APPLICATIONS OF DYNAMIC PROGRAMMING
to process engineering

by

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B. S., National Taiwan University, 1955

A MASTER'S THESIS

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Chemical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

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ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Dr. Liang-tseng Fan for his constant enthusiasm and advice and for helpful guidance throughout the process of this work; Drs. Larry E. Erickson and Ching-lai Hwang for reading the manuscript; William H. Honstead, Head of the Department of Chemical Engineering, for his assistance and encouragement; the Kansas State University Computing Center for the use of their facilities; Kansas State Water Resources Research Institute and the Office of Saline Water for their financial assistance.
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CHAPTER I

INTRODUCTION

1. THE CONCEPT OF OPTIMIZATION

In recent years, chemical engineers have become increasingly concerned with the problems of choosing the design and operating variables of chemical plants such that some performance criteria are optimized. The intensive competition in the industry in general and in the chemical process industry in particular necessitates the improvement of technologies, since even marginal savings in production costs may be of vital importance in this competitive community. The act of optimization is essentially that of designing the most economical system of equipment and obtaining the best performance either under given conditions or subjected to certain restrictions.

A single optimization problem corresponds to the seeking of the extreme value of a function by differential calculus. But it is often the case that the optimal problems in engineering and industry cannot be solved by direct applications of conventional methods of optimization. Various schemes more sophisticated than the conventional methods have been proposed. Among them are the maximum principle, and dynamic programming.

2. THE FEATURES OF DYNAMIC PROGRAMMING

Dynamic programming provides a powerful tool for solving multistage decision processes, which arise in various fields. It is based upon the principle of optimality and employs the techniques of invariant imbedding [1, 2]. The concept upon which the dynamic programming technique is based is a new and versatile mathematical tool for the treatment of many novel and interesting problems.
Because of the simplicity and versatility of the principle of optimality, the method of dynamic programming can be used to deal with a process for which the transformation or transition at each stage is too complex to be expressed explicitly and analytically. Application of this principle also guarantees that the decision made at each stage is the best one in light of the entire process. While the number of stages in a process may be numerous, a small number of decisions are usually involved at each stage, and, with the aid of modern computers, one can often easily solve a multistage decision problem by using dynamic programming. A continuous process can also be optimized using dynamic programming by treating the process as one with a large number of infinitesimal stages.

3. THE APPLICATIONS OF DYNAMIC PROGRAMMING

Dynamic programming has been applied extensively to solve various practical and theoretical problems. Aris [3] has written a book dealing specifically with chemical reactor design. He has also written two other books related to the applications of dynamic programming [4, 5]. Roberts [6] has made optimal catalyst replacement studies. Aris et al. [7] studied optimum crosscurrent extraction. Rudd [8] has investigated a reliability problem in chemical system design and the optimal allocation of limited resources. A model for the optimization of countercurrent flow processes has been developed by Dranoff et al. [9]. Mitten and Nemhauser [10] applied the dynamic programming technique to determine the optimal design of a process consisting of a sequence of heterogeneous stages. Optimization methods applicable for processes with bypass and recycle streams have also been developed [11].
4. THE ALGORITHM OF DYNAMIC PROGRAMMING AND THE PRINCIPLE OF OPTIMALITY

The essential notions of dynamic programming are linked to a serial structure. It is of the nature of serial optimization. Its cornerstone is the principle of optimality founded by Bellman [1]. It states that "An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

The mathematical transliteration of this principle yields a new class of functional equations, and the mathematical viewpoint developed in treating these problems enables us to approach some classical as well as new problems.

5. THE INVERSE PRINCIPLE OF OPTIMALITY

If we stipulate the output $x$ of the sequential process of Fig. 1, but not the input $x_{n+1}$, then there is an inverse principle of optimality for such a process, provided the stage transformation may be inverted. The inverse of the transformation at stage $n$ can be written as

$$x_{n+1} = \bar{T}_n(x^n: \Theta^n), \quad n = 1, 2, \ldots, N-1.$$  

The inverse principle is stated as follows [5]:

In a sequential process, the optimal policy for a fixed final and free initial state has the property that whatever the final state and decision may be, the proceeding decisions constitute an optimal policy with respect to the state resulting from them.

6. MULTISTAGE DECISION PROCESSES

A typical multistage stagewise process is shown in Fig. 1 where $x_n$ is a state vector which represents the state variables from stage $n$, and $\Theta_n$ is a vector which stands for the decision variables in stage $n$. Note that the
Fig. 1. A typical multi-stage process (backward numbering).
stages have been numbered in the backward direction. They can equally well be numbered in the forward manner, which ever is more convenient.

If these stages are series of continuous flow stirred tank reactors (CSTR system), then the state variables may be concentrations of the various species--reactants and/or products. Decision variables of a CSTR system may be the temperature and pressure of the reaction mixture, but the specification of a decision variable is quite arbitrary. It could be the flow rate or equivalently the residence time of each stage.

The function of each stage is to transform the state variables from the input state to the output state. This transformation can generally be expressed as

$$x_n = T_n(x_{n+1}; \theta_n), \quad n = N, N-1, \ldots, 2, 1.$$  (1)

Equation (1) is in vector form. If there are $s$ state variables and one decision variable, equation (1) can be written as

$$x_{i,n} = T_{i,n}(x_{1,n+1}, x_{2,n+1}, \ldots, x_{s,n+1}; \theta_n), \quad i = 1, 2, \ldots, s$$

$$n = N, \ldots, 2, 1.$$  (2)

The objective of optimization is to seek a set of admissible values of $\theta_1, \theta_2, \ldots, \theta_N$ so that a desired performance criterion or a return function which is usually called an objective function is maximized (or minimized). The characteristic feature of a multistage decision process is that there is an interval profit or return associated with each stage of the process and the objective function can be expressed as the summation of the interval profits.
The value of the objective function depends on the initial state and a sequence of decisions \( \theta_N, \ldots, \theta_1 \). If we represent the maximum return function by \( f_N(x_{N+1}) \), then

\[
f_N(x_{N+1}) = f_N(x_{1,N+1}, x_{2,N+1}, \ldots, x_{s,N+1})
= \max \ S(x_{N+1}; \theta_N, \ldots, \theta_1)
= \max \ \frac{1}{\theta_n} \sum_{n=N}^s g_n(x_{n+1}; \theta_n).
\]

Thus, in general, \( f_n(x_{n+1}) \) is the maximum return obtainable from the operation of an \( n \)-stage process if an optimal policy is followed starting with the initial state \( x_{n+1} \).

If there is one decision variable in each stage, equation (4) expresses an \( N \)-dimensional optimization problem because this problem must be optimized with respect to all the \( N \) decision variables. The dynamic programming technique treats this problem as \( N \) one-dimensional problems. For a one-stage process, equation (4) becomes

\[
f_1(x_2) = \max_{\theta_1} g_1(x_2; \theta_1)
\]

which is the simplest optimization problem among the sequence of problems for \( n = 1, 2, \ldots, N \). The other members of this sequence can be obtained by writing equation (4) in the form:

\[
f_n(x_{n+1}) = \max_{\theta_n} \max_{\theta_{n-1}} \ldots \max_{\theta_1} \left[ g_n(x_{n+1}; \theta_n) + \ldots + g_1(x_2; \theta_1) \right]
\]
or

\[ f_n(x_{n+1}) = \max_{\theta_n} \left\{ g_n(x_{n+1}; \theta_n) + \max_{\theta_{n-1}} \cdots \max_{\theta_1} g_{n-1}(x_{n}; \theta_{n-1}) + \cdots + g_1(x_2; \theta_1) \right\} \]  

(6)

The expression

\[ \max_{\theta_{n-1}} \cdots \max_{\theta_1} \left[ g_{n-1}(x_{n}; \theta_{n-1}) + \cdots + g_1(x_2; \theta_1) \right] \]

stands for the maximum return (the objective function) from an (n-1)-stage process with initial stage \( x_n \). Hence, we can also write

\[ f_{n-1}(x_{n}) = \max_{\theta_{n-1}} \cdots \max_{\theta_1} \left[ g_{n-1}(x_{n}; \theta_{n-1}) + \cdots + g_1(x_2; \theta_1) \right] \]  

(7)

Thus, equation (6) can be simplified to

\[ f_n(x_{n+1}) = \max_{\theta_n} \left[ g_n(x_{n+1}; \theta_n) + f_{n-1}(x_{n}) \right] \]  

(8)

or

\[ f_n(x_{n+1}) = \max_{\theta_n} \left[ g_n(x_{n+1}; \theta_n) + f_{n-1}(T(x_{n+1}; \theta_n)) \right] . \]

This is the so-called functional equation of dynamic programming. It gives a recursion relationship between an N stage process and an N-1 stage process. The solution of the functional equation yields the value of the maximum return and the corresponding optimal policy, which belongs to the set \( \{ \theta_n \} \).

If we consider the process with stages numbered in the forward order (see Fig. 2), the transformation equation is usually in the form of

\[ x_n = T_n(x_{n-1}; \theta_n), \quad n = 1, 2, \ldots, N, \]
Fig. 2. A typical multi-stage process (forward numbering).
and the objective of maximum return function is

\[ f_N(x_0) = \max S(x_0; \theta_1, \theta_2, \ldots, \theta_N) \]

\[ = \max \sum_{n=1}^{N} S_n(x_{n-1}; \theta_n). \]

Therefore, the functional equation can be written as

\[ f_N(x_0) = \max \{ S_1(x_0; \theta_1) + f_{N-1}(x_1) \}. \]

Another form for expressing the transformation equation is

\[ x^n = T^n(x^{n-1}; \theta^n), \quad n = 1, 2, \ldots, N \]

and the objective or maximum return function is

\[ f_N(x^0) = \max S(x^0; \theta^1, \theta^2, \ldots, \theta^N) \]

\[ = \max \sum_{n=1}^{N} S_n(x^{n-1}; \theta^n). \]

so that the functional equation takes the form

\[ f_N(x^0) = \max \{ S_1(x^0; \theta^1) + f_{N-1}(x^1) \}. \]

Another way of interpreting the principle of optimality is imbedding a smaller system into a bigger system as shown in Fig. 3. The functional equation is of the form

*The superscripts denote the stage number. The power of n is represented by putting n outside of the parenthesis.*
Fig. 3. A schematic diagram showing the imbedding of a smaller system into a bigger system. The arrows indicate the direction of imbedding.
\[ f_1(x_0) = \max_{\theta_1} g_1(x_0; \theta_1) \]

\[ f_2(x_0) = \max_{\theta_1} g_1(x_0; \theta_1) + f_1(T_1(x_0; \theta_1)) \]

and in general

\[ f_N(x_0) = \max_{\theta_1} \left\{ g_1(x_0; \theta_1) + f_{N-1}(T_{N-1}(x_{N-2}; \theta_{N-1})) \right\} \]

Generally speaking, the evaluation of the functional equation seldom appears in an analytical form. Instead, the equation is evaluated by numerical methods on a high speed computer.

7. THE ADVANTAGES AND DISADVANTAGES OF DYNAMIC PROGRAMMING

Dynamic programming provides a new tool for solving problems which were formerly considered too complicated to be solved. It can be applied to a series of stages which have different operating and state variables at each stage. Furthermore, by the use of the principle of optimality, a chemical process can be separated into different segments (subprocesses) and each segment can be optimized by a different technique. Another important feature of dynamic programming lies in its ability to handle any kind of constraints. Many types of constraints cause trouble when other techniques are used [12].

The principle disadvantage of dynamic programming is the problem of high dimensionality. Since we imbed a whole family of decisions, enormous time and space are required, and the available memory in a modern computer still limits the use of the dynamic programming technique to systems of several variables. This is the so-called "curse of dimensionality" [1]. Any reduction in dimensionality achieved by the use of mathematical knowledge will
save tremendously in computational cost.

Another disadvantage of dynamic programming is that in using it to optimize continuous processes it leads to a set of partial differential equations. However, it is often possible to convert this set of partial differential equations into a system of finite difference equations. Thus, the method of dynamic programming can be used to optimize continuous processes [4].

A third disadvantage of dynamic programming is that it cannot be applied easily to processes in which the optimum conditions at any stage can be disturbed by conditions at a following stage. This is a rather serious limitation if we consider the various feedback and countercurrent operations which are often employed in the chemical and petroleum industries [12].

8. THE OBJECTIVES AND SCOPE OF THIS WORK

In this work, a comprehensive study of the theories and applications of dynamic programming together with some techniques for reducing dimensionality are presented. The wide applicability of dynamic programming in optimizing multistage decision processes is illustrated by using it to optimize several process engineering problems.
References


CHAPTER II

GENERAL DESCRIPTIONS AND APPLICATIONS TO SIMPLE PROCESSES

1. COMPUTATIONAL SCHEME FOR THE FUNCTIONAL EQUATION [1].

As stated in Chapter I, the functional equation is of the form

\[ f_n(x_{n+1}) = \max_{\theta_n} \left\{ g_n(x_{n+1}; \theta_n) + f_{n-1}(x_n) \right\}, \]

for an n-stage process starting with an initial state \( x_{n+1} \) as shown in Fig. 1 (backward numbering system) in Chapter I.

The significance of the functional equation is that the optimal decision at stage \( n \) is so chosen that the sum of \( f_{n-1}(x_n) \) and \( g_n(x_{n+1}; \theta_n) \) is maximized. Since the states of the intermediate stages are unknown before the problem is completely solved, the functional equation cannot become immediately useful in solving an n-stage optimization problem. The so-called imbedding technique is carried out in two steps to establish the recurrence relationships. The first step is to construct a table for each stage relating the corresponding optimal decision to the objective function for various values of the state variable entering each stage. The second step is to determine the optimal policy for the whole process by means of the table entry technique utilizing all the optimal tables constructed [2].

In constructing the optimal tables, the computation is initiated with the last stage, proceeding backwards to the initial stage. In each step of tabulation, the stage for which the optimal table is to be tabulated is regarded as the initial stage. For example, if we are constructing the optimal table for stage \( n \), it is taken as the initial stage, whereas all the downstream stages are included in an (n-1)-stage process for which the
capacities of only about $3.2 \times 10^4$ locations, three-dimensional problems cannot be solved routinely. This is known as the dimensionality difficulty of the dynamic programming approach.

It is worthwhile to mention that there is also a functional equation of the form [3]

$$f_{n+1}(x) = \max_{\theta_n} \left\{ g_n(x, \theta_n) \cdot f_n(x) \right\}.$$  

A simple example of its use is to divide a distance $\ell$ into $n$ parts in such a way that the product of the $n$ parts is a maximum. In this case, we let $f_n(\ell)$ be the maximum attainable product, $x$ be the length of the first subdivision, and $(\ell-x)$ be the length of the remaining $(n-1)$ parts. Then, the functional equation takes the form [4]

$$f_n(\ell) = \max_{0 \leq x \leq \ell} \left\{ x f_{n-1}(\ell-x) \right\}.$$  

2. ILLUSTRATIVE PROBLEMS IN ENGINEERING  

EXAMPLE 1, ALLOCATION OF PRODUCTION LEVEL.  

Let us consider a typical linear programming problem in which $N$ grades of paper are produced on a paper machine. Due to the restrictions on the raw materials, not more than $a_i$ tons of grade $i$ can be produced in a week.

Let

$$x_i = \text{number of tons of grade } i \text{ produced in the week}$$

$$b_i = \text{number of hours required to produce a ton of grade } i$$

$$P_i = \text{net profit per ton of grade } i.$$  

The problem in hand can be stated as

$$0 \leq x_i \leq a_i.$$
\[ \sum_{i=1}^{N} b_{i}x_{i} \leq c = \text{total production hours} \]

\[ \max S = \max P(x_{1}, \ldots, x_{N}) = \max \sum_{i=1}^{N} P_{i}x_{i} \]

We shall consider this problem as an \( N \)-stage process with one decision variable at each stage. According to the notion of dynamic programming, we define a maximum return function

\[ f_{N}(c) = \text{maximum return obtainable from the } N \text{-stage process with} \]
\[ c \text{ production hours per week.} \]

so that

\[ f_{N}(c) = \max \left( \left\{ x_{i} \right\} \right) P(x_{1}, \ldots, x_{N}) = \max \sum_{i=1}^{N} P_{i}x_{i} \]

where \( \left\{ x_{i} \right\} \) indicates a series of decision variables \( x_{1}, \ldots, x_{N} \).

For ease of discussion, let us change the inequality constraint to the following equality constraint.

\[ \sum_{i=1}^{N} b_{i}x_{i} = c . \]

This is a typical allocation problem \([1]\). Many other types of allocation problems can be found in any book treating dynamic programming \([5, 6]\).

We wish to allocate the total of \( c \) working hours to produce \( N \)-different grades of papers in an optimal way.

For a one-stage process, i.e. only one grade of paper is produced, the maximum return becomes

\[ f_{1}(c) = \max_{0 \leq x_{1} \leq a_{1}} P_{1}x_{1} = P_{1} \frac{c}{b_{1}} \]
with \( 0 \leq \frac{c}{b_1} \leq a_1 \).

For a two-stage process, the maximum return \( f_2(c) \) is written as

\[
f_2(c) = \max_{0 \leq x_2 \leq a_2} \left[ P_2 x_2 + f_1(c - b_2 x_2) \right].
\]

It is understood that whatever the amount of \( c \) which has already been allocated to the second stage, the remaining amount \( c - b_2 x_2 \) must be used in an optimal way for the remaining one-stage process.

Extending this notion, we see that whatever the amount of \( c \) which has already been allocated to the \( N \)-th stage for an \( N \)-stage process, the remaining amount \( c - b_N x_N \) must be allocated in the optimal way for the remaining \( (N-1) \)-stage process.

Thus, the maximum return for an \( N \)-stage process is

\[
f_N(c) = \max_{0 \leq x_N \leq a_N} P_N x_N + f_{N-1}(c - b_N x_N).
\]

It can be seen that this algorithm can be extended to the multidimensional case.

For illustration, let us consider the case of \( N = 2 \) and

\[
x_1 \leq 500, \quad x_2 \leq 250,
\]

\[
0.2 x_1 + 0.4 x_2 \leq 140,
\]

\[
P(x_1, x_2) = 20 x_1 + 45 x_2.
\]

Equation (2) can be written as

\[
x_1 + 2x_2 \leq 700 = c.
\]
Note that \( a_1 = 500, a_2 = 250, b_1 = 1, b_2 = 2, P_1 = 20 \) and \( P_2 = 45 \).

The maximum return for a one-stage process can be rewritten as

\[
 f_1(c) = \max_{0 \leq x_1 \leq a_1} p_1 x_1(c) .
\]  

(5)

Due to the constraint on \( x_1 \), the maximum return is

\[
 f_1(c) = p_1 \frac{c}{b_1} = 20c, \quad c \leq a_1 = 500
\]  

(6)

\[
 f_1(c) = 20 a_1 = 10,000, \quad c \geq a_1 = 500 .
\]  

(7)

The optimal table with a grid value of \( c = 100 \) is given in Table 1.

The maximum return for a two stage process is

\[
 f_2(c) = \max_{0 \leq x_2 \leq a_2} \left[ p_2 x_2(c) + f_1(c - b_2 x_2(c)) \right] .
\]  

(8)

We define

\[
 r_2(c) = p_2 x_2(c) + f_1(c - b_2 x_2(c)) ,
\]  

(9)

then it follows that

\[
 f_2(c) = \max_{x_2} r_2(c) .
\]  

(10)

Values of \( r_2(c) \) with \( c = 100 \) is given in Table 2.

It is seen that \( f_2(c) = 2250 \) for \( c = 100 \). In a like manner, we can find \( f_2(c) \) for other grid values of \( c \), and the optimal table for the two-stage process is tabulated in Table 3.

It is concluded that the maximum return for the two-stage process will be $15,250/week with an optimal allocation of production hours to produce 200 and 250 tons per week of grades 1 and 2 paper respectively. This
<table>
<thead>
<tr>
<th>c</th>
<th>$x_1 = \theta_1$</th>
<th>$p_1 x_1$</th>
<th>$f_1(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>hr/week</td>
<td>ton/week</td>
<td>$$/week$</td>
<td>$$/week$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>2,000</td>
<td>2,000</td>
</tr>
<tr>
<td>200</td>
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<td>4,000</td>
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</tr>
<tr>
<td>700</td>
<td>500</td>
<td>10,000</td>
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TABLE 2
VALUES OF $r_2(c)$ FOR TWO STAGE PROCESS

<table>
<thead>
<tr>
<th>$c$ hr/week</th>
<th>$b_2 x_2$ hr/week</th>
<th>$x_2 = \theta_2$ ton/week</th>
<th>$p_2 x_2$ $$/\text{week}$</th>
<th>$f_1(c-b_2 x_2)$ $$/\text{week}$</th>
<th>$r_2(c)$ $$/\text{week}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2,000</td>
<td>2,000</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>10</td>
<td>450</td>
<td>1,600</td>
<td>2,050</td>
</tr>
<tr>
<td>100</td>
<td>40</td>
<td>20</td>
<td>900</td>
<td>1,200</td>
<td>2,100</td>
</tr>
<tr>
<td>100</td>
<td>60</td>
<td>30</td>
<td>1,350</td>
<td>800</td>
<td>2,150</td>
</tr>
<tr>
<td>100</td>
<td>80</td>
<td>40</td>
<td>1,800</td>
<td>400</td>
<td>2,200</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>50</td>
<td>2,250</td>
<td>0</td>
<td>2,250</td>
</tr>
</tbody>
</table>

TABLE 3
OPTIMAL TABLE (TWO STAGE PROCESS)

<table>
<thead>
<tr>
<th>$c$ hr/week</th>
<th>$x_2 = \theta_2$ ton/week</th>
<th>$p_2 x_2$ $$/\text{week}$</th>
<th>$f_1(c-b_2 x_2)$ $$/\text{week}$</th>
<th>$f_2(c)$ $$/\text{week}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>50</td>
<td>2,250</td>
<td>0</td>
<td>2,250</td>
</tr>
<tr>
<td>200</td>
<td>100</td>
<td>4,500</td>
<td>0</td>
<td>4,500</td>
</tr>
<tr>
<td>300</td>
<td>150</td>
<td>6,750</td>
<td>0</td>
<td>6,750</td>
</tr>
<tr>
<td>400</td>
<td>200</td>
<td>9,000</td>
<td>0</td>
<td>9,000</td>
</tr>
<tr>
<td>500</td>
<td>250</td>
<td>11,250</td>
<td>0</td>
<td>11,250</td>
</tr>
<tr>
<td>600</td>
<td>250</td>
<td>11,250</td>
<td>2,000</td>
<td>13,250</td>
</tr>
<tr>
<td>700</td>
<td>250</td>
<td>11,250</td>
<td>4,000</td>
<td>15,250</td>
</tr>
</tbody>
</table>
example is essentially that given by Lee [7].

EXAMPLE 2. DIRECTED NETWORK PROBLEM

(a) Both ends fixed (boundary value problem)

In Fig. 1, the circles represent the nodes, and the numbers on the lines connecting nodes denote the distance between the nodes. Suppose that the problem is to look for the minimum length or the shortest path starting from left to right.

According to the classical method, all possibilities must be enumerated. But the enumeration of all the possibilities would be prohibitive for a large size problem. We work backwards employing the dynamic programming approach in the following.

Stage III

<table>
<thead>
<tr>
<th>path</th>
<th>distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>

There is no choice of path for each initial state.

Stage II (III)

<table>
<thead>
<tr>
<th>path</th>
<th>distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6+1 = 7</td>
</tr>
<tr>
<td>2</td>
<td>7+10 = 17</td>
</tr>
<tr>
<td>3</td>
<td>10+1 = 11</td>
</tr>
<tr>
<td>4</td>
<td>9+10 = 19</td>
</tr>
</tbody>
</table>

Paths 2 and 4 are eliminated from our consideration.
Fig. 1. Directed network with both ends fixed.
maximum values of the objective function denoted as \( f_{n-1}[T_n(x_{n+1}; \theta_n)] \) are already obtained and tabulated. By means of the functional equation and the optimal table of stage \((n-1)\), we can obtain the optimal decision at stage \(n\) for each possible value of the state variable entering that stage. The decisions so obtained and the corresponding value of \( f_n(x_{n+1}) \) are listed in the table, which in turn is used to construct the optimal table for the preceding stage.

For comparison, we consider the computing time required for both the method of exhaustive search and dynamic programming. For a process with both ends free, there are \( N(M)^N \) possible paths if there are \( N \) stages with \( M \) grid points of the state variable at each stage. Assuming that it takes approximately \( 10^{-4} \) seconds to compute one path, then the total computing time would be \( 5 \times 10^9 \) years in the former case (\( N = 20, M = 10 \)). But in the latter case, the number of computations is \((2N-1)M^2\) and it takes only 1 second and hence tremendous saving in computing time is obtained.

It may be noted that because we can compute the values of \( f_n(x_{n+1}) \) for only a finite number of admissible values of \( x_{n+1} \), the methods of interpolation or extrapolation must be used to obtain the values of \( f_n(x_{n+1}) \) for those values of \( x_{n+1} \) which fall between two neighboring values of \( x_{n+1} \) listed in the optimal table.

The above computational procedure is only applicable to a one-dimensional multistage decision process, but this scheme can be extended to the multidimensional case. It is seen that the storage requirement of a computer increases exponentially with the dimensionality of the initial state variables. Usually, the storage requirement of a problem is about \((100)^s\) where \( s \) is the number of state variables. Since current computers have fast memory
Stage I (II & III)

<table>
<thead>
<tr>
<th>path</th>
<th>distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7+7 = 14</td>
</tr>
<tr>
<td>2</td>
<td>6+11 = 17</td>
</tr>
</tbody>
</table>

We see that the minimum distance is 14 with the path $7 \rightarrow 6 \rightarrow 1$.

(b) Both ends free

We shall find the minimum distance starting from left to right by means of dynamic programming (Fig. 2).

Stage III

<table>
<thead>
<tr>
<th>path</th>
<th>distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Paths 2 and 3 are eliminated from our consideration.

Stage II(& III)

<table>
<thead>
<tr>
<th>path</th>
<th>distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6+1 = 7</td>
</tr>
<tr>
<td>2</td>
<td>7+3 = 10</td>
</tr>
<tr>
<td>3</td>
<td>10+1 = 11</td>
</tr>
<tr>
<td>4</td>
<td>9+3 = 12</td>
</tr>
</tbody>
</table>

Paths 2 and 4 are rejected.
Fig. 2. Directed network with both ends free.
Stage I (II & III)

<table>
<thead>
<tr>
<th>path</th>
<th>distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7+7 = 14</td>
</tr>
<tr>
<td>2</td>
<td>6+11 = 17</td>
</tr>
<tr>
<td>3</td>
<td>4+7 = 11</td>
</tr>
<tr>
<td>4</td>
<td>1+11 = 12</td>
</tr>
</tbody>
</table>

We see that the minimum distance is 12 corresponding to the path 4→6→1.

(c) If the final state is fixed, it becomes a final value problem, whereas if the initial state is specified, it becomes an initial value problem. The computational scheme is essentially the same as that of cases (a) and (b).

EXAMPLE 3. ALLOCATION OF REACTOR VOLUMES IN A CSTR SYSTEM

Let us consider a liquid-phase chemical reaction between two immiscible reactants A and B. The reaction mechanism is

$$A + B \rightarrow C + D$$

where C is miscible with A and D is miscible with B. The overall kinetics is assumed to be

$$-r_A = \frac{1}{V} \frac{dN_A}{dt} = kC_A C_B$$

It is assumed that there is complete mixing in each reactor and that the rate is controlled by the chemical reaction only. In the above equation $C_A$ and $C_B$ are the concentrations of A and B in the completely mixed system.

