2z_2}{N} [C + A(t)]
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & -P_{11} + P_{12} [C + A(t)] \\
-\frac{P_{11} + P_{12} [C + A(t)]}{1 - \frac{2Q(t)}{N}}, -2P_{12} + 2P_{22} [C + A(t)] \\
\end{pmatrix}
\]

Hence Equation (28) is represented by the following three equations:

\[
\frac{dP_{11}}{dt} = 2C_1 + R_1^2 T \tag{73}
\]

\[
\frac{dP_{12}}{dt} = P_{11} - P_{12} [C + A(t)] \left(1 - \frac{2Q(t)}{N}\right) + R_1 R_2 T \tag{74}
\]

\[
\frac{dP_{22}}{dt} = \frac{2z_2}{N} [C + A(t)] + 2P_{12} - 2P_{22} [C + A(t)] \left(1 - \frac{2Q(t)}{N}\right) + R_2^2 T \tag{75}
\]

To avoid confusion, the \( q \) in the derivation of the method given in Equation (48) is denoted by \( Q_F \) here. Thus \( Q \) still represents the sales for this
problem.

Equation (48) is given by

\[
\frac{dQ_F}{dt} = \begin{pmatrix}
\frac{R_1}{T} \\
\frac{R_2}{T}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{T} \\
\frac{1}{T}
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
+ Q(t) \left[ 1 - \frac{Q(t)}{N} \right]
\begin{pmatrix}
Q_F_1 \\
Q_F_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{R_1 s}{T} \\
\frac{R_2 s}{T}
\end{pmatrix}
\begin{pmatrix}
\frac{R_1}{T} \\
\frac{R_2}{T}
\end{pmatrix}
\begin{pmatrix}
Q_F_2 \cdot Q(t) \cdot \left[ 1 - \frac{Q(t)}{N} \right] \\
Q_F_2 \cdot Q(t) \cdot \left[ 1 - \frac{Q(t)}{N} \right]
\end{pmatrix}
\]

\[
\begin{pmatrix}
- Q_F_2 \\
Q_F_2 \left[ C + A(t) \right] \left[ 1 - \frac{2Q(t)}{N} \right]
\end{pmatrix}
\]

\[
\frac{dQ_F}{dt} = \frac{R_1 s}{T} + \frac{R_1}{T} \cdot Q_F_2 \cdot Q(t) \cdot \left[ 1 - \frac{Q(t)}{N} \right] + Q_F_2
\]

(76)
<table>
<thead>
<tr>
<th>t</th>
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<th>Q(t)</th>
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Table 4 (continued)

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\[
\frac{dQF_2}{dt} = \frac{R_2s}{T} + \frac{R_2}{T} \cdot QF_2 \cdot Q(t) \cdot \left[1 - \frac{Q(t)}{N}\right] - QF_2 \left[C + A(t)\right] \cdot \left[1 - \frac{2Q(t)}{N}\right]
\]  
\hspace{1cm} (77)

Thus Equations (76) and (77) represent Equation (48). The equation for improving the control variable becomes

\[
A(t)^{(j+1)} = A(t)^{(j)} + \frac{C}{T} \left[s + QF_2 \cdot Q(t) \cdot \left[1 - \frac{Q(t)}{N}\right]\right] + \frac{1}{T} \left[R_1(x^{(j+1)} - x^{(j)}) + R_2(Q^{(j+1)} - Q^{(j)})\right]
\]

Equation (78) represents Equation (50)  
\hspace{1cm} (78)

This problem illustrates how tedious the calculations become when the number of variables increases.

**Numerical Results**

In here, the starting trajectories of the two state variables, inventory \(I(t)\) and the sales \(Q(t)\), were fed from the results of the solution of the same problem by dynamic programming. These values are listed in Table 4. Actually these values are obtained after dividing the original results by 100. This was required to prevent the exponential overflow of the system of equations. The starting trajectory of the control variable was tried in the range of 0.001 to 6.0. It was found that all these values would work; however, the best value was found to be \(\theta(t) = 0.5\), \(0 < t < t_f\).
Table 5

Effect of $\varepsilon$ on the Rate of Convergence of $I(t_f)$ with $A_0(t) = 0.5$.

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<th>$\varepsilon = 0.7$</th>
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*The Values of $I_0(t) \& Q_0(t)$ are obtained from Table 4.*
Fig. 10  Convergence rate of Inventory.
Fig. 11 Convergence rate of Inventory.
Fig. 12 Convergence rate of Inventory.
Table 6

Effect of \( \varepsilon \) on the Rate of Convergence of \( Q(t) \), with \( A_0(t) = 0.5 \).

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*The Values of \( I_0(t) \) & \( Q_0(t) \) are obtained from Table 4.*
Fig. 13 Convergence rate of Sales.
Fig. 14 Convergence rate of Sales.

$E = 0.3$

- ZEROETH
- $5^{th}$ IT
- $8^{th}, 9^{th} & OPT.$

SALES

$t$
Fig. 15 Convergence rate of Sales.
Table 7

Effect of \( \epsilon \) on the Rate of Convergence
of Total Profit, with \( A_0(t) = 0.5 \).

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*The Values of \( I_0(t) \) & \( Q_0(t) \) are obtained from Table 4.*
Fig. 16 Convergence rate of Profit
Fig. 17 Convergence rate of Profit.
Fig. 18 Convergence rate of Profit.
Table 8

Effect of $\epsilon$ on the Rate of Convergence
of $A(t_0)$, with $A_0(t) = 0.5$.

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*The Values of $I_0(t)$ & $Q_0(t)$ are obtained from Table 4.*
Fig. 19 Convergence rate of Advertisement.
Fig. 20 Convergence rate of Advertisement.
Fig. 21 Convergence rate of Advertisement.
Table 9

Starting Trajectories for Inventory, Sales and Advertisement, 0 \leq t \leq t_f.

<table>
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<tr>
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<th>( A_0(t) )</th>
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Table 10

Effect of $\varepsilon$ on Rate of Convergence of $I(t_f)$ with $I_0(t) = Q_0(t) = 0.2$, $A_0(t) = 0.5$, $0 \leq t \leq t_f$.

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<th>$\varepsilon = 0.5$</th>
<th>$\varepsilon = 0.7$</th>
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<td>0.8524</td>
<td>0.8524</td>
<td>0.8524</td>
<td>0.8524</td>
</tr>
<tr>
<td>5</td>
<td>0.7264</td>
<td>0.6343</td>
<td>0.6114</td>
<td>0.6322</td>
</tr>
<tr>
<td>10</td>
<td>0.6624</td>
<td>0.5999</td>
<td>0.5940</td>
<td>0.5928</td>
</tr>
<tr>
<td>15</td>
<td>0.6309</td>
<td>0.5945</td>
<td>0.5935</td>
<td>0.5935</td>
</tr>
<tr>
<td>20</td>
<td>0.6142</td>
<td>0.5936</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>25</td>
<td>0.6051</td>
<td>0.5935</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>
Fig. 22 Convergence rate of Inventory.
Fig. 23 Convergence rate of Inventory.
Table 11

Effect of $\varepsilon$ on Rate of Convergence of

$Q(t_f), I_0(t) = Q_0(t) = 0.2, \Lambda_0(t) = 0.5, 0 \leq t \leq t_f$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$\varepsilon = 0.1$</th>
<th>$\varepsilon = 0.3$</th>
<th>$\varepsilon = 0.5$</th>
<th>$\varepsilon = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9781</td>
<td>0.9781</td>
<td>0.9781</td>
<td>0.9781</td>
</tr>
<tr>
<td>5</td>
<td>1.083</td>
<td>1.178</td>
<td>1.198</td>
<td>1.165</td>
</tr>
<tr>
<td>10</td>
<td>1.153</td>
<td>1.215</td>
<td>1.222</td>
<td>1.223</td>
</tr>
<tr>
<td>15</td>
<td>1.185</td>
<td>1.221</td>
<td>&quot;</td>
<td>1.222</td>
</tr>
<tr>
<td>20</td>
<td>1.202</td>
<td>1.222</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>25</td>
<td>1.211</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>
Fig. 24 Convergence rate of Sales.
Fig. 25 Convergence rate of Sales.
Table 12

Effect of $\varepsilon$ on Rate of Convergence of Total Profit, $I_0(t) = Q_0(t) = 0.2$, $A_0(t) = 0.5$, $0 \leq t \leq t_f$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$\varepsilon = 0.1$</th>
<th>$\varepsilon = 0.3$</th>
<th>$\varepsilon = 0.5$</th>
<th>$\varepsilon = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.298</td>
<td>5.298</td>
<td>5.298</td>
<td>5.298</td>
</tr>
<tr>
<td>10</td>
<td>6.518</td>
<td>6.626</td>
<td>6.626</td>
<td>6.626</td>
</tr>
<tr>
<td>15</td>
<td>6.595</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>20</td>
<td>6.617</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>25</td>
<td>6.624</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>
Fig. 26 Convergence rate of Profit
Fig. 27 Convergence rate of Profit.
Table 13

Effect of $\epsilon$ on Rate of Convergence of

\[ A(t_0), I_0(t) = Q_0(t) = 0.2, A_0(t) = 0.5, 0 \leq t \leq t_f. \]

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$\epsilon = 0.1$</th>
<th>$\epsilon = 0.3$</th>
<th>$\epsilon = 0.5$</th>
<th>$\epsilon = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.320</td>
<td>2.960</td>
<td>4.599</td>
<td>6.239</td>
</tr>
<tr>
<td>5</td>
<td>2.973</td>
<td>4.804</td>
<td>5.223</td>
<td>5.284</td>
</tr>
<tr>
<td>10</td>
<td>4.005</td>
<td>5.169</td>
<td>5.223</td>
<td>5.220</td>
</tr>
<tr>
<td>15</td>
<td>4.549</td>
<td>5.215</td>
<td>5.221</td>
<td>5.221</td>
</tr>
<tr>
<td>20</td>
<td>4.847</td>
<td>5.220</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>25</td>
<td>5.012</td>
<td>5.221</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>
Fig. 28 Convergence rate of Advertisement.
Fig. 29 Convergence rate of Advertisement.
Table 14

Effect of Different Starting Trajectories on $I(t_f)$ with $\epsilon = 0.4$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$I_0(t)$ = 0.5</th>
<th>$I_0(t)$ = 0.5</th>
<th>$I_0(t)$ = 0.6</th>
<th>$I_0(t)$ = 0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Q_0(t)$ = 0.5</td>
<td>$Q_0(t)$ = 1.0</td>
<td>$Q_0(t)$ = 1.3</td>
<td>$Q_0(t)$ = 1.3</td>
</tr>
<tr>
<td></td>
<td>$A_0(t)$ = 2.0</td>
<td>$A_0(t)$ = 2.0</td>
<td>$A_0(t)$ = 2.0</td>
<td>$A_0(t)$ = 5.0</td>
</tr>
<tr>
<td>1</td>
<td>0.6134</td>
<td>0.6134</td>
<td>0.6134</td>
<td>0.3305</td>
</tr>
<tr>
<td>5</td>
<td>0.6024</td>
<td>0.5922</td>
<td>0.5876</td>
<td>0.5356</td>
</tr>
<tr>
<td>10</td>
<td>0.5945</td>
<td>0.5939</td>
<td>0.5936</td>
<td>0.5887</td>
</tr>
<tr>
<td>15</td>
<td>0.5936</td>
<td>0.5936</td>
<td>0.5935</td>
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<td>20</td>
<td>0.5935</td>
<td>0.5935</td>
<td>&quot;</td>
<td>0.5934</td>
</tr>
</tbody>
</table>
Table 15