The reaction is carried out in a sequence of three stirred tank reactors connected in series with equimolar feed of A and B flowing either cocurrently or countercurrently to achieve a prescribed degree of conversion. The process flow diagram is shown in Fig. 1a for the cocurrent flow system.
Fig. 1a. Schematic diagram of the two-phase cocurrent reactor system.
and in Fig. 1b for the countercurrent flow system. The volume of stage n (the nth reactor) can be obtained from the material balance as follows:

\[ V^n = \frac{F_{AO}(x^n_A - x_A^{n-1})}{-r^n_A} = \frac{F_{AO}(x^n_A - x_A^{n-1})}{kC_{A1}C_{B1}(1-x^n_A)(1-x^n_B)} \]

with \( C_{A1} \) and \( C_{B1} \) defined as

\[ C_{A1} = \frac{F_{AO}}{F_{AO} + \frac{F_{BO}}{C_{A0} + C_{BO}}} \]
\[ C_{B1} = \frac{F_{BO}}{F_{AO} + \frac{F_{BO}}{C_{A0} + C_{BO}}} \]

where

- \( F_{AO} \) = feed rate of A based on the inlet condition, lb-mole/sec.
- \( F_{BO} \) = feed rate of B based on the inlet condition, lb-mole/sec.
- \( k \) = specific rate constant, ft\(^3\)/sec. lb-mole,
- \( C_{A0} \) : initial concentration of reactant A, lb-mole/ft\(^3\),
- \( C_{B0} \) : initial concentration of reactant B, lb-mole/ft\(^3\),
- \( x_A \) : degree of conversion of reactant A,
- \( x_B \) : degree of conversion of reactant B.

For isothermal operation,
Fig. 1b. Schematic diagram of the two-phase counter-current reactor system.
\[ \frac{F_{AO}}{kC_{A1}C_{B1}} \]

is a constant.

For simplicity, we shall consider the case for which

\[ \frac{F_{AO}}{kC_{A1}C_{B1}} = 1 \]

and

\[ F_{AO} = F_{BO}. \]

Furthermore, since pure A and pure B are fed to the system, we can write

\[ x^n_A = x^n_B \] (12a)

for the cocurrent system, and the equation

\[ x^{n-1}_A + x^n_B = \text{final conversion} = \gamma \] (12b)

holds for each stage in the case of countercurrent flow. Hence equation (11) can be rewritten as

\[ V^n = \frac{x^n_A - x^{n-1}_A}{(1 - x^n_A)^2} \] (13a)

for cocurrent flow and

\[ V^n = \frac{x^n_A - x^{n-1}_A}{(1-x^n_A)(1-\gamma + x^{n-1}_A)} \] (13b)

for countercurrent flow.
The objective is to minimize the total reactor volume for a prescribed degree of conversion of reactant A by proper choices of conversion at stages 1 and 2 or equivalently by proper allocations of the reactor volume to individual reactors. That is,

\[ \min S = \min \sum_{n=1}^{3} V^n, \quad n = 1, 2, 3. \]

We define conversion of reactant A at stage \( n-1 \) as the state variable, \( x_{1}^{n-1} \), and the conversion of reactant A at the exit of stage \( n \) as the decision variable \( \theta^n \), i.e.

\[ x_{1}^{n} = \theta^n. \quad (14) \]

Thus equation (13a) for the cocurrent system can be rewritten as

\[ V^n = g_1(x_{1}^{n-1}; \theta^n) = \frac{\theta^n - x_{1}^{n-1}}{(1 - \theta^n)^2}, \quad n = 1, 2, 3. \quad (15a) \]

and the objective function becomes

\[ \min S = \min \sum_{n=1}^{3} V^n \]

\[ = \min \sum_{\theta^n}^{\{\theta^n\}} g_1(x_{1}^{n-1}; \theta^n), \quad n = 1, 2, 3. \quad (16a) \]

Similarly, equation (13b) for the countercurrent system becomes

\[ V^n = g_2(x_{1}^{n-1}; \theta^n) = \frac{\theta^n - x_{1}^{n-1}}{(1-\theta^n)(1-\gamma + x_{1}^{n-1})}, \quad n = 1, 2, 3. \quad (15b) \]

and the objective function takes the form
\[
\min S = \min \sum_{n=1}^{3} v^n \\
= \min \sum_{n=1}^{3} g_2(x_{n-1}^n; \theta^n), \quad n = 1, 2, 3. \quad (16b)
\]

Therefore, the functional equations of dynamic programming are as follows:

(A) Co concurrent system.

\[
f_1(x_1^2) = \min_{\theta^3} \left\{ \frac{\theta^3 - x_1^2}{(1-\theta^3)^2} \right\} = \min_{\theta^3} g_1(x_1^2; \theta^3)
\]

\[
= \frac{\gamma - x_1^2}{(1-\gamma)^2}
\]

for a one-stage process including stage 3 (the third reactor),

\[
f_2(x_1^1) = \min_{\theta^2} \left\{ g_1(x_1^1; \theta^2) + f_1(x_1^2) \right\}
\]

\[
= \min_{\theta^2} \left\{ \frac{\theta^2 - x_1^1}{(1-\theta^2)^2} + f_1(x_1^2) \right\}
\]

for a two-stage process including stage 2 (the second reactor) and stage 3,

\[
f_3(x_1^0) = \min_{\theta^1} \left\{ g_1(x_1^0; \theta^1) + f_2(x_1^1) \right\}
\]

\[
= \min_{\theta^1} \left\{ \frac{\theta^1}{(1-\theta^1)^2} + f_2(x_1^1) \right\}
\]
for a three-stage process including all reactors.

It is worth noting that

\[
x_1^0 = 0, \\
x_1^3 = \frac{\theta^3}{\gamma}.
\]

(B) Countercurrent system.

\[
f_1(x_1^2) = \min_{\theta^1} g_2(x_1^2; \theta^3)
\]

\[
= \min_{\theta^1} \left\{ \frac{\theta^3 - x_1^2}{(1-\theta^3)(1-\gamma + x_1^2)} \right\}
\]

\[
= \frac{\gamma - x_1^2}{(1-\gamma)(1-\gamma + x_1^2)}
\]

for a one-stage process including stage 3 (the third reactor),

\[
f_2(x_1^1) = \min_{\theta^2} \left\{ g_2(x_1^1; \theta^2) + f_1(x_1^2) \right\}
\]

\[
= \min_{\theta^2} \left\{ \frac{\theta^2 - x_1^1}{(1-\theta^2)(1-\gamma + x_1^1)} + f_1(x_1^2) \right\}
\]

for a two-stage process including stage 2 (the second reactor) and stage 3,
\[ f_3(x^0_1) = \min_{\theta^1} \left\{ g_2(x^0_1; \theta^1) + f_2(x^1_1) \right\} \]

\[ = \min_{\theta^1} \left\{ \frac{\theta^1}{(1 - \gamma)(1 - \theta^1)} + f_2(x^1_1) \right\} \]

for a three-stage process including all reactors.

The block diagram representing these functional relationships is given in Fig. 2.

**COMPUTATION AND RESULTS**

The computations have been performed on an IBM 1620 computer. The computer symbols and program are given in Tables 1 and 2, and the computer flow diagram is shown in Fig. 3. For purposes of illustration optimal tables are given in Tables 3a, 3b, and 3c for the cocurrent system and Tables 4a, 4b, and 4c for the countercurrent system for a coarse grid. More exact values of the optimal reactor volume for each stage in both cases are listed in Table 5. The optimal total reactor volume is plotted as a function of the conversion of reactant A in Fig. 4.

It can be concluded that:

1. The optimal volume of each stage is the same regardless of the conversion of reactant A in the countercurrent reaction system. The small variations in reactor volume allocation are due to the discretization error incurred by choosing a finite number of grid prints for computation.

2. For the same degree of conversion of reactant A, the countercurrent flow reaction scheme is more economical than the cocurrent flow reaction scheme if we neglect the cost of separators for the countercurrent flow
Fig. 2. Block diagram representation of the functional equations for the reactor system.
### TABLE 1
PROGRAM SYMBOLS AND EXPLANATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanations</th>
</tr>
</thead>
<tbody>
<tr>
<td>DELX1</td>
<td>increment of $x_1^1$</td>
</tr>
<tr>
<td>DELX2</td>
<td>increment of $x_1^2$</td>
</tr>
<tr>
<td>DELT1</td>
<td>increment of $\theta_1^1$</td>
</tr>
<tr>
<td>DELT2</td>
<td>increment of $\theta_2^2$</td>
</tr>
<tr>
<td>FC</td>
<td>final conversion of reactant A</td>
</tr>
<tr>
<td>F1X12</td>
<td>minimum volume for one-stage process</td>
</tr>
<tr>
<td>F2X11</td>
<td>minimum volume for two-stage process</td>
</tr>
<tr>
<td>G1X10</td>
<td>volume of stage 1</td>
</tr>
<tr>
<td>G2X11</td>
<td>volume of stage 2</td>
</tr>
<tr>
<td>G3X12</td>
<td>volume of stage 3</td>
</tr>
<tr>
<td>SG2F1</td>
<td>sum of the volumes of stages 2 and 3</td>
</tr>
<tr>
<td>TH1</td>
<td>conversion of reactant A at the outlet of stage 1</td>
</tr>
<tr>
<td>TH2</td>
<td>conversion of reactant A at the outlet of stage 2</td>
</tr>
<tr>
<td>X11</td>
<td>conversion of reactant A at the inlet of stage 2</td>
</tr>
<tr>
<td>X12</td>
<td>conversion of reactant A at the inlet of stage 3</td>
</tr>
</tbody>
</table>

- $\Delta x_1^1$
- $\Delta x_1^2$
- $\Delta \theta_1^1$
- $\Delta \theta_2^2$
- $\gamma$
- $f_1(x_1^2)$
- $f_2(x_1^1)$
- $g(x_1^1; \theta_1^1)$
- $g(x_1^1; \theta_2^2)$
- $g(x_1^2; \theta_3^3)$
- $g(x_1^1; \theta_2^2) + f_1(x_1^2)$
- $\theta_1^1$
- $\theta_2^2$
- $x_1^1$
- $x_1^2$
TABLE 2A
OPTIMIZATION OF REACTOR VOLUME WITH TWO PHASE REACTION (COCURRENT)

| DIMENSION X12(21),X11(21),TH2(21),TH1(21),G3X12(21),F1X12(21) |
| DIMENSION G2X11(21),SG2F1(21),F2X11(21),G1X10(21),SG1F2(21) |

1 FORMAT(9F8.3)
12 FORMAT(5F1(4))
READ 1,FC,X12(1),X11(1),TH2(1),TH1(1),DELX2,DELX1,DELT2,DELT1
CALC,G3X12(1)
DO 11 I=2,21
X12(I)=X12(1)+DELX2
G3X12(I)=(FC-X12(I))/((1.-FC)*(1.-FC))
F1X12(I)=G3X12(I)
PUNCH 12,X12(I),G3X12(I),F1X12(I)
11 CONTINUE
CALC,F2X11(J)
DO 21 J=2,21
X11(J)=X11(J-1)+DELX1
DO 22 I=J,21
TH2(I)=TH2(I-1)+DELT2
G2X11(I,J)=(TH2(I)-X11(J))/(1.-TH2(I))/(1.-TH2(I))
SG2F1(I,J)=G2X11(I,J)+F1X12(I)
IF (I-J)33,23,24
23 F2X11(J)=SG2F1(J,I)
PUNCH 12,X11(J),TH2(I),G2X11(I,J),F1X12(I),F2X11(I)
24 IF(F2X11(J))=SG2F1(J,I)22,27,25
25 F2X11(J)=SG2F1(J,I)
PUNCH 12,X11(J),TH2(I),G2X11(I,J),F1X12(I),F2X11(I)
22 CONTINUE
21 CONTINUE
CALC,F3X10
DO 31 J=2,21
TH1(J)=TH1(J-1)+DELT1
G1X10(J)=TH1(J)*(1.-TH1(J))*(1.-TH1(J))
SG1F2(J)=G1X10(J)+F2X11(J)
IF(J-1)33,32,33
32 F3X10=SG1F2(J)
PUNCH 12,TH1(J),G1X10(J),F2X11(J),F3X10
33 IF(F3X10)=SG1F2(J)31,31,34
34 F3X10=SG1F2(J)
PUNCH 12,TH1(J),G1X10(J),F2X11(J),F3X10
31 CONTINUE
END
Fig. 3. Computer flow diagram.
TABLE 3a

OPTIMAL TABLE (ONE STAGE COCURRENT PROCESS
WITH THE FINAL CONVERSION, \( \gamma = 0.2 \))

<table>
<thead>
<tr>
<th>( x_1^2 )</th>
<th>( g(x_1^2; \theta^3) )</th>
<th>( f_1(x_1^2) )</th>
<th>( x_1^3 = \theta^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.3125</td>
<td>0.3125</td>
<td>0.200</td>
</tr>
<tr>
<td>0.01</td>
<td>0.2969</td>
<td>0.2969</td>
<td>0.200</td>
</tr>
<tr>
<td>0.02</td>
<td>0.2813</td>
<td>0.2813</td>
<td>0.200</td>
</tr>
<tr>
<td>0.03</td>
<td>0.2656</td>
<td>0.2656</td>
<td>0.200</td>
</tr>
<tr>
<td>0.04</td>
<td>0.2500</td>
<td>0.2500</td>
<td>0.200</td>
</tr>
<tr>
<td>0.05</td>
<td>0.2344</td>
<td>0.2344</td>
<td>0.200</td>
</tr>
<tr>
<td>0.06</td>
<td>0.2188</td>
<td>0.2188</td>
<td>0.200</td>
</tr>
<tr>
<td>0.07</td>
<td>0.2031</td>
<td>0.2031</td>
<td>0.200</td>
</tr>
<tr>
<td>0.08</td>
<td>0.1875</td>
<td>0.1875</td>
<td>0.200</td>
</tr>
<tr>
<td>0.09</td>
<td>0.1719</td>
<td>0.1719</td>
<td>0.200</td>
</tr>
<tr>
<td>0.10</td>
<td>0.1563</td>
<td>0.1563</td>
<td>0.200</td>
</tr>
<tr>
<td>0.11</td>
<td>0.1406</td>
<td>0.1406</td>
<td>0.200</td>
</tr>
<tr>
<td>0.12</td>
<td>0.1250</td>
<td>0.1250</td>
<td>0.200</td>
</tr>
<tr>
<td>0.13</td>
<td>0.1094</td>
<td>0.1094</td>
<td>0.200</td>
</tr>
<tr>
<td>0.14</td>
<td>0.0938</td>
<td>0.0938</td>
<td>0.200</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0781</td>
<td>0.0781</td>
<td>0.200</td>
</tr>
<tr>
<td>0.16</td>
<td>0.0625</td>
<td>0.0625</td>
<td>0.200</td>
</tr>
<tr>
<td>0.17</td>
<td>0.0469</td>
<td>0.0469</td>
<td>0.200</td>
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</table>
TABLE 3b  
OPTIMAL TABLE (TWO STAGE COCURRENT PROCESS WITH THE FINAL CONVERSION, $\gamma = 0.2$)

<table>
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<tr>
<th>$x_1^1$</th>
<th>$x_1^2 = \theta^2$</th>
<th>$g(x_1^1; \theta^2)$</th>
<th>$f_1(x_1^2)$</th>
<th>$f_2(x_1^1)$</th>
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<tbody>
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</table>
TABLE 3c

OPTIMAL TABLE (THREE STAGE COCURRENT PROCESS WITH THE FINAL CONVERSION, $\gamma = 0.2$)

<table>
<thead>
<tr>
<th>$x_1^0$</th>
<th>$x_1^1 = \theta_1$</th>
<th>$x_1^2 = \theta_2$</th>
<th>$x_1^3 = \theta_3$</th>
<th>$g(x_1^0; \theta_1)$</th>
<th>$g(x_1^1; \theta_2)$</th>
<th>$g(x_1^2; \theta_3)$</th>
<th>$f(x_1^0)$</th>
</tr>
</thead>
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<tr>
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TABLE 4a

OPTIMAL TABLE (ONE STAGE COUNTERCURRENT PROCESS WITH THE FINAL CONVERSION, χ = 0.2)

<table>
<thead>
<tr>
<th>x₁²</th>
<th>g(x₁²; θ³)</th>
<th>f₁(x₁²)</th>
<th>x₁³ = θ³</th>
</tr>
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<tr>
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TABLE 4b
OPTIMAL TABLE (TWO STAGE COUNTERCURRENT PROCESS WITH THE FINAL CONVERSION, $\gamma = 0.2$)

<table>
<thead>
<tr>
<th>$x_1^1$</th>
<th>$x_1^2 = \theta^2$</th>
<th>$g(x_1^1, \theta^1)$</th>
<th>$f_1(x_1^2)$</th>
<th>$f_2(x_1^1)$</th>
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### TABLE 4c

Optimal Table (Three Stage Countercurrent Process with the Final Conversion, \( \gamma = 0.2 \))

<table>
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<th>( x_1^0 )</th>
<th>( x_1^1 = \theta^1 )</th>
<th>( x_1^2 = \theta^2 )</th>
<th>( x_1^3 = \theta^3 )</th>
<th>( g(x_1^0; \theta^1) )</th>
<th>( g(x_1^1; \theta^2) )</th>
<th>( g(x_1^2; \theta^3) )</th>
<th>( f_3(x_1^0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
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### TABLE 5

OPTIMAL REACTOR VOLUMES

(1) Cocurrent system

<table>
<thead>
<tr>
<th>Conversion, %</th>
<th>Volume $V^1$ (ft³)</th>
<th>Volume $V^2$ (ft³)</th>
<th>Volume $V^3$ (ft³)</th>
<th>Total (ft³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0865</td>
<td>0.0897</td>
<td>0.0931</td>
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</tr>
<tr>
<td>40</td>
<td>0.2427</td>
<td>0.2638</td>
<td>0.2833</td>
<td>0.7898</td>
</tr>
<tr>
<td>80</td>
<td>1.775</td>
<td>2.235</td>
<td>2.700</td>
<td>6.710</td>
</tr>
<tr>
<td>95</td>
<td>10.947</td>
<td>16.000</td>
<td>20.000</td>
<td>46.947</td>
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</table>

(2) Countercurrent system

<table>
<thead>
<tr>
<th>Conversion, %</th>
<th>Volume $V^1$ (ft³)</th>
<th>Volume $V^2$ (ft³)</th>
<th>Volume $V^3$ (ft³)</th>
<th>Total (ft³)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0894</td>
<td>0.0890</td>
<td>0.2674</td>
</tr>
<tr>
<td>40</td>
<td>0.2535</td>
<td>0.2538</td>
<td>0.2535</td>
<td>0.7607</td>
</tr>
<tr>
<td>80</td>
<td>1.610</td>
<td>1.590</td>
<td>1.610</td>
<td>9.810</td>
</tr>
<tr>
<td>95</td>
<td>6.076</td>
<td>6.043</td>
<td>6.076</td>
<td>18.195</td>
</tr>
<tr>
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<td>20.482</td>
<td>20.062</td>
<td>20.482</td>
<td>61.026</td>
</tr>
</tbody>
</table>
Fig. 4. Optimal total reactor volume vs conversion for two continuous flow stirred tank reactor systems.
reaction scheme because the former requires less reactor volume to achieve the same degree of conversion. As seen in Fig. 5, a considerable difference in the optimal total reactor volumes between the two systems occurs, especially at high conversions.

(3) In the cocurrent flow reaction scheme, the volume ratios of stages 2 and 3 to stage 1 are comparable with those given in Ref. 2 for the case of a second order irreversible reaction. For comparison, the numerical values are listed in Table 6. It can be seen that the differences become greater as the conversion increases because a slight increase in conversion requires a large increase in reactor volume at a higher conversion. Since finite numbers of grid points are chosen for computation, considerable interpolation is required, especially at higher conversions. The optimal volume ratios vs. conversion of reactant A for both cases are plotted in Fig. 6.

A comparison of the optimal and non-optimal values for four cases are given in Table 7.
Fig. 5. Difference in the optimum total volume between the cocurrent and countercurrent systems.
TABLE 6
RATIOS OF INDIVIDUAL OPTIMAL REACTOR VOLUMES FOR THE COCURRENT SYSTEM

<table>
<thead>
<tr>
<th>Conversion, %</th>
<th>$V_2/V_1$</th>
<th>This Study</th>
<th>Ref [2]*</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
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<td>1.037</td>
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<tr>
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<td>95</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Conversion, %</th>
<th>$V_3/V_1$</th>
<th>This Study</th>
<th>Ref [2]*</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
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</tr>
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<td>95</td>
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</table>

*Obtained by interpolation of the results of the Lagrangian multiplier solution given in Ref. [2].
Fig. 6. Optimal ratio of individual reactor volume.
### TABLE 7

**COMPARISON OF OPTIMAL AND NON-OPTIMAL VALUES**

(1) Co-current System

<table>
<thead>
<tr>
<th>Conversion, %</th>
<th>Case</th>
<th>$x_1^1 = \theta_1$</th>
<th>$x_1^2 = \theta_2$</th>
<th>$V_1^1$ ft$^3$</th>
<th>$V_2^2$ ft$^3$</th>
<th>$V_3^3$ ft$^3$</th>
<th>Total ft$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>Optimal</td>
<td>0.168</td>
<td>0.298</td>
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<td>0.2638</td>
<td>0.2833</td>
<td>0.7898</td>
</tr>
<tr>
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<td>0.162</td>
<td>0.294</td>
<td>0.2307</td>
<td>0.2539</td>
<td>0.3056</td>
<td>0.7902</td>
</tr>
<tr>
<td>95</td>
<td>Optimal</td>
<td>0.740</td>
<td>0.900</td>
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<td>16.000</td>
<td>20.000</td>
<td>46.947</td>
</tr>
<tr>
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<td>0.730</td>
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<td>10.014</td>
<td>13.223</td>
<td>24.000</td>
<td>47.237</td>
</tr>
</tbody>
</table>

(2) Countercurrent System

<table>
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<tr>
<th>Conversion, %</th>
<th>Case</th>
<th>$x_1^1 = \theta_1$</th>
<th>$x_1^2 = \theta_2$</th>
<th>$V_1^1$ ft$^3$</th>
<th>$V_2^2$ ft$^3$</th>
<th>$V_3^3$ ft$^3$</th>
<th>Total ft$^3$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.132</td>
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<td>0.2538</td>
<td>0.2535</td>
<td>0.7607</td>
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<tr>
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<td>Non-optimal</td>
<td>0.124</td>
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<td>0.2359</td>
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<td>0.233</td>
<td>0.717</td>
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<td>6.043</td>
<td>6.076</td>
<td>18.195</td>
</tr>
<tr>
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<td>6.688</td>
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<td>18.233</td>
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</tbody>
</table>
References:

NOMENCLATURE

\[ b_i \] : Numbers of hours required to produce a ton of grade \( i \), \( \text{hr ton} \).

\[ c \] : Total production hours, hr.

\[ N_A \] : Number of moles of reactant A, 16-mole.

\[ P_i \] : Net profit per ton of grade \( i \), \( \frac{\$}{\text{ton}} \).

\[ -r_A \] : Reaction rate based on reactant A, \( \frac{16\text{-mole}}{\text{ft}^3 \text{sec}} \).

\[ V_i \] : Reactor volume, \( \text{ft}^3 \).

\[ x^n_1 \] : Conversion of reactant A at the exit of stage \( n \).

\[ x_i \] : Number of tons of grade \( i \) produced in a week, \( \frac{\text{ton}}{\text{week}} \).

\[ S = P \] : Maximum return or net profit, \( \frac{\$}{\text{week}} \).

Greek Letters

\[ \theta^n \] : Conversion of reactant A at the inlet of stage \( n \).

\[ \gamma \] : Final conversion of reactant A.
CHAPTER III

OPTIMIZATION OF A MULTIEFFECT MULTISTAGE EVAPORATION SYSTEM BY DYNAMIC PROGRAMMING

The optimization of a multieffect multistage evaporator system with a nominal capacity of 50 MM gallons per day is considered here [1, 2].

There are 23 stages each in the first effect and second effect, and 22 stages in the third effect. The production of distillate in each stage is achieved by adiabatic flash vaporization such that the temperature of the flashing brine decreases as it cascades down stage by stage. A portion of the flashing brine from each effect is recycled back through the condenser tubes on which water vapor is condensing, so that the make-up feed and the recycle brine are preheated prior to mixing. The make-up feed together with the recycle brine in the first effect is finally heated to the maximum flashing temperature by a shell and tube brine heater with low pressure steam from an adjacent power plant. The diagram of the process is shown in Fig. 1.

The maximum brine temperature is limited by scale formation and the lower temperature end is slightly higher than the temperature of the make-up feed because a certain temperature difference for heat transfer must be maintained for heating the make-up and recycle brine.

This scheme is more efficient thermodynamically than recycling the blowdown from the last effect directly to the first effect, because the mixing streams are at the same temperature and the concentration differences between the mixing streams are reduced as the whole system is divided into three effects. The free energy decrease due to mixing, the thermodynamic irreversibility, is thus reduced.

Furthermore, the heat recovery becomes more efficient as the number of
Fig. 1. Process flow diagram of the multistage multieffect flash evaporation system.
stages used for flashing and preheating increases; inter-effect recycling makes it possible to add more stages per temperature gradient. Theoretically, the larger the number of stages, the higher the performance ratio (ratio of distillate production to steam consumption), but the number of stages cannot be increased without limit because a certain pressure gradient should be maintained for the flow of the flashing brine. The optimization involves the following factors:

1. number of total stages
2. total temperature differences (temperature differences between the flashing and recycle brine)
3. number of stages allocated to each effect
4. exit temperature and concentration of each effect
5. allocation of distillate production in each effect
6. brine velocities of various concentrations which affect the overall heat transfer coefficients of heater and condenser tubes.

We shall, however, take only the role of the total temperature differences and exit brine temperatures of each effect into account.

In this problem, the temperatures of the flashing brine is chosen as the state variable and the recycle ratio of each effect as the decision variable. The ratio of energy input to the make-up feed, q/F, is taken as the parameter. For clarity, performance and cost equations of this process are listed in Table 1 (see Appendices I and II for derivation).

1. STATEMENT OF PROBLEM

We shall employ dynamic programming to perform the optimization. The following conditions and assumptions are made in formulating this
optimization problem:

(1) Equal distillate production for each effect.

(2) Equal temperature drops of the flashing brine in each stage of the heat recovery section of each effect.

(3) Equal boiling point elevations for each stage such that the temperature profiles of the flashing brine and distillate are parallel.

(4) The temperature difference between the flashing and recycle brine is constant for the heat recovery section of each effect.

From the performance and cost equations given in Table 1, the following items and relations which are necessary for developing the functional relationships of dynamic programming are obtained.

(1) The steam cost is

\[ \frac{2C_1 q}{F \lambda_S} \]

(2) The fixed charge cost for the brine heater (see equation (3)) is

\[ \frac{1.92 C_2 (1+\theta^1)}{U_B} \left\{ \ln \left( 24.4 + \frac{q}{0.96 F(1+\theta^1)} \right) - \ln (24.4) \right\} \]

where \( \theta^1 \) is the recycle ratio of the first effect which is related to the maximum brine and first blowdown temperatures \( x_0^1 \) and \( x_1^0 \) as given in equation (1) by taking material and energy balances for the first effect.

\[ \theta^1 = \frac{1065 + 4.986x_1^1 - 5.594x_0^1}{5.748(x_0^1 - x_1^1)} \]

(3) The fixed charge cost for the condenser tubes in the heat recovery section of the first effect is
### TABLE 1

**SYSTEM EQUATIONS**

The first stage

1. Transformation equation
   
   (material and energy balance)
   
   \[ x_1 = \frac{(5.748 \theta_1^1 + 5.594)x_1^0 - 1065}{5.748 \theta_1^1 + 4.986} \]  
   
   \[ \theta_1^1 \text{ given} \]
   
   \[ X_1^0 \text{ given} \]
   
   \[ Q/F \text{ assumed} \]

2. Steam cost
   
   \[ \frac{2C_1q}{F \lambda_s} \]
   
   \[ C_1 \text{ given} \]
   
   \[ \lambda_s \text{ given} \]
   
   \[ q/F \text{ assumed} \]

3. Fixed charge cost for the brine heater
   
   \[ \frac{1.92C_2(1+\theta_1^1)}{U_B} \ln \left( \frac{24.4 + \frac{q}{0.96F(1+\theta_1^1)}}{24.4} \right) \]
   
   \[ C_2 \text{ given} \]
   
   \[ U_B \text{ given} \]

4. Fixed charge cost for the condenser tubes, heat recovery section
   
   \[ \frac{76.8C_3(1+\theta_1^1)(0.96(1+\theta_1^1)(1065-0.608x_1^0) - \frac{q}{F}(5.748 \theta_1^1 + 4.986))}{(U_1)_{av}(41q/F(5.748 \theta_1^1 + 4.986)-0.96(1+\theta_1^1)(1463.9-0.608x_1^0+459.8 \theta_1^1))} \]
   
   \[ C_3 \text{ given} \]
   
   \[ (U_1)_{av} \text{ given} \]
TABLE 1 (Cont'd)

(5) Fixed charge cost for the condenser tubes, heat rejection section

\[
(\frac{12c_3}{(u_1)_{av}})_{av} \left\{ \frac{3}{0.952(1+\theta^2)} + \frac{2}{0.96(1+\theta^1)} - \frac{12f}{q} \right\} \tag{5}
\]

(6) Recycle cost

\[
\frac{288c_4 \times \theta^1(P_0 - P_1)}{62.5 \times 550 \times 3600 \times 4.85} \tag{6}
\]

\[c_4: \text{given}\]

\[P_0: \text{from steam table}\]

\[P_1 = \exp \left\{ 14.07 - \frac{9 \times 3600}{5(x_1 - 32) + 1989} \right\} \]

The second stage

(1) Transformation equation
(material and energy balance)

\[x_1^2 = \frac{(5.685\theta^2 + 4.594)x_1^1 - 1065}{5.685\theta^2 + 3.986} \tag{7}\]

(2) Fixed charge cost for the condenser tubes, heat recovery section

\[
\frac{76.16c_3(1+\theta^2)}{(u_2)_{av}} \left\{ \frac{0.952(1+\theta^2)(1065 - 0.608x_1^1)}{5.685\theta^2 + 3.986} - \frac{a}{4f} (5.685\theta^2 + 3.986) \right\} \tag{8}
\]

\[(u_2)_{av} + \text{given}\]
TABLE 1 (Cont’d)

(4) Recycle cost

\[ \frac{288C_4 \theta^2(P_1 - P_2)}{62.5 \times 550 \times 3600 \times 0.85} \]  \hspace{1cm} (10)

\[ P_2 = \exp \left\{ 14.07 - \frac{9 \times 3600}{5(x_1^2-32) + 1989} \right\} \]

The third stage

(1) Transformation equation (material and energy balance)

\[ x_1^3 = \frac{(5.58\theta^3 + 3.594)x_1^2 - 1065}{5.58\theta^3 + 2.986} \]  \hspace{1cm} (11)

(2) Fixed charge cost for the condenser tubes, heat recovery section

\[ \frac{75.2C_3(1+\theta^3)\cdot 0.94(1+\theta^3)(1065 - 0.608x_1^2) - q \cdot (5.58\theta^3 + 2.986)}{(U_3)_{av}\left[\frac{41q}{F}(5.58\theta^3 + 2.986) - 0.94(1+\theta^3)(1303.9 - 0.609x_1^2 + 446.4\theta^3)\right]} \]  \hspace{1cm} (12)

\[(U_3)_{av} : \text{given}\]

(3) Fixed charge cost for the condenser tubes, heat rejection section

\[ \frac{7.52C_3(1+\theta^3)(5.58\theta^3 + 2.986)}{(U_3)_{av}\left[15.58\theta^3 + 2.986 + \frac{2F}{q}(5.58\theta^3 + 3.594)x_1^2 - 1065\right]0.94(1+\theta^3) - \frac{0.94F}{q}(1+\theta^3)(2T_F+8)(5.58\theta^3 + 2.986)} \]  \hspace{1cm} (13)

(4) Recycle cost

\[ \frac{288C_4 \theta^3(P_2 - P_3)}{62.5 \times 550 \times 3600 \times 0.85} \]
TABLE 1 (Cont'd)

\[ P_3 = \exp \left( 14.07 - \frac{9 \times 3600}{5(x_3 - 32) + 1989} \right) \]

**Others**

1. Dumping cost for the distillate and blowdown

\[
\frac{288C_4(14.7 - P_3)}{62.5 \times 550 \times 3600 \times 0.85}
\]  

2. Pumping cost for the make-up feed

\[
\frac{2.36C_4(\Delta H_0 + 50)}{550 \times 3600}
\]

\[
\Delta H_0 = \frac{144(P_0 - 14.7)}{62.5}
\]

3. Structure cost

\[
\frac{43.2 \times 7.48 \times 24}{62.4 \times 5 \times 10^7}
\]
It should be noted that $x_1^0$ is eliminated so that the fixed charge cost is expressed as a function of $x_1^1$ and $\theta^1$.

(4) The fixed charge cost for the condenser tubes in the heat recovery section of the second effect is

$$\frac{76.8C_3(1+\theta^1)[0.96(1+\theta^1)(1065-0.608x_1^1)-\frac{q}{F}(5.748\theta^1+5.594)]}{(U_1)_{av}\left\{\frac{q}{F}(5.748\theta^1+5.594)-0.96(1+\theta^1)(1512.5-0.608x_1^1+459.8\theta^1)\right\}}$$

where $\theta^2$ is the recycle ratio of the second effect. The recycle ratio is related to the first and second blowdown temperatures $x_1^1$ and $x_2^1$ as given in equation (7) of Table 1. Equation (7), which is obtained from the material and energy balances around the second effect, is

$$\theta^2 = \frac{1065 + 3.986x_2^1 - 4.594x_1^1}{5.685(x_2^1 - x_1^1)}$$

(5) The fixed charge cost for the condenser tubes in the heat recovery section of the third effect is

$$\frac{75.2C_3(1+\theta^2)[0.94(1+\theta^2)(1065-0.608x_1^2)-\frac{q}{F}(5.58\theta^2+3.594)]}{(U_3)_{av}\left\{\frac{q}{F}(5.58\theta^2+3.594)-0.94(1+\theta^2)(1452.5-0.608x_1^2+446.4\theta^2)\right\}}$$

where $\theta^3$ is the recycle ratio of the third effect, which is related to the second and final blowdown temperatures $x_1^2$ and $x_1^3$ as given in equation (11). By taking material and energy balances around the third effect, equation (11) is obtained as
The fixed charge cost for the condenser tubes in the heat rejection section of the third effect is

\[
\theta^3 = \frac{1065 + 2.986x_1^3 - 3.594x_1^2}{5.58(x_1^2 - x_1^3)}
\]

With these cost functions in hand, we can write the unit production cost for each stage, which, in the language of dynamic programming, is the so-called return function as

\[
G^1(x_1^1; \theta^1) = \text{eq. (2)} + \text{eq. (3)} + \text{eq. (4)}
\]

\[
G^2(x_1^2; \theta^2) = \text{eq. (8)}
\]

\[
G^3(x_1^3; \theta^3) = \text{eq. (12)} + \text{eq. (13)}
\]

The fixed charge costs for the condenser tubes in the heat rejection sections of the first and second effects and the recycle costs for each effect are not included in the optimization because

1. the determination of the optimal tube area for the heat rejection section involves decisions in memory.
2. the total recycle cost is not only approximately constant but is of minor significance to the total production cost (see Appendix II).

Therefore the cost items in Table I which are excluded from the present consideration will be added to the optimal production cost after the optimization has been carried out.

Now, let us consider the problem in more detail so that it will enable
us to choose the computational ranges of variables.

First of all, we may observe that in computing $G^1(x_1^1;\theta^1)$, the steam cost is constant for a given $q/F$ and the fixed charge cost for the brine heater is nearly proportional to $\theta^1$. Furthermore, from the relation of the energy input to the brine heater,

$$q/F = (1+\theta^1)(C_p)_{av} \Delta t_1.$$  
With a given $q/F$, the larger the value $\theta^1$ is, the smaller the temperature rise through the heater $\Delta t_1$, and the larger the fixed charge cost for the condenser tubes. It is obvious that $\Delta t_1$ directly affects the average temperature difference for heat transfer of the condenser tubes in the heat recovery section of the first effect.

In the derivation in Appendix II, we have shown that the condenser tube areas are approximately equal to

$$A_n = \frac{q_n}{(U_n)_{av} \Delta t_n} = \frac{F}{6} \frac{\theta_n \Delta t_n}{(U_n)_{av} \Delta t_n} \quad n = 1, 2, 3$$

and we also have the formula

$$\frac{q}{F} = (1 + r_n)(C_p)_{av} \Delta t_n \quad n = 1, 2, 3$$

where $r_n$ is replaced by $\theta^n$ in this formulation.

For a given $q/F$, $\frac{1}{\Delta t_n}$ is an approximately linear function of $\theta^n$ and so is $A_n$. With $q/F$ fixed, $G^1(x_1^1;\theta^1)$ is approximately a linear function of $\theta^1$ since the fixed charge cost for the condenser tubes takes on the major part of the variable cost in $G^1(x_1^1;\theta^1)$. It is also true of $G^2(x_1^2;\theta^2)$ and $G^3(x_1^3;\theta^3)$ because they contain only the fixed charge cost for the condenser
tubes.

From the above considerations, it is intuitively seen that the smaller the \( \theta^n \)'s, the smaller the unit production cost.

If we consider the limiting case where \( \theta^n \) goes to zero, the process turns out to be unfeasible.

Actually, from the material and energy balance

\[
\frac{F}{2} x (\lambda)_{av} = F(1+\theta)(C_{p,av}) (x_1^0 - x_1^3)
\]

for a one stage process, we see that the magnitude of \( \theta \) is approximately 2.36 as follows

\[
\frac{F}{2} (1000) = F(1+\theta)(0.96)(250-95)
\]

\[
\theta \approx 2.36
\]

For a three stage process, this is still true because \( \frac{F}{2} \) becomes \( \frac{F}{6} \), and \( x_1^0 - x_1^3 \) becomes approximately \( \frac{1}{3}(x_1^0 - x_1^3) \) and thus the magnitudes of \( \theta^n \) remain approximately 2.36. It does not mean that \( \theta^n \) must always take on this value, one of the \( \theta^n \) can take on a value much smaller than 2.36, but the other two \( \theta^n \) will become larger at the expense of its reduction.

Since

\[
q_1 = F(1+\theta^1)(C_{p, f1}) (x_1^0 - x_1^1) = \frac{F}{6} (\lambda_1)_{av}
\]

\[
q_2 = F(1+\theta^2)(C_{p, f2}) (x_1^1 - x_1^2) = \frac{F}{6} (\lambda_2)_{av}
\]

\[
q_3 = F(1+\theta^3)(C_{p, f3}) (x_1^2 - x_1^3) = \frac{F}{6} (\lambda_3)_{av}
\]

and
\[ (\lambda_1)_{av} < (\lambda_2)_{av} < (\lambda_3)_{av}. \]

we can see that

\[ q_1 < q_2 < q_3 \]

and hence

\[ x_1^0 - x_1^1 < x_1^1 - x_1^2 < x_1^2 - x_1^3 \]

for approximately equal \( \theta^n \), \( n = 1, 2, 3 \). This observation suggests the choices

\[ x_1^0 - x_1^1 < \frac{1}{3}(x_1^0 - x_1^3), \]

\[ x_1^1 - x_1^2 = \frac{1}{3}(x_1^0 - x_1^3), \]

\[ x_1^2 - x_1^3 > \frac{1}{3}(x_1^0 - x_1^3). \]

According to the dynamic programming algorithm [Ref. 3], the functional equations are written as:

(1) For a one-stage process including only the first stage (see Fig. 2)

\[ f_1(x_1^1) = \min_{\theta^1} \left\{ G^1(x_1^1; \theta^1) \right\} \]

(2) For a two-stage process including the first and second stages (see Fig. 2)

\[ f_2(x_1^2) = \min_{\theta^2} \left\{ G^2(x_1^2; \theta^2) + f_1(x_1^1) \right\} \]

(3) For a three stage process including all the stages (see Fig. 2)
Fig. 2. Schematic diagram showing the numbering of stages used for the imbedding scheme.
\[ f_3(x^3) = \min_{\theta^3} G^3(x^3_1; \theta^3) + f_2(x^2_1). \]

2. COMPUTATIONAL SCHEME

The computational procedure is summarized below:

(A) Computation of \( f_1(x^1_1) \)

1. Choose \( x^0_1 = 250, \quad q/F = 26.7 \)
2. Choose ten grid points for \( x^1_1 \) with increment 0.3

\[ x^1_1 = 201.7 \sim 204.4 \]
3. Compute for each path

\[ G^1(x^1_1; \theta^1) \]
4. Set \( f_1(x^1_1) \) equal to \( G^1(x^1_1; \theta^1) \) corresponding to each \( x^1_1 \).

(B) Computation of \( f_2(x^2_1) \)

1. Choose \( x^2_1 = 148.8, \quad q/F = 26.7 \)
2. Compute \( G^2(x^2_1; \theta^2) \) for each path connecting

\[ x^2_1 = 148.8 \text{ to } x^1_1 = 201.7 \sim 204.4 \]
3. Add the previously calculated \( f_1(x^1_1) \) to \( G^2(x^1_1; \theta^2) \) and find

\[ f_2(x^2_1) \]
4. Compute \( f_2(x^2_1) \) by varying \( x^2_1 \) with increment 0.3 for the range

\[ 148.8 \leq x^2_1 \leq 151.5. \]
(C) Computation of $f_3(x_1^3)$

1. Choose $x_1^3 = 90, \quad q/F = 26.7$

2. Compute $G^3(x_1^3; \theta^3)$ for each path connecting $x_1^3 = 90$ to $x_2^3 = 148.8 \sim 151.5$.

3. Add the previously calculated $f_2(x_2^2)$ to $G^3(x_1^3; \theta^3)$ and find $f_3(x_1^3)$.

4. Compute $f_3(x_1^3)$ by varying $x_1^3$ with increment 0.5 for the range $90 \leq x_1^3 \leq 94.5$.

Repeat this computational procedure by varying q/F with increment 0.3 for $26.7 \leq q/F \leq 27.9$. The computational paths are shown in Fig. 3. The computer flow diagram is shown in Fig. 4 and the computer symbols and program are given in Tables 2a and 2b.

3. RESULTS AND DISCUSSION

The computations were carried out on an IBM 1620 computer, and the optimal tables are shown in Tables 3, 4 and 5. It is noted that for each q/F, the production cost becomes lower and lower as the final blowdown temperature $x_1^3$ goes down.

From the temperature profiles for the heat rejection section of the third effect, it is seen that (Fig. 5)

$$\frac{\Delta t_{3r} + \Delta t_{3i}}{2} \geq \frac{\Delta t_{3r}}{2} + 2$$

or

$$\Delta t_{3j} \geq 4$$

where $\Delta t_{3r}$ = temperature drop of the flashing brine across the heat
Fig. 3. Grid points of the state variable taken for computation. The connected line is the optimal path. \( q_F = 27.0 \)
Fig. 4. Computer flow diagram.
### TABLE 2a

**EXPLANATION OF COMPUTER PROGRAM SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CI</td>
<td>unit steam cost, $C_1$</td>
</tr>
<tr>
<td>C2</td>
<td>unit fixed charge cost, brine heater, $C_2$</td>
</tr>
<tr>
<td>C3</td>
<td>unit fixed charge cost, condenser tubes, $C_3$</td>
</tr>
<tr>
<td>F1X1</td>
<td>maximum return function, one-stage process, $f_1(x^1)$</td>
</tr>
<tr>
<td>F2X2</td>
<td>maximum return function, two-stage process, $f_2(x^2)$</td>
</tr>
<tr>
<td>F3X3</td>
<td>maximum return function, three-stage process, $f_3(x^3)$</td>
</tr>
<tr>
<td>G1X1</td>
<td>return function, stage 1, $G^1(x^1;\theta^1)$</td>
</tr>
<tr>
<td>G2X2</td>
<td>return function, stage 2, $G^2(x^2;\theta^2)$</td>
</tr>
<tr>
<td>G3X3</td>
<td>return function, stage 3, $G^3(x^3;\theta^3)$</td>
</tr>
<tr>
<td>QDF</td>
<td>ratio of the energy input to the make-up feed, $q/F$</td>
</tr>
<tr>
<td>RMS</td>
<td>latent heat of the live steam, $\lambda_s$</td>
</tr>
<tr>
<td>SG12</td>
<td>sum of the return function of stage 2 and the maximum return function of the one-stage system, $G^2(x^2;\theta^2) + f_1(x^1)$</td>
</tr>
<tr>
<td>SG23</td>
<td>sum of the return function of stage 3 and the maximum return function of the two-stage system, $G^3(x^3;\theta^3) + f_2(x^2)$</td>
</tr>
<tr>
<td>TF</td>
<td>temperature of the make-up feed, $T_F$</td>
</tr>
<tr>
<td>TH1</td>
<td>ratio of the recycle brine in the first stage to the make-up feed, $\theta^1$</td>
</tr>
<tr>
<td>TH2</td>
<td>ratio of the recycle brine in the second stage to the make-up feed, $\theta^2$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Explanation</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>TH3</td>
<td>ratio of the recycle brine in the third stage to the make-up feed, $\theta^3$</td>
</tr>
<tr>
<td>UB</td>
<td>overall heat transfer coefficient of the heater tubes, $U_B$</td>
</tr>
<tr>
<td>U1AV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the first stage, $(U_1)_{av}$</td>
</tr>
<tr>
<td>U2AV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the second stage, $(U_2)_{av}$</td>
</tr>
<tr>
<td>U3AV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the third stage, $(U_3)_{av}$</td>
</tr>
<tr>
<td>X1</td>
<td>blowdown temperature, the first stage, $x_1$</td>
</tr>
<tr>
<td>X2</td>
<td>blowdown temperature, the second stage, $x_2$</td>
</tr>
<tr>
<td>X3</td>
<td>blowdown temperature, the third stage, $x_3$</td>
</tr>
</tbody>
</table>
TABLE 2B

OPTIMIZATION OF THE MULTIFLASH EVAPORATION PROCESS

DIMENSION X1(11), X2(11), X3(11), G1X1(11), F1X1(11), TH1(11)
DIMENSION F2X2(11), F3X3(11), TH2(11, 11), TH3(11, 11), G2X2(11, 11)
DIMENSION G3X3(11, 11), SG12(11, 11), SG23(11, 11)
1 FORMAT(6X, 6E11.4)
12 FORMAT(1H, 2F10.2, 2X, 2E12.6)
13 FORMAT(1H, 3F10.2, 2F11.5, 2F12.6)
READ 1, C1, C2, C3, RMS, U5, U1AV
READ 1, U2AV, U3AV, TF, QDF
PRINT 1, C1, C2, C3, RMS, U5, U1AV
PRINT 1, U2AV, U3AV, TF, QDF
CALC.TH1(J), G1X1(J)
X0 = 250.
X1(1) = 201.4
DC 11 J = 2, 11
X1(J) = X1(J-1) + 3.0
TH1(J) = (1.65 + 4.986*X1(J) - 5.594*X1(J))/(5.748*(X0 - X1(J)))
G1X1(J) = 2*C1*QDF/RMS + 0.92*C2*(1.0 + TH1(J))*(LOG(24.4 + QDF/(0.96*(1.0 +
TH1(J)) - LOG(24.4)))/1065.8*C3*(1.0 + TH1(J)))*(QDF/(1.0 + TH1(J)))*(1065.8
- 608*X1(J)) - QDF*(5.748*TH1(J) + 5.594)/U1AV/(41.0*QDF*(5.748*TH1(J)
+ 5.594) - 96*(1.0 + TH1(J))*1512.5 - 608*X1(J) + 45.9*R*TH1(J))
F1X1(J) = G1X1(J)
PUNCH 12, X1(J), TH1(J), G1X1(J), F1X1(J)
11 CONTINUE
CALC.TH2(K, J), F2X2(K)
X2(1) = 148.5
DC 21 K = 2, 11
X2(K) = X2(K-1) + 3.0
DC 22 J = 2, 11
TH2(K, J) = (1.65 + 3.086*X2(K) - 4.94*X2(K))/(5.685*(X1(J) - X2(K)))
G2X2(K, J) = 76.16*C3*(1.0 + TH2(K, J)))*(0.952*(1.0 + TH2(K, J)))*(1065.8
- 608*X2(K)) - QDF*(5.685*TH2(K, J) + 4.594)/U2AV/(41.0*QDF*(5.685*TH2(K,
J) + 4.594) - 952*(1.0 + TH2(K, J))*(1432.5 - 608*X2(K) + 454.8*R*TH2(K,
J)))
SG12(K, J) = G2X2(K, J) + F1X1(J)
IF(J-2) = 23, 24
23 F2X2(K) = SG12(K, J)
PUNCH 13, X2(K), TH2(K, J), X1(J), G2X2(K, J), F1X1(J), F2X2(K)
24 IF(F2X2(K) - SG12(K, J)) = 22, 26
26 F2X2(K) = SG12(K, J)
PUNCH 13, X2(K), TH2(K, J), X1(J), G2X2(K, J), F1X1(J), F2X2(K)
27 CONTINUE
21 CONTINUE
CALC. TH3(L,K), F3X3(L)
X2(1) = 89.5
DO 31 L = 2, 11
   X3(L) = X3(L-1) + .5
   DO 32 K = 2, 11
      TH3(L,K) = (1.65 + 7.986*X3(L) - 3.594*X2(K)) / (5.58*(X2(K)-X3(L)))
      G3X3(L,K) = 75.2*C3*(1+TH3(L,K))*2.94*(1+TH3(L,K))*1065 - 608*X3
      1(L)) - QDF * (5.58*TH3(L,K) + 3.594) / U3AV / (41 * QDF * (5.58*TH3(L,K) 3.594
      2) - 94*(1 + TH3(L,K)) * (1352 - 608*X3(L) + 446.4*TH3(L,K)) + 7.52*C3*(
      31 + TH3(L,K)) / U3AV / (1 + 1.88*QDF*(1 + TH3(L,K)) * X3(L) - 1.88*QDF*(1 +
      4*TH3(L,K)) * (TF + 4.))
      SG23(L,K) = G3X3(L,K) + F2X2(K)
      IF(K-2) 33, 33
      32 F3X3(L) = SG23(L,K)
      PUNCH 13, X3(L), TH3(L,K), X2(K), G3X3(L,K), F2X2(K), F3X3(L)
      34 IF(F3X3(L) - SG23(L,K)) 32, 32, 36
      36 F3X3(L) = SG23(L,K)
      PUNCH 13, X3(L), TH3(L,K), X2(K), G3X3(L,K), F2X2(K), F3X3(L)
32 CONTINUE
31 CONTINUE
END

(Cont'd)
### TABLE 3

**ONE STAGE OPTIMAL TABLE**

\( (x_1^0 = 250, \ q/F = 26.7) \) \hspace{1cm} \( (x_1^0 = 250, \ q/F = 27.0) \)

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<th>( f_1(x_1) \times 10^5 )</th>
<th>( x_1 \times 10^5 )</th>
<th>( f_1(x_1) \times 10^5 )</th>
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<td>$/lb</td>
<td>$/lb</td>
<td>( ^0F )</td>
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<td>1.69194</td>
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TABLE 3 (Cont'd)

ONE STAGE OPTIMAL TABLE

\((x_1^0 = 250, \ q/F = 27.3)\)

\((x_1^0 = 250, \ q/F = 27.6)\)

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<th>(f_1(x_1^0))</th>
<th>(x_1^0)</th>
<th>(x_1^1)</th>
<th>(g_1^1(x_1^0; \theta_1^0))</th>
<th>(f_1(x_1^0))</th>
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<td>1.71307 (x10^5)</td>
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<td>1.72484 (x10^5)</td>
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<td>1.71491 (x10^5)</td>
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<td>1.72664 (x10^5)</td>
<td>1.72664 (x10^5)</td>
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<td>1.71677 (x10^5)</td>
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TABLE 3 (Cont'd)

ONE STAGE OPTIMAL TABLE

\( (x_1^0 = 250, \ q/F = 27.9) \)

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TABLE 4
TWO STAGE OPTIMAL TABLE

$q/F = 26.7$

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<td>°F</td>
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**TABLE 4 (Cont'd)**

**TWO STAGE OPTIMAL TABLE**

\( q/F = 27.0 \)

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<th>( f_2(x_1^2) \times 10^5 )</th>
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</thead>
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<td>$/lb</td>
<td>$/lb</td>
<td>$/lb</td>
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**TWO STAGE OPTIMAL TABLE**

\( q/F = 27.3 \)

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### TABLE 4 (Cont'd)

**TWO STAGE OPTIMAL TABLE**

\[ q/F = 27.6 \]

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<th>( f_1(x_1^2) \times 10^5 )</th>
<th>( f_2(x_1^2) \times 10^5 )</th>
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<td>$/lb</td>
<td>( ^\circ F )</td>
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THREE STAGE OPTIMAL TABLE

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Conclusion:

(1) Optimal path

\[
x_1^0, \quad \theta^1, \quad x_1^1, \quad \theta^2, \quad x_1^2, \quad \theta^3, \quad x_1^3
\]

250.0, 2.42, 201.7, 2.45, 149.1, 2.47, 91.0

(2) $q/F = 27.1$

(3) Optimal production cost: $2.27654/lb distillate
Fig. 5. Temperature profiles of the flashing brine, distillate and make-up feed + coolant in the heat rejection section of the third effect.
rejection section.

\[ \Delta t_{3j} = \text{temperature difference between the spent and make-up brine.} \]

For this reason, if a make-up feed of 85°F is used, the flashing brine temperature should be at least 89°F. Actually, from the mechanical point of view, there is a restriction on the final blowdown temperature because there is a limitation to the highest vacuum maintainable.

From the computed results we can see that, for a final blowdown temperature, \( x_1 \), below 91°F, the production cost is strictly monotonically decreasing, but this does not necessarily guarantee that the production cost is really decreasing because we have not included the pumping cost of the coolant in this study. If we take an energy balance for the heat rejection section of the third effect, the following approximate relationship holds.

\[
F(1+r_3)(C_p)_{f3} \Delta t_{3r} = F(1+r_c)(C_p)_{F} \Delta t_{3j} .
\]

Since \((C_p)_{f3} = (C_p)_{F}\) and \( r_3 \) is of the magnitude of 2.5, we have, by taking \( \Delta t_{3r} = 8 \) and \( \Delta t_{3j} = 4 \)

\[
8(1+r_3) = (1+r_c)^4
\]

\[
r_c = 2r_3 + 1 = 6 .
\]

where \( r_c \) is the ratio of the coolant and make-up feed. It is seen that a large quantity of coolant is required and hence, there is an increase in the pumping cost as the blowdown temperature decreases which we did not take into account.

It is concluded that the decrease in production cost by lowering the final blowdown temperature does not necessarily compensate for the increase due to the pumping cost of the coolant. Therefore, it would probably be
better that the final blowdown temperature be fixed at 91°F or slightly above this.

In Fig. 6 a plot of the production cost vs q/F is given using the final blowdown temperature $x_1^3$ as a parameter. It is seen that there is a minimum point corresponding to each final blowdown temperature and that it shifts in the direction of decreasing q/F as the final blowdown temperature decreases. From Fig. 7, it can be seen that the optimal value of q/F is located at a point approximately equal to 27.1 and that the corresponding optimal production cost is equal to $2.27654 \times 10^{-5}$/lb. It is to be noted that there is a minor error involved in this estimation due to the discretization of the grid points which is inherent in dynamic programming. A value more accurate than this for the optimal production cost can be obtained by linear interpolation. Since the production cost curve is so flat, this interpolation was not carried out here.