Effect of Different Starting Trajectories on Rate of Convergence of $Q(t_f)$ with $\varepsilon = 0.4$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$I_0(t) = 0.5$</th>
<th>$I_0(t) = 0.5$</th>
<th>$I_0(t) = 0.6$</th>
<th>$I_0(t) = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Q_0(t) = 0.5$</td>
<td>$Q_0(t) = 1.0$</td>
<td>$Q_0(t) = 1.3$</td>
<td>$Q_0(t) = 1.3$</td>
</tr>
<tr>
<td></td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 5.0$</td>
</tr>
<tr>
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<td>1.340</td>
<td>1.340</td>
<td>1.491</td>
</tr>
<tr>
<td>5</td>
<td>1.219</td>
<td>1.243</td>
<td>1.255</td>
<td>1.316</td>
</tr>
<tr>
<td>10</td>
<td>1.222</td>
<td>1.224</td>
<td>1.225</td>
<td>1.232</td>
</tr>
<tr>
<td>15</td>
<td>&quot;</td>
<td>1.222</td>
<td>1.222</td>
<td>1.223</td>
</tr>
<tr>
<td>20</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>1.222</td>
</tr>
</tbody>
</table>
Table 16

Effect of Different Starting Trajectories on Rate of Convergence of Total Profit with $\varepsilon = 0.4$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$I_0(t) = 0.5$</th>
<th>$I_0(t) = 0.5$</th>
<th>$I_0(t) = 0.6$</th>
<th>$I_0(t) = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Q_0(t) = 0.5$</td>
<td>$Q_0(t) = 1.0$</td>
<td>$Q_0(t) = 1.3$</td>
<td>$Q_0(t) = 1.3$</td>
</tr>
<tr>
<td></td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 5.0$</td>
</tr>
<tr>
<td>1</td>
<td>4.668</td>
<td>4.668</td>
<td>4.668</td>
<td>-16.120</td>
</tr>
<tr>
<td>5</td>
<td>6.621</td>
<td>6.588</td>
<td>6.551</td>
<td>6.249</td>
</tr>
<tr>
<td>10</td>
<td>6.626</td>
<td>6.625</td>
<td>6.625</td>
<td>6.621</td>
</tr>
<tr>
<td>15</td>
<td>&quot;</td>
<td>6.626</td>
<td>6.626</td>
<td>6.626</td>
</tr>
<tr>
<td>20</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>
Table 17

Effect of Different Starting Trajectories on Rate of Convergence of $A(t_0)$ with $\varepsilon = 0.4$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$I_0(t) = 0.5$</th>
<th>$I_0(t) = 0.5$</th>
<th>$I_0(t) = 0.6$</th>
<th>$I_0(t) = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Q_0(t) = 0.5$</td>
<td>$Q_0(t) = 1.0$</td>
<td>$Q_0(t) = 1.3$</td>
<td>$Q_0(t) = 1.3$</td>
</tr>
<tr>
<td></td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 2.0$</td>
<td>$A_0(t) = 5.0$</td>
</tr>
<tr>
<td>1</td>
<td>2.330</td>
<td>1.442</td>
<td>-1.176</td>
<td>1.823</td>
</tr>
<tr>
<td>5</td>
<td>4.910</td>
<td>4.677</td>
<td>4.556</td>
<td>4.476</td>
</tr>
<tr>
<td>10</td>
<td>5.210</td>
<td>5.182</td>
<td>5.173</td>
<td>5.151</td>
</tr>
<tr>
<td>15</td>
<td>5.220</td>
<td>5.218</td>
<td>5.217</td>
<td>5.215</td>
</tr>
<tr>
<td>20</td>
<td>5.221</td>
<td>5.221</td>
<td>5.221</td>
<td>5.220</td>
</tr>
</tbody>
</table>
Fig. 30  Convergence rate of Advertisement.
Fig. 31 Convergence rate of Advertisement.
Fig. 32 Convergence rate of Advertisement.
Fig. 33 Convergence rate of Advertisement.

\[ E = 0.4, \quad I_0(t) = 0.6 \]
\[ Q_0(t) = 1.3, \quad A_0(t) = 5.0 \]

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**9th, 10th, & OPT.**
The results starting with this trajectory were explored in detail with different value of \( c \). Tables 5 through 8 show the convergence rate of inventory, sales, profit function and advertisement, respectively, for the different values of \( c \).

Figures 10 through 12 show the convergence rate of inventory for different values of \( \varepsilon \). Similarly, Figs. 13 through 15 show the convergence rate of sales, Figs. 16 through 18 of profit function and Figs. 19 through 21 of advertisement for different values of \( \varepsilon \). The maximum value of \( \varepsilon \) that would lead to convergence in this case was found to be 0.7. \( \varepsilon = 1.0 \) would lead to exponential overflow in this situation. Another interesting point noted was that almost the same convergence rate was obtained with \( \varepsilon = 0.5 \) and with \( \varepsilon = 0.7 \). Thus a higher \( \varepsilon \) did not increase the convergence rate.

In another approach to this problem, a number of different starting trajectories for inventory, sales and advertisement were used. These are listed in Table 9. Set (1) of the various trajectories listed in Table 9 was explored in detail with different values of \( \varepsilon \).

Tables 10 through 13 show the convergence rate of inventory, sales, profit function and advertisement respectively, for different values of \( \varepsilon \). Figures 22 and 23 show the convergence rate of inventory for different values of \( \varepsilon \). Similarly Figs. 24 and 25 show the convergence rate of sales, Figs. 26 and 27 of profit function and Figs. 28 and 29 of advertisement for different values of \( \varepsilon \).

The remaining starting trajectories from Table 9, namely sets (1) through (5) were tried with \( \varepsilon = 0.4 \). Tables 14 through 17 list the convergence rate of advertisement for these trajectories. Figs. 30 through 33 show the convergence rate of advertisement for these different trajectories.
The starting trajectories (1) through (5) from Table 9 led to convergence almost in the same number of iterations. The maximum value of $\epsilon$ that would lead to convergence was found to be 0.7 in this case also.

Thus it is seen that the problem is very stable and that the optimum can be reached almost with any reasonable values of the starting trajectories.

Another computational feature that was encountered in the solution of this problem was regarding the numerical solution of the differential equations. As their number increased it was found advisable to use the IBM subroutine "RKGS" for their numerical solution. However, this subroutine imposed the problem of accuracy which has to be specified by the user. This is the accuracy against which the results are checked after each integration step. If the accuracy is too low, the integration step size is halved and this continues until the specified accuracy is obtained. Thus, if the accuracy is not appropriate, the grid points may not be the ones desired by the user. The calculations of $R, S, and T$ should be done both in subroutine "FCT" and "OUTP". (See appendix 7.2) Also, to test the fact that this method would lead to convergence at the nearest stationary point regardless of whether it is a maximization or a minimization problem, the objective function was made negative and the same problem solved again. The results agree in both the cases. Thus whether a maximum or minimum will be reached all depends on the nature of the curve of the objective function.
3.3 A Chemical Manufacturing Problem with Advertisement

The Model

Figure 34 represents a chemical manufacturing process and stages 1 and 2 represent two reactors. The raw material entering the first reactor is a mixture of A and B. After the second stage, the product A and product B are separated, as is the remaining raw material, product C. Product B is the more valuable of the three products and, to enhance its sale, it has to be advertised. Also, to meet the fluctuations in its demand, a certain amount of inventory has to be kept. It shall be assumed that the demands for products A and C are unlimited.
Let $x_0$ and $y_0$ represent the concentration of A and B in the original raw material before it enters the first stage or reactor. Similarly, let $x_1$, $y_1$ and $x_2$, $y_2$ represent the concentrations of A and B before and after the second stage, respectively. To bring about this reaction, temperatures $T_1$ and $T_2$ have to be applied to the two reactors. The reactions in the reactor can be represented by the following equations:

Let $q = \text{flow rate}$

$v_1 = \text{volume of the first reactor}$

$v_2 = \text{volume of the second reactor}$.

Then,

$$\frac{dx_1}{dt} = \frac{dx_1}{v_1} = q(x_0 - x_1) - v_1 K_a x_1$$

$$\frac{dy_1}{dt} = \frac{dy_1}{v_1} = q(y_0 - y_1) - v_1 K_b y_1 + v_1 K_a x_1$$

$$\frac{dx_2}{dt} = \frac{dx_2}{v_2} = q(x_1 - x_2) - v_2 K_a x_2$$

$$\frac{dy_2}{dt} = \frac{dy_2}{v_2} = q(y_1 - y_2) - v_2 K_b y_2 + v_2 K_a x_2$$

where

$K_a = G_a \exp \left( -\frac{E_a}{RT_1} \right)$

$K_a = G_a \exp \left( -\frac{E_a}{RT_2} \right)$

$K_b = G_b \exp \left( -\frac{E_b}{RT_1} \right)$

$K_b = G_b \exp \left( -\frac{E_b}{RT_2} \right)$.
This completes the production part of the system. Now consider the inventory. The rate of change of inventory is the difference between the rate of production of B and its rate of sale. If \( I(t) \) represents the inventory at time \( t \), then

\[
\frac{dI(t)}{dt} = qy_2 - C_a K(t)
\]  

(83)

The sales equation is assumed similar to the problem in Para. 3.2.

\[
\frac{dK(t)}{dt} = [C + A(t)] \cdot K(t) \cdot \left[1 - \frac{K(t)}{N}\right]
\]  

(84)

Equations (79) through (84) represents the performance equations of the whole system under consideration.

This problem has six state variables, namely \( x_1, y_1, x_2, y_2, I(t) \), \( K(t) \) and three control variables namely \( T_1, T_2 \) and \( A(t) \).

The profit function can be formulated as:

Profit = (sales revenue from A,B,C)-(cost of holding the inventory for B) - (cost of advertising for B) - (cost of production)

Sales revenue from A, B and C is \( = C_1 q K(t) + C_2 q x_2 + C_3 q (1 - x_2 - y_2) \)

where, \( C_1, C_2, C_3 \) represent the unit sales prices for A, B, C respectively.

Cost of holding the inventory of B = \( C_I (I_M - I(t))^2 \) where \( I_M \) is the capacity of the warehouse and \( C_I \) = inventory carrying cost.

Cost of advertising = \( C_A A^2(t) K^2(t) \).

Cost of production comes from the fact that the two reactors have to be supplied with heat energy in order to obtain the desired temperature. Let \( C_T \) represent the cost of raising the reactor temperature by a unit degree. Then the cost of production becomes
where $T_{lm}$ is the temperature of the entering raw material. Thus the function to be maximized is

$$\begin{aligned}
J &= \int_{0}^{t_f} \left[ C_1 C_q K(t) + C_2 q x_2 + C_3 q (1-x_2-y_2) - C_1 [I_m - I(t)]^2 \
&- C_A A^2(t) K^2(t) - C_T [(T_{lm} - T_1)^2 + (T_1 - T_2)^2] \right] \, dt \\
&- C_A A^2(t) K^2(t) - C_T [(T_{lm} - T_1)^2 + (T_1 - T_2)^2] .
\end{aligned}$$

(85)

Recursive Relations

The necessary relations for the second variation can be obtained in the following manner. The various derivatives can be obtained as follows

$$\begin{align*}
\frac{\partial J}{\partial x_1} &= 0 \\
\frac{\partial J}{\partial y_1} &= 0 \\
\frac{\partial J}{\partial x_2} &= q(C_2 - C_3) \\
\frac{\partial J}{\partial y_2} &= -C_3 q \\
\frac{\partial J}{\partial I} &= 2C_1 (I_m - I(t)) \\
\frac{\partial J}{\partial K} &= C_1 C_q - C_A A^2(t) \cdot 2K(t)
\end{align*}$$
\[
\frac{\partial^2 J}{\partial x^2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2C_I & 0 \\
0 & 0 & 0 & 0 & 0 & -2C_A A^2(t)
\end{pmatrix}
\]

\[
\frac{\partial^2 J}{\partial \theta \partial x} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\frac{\partial J}{\partial \theta} = \begin{pmatrix}
2C_T (T_{1m} - T_1) + 2C_T (T_1 - T_2) \\
2C_T (T_1 - T_2) \\
-2C_A A(t) K^2(t)
\end{pmatrix}
\]

\[
\frac{\partial^2 J}{\partial \theta^2} = \begin{pmatrix}
0 & 2C_T & 0 \\
0 & 2C_T & 0 \\
0 & 0 & -2C_A K^2(t)
\end{pmatrix}
\]
\[
\frac{\partial f'}{\partial x} = \begin{bmatrix}
BM_1 & Ga \cdot EAT_1 & (q/V_2) & 0 & 0 & 0 \\
0 & BM_2 & 0 & q/V_2 & 0 & 0 \\
0 & 0 & BM_3 & Ga \cdot EAT_2 & 0 & 0 \\
0 & 0 & 0 & BM_4 & q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -C_q BM_5
\end{bmatrix}
\]

where,

\[
BM_1 = -(q/V_1) - Ga \cdot EAT_1, \quad EAT_1 = \exp\left(-\frac{EA}{RT_1}\right)
\]

\[
BM_2 = -(q/V_1) - Gb \cdot EBT_1, \quad EBT_1 = \exp\left(-\frac{EB}{RT_1}\right)
\]

\[
BM_3 = -(q/V_2) - Ga \cdot EAT_2, \quad EBT_1 = \exp\left(-\frac{EB}{RT_2}\right)
\]

\[
BM_4 = -(q/V_2) - Gb \cdot EBT_2, \quad EBT_1 = \exp\left(-\frac{EB}{RT_2}\right)
\]

\[
BM_5 = (C+A(t)) - Gb \cdot EBT_2
\]
\[
\left(\frac{\partial f'}{\partial x}\right)'
= \begin{pmatrix}
BM_1 & 0 & 0 & 0 & 0 & 0 \\
Ga \cdot EAT_1 & BM_2 & 0 & 0 & 0 & 0 \\
q/V_2 & 0 & BM_3 & 0 & 0 & 0 \\
0 & q/V_2 & Ga \cdot EAT_2 & BM_4 & 0 & 0 \\
0 & 0 & 0 & q & 0 & -c_q \\
0 & 0 & 0 & 0 & 0 & BM_5
\end{pmatrix}
\]

\[
\frac{\partial f'}{\partial \theta} = \begin{pmatrix}
FT_1 & FT_2 & 0 & 0 & 0 & 0 \\
0 & 0 & FT_3 & FT_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & FT_5
\end{pmatrix}
\]

where

\[
FT_1 = - \frac{Ea}{RT_1^2} G_x e - \frac{Ea}{RT_1}
\]

\[
FT_2 = - \frac{Eb}{RT_1^2} G_y e - \frac{Eb}{RT_1} + \frac{Ea}{RT_1^2} G_x e - \frac{Ea}{RT_1}
\]

\[
FT_3 = - \frac{Ea}{RT_2^2} G_x e - \frac{Ea}{RT_2}
\]
\[ \frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f_2}{\partial x^2} = \frac{\partial^2 f_3}{\partial x^2} = \frac{\partial^2 f_4}{\partial x^2} = \frac{\partial^2 f_5}{\partial x^2} = \left( \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \]

\[ \frac{\partial^2 f_6}{\partial x^2} = \left( \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) - \frac{2}{N} [C+A(t)] \]
\[ \frac{\partial f_1}{\partial x} = \begin{pmatrix} \mathbf{Bm}_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \frac{\partial^2 f_1}{\partial x \partial \theta} = \begin{pmatrix} \mathbf{G}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \frac{\partial f_2}{\partial x} = \begin{pmatrix} \mathbf{G}_a \cdot \mathbf{EAT}_1 \\ \mathbf{Bm}_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \frac{\partial^2 f_2}{\partial x \partial \theta} = \begin{pmatrix} -\mathbf{G}_1 & \mathbf{G}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \frac{\partial f_3}{\partial x} = \begin{pmatrix} \frac{q}{\mathbf{V}_2} \\ 0 \\ -\frac{q}{\mathbf{V}_2} - \mathbf{G}_a \cdot \mathbf{EAT}_2 \\ 0 \\ 0 \end{pmatrix} \quad \frac{\partial^2 f_3}{\partial x \partial \theta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{G}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]
\[
\frac{\partial^2 f_4}{\partial x \partial \theta} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -G_3 & G_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\frac{\partial^2 f_5}{\partial x \partial \theta} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\frac{\partial f_6}{\partial x} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\frac{\partial^2 f_6}{\partial x \partial \theta} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
where,

\[ G_1 = -\frac{EA}{RT_1} \cdot G_a \exp\left(-\frac{EA}{RT_1}\right) \]

\[ G_3 = -\frac{EA}{RT_2} \cdot G_a \exp\left(-\frac{EA}{RT_2}\right) \]

\[ G_2 = -\frac{EB}{RT_1} \cdot G_b \exp\left(-\frac{EB}{RT_1}\right) \]

\[ G_4 = -\frac{EB}{RT_2} \cdot G_b \exp\left(-\frac{EB}{RT_2}\right) \]

\[
\frac{af_1}{\theta} = \begin{pmatrix}
-\frac{E_a}{RT_1} \cdot G_a x_1 e^\frac{t}{T_1} & \frac{E_a}{RT_1} \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

\[
\frac{\partial^2 f_1}{\partial \theta^2} = \begin{pmatrix}
-G_1 x_1 \frac{2}{T_1} \cdot \frac{E_a}{RT_1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\frac{af_2}{\theta} = \begin{pmatrix}
-\frac{E_b}{RT_1} \cdot G_b y_1 e^\frac{t}{T_1} & \frac{E_b}{RT_1} \\
+\frac{E_a}{RT_1} \cdot G_a x_1 e^\frac{t}{T_1} & -\frac{E_a}{RT_1} \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

\[
\frac{\partial^2 f_2}{\partial \theta^2} = \begin{pmatrix}
-G_2 y_1 \frac{2}{T_1} \cdot \frac{E_b}{RT_1} & 0 & 0 \\
+G_1 x_1 \left(-\frac{2}{T_1} + \frac{E_a}{RT_1}\right) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
\[
\begin{align*}
\frac{\partial f_3}{\partial \theta} &= \begin{pmatrix} 0 & -\frac{E_a}{RT^2} \end{pmatrix} \begin{pmatrix} G_a x_2 e \\ 0 \end{pmatrix}
\end{align*}
\]
\[
\begin{align*}
\frac{\partial^2 f_3}{\partial \theta^2} &= \begin{pmatrix} 0 & -G_3(x_2)(\frac{2}{T^2} - \frac{E_a}{RT^2}) \\ 0 & 0 \end{pmatrix}
\end{align*}
\]
\[
\begin{align*}
\frac{\partial f_4}{\partial \theta} &= \begin{pmatrix} 0 & -\frac{E_b}{RT^2} \\ -\frac{E_b}{RT^2} & -\frac{E_a}{RT^2} \\ +\frac{E_a}{RT^2} G_a x_2 e & 0 \end{pmatrix}
\end{align*}
\]
\[
\begin{align*}
\frac{\partial^2 f_4}{\partial \theta^2} &= \begin{pmatrix} 0 & -G_4 y_2 \left(\frac{2}{T^2} - \frac{E_b}{RT^2}\right) \\ 0 & 0 \end{pmatrix}
\end{align*}
\]
\[
\begin{align*}
\frac{\partial f_5}{\partial \theta} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{align*}
\]
\[
\begin{align*}
\frac{\partial^2 f_5}{\partial \theta^2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]
\[
\frac{\partial f_6}{\partial \theta} = \left\{ \begin{array}{ccc}
0 \\
0 \\
K(t)(1 - \frac{\dot{K}(t)}{N})
\end{array} \right\} \quad \frac{\partial^2 f_6}{\partial \theta^2} = \left\{ \begin{array}{ccc}
0 \\
0 \\
0
\end{array} \right\}
\]

\[
P = \left( \begin{array}{cccccc}
P_{11} & P_{21} & P_{31} & P_{41} & P_{51} & P_{61} \\
P_{21} & P_{22} & P_{32} & P_{42} & P_{52} & P_{62} \\
P_{31} & P_{32} & P_{33} & P_{43} & P_{53} & P_{63} \\
P_{41} & P_{42} & P_{43} & P_{44} & P_{54} & P_{64} \\
P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} \\
P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66}
\end{array} \right)
\]

Now the expressions for \( R, S, T \) which are required for obtaining the second variational Equations (19), (28) and (48) may be determined.

Thus

\[
P = \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial f}{\partial \theta} \cdot P + \sum_{i=1}^{6} \dot{z}_i \frac{\partial^2 f_i}{\partial \theta \partial x}
\]

Let

\[
R = \left( \begin{array}{cccccc}
R_1 & R_4 & R_7 & R_{10} & R_{13} & R_{16} \\
R_2 & R_5 & R_8 & R_{11} & R_{14} & R_{17} \\
R_3 & R_6 & R_9 & R_{12} & R_{15} & R_{18}
\end{array} \right)
\]
where

\[ R_1 = P_{11} FT1 + P_{21} FT2 + z_1 G_1 - z_2 G_1 \]
\[ R_2 = P_{31} FT3 + P_{41} FT4 \]
\[ R_3 = P_{61} FT5 \]
\[ R_4 = P_{21} FT1 + P_{22} FT2 + z_2 G_2 \]
\[ R_5 = P_{32} FT3 + P_{42} FT4 \]
\[ R_6 = P_{62} FT5 \]
\[ R_7 = P_{31} FT1 + P_{32} FT2 \]
\[ R_8 = P_{33} FT3 + P_{43} FT4 + z_3 G_3 - z_4 G_3 \]
\[ R_9 = P_{63} FT5 \]
\[ R_{10} = P_{41} FT1 + P_{42} FT2 \]
\[ R_{11} = P_{43} FT3 + P_{44} FT4 + z_4 G_4 \]
\[ R_{12} = P_{64} FT5 \]
\[ R_{13} = P_{51} FT1 + P_{52} FT2 \]
\[ R_{14} = P_{53} FT3 + P_{54} FT4 \]
\[ R_{15} = P_{65} FT5 \]
\[ R_{16} = P_{61} FT1 + P_{62} FT2 \]
\[ R_{17} = P_{63} FT3 + P_{64} FT4 \]
\[ R_{18} = P_{66} FT5 + z_6 \left[ 1 - \frac{2K(t)}{N} \right] - 4C_A(t) K(t) \]

From Equation (46)

\[ S = \frac{\partial J}{\partial \theta} + \sum_{i=1}^{6} z_i \frac{\partial f_i}{\partial \theta} \]
Let

\[
S = \begin{bmatrix}
S_1 \\
S_2 \\
S_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2C_T(T_{1m} - T_1) + 2C_T(T_1 - T_2) + z_1 G_1 x_1 + (G_2 y_1 - G_1 x_1) z_2 \\
2C_T(T_1 - T_2) + z_3 x_2 G_3 + z_4 (y_2 G_4 - x_2 G_3) \\
z_6 K(t) [1 - \frac{K(t)}{N}] - 2C_A(t) K^2(t)
\end{bmatrix}
\]

From Equation 47

\[
T = \frac{\partial^2 f}{\partial \theta^2} + \sum_{i=1}^{6} z_i \frac{\partial^2 f_i}{\partial \theta^2}
\]

Let

\[
T = \begin{bmatrix}
DT_1 & DT_4 & DT_7 \\
DT_2 & DT_5 & DT_9 \\
DT_3 & DT_6 & DT_9
\end{bmatrix}
\]

where

\[
DT_1 = - z_1 G_1 x_1 \left( - \frac{2}{T_1} - \frac{E_a}{RT_1^2} \right) - G_2 z_2 y_1 \left( \frac{2}{T_1} - \frac{E_b}{RT_1^2} \right)
\]

\[
- z_2 G_1 x_1 \left( - \frac{2}{T_1} + \frac{E_a}{RT_1^2} \right)
\]

\[
DT_2 = - 2C_T
\]

\[
DT_3 = 0
\]
\[ DT4 = 2C_T \]
\[ DT5 = - 2C_T - z_3 G_3 x_2 \left( \frac{2}{T_2} - \frac{Ea}{RT_2} \right) - z_4 G_4 y_2 \left( \frac{2}{T_2} - \frac{Eb}{RT_2} \right) \]
\[ - G_3 z_4 x_2 \left( - \frac{2}{T_2} + \left( - \frac{2}{T_2} + \frac{Ea}{RT_2} \right) \right) \]

\[ DT6 = 0 \]
\[ DT7 = 0 \]
\[ DT8 = 0 \]
\[ DT9 = - 2C_A K^2(t) \]