Finally, we add the general expenses and administrative expenditures of $0.172/10^3$ gallons so that the optimal value of the total production cost, $0.3993/10^3$ gallons, or approximately $0.4/10^3$ gallons at q/F of 27.1, can be obtained.

In Fig. 8, the production costs are plotted against the final blowdown temperatures at the computed optimal value of q/F = 27.0, which is closest to the true optimal value of q/F = 27.1. It shows that the production cost is approximately a linear function of the final blowdown temperature for a given q/F. This is consistent with the fact that for a given heat input to the system, the lower the final blowdown temperature, the better the heat economy and the lower the production cost. In Fig. 8, a plot of the production cost vs. the second blowdown temperature $x_1^2$ is also given with $x_1^1$.
Fig. 6. Production cost vs. the ratio of energy input to the make-up feed with the final blowdown temperature $X_1^3$ as parameter.
Fig. 7. Production cost plotted against the ratio of energy input to the make-up feed with the final blowdown temperature $x_1^3 = 91^\circ F$. 
Fig. 8. Production cost vs. final blowdown temperature $x^3_{i_1}$ and vs. second blowdown temperature $x^2_{i_1}$ with $x^1_{i_1} = 201.7$, $x^3_{i_1} = 91$ and $\gamma_F = 27$. 
and $x_1^3$ fixed in the optimal path of $201.7^\circ F$ and $91.0^\circ F$ respectively. The optimal production costs in both plots agree with each other.

In Table 6, comparison of the production cost among cases with optimal and non-optimal policies is given. The optimal production cost is equal to $0.2273/10^3$ gallon. There is another case which gives the same value due to the truncation error.

REFERENCES


TABLE 6
COMPARISON OF THE PRODUCTION COST AMONG THE CASES WITH OPTIMAL AND NON-OPTIMAL POLICIES

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Results of the optimal policy by D. P.

| 27.1 | 2.42 | 201.7 | 2.45 | 149.1 | 2.47 | 91.0 | 0.2273 |

Results of the non-optimal policy by P. P.

| 27.0 | 2.44 | 202.0 | 2.44 | 149.4 | 2.48 | 91.5 | 0.2277 |
| 27.0 | 2.46 | 202.3 | 2.46 | 150.0 | 2.49 | 92.5 | 0.2281 |
| 27.0 | 2.42 | 201.7 | 2.46 | 149.4 | 2.48 | 91.5 | 0.2275 |
| 27.0 | 2.42 | 201.7 | 2.43 | 148.8 | 2.48 | 91.0 | 0.2273* |
| 27.0 | 2.44 | 202.0 | 2.44 | 149.4 | 2.48 | 91.5 | 0.2276 |
| 27.0 | 2.44 | 202.0 | 2.48 | 150.0 | 2.49 | 92.5 | 0.2281 |

Results of simulation

<table>
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<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>Production cost ($/10^3$ gallon)</th>
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<tr>
<td>26.4</td>
<td>2.441</td>
<td>202.0</td>
<td>2.523</td>
<td>150.6</td>
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<td>2.481</td>
<td>152.0</td>
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<td>204.0</td>
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<td>152.0</td>
<td>2.369</td>
</tr>
</tbody>
</table>

*This is due to the truncation error in the computer printing.
NOMENCLATURE

c_1 \quad \text{Unit steam cost,} \quad \frac{\$}{\text{lb}}

c_2 \quad \text{Unit fixed charge cost, brine heater,} \quad \frac{\$}{\text{ft}^2 \cdot \text{hr}}

c_3 \quad \text{Unit fixed charge cost, condenser tubes,} \quad \frac{\$}{\text{ft}^2 \cdot \text{hr}}

c_4 \quad \text{Unit pumping cost,} \quad \frac{\$}{\text{HPhr}}

f_1(x_1^1) \quad \text{Maximum return function, one-stage process,} \quad \frac{\$}{\text{lb}}

f_2(x_1^2) \quad \text{Maximum return function, two-stage process,} \quad \frac{\$}{\text{lb}}

f_3(x_1^3) \quad \text{Maximum return function, three-stage process,} \quad \frac{\$}{\text{lb}}

G(x_1^1; \theta_1) \quad \text{Return function, stage 1,} \quad \frac{\$}{\text{lb}}

G^2(x_1^2; \theta_2) \quad \text{Return function, stage 2,} \quad \frac{\$}{\text{lb}}

G^3(x_1^3; \theta_3) \quad \text{Return function, stage 3,} \quad \frac{\$}{\text{lb}}

\frac{q}{F} \quad \text{Ratio of the energy input to the make-up feed,} \quad \frac{\text{lb}}{\text{lb}}

T_F \quad \text{Temperature of the make-up feed,} \quad ^\circ F

U_B \quad \text{Overall heat transfer coefficient of the heater tubes,} \quad \frac{\text{BTU}}{\text{ft}^2 \cdot \text{hr}}

(U_n^\text{av}) \quad \text{Overall heat transfer coefficient of the condenser tubes, subscript } n \text{ is the stage number,} \quad \frac{\text{BTU}}{\text{ft}^2 \cdot \text{hr}}

x_1 \quad \text{Blowdown temperature of the first stage,} \quad ^\circ F

x_2 \quad \text{Blowdown temperature of the second stage,} \quad ^\circ F
\( \alpha_3 \)  Blowdown temperature of the third stage, \(^{\circ}\)F

Greek Letters

\( s \)  Latent heat of process steam, BTU/\(\text{lb}\)

\( \gamma^n \)  Ratio of the recycle brine to the make-up feed, \(\text{lb}\)/\(\text{lb}\), superscript is the stage number,
CHAPTER IV

REFINEMENT OF SOLUTIONS AND REDUCTION IN DIMENSIONALITY

As stated in Chapter I, the main disadvantage of dynamic programming is the dimensionality difficulty which occurs as the number of state variables increases. Various schemes have been proposed to overcome the dimensionality difficulty [1, 2, 3, 4]. Most of them essentially trade computer time for computer memory. One way of reducing the memory requirement is to increase the interval size, or to decrease the number of grid points. But the discretization error will be magnified and hence there results a solution which is not very accurate. Either linear interpolation or extrapolation must be used to refine the solution for the case in which the transformation functions are almost linear or only approximate solutions are desired. Furthermore, iteration can also be used to obtain more accurate results.

In this chapter, a more sophisticated interpolation technique, namely the polynomial approximation, and several methods for reducing the dimensionality of a process such as the use of the Lagrangian multipliers, the ratio of state variables and the method of continuous independent variable are presented.

1. POLYNOMIAL APPROXIMATION

Instead of listing a table of grid points for the state variables, the table which represents the maximum return function $f_{N-1}(x)$ is correlated into a polynomial during the computations [2]. Thus each polynomial stands for a maximum return function, $f_n(x)$, for $n = 1, \ldots, N$. In the present approach, we shall use the least-square method and represent the optimal table by means of orthogonal Legendre polynomials.
The function \( f_n(x) \) is a polynomial of degree \( R \), and \( P_k(x) \) is a polynomial of degree \( K \) on the interval \([-1, 1]\).

From the orthogonal properties of these polynomials

\[
\int_{-1}^{1} P_k(x) P_j(x) \, dx = \begin{cases} 
0 & k \neq j \\
\frac{2}{2k+1} & k = j
\end{cases}
\]

one is able to obtain

\[
a_{kn} = \frac{2k+1}{2} \int_{-1}^{1} f_n(x) P_k(x) \, dx .
\]

The integration on the right of equation (2) may be performed by use of Gaussian quadrature

\[
\int_{-1}^{1} f_n(x) P_k(x) \, dx = \sum_{j=1}^{s} b_j f_n(x_j) P_k(x_j)
\]

which is exact when \( f_n(x) P_k(x) \) is a polynomial of degree \( 2s-1 \), and the \( x_j \)'s are the zeros of the Legendre polynomial of degree \( s \) and the \( b_j \)'s are the Christoffel numbers (weights). For this functional approximation and integration the optimum grid spacing is not equidistant and we require the storage of \( R+1 \) coefficients [4].

Now, we consider the system with the following functional equations

\[
f_1(c) = \max_{\theta} g(c; \theta)
\]

\[
f_N(c) = \max_{\theta} \left\{ g(c; \theta) + f_{N-1}(T(c; \theta)) \right\}
\]

(4)
where $c$ is the initial state and $T(c; \Theta)$ is the transformation equation.

In order to obtain $f_n(x_j)$ we must first evaluate $f_1(x_j)$ for $x_1, x_2, \ldots, x_s$, the zeros of the Legendre polynomial of degree $s$. We evaluate $f_1(x_j)$ as usual for a one-stage process by the proper choice of $\Theta$; with the value of $f_1(x_j)$ obtained, we evaluate $f_1(x)$ for any value of $x$ in the range of $(-1, 1)$. The $b_j$ and $P_k(x_j)$ are found from tables or the Legendre polynomial recursion equations. Then $a_k,1$ is found for $k = 1, 2, \ldots, R$. Once the $a_k,1$ are known, we can evaluate $f_1(x)$ for any $x$ within $(-1, 1)$. In such a manner, $f_1(x)$ is curve fitted from the values of the $f_1(x_j)$ terms.

For the two-stage process, we evaluate $f_2(x_j)$ by equation (4). The choice of the $\Theta$ that maximizes the right hand side of equation (4) transforms the state of the system from $x_j$ into $(x_j + \Delta x_j)$. Since the new state $x_j + \Delta x_j$ will, in general, not be a zero of the Legendre polynomial of degree $s$, the term $f_1(x_j + \Delta x_j)$ can be evaluated from the coefficients $a_k,1$, which were developed from the one-stage process. The coefficients $a_{k,2}$ for $k = 0, 1, \ldots, R$ can be obtained from equation (3) as soon as $f_2(x_j)$ are evaluated. From $a_{k,2}$ and equation (1), $f_2(x)$ is found for general values of $x$. Again, $f_2(x)$ is curve fitted from $f_2(x_j)$. In this manner, we develop $f_N(x)$. It is noted that the Legendre polynomials $P_k(x_j)$ and the coefficient $b_j$ can be calculated independently of $f_N(x_j)$ [5].

The following is an outline of this scheme applied to the optimum allocation of residence time in a continuous stirred tank reactor system.

Let us consider an isothermal first order reversible reaction

$$
A \xrightarrow{k_1} B \xrightarrow{k_2}
$$
carried out in a sequence of three CSTR's as shown in Fig. 1.

We take
\[ k_1 = \text{forward reaction rate} = 1.19, \text{ min}^{-1} \]
\[ k_2 = \text{backward reaction rate} = 0.02, \text{ min}^{-1} \]
\[ c_e = \text{equilibrium concentration of B} = 0.95, \text{ lb-mole/ft}^3 \]
\[ r = \text{final concentration of B} = 0.9, \text{ lb-mole/ft}^3. \]

Our objective is to minimize the total residence time by proper choice of \( \theta_n, n = 1, 2 \) (\( \theta_3 \) is fixed since the final concentration of B is given).

According to [6], the least total holding time required for an N-stage process is

\[
f_N(c_{N+1}) = \frac{N}{k_1 + k_2} \left[ \frac{c_e - c_{N+1}}{c_e - r} \right]^{1/N - 1}
\]

where \( c_{N+1} \) is the initial concentration of B. We shall consider the case \( c_{N+1} = 0 \), and perform the approximation by use of a Legendre polynomial of degree 3.

Then it takes the form

\[
f_n(c) = \sum_{k=0}^{3} a_{kn} P_k(c)
\]

since the interval of \( c \) is \( 0 \leq c \leq 0.9 \), we have to transform the interval of integration by the formula (7)

\[
c = \frac{(b-a)x + b + a}{2}
\]

in which \(-1 \leq x \leq 1\), and \( a \leq c \leq b \). In this case \( a = 0, b = 0.9 \), so that

\[
c = \frac{0.9x + 0.9}{2} = 0.45(x+1)
\]
Fig. 1. Schematic diagram for an isothermal, first-order, reversible reaction \( A \xrightarrow{k_1} \frac{k_1}{k_2} B \).
or

\[ x = \frac{c - 0.45}{0.45} \]

The corresponding polynomials

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2} (3x^2 + 1) \]
\[ P_3(x) = \frac{1}{2} (5x^3 - 3x) \]

are transformed into functions of \( c \).

Since we take \( K = 3, \ s = 4 \) will give an exact formulation. Starting with a one stage process, we write

\[ f_1(c_2) = \sum_{k=0}^{3} a_{k,1} P_k(c_2) \]
\[ f_n(c) = \sum_{k=0}^{3} a_{kn} P_k(c) . \]

Since the interval of \( c \) is \( 0 \leq c \leq 0.9 \), we have to transform the interval of integration by the formula (7)

\[ c = \frac{(b-a)x + b + a}{2} \]

in which \(-1 \leq x \leq 1\), and \( a \leq c \leq b \). In our case \( a = 0, \ b = 0.9 \), so that

\[ c = \frac{0.9x + 0.9}{2} = 0.45(x+1) \]

or
\[ x = \frac{c - 0.45}{0.45} \]

The procedure is:

1. Find the roots \( x_j \) and weights \( b_j \) of the Legendre polynomial of degree 4 from a mathematical table (\( j = 1, 2, 3, 4 \)).

2. Transform \( P_k(x) \) to \( P_k(c_j) \) by \( x = \frac{c - 0.45}{0.45} \) for \( k = 0, 1, 2, 3 \).

3. Evaluate \( P_k(c_{2j}) \)
   \[ c_{2j} = 0.45(x_j + 1) \]

4. Evaluate \( f_1(c_{2j}) \).

5. Replace \( b_j \) by \( 0.45b_j \) for change in interval from \(-1 \leq x \leq 1\) to \( 0 \leq c_2 \leq 0.9 \).

6. Evaluate \( a_{k1} \) by
   \[ a_{k1} = \frac{2k+1}{2} \sum_{j=1}^{4} b_j f_1(c_{2j}) P_k(c_{2j}), \quad k = 0, 1, 2, 3 \]

7. Find \( f_1(c_2) \) for any value of \( c_2 \).

For a two-stage process, we can proceed in the same manner.

Since the functional equation is of the form
\[ f_2(c_3) = \min_{\theta_2} \{ g(c_3; \theta_2) + f_1(c_2) \} \]
or
\[ f_2(c_{3j}) = \min_{\theta_{2j}} \{ g(c_{3j}; \theta_{2j}) + f_1(c_{3j} + \Delta c_{3j}) \} \]
we have to choose the optimal value of \( \theta_{2j} \) corresponding to \( c_{3j} \) so that \( f_2(c_{3j}) \) are obtained for \( j = 1, 2, 3, 4 \).
It should be noted that the first and second label of the subscript are the stage and grid point number (root number of the Legendre polynomial) respectively.

2. LAGRANGIAN MULTIPLIERS

Lagrangian multipliers have been extensively used for the optimization of non-linear functions with some types of constraints. Aris [8] used the technique to consider constraints in dynamic programming problems. Kuo and Rubin [9] applied the method to determine inlet conditions of a process in order to extremize characteristics of the chemical reactions and simultaneously meet certain constraints such as specified product yield or temperature.

Let a function \( f(x_1, \ldots, x_n) \) be minimized subject to the constraints

\[
g_j(x_1, \ldots, x_n) = 0, \quad j = 1, \ldots, m \quad m \leq n. \tag{5}
\]

Then, it is required to find the minima of

\[
f(x_1, \ldots, x_n) + \sum_{j=1}^{M} \lambda_j g_j(x_1, \ldots, x_n) \tag{6}
\]

where \( \lambda_j \) are called Lagrangian multipliers. This involves setting the partial derivatives of this function equal to zero, but this in itself is not a sufficient condition for a relative minimum. For example, in the case of \( f(x_1, x_2) = x_1 x_2 \), the origin is neither a minimum nor a maximum and is called a saddle point. In any case, setting the partial derivatives of this function equal to zero is a necessary condition for a stationary value of the function. The sufficient conditions for relative maxima and minima involve, in addition to the first partial derivative being zero, the
inequalities in the second order derivatives.

When the variables are constrained by inequalities rather than by equations, or by both types, then the Lagrange method cannot be directly applied. However, an appreciable extension has been developed by Kuhn and Tucker [10].

For the case of handling inequality constraints, such as \( T_{\text{min}} \leq T \leq T_{\text{max}} \), we introduce so-called slack variables and let \( u^2 = (T - T_{\text{min}})(T_{\text{max}} - T) \geq 0 \), when \( T = T_{\text{min}} \) or \( T_{\text{max}} \). This is suggested by Valentine [11].

Next, we will consider the formal application of the Lagrangian multipliers in the dynamic programming formulation. Suppose we wish to maximize a sum of \( N \) functions \( \sum_{n=1}^{N} g_n(x_n) \) of the \( n \) positive variables \( x_1, \ldots, x_N \), subject to restrictions such as

\[
\sum_{n=1}^{N} k_{mn} x_n \leq k_m, \quad m = 1, \ldots, M.
\]  

(7)

Then the maximum we obtain will be a function of the set of quantities \( k_m \), and we write

\[
\max \sum_{n=1}^{N} g_n(x_n) = f_N(k_1, \ldots, k_N).
\]  

(8)

If we make an allocation \( x_N \) then the remaining allocations \( x_{N-1}, \ldots, x_1 \) are subject to

\[
\sum_{n=1}^{N-1} k_{mn} x_n = K_m - k_m(x_N).
\]  

(9)

The maximum of \( \sum_{n=1}^{N} g_n(x_n) \) subject to these restrictions is
Thus writing

\[ f_{N-1}(k_1, k_2, \ldots, k_M) = \sum_{n=1}^{N} g_n(x_n) \]

and applying the principle of optimality, we have

\[ f_N(k_1, \ldots, k_M) = \max \left[ g_N(x_N) + f_{N-1}(k_1 - k_{\text{MN}}(x_N), \ldots, k_M - k_{\text{MM}}(x_M)) \right] \]  (10)

This is a typical dynamic programming formulation but its dimensionality is \( M \) and the storage requirements increase exponentially with \( M \).

If we modify the objective function to

\[ \sum_{n=1}^{N} g_n(x_n) - \lambda M \sum_{n=1}^{N} k_{\text{MN}}(x_n) \]  (11)

we can solve the problem for fixed \( \lambda_M \) and consider only the first (M-1) restrictions in equation (7). Thus we can write

\[ f_N(k_1, \ldots, k_{M-1}, \lambda_M) = \max \left[ \sum_{n=1}^{N} \left[ g_n(x_n) - \lambda M k_{\text{MN}}(x_n) \right] \right] \]  (12)

and its dimensions are only (M-1). When it has been solved, we have only to find the \( \lambda_M \) which satisfies the condition in (6). In general, we can eliminate the (M-1) restrictions by introducing Lagrangian multipliers and write

\[ f_N(k_1, \ldots, k_L; \lambda_{L+1}, \ldots, \lambda_M) = \max \sum_{n=1}^{N} \left[ g_n(x_n) - \sum_{m=L+1}^{M} \lambda_m k_{mn}(x_n) \right] \]  (13)

We now have dimension \( L \) but must look for the set of (M-L) Lagrangian
multipliers that satisfy the last (M-L) restrictions. This is in effect, exchanging computer memory for time.

In regard to the physical significance of the Lagrangian multiplier, one may consider it the operating cost per unit volume for the CSTR sequence when one considers the optimal allocation of the reactor volume or equivalently of the residence (or holding) time with restriction on the total volume [6].

A Lagrangian multiplier may also be interpreted as the relative cost or cost ratio when one considers the allocation of a limited resource, an illustrative example using a Lagrangian multiplier will be presented in Chapter VI for the solvent allocated to the cross-current extraction system with a recycle loop.

3. RATIOS OF STATE VARIABLES

If the transformation functions of the state variables are homogeneous in nature and are linear with respect to each other, this linear property simplifies the problem in that the ratio of two state variables may be used as a new state variable. This reduces the dimensionality difficulty in that it results in a reduction in the number of state variables [8].

As an illustration, let us consider an isothermal first order consecutive reaction,

\[ \frac{k_1}{A} = 0.1 \quad \frac{k_2}{R} = 0.1 \]

\[ A \rightarrow R \rightarrow S, \]

carried out in a series of N CSTR as shown in Fig. 2. If we define

\[ x_1^n = \text{concentration of species } A \text{ at stage } n \]
\[ x_2^n = \text{concentration of species } R \text{ at stage } n \]

then the reaction rates of A and R are
Fig. 2. Schematic diagram for an isothermal, first-order, consecutive reaction $A \xrightarrow{k_1} R \xrightarrow{k_2} S$. 
\[ (-r_A^n) = k_1 x_1^n \]
\[ (-r_R^n) = k_2 x_2^n - k_1 x_1^n \, . \]

The material balance of each species at stage \( n \) is

\[ v x_1^{n+1} = v x_1^n + (-r_A^n) v^n \]

or

\[ x_1^{n+1} = x_1^n + 0.1 x_1^n x_1^n \]  \hspace{1cm} (14)

and

\[ v x_2^{n+1} = v x_2^n + (-r_R^n) v^n \]

or

\[ x_2^{n+1} = x_2^n + (0.1 x_2^n - 0.1 x_1^n) x_1^n \]  \hspace{1cm} (15)

where

\[ v = \text{volumetric flow rate, } \text{ft}^3/\text{sec.} \]

The production of \( R \) for a definite feed condition is to be maximized by choosing optimal decisions \( \theta^n \), \( n = 1, 2, \ldots, N \). This is a two-dimensional problem which not only requires a complex computational scheme but also demands a great deal of computer memory. However, the problem can be simplified by defining a new state variable

\[ x_3^n = \frac{x_2^n}{x_1^n} \]  \hspace{1cm} (16)

because of the linear transformation equations involved.
Dividing equation (15) by equation (14)

\[
\frac{x_{2}^{n+1}}{x_{1}^{n+1}} = \frac{x_{2}^{n} + \theta^{n}(0.1x_{2}^{n} - 0.1x_{1}^{n})}{x_{1}^{n}(1 + 0.1\theta^{n})}
\]

\[
x_{3}^{n+1} = \frac{x_{3}^{n} + \theta^{n}(0.1x_{3}^{n} - 0.1)}{1 + 0.1\theta^{n}}
\]

or

\[
x_{3}^{n} = \frac{x_{3}^{n+1}(1 + 0.1\theta^{n}) + 0.1\theta^{n}}{1 + 0.1\theta^{n}}
\]  

(17)

This equation gives a functional relation for determining the output ratio \(x_{3}^{n}\) at any stage in terms of the feed ratio \(x_{3}^{n+1}\).

Since the production of \(R\) is to be maximized, the objective function is

\[
x_{2}^{1} - x_{2}^{N+1} = \sum_{1}^{N} (x_{2}^{n} - x_{2}^{n+1})
\]  

(18)

and the maximum return function is

\[
f_{N}(x_{1}^{N+1}, x_{2}^{N+1}) = \max_{\{\theta^{n}\}} \sum_{1}^{N} (x_{2}^{n} - x_{2}^{n+1})
\]  

(19)

Because of the linear nature of the equations, we can write

\[
f_{N}(x_{1}^{N+1}, x_{2}^{N+1}) = x_{1}^{N+1} \frac{x_{1}^{N+1} x_{2}^{N+1}}{x_{1}^{N+1} x_{1}^{N+1}}
\]

\[
= x_{1}^{N+1} g(x_{3}^{N+1})
\]  

(19)

Applying the dynamic programming algorithm, we have
\[ f_N(x_1^{N+1}, x_2^{N+1}) = \max_{\theta^N} \left\{ (x_2^N - x_2^{N+1}) + f_{N-1}(x_1^N, x_2^N) \right\} \]

\[ = \max_{\theta^N} x_1^{N+1} \left\{ \frac{x_2^N - x_2^{N+1}}{x_1^N + \theta^N} - \frac{x_1^N}{x_1^{N+1}} g(x_3^{N-1}) \right\} , \quad (20) \]

and the functional equation becomes

\[ g_N(x_3^{N+1}) = \max_{\theta^N} \left\{ \frac{0.1\theta^N}{1 + 0.1\theta^N} - \frac{0.1\theta^N x_3^N}{1 + 0.1\theta^N} + \frac{1}{1 + 0.1\theta^N} g_{N-1}(x_3^N) \right\} . \quad (21) \]

Equation (21) can be solved recursively in conjunction with equation (17).

It is worth noting that there is "the menace of an expanding grid" as Bellman stated if the ratio of two state variables is greater than one. In this case, one can define an inverse ratio of two state variables which is less than one.

Therefore, one can write

\[ f_N(x_1^{N+1}, x_2^{N+1}) = x_1^{N+1} g(x_3^{N+1}) = x_2^{N+1} h(x_4^{N+1}) \quad (22) \]

where

\[ x_3^{N+1} = \frac{x_2^{N+1}}{x_1^N} \]

and

\[ x_4^{N+1} = \frac{x_1^{N+1}}{x_2^N} . \]

In this way one can use whichever ratio is suitable, and all the functions
can be tabulated in the finite range between zero and one [12].

4. DYNAMIC PROGRAMMING WITH CONTINUOUS INDEPENDENT VARIABLE
   (STATE INCREMENT DYNAMIC PROGRAMMING)

   A useful technique in reducing the high speed memory requirements needed
to solve optimization problems by dynamic programming has been developed by
Larson [13]. This new computational procedure is called dynamic programming
with continuous independent variable, or state increment dynamic programming.
It is based on the iterative application of Bellman's principle of optimality.
It differs from the conventional method in the choice of the time interval
of control. Instead of using a fixed interval, the new procedure determines
the time interval as the minimum time required for at least one of the state
variables to change by one increment. As a consequence of this choice of
interval, the next state for any given control is known to be within a small
neighborhood of the point at which control is applied. By using this result,
it is possible to compute optimal control in units called blocks that cover
a relatively long time interval but a small distance along each state
variable. By using only one or two high speed memory locations per state in
the block, it is possible to compute the optimal control throughout the
block. Specialized computations near the boundaries of the block allow the
optimal trajectories to pass from block to block.

   This technique has been applied in a computer program which calculates
minimum fuel trajectories for supersonic (Mach 3) aircraft under a variety
of conditions and constraints. The basic program can be used both for a
detailed evaluation of an aircraft design and for real-time control of an
aircraft.
A. General problem.

Let us consider the non-autonomous system of the form

\[ \frac{dx}{dt} = g(x, \theta, t) \] (23)

\[ x = s \text{ dimensional state variable} \]
\[ \theta = r \text{ dimensional control variable} \]
\[ t = \text{ independent variable} \]
\[ g = s \text{ dimensional vector function.} \]

Equation (23) is a set of non-linear time varying differential equations.

The system equations can be written as

\[ x(t+\Delta t) = x(t) + g(x(t), \theta(t), t)\Delta t \] (24)

where \( \Delta t \) is a small incremental independent variable.

The cost function \( S \) is to be minimized by proper choice of a set of admissible control variables \( u \).