The equations for \( \frac{dz}{dt} \), i.e. Equation (19), take the following form:

\[ \frac{dz_1}{dt} = - z_1 BM1 - z_2 G \ EAT1 - (q/V_2) z_3 \] (86)

\[ \frac{dz_2}{dt} = - z_2 BM2 - z_4 (q/V_2) \] (87)

\[ \frac{dz_3}{dt} = - q(C_2 - C_3) - z_3 BM3 - G \ EAT2 z_4 \] (88)

\[ \frac{dz_4}{dt} = C_3 q - BM4 z_4 - z_5 q \] (89)

\[ \frac{dz_5}{dt} = 2C_I (I_M - I(t)) \] (90)

\[ \frac{dz_6}{dt} = - C_1 q + C_a A^2(t) \ 2K(t) + z_5 C q - z_6 BM5 \] (91)
Let

\[
\begin{bmatrix}
R'TR_1 \\
R'TR_2 \\
R'TR_3 \\
R'TR_4 \\
R'TR_5 \\
R'TR_6
\end{bmatrix}
= \begin{bmatrix}
R'TR_1 & R'TR_2 & R'TR_3 & R'TR_4 & R'TR_5 & R'TR_6 \\
R'TR_2 & R'TR_7 & R'TR_8 & R'TR_9 & R'TR_{10} & R'TR_{11} \\
R'TR_3 & R'TR_8 & R'TR_{12} & R'TR_{13} & R'TR_{14} & R'TR_{15} \\
R'TR_4 & R'TR_9 & R'TR_{13} & R'TR_{16} & R'TR_{17} & R'TR_{18} \\
R'TR_5 & R'TR_{10} & R'TR_{14} & R'TR_{17} & R'TR_{19} & R'TR_{18} \\
R'TR_6 & R'TR_{11} & R'TR_{15} & R'TR_{18} & R'TR_{20} & R'TR_{21}
\end{bmatrix}
\]

Equation (28) now becomes

\[
\frac{dP_{11}}{dt} = -2P_{11}BM1 - 2P_{21}Ga\text{EAT1} - 2P_{31} \left(\frac{q}{V_2}\right) + RTR_{11} \tag{92}
\]

\[
\frac{dP_{21}}{dt} = -P_{21} \left(BM1 + BM2\right) - P_{22}Ga\text{EAT1} - \left(P_{32} + P_{41}\right) \left(\frac{q}{V_2}\right) + RTR_{21} \tag{93}
\]

\[
\frac{dP_{31}}{dt} = -P_{31} \left(BM1 + BM3\right) - P_{41}Ga\text{EAT2} - P_{32}Ga\cdot\text{EAT1} - P_{33} \left(\frac{q}{V_2}\right) + RTR_{31} \tag{94}
\]

\[
\frac{dP_{41}}{dt} = -P_{41} \left(BM1 + BM4\right) - P_{42}Ga\text{EAT1} - P_{43} \left(\frac{q}{V_2}\right) - P_{51}q + RTR_{41} \tag{95}
\]

\[
\frac{dP_{51}}{dt} = -P_{51}BM1 - P_{52}Ga\text{EAT1} - P_{53} \left(\frac{q}{V_2}\right) + RTR_{51} \tag{96}
\]

\[
\frac{dP_{61}}{dt} = P_{51}C_q - P_{61} \left(BM1 + BM5\right) - P_{62}Ga\text{EAT1} - P_{63} \left(\frac{q}{V_2}\right) + RTR_{61} \tag{97}
\]

\[
\frac{dP_{22}}{dt} = -2P_{22}BM2 - 2P_{42} \left(\frac{q}{V_2}\right) + RTR_{22} \tag{98}
\]

\[
\frac{dP_{32}}{dt} = -P_{32} \left(BM2 + BM3\right) - P_{42}Ga\text{EAT2} - P_{43} \left(\frac{q}{V_2}\right) + RTR_{32} \tag{99}
\]
\[
\frac{dp_{42}}{dt} = -P_{42} (BM2 + BM4) - P_{44} (q/V_2) - P_{52} q + RTR9 \quad (100)
\]

\[
\frac{dp_{52}}{dt} = -P_{52} BM2 - P_{54} (q/V_2) + RTR10 \quad (101)
\]

\[
\frac{dp_{62}}{dt} = P_{52} Cq - P_{62} (BM2 + BM5) - P_{64} (q/V_2) + RTR11 \quad (102)
\]

\[
\frac{dp_{33}}{dt} = -2P_{33} BM3 - 2P_{43} Ga EAT2 + RTR12 \quad (103)
\]

\[
\frac{dp_{43}}{dt} = -P_{43} (BM3 + BM4) - P_{44} Ga EAT2 - P_{53} q + RTR13 \quad (104)
\]

\[
\frac{dp_{53}}{dt} = -P_{53} BM3 - P_{54} Ga EAT2 + RTR14 \quad (105)
\]

\[
\frac{dp_{63}}{dt} = P_{53} Cq - P_{63} (BM3 + BM5) - P_{64} Ga EAT2 + RTR15 \quad (106)
\]

\[
\frac{dp_{44}}{dt} = -2P_{44} BM4 - 2P_{54} q + RTR16 \quad (107)
\]

\[
\frac{dp_{54}}{dt} = -P_{54} BM4 - P_{55} q + RTR17 \quad (108)
\]

\[
\frac{dp_{64}}{dt} = P_{54} Cq - P_{64} (BM4 + BM5) - P_{65} q + RTR18 \quad (109)
\]

\[
\frac{dp_{55}}{dt} = 2C_I + RTR19 \quad (110)
\]

\[
\frac{dp_{65}}{dt} = P_{55} Cq - P_{65} BM5 + RTR20 \quad (111)
\]

\[
\frac{dp_{66}}{dt} = 2z_6 [C + A(t)] + 2Ca (A^2(t)) + 2P_{65} Cq - 2P_{66} BM5 + RTR21 \quad (112)
\]
Equation (48) is given by

\[
\frac{dQ}{dt} = R' T^{-1} S + R' T^{-1} \left( \frac{\partial f'}{\partial q} \right) Q - \left( \frac{\partial f'}{\partial x} \right) Q
\]

where \( Q \) is six dimensional. Here all the terms were obtained by the matrix multiplication. The new control is calculated as given by Equation (50), i.e.

\[
\begin{bmatrix}
T_1 \\
T_2 \\
A(t)
\end{bmatrix}^{(j+1)} = \begin{bmatrix}
T_1 \\
T_2 \\
A(t)
\end{bmatrix}^{(j)} - \left( \epsilon T^{-1} (S + \frac{\partial f'}{\partial q} Q) \right)^{(j)} - (T^{-1} R)^j \cdot (x^{(j+1)} - x^{(j)})
\]
Table 18

Numerical Values of the Constants

**Set # 1**

\[
\begin{align*}
q &= 60 \\
C_q &= 1.0 \\
Im &= 20.0 \\
C_2 &= 0.0 \\
C_A &= 0.0002 \\
E_A &= 18000.0 \\
y_I &= 0.430 \\
v_1 &= 12.0 \\
C &= 1.0 \\
T_{1m} &= 340. \\
C_3 &= 0.0 \\
C_T &= 0.0005 \\
E_B &= 30000.0 \\
Ga &= 0.535 \times 10^{11} \\
v_2 &= 12.0 \\
N &= 100.0 \\
C_I &= 5.0 \\
R &= 2.000 \\
x_I &= 0.530 \\
Ga &= 0.461 \times 10^{18}
\end{align*}
\]

**Set # 2**

\[
\begin{align*}
q &= 60. \\
C_q &= 1.0 \\
Im &= 10. \\
C_2 &= 0. \\
C_A &= 0.01 \\
E_A &= 18000 \\
y_I &= 0.43 \\
v_1 &= 12.0 \\
C &= 1.0 \\
T_{1m} &= 340. \\
C_3 &= 0 \quad (\text{same as } C_3 = 0 \text{ in Set # 1}) \\
C_T &= 0.001 \\
E_B &= 30000 \quad (\text{same as } E_B = 30000 \text{ in Set # 1}) \\
Ga &= 0.535 \times 10^{11} \\
v_2 &= 12.0 \\
N &= 100 \quad (\text{same as } N = 100 \text{ in Set # 1}) \\
C_I &= 5.0 \\
R &= 2.0 \quad (\text{same as } R = 2.0 \text{ in Set # 1}) \\
x_I &= 0.53 \quad (\text{same as } x_I = 0.53 \text{ in Set # 1}) \\
Ga &= 0.461 \times 10^{18} \quad (\text{same as } Ga = 0.461 \times 10^{18} \text{ in Set # 1})
\end{align*}
\]

**Set # 3**

Same as Set # 2 except

- Im = 20.
- \( C_T = 0.0005 \)
- \( K(t_0) = 1.0 \)
Discussion

This particular problem reveals how the theoretical attractiveness of the second variation method is more than offset by both the complexity and by the number of equations to be integrated.

In this problem \((6+1)\) or seven equations are to be integrated in the forward direction and \((6+6(7)/2+6)\) or 33 equations in the backward direction. In addition, the calculations of \(R\), \(S\), and \(T\) are in terms of matrix multiplications and \(T^{-1}\) has to be calculated at each step of the integration in Equations (28) and (48).

This program was tried with three different sets of numerical values which are shown in Table 18. These values were taken from the solution of the same problem first by variation and quasilinearization respectively.

This problem was found to be unstable as far as its solution by the second variation is concerned. With all the various values tried, the program could make a complete iteration. However, it fails in the backward integration of the second variational equations because of exponential overflow.
4. CONCLUSION

The second variation method has been shown to be a fairly useful tool for attacking the complex practical optimization problems involving a fairly large number of variables. The convergence is very fast, provided the initial or starting guess is sufficiently close to the optimal trajectory. This, however, becomes more and more difficult when more than one control variable are involved. In that case, the number of combinations that could be used as the starting trajectory is quite large and makes the initial guess a difficult task. This can be overcome by using the first variation method for the first few iterations and then switching to the second variation method. This combination provides rapid convergence to the optimum from almost any realistic starting trajectory. The theoretical attractiveness of this method is removed by its disadvantages like the guess of the initial trajectory for the state variables in addition to that of control variables. Also the number of equations and their complexity make the use of this technique tedious.

The first variation method, of which the second variation method is a natural evolution, should be used in combination with the second variation. The first variation method, unlike the second variation, will approach optimum from almost any realistic starting trajectory. The results of the first variation method could then be used as the starting trajectories for the second variation. In this way, the convergence problem of the second variation can be partly overcome. This combination provides a rapid convergence from almost any realistic starting trajectory for most engineering problems. While evaluating the merits and demerits of this technique, it should be borne in mind that no single optimization
technique is suitable for all classes of problems that will be encountered. Each technique will be most efficient only for a particular type or types of problems. It is left to the decision of the engineer to select any one or a combination of these techniques for the problem he is facing.
5. ACKNOWLEDGMENT

The author is deeply indebted to Dr. E.S. Lee, major professor, for his patient guidance, valuable comments, constructive criticism and deep interest in preparing this thesis.
6. BIBLIOGRAPHY

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7. APPENDIX

7.1 Computer Program for the Inventory Model

7.2 Computer Program for the Inventory and Advertising Model

7.3 Computer Program for the Chemical Manufacturing Problem with Advertisement
C SECOND VARIATIONAL GRADIENT TECHNIQUE—RANGNEKAR

100 FORMAT (1H-, 'NO.', X1, X2, \( \theta \), \( T \) )

101 FORMAT (1H-, (1H, 1, 2, X, E15.8, 2, X, E15.8, 2, X, E15.8, 2, X, E15.8, 2, X, E15.8, 2

102 FORMAT (9F9.7)

103 FORMAT (1H, 9(F7.3))

104 FORMAT (1H-, '**********************************', 11)

ITMAX=58
CP=-.1
PRINT 103, PA, CP, XM, CI, D, A, X1(I), X2(I)
CP=.001
DO 12 I=1,101
I(I)=9.
K(I)=5.
DO 1 N=1, ITMAX
L(N)=N
DO 6 I=1,101
M(I)=I
C********** FOR ward INTEGRATION OF Y1
P1(I)=D*(T(I)-A-B*D*(I-1))
P2(I)=D*(T(I)-A-B*(D*(I-1)+.5*D))
P3(I)=P2(I)
P4(I)=D*(T(I)-A-B*(D*(I-1)+D))
DX1(I)=(1./6.)*(P1(I)+2.*P2(I)+2.*P3(I)+P4(I))
X1(I+1)=X1(I)+DX1(I)
C********** FOR ward INTEGRATION OF X2
Q1(I)=D*(CI*(XM-X(I))*XM-X(I))+(CP*EXP((PA-T(I)))*(PA-T(I))))
Q2(I)=D*(CI*(XM-X(I))+.5*P1(I))**2+CP*EXP((PA-T(I))**2))
Q3(I)=D*(CI*(XM-X(I))+.5*Q2(I))**2+CP*EXP((PA-T(I))**2))
Q4(I)=Q3(I)+P4(I)
Q5(I)=Q4(I)
Q6(I)=Q5(I)
Q7(I)=Q6(I)
Q8(I)=Q7(I)
Q9(I)=Q8(I)
Q10(I)=Q9(I)
Q11(I)=Q10(I)
Q12(I)=Q11(I)
Q13(I)=Q12(I)
Q14(I)=Q13(I)
Q15(I)=Q14(I)
Q16(I)=Q15(I)
Q17(I)=Q16(I)
Q18(I)=Q17(I)
Q19(I)=Q18(I)
Q20(I)=Q19(I)
Q21(I)=Q20(I)
Q22(I)=Q21(I)
Q23(I)=Q22(I)
Q24(I)=Q23(I)
Q25(I)=Q24(I)
Q26(I)=Q25(I)
Q27(I)=Q26(I)
Q28(I)=Q27(I)
Q29(I)=Q28(I)
Q30(I)=Q29(I)
Q31(I)=Q30(I)
Q32(I)=Q31(I)
Q33(I)=Q32(I)
Q34(I)=Q33(I)
Q35(I)=Q34(I)
Q36(I)=Q35(I)
Q37(I)=Q36(I)
Q38(I)=Q37(I)
Q39(I)=Q38(I)
Q40(I)=Q39(I)
Q41(I)=Q40(I)
Q42(I)=Q41(I)
Q43(I)=Q42(I)
Q44(I)=Q43(I)
Q45(I)=Q44(I)
Q46(I)=Q45(I)
Q47(I)=Q46(I)
Q48(I)=Q47(I)
Q49(I)=Q48(I)
Q50(I)=Q49(I)
Q51(I)=Q50(I)
Q52(I)=Q51(I)
Q53(I)=Q52(I)
Q54(I)=Q53(I)
Q55(I)=Q54(I)
Q56(I)=Q55(I)
Q57(I)=Q56(I)
Q58(I)=Q57(I)
Q59(I)=Q58(I)
Q60(I)=Q59(I)
Q61(I)=Q60(I)
Q62(I)=Q61(I)
Q63(I)=Q62(I)
Q64(I)=Q63(I)
Q65(I)=Q64(I)
Q66(I)=Q65(I)
Q67(I)=Q66(I)
Q68(I)=Q67(I)
Q69(I)=Q68(I)
Q70(I)=Q69(I)
Q71(I)=Q70(I)
Q72(I)=Q71(I)
Q73(I)=Q72(I)
Q74(I)=Q73(I)
Q75(I)=Q74(I)
Q76(I)=Q75(I)
Q77(I)=Q76(I)
Q78(I)=Q77(I)
Q79(I)=Q78(I)
Q80(I)=Q79(I)
Q81(I)=Q80(I)
Q82(I)=Q81(I)
Q83(I)=Q82(I)
Q84(I)=Q83(I)
Q85(I)=Q84(I)
Q86(I)=Q85(I)
Q87(I)=Q86(I)
Q88(I)=Q87(I)
Q89(I)=Q88(I)
Q90(I)=Q89(I)
Q91(I)=Q90(I)
Q92(I)=Q91(I)
Q93(I)=Q92(I)
Q94(I)=Q93(I)
Q95(I)=Q94(I)
Q96(I)=Q95(I)
Q97(I)=Q96(I)
Q98(I)=Q97(I)
Q99(I)=Q98(I)
Q100(I)=Q99(I)
DO 7 I=1, 100
K=101
Z(I(101))=0.
P(I(101))=0.
C(I(101))=0.
C*M1=-D*2.*CI*(XM-X(K+1-I))
43  \( ZM2 = ZM1 \)
44  \( ZM3 = ZM1 \)
45  \( ZM4 = ZM1 \)
46  \( Z(K+1-I) = Z(K+1-I) + (1./6.)* \) (\( ZM1 + 2.*ZM2 + 2.*ZM3 + ZM4 \) )
47  \( PM1 = D*(-2.*CI + TX(K+1-I)*(P(K+1-I)**2)) \)
48  \( PM2 = D*(-2.*CI + TX(K+1-I)*(P(K+1-I)*PM1/2.**2)) \)
49  \( PM3 = D*(-2.*CI + TX(K+1-I)*(P(K+1-I)*PM2/2.**2)) \)
50  \( PM4 = D*(-2.*CI + TX(K+1-I)*(P(K+1-I)*PM3**2)) \)
51  \( P(K-I) = P(K+1-I) + (1./6.)* \) (\( PM1 + 2.*PM2 + 2.*PM3 + PM4 \) )
52  \( DQ/DT = (P(Z+Q-S))/TX \)
53  \( QM1 = D*((P(K+1-I)*Z(K+1-I) + Q(K+1-I) - S(K+1-I)))/TX(K+1-I)) \)
54  \( QM2 = D*((P(K+1-I)*5*PM1 + Z(K+1-I) + 5*ZM1 + Q(K+1-I) + 5*QM1 - S(K+1-I))/TX(K+1-I)) \)
55  \( QM3 = D*((P(K+1-I)*5*PM2 + Z(K+1-I) + 5*ZM2 + Q(K+1-I) + 5*QM2 - S(K+1-I))/TX(K+1-I)) \)
56  \( QM4 = D*((P(K+1-I)*5*PM3 + Z(K+1-I) + 5*ZM3 + Q(K+1-I) + 5*QM3 - S(K+1-I))/TX(K+1-I)) \)
57  \( C(K-I) = Q(K+1-I) + (1./6.)* \) (\( QM1 + 2.*QM2 + 2.*QM3 + QM4 \) )
58  \( DO 8 I = 1,101 \)
59  \( 8 T1(I) = T(I) - EP*((Z(I) - S(I) + Q(I))/TX(I)) - (P(I)/TX(I))*(X1(I) - X(I)) \)
60  \( PRINT 100 \)
61  \( PRINT 101, (M(I), X1(I), X2(I), Z(I), P(I), Q(I), T1(I), TX(I)), I = 1,101 \)
62  \( DO 9 I = 1,101 \)
63  \( 9 X(I) = X1(I) \)
64  \( 1 CONTINUE \)
65  \( STOP \)
66  \( END \)
--- READ THE VARIOUS CONSTANTS ---

100 FORMAT (9F8.3)
REAC 100, A, B, C, AN, F, CI, PI, CA, EP

101 FORMAT (214)
READ 101, NK, ITMAX

102 FORMAT ('FOLLOWING VALUES OF THE VARIOUS CONSTANTS ARE READ IN')
103 FORMAT ('OA=',F8.3,' B=',F8.3,' C=',F8.3,' AN
104 FORMAT ('OF=',F8.3,' CI=',F8.3,' PI=',F8.3,' LA
105 FORMAT ('OEP=',F8.3,' NK=',I4,' ITMAX=',I4)

PRINT 102
PRINT 103, A, B, C, AN
PRINT 104, F, CI, PI, CA
PRINT 105, EP, NK, ITMAX

--- VALUES OF THE STATE AND CONTROL VARIABLES AT 101 GRID POINTS ---

200 FORMAT ('THE FOLLOWING VALUES OF STATE AND CONTROL VARIABLES AT 1
101 GRID POINTS ARE READ IN')

PRINT 200
PRINT 201 FORMAT (12F6.3)
READ 201, (YO(I,NM), NM=1,NK)
READ 201, (YO(2,NM), NM=1,NK)
READ 201, (AT(NM), NM=1,NK)

203 FORMAT ('NO. INVENTORY SALES ADVT.')
PRINT 203

202 FORMAT (1H-I, 1H, 1H, 1H, 1H, 1H, 1H, 1H, 1H, 1H)
PRINT 202, (NM, YO(1,NM), YO(2,NM), AT(NM), NM=1,NK)

--- MAIN DO LOOP FOR ITERATIONS ---

DO 2 IJ=1, ITMAX

3 FORMAT ('O#################################################### ITERATION
1 NO. I, I4, '####################################################')
PRINT 3, IJ

--- VARIOUS PARAMETERS FOR FORWARD INTEGRATION ---

PRMT(1)=0.
PRMT(2)=1.
PRMT(3)=.01
PRMT(4)=.001
NDIM=3
DERY(1)=NDIM
DERY(1)=1./DERY(1)
DO 4 I=2, NDIM
37  
38  \text{DERY}(1)=\text{DERY}(1)
39  
40  Y(1)=.2
41  Y(2)=.2
42  Y(3)=0.
43  K=1
44  \text{--- CALL RKGS FOR THE FORWARD INTEGRATION ---}
45  \text{CALL RKGS(PRMT, Y, DERY, NDIM, IHLF, FCT, OUTP, AUX)}
46  \text{--- VARIOUS PARAMETERS FOR THE BACKWARD INTEGRATION ---}
47  \text{PRMT}(1)=1.
48  \text{PRMT}(2)=0.
49  \text{PRMT}(3)=-.01
50  \text{PRMT}(4)=.01
51  NDIM=7
52  \text{DERY}(1)=\text{NDIM}
53  \text{DERY}(1)=1./\text{DERY}(1)
54  \text{DO 5 I=2,NDIM}
55  \text{5 DERY}(I)=\text{DERY}(1)
56  \text{DO 6 I=1,7}
57  \text{6 Y(I)=0.}
58  K=2
59  \text{--- CALL RKGS FOR BACKWARD INTEGRATION ---}
60  \text{CALL RKGS(PRMT, Y, DERY, NDIM, IHLF, FCT, OUTP, AUX)}
61  \text{DO 7 L=1,NK}
62  \text{YO(1,L)=Yl(1,L)}
63  \text{YO(2,L)=Yl(2,L)}
64  \text{AT(L)=ATNEW(L)}
65  \text{2 CONTINUE}
66  \text{STOP}
67  \text{END}
SUBROUTINE FCT(X,Y,DERY)
DIMENSION PRMT(10),DERY(10),AUX(8,12),YO(2,101),YL(9,101),Y(10),AT(1(101),ATNEW(101))
COMMON Y1,AT,ATNEW,VO,A,B,C,AN,F,CI,PI,CA,K,NK,EP,R1,R2,S,T,J,N
C DEPENDING ON THE VALUE OF K, EITHER THE FIRST PART OR THE SECOND PART
C OF THIS SUBROUTINE IS USED FOR THE FORWARD AND THE BACKWARD
C INTEGRATION RESPECTIVELY.
C --- THIS PART OF SUBROUTINE IS FOR FORWARD INTEGRATION ONLY
C IF (K.EQ.2) GO TO 10
C IF (X.NE.0) GO TO 11
J=1
C Y(1),Y(2) DENOTE X AND Q IN THE PROBLEM
C INDEPENDENT VARIABLE T IN THE ORIGINAL EGNs. IS DENOTED BY X IN THE
C PROGRAM
11 DERY(1)=A*B*X-Y(2)
7C DERY(2)=Y(2)*(C+AT(J))*(1.-Y(2)/AN)
7C DERY(3)=Y(2)*F-CI*(PI-Y(1)**2)-CA*Y(2)*(AT(J)**2)
7C RETURN
C --- FIRST PART FOR FORWARD INTEGRATION ENDS
C --- SECOND PART FOR BACKWARD INTEGRATION
C 10 IF(X.NE.1) GO TO 12
11 N=NK
12 R1=Y(4)*Y1(2,N)/(1.-Y1(2,N)/AN)
13 R2=-2.*CA*AT(N)+Y(5)*Y1(2,N)*(1.-Y1(2,N)/AN)+Y(2)*Y(1)-2.*Y1(2,N)/AN)
14 S=-2.*CA*Y1(2,N)*AT(N)+Y(2)*Y1(2,N)*(1.-Y1(2,N)/AN)
15 T=-2.*CA*Y1(2,N)
16 DERY(1)=S-2.*CA*Y1(2,N)*AT(N)+Y(2)*Y1(2,N)*(1.-Y1(2,N)/AN)
17 DERY(2)=F+CA*AT(N)**2)+Y(1)-Y(2)*(C+AT(N))*Y1(2,N)/(1.-2.*Y1(2,N)/AN)
C --- BACKWARD INTEGRATION OF DP/DT ------- 3 EQUATIONS
18 DERY(3)=2.*CI+R1**2)*T
19 DERY(4)=Y(3)-Y(4)*(C+AT(N))*(1.-2.*Y1(2,N)/AN)+R1*R2*T
20 DERY(5)=(2.*Y(2)/AN)**2*(C+AT(N)+2.*Y(4)-2.*Y(5)*(C+AT(N))*(1.-2.*Y1(2,N)/AN)
C --- BACKWARD INTEGRATION OF DQF/DT -------
C --- THE SCALAR FUNCTION Q IN THE ORIGINAL DERIVATION IS DENOTED BY Y(6)
C AND Y(7) RESP., IN THIS PROGRAM
24 DERY(6)=(R1*S)/T+(R1/T)*Y(7)*Y1(2,N)*(1.-Y1(2,N)/AN)+Y(7)
25 DERY(7)=(R2*S)/T+(R2/T)*Y(7)*Y1(2,N)*(1.-Y1(2,N)/AN)-Y(7)*Y1(2,N)/AN)
26 RETURN
C --- SECOND PART ENDS
C END
SUBROUTINE OUTF(X,Y,DERY,HLF,NDIM,PRMT)