The cost function is given as

\[ S(x(t_0), \theta, t_0) = \int_{t_0}^{T} \ell(x(\tau), \theta(\tau), \tau) d\tau \] (25)

\[ t_0 = \text{ initial time} \]
\[ T = \text{ final time} \]
\[ \tau = \text{ dummy variable of integration} \]
\[ \ell = \text{ loss function.} \]

B. The principle of optimality

We define

\[ f(x, t) = \min_{\theta(\tau) \in U, t \leq \tau \leq T} \left[ \int_{t}^{T} \ell(x(\tau), \theta(\tau), \tau) d\tau \right] \] (26)
The admissible control function that achieves this minimum is called the optimal control and is denoted as $\bar{\theta}(x,t)$. Invoking the principle of optimality, we write

$$f(x,t) = \min_{\{\theta\}} \left[ \ell(x,\theta,t)\Delta t + f(x) + g(x,\theta,t) t, t + \Delta t \right]$$

(27)

C. The example of supersonic flight.

The notion of dynamic programming with a continuous independent variable can best be presented by considering a simple, physical example. The problem under consideration is that of calculating the optimal control of the altitude of a supersonic air transport flying at a constant velocity. The altitude $h$ and the flight path angle $\alpha$ define the state and control variables respectively, and the system equation can be written as

$$\frac{dh}{dt} = g(h,\alpha,t) = v \sin \alpha$$

(28)

which becomes

$$\frac{dx}{dt} = g(x,\theta,t) = v \sin \theta$$

(29)

upon introduction of the above defined state and control variables.

Where $v$ is the magnitude of the constant velocity vector, the cost function is the total fuel consumed, so that the loss function becomes

$$\ell(x,\theta,t) = \frac{dw}{dt} = W_F(x,\theta,w,t)$$

(30)

where $W = \text{weight of aircraft plus fuel}$

$W_F = \text{fuel flow per unit time}$.
D. The concept of a continuous independent variable.

In the conventional dynamic programming the state variable and the independent variable, in this case time, are quantized according to the following.

The state variable, altitude, is divided into uniform increments, $\delta h$, over a finite range $0 \leq h \leq \beta$. Therefore, the state variable, $x$, assumes the form

$$ x = j \delta h, \quad j = 0, 1, \ldots, N \quad (31) $$

so that

$$ \beta = N \delta h. $$

The independent variable is also quantized into uniform increments, $\delta t$. At every point on the h-t plane, the control variable, $U$, assumes values of the set of admissible controls given by

$$ U = \{ u(1), u(2), \ldots, u(k) \} \quad (32) $$

In the continuous independent variable concept, the increments, $\delta t$, are not fixed but are determined by

$$ \delta t = \frac{\delta h}{v \sin u} \quad (33) $$

Equation (33) then allows $\delta t$ to vary over a wide range dependent upon the control variable, $U$. 

REFERENCES


CHAPTER V

THE K-TH BEST POLICY

It is natural that, whenever possible, the optimal policy should be used. However, it is of importance to set up the best alternative to the optimal policy for the immediate adoption in case that the optimal policy is inaccessible. Furthermore, a knowledge of the structure of the optimal and suboptimal, i.e., second best, third best,..., policies will give us a better understanding of the process. The determination of suboptimal policies also has significance in connection with sensitivity analysis in the numerical solution of an optimization problem. To solve a problem numerically, finite difference approximation is used together with digital computers. Because of the limited memory capacities of computers, we are often forced to use coarse grids of variables. Under this circumstance, a study of the neighborhood of the optimal policy can facilitate the evaluation of the meaningfulness of the solution. Thus a significant difference between the optimal and suboptimal policies may tell us that the approximation is too crude. If the suboptimal policies differ slightly from the optimal policy, we are assured that the solution is reliable.

Based on the principle of optimality, Bellman and Kalaba [1] presented an elegant method to obtain the suboptimal policies. Recently, Fan et al. [2] developed a detailed computational technique, which may be employed in applying the above-mentioned method to practical problems. In this chapter, the detailed computational technique is presented and illustrated by a simple directed network problem. An example in the optimal design of a multistage process with parallel redundancy is worked out in detail to show
how such a technique can be applied to practical problems.

1. THE ALGORITHM

For a multistage decision process (see Fig. 1 in Chapter I) in which the state of the process stream is transformed at each stage by the decision made at that stage, a typical optimization problem is to determine the decision at each stage, subject to certain constraints, so that the objective function which is the performance criterion of the process, is maximized.

The $k$-th best policy is defined as the sequence of decisions which gives the objective function a value which is smaller than all those values given by 1st, 2nd, ..., $(k-1)$-th best policies, but is at least as great as the values of the objective functions given by all other suboptimal policies, i.e. $(k+1)$-th, $(k+2)$-th, ....

The algorithm given below can be used to determine the first $k$ best policies, all at a time, for the processes with a finite difference of admissible decisions at each stage.

We define:

$\{ \theta_s \} = \text{the set of admissible decisions}$

$p_n(x, \theta_n) = \text{the return function at stage } n \text{ with the state of the entering stream } x \text{ and the decision } \theta_n$

$f_N = \sum_{n=1}^{N} p_n(x, \theta_n) = \text{the objective function for an } N\text{-stage process}$

$T(x, \theta_n) = \text{transformation of the state } x \text{ resulting from the decision } \theta_n$

$\text{Max}_{s} \{ r_s \} = \text{the } k\text{-th largest value of the quantities } r_s, s = 1, 2, ...$
\[ f^{(k)}_n(x) = \text{the value of the objective function for an } n\text{-stage process} \]
when the \( k \)-th best policy is followed and the state of the stream entering the initial stage is \( x \).

\[ f^{i,j}_n = \max_s \left\{ f^{(i)}_{n-1}(T(x, \theta_s)) + p_n(x, \theta_s) \right\} \]

To determine the optimal and suboptimal policies all at a time, we make a straightforward extension of the principle of optimality as follows [2]:

"The \( k \)-th best policy has the property that, whatever the initial state and decision are, the remaining decisions must be one of these sequences of decisions which will constitute the first \( k \) best policies with regard to the state resulting from the initial decision."

Thus, the decision at each stage in a process following the \( k \)-th best policy can be determined one by one, starting from the last stage. If we consider a certain stage at which a decision is to be made as the initial stage, it follows from the above-stated principle that only the first \( k \) best policies for the remaining part of the process need to be considered in determining the decision at the stage under consideration.

The construction of optimal tables follow the procedures stated in Section 1 of Chapter II.

Since the decision at stage \( n \) for the \( k \)-th best policy is that which gives \( f^{(k)}_n(x) \), it is required to find the \( f^{(k)}_n(x) \) from all possible values of the objective function. The following is a convenient way to carry it out:

1. Compute \( f^{i,j}_n \)
2. Construct the following ordered array
\[
[f_{i,j}^n] = \begin{bmatrix}
  f_{1,1}^1 & f_{1,2}^1 & f_{1,3}^1 \\
  f_{1}^2 & f_{2}^2 & f_{3,3}^2 \\
  f_{3}^2 & f_{3}^3 & f_{3}^3 \\
  & & \\
  & & \\
  & & \\
  & & \\
  & & \\
  & & \\
  & & \\

dl'.
\]

(3) \( f^{(k)}_n(x) \) is then determined as the k-th largest element in \([f_{i,j}^n]\),
i.e., \( f^{(k)}_n(x) = \max_{i,j,k} \{f_{i,j}^n\} \). This can be conveniently carried out
as follows: Let \([f_{i,j}^n]\) be the remaining ordered array obtained
by eliminating from \([f_{i,j}^n]\) all those elements which constitute
\( f^{(1)}_n(x), f^{(2)}_n(x), \ldots, f^{(k-1)}_n(x) \). Then

\[
f^{(k)}_n(x) = \max_{i,j} \{ \text{elements of } [f_{i,j}^n] \}
\]

To determine the maximum elements in \([f_{i,j}^n]\), the first elements of each
row in \([f_{i,j}^n; k^j]\) is compared until the row whose first element is \( f_{i,1}^n \).

2. A MULTISTAGE PROCESS WITH PARALLEL REDUNDANCY

The following example shows the application of the algorithm to the
optimal design of a multistage process with parallel redundancy [2]. Fig.
1 shows a multistage process in which a primary raw material is reacted with
a secondary specie in the initial stage to produce an intermediate product
which is then fed to the next stage and reacted with another secondary specie
Fig. 1  Series processing sequence.
and so on through the entire process. Suppose that the secondary species are all quite unstable and cannot be stored and, therefore, must be produced upon demand by special reactions. All the intermediate products are also assumed to be unstable. Then it is clear that if a secondary specie is not available on time at any stage, the entire process will fail.

Such a failure is a stochastic phenomenon and, therefore, can be considered from a probabilistic point of view. The probability that the n-th secondary specie will be available on time is called the reliability of stage n and represented by $R_n$. The reliability of the whole process, $R$, is the probability that all the $N$ secondary species are available on time and thus

$$R = \prod_{n=1}^{N} R_n$$

If the process which produces a certain secondary specie fails frequently, it would be desirable to produce more than one batch of that specie to increase the probability that it will be available on time. The production of more than one batch to reduce the effects of failure is named the parallel redundancy.

Suppose that $b_n$ batches of the n-th secondary specie are prepared. Since only one batch is needed, $(b_n - 1)$ batches are redundant. The probability that all $b_n$ batches will fail is equal to $(1 - R_n)^{b_n}$. Hence, the probability that at least one batch will succeed is $[1 - (1 - R_n)^{b_n}]$ which is by definition the reliability of stage n with its redundancies. Thus, the reliability of the entire process with redundancy can be represented by
\[ R = \frac{N}{\sum_{n=1}^{N} \left[ 1 - (1 - R_n)^b_n \right]} \]  

(1)

Since \( b_n \geq 1 \) and \( R_n \leq 1 \), it can be shown that

\[ \left[ 1 - (1 - R_n)^b_n \right] \geq R_n \]

which indicates that the reliability of the process is increased by the use of the parallel redundancy.

Let \( P_g \) be the unit price of the final product. The expected return for the system is then expressed by \( P_g R \). Let \( C_n \) be the construction cost of one reactor for the production of the \( n \)-th secondary specie (the cost is properly distributed over the life of the process), and \( O_n \) be the operating cost. Then, the net profit of the entire process, \( P \), is

\[ P = P_g R - \sum_{n=1}^{N} (C_n + O_n) b_n \]  

(2)

The optimal parallel redundancy is the design which maximizes \( P \) given in equation (2). Rudd [3] has applied the method of dynamic programming in the optimal design of such a process. Now let us use the algorithm described in Section 1 to find the first four best designs, all at a time.

For the sake of definiteness, we will consider a three-stage process with the following data:

<table>
<thead>
<tr>
<th>Stage</th>
<th>( C_n )</th>
<th>( O_n )</th>
<th>( R_n ) for one batch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 3</td>
<td>0.1</td>
<td>0.1</td>
<td>1/3</td>
</tr>
<tr>
<td>Stage 2</td>
<td>0.5</td>
<td>0.5</td>
<td>1/2</td>
</tr>
<tr>
<td>Stage 1</td>
<td>0.5</td>
<td>0.5</td>
<td>3/4</td>
</tr>
</tbody>
</table>

The unit price associated with the final product, \( P_g \), is 10 units. Now,
if \( P \) in equation (2) is considered as the objective function, \( R \) in equation (1) as the state variable which is denoted by \( x \), and \( b_n \) as decision, the following relations can be obtained:

\[
f_{1,}^{i, j} = \max_{b_1} \left\{ 10x_1 - b_1 \right\}, \quad b_1 = 1, 2, \ldots \quad (3)
\]

\[
x_1 = x_2 \left[ 1 - \left( \frac{1}{4} \right)^{b_1} \right] \quad (4)
\]

\[
f_{2,}^{i, j} = \max_{b_2} \left\{ f_{1}^{i}(x_2) - b_2 \right\}, \quad b_2 = 1, 2, \ldots \quad (5)
\]

\[
x_2 = x_3 \left[ 1 - \left( \frac{1}{2} \right)^{b_2} \right] \quad (6)
\]

\[
f_{3,}^{i, j} = \max_{b_3} \left\{ f_{2}^{i}(x_3) - 0.2 \cdot b_3 \right\}, \quad b_3 = 1, 2, \ldots \quad (7)
\]

\[
x_3 = x_4 \left[ 1 - \left( \frac{2}{3} \right)^{b_3} \right] = 1 - \left( \frac{2}{3} \right)^{b_3} \quad (8)
\]

Here \( x_n \) is the reliability of the process consisting of stage \( n \) and all upstream stages. Since the primary specie is assumed to be always available, \( x_4 = 1 \).

For the one-stage process consisting of stage 1 only, we have

\[
f_{1}^{(k)}(x) = f_{1,}^{1, k}
\]

This can be readily obtained and therefore the construction of the ordered array of \( f_{1,}^{i, j} \) is not necessary. A portion of the numerical results is given in Table 5.

For the two-stage process including stage 1 and stage 2, all values of \( f_{2,}^{i, j} \) are calculated from equation (5) for each possible values of \( x_3 \). The
### TABLE 5

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>$f_1^{(1)}(x_2)$</th>
<th>$b_1$</th>
<th>$f_1^{(2)}(x_2)$</th>
<th>$b_1$</th>
<th>$f_1^{(3)}(x_2)$</th>
<th>$b_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.65</td>
<td>4.093</td>
<td>2</td>
<td>3.875</td>
<td>1</td>
<td>3.398</td>
<td>3</td>
</tr>
<tr>
<td>0.70</td>
<td>4.562</td>
<td>2</td>
<td>4.250</td>
<td>1</td>
<td>3.890</td>
<td>3</td>
</tr>
<tr>
<td>0.75</td>
<td>5.031</td>
<td>2</td>
<td>4.625</td>
<td>1</td>
<td>4.382</td>
<td>3</td>
</tr>
<tr>
<td>0.80</td>
<td>5.500</td>
<td>2</td>
<td>5.000</td>
<td>1</td>
<td>4.875</td>
<td>3</td>
</tr>
<tr>
<td>0.85</td>
<td>5.968</td>
<td>2</td>
<td>5.375</td>
<td>1</td>
<td>5.367</td>
<td>3</td>
</tr>
<tr>
<td>0.90</td>
<td>6.437</td>
<td>2</td>
<td>5.859</td>
<td>3</td>
<td>5.750</td>
<td>1</td>
</tr>
</tbody>
</table>

### TABLE 6

<table>
<thead>
<tr>
<th>$x_3$</th>
<th>$f_2^{(1)}(x_3)$</th>
<th>$b_2$</th>
<th>$x_2$</th>
<th>$i^*$</th>
<th>$f_2^{(2)}(x_3)$</th>
<th>$b_2$</th>
<th>$x_2$</th>
<th>$i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90</td>
<td>2.382</td>
<td>3</td>
<td>0.7875</td>
<td>1</td>
<td>2.327</td>
<td>2</td>
<td>0.6750</td>
<td>1</td>
</tr>
<tr>
<td>0.95</td>
<td>2.792</td>
<td>3</td>
<td>0.8312</td>
<td>1</td>
<td>2.679</td>
<td>2</td>
<td>0.7125</td>
<td>1</td>
</tr>
<tr>
<td>1.00</td>
<td>3.202</td>
<td>3</td>
<td>0.8750</td>
<td>1</td>
<td>3.031</td>
<td>2</td>
<td>0.7500</td>
<td>1</td>
</tr>
</tbody>
</table>

* $i$ represents the number $i$ in $f_n^{(i)}(T(x, \theta_n))$, which is involved in computing $f_n^{(k)}(x)$.
The corresponding \( f_{2}^{\text{i,j}} \) array is then constructed. For example, the ordered array for \( x_{3} = 0.9 \) is

\[
\begin{bmatrix}
2.382 & 2.062 \\
2.327 & 1.906 \\
1.909 & \\
\end{bmatrix}
\]

The values of \( f_{2}^{(k)}(x_{3}) \) for \( x_{3} = 0.9 \) are then obtained from the above array. Similarly, we can find \( f_{2}^{(k)}(x_{3}) \) for all other values of \( x_{3} \). A part of the results is shown in Table 6.

The construction of the table for the three-stage process is similar to that for the two-stage process. However, for the three-stage process it suffices to construct only for \( x_{4} = 1 \). The numerical results are given in Table 7. The first four highest profits for the whole process are represented by those \( f_{3}^{(k)}(x_{4}) \), \( k = 1, 2, 3, 4 \) listed in Table 7. The corresponding optimal and suboptimal policies are then determined by the table entry technique utilizing Tables 5 through 7. For example, the optimal policy is obtained as follows:

Starting from Table 7, we find \( f_{3}^{(1)}(x_{4}) = 1.322 \) with \( b_{3} = 7, x_{3} = 0.9414 \) and \( i = 1 \). By means of linear interpolation, the values of \( b_{2}, x_{2} \) and \( i \) for \( x_{3} = 0.9414 \) can be obtained from Table 6. The results are

\[
b_{2} = 3, \quad x_{2} = 0.8237, \quad i = 1
\]

Similarly, from Table 5 we find \( b_{1} = 2 \). The optimal and suboptimal policies thus obtained are summarized in Table 8.
### TABLE 7

<table>
<thead>
<tr>
<th>$k$</th>
<th>$f_3^k(x_4)$</th>
<th>$b_3$</th>
<th>$x_3$</th>
<th>$i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.322</td>
<td>7</td>
<td>0.9414</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1.282</td>
<td>6</td>
<td>0.9122</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1.282</td>
<td>8</td>
<td>0.9609</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1.218</td>
<td>7</td>
<td>0.9414</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1.212</td>
<td>6</td>
<td>0.9122</td>
<td>2</td>
</tr>
</tbody>
</table>

### TABLE 8

<table>
<thead>
<tr>
<th>Profit</th>
<th>Policies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$b_1$</td>
</tr>
<tr>
<td>1-st highest = 1.322</td>
<td>2</td>
</tr>
<tr>
<td>2-nd highest = 1.282</td>
<td>2</td>
</tr>
<tr>
<td>3-rd highest = 1.218</td>
<td>2</td>
</tr>
<tr>
<td>4-th highest = 1.212</td>
<td>2</td>
</tr>
</tbody>
</table>
References


NOMENCLATURE

$\Theta_s$ the set of admissible decisions.

$f_{n-1}$ the optimal value of the return function for the (n-1) stages.

$f_n^{(k)}$ the value of the return function for an n-stage process in the kth best policy.

$f_n$ return function for an N-stage process.

$P$ return function of the whole process.

$P_g$ unit price of the final product.

$P_n(x, \theta_n)$ return of stage n with a state x and a decision $\theta_n$.

$R_n$ reliability of stage n.

$T(x; \theta_n)$ transformation of the state x by the decision $\theta_n$. 
CHAPTER VI

OPTIMIZATION OF COMPLEX MULTISTAGE PROCESSES BY
THE DYNAMIC PROGRAMMING TECHNIQUE

In this chapter, an application of the dynamic programming technique to
the optimization of geometrically complex multistage processes is presented.

Based on the dynamic programming technique, Mitten and Nemhauser [1],
and Aris [2] have presented several schemes for the optimization of certain
types of geometrically complex processes. Aris, Nemhauser and Wilde [4]
presented a method for formulating cyclic and branching problems. Compara-
tive advantages and disadvantages of both methods are discussed in Ref. [5].

1. COMPLEX MULTISTAGE PROCESSES

Since a multistage process is made up by a number of stages, its features
are solely determined by the types of stages of which it is composed and the
way in which the stages are linked [5].

A stage is a unit consisting of at least one input and one output
stream. The quantities of an input or output stream of a deterministic pro-
cess are uniquely expressed by a state vector \( x_n \) and each stage is associated
with a decision vector \( \theta_n \) where the subscript \( n \) is the label for the stage.
The state vector of the output streams from a stage is a function of the
state vector of the input streams and the decision vector of that stage. As
mentioned in the previous chapters, the functional relationship can be
expressed in vector form as

\[
x_n = T_n(x_{n-1}; \theta_n).
\]

where \( T_n(x_{n-1}; \theta_n) \) is the transformation function, \( x_{n-1} \) and \( x_n \) are the input
and output state vectors of stage n respectively.

Stages can be conveniently classified according to the numbers of input and output streams associated with them. A stage with \( m \) input streams and \( n \) output streams will be called a \((m, n)\) stage. Fig. 1 shows four basic types of stages, that is, \((1,1)\), \((1,2)\) stage, \((2,1)\) stage and \((2,2)\) stage, which are also given the names of linking stage, separating stage, combining stage and complex stage, respectively [5].

A geometrically complex process can be decomposed into a primary main process and one or more than one primary side processes. The primary main process can be any combination of stages with the following properties.

1. All of the stages are connected in series.
2. All of the streams connecting these stages flow in the same direction.
3. The initial stage of this combination must be one of the initial stages of the whole process, which will be called the global initial stages.
4. The final stage of this combination must be one of the final stages of the whole process, which will be called the global final stages.

In selecting a primary main process of any process, it is desirable to include in it as many stages as possible. Stages outside the primary main process may be conveniently combined into several primary side processes. A primary side process may have one or more junctions with the primary main process. A side process connected to the main process by its final stream only will be called a parallel side process. A side process which has at least one of its initial streams linked to the main process will be called a
Fig. 1. Four basic types of stages.
subordinate side process or simply a subprocess.

Both the main and side processes will be denoted by the labels of their initial and final streams. For example, the complex process shown in Fig. 2 may be decomposed into a main process \((0\alpha, 9, 4\xi, 7)\), a parallel side process \((0\beta, 9)\) and a subprocess \((4\beta, 11)\).

It may be noted that a main process or a side process can have several initial and/or final streams. An initial stream of a main process or a side process will be called a global initial stream if it is also an initial stream of the whole process, or will be called a local initial stream if it is not an initial stream of the whole process. Similarly, a final stream will be called a global final stream if it is also a final stream of the entire process. Otherwise it is called a local final stream.

The process shown in Fig. 2 consists of two primary side processes which are themselves of the type of a simple process, that is, a process being solely composed of (1,1) stages. A primary side process may itself be geometrically a very complex process. Such a complex primary side process can be further decomposed into a secondary main process and one or several secondary side processes. A secondary main process is chosen to be a serial process. A secondary main process selected from a primary parallel side process should have a global initial stream as its initial stream and one of the local initial streams of the primary main process as its final stream. A secondary main process selected from a primary subprocess should have one of the local final streams of the primary main process as its initial stream, and a global final stream, if it has one, as its final stream. For instance, the process shown in Fig. 3 has a complex primary subprocess \((2\beta, 10)\) which can be decomposed into a secondary main process \((2\beta, 11, 9\beta, 10)\) and a
Fig. 2. A complex process with two simple side processes.
Fig. 3. A complex process with a complex side process.
secondary subprocess (9^, 11).

It is observed that any complex process can be eventually decomposed into several main processes and side processes of different ranks, all of them being serial processes [6].

2. A MULTISTAGE OPTIMIZATION PROBLEM

When a process is to be optimized, each stage is associated with a return function which is a function of the input state vectors and the decision vector. The sum of the return functions over all stages is called the objective function, that is, if the return function of stage \( n \) is \( r(x_{n-1}; \theta_n) \), the objective function of the process is \( \sum_{n} r(x_{n-1}; \theta_n) \). It can be seen that when all of the transformation functions are given, the objective function of the process can be expressed as a function of the global initial states and the decision vector at each stage.

The optimization problem is to find the value of the decision vector at each stage so as to maximize the objective function. All of the transformation functions are given, whereas the global initial states and final states may be either fixed or free.

Since the global initial and final states play an important role in the optimization problem, it is convenient to define a "maximum return function," (MRF) which stands for the total return of a process as a function of its global initial and final states. For example, if a process of \( N \) stages has an initial state \( x_0 \) and a final state \( x_N \), its MRF is written as

\[
E(x_0, x_N) = \max_{\theta_n} \left\{ \sum_{n=1}^{N} r(x_{n-1}; \theta_n) \right\}, \quad n = 1, 2, \ldots, N \tag{2}
\]

with
initial state = x_0
final state = x_N

where x_0 and x_N are considered as parameters.

Once the MRF of a process has been obtained, an optimization problem with any kind of boundary conditions can be readily solved. For instance, the maximum return of a process with given initial state and free final state can be written as

\[ f(\bar{x}_0, x_N) = \text{Max} \ f(x_0, x_N) \]

with

\[ x_0 = \text{a given value} \]

where the single bar over x_N indicates that this particular value of x_N is obtained by maximizing the MRF with respect to the parameter x_N; the double bar over x_0 indicates that it is a specified value.

It is important to check the degree of freedom of an optimization problem. When both the global initial and final states are free, the degrees of freedom are equal to the sum of the global initial variables and the decision variables over all stages. When some or all of the initial and final states are specified the degree of freedom is decreased by the number of the specified state variables.

If a process has D_n decision variables at stage n, S^I_n global initial state variables ans S^F_n final state variables, then the highest degree of freedom is \((\sum_n D_n + S^I_n)\), and the MRF is a function of \((S^I_n + S^F_n)\) variables. It may happen that the number \((\sum_n D_n + S^I_n)\) is smaller than the number \((S^I_n + S^F_n)\). For such cases, the MRF can only be a function \((\sum_n D_n + S^I_n)\)
variables; hence we must arbitrarily assign \( (S_F - \sum_{n} D_n) \) final state variables as dependent variables. The MRF will then be written as \( f(x_L, x_F | x_F) \), in which \( x_L \) denotes the initial state variables; \( x_F \) represents those final state variables which are considered as independent, and \( x_F \), separated from \( x_L \) and \( x_F \) by a vertical bar, stands for those final state variables which are considered as dependent. For example, a one-stage process with \( D_1 = 1 \), \( S_1 = S_F = 2 \) will have its MRF written as \( f(x_{0,1}, x_{0,2}, x_{1,1} | x_{1,2}) \), in which the first subscript of \( x \) represents the label of the stream whereas the second subscript denotes the label of a component of a state vector.

3. PRINCIPLE OF OPTIMALITY AND FUNCTIONAL EQUATIONS

Since this chapter is for the purpose of extending the principle of optimality to deal with geometrically complex processes, a careful consideration of the original version of this principle will be helpful.

The principle is based on an implicit assumption that the maximum return of a process is a function of its initial state only. It is important to note that the principle of optimality was originally formulated for a simple sequential process. The final state of such a process is either specified in the problem or can be readily determined by maximizing the return of the final stage with respect to the state of the input stream of this stage, since the final state does not have any effect on other stages. Making use of this property, Bellman formulated the dynamic programming algorithm which can be represented by the following functional equation

\[
f(x_{n-1}) = \max_{\theta_n} \left[ r(x_{n-1}; \theta_n) + f(x_n) \right]
\]

where \( x_{n-1} \) and \( x_n \) represent the input and output of stage \( n \), respectively;
\( \theta_n \) represents the decision of stage \( n \); \( f(x_n) \) is the MRF of the downstream subprocess \((n,N)\), that is, the subprocess consisting of all the stages downstream to stage \( n \), whereas \( f(x_{n-1}) \) is the MRF of the resulting process \((n-1,N)\). As explained earlier, the MRF of a simple process can be expressed as a function of its initial state only.

Equation (3) can be interpreted as absorbing (imbedding) the MRF of the downstream subprocess \((n,N)\) into the MRF of the resulting process \((n-1,N)\). The variable \( x_n \) is an intermediate state variable of the resulting process \((n-1,N)\) and hence must be matched when \( f(x_n) \) is imbedded into \( f(x_{n-1}) \). The decision variable \( \theta_n \) is an independent variable and must be so chosen that \( f(x_{n-1}) \) be maximum with respect to \( x_{n-1} \). Thus during the process of imbedding, all the intermediate state variables must be matched, whereas all the independent variables, except those which are used as parameters for the resulting MRF, must be so chosen that the resulting MRF is maximized. When the functional equation is interpreted in this manner, its extension to dealing with a complex process will be a straightforward matter.

Now let us consider any serial process obtained from the decomposition of a complex process. If the serial process has a local final stream, the value of this local state will affect the return of its downstream subprocesses. Since such a local final state cannot be determined in the same way as that of the global final state of a simple process, it must be so chosen that the sum of the return from this serial process and the return from its downstream subprocess be maximum. A convenient way to overcome this difficulty is to express the MRF in terms of the initial and final states. Since the final state is considered as a variable, its effect on the return of its downstream subprocess can be taken into account when its value is to be fixed.
Equation (3) can then be modified to the following forms for different types of stages.