DIMENSION PRMT(10), CERY(10), AUX(8,12), YO(2,101), Y1(9,101), Y(10), AT(101), ATNEW(101)

COMMON Y1, AT, ATNEW, YO, A, B, C, AN, F, CI, PI, CA, K, NK, EP, R1, R2, S, T, J, N

IF(K.EQ.2) GO TO 2C

C ---- THIS PART OF THE SUBROUTINE IS FOR THE FORWARD INTEGRATION ONLY

IF (X.NE.0) GO TO 21

J=0

23 FORMAT ('- NO. GRID PT. INVENTORY SALES
1 PROFIT ADVT.')

PRINT 23

21 J=J+1

C ---- STORING THE VALUES OF STATE VARIABLES AT EACH GRID POINT, TO BE
C USED IN THE SECOND PART OF THIS SUBROUTINE FOR THE CALCULATION OF
C THE NEW VALUES OF THE CONTROL VARIABLE AT 101 GRID POINTS

DO 22 M=1,2

22 Y1(M,J)=Y(M)

ABC=-Y(3)

24 FORMAT (1H14,4X,F6.2,5X,4(E12.4,7X))

PRINT 24, J, X, Y(1), Y(2), ABC, AT(J)

2C IF (J.EQ.101) PRMT(5)=1.

RETURN

C ----- FIRST PART FOR FORWARD INTEGRATION ENDS

C ----- SECOND PART FOR BACKWARD INTEGRATION ONLY

20 IF (X.NE.1) GO TO 25

306 FORMAT ('-NO. GRID PT. Z1 Z2 P1 P2 QF1 QF2 S T')

PRINT 306

N=NK+1

25 N=N-1

R1=Y(4)*Y1(2,N)*((1.-Y1(2,N)/AN)

R2=-2.*CA*AT(N)+Y(5)*Y1(2,N)*((1.-Y1(2,N)/AN)+Y(2)*((1.-2.*Y1(2,N)/AN))

S=-2.*CA*Y1(2,N)*AT(N)+Y(2)*Y1(2,N)*((1.-Y1(2,N)/AN)

T=-2.*CA*Y1(2,N)

C ----- BACKWARD INTEGRATION OF DZ/DT ------ 2 EQUATIONS

C ----- CALCULATION OF THE NEW VALUE OF THE CONTROL VARIABLE AT

C N TH GRID POINT

30 XX1=Y1(1,N)-YO(1,N)

XX2=Y1(2,N)-YO(2,N)

ATNEW(N)=AT(N)-(EP/T)*((S+Y(7)*Y1(2,N)*((1.-Y1(2,N)/AN)))-(R1*XX1+R2*XX2)/T

308 FORMAT (1H14,1X,F5.2,1X,9(E12.4,1X))

PRINT 308, N, X, (Y(IM), IM=1,7), S, T

2C IF (N.EQ.1) PRMT(5)=1.

310 RETURN

C ----- SECOND PART FOR THE BACKWARD INTEGRATION ENDS

END
SUBROUTINE RKGS(PRMT,Y,DERY,NDIM,HLF,FCT,OUTP,AUX)

DIMENSION Y(1),DERY(1),AUX(8,1),A(4),B(4),C(4),PRMT(1)

X=PRMT(1)
H=PRMT(3)
PRMT(5)=0.
CALL FCT(X,Y,DERY)

PREPARATIONS FOR RUNGE-KUTTA METHOD
2 A(1)=.5
A(2)=.2928932
A(3)=1.707107
A(4)=.1666667
B(1)=2.
B(2)=1.
B(3)=1.
B(4)=2.
C(1)=.5
C(2)=.2928932
C(3)=1.707107
C(4)=.5

PREPARATIONS OF FIRST RUNGE-KUTTA STEP
DO 3 I=1,NDIM
AUX(1,1)=Y(I)
AUX(2,1)=DERY(I)
AUX(3,1)=0.
3 AUX(6,1)=0.

RECORDING OF INITIAL VALUES OF THIS STEP
7 CALL CUTP(X,Y,DERY,IREC,NDIM,PRMT)
IF(PRMT(5))40,8,40

START OF INNERMOST RUNGE-KUTTA LOOP
8 J=1
10 AJ=A(J)
BJ=B(J)
CJ=C(J)
DO 11 I=1,NDIM
R1=H*DERY(I)
R2=AJ*(R1-BJ*AUX(6,I))
Y(I)=Y(I)+R2
R2=R2+R2+R2
11 AUX(6,I)=AUX(6,I)+R2-CJ*R1
12 J=J+1
IF(J-4)12,15,15
IF(J-3)13,14,13
13  X = X + .5 * H
14  CALL FCT(X, Y, DERY)
15  GOTO 10
   C   END OF INNERMOST RUNGE-KUTTA LOOP
   C
16  DO 29 I = 1, NDIM
17     AUX(1, I) = Y(I)
18     AUX(2, I) = DERY(I)
19   29 AUX(6, I) = AUX(3, I)
20    CALL CUTP(X, Y, DERY, IHLF, NDIM, PRMT)
21    IF(PRMT(5)) 40, 30, 40
22   30 DO 31 I = 1, NDIM
23     Y(I) = AUX(1, I)
24   31 DERY(I) = AUX(2, I)
25  40 RETURN
26  END
38 READ 60, (AT(I), I = 1, NK)
39 DU 515 JK = 1, 101
40 515 Y0(6, JK) = Y0(6, JK) / 100.
41 63 FORMAT (' FOLLOWING VALUES OF THE 6 STATE VARIABLES AND 3 CONTROL
1 VARIABLES ARE READ IN')
42 PRINT 63
43 61 FORMAT (1H , (1H , I3, 1X, 9(F10.5, 1X)) )
44 PRINT 61, (I, Y0(I, I), Y0(2, I), Y0(3, I), Y0(4, I), Y0(5, I), Y0(6, I), T(I))
45 IF AT(I), AT(I), 1 = 1, NK)

C ---------------- MAIN DO LOOP FOR ITERATIONS ----------------

45 DO 100 IJ = 1, ITMAX
46 200 FORMAT (' ', I4, ' ')
47 PRINT 200, IJ
48 PRMT(1) = 0.0
49 PRMT(2) = 1.
50 PRMT(3) = .01
51 PRMT(4) = .1
52 NDIM = 7
53 DERY(1) = NDIM
54 DERY(1) = 1.0 / DERY(1)
55 DO 1 I = 2, NDIM
56 DERY(I) = DERY(1)
57 Y(1) = .53
58 Y(2) = .43
59 Y(3) = .53
60 Y(4) = .43
61 Y(5) = 8.
62 Y(6) = .01
63 Y(7) = 0
64 KSL = 1
65 CALL RKGS (PRMT, Y, DERY, NDIM, IHLF, FCT, OUTP, AUX)
66 PRMT(1) = 1.
67 PRMT(2) = 0.
68 PRMT(3) = -.01
69 PRMT(4) = 10.
70 NDIM = 33
71 DERY(1) = NDIM
72 DERY(1) = 1.0 / DERY(1)
73 DO 3 I = 2, NDIM
74 DERY(I) = DERY(1)
75 DO 4 I = 1, 33
76 4 Y(I) = 0.
77 KSL = 2
78 CALL RKGS (PRMT, Y, DERY, NDIM, IHLF, FCT, OUTP, AUX)
79 DO 120 L = 1, NK
80 Y0(1, L) = Y1(1, L)
81 Y0(2, L) = Y1(2, L)
82 Y0(3, L) = Y1(3, L)
83 Y0(4, L) = Y1(4, L)
84 \[ Y_0(5,L) = Y_1(5,L) \]
85 \[ Y_0(6,L) = Y_1(6,L) \]
86 \[ T_1(L) = T_{1\text{NEW}}(L) \]
87 \[ T_2(L) = T_{2\text{NEW}}(L) \]
88 \[ 120 \quad A_1(L) = A_{1\text{NEW}}(L) \]
89 \[ 100 \quad \text{CONTINUE} \]
90 \[ \text{STOP} \]
91 \[ \text{END} \]
SUBROUTINE FCT(X,Y,DERY)
DIMENSION PRMT(10), Y(42), DERY(42), AUX(8,43), Y1(40,202), Y2(202), AT(202), YU(6,202), A(10), L(1
2), M(10)

COMMON Y1, T1, NEW, T2, ATNEW, T12, T2(202), AT(202), YU(6,202), A(10), L(1
2), M(10)

COMMON Y1, T1, NEW, T2, ATNEW, T12, T2(202), AT(202), YU(6,202), A(10), L(1
2), M(10)

42 DERY(1)=DIS/V1*(X-Y(1))-GA*EXP(-EA/(R*T1(J)))*Y(1)
1CC DERY(2)=(DIS/V2)*(Y(1)-Y(2))-GA*EXP(-EB/(R*T1(J)))*Y(2)+GA*EXP(-EA/(R*T1(J)))*Y(1)
1C1 DERY(3)=(DIS/V2)*(Y(1)-Y(3))-GA*EXP(-EA/(R*T2(J)))*Y(3)
1C2 DERY(4)=(DIS/V2)*(Y(2)-Y(4))-GA*EXP(-EB/(R*T2(J)))*Y(4)+GA*EXP(-EA/(R*T2(J)))*Y(3)
1C3 DERY(5)=DIS*(Y(4)-CQ*Y(6))
1C4 DERY(6)=(C+AT(J))*Y(6)-((Y(6)*2)/AN)
1C5 DERY(7)=CQ*C1*Y(6)+C2*DIS*(Y(3)+C3*DIS*(1-Y(3)-Y(4)))-C1*((AIM-Y(5)-1)*C+AT(J)*Y(5)-Y(6))
1C6 RETURN
1C7 25 IF (X.NE.0) GO TO 47
1C8 N=NK
1C9 47 CALL CALCL(X,Y,DERY)

C ------------------ INTEGRATION OF DZ/DT ------------------
11C DERY(1)=-B*M1*Y(1)-Y(2)*GA*EAT1-(DIS/V2)*Y(3)
111 DERY(2)=-B*M2*Y(2)-Y(4)*(DIS/V2)
112 DERY(3)=DIS*(C3-C2)-B*M3*Y(3)-Y(4)*GA*EAT2
113 DERY(4)=C3*DIS-Y(4)*B*M4-DIS*Y(5)
114 DERY(5)=2.*C1*(Y15,N)-AIM
115 DERY(6)=C1*CQ+2.*C1*(AT(N)**2)*Y1(6,N)*Y(6)*CQ*(C+AT(N))*Y(1)**2*Y(6,N)/AN)

C ------------------ INTEGRATION OF DP/DT ------------------
116 DERY(7)=-2.*Y(7)*(B*M1-2.*Y(8)*GA*EAT1-2.*Y(9)*(DIS/V2)+RTR1
117 DERY(8)=-Y(8)*(B*M1+B*M2)-Y(13)*GA*EAT1-(Y(14)+Y(10))*(DIS/V2)+RTR2
118 DERY(9)=-Y(9)*(B*M1+B*M3)-Y(10)*GA*EAT2-Y(14)*GA*EAT1-Y(18)*(DIS/V2)+RTR3
119 DERY(10)=-Y(10)*(B*M1+B*M4)-Y(15)*GA*EAT1-Y(19)*(DIS/V2)-Y(11)*(DIS/RTR4
12C DERY(11)=-Y(11)*(B*M1-16)*GA*EAT1-Y(20)*(DIS/V2)+RTR5
121 DERY(12)=-Y(12)*(CQ-Y(12))*(B*M1+B*M5)-Y(17)*GA*EAT1-Y(21)*(DIS/V2)+RTR6
122 DERY(13)=-2.*Y(13)*(B*M2-2.*Y(15)*(DIS/V2)+RTR7
123 DERY(14)=-Y(14)*(B*M2+B*M3)-Y(15)*GA*EAT2-Y(19)*(DIS/V2)+RTR8
124 DERY(15)=-Y(15)*(B*M2+B*M4)-Y(22)*(DIS/V2)-Y(16)*(DIS+RTR9
DERY(16) = -Y(16) * BM2 - Y(23) * (DIS/V2) + RTR10
DERY(17) = Y(16) * CQ - Y(17) * (BM2 + BM5) - Y(24) * (DIS/V2) + RTR11
DERY(18) = -2 * Y(18) * BM3 - 2 * Y(19) * GA * EAT2 + RTR12
DERY(19) = -Y(19) * (BM3 + BM4) - Y(22) * GA * EAT2 - Y(20) * DIS + RTR13
DERY(20) = -Y(20) * BM3 - Y(23) * GA * EAT2 + RTR14
DERY(21) = Y(20) * CQ - Y(21) * (BM3 + BM5) - Y(24) * GA * EAT2 + RTR15
DERY(22) = -2 * Y(22) * BM4 - 2 * Y(23) * DIS + RTR16
DERY(23) = -Y(23) * BM4 - Y(25) * DIS + RTR17
DERY(24) = Y(23) * CQ - Y(24) * (BM4 + BM5) - Y(26) * DIS + RTR18
DERY(25) = RTR19 + 2 * C1
DERY(26) = Y(25) * CQ - Y(26) * BM5 + RTR20
DERY(27) = 2 * Y(26) * CN + 2 * CA * (AT(N) / AN + 2 * CA * (AT(N) + 2) + 2 * Y(26) * CO - 2 * Y(27) * BM5 + RTR21

C --------------- INTEGRATION OF DQ/DT ---------------

DERY(28) = RTS1 + RTFQ1 - FQ1
DERY(29) = RTS2 + RTFQ2 - FQ2
DERY(30) = RTS3 + RTFQ3 - FQ3
DERY(31) = RTS4 + RTFQ4 - FQ4
DERY(32) = RTS5 + RTFQ5 - FQ5
DERY(33) = RTS6 + RTFQ6 - FQ6
RETURN
END
SUBROUTINE OUTP(X, Y, DERY, IHLF, NDIM, PRMT)
DIMENSION PRMT(10), Y(42), DERY(42), AUX(8, 43), Y1(40, 202), T1NEW(202)
T2NEW(202), ATNEW(202), T1(202), T2(202), AT(202), YD(6, 202), A(10)
COMMON Y1, T1NEW, T2NEW, ATNEW, T1, T2, AT, NK, XX1, XX2, XX3, XX4, XX5, XX6, Y1, FT1, FT2, FT3, FT4, FT5, S1, S2, S3, R1, R2, R3, R4, R5, R6, R7, R8, R9, R10, R11, R12, R13, R14, R15, R16, R17, R18, A, J, N, EP, KSL, R, DIS, V1, V2, C0, C1, AN, AIM, T3, M, C1, C2, C3, CT, CA, CT, EA, EB, XI, Y1, GA, GB, RTR1, RTR2, RTR3, RTR4, RTR5, RTR6, RTR7, RTR8, RTR9, RTR10, RTR11, RTR12, RTR13, RTR14, RTR15, RTR16, RTR17
COMMON RTR18, RTR19, RTR20, RTR21, RTS1, RTS2, RTS3, RTS4, RTS5, RTS6, RTRQ1, RTRQ2, RTRQ3, RTRQ4, RTRQ5, RTRQ6, RTRQ7, RTRQ8, RTRQ9, RTRQ10, RTRQ11, RTRQ12, RTRQ13, RTRQ14, RTRQ15, RTRQ16, RTRQ17
139 IF (KSL.EQ.2) GO TO 36
140 IF (X.NE.0) GO TO 31
141 J = 0
142 31 J = J + 1
143 DO 32 K = 1, 6
144 32 Y1(K, J) = Y(K)
145 15 FORMAT (1H, 13, 3X, F6.3, 2X, 7(E12.4, 3X))
146 16 PRINT 15, J, X, (Y(I), I = 1, 7)
147 IF (J.EQ.NK) PRMT(5) = 1.
148 RETURN
149 36 IF (X.NE.1) GO TO 33
150 N = NK + 1
151 33 N = N - 1
152 CALL CALCL(X, Y, DERY)
153 20 FORMAT (1H, 13, 1X, F5.3, 3X, 8(E12.4, 3X))
154 PRINT 20, N, X, (Y(I), I = 1, 8)
155 PRINT 20, N, X, (Y(I), I = 9, 16)
156 PRINT 20, N, X, (Y(I), I = 17, 24)
157 PRINT 20, N, X, (Y(I), I = 25, 32)
C ------- ----------------- (DF'/DTHETA).Q -------
158 DO 91 I = 1, 33
159 91 Y1(I) = Y(I)
160 75 FDTQ1 = FT1*Y1(35, N) + FT2*Y1(36, N)
161 FDTQ2 = FT3*Y1(37, N) + FT4*Y1(38, N)
162 FDTQ3 = FT5*Y1(40, N)
C ------- S+(DF'/DTHETA).Q -------
163 SFD1 = S1 + FDTQ1
164 SFD2 = S2 + FDTQ2
165 SFD3 = S3 + FDTQ3
C ------- T(INVERSE). (S+(DF'/DTHETA).Q) -------
166 TISFC1 = A(1)*SFD1 + A(4)*SFD2 + A(7)*SFD3
167 TISFC2 = A(2)*SFD1 + A(5)*SFD2 + A(8)*SFD3
168 TISFC3 = A(3)*SFD1 + A(6)*SFD2 + A(9)*SFD3
169 EP = 1
C ------- T(INVERSE).R -------
170 TIR1 = A(1)*R1 + A(4)*R2 + A(7)*R3
171 TIR2 = A(1)*R4 + A(4)*R5 + A(7)*R6
172 TIR3 = A(1)*R7 + A(4)*R8 + A(7)*R9
173 TIR4 = A(1)*R10 + A(4)*R11 + A(7)*R12
184 \[ TIR13 = A(1) \times R13 + A(4) \times R14 + A(7) \times R15 \]
185 \[ TIR16 = A(1) \times R16 + A(4) \times R17 + A(7) \times R18 \]
186 \[ TIR2 = A(2) \times R1 + A(5) \times R2 + A(8) \times R3 \]
187 \[ TIR5 = A(2) \times R4 + A(5) \times R5 + A(8) \times R6 \]
188 \[ TIR8 = A(2) \times R7 + A(5) \times R8 + A(8) \times R9 \]
189 \[ TIR11 = A(2) \times R10 + A(5) \times R11 + A(8) \times R12 \]
190 \[ TIR14 = A(2) \times R13 + A(5) \times R14 + A(8) \times R15 \]
191 \[ TIR17 = A(2) \times R16 + A(5) \times R17 + A(8) \times R18 \]
192 \[ TIR3 = A(3) \times R1 + A(6) \times R2 + A(9) \times R3 \]
193 \[ TIR6 = A(3) \times R4 + A(6) \times R5 + A(9) \times R6 \]
194 \[ TIR9 = A(3) \times R7 + A(6) \times R8 + A(9) \times R9 \]
195 \[ TIR12 = A(3) \times R10 + A(6) \times R11 + A(9) \times R12 \]
196 \[ TIR15 = A(3) \times R13 + A(6) \times R14 + A(9) \times R15 \]
197 \[ TIR18 = A(3) \times R16 + A(6) \times R17 + A(9) \times R18 \]

\[ \text{C} \quad \text{----------------------------- } x(j+1) - x(j) \quad \text{-----------------------------} \]

198 \[ XX1 = Y1(1,N) - Y0(1,N) \]
199 \[ XX2 = Y1(2,N) - Y0(2,N) \]
200 \[ XX3 = Y1(3,N) - Y0(3,N) \]
201 \[ XX4 = Y1(4,N) - Y0(4,N) \]
202 \[ XX5 = Y1(5,N) - Y0(5,N) \]
203 \[ XX6 = Y1(6,N) - Y0(6,N) \]
204 \[ TIRX1 = XX1 \times TIR1 + XX2 \times TIR2 + XX3 \times TIR3 + XX4 \times TIR4 + XX5 \times TIR5 + XX6 \times TIR6 \]
205 \[ TIRX2 = XX1 \times TIR2 + XX2 \times TIR3 + XX3 \times TIR4 + XX4 \times TIR5 + XX5 \times TIR6 + XX6 \times TIR7 \]
206 \[ TIRX3 = XX1 \times TIR3 + XX2 \times TIR4 + XX3 \times TIR5 + XX4 \times TIR6 + XX5 \times TIR7 + XX6 \times TIR8 \]

\[ \text{C} \quad \text{----------------------------- } \text{IMPROVED VALUES OF CONTROL VARIABLES} \quad \text{-----------------------------} \]