(a) for (1,1) stages,

\[ f(x_{n-1}, x_F) = \max_{\Theta_n} \left\{ r(x_{n-1}; \Theta_n) + f(x_n, x_F) \right\} \]  

where \( x_F \) represents the local final state.

(b) for (1,2) stages,

\[ f(x_{n-1}, x_{F\alpha}, x_{FB}) = \max_{\Theta_n} \left\{ r(x_{n-1}; \Theta_n) + f(x_{n\alpha}, x_{F\alpha}) \right. \]
\[ \quad + f(x_{n\beta}, x_{FB}) \]  

where \( x_{F\alpha} \) and \( x_{FB} \) are the local final states of downstream subprocess \((n\alpha, F\alpha)\) and \((n\beta, F\beta)\), respectively.

(c) for (2,1) stages,

(1) if none of the input streams of stage \( n \) is one of the local final streams of the downstream subprocess:

\[ f(x_{(n-1)\alpha}, x_{(n-1)\beta}, x_F) = \max_{\Theta_n} \left\{ r(x_{(n-1)\alpha}; x_{(n-1)\beta}, \Theta_n) \right. \]
\[ \quad + f(x_n, x_F) \]  

(2) if one of the input streams of stage \( n \), say stream \((n-1)\beta\), is one of the local final streams of the downstream subprocess:

\[ f(x_{(n-1)\alpha}, x_F) = \max_{\Theta_n} \left\{ r(x_{(n-1)\alpha}; x_{(n-1)\beta}, \Theta_n) \right. \]
\[ \quad + f(x_n, x_{(n-1)\beta}, x_F) \]  

where the downstream subprocess is labelled by \((n,(n-1)\beta,F')\)

(d) for (2,2) stages
(1) if none of the input streams of stage $n$ is one of the local final streams of the downstream subprocess:

$$f(x_{(n-1)\lambda}, x_{(n-1)\beta}, x_{F_\lambda}, x_{F_\beta}) = \max_{\theta_n} \{ r(x_{(n-1)\lambda}; x_{(n-1)\beta}, \theta_n) + f(x_{n\lambda}, x_{F_\lambda}) + f(x_{n\beta}, x_{F_\beta}) \}$$  \hspace{1cm} (8)

(2) if one of the input streams of stage $n$, say $x_{(n-1)\beta}$, is one of the local final streams of a downstream subprocess, say subprocess $(n_\beta, (n-1)\beta, F'_\beta)$

$$f(x_{(n-1)\lambda}, x_{F_\lambda}, x_{F'_\beta}) = \max_{\theta_n, x_{(n-1)\beta}} \{ r(x_{(n-1)\lambda}; x_{(n-1)\beta}, \theta_n) + f(x_{n\lambda}, x_{F_\lambda}) + f(x_{n\beta}, x_{(n-1)\beta}, x_{F'_\beta}) \}$$  \hspace{1cm} (9)

It can be seen that the functional equation is nothing more than absorbing (imbedding) the MRF's of the downstream subprocesses into the MRF of the process resulting from connecting the stage under consideration, the stage $n$, to all of its downstream stages. During the process of absorbing, the decision variable and the input states of the stage under consideration are considered as independent variables, which, except for those which are used as parameters, must be so chosen that the resulting MRF be maximum. Those initial and final states of the downstream subprocesses which become the intermediate states of the resulting process must be matched.

When the functional equation is applied to a global final-stage of the type of $(1,1)$ stage, it becomes

(1) if the output $x_n$ is free

$$f(x_{n-1}, x_n) = \max_{\theta_n} \{ r(x_{n-1}; \theta_n) \}$$  \hspace{1cm} (10)
(2) if the output $x_n$ is fixed

$$f(x_{n-1}; x_n) = \max_{\theta_n} \{ r(x_{n-1}; \theta_n) \}$$  \hspace{1cm} (11)

with $\bar{x}$ specified.

When the functional equation is applied to local final states, it reduces to a form similar to equation (11), except that the outputs be considered as parameters. It is implicitly assumed in equations (10) and (11) that the number of the decision variables at stage $n$ is larger than the number of its output state variables. If this is not true, the degree of freedom must be checked carefully as discussed before, and equations (10) and (11) must be changed accordingly.

Relations similar to equations (10) and (11) can also be written for the final stages which are not of the type of (1,1) stage.

4. PROCEDURES FOR OPTIMIZING COMPLEX PROCESSES

The optimal policy for a geometrically complex process may be obtained by the following procedures.

(1) Decompose the process into several main processes and side processes of different ranks as discussed in Section 1.

(2) Apply the functional equations given in the previous section to each stage to obtain the optimal decisions as functions of the initial and final states. The order of obtaining the MRF's for the side and main processes is in reverse to that of decomposing the process. In other words, the side process of the highest rank, is considered first. The MRF of a subprocess is absorbed by the main process of the same rank, and the MRF of a main process is
absorbed by the side process of the next lower rank. For example, the MRF of a secondary main process is absorbed by a primary side process. It may be noted that the MRF of a parallel side process is not absorbed by the aforementioned functional equations, but will be combined into the MRF of the main process by

\[ f(x_{L\alpha}, x_{I\beta}, x_{F\alpha}, x_{F\beta}) = \max_{x_n} \left[ f(x_{L\alpha}, x_{F\alpha}, x_n) + f(x_{I\beta}, x_{F\beta}, x_n) \right] \]  

(12)

where \((L\alpha, F\alpha, n)\) is the main process and \((I\beta, F\beta, n)\) is a parallel side process.

A parallel side process with a global initial stream as its initial stream can be converted into a subordinate side process, if it is of the type of simple process, and its transformation equation can be inverted [4]. This procedure will reduce the time of computation.

If a main or a side process has a portion which is itself of a type of simple process, the time of computation may be reduced by obtaining the MRF for this portion first and then determining its end conditions by an equation similar to equation (12). For example, the MRF for the subprocess \((0\alpha, 2)\) in Fig. 2 can be obtained first and then combined with the MRF of subprocess \((2, 9, 7, 11)\) to obtain the MRF of subprocess \((0\alpha, 9, 7, 11)\) as

\[ f(x_{0\alpha}, x_9, x_7, x_{11}) = \max_{x_2} \left[ f(x_{0\alpha}, x_2) + f(x_2, x_9, x_7, x_{11}) \right] \]  

(13)

Eventually we obtain the MRF of the whole process with some fixed or chosen values of the global final states, depending on whether the final states are specified in the problem or not. The initial states can then be chosen so that the total return is maximum. These values of the initial states are then used to obtain the
optimal decisions which have been expressed as functions of the
global initial states. If the global initial states are specified
in the problem, the decisions of the global initial states are
determined by

$$f(x_I, x_F) = \max_{\theta_I} \left\{ r(x_I; \theta_I) + f(x_I, x_F) \right\}$$

with

$$x_I = x_I = \text{a given value}.$$ 

EXAMPLE 1. A PROCESS WITH FEED BACK LOOPS

Let us consider the process with two staggered feedback loops shown in
Fig. 4. For simplicity, we shall assume that all of the state and decision
vectors are one-dimensional. The process can be decomposed into a main pro-
cess (0, b, d, 8β, 11β, 12) and two subprocesses (8β, b) and (11β, d).

The MRF of the subprocess (11β, d) can be obtained by the following
equations.

$$f(x_c, x_d) = r(x_c; \theta_d) \quad (14)$$

$$f(x_{11\beta}, x_d) = \max_{\theta_c} \left\{ r(x_{11\beta}; \theta_c) + f(x_c, x_d) \right\} \quad (15)$$

Similarly, we may find $f(x_{8\beta}, x_1), f(x_{8\beta}, x_b), f(x_5, x_7),$ and $f(x_2, x_4)$. If
the global final state is free, we have

$$f(x_{11\alpha}, x_{12}) = \max_{\theta_{12}} \left\{ r(x_{11\alpha}; \theta_{12}) \right\} \quad (16)$$

The MRF of the whole process can then be obtained by the following equations.
Fig. 4. A process with two staggered feedback loops.
EXAMPLE 2. A STAGEWISE CROSS-CURRENT EXTRACTION PROCESS

Let us consider the cross-current extraction process with recycle as shown in Fig. 5 [8, 9]. The process consists of 3 equilibrium stages through which a solvent containing a solute passes. The solvent is fed to the system at a flow rate of q. The solute is extracted from the solvent by the addition of wash water at each stage. The solvent and wash water are assumed immiscible. A portion of the solvent from the last extractor is fed back to the first extractor at a flow rate r. Thus, the solvent flows from stage to stage at a rate, q+r. The problem is to maximize the objective function which is the net profit per unit time by suitable choices of flow rates of wash water, w_1, w_2, and w_3.
Fig. 5. Schematic flow diagram for the cross-current extraction process with recycle
\[ S' = \alpha q(x_0 - x_3) - \beta (w_1 + w_2 + w_3) \]
\[ = \alpha \left[ (q+r)x_{m1} - rx_3 - qx_3 \right] - \beta (w_1 + w_2 + w_3) \]
\[ = \alpha \left( q+r \right) \left( x_{m1} - x_3 \right) - \beta (w_1 + w_2 + w_3) \]

where
\[ x_{m1} = \frac{q x_0 + r x_3}{q + r} \]

and where \( \alpha \) and \( \beta \) are unit prices of the solute and wash water respectively.

Basing the return on the unit price of the solute, we can rewrite this as
\[ S = \frac{S'}{\alpha} = q(x_0 - x_3) - \lambda (w_1 + w_2 + w_3) \]
\[ = (q + r)(x_{m1} - x_3) - \lambda (w_1 + w_2 + w_3) \]

where
\[ \lambda = \frac{\beta}{\alpha} \]

is the dimensionless price ratio of the wash water and solute. Note that the unit of \( S \) becomes weight per unit time according to this definition since
\[ S = \frac{S'}{\alpha} = \frac{S}{\text{time}} \times \frac{1 \text{ lb.}}{\text{lb.}} = \frac{1 \text{ lb.}}{\text{time}} \]

The state variable is the concentration of solute \( x \), based on the weight fraction. The decision variable at each stage is the flow rate of wash water \( w \).

The phase equilibrium relation is given by [8],
\[ y = h(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5 \]  \quad (25)

with
\[\begin{align*}
a &= 0.00099, \\
b &= 1.7971, \\
c &= 35.196, \\
d &= -633.84, \\
e &= 3371.3, \\
f &= -5916.0.
\end{align*}\]

First we let \( w_n = \theta_n \) and rewrite the objective function (or the maximum return function) as

\[
\begin{align*}
\text{Max } S &= f(x_0, x_3) = \text{Max} \left\{ (q + r)(x_{n-1} - x_n) - \lambda \theta_n \right\} \\
\lambda &= \text{Max} \left\{ \sum_{n=1}^{3} r(x_{n-1}; \theta_n) \right\}
\end{align*}
\]

where \( r(x_{n-1}; \theta_n) \) is the return function of stage \( n \), which stands for the net profit per unit time at stage \( n \) based on the unit price of solute and hence it has units of weight per unit time. Applying a generalized version of the dynamic programming algorithm given in this chapter, we can write the functional equation as follows:

(1) for a one stage process including stage 3 (the third extractor) with a branching point downstream

\[
f(x_2, x_3) = \text{Max} \left\{ r(x_2; \theta_3) \right\}
\]

(2) for a two stage process including stage 2 (the second extractor) with a branching point downstream

\[
f(x_1, x_3) = \text{Max} \left\{ r(x_1; \theta_2) + f(x_2, x_3) \right\}
\]

(3) for a three stage process including all stages (all extractors) with a branching point and a combining point
\[ f(x_0, x_3) = \max \left\{ r(x_0, x_3, \theta_1) + f(x_1, x_3) \right\} \]

We shall consider the case with

\[ q = 1 \frac{\text{lb.}}{\text{min.}} \]
\[ r = 1 \frac{\text{lb.}}{\text{min.}} \]
\[ \lambda_0 = 0.05 \]
\[ x_6 = 0.2 \]

We shall use \( y_n \) to designate the concentration of wash water flowing out of each extractor. The concentration in the fresh wash water is negligible.

The return function \( r(x_2; \theta_3) \) of stage 3 is

\[ r(x_2; \theta_3) = (q + r)(x_2 - x_3) - \lambda \theta_3 . \quad (26) \]

From the material balance of stage 3 (the third extractor) we have

\[ (q + r)(x_2 - x_3) = \theta_3 y_3 , \]

or

\[ \theta_3 = \frac{(q + r)(x_2 - x_3)}{y_3} \quad (27) \]

Substitution of equation (27) and the equilibrium relation equation (25) into equation (26), gives

\[ r(x_2; \theta_3) = (q + r)(x_2 - x_3) \left\{ 1 - \frac{\lambda}{h(x_3)} \right\} \]

\[ = 2(x_2 - x_3) \left\{ 1 - \frac{0.05}{h(x_3)} \right\} . \]

Similarly, the return function \( r(x_1; \theta_2) \) of stage 2 (the second extractor)
\[ r(x_1; \theta_2) = 2(x_1 - x_2) - \frac{0.05}{h(x_2)} \]

and the return function \( r(x_0, x_3, \theta_1) \) of stage 1 (the first extractor) is

\[ r(x_0, x_3, \theta_1) = qx_0 + rx_3 - (q + r)x_1 - \theta_1 \]

\[ = x_0 + x_3 - 2x_1 - 0.05\theta_1 \]

\[ = x_0 + x_3 - 2x_1 - \frac{0.05(x_0 + x_3 - 2x_1)}{h(x_1)} \]

\[ = (0.2 + x_3 - 2x_1) - \frac{0.05}{h(x_1)} \]

RESULTS AND CONCLUSION. The computations have been performed on an I.B.M. 1620 computer. Each time, a value of the final exit concentration, \( x_3 \), is read in as a parameter as shown in the functional relations developed.

A sketch of the grid points of the state variables employed in computation and the computer flow diagram are shown in Figs. 6 and 7 respectively. The computer symbols and program are given in Tables 1 and 2.

The optimal values, the state variables and return functions obtained are as follows:

\[ \bar{x}_0 = 0.1215, \bar{x}_1 = 0.081, \bar{x}_2 = 0.058, \bar{x}_3 = 0.043 \]

\[ \bar{r}(x_2; \theta_3) = 0.01552 \text{ lb./min.} \]
Fig. 6. A portion of the grid points taken for computation. The connected line segments show the optimal path.
Fig. 7. Computer flow diagram for the cross-current extraction process with recycle.
TABLE 1
EXPLANATION OF PROGRAM SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>weight fraction of raffinate at the exit of stage 1, $x_1$</td>
</tr>
<tr>
<td>X2</td>
<td>weight fraction of raffinate at the exit of stage 2, $x_2$</td>
</tr>
<tr>
<td>X3</td>
<td>weight fraction of raffinate at the exit of stage 3, $x_3$</td>
</tr>
<tr>
<td>HX1</td>
<td>phase equilibrium function for stage 1, $h(x_1)$</td>
</tr>
<tr>
<td>HX2</td>
<td>phase equilibrium function for stage 2, $h(x_2)$</td>
</tr>
<tr>
<td>HX3</td>
<td>phase equilibrium function for stage 3, $h(x_3)$</td>
</tr>
<tr>
<td>RX2T3</td>
<td>return function of stage 3, $r(x_2; \theta_3)$</td>
</tr>
<tr>
<td>RX1T2</td>
<td>return function of stage 2, $r(x_1; \theta_2)$</td>
</tr>
<tr>
<td>RX3T1</td>
<td>return function of stage 1, $r(x_0, x_3; \theta_1)$</td>
</tr>
<tr>
<td>F1X2</td>
<td>maximum return function for a one stage process, $f(x_2, x_3)$</td>
</tr>
<tr>
<td>F2X1</td>
<td>maximum return function for a two stage process, $f(x_1, x_3)$</td>
</tr>
<tr>
<td>SR2F1</td>
<td>sum of the return function of stage 2 and the maximum return function of stage 3, $r(x_1; \theta_2) + f(x_2, x_3)$</td>
</tr>
<tr>
<td>SMRF</td>
<td>maximum return function for a three stage process, $f(x_0, x_3)$</td>
</tr>
</tbody>
</table>
TABLE 2
CROSSCURRENT EXTRACTION WITH RECYCLE

DIMENSION X1(15),X2(15),HX1(15),HX2(15),RX2T3(15),F1X2(15)
DIMENSION F2X1(15),SR2F1(15,15),SR1F2(15),RX1T2(15,15),RX3T1(15)

1 FORMAT(6X,F8.3)
2 FORMAT(6X,F11.5)
3 FORMAT(6X,2F8.3,4F11.5)

READ 1,X3
A=1.0099
B=1.7971
C=35.19
D=-633.84
E=3371.3
F=-5916.6
HX3=A+B*X3+C*X3**2+D*X3**3+E*X3**4+F*X3**5
PUNCH 2,HX3
CALC.F1X2(J)
X2(1)=.049
DC 11 J=2,15
X2(J)=X2(J-1)+.001
HX2(J)=A+B*X2(J)+C*X2(J)**2+D*X2(J)**3+E*X2(J)**4+F*X2(J)**5
RX2T3(J)=2.*(X2(J)-X3)*(1.-.05/HX3)
F1X2(J)=RX2T3(J)
PUNCH 3,X2(J),X3,HX2(J),RX2T3(J),F1X2(J)
11 CONTINUE
CALC.F2X1(K)
X1(1)=.071
DC 16 K=2,15
X1(K)=X1(K-1)+.001
HX1(K)=A+B*X1(K)+C*X1(K)**2+D*X1(K)**3+E*X1(K)**4+F*X1(K)**5
DC 17 J=2,15
RX1T2(K,J)=2.*(X1(K)-X2(J))*(1.-.05/HX2(J))
SR2F1(K,J)=RX1T2(K,J)+F1X2(J)
IF(J-2)18,18,19
18 F2X1(K)=SR2F1(K,J)
19 IF(SR2F1(K,J)-F2X1(K))17,17,20
20 F2X1(K)=SR2F1(K,J)
PUNCH 3,X1(K),X2(J),HX1(K),RX1T2(K,J),F1X2(J),F2X1(K)
17 CONTINUE
16 CONTINUE
CALC.SMRF
DC 21 K=2,15
RX3T1(K)=1.*(2+X3-2.*X1(K))*(1.-.05/HX1(K))
SR1F2(K)=RX3T1(K)+F2X1(K)
IF(K-2)22,22,23
22 SMRF=SR1F2(K)
23 IF(SR1F2(K)-SMRF)21,21,24
24 SMRF=SR1F2(K)
PUNCH 3,X3,X1(K),RX3T1(K),F2X1(K),SMRF
21 CONTINUE
END
\[
\begin{align*}
\bar{r}(x_1; \theta_2) &= 0.02886 \text{ lb./min.}, \\
\bar{r}(x_0, x_3, \theta_1) &= 0.05647 \text{ lb./min.}, \\
f(x_2, x_3) &= 0.01552 \text{ lb./min.} \\
f(x_1, x_3) &= 0.04439 \text{ lb./min.} \\
f(x_0, x_3) &= 0.10086 \text{ lb./min.}
\end{align*}
\]

The optimal value of the decision variable at the first extractor is

\[
\bar{\theta}_1 = \frac{\bar{r}(x_0, x_3, \theta_1)}{h(x_1) - \lambda} = \frac{0.05647}{0.16512 - 0.05} = 0.491 \text{ lb./min.}
\]

Similarly, we have

\[
\bar{\theta}_2 = 0.343 \text{ lb./min.}
\]

and

\[
\bar{\theta}_3 = 0.290 \text{ lb./min.}
\]

It must be understood that there are small errors in these results due to discretization.

We see that the allocation of wash water is in the decreasing order as indicated in Refs. [9] and [5]. The small differences among the optimal values obtained from this problem and those given in Refs. [9] and [5] are due to the slight differences in phase equilibrium relationship and the
differences in the errors due to the discretization. The optimal values of \( \theta_n \) and the maximum return functions of Refs. [9] and [5], and from this study are given in Table 3.

The optimal results and the best results corresponding to each final exit concentration \( x_3 \) are listed in Table 4. Some non-optimal results are also shown in Table 5 for comparison. It is clear that the profit based on unit price of the solute in each non-optimal case is smaller than the optimal value of 0.10086 \( \frac{10^6}{\text{min.}} \) corresponding to the value of \( x_3 \) at 0.043, as given in Table 3.
### TABLE 3

**COMPARISON OF OPTIMAL VALUES**

<table>
<thead>
<tr>
<th>Case</th>
<th>$\theta_1$ (lb., min.)</th>
<th>$\theta_2$ (lb., min.)</th>
<th>$\theta_3$ (lb., min.)</th>
<th>RMF (lb., min.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. 9</td>
<td>0.518</td>
<td>0.335</td>
<td>0.274</td>
<td>0.1018</td>
</tr>
<tr>
<td>Ref. 5</td>
<td>0.610</td>
<td>0.440</td>
<td>0.420</td>
<td>0.0974</td>
</tr>
<tr>
<td>this study</td>
<td>0.491</td>
<td>0.343</td>
<td>0.290</td>
<td>0.1009</td>
</tr>
</tbody>
</table>
# Table 4

Comparison of Optimal Results for Various Values of Exit Concentration, \(x_3\)

<table>
<thead>
<tr>
<th>(x_{ml} = \frac{x_0 + x_3}{2})</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(f(x_2, x_3))</th>
<th>(f(x_1, x_3))</th>
<th>(f(x_0, x_3))</th>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
<th>(\theta_3)</th>
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<tbody>
<tr>
<td>wt. fraction</td>
<td>wt. fraction</td>
<td>wt. fraction</td>
<td>lb. min.</td>
<td>lb. min.</td>
<td>lb. min.</td>
<td>lb. min.</td>
<td>lb. min.</td>
<td>lb. min.</td>
<td>lb. min.</td>
</tr>
<tr>
<td>0.1180</td>
<td>0.075</td>
<td>0.051</td>
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<td>4.091</td>
<td>9.986</td>
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</tr>
<tr>
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<td>0.052</td>
<td>0.037</td>
<td>1.323</td>
<td>4.170</td>
<td>10.016</td>
<td>0.531</td>
<td>0.391</td>
<td>0.336</td>
</tr>
<tr>
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<td>0.038</td>
<td>1.368</td>
<td>4.126</td>
<td>10.040</td>
<td>0.538</td>
<td>0.368</td>
<td>0.326</td>
</tr>
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<td>0.039</td>
<td>1.409</td>
<td>4.195</td>
<td>10.058</td>
<td>0.528</td>
<td>0.363</td>
<td>0.318</td>
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<td>0.040</td>
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<td>4.261</td>
<td>10.072</td>
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<tr>
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<td>0.057</td>
<td>0.042</td>
<td>1.519</td>
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<td>0.043*</td>
<td>1.552*</td>
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<td>10.086*</td>
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<td>0.343*</td>
<td>0.290*</td>
</tr>
<tr>
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<td>0.059</td>
<td>0.044</td>
<td>1.583</td>
<td>4.492</td>
<td>10.082</td>
<td>0.481</td>
<td>0.338</td>
<td>0.284</td>
</tr>
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<td>0.045</td>
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</table>

*Optimal results
<table>
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<tr>
<th>$x_{ml}$ = $\frac{x_1 + x_3}{2}$</th>
<th>$\theta_1$</th>
<th>$r(x_0, x_3, \theta_1)$</th>
<th>$x_1$</th>
<th>$\theta_2$</th>
<th>$r(x_1; \theta_2)$</th>
<th>$x_2$</th>
<th>$\theta_3$</th>
<th>$r(x_2; \theta_3)$</th>
<th>$x_3$</th>
<th>Total $\times 10^2$</th>
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<td>Fraction</td>
<td>lb. min.</td>
<td>Fraction</td>
<td>lb. min.</td>
<td>Fraction</td>
<td>lb. min.</td>
<td>Fraction</td>
<td>lb. min.</td>
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<tr>
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<td>0.05949</td>
<td>0.078</td>
<td>0.336</td>
<td>0.02715</td>
<td>0.056</td>
<td>0.276</td>
<td>0.01418</td>
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<td>0.02697</td>
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<td>0.01185</td>
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<td>10.035</td>
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<td>0.549</td>
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<td>0.078</td>
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<td>0.02510</td>
<td>0.058</td>
<td>0.241</td>
<td>0.01391</td>
<td>0.045</td>
<td>10.064</td>
</tr>
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References:


APPENDICES
APPENDIX I

ANALYSIS, DESIGN, AND SIMULATION OF A MULTIEFFECT, MULTISTAGE EVAPORATOR SYSTEM

Water administrators throughout the world are greatly concerned about supplementing water supplies as the demands for fresh water increase year by year. For many years, fresh water has been obtained from salt water by flash evaporation due to the comparative lower cost of converting sea water into potable water by means of large capacity flash evaporators. More water is needed than can be provided by the natural sources of supply and it seems likely that the desalination of sea water is turning out to be a must. As a consequence, much recent effort has been directed to the development of desalting plants with the purpose of producing large quantities of fresh water at low costs. It is anticipated that multieffect multistage (MEMS) flash evaporator systems can economically provide fresh water to areas with large population. This study is essentially concerned with the design and simulation of a multieffect multistage flash evaporation system with a nominal capacity of 50 MM (50 million) gallons per day. The process model is similar to that of the proposed pilot plant at Millstone (1).

The outstanding feature of this process which distinguishes it from other flash evaporation processes is that a portion of the flashing brine stream from each effect is recycled back to the inlet of that effect. This scheme is more efficient thermodynamically than recycling the spent brine from the last effect directly to the first effect. The recycle stream is passed through condenser tube bundles on which water vapor is condensing in order to preheat the recycle brine prior to mixing it with the brine stream coming from the previous effect. Since heat recovery becomes more efficient as the number of
stages used for flashing and preheating increases, inter-effect recycling makes it possible to add more stages per temperature gradient as pressure difference increases.

There are many design variables to consider and many ways to carry out the simulation. The number of stages, total temperature difference (temperature difference between the flashing and recycle brine), blowdown temperature or concentration at the outlet of each effect, allocation of number of stages in each effect, and selection of brine velocities are all important, but the most important factors are the first two. In this simulation study, we chose a 68 stage model with 23 stages in the first and the second effects and 22 stages in the third effect.

Two models of the system a "micro stage model" and a "macro stage model" are used. In the micro stage model the performance of each stage within the heat recovery section of each effect is considered, while in the macro stage model the performance of each heat recovery section is considered as a unit. In both models, we treat each heat rejection section as a unit even though we assume that there are 3 stages in the heat rejection section of the first and second effects and 2 stages in the third. Since each heat rejection section also acts as a brine heater for the next effect, it is much more convenient to treat them as a whole unit. In both models, we assume equal distillate production for each effect (this automatically fixes the blowdown concentration of each effect). In the former, we examine the effect of changing the total temperature difference and the blowdown temperature for each effect, while in the latter we fix the blowdown temperature and study the effect of changing the total temperature difference.
1. PROCESS DESCRIPTION

Assume that the MEMS plant is located on the coast about 50 feet above the sea level. The sea water is usually pumped from the ocean via forebays or sumps. It passes through the heat rejection section of the third effect and its pH value is adjusted by adding sulfuric acid. Then it goes to the deaerator system where the dissolved oxygen and nitrogen (inert gases) and a small amount of carbon dioxide are removed. The pretreated raw feed is heated by passage through the condenser tubes of the heat recovery section of the third effect and all of the stages of the second and the first effects. It is finally heated to the desired flashing temperature by a shell and tube brine heater with low pressure steam from an adjacent power plant. Before entering the flashing chamber of the first effect, the heated feed is mixed with the recycle brine from the outlet of the same effect. The recycle brine is heated to the same temperature as the heated feed by passing it through the brine heater also. Under these circumstances, the mixing streams are at the same temperature and the concentration differences are reduced because the whole system has been divided into three effects. The free energy decreases due to mixing, that is, the thermodynamic irreversibility has thus been reduced. The flashing brine then cascades from stage to stage as a result of the pressure differential maintained. In each stage, a portion of the water flashes from the brine solution. It is condensed on the condenser tubes via demisters and is caught in troughs. The distillate also cascades from stage to stage and is finally pumped to the storage tanks as product from the lowest pressure end, i.e., the last stage of the third effect. The brine in excess of that required for recycle in the third effect is pumped out of the system and discharged to the ocean as a blowdown. A schematic diagram of the process is shown in Figs. 1 and 2.
Fig. 1. Process flow diagram of the multieffect multistage flash evaporator system.
Fig. 2. Temperature profiles of the multi-effect multi-stage flash evaporator system (The micro stage model).
2. FORMULATION OF THE DESIGN EQUATIONS ACCORDING TO THE MICROSTAGE MODEL

By making an analysis of the process, the design equations for the simulation study may be developed. In this simulation study a 68 stage, 50 million gallons per day plant is investigated. It is assumed that the salt water fed to the system contains 3.5 weight per cent salt and that the concentrated brine leaving the system (blowdown) contains 7 weight per cent salt. This information is presented together with other numerical data in Table 7.