207 \[ T1\text{NEW}(N) = T1(N) - \text{EP} \times T1\text{SFQ1} - TIRX1 \]
208 \[ T2\text{NEW}(N) = T2(N) - \text{EP} \times T1\text{SFQ2} - TIRX2 \]
209 \[ AT\text{NEW}(N) = AT(N) - \text{EP} \times T1\text{SFQ3} - TIRX3 \]
210 \[ 21 \text{ FORMAT } (1H13,3X,F5.2,2X,4(F12.4,2X)) \]
211 \[ \text{PRINT } 21,N,X,Y(33),T1\text{NEW}(N),T2\text{NEW}(N),AT\text{NEW}(N) \]
212 \[ \text{IF } (N \neq 1) \text{ PRMT}(5) = 1. \]
213 \[ 19 \text{ RETURN} \]
214 \[ \text{END} \]
215 **SUBROUTINE CALCL(X,Y,DERY)**
216 **DIMENSION PMT(10),Y(42),DERY(42),AUX(8,43),Y1(40,202),T1NEW(202)**
217 **T2NEW(202),T1(202),T2(202),AT(202),Y0(6,202),A(10),L(10)**
218
219 **COMMON Y1,T1NEW,T2NEW,ATNEW,T1,T2,AT,NK,XX1,XX2,XX3,XX4,XX5,XX6,Y0**
223 **,F63,F64,F65,F66,F67,F68,F69,F70,F71,F72,F73,F74,F75,F76,F77,F78,F79**
224 **,F80,F81,F82,F83,F84,F85,F86,F87,F88,F89,F90,F91,F92,F93,F94,F95,F96**
228 **,F140,F141,F142,F143,F144,F145,F146,F147,F148,F149,F150,F151,F152,F153**
244 **,F322,F323,F324,F325,F326,F327,F328,F329,F330,F331,F332,F333,F334,F335**
249 **,F392,F393,F394,F395,F396,F397,F398,F399,F400,F401,F402,F403,F404,F405**
255 **,R1=R(7)*FT1+R(8)*FT2+R(1)*G1-R(2)*G1**
256 **,R2=R(9)*FT3+R(10)*FT4**
257 **,R3=R(12)*FT5**
258 **,R4=R(8)*FT1+R(13)*Y(2)*G2**
259 **,R5=R(14)*FT3+Y(15)*FT4**
260 **,R6=R(17)*FT5**
261 **,R7=R(9)*FT1+Y(14)*FT2**
262 **,R8=R(18)*FT3+Y(19)*FT4-G3*Y(4)+G3*Y(3)**
263 **,R9=R(21)*FT5**
264 **,R10=R(10)*FT1+Y(15)*FT2**
265 **,R11=R(19)*FT3+Y(22)*FT4+G4*Y(4)**
266 **,R12=R(24)*FT5**
267 **,R13=Y(11)*FT1+Y(16)*FT2**
268 **,R14=R(20)*FT3+Y(23)*FT4**
269 **,R15=Y(26)*FT5**
270 **,R16=Y(12)*FT1+Y(17)*FT2**
271 **,R17=Y(21)*FT3+Y(24)*FT4**
272 **,R18=Y(27)*FT5+Y(6)*(1.2.*Y(6,N)/AN)-4.*CA*AT(N)*Y(6,N)**
\[ S_1 = 2 \cdot CT \cdot (T_{11} + T_{21} + T_{31} + T_{41}) + Y_{11} \cdot (L_{11} + L_{12} + L_{13} + L_{14}) + Y_{21} \cdot (L_{21} + L_{22} + L_{23} + L_{24}) + Y_{31} \cdot (L_{31} + L_{32} + L_{33} + L_{34}) + Y_{41} \cdot (L_{41} + L_{42} + L_{43} + L_{44}) \]

\[ S_2 = 2 \cdot CT \cdot (T_{12} + T_{22} + T_{32} + T_{42}) + Y_{12} \cdot (L_{11} + L_{12} + L_{13} + L_{14}) + Y_{22} \cdot (L_{21} + L_{22} + L_{23} + L_{24}) + Y_{32} \cdot (L_{31} + L_{32} + L_{33} + L_{34}) + Y_{42} \cdot (L_{41} + L_{42} + L_{43} + L_{44}) \]

\[ S_3 = 2 \cdot CT \cdot (T_{13} + T_{23} + T_{33} + T_{43}) + Y_{13} \cdot (L_{11} + L_{12} + L_{13} + L_{14}) + Y_{23} \cdot (L_{21} + L_{22} + L_{23} + L_{24}) + Y_{33} \cdot (L_{31} + L_{32} + L_{33} + L_{34}) + Y_{43} \cdot (L_{41} + L_{42} + L_{43} + L_{44}) \]

\[ S_4 = 2 \cdot CT \cdot (T_{14} + T_{24} + T_{34} + T_{44}) + Y_{14} \cdot (L_{11} + L_{12} + L_{13} + L_{14}) + Y_{24} \cdot (L_{21} + L_{22} + L_{23} + L_{24}) + Y_{34} \cdot (L_{31} + L_{32} + L_{33} + L_{34}) + Y_{44} \cdot (L_{41} + L_{42} + L_{43} + L_{44}) \]
C --------------- CALCULATION OF T-INVERSE ---------------

C --------------- MATRIX R^T(INVERSE) (6x3) ---------------

C --------------- MATRIX R^T(INVERSE).S (6x1) ---------------

C --------------- MATRIX R^T(INVERSE) ---------------
SUBROUTINE MINV

PURPOSE
    INVERT A MATRIX

USAGE
    CALL MINV(A,N,D,L,M)

DESCRIPTION OF PARAMETERS

A - INPUT MATRIX, DESTROYED IN COMPUTATION AND REPLACED BY
    RESULTANT INVERSE.
N - ORDER OF MATRIX A
D - RESULTANT DETERMINANT
L - WORK VECTOR OF LENGTH N
M - WORK VECTOR OF LENGTH N

REMARKS

MATRIX A MUST BE A GENERAL MATRIX

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
    NONE

METHOD

THE STANDARD GAUSS-JORDAN METHOD IS USED. THE DETERMINANT
IS ALSO CALCULATED. A DETERMINANT OF ZERO INDICATES THAT
THE MATRIX IS SINGULAR.

RETURN
SUBROUTINE MINV(A, N, D, L, M)

DIMENSION A(1), L(1), M(1)

C

C IF A DOUBLE PRECISION VERSION OF THIS Routines IS DESIRED. THE
C C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION
C STATEMENT WHICH FOLLOWS.
C
C DOUBLE PRECISION A, D, BIGA, HOLD
C
C THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS
C APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS
C ROUTINE.
C
C THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO
C CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS. ABS IN STATEMENT
C 10 MUST BE CHANGED TO DABS.
C
C SEARCH FOR LARGEST ELEMENT
C
D = L. C
N = N - N
DO 8 C K = 1, N
NK = NK + N
L(K) = K
M(K) = K
KK = NK + K
BIGA = A(KK)
DO 2 C J = K, N
IZ = N * (J - 1)
DO 20 I = K, N
IJ = I + 1
10 IF(ABS(BIGA) - ABS(A(IJ))) 15, 20, 20
BIGA = A(IJ)
L(K) = I
M(K) = J
20 CONTINUE
C
C INTERCHANGE ROWS
C
J = L(K)
IF(J = K) 35, 35, 25
KI = K - N
DO 30 I = 1, N
30 KI = KI + N
HOLD = -A(KI)
JI = KI - K + J
A(KI) = A(JI)
A(JI) = HOLD

INTERCHANGE COLUMNS

IF(I-K) 45, 45, 38
JP = N*(I-1)
DO 4C J = 1, N
JK = NK + J
JI = JP + J
HOLD = -A(JK)
A(JK) = A(JI)
A(JI) = HOLD

DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS CONTAINED IN BIGA)

IF(BIGA) 48, 46, 48
D = 0. C
RETURN
DO 55 I = 1, N
IF(I-K) 50, 55, 50
IK = NK + I
A(IK) = A(IK) / (-BIGA)
CONTINUE

REDUCE MATRIX

DO 65 I = 1, N
IK = NK + I
HOLD = A(IK)
IJ = I - N
DO 65 J = 1, N
IJ = IJ + N
IF(I-K) 60, 65, 60
IF(J-K) 62, 65, 62
KJ = IJ - I + K
A(IJ) = HOLD*A(KJ) + A(IJ)
CONTINUE

DIVIDE ROW BY PIVOT

KJ = K - N
DO 75 J = 1, N
KJ = KJ + N
IF(J-K) 70, 75, 70
A(KJ) = A(KJ) / BIGA
CONTINUE
PRODUCT OF PIVOTS

D = 0 * BIGA

REPLACE PIVOT BY RECIPROCAL

\[ A(KK) = 1.0 / BIGA \]

80 CONTINUE

FINAL ROW AND COLUMN INTERCHANGE

K = N

100 K = (K - 1)

IF (K) GO TO 150, 150, 105

105 I = L(K)

120, 120, 108

JQ = N * (K - 1)

JR = N * (I - 1)

DO 110 J = 1, N

JK = JQ + J

HOLD = A(JK)

JI = JR + J

A(JK) = -A(JI)

110 A(JI) = HOLD

120 J = M(K)

IF (J - K) GO TO 100, 100, 125

125 KI = K - N

DO 130 I = 1, N

KI = KI + N

HOLD = A(KI)

JI = KI - K + J

A(KI) = -A(JI)

130 A(JI) = HOLD

GO TO 100

150 RETURN

END
SUBROUTINE RKGS(PRMT,Y,DERY,NDIM,IHLF,FCT,CUTP,AUX)

DIMENSION Y(1),DERY(1),AUX(8,1),A(4),B(4),C(4),PRMT(1)

X=PRMT(1)
H=PRMT(3)

PRMT(5)=0.

CALL FCT(X,Y,DERY)

C

PREPARATIONS FOR RUNGE-KUTTA METHOD

A(1)=.5
A(2)=.2928932
A(3)=1.707107
A(4)=.1666667

B(1)=2.
B(2)=1.
B(3)=1.
B(4)=2.

C(1)=.5
C(2)=.2928932
C(3)=1.707107
C(4)=.5

PREPARATIONS OF FIRST RUNGE-KUTTA STEP

DO 3 I=1,NDIM
AUX(I,1)=Y(I)

3 AUX(2,1)=DERY(I)

AUX(3,1)=0.

AUX(6,1)=0.

RECORDING OF INITIAL VALUES OF THIS STEP

CALL CUTP(X,Y,DERY,IREC,NDIM,PRMT)

IF(PRMT(5))40,8,40

START OF INNERMOST RUNGE-KUTTA LOOP

8 J=1
10 AJ=A(J)

BJ=B(J)

CJ=C(J)

DO 11 I=1,NDIM

R1=H*DERY(I)

R2=AJ*(R1-BJ*AUX(6,I))

Y(I)=Y(I)+R2

R2=R2+R2*R2

AUX(6,I)=AUX(6,I)+R2-CJ*R1

11 IF(J-4)12,15,15

12 J=J+1

13 IF(J-3)13,14,13
515       13  X = X + .5 * H
520       14  CALL FCT(X, Y, DERY)
521      GOTO 10

C       END OF INNERMOST RUNGE-KUTTA LOOP
C
C
522        15  DO 29  I = 1, NDIM
523        16  AUX(1, I) = Y(I)
524        17  AUX(2, I) = DERY(I)
525        19  AUX(6, I) = AUX(3, I)
526        20  CALL OUTP(X, Y, DERY, IHLF, NDim, PRMT)
527        21  IF(PRMT(5)) 40, 30, 40
528        30  DO 31  I = 1, NDim
529        31  Y(I) = AUX(1, I)
530        33  DERY(I) = AUX(2, I)
531      GO TO 8
532        40  RETURN
533      END
OPTIMIZATION OF MANAGEMENT SYSTEMS

BY SECOND VARIATION

by

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ABSTRACT

There are many difficulties in using either the classical multistage optimization techniques or dynamic programming for solving nonlinear complex problems involving a fairly large number of variables. The former gives boundary value difficulties while the latter has the difficulty of dimensionality. The methods of gradients and other techniques such as quasilinearization partially overcome these difficulties.

The basic philosophy of the methods of gradients is fairly simple. First a sequence of values of the control vector is selected. Then the gradient of the performance index with respect to each of the control vector is calculated. Finally each control vector is improved by moving it in the direction of the gradient. This improved sequence of control vectors then becomes the basis for the next iteration.

The functional gradient technique, one of the many versions of the gradient methods, has been developed for optimal control problems. The second variation method overcomes certain difficulties of the functional gradient technique. The convergence rate of the second variation method, provided the method converges, is very fast. However, the initial guess of the trajectory for the control variable has to be near the optimal trajectory in order to obtain convergence. Too, the number of equations to be integrated and their complexity tend to suppress its advantage of rapid convergence.

First, the method of second variation is discussed in detail. Then the method is applied to three problems in the field of production and inventory control to illustrate the approach.
The first application is a simple inventory model involving one state variable and one control variable. The objective function is the cost function, which is to be minimized. The second application is an inventory and advertising model where it is desired to maximize the profit function. This problem has two state variables and one control variable. The last application is that of a chemical manufacturing problem with advertisement. It has six state variables and three control variables.

These examples suggest that the first variation method, of which the second variation method is a natural evolution, should be used in combination with the second variation. The first variation method, unlike the second variation, will approach optimum from almost any realistic starting trajectory. The results of the first variation method could then be used as the starting trajectories for the second variation. In this way, the convergence problem of the second variation can be partly overcome. Furthermore, this combination provides a rapid convergence from almost any realistic starting trajectory for most engineering problems.