(A) Material and Energy Balances for the nth Stage of the Heat Recovery Section of the First Effect. From Fig. 3, we may develop the following material and energy balance relationships. A distillate material balance gives *

\[ x_n^2 = x_{n-1}^2 + \Delta W^n, \]  

where \( \Delta W^n \) depends on the energy balance

\[ \left[ F(1\!+\!r_1) - x_2^{n-1} \right] (C_p^{n-1}) f_1 T_f^{n-1} = \Delta W^n H_v^n + \left[ F(1\!+\!r_1) - x_2^n \right] (C_p^n) f_1 T_f^n, \]  

where

- \( T_f \) = temperature of flashing brine (\( ^\circ F \)),
- \( \Delta W \) = amount of water vapor formed (lb/hr),
- \( H_v \) = enthalpy of water vapor relative to liquid to \( 0^\circ F \) (BTU/lb),
- \( (C_p) f_1 \) = heat capacity of flashing brine (BTU/lb\( ^\circ F \)),
- \( r_1 \) = ratio of the recycle brine in the first effect to the make-up feed (lb/lb),
- \( x_2 \) = amount of distillate (lb/hr).

*All superscripts denote stage numbers. A complete list of nomenclature is given in Table 2 on page 200.
Fig. 3. Block diagram for the flashing brine at the \( n \)-th stage, heat recovery section of the first effect.

Fig. 4. Block diagram for the operation of the \( n \)-th stage, heat recovery section of the first effect.
Equation (2) can be simplified if we put

\[(C_p^{-1})_{fl} = (C_p^n)_{fl} = (C_p)_{fl} \text{ = average heat capacity of the}
\]
\[\text{flashing brine in the first effect (BTU/lb}^\circ\text{F).}
\]

Then we have,

\[
(C_p)_{fl} = \frac{(C_p)_F + r_1(C_p^{23})_{fl} + (C_p^{23})_{fl}}{1 + r_1} = \frac{(C_p)_F + (1 + 2r_1)(C_p^{23})_{fl}}{2(1 + r_1)}
\]

where

\[(C_p)_F \text{ = heat capacity of the make-up feed (BTU/lb}^\circ\text{F),}
\]
\[(C_p^{23})_{fl} \text{ = heat capacity of the flashing brine at stage 23}
\]
\[(\text{BTU/lb}^\circ\text{F}).
\]

According to the Bechtel Corporation (3), heat capacities of brine at various concentrations can be expressed as

\[
(C_p) = 1 - c
\]

in which c is the concentration of the brine in weight fraction. The enthalpy of water vapor can be written as

\[
H_v^n = (C_p)_W x_1^n + \lambda^n
\]

where

\[(C_p)_W = 1.0 = \text{heat capacity of distillate (BTU/lb}^\circ\text{F),}
\]
\[x_1 = \text{temperature of distillate (}^\circ\text{F),}
\]
\[\lambda^n = \text{latent heat of vaporization (BTU/lb).}
\]

Thus, equation (2) becomes
\[
\{F(1+r_1) - x_2^{n-1}\}(C_p)f_1 T_f^{n-1} = \Delta w^n (x_1^n + \lambda^n) + \{F(1+r_1) - x_2^{n-1} - \Delta w^n\}
\]
\[
(C_p)f_1 T_f^n
\]

Solving for \(\Delta w^n\) gives
\[
\Delta w^n = \frac{F(1+r_1) - x_2^{n-1} (C_p)f_1 (T_f^{n-1} - T_f^n)}{\lambda^n + (x_1^n - (C_p)f_1 T_f^n)}
\]

Substituting equation (6) into equation (1), we have (see Fig. 4)
\[
x_2^n = x_2^{n-1} + \frac{\{F(1+r_1) - x_2^{n-1}\}(C_p)f_1 (T_f^{n-1} - T_f^n)}{\lambda^n + (x_1^n - (C_p)f_1 T_f^n)}.
\]

The temperature of the distillate in each stage \(x_1^n\) depends upon the pressure that is maintained in that stage. The temperature difference
\[
\theta^n = x_1^{n-1} - x_1^n
\]
depends on the difference in pressure that is maintained. Noting that
\[
T_f^{n-1} = x_1^{n-1} + \Delta t_{BPE}^{n-1},
\]
\[
T_f^n = x_1^n + \Delta t_{BPE}^n,
\]
we have
\[
T_f^{n-1} - T_f^n = \theta^n + \Delta t_{BPE}^{n-1} - \Delta t_{BPE}^n.
\]

According to reference (3), the boiling point elevation of the flashing brine at each stage is approximately equal to 2°F. Thus, we put
\[
\Delta t_{BPE}^{n-1} = \Delta t_{BPE}^n = 2.
\]
Furthermore, since the term $x_1^n - (C_p)_{pl} T_f^n$ is negligibly small compared with $\chi^n$, equation (7) can be simplified further to obtain

$$x_2^n = x_2^{n-1} + \frac{\{F(1+r_1) - x_2^{n-1}\} (C_p)_{pl} \theta^n}{\chi^n}$$  \hspace{1cm} (9)$$

with $x_2^0 = 0$.

An energy balance for the flash chamber of the nth stage gives

$$q^n = x_2^{n-1} (C_p)_w x_1^{n-1} + \{F(1+r_1) - x_2^{n-1}\} (C_p)_{pl} T_f^{n-1}$$

$$- x_2^{n-1} (C_p)_w x_1^n - \{F(1+r_1) - x_2^n\} (C_p)_{pl} T_f^n,$$

or

$$q^n = x_2^{n-1} x_1^{n-1} + \{F(1+r_1) - x_2^{n-1}\} (C_p)_{pl} (x_1^{n-1} + \Delta t_{BPE})$$

$$- (x_2^{n-1} + \Delta w^n) (x_1^{n-1} - \theta^n)$$

$$- \{F(1+r_1) - x_2^n - \Delta w^n\} (C_p)_{pl} (x_1^{n-1} - \theta^n + \Delta t_{BPE}^n),$$

The above expression may be further simplified to give

$$q^n = x_2^{n-1} \theta^n + \{F(1+r_1) - x_2^{n-1}\} (C_p)_{pl} (\theta^n + \Delta t_{BPE}^{n-1} - \Delta t_{BPE}^n)$$

$$+ \Delta w^n \left( (C_p)_{pl} (x_1^n + \Delta t_{BPE}^n) - x_1^n \right)$$

Since

$$\Delta t_{BPE}^{n-1} = \Delta t_{BPE}^n$$

and the last term is negligibly small, we can write
\[ q^n = x_2^{n-1} \theta^n + \{F(1+r_1) - x_2^{n-1}\} (C_p)_f l \theta^n. \]  

(10)

For the fluid inside the condenser tubes we have the following energy balance

\[ q^n = F(1+r_1) (C_p)_r l (x_3^{n-1} - x_3^n) \]  

(11)

with

\[ x_3^0 = T_f^o - \Delta t_1 \]

where

- \(x_3\) = temperature of recycle brine (\(^{\circ}\)F),
- \(T_f^o\) = maximum brine temperature (\(^{\circ}\)F),
- \(\Delta t_1\) = temperature rise by passage through the brine heater (\(^{\circ}\)F),
- \((C_p)_r l\) = heat capacity of recycle brine (BTU/lb\(^{\circ}\)F).

\((C_p)_r l\) is calculated as

\[ (C_p)_r l = \frac{(C_p)_F + r_1 (C_p^23)}{1 + r_1}. \]

The condenser tube area needed for each stage may be determined from the equation

\[ q^n = U^n A^n (\Delta t^n)^{L.M.T.D.}. \]  

(12)

where

- \(U^n\) = overall heat transfer coefficient at stage \(n\) (BTU/ft\(^2\)hr),
- \((\Delta t^n)^{L.M.T.D.}\) = logarithmic mean temperature difference for heat transfer at stage \(n(^{\circ}\)F),
\[(\Delta t^n)_{L.M.T.D.} = \frac{(x^n_1 - x^n_3) - (x^n_1 - x^n_{1-1})}{\ell n \frac{x^n_1 - x^n_3}{x^n_1 - x^n_{1-1}} = \frac{x^n_{1-1} - x^n_3}{\ell n \frac{x^n_1 - x^n_3}{x^n_1 - x^n_{1-1}}}} \quad (13)\]

Combining equations (10) and (11), we have

\[x^n_3 = x^n_{1-1} - \frac{\theta^n (F(l+r_1)(C_p) + x^n_{1-1} (1 - (C_p)_{f1}))}{F(l+r_1)(C_p)_{r1}} \quad (14)\]

From equations (8), (11), (12) and (13), we can obtain

\[A^n = \frac{q^n}{u^n} = \frac{F(l+r_1)(C_p)_{r1}}{u^n} \ell n \frac{x^n_{1-1} - x^n_3}{x^n_1 - x^n_{1-1}} \]

\[= \frac{F(l+r_1)(C_p)_{r1}}{u^n} \left(\ell n (x^n_{1-1} - \theta^n - x^n_3) - \ell n (x^n_{1-1} - \theta^n - x^n_{1-1})\right) \quad (15)\]

Taking \(u^n = (U_1)_{av}\) as the average overall heat transfer coefficient for the condenser tubes in the first effect, and combining equations (14) and (15), we obtain

\[A^n = \frac{F(l+r_1)(C_p)_{r1}}{(U_1)_{av}} \left(\ell n (x^n_{1-1} - \theta^n - x^n_3) + \frac{\theta^n (F(l+r_1)(C_p)_{f1} + x^n_{1-1} (1 - (C_p)_{f1}))}{F(l+r_1)(C_p)_{r1}} \right) \]

\[ - \ell n (x^n_{1-1} - \theta^n - x^n_{1-1})\right), \quad n = 1, 2, \ldots, 20 \quad (16)\]
(B) **Cost Equations for the Brine Heater and the Heat Recovery Section of the First Effect.** In the following, we shall consider the unit production cost in dollars per pound of distillate term by term. The total production is \( \frac{F}{2} \) so that each cost item is divided by \( \frac{F}{2} \) to give the unit production cost in dollars per pound of distillate.

(a) **Steam cost.** The amount of steam consumed, \( S \) is

\[
S = \frac{F(1+r_1)(C_p)_{rl} \Delta t_1}{\lambda_s}
\]

The steam cost in dollars per pound of distillate produced, \((x_4)_s\) is

\[
(x_4)_s = \frac{C_1 F(1+r_1)(C_p)_{rl} \Delta t_1}{\frac{F}{2} \lambda_s} = \frac{2C_1 (1+r_1)(C_p)_{rl} \Delta t_1}{\lambda_s}
\]

where

\[
\lambda_s = \text{latent heat of steam (BTU/lb)}
\]

\[C_1 = \text{unit steam cost ($/lb)}^*
\]

(b) **Fixed charge cost for the brine heater.** The amount of heat transferred in the brine heater, \( q_B \), is

\[
q_B = F(1+r_1)(C_p)_{rl} \Delta t_1 = U_{AB} \Delta t \text{ L.M.T.D.}
\]

where

\*(All unit cost items are from (3).)
\[
\Delta t_{\text{L.M.T.D.}} = \frac{(T_s - 250 + \Delta t_1) - (T_s - 250)}{\ln \frac{T_s - 250 + \Delta t_1}{T_s - 250}} = \frac{\Delta t_1}{\ln \frac{T_s - 250 + \Delta t_1}{T_s - 250}}
\]

\[T_s = \text{temperature of the steam.}\]

In case steam of 40 psi is used, \(T_s = 274.4^\circ F\) and

\[
\Delta t_{\text{L.M.T.D.}} = \frac{\Delta t_1}{\ln \frac{24.4 + \Delta t_1}{24.4}}
\]

Using the above equations, the heat transfer area in the brine heater, \(A_B\), becomes

\[
A_B = \frac{F(1+r_1)(C_p)_{rl}}{U_B} \frac{\Delta t_1}{\ln \frac{24.4 + \Delta t_1}{24.4}}
\]

and the fixed charge cost for the brine heater per pound of distillate produced, \((x_4)_B\), is written as

\[
(x_4)_B = \frac{C_2 F(1+r_1) (C_p)_{rl}}{2 \ln 2} \left\{ \ln(24.4 + \Delta t_1) - \ln(24.4) \right\}
\]

or

\[
(x_4)_B = \frac{2C_2 (1+r_1)(C_p)_{rl}}{U_B} \left\{ \ln(24.4 + \Delta t_1) - \ln(24.4) \right\} \tag{18}
\]

where \(C_2\) is the unit fixed charge cost for the brine heater ($/ft^2 \text{ hr}$).

(c) Fixed charge cost for the condenser tubes.

From equation (16), the fixed charge cost for the condenser tubes per
pound of distillate product, \((x_4^n)_{c1}\), becomes

\[
(x_4^n)_{c1} = \frac{C_3 F(1+r_1)(C_p) r_1}{2 (U_1)_{av}} \left[ \ln (x_1^{n-1} - \theta^n - x_3^{n-1}) + \frac{\theta^n}{F(1+r_1)(C_p) r_1} \ln (x_1^{n-1} - \theta^n - x_3^{n-1}) \right] + \frac{x_2^{n-1}(1-(C_p f_1))}{(1-(C_p f_1))} - \frac{\ln (x_1^{n-1} - \theta^n - x_3^{n-1})}{(1-(C_p f_1))}
\]

\[
(x_4^n)_{c1} = \frac{2C_3 (1+r_1)(C_p) r_1}{(U_1)_{av}} \left[ \ln (x_1^{n-1} - \theta^n - x_3^{n-1}) + \frac{\theta^n}{F(1+r_1)(C_p) r_1} \ln (x_1^{n-1} - \theta^n - x_3^{n-1}) \right] + \frac{x_2^{n-1}(1-(C_p f_1))}{(1-(C_p f_1))} - \frac{\ln (x_1^{n-1} - \theta^n - x_3^{n-1})}{(1-(C_p f_1))}
\]  

(19)

where \(C_3\) is the unit fixed charge cost for the condenser tubes ($/ft^2$ hr).

(d) Cost of recycle for the first effect. The cost of recycle which we wish to consider results because of the pressure differences created in order to produce flashing at each stage. In order to recycle some of the brine which leaves stage 23, we must increase its pressure to that of the brine stream flowing from the brine heater to the first stage. The work that must be done to accomplish this, \(W_{r_1}\), is

\[
W_{r_1} = \frac{\Delta P_1}{\rho} \times \frac{144}{3600 \times 550} \text{ HP-hr.}
\]

where

\[
\rho = \text{density of the flashing brine} = 62.5 \text{ (lb/ft}^3\text{)}
\]

\[
\Delta P_1 = P_o - P_1,
\]

\[
P_o = \text{vapor pressure of brine entering stage one at } T_f^o \text{ (psi),}
\]

The vapor pressure of the brine leaving stage 23, \(P_1\), is calculated by an empirical formula [4], that is,
\[ P_1 = \exp \left( 14.07 - \frac{9 \times 3660}{5(x_3^{20} - 32) + 1989} \right) \text{ (psi)} \]

We can write the cost of recycle, \((x_4)_{r1}\) as

\[
(x_4)_{r1} = \frac{C_4 r_1 (P_0 - P_1) \times 144}{\frac{F}{2} \times 62.5 \times 550 \times 3600 \times 0.85} = \frac{2C_4 r_1 (P_0 - P_1) \times 144}{62.5 \times 550 \times 3600 \times 0.85}
\]

where \(C_4\) is the unit pumping cost including the electric power and depreciation cost for the pumps and the correction factor 0.85, which accounts for the efficiency of the pumps, is taken from reference (3).

For the heat recovery section of the first effect, the accumulated cost per pound of distillate produced, \(x_4^n\), is

\[
x_4^n = (x_4)_s + (x_4)_B + (x_4)_c l^*_1, \quad n = 1, 2, \ldots, 20
\]

or

\[
x_4^n = (x_4)_s + (x_4)_B + \frac{2C_3 (1+r_1) (C)}{(U_1)_av} \left[ \ln \left( x_1^{n-1} - \theta^n - x_3^{n-1} \right) + \frac{\theta^n}{F(1+r_1)(C)} \right] r_1
\]

\[
= \left[ F(1+r_1)(C) + x_2^{n-1}(1 - (C)_{f1}) \right] - \ln \left( x_1^{n-1} - \theta^n - x_3^{n-1} \right)
\]

\[ n = 1, \ldots, 20. \]

(C) **Material and Energy Balances and Cost Equations for the Heat Rejection Section of the First Effect.** The heat rejection section of the first effect also acts as the brine heater for the second effect. In this study, the

*The cost of recycle for the first effect will be included when the cost of the heat rejection section of the first effect is considered.*
temperature of the brine leaving the flash chamber of the last stage of the heat rejection section is assumed to be equal to the temperature of the feed stream which enters the heat recovery section, that is,

$$x_3^{20} = T_{f}^{23} = x_1^{23} + 2$$

An energy balance around the heat rejection section of the first effect gives (see Fig. 5)

$$F(1 + r_1) (C_p)^{f1} (x_1^{20} + \Delta t_{BPE}^2 - x_3^{20}) = \{F(C_p)_{f} + F_{r2}(C_p)^{46}_{f2}\} (x_3^{20} - x_3^{23})$$

According to reference (3)

$$(C_p)^{f1}_{F} = 0.965$$

and

$$(C_p)^{46}_{f2} = 1 - 0.035 \times \frac{3}{2} = 1 - 0.0525 = 0.9475$$

Solving for $x_3^{23}$, we have

$$x_3^{23} = x_3^{20} - \frac{(1 + r_1) (C_p)^{f1}_{f1}}{0.965 + r_2 \times 0.9475} (F(1 + r_1)(C_p) f1 (x_1^{20} + 2 - x_3^{20}))$$

$$= x_3^{20} - \frac{(1 + r_1)(C_p) f1 (x_1^{20} + 2 - x_3^{20})}{0.965 + r_2 \times 0.9475} \tag{22}$$

The amount of distillate produced in the heat rejection section of the first effect is given by

$$\Delta W_{lj} = \frac{F(1 + r_1)(C_p) f1 (x_1^{20} + 2 - x_3^{20})}{\lambda_{avlj}}$$

or

$$\Delta W_{lj} = \frac{1098.3 - 0.3055 (x_1^{20} + x_3^{20} + 2)}{0.965 + r_2 \times 0.9475} \tag{23}$$

where the formula for $\lambda$ is obtained by curve fitting the steam table
Fig. 5. Schematic flow diagram for the first effect.
\[ \lambda = 1098.3 - 0.611 x_1 \]

Thus

\[
(\lambda)_{\text{avlj}} = \frac{1098.3 - 0.611(x_1^{20} + 2) + 1098.3 - 0.611 x_3^{20}}{2} = 1098.3 - 0.3055(x_1^{20} + 2 + x_3^{20})
\]

The heat transfer area required for the condenser tubes of the heat rejection section of the first effect is

\[
A_{l1} = \frac{q_{l1}}{(U_1)_{\text{av}}^{(\Delta t)}_{\text{avlj}}} = \frac{F(1+r_1)(C_p f_1(x_1^{20} + 2 - x_3^{20}))}{(U_1)_{\text{av}}^{(\Delta t)}_{\text{avlj}}}
\]

where \((\Delta t)_{\text{avlj}}\) is evaluated as follows

\[
\Delta t_{\text{avlj}} = \frac{(x_1^{21} - x_3^{20}) + (x_1^{23} - x_3^{23})}{2}
\]

From Fig. 7-a, we see that

\[
x_1^{21} = x_1^{20} + \frac{x_1^{20} + 2 - x_3^{20}}{3}
\]

Using the above equation together with the assumption that

\[
x_1^{23} = x_3^{20} - 2
\]

we obtain

\[
\Delta t_{\text{avlj}} = \frac{3(x_3^{20} - x_3^{23}) + 2(x_1^{20} - x_3^{20}) - 8}{6}
\]

Substitution of equation (25) into equation (24) yields
\[ A_{1j} = \frac{6 F(1+r_1)(C_p f_1)(x_{1}^{20} + 2 - x_{3}^{20})}{(U_1)_{av} \{3(x_{3}^{20} - x_{3}^{23}) + 2(x_{1}^{20} - x_{3}^{20}) - 8\}} \]

Hence, the unit production cost is obtained as

\[ (x_{4})_{1j} = \frac{6 F(C_3(1+r_1))(C_p f_1)(x_{1}^{20} + 2 - x_{3}^{20})}{2 (U_1)_{av} \{3(x_{3}^{20} - x_{3}^{23}) + 2(x_{1}^{20} - x_{3}^{20}) - 8\}} \]

\[ = \frac{12C_3(1+r_1)(C_p f_1)(x_{1}^{20} + 2 - x_{3}^{20})}{(U_1)_{av} \{3(x_{3}^{20} - x_{3}^{23}) + 2(x_{1}^{20} - x_{3}^{20}) - 8\}} \]  

(26)

(D) Material and Energy Balances and Cost Equations for the Heat Recovery Section of the Second Effect. Using Fig. 6, we can carry out an analysis for the heat recovery section of the second effect. Equations analogous to those obtained for the first effect can be derived. The temperature of distillate, \( x_1^n \), may still be determined from equation (8), that is,

\[ x_1^n = x_1^{n-1} - \theta^n, \quad n = 24, \ldots, 43 \]  

(27)

with

\[ x_1^{23} = x_1^{20} - 2 \]

where we take \( \Delta t_{BPE} \) as \( 2^\circ F \).

The amount of distillate can be determined from

\[ x_2^n = x_2^{n-1} + \frac{\{F(\frac{5}{6} + r_2) - x_2^{n-1}\}(C_p f_2 \theta^n)}{\lambda^n}, \quad n = 24, \ldots, 43 \]  

(28)

with

\[ x_2^{23} = 0 \]
Fig. 6. Schematic flow diagram for the second effect.
which is analogous to equation (9) for the first effect expect that $F$ is replaced by $\frac{5}{6} F$ as $\frac{1}{6}$ of the make-up feed has been flashed out in the previous effect. It should also be noted that the amount of distillate refers only to the distillate formed after entering the second effect. As a matter of fact, the distillate produced in the first effect just gives up its sensible heat when it cascades down the subsequent effects, and it does not affect the quantity of water that is flashed from the brine.

By using a procedure similar to that used to obtain equation (14), we can obtain the recursive equation for $x_3$ as

$$x_3^n = x_3^{n-1} - \frac{\{x_2^n + r_2^n x_2^{n-1} (1 - \frac{C_i}{p} f_2) + \frac{5}{6} \theta^n\}}{F(0.965 + r_2 x 0.9475)}$$

$$n = 24, \ldots, 43.$$ 

Similarly, the heat transfer area for the condenser tubes that is required at each stage may be obtained by analogy. The equation which corresponds to equation (16) for the first effect is

$$A^n = \frac{F(0.965 + r_2 x 0.9475)}{(U_2')_{av}} x \left[ \ln(x_1^{n-1} - \theta^n - x_3^n) - \ln(x_1^{n-1} - \theta^n - x_3^{n-1}) \right]$$

$$= \frac{F(0.965 + r_2 x 0.9475)}{(U_2')_{av}} x \left[ \ln(x_1^{n-1} - \theta^n - x_3^{n-1}) - \ln(x_1^{n-1} - \theta^n - x_3^{n-1}) \right]$$

$$+ \frac{\{x_2^n + r_2^n x_2^{n-1} (1 - \frac{C_i}{p} f_2) + \frac{5}{6} \theta^n\}}{F(0.965 + r_2 x 0.9475)} - \ln(x_1^{n-1} - \theta^n - x_3^{n-1})$$

$$n = 24, \ldots, 43$$

Multiplication of $A^n$ by $\frac{2C_3}{F}$ gives the fixed charge cost. The recycle cost is obtained by using exactly the same procedure as that used for the first
effect. We obtain

$$\left( x_4 \right)_{r2} = \frac{2 \times C_4 \times r_2 \times 144(P_1 - P_2)}{62.5 \times 550 \times 3600 \times 0.85}$$

(31)

where

$$P_2 = \exp \left( 14.07 - \frac{9 \times 3660}{5(x_3 - 32) + 1989} \right)$$

Thus, the unit production cost $x_4^n$ becomes

$$x_4^n = x_4^{23} + \frac{2C_2(0.965 + r_2 \times 0.9475)}{(U_2)_{av}} \left( \ln \left( x_1^{n-1} - e^n - x_3^{n-1} \right) \right)$$

$$+ \left\{ \frac{F(\frac{5}{6}r_2)(C_2)p f_2 + x_2^{n-1}(1-(C_2)p f_2 + \frac{5}{6})e^n}{F(0.965 + r_2 \times 0.9475)} \right\} - \ln(x_1^{n-1} - e^n - x_3^{n-1})$$

$$n = 24, ..., 43,$$

(32)

where

$$x_4^{23} = x_4^{20} + (x_4)^{1j} + (x_4)^{r1}.$$

which are the accumulated cost up through stage 20, the cost of the condenser tube area for the heat rejection section of the first effect, and the cost of recycle in the first effect, respectively.

(E) Material and Energy Balances and Cost Equations for the Heat Rejection Section of the Second Effect. The heat rejection section of the second effect also acts as the brine heater for the third effect. As before, we assume that the operation is such that

$$x_4^{43} = \frac{T_f}{46} = x_1^{46} + 2$$
An energy balance for the heat rejection section of the second effect gives

\[ F(\frac{5}{6} + r_2)(C_p f_2(T^f - T_f) + F(\frac{43}{6} - x_1) = \{F(C_p F + Fr_3(C_p f_3)(x_3 - x^3_3) \}

The following approximate form of this equation will be used here.

\[ F(1 + r_2)(C_p f_2(x_1^{43} + 2 - x_3^{43}) = \{F(C_p F + Fr_3(C_p f_3)(x_3 - x_3) \}

According to (3)

\[ (C_p F) = 0.965 \]

and

\[ (C^{68}_p f_3) = 0.930. \]

Solving for \( x_3^{46} \), we have

\[ x_3^{46} = x_3^{43} - \frac{(1 + r_2)(C_p f_2)(x_1^{43} + 2 - x_3^{43})}{0.965 + r_3 x 0.93} \]

(33)

The production of distillate in the heat rejection section of the second effect may be calculated as follows:

\[ (\Delta W)_{2j} = \frac{[F(\frac{5}{6} + r_2) - x_2^{43}](C_p f_2(T^f - T_f)}}{(\lambda)_{av2j}} \]

Since

\[ (\lambda)_{av2j} = 1098.3 - 0.3055(x_1^{43} + 2 + x_3^{43}) \]

we obtain

\[ (\Delta W)_{2j} = \frac{[F(\frac{5}{6} + r_2) - x_2^{43}](C_p f_2(x_1^{43} + 2 - x_3^{43})}{1098.3 - 0.3055(x_1^{43} + 2 + x_3^{43})} \]

(34)
The heat transfer area required for the condenser tubes of the heat rejection section of the second effect is

\[ A_{2j} = \frac{q_{2j}}{(U_2)_{av}(\Delta t)_{av2j}} = \frac{F(1+r_2)(C_p f_2(x_1^{43} + 2 - x_3^{43})}{(U_2)_{av}(\Delta t)_{av2j}} \tag{35} \]

where (see Fig. 7b)

\[ (\Delta t)_{av2j} = \frac{3(x_3^{43} - x_3^{46}) + 2(x_1^{43} - x_3^{43}) - 8}{6} \tag{36} \]

Substitution of equation (36) into equation (35) yields

\[ A_{2j} = \frac{6F(1+r_2)(C_p f_2(x_1^{43} + 2 - x_3^{43})}{(U_2)_{av}[3(x_3^{43} - x_3^{46}) + 2(x_1^{43} - x_3^{43}) - 8]} \]

Hence, the unit production cost will be

\[ (x_4^{2j}) = \frac{12C_3(1+r_2)(C_p f_2(x_1^{43} + 2 - x_3^{43})}{(U_2)_{av}[3(x_3^{43} - x_3^{46}) + 2(x_1^{43} - x_3^{43}) - 8]} \tag{37} \]

(F) Material and Energy Balances and Cost Equations for the Heat Recovery Section of the Third Effect. Using Fig. 8 and an analysis which is analogous to that used in Sections (A) and (D), equations which describe the heat recovery section of the third effect can be obtained. Equation (8) for the temperature of the distillate may still be used, that is,

\[ x_1^n = x_1^{n-1} - \theta^n, \quad n = 47, \ldots, 66 \tag{38} \]

where

\[ x_1^{46} = x_3^{43} - 2.0 \]
Fig. 8. Schematic flow diagram for the third effect.
The amount of distillate can be determined from

\[ x_2^n = x_2^{n-1} + \frac{[F(\frac{2}{3} + r_3) - x_2^{n-1}](C_p f_3) \theta^n}{\lambda^n}, \quad n = 47, \ldots, 66, \quad (39) \]

with \( x_2^{46} = 0. \)

which is analogous to equation (9) for the first effect except that \( F \) is replaced by \( \frac{2}{3} F \) as \( \frac{1}{3} \) of the make-up feed has been flashed in the previous two effects. Again, note that the amount of distillate refers only to the new distillate which is obtained after entry of the flashing brine to the third effect.

By using a procedure similar to that used to obtain equations (14) and (26), we can obtain

\[ x_3^n = x_3^{n-1} - \frac{[F(\frac{2}{3} + r_3)(C_p f_3) + x_2^{n-1}(1-(C_p f_3)) + \frac{F}{3}] \theta^n}{F(0.965 + r_3 x 0.930)}, \quad n = 47, \ldots, 66, \quad (40) \]

with \( x_3^{46} \) calculated from equation (33).

The heat transfer area, \( A^n \), may be obtained by using a procedure like that used to obtain equation (16). It is given by

\[ A^n = \frac{F(0.965+r_3 x 0.93)}{(U_3)_{av}} \left[ \ln(x_1^{n-1}-\theta^n-x_3^{n}) - \ln(x_1^{n-1}-\theta^n-x_3^{n-1}) \right] \]

\[ \quad = \frac{F(0.965+r_3 x 0.93)}{(U_3)_{av}} \left[ \ln(x_1^{n-1}-\theta^n-x_3^{n-1}) + \frac{[F(\frac{2}{3} + r_3)(C_p f_3) + x_2^{n-1}(1-(C_p f_3)) + \frac{F}{3}] \theta^n}{F(0.965 + r_3 x 0.93)} - \ln(x_1^{n-1}-\theta^n-x_3^{n-1}) \right], \quad n = 47, \ldots, 66, \quad (41) \]
Multiplication of $A^n$ by $\frac{2C_3}{F}$ gives the fixed charge cost. The recycle cost is obtained by using exactly the same procedure as that used for the first effect. We obtain

$$(x_4)_{r3} = \frac{2 \times C_4 \times r_3 \times 144 \times (P_2 - P_3)}{62.5 \times 550 \times 3600 \times 0.85}.$$  \hspace{1cm} (42)$$

Where

$$P_3 = \exp\left(14.07 - \frac{9 \times 3660}{5(x_3^3 - 32) + 1989}\right) \text{ in (lb/in}^2\).$$

We are now able to write the accumulated unit production cost as

$$x_4^n = x_4^{46} + \frac{2C_3(0.965 + r_3 \times 0.93)}{(U_3)_{av}} \left(\ln(x_1^{n-1} - \theta^n - x_3^n) - \ln(x_1^{n-1} - \theta^n - x_3^{n-1})\right), \hspace{0.5cm} n = 47, \ldots, 66 \hspace{1cm} (43)$$

where

$$x_4^{46} = x_4^{43} + (x_4)_{2j} + (x_4)_{r2}$$

which are the accumulated cost up through stage 43, the cost of the condenser tube area for the heat rejection section of the second effect, and the cost of recycle in the second effect, respectively.

**G** Material and Energy Balances and Cost Equations for the Heat Rejection Section of the Third Effect. We again assume that the operation is such that

$$x_3^{66} = T_1^{68} = x_1^{68} + 2$$

An energy balance for the heat rejection section of the third effect gives
\[ q_{3j} = F(\frac{2}{3} + r_3)(C_p)_{f3}(T_{f}^{66} - T_{f}^{68}) + \frac{F}{3}(x_1^{66} - x_1^{68}) \]

The following approximate form of this equation will be used.

\[ q_{3j} = F(1+r_3)(C_p)_{f3}(x_1^{66} + 2 - x_3^{66}) \]

The production of distillate in the heat rejection section of the third effect may be calculated from the equations

\[ (\Delta W)_{3j} = \frac{(\lambda)_{av3j}}{\frac{F(\frac{2}{3} + r_3) - x_2^{66}}{(C_p)_{f3}(x_1^{66} + 2 - x_3^{66})}} \]

Since

\[ (\lambda)_{av3j} = 1098.3 - 0.3055 (x_1^{66} + 2 + x_3^{66}) \]

we can obtain

\[ (\Delta W)_{3j} = \frac{(\lambda)_{av3j}}{1098.3 - 0.3055(x_1^{66} + 2 + x_3^{66})} \]

The heat transfer area required by the condenser tubes of the heat rejection section of the third effect is

\[ A_{3j} = \frac{q_{3j}}{(U_{3j})_{av}(\Delta t)_{av3j}} = \frac{F(1+r_3)(C_p)_{f3}(x_1^{66} + 2 - x_3^{66})}{(U_{3j})_{av}(\Delta t)_{av3j}} \]

From Fig. 7c, we have
\[ x_1 - x_3 = x_1 - \frac{x_1 + 2 - x_3}{2} - x_3, \]
\[ x_1 - T_F = x_3 - 2 - T_F, \]
\[ (\Delta t)_{av} = \frac{(x_1 - x_3) + (x_1 - T_F)}{2} = \frac{x_1 + x_3 - 2T_F - 6}{4} \]  

where \( T_F \) = temperature of the make-up feed.

Substituting equation (46) into equation (45), we obtain
\[ A_{3j} = \frac{4F(1+r_3)(C_p)_{f3}(x_1^6 + 2 - x_3^6)}{(U_3)_{av}(x_1^6 + x_3^6 - 2T_F - 6)} \]

Again, multiplication of \( A_{3j} \) by \( 2C_3/F \) yields
\[ (x_4)_{3j} = \frac{8C_3(1+r_3)(C_p)_{f3}(x_1^6 + 2 - x_3^6)}{(U_3)_{av}(x_1^6 + x_3^6 - 2T_F - 6)} \]  

Since the final stage of the third effect is under vacuum, pumping work is required for discharging the distillate and blowdown, the minimum pumping head required is,
\[ \frac{\Delta P}{\rho} = \frac{P_{atm} - P_3}{\rho} \]

in which \( P_{atm} \) stands for the atmospheric pressure. The pumping cost is
\[ (x_4)_{DC} = \frac{2C_4 \times (14.7 - P_3) \times 144}{62.5 \times 550 \times 3600 \times 0.85} \]  

Since the plant is assumed to be located 50 feet above sea level, the theoretical pumping head required is
\[
\frac{\Delta P_0}{\rho} + 50 = \frac{P_0 - 14.7}{\rho} + 50 = \Delta H_0 + 50 \tag{49}
\]

where the first term is the head difference which should be overcome in order to pump the make-up feed to the point corresponds to the maximum brine temperature. A correction factor of 1.18 should be attached to that to take account of the pumping efficiency. \( \frac{1}{0.85} = 1.18 \).

Therefore, the pumping cost is written as

\[
(x_4)_{FP} = \frac{2C_4 \times 1.18(\Delta H_0 + 50)}{550 \times 3600} \tag{50}
\]

The fixed charge cost for the structure for \( 50 \times 10^6 \) gallons/day is

- unit structure cost = \( \frac{2.3}{34} \times 10^6 = 0.0676 \times 10^6 \) \$/stage,
- total structure cost = \( 0.0676 \times 68 \times 10^6 \) \$

or

- capital charged cost = \( 0.0676 \times 68 \times 10^6 \times 9.4 \times 10^{-6} \) \$/hr

= 43.2 \$/hr

where \( 9.4 \times 10^{-6} \) is the capital charge constant based on the plant life of 20 years. Hence

\[
(x_4)_{\text{Str}} = \frac{43.2}{62.4 \times 5 \times 10^7} = \frac{(43.2)(7.48)(24)}{(62.4)(5)(10^7)} \quad \text{\$/lb} \tag{51}
\]

We are now able to write the accumulated cost as

\[
x_4^{68} = x_4^{66} + (x_4)_3^j + (x_4)_{\text{DC}} + (x_4)_{\text{r3}}.
\]

where \( x_4^{66} \) is the accumulated cost for 66 stages, \( (x_4)_3^j \) is the cost of condenser tube area for the heat rejection section of the third effect, \( (x_4)_{\text{DC}} \) is the cost required to discharge the distillate and blowdown, and \( (x_4)_{\text{r3}} \) is
the cost of recycle in the third effect. In addition to this, we must consider the pumping cost of the feed, \((x_4)_\text{FP}\), and the cost of the structure, \((x_4)_\text{Str}\), that is,

\[
(x_4)_T = x_4 + (x_4)_\text{FP} + (x_4)_\text{Str}
\]  

(52)

The above cost is in dollars per pound. In units of dollars per 1000 gallons we can write

\[
C_T = \frac{(x_4)_T \times 10^3 \times 62.4}{7.48}
\]

The above mentioned cost items include only the in-plant operating and capital charge cost. Additional cost items should be included to calculate the total production cost. They are obtained from Ref. (3) and are listed in Table 1.

The design equations are presented in table form in Table 3.

3. RESULTS OF THE SIMULATION ACCORDING TO THE MICRO STAGE MODEL.

The simulation was carried out for two sets of variables on an IBM 1620 computer. The ranges of variables are listed in Table 4. For the first simulation study, the set of variables that was used is

\[
\Delta t_1 = 8.00^\circ F, \\
\theta_1 = 2.00^\circ F, \\
\theta_2 = 2.17^\circ F, \\
\theta_3 = 2.38^\circ F.
\]

For the second simulation study the set of variables is

---

*We assumed equal \(\theta^n\) for each stage within an effect.
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<th>Item</th>
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<td>Chemicals for pretreatment</td>
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<tr>
<td>Supplies &amp; Maintenance materials</td>
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TABLE 2
NOMENCLATURE (THE MICROSTAGE MODEL)

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<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
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<tbody>
<tr>
<td>$C_1$</td>
<td>unit steam cost</td>
<td>$/lb</td>
</tr>
<tr>
<td>$C_2$</td>
<td>unit fixed charge cost for the brine heater</td>
<td>$/ft^2\ hr$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>unit fixed charge cost for the condenser tubes</td>
<td>$/ft^2\ hr$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>unit pumping cost</td>
<td>$/$H.P.\ hr</td>
</tr>
<tr>
<td>$(C_p)^{f1}$</td>
<td>heat capacity of the flashing brine, the first effect</td>
<td>BTU/lb\ $^oF$</td>
</tr>
<tr>
<td>$(C_p)^{f2}$</td>
<td>heat capacity of the flashing brine, the second effect</td>
<td>BTU/lb\ $^oF$</td>
</tr>
<tr>
<td>$(C_p)^{r1}$</td>
<td>heat capacity of recycle brine, the first effect</td>
<td>BTU/lb\ $^oF$</td>
</tr>
<tr>
<td>$F$</td>
<td>amount of make-up feed</td>
<td>lb/hr</td>
</tr>
<tr>
<td>$P_0$</td>
<td>initial flashing pressure</td>
<td>lb/in$^2$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>blowdown pressure of the first effect</td>
<td>lb/in$^2$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>blowdown pressure of the second effect</td>
<td>lb/in$^2$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>final blowdown pressure</td>
<td>lb/in$^2$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>ratio of recycle brine in the first effect to the make-up feed</td>
<td>lb/lb</td>
</tr>
<tr>
<td>$r_2$</td>
<td>ratio of recycle brine in the second effect to the make-up feed</td>
<td>lb/lb</td>
</tr>
<tr>
<td>$r_3$</td>
<td>ratio of recycle brine in the third effect to the make-up feed</td>
<td>lb/lb</td>
</tr>
<tr>
<td>$U_B$</td>
<td>overall heat transfer coefficient of the brine heater</td>
<td>BTU/ft$^2\ hr^oF$</td>
</tr>
<tr>
<td>$(U_{1av})$</td>
<td>average overall heat transfer coefficient of the condenser tubes, the first effect</td>
<td>BTU/ft$^2\ hr^oF$</td>
</tr>
<tr>
<td>$(U_{2av})$</td>
<td>average overall heat transfer coefficient of the condenser tubes, the second effect</td>
<td>BTU/ft$^2\ hr^oF$</td>
</tr>
<tr>
<td>$(U_{3av})$</td>
<td>average overall heat transfer coefficient of the condenser tubes, the third effect</td>
<td>BTU/ft$^2\ hr^oF$</td>
</tr>
<tr>
<td>$\Delta H_0$</td>
<td>pumping head required for the make-up feed</td>
<td>ft</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td>Unit</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>$\Delta t_1$</td>
<td>temperature rise through the brine heater</td>
<td>°F</td>
</tr>
<tr>
<td>$\Delta t_{BPE}$</td>
<td>boiling point elevation of the flashing brine</td>
<td>°F</td>
</tr>
<tr>
<td>$x_1$</td>
<td>temperature of distillate</td>
<td>°F</td>
</tr>
<tr>
<td>$x_2$</td>
<td>amount of distillate</td>
<td>lb/hr</td>
</tr>
<tr>
<td>$x_3$</td>
<td>temperature of recycle brine</td>
<td>°F</td>
</tr>
<tr>
<td>$x_4$</td>
<td>unit production cost</td>
<td>$/lb</td>
</tr>
<tr>
<td>$\Lambda_s$</td>
<td>latent heat of steam</td>
<td>BTU/lb</td>
</tr>
<tr>
<td>$0$</td>
<td>temperature drop of distillate at each stage</td>
<td>°F</td>
</tr>
</tbody>
</table>
### TABLE 3
**DESIGN EQUATIONS (THE MICROSTAGE MODEL)**

**Brine heater**

1. **Steam cost**

   \[ \frac{2C_1 (1+r_1) (C_p r_1 l x \Delta t_1)}{\lambda_s} \]

   \( (C_p r_1 l, C_1, \lambda_s); \text{given} \)

   \( \Delta t_1 : \text{assumed} \)

   \( r_1 : \text{calculated} \)

2. **Fixed charge cost for the brine heater**

   \[ \frac{2C_2 (1+r_1) (C_p r_1 l (\ln(24.4 + \Delta t_1) - \ln(24.4)))}{U_B} \]

   \( C_2 : \text{given} \)

   \( U_B : \text{given} \)

**The first effect**

1. **Temperature of distillate**

   \( x_1^n = x_1^{n-1} - \theta^n, \quad n = 1, 2, \ldots, 20 \)

   \[ x_1^0 = T_f^0 - \Delta t_{B,P,E} \]

   \( T_f^0 : \text{assumed} \)

   \( \Delta t_{B,P,E} : \text{given} \)

   \( \theta^n : \text{assumed} \)

**Heat recovery section**
TABLE 3 (Cont’d)

(2) Amount of distillate

\[ x_2^n = x_2^{n-1} + \frac{\{F(1+r_1) - x_2^{n-1}\} (C_p)_{f_1} \theta^n}{x_2^n} \]

\[ n = 1, 2, \ldots, 20 \] (9)

\[ x_2^0 = 0 : \text{given} \]

\[ F : \text{given} \]

\[ (C_p)_{f_1} : \text{given} \]

\[ \lambda^n = 1098.3 - 0.611x_1^n \]

(3) Temperature of recycle brine

\[ x_3^n = x_3^{n-1} - \frac{\{F(1+r_1)(C_p)_{f_1} + x_2^{n-1}(1-(C_p)_{f_1})\} \theta^n}{F(1+r_1)(C_p)_{rl}} \]

\[ n = 1, 2, \ldots, 20 \] (14)

\[ x_3^0 = (T_f)_\text{max} - \Delta t_1 \]

(4) Fixed charge cost for condenser tubes

\[ 2C_3(1+r_1)(C_p)_{rl} \frac{\{\ln(x_1^{n-1} - \theta^n - x_3^n) - \ln(x_1^{n-1} - \theta^n - x_3 \! - x_3^{n-1})\}}{(U_1)_{av.}} \]

\[ n = 1, 2, \ldots, 20 \] (19)

\[ (U_1)_{av.} : \text{given} \]

\[ C_3 : \text{given} \]
TABLE 3 (Cont'd)

(2) Amount of distillate

\[ x_2^n = x_2^{n-1} + \frac{\{F(5/6 + r_2) - x_2^{n-1}\}(C_p f_2) e^n}{1098.3 - 0.611 x_1^n} \]  

\[ n = 24, \ldots, 43 \]

\[ x_2^{23} = 0 : \text{given} \]

\[ (C_p f_2) : \text{given} \]

\[ r_2 : \text{calculated} \]

(3) Temperature of feed and recycle brine

\[ x_3^n = x_3^{n-1} - \frac{\{F(5/6 + r_2)(C_p f_2) x_2^{n-1}(1-(C_p f_2) + F/6) e^n\}}{F(0.965 + r_2 x 0.9475)} \]

\[ n = 24, \ldots, 43 \]

\[ x_3^{23} = x_3^{20} - \frac{(1+r_1)(C_p f_1)(x_1^{20} + 2 - x_3^{20})}{0.965 + r_2 x 0.9475} \]

(4) Fixed charge cost for condenser tubes

\[ 2C_3(0.965 + r_2 x 0.9475)(\ln(x_1^{n-1} - \theta^n x_3^n) - \ln(x_1^{n-1} - \theta^n x_3^{n-1})) \]

\[ (U_2)_{av.} \]

\[ n = 24, \ldots, 43 \]

\[ (U_2)_{av.} : \text{given} \]
TABLE 3 (Cont'd)

**The first effect**

| (1) Total amount of distillate | \[
\begin{array}{c}
F(1+r_1) - x_{21}^{20}\frac{f_1}{p} (x_1^{20} + 2 - x_3^{20}) \\
1098.3 - 0.3055(x_1^{20} + x_3^{20} + 2)
\end{array}
\] (23) |
|---|---|
| (2) Fixed charge cost for condenser tubes | \[
\begin{array}{c}
\frac{12C_3(1+r_1)(C_1 f_1(x_1^{20} + 2 - x_3^{20}))}{3(x_3^{20} - x_3^{23}) + 2(x_1^{20} - x_3^{20} - 8) (U_1)_{\text{av.}}}
\end{array}
\] (26) |
| (3) Cost of recycle | \[
\frac{2 \times C_4 \times r_1 \times 144 (P_0 - P_1)}{62.5 \times 550 \times 3600 \times 0.85}
\] (20) |

where

- \(P_0\) : given

- \(P_1 = \exp{(14.07 - \frac{9 \times 3660}{5(x_3^{20} - 32) + 1989}})\)

**The second effect**

<table>
<thead>
<tr>
<th>(1) Temperature of distillate</th>
<th>[x_1^n = x_1^{n-1} - \theta^n, \quad n = 24, \ldots, 43] (27)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta^n) : assumed</td>
<td>---</td>
</tr>
</tbody>
</table>
### TABLE 3 (Cont'd)

#### The second effect

| (1) Total amount of distillate | \[
\begin{align*}
\{ & F(\frac{5}{6} + r_2) - x_2^{43}\} (C_p f_2) (x_1^{43} + 2 - x_3^{43}) \\
& 1098.3 - 0.3055(x_1^{43} + 2 + x_3^{43})
\end{align*}
\]
| \[ (34) \] |
| (2) Fixed charge cost for condenser tubes | \[
\begin{align*}
& \frac{12C_3 (1 + r_2) (C_p f_2)(x_1^{43} + 2 - x_3^{43})}{3(x_3^{43} - x_3^{46}) + 2(x_1^{43} - x_3^{43}) - 8} (U_2)_{av.}
\end{align*}
\]
| \[ (37) \] |
| (3) Cost of recycle | \[
2 \times C_4 \times r_2 \times \frac{144}{62.5} \times \frac{(P_1 - P_2)}{550 \times 3600 \times 0.85}
\]
| \[ (31) \] |

\[
P_2 = \exp \left(14.07 - \frac{9 \times 3660}{5(x_3^{43} - 32) + 1989}\right)
\]

#### The third effect

| (1) Temperature of distillate | \[ x_1^n = x_1^{n-1} - \theta^n, \quad n = 47, \ldots, 66 \] |
| \[ \theta^n : \text{assumed} \] |

| (2) Amount of distillate | \[
\begin{align*}
\frac{x_2^n}{x_2^{n-1}} = & \frac{\{F(\frac{2}{3} + r_3) - x_2^{n-1}\} (C_p f_3) \theta^n}{1098.3 - 0.611 x_1^n} \\
& n = 47, \ldots, 66
\end{align*}
\]
| \[ (39) \] |

\[
(C_p f_3) : \text{given}
\]
TABLE 3 (Cont'd)

\[ r_3 : \text{calculated} \]
\[ x_2^{46} = 0 : \text{given} \]

(3) Temperature of feed and recycle brine

\[ x_3^n = x_3^{n-1} - \frac{\{F(\frac{2}{3} + r_3)(C_p)f_3 + x_2^{n-1}(1-(C_p)f_3/s) + F/s\}e^n}{F(0.965 + r_3 x 0.93)} \]
\[ n = 47, \ldots, 66 \quad (40) \]

(4) Fixed charge cost for condenser tubes

\[ x_3^{46} = x_3^{43} - \frac{(1+r_3)(C_p)f_2(x_1^{43} + 2 - x_3^{43})}{0.965 + r_3 x 0.93} \quad (33) \]

\[ 2C_3(0.965 + r_3 x 0.93)(1n(x_1^{n-1} - \theta^n - x_3^{n-1}) - 1n(x_1^{n-1} - \theta^n - x_3^{n-1})) \]
\[ (U_3^3)_{av.} \]
\[ n = 47, \ldots, 66 \quad (41) \]

\[(U_3^3)_{av.} : \text{given} \]

The third effect

heat rejection section

(1) Total amount of distillate produced

\[ \frac{\{F(\frac{2}{3} + r_3) - x_2^{66}\}(C_p)f_3(x_1^{66} + 2 - x_3^{66})}{1098.3 - 0.3055(x_1^{66} + 2 - x_3^{66})} \quad (44) \]

(2) Fixed charge cost for condenser tubes

\[ \frac{8C_3(1+r_3)(C_p)f_3(x_1^{66} + 2 - x_3^{66})}{(x_1^{66} + x_3^{66} - 172)(U_3^3)_{av.}} \quad (47) \]
TABLE 3 (Cont'd)

(3) Cost of recycle

\[ 2 \times C_4 \times r_3 \times \frac{144}{62.5} \times \frac{P_2 - P_3}{550 \times 3600 \times 0.85} \]  

\[ P_3 = \exp \left(14.07 - \frac{9 \times 3660}{5(x_3^6 - 32) + 1989}\right) \]  

Others

(1) Pumping cost for the make-up feed

\[ \frac{H_0}{2C_4\left(\frac{0.85}{0.85} + \frac{50}{0.85}\right)} \]  

\[ \frac{550 \times 3600}{H_0} \]  

\[ H_0 : \text{given (steam table)} \]  

(2) Pumping cost for the distillate and blowdown

\[ 2 \times C_4 \times \frac{144}{62.5} \times \frac{14.7 - P_3}{550 \times 3600 \times 0.85} \]  

(3) Structure cost for the whole system

Total structure cost = \( c_5 \times \text{No. of stages} \)

\[ \text{fixed charge cost} = \frac{\text{total structure cost} \times 9.4 \times 10^{-6}}{62.4 \times 5 \times 10^7} \]  

\[ 7.48 \times 24 \]  

\[ c_5 : \text{given} \]
Summary

Unit production cost includes:

(1) steam cost
(2) fixed charge cost for the brine heater
(3) fixed charge cost for the condenser tubes in each effect
(4) cost of recycle for each effect (includes power and depreciation of pumps)
(5) pumping cost for the make-up feed (includes power and depreciation)
(6) pumping cost for the distillate and blowdown (includes power and depreciation)
(7) structure cost for the whole system
# Table 4

## Unit Costs Used in This Study

1. **Steam cost**
   
   \[
   \text{Steam cost} = 0.25/1000 \text{ lb} \\
   \text{or} \quad c_1 = 2.5 \times 10^{-4}/\text{lb}
   \]

2. **Fixed charge cost for the brine heater**
   
   \[
   \text{Unit price of the brine heater} = 4/\text{ft}^2 \\
   c_2 = 4 \times 9.4 \times 10^{-6} = 3.76 \times 10^{-5}/\text{ft}^2 \text{ hr}
   \]

3. **Fixed charge cost for the condenser tubes**
   
   \[
   \text{Unit price of the condenser tubes} = 2.55/\text{ft}^2 \\
   c_3 = 2.55 \times 9.4 \times 10^{-6} = 2.397 \times 10^{-5}/\text{ft}^2 \text{ hr}
   \]

4. **Pumping cost**
   
   \[
   \text{Unit price of pumps} = 85/\text{HP} \\
   \text{Pump depreciation cost} = 85 \times 9.4 \times 10^{-6} = 8 \times 10^{-4}/\text{HP hr} \\
   \text{Power cost} = 0.69c/\text{KWH} = 5.2 \times 10^{-3}/\text{HP hr} \\
   c_4 = \text{pump depreciation + power cost} = 6.0 \times 10^{-3}/\text{HP hr}
   \]

---

*9.4 \times 10^{-6} is the proportionality constant of depreciation based on the plant life of 20 years with 90% stream days and an annual interest of 4%.*
### TABLE 5

**RANGE OF VARIABLES (THE MACROSTAGE MODEL)**

<table>
<thead>
<tr>
<th>Variable name</th>
<th>Unit</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Temperature of flashing brine</td>
<td>°F</td>
<td>87 &lt; T_F &lt; 274.4</td>
</tr>
<tr>
<td>2. Temperature of distillate</td>
<td>°F</td>
<td>85 &lt; x_1 &lt; 272.4</td>
</tr>
<tr>
<td>3. Temperature of recycle brine</td>
<td>°F</td>
<td>85 &lt; x_3 &lt; 272.4</td>
</tr>
<tr>
<td>4. Temperature difference between the flashing and recycle brine (before flashing)</td>
<td>°F</td>
<td>2 &lt; ΔT_1 &lt; 15</td>
</tr>
<tr>
<td>5. Concentration of the flashing brine</td>
<td>wt.%</td>
<td>3.5 &lt; C_f ≤ 7.0</td>
</tr>
<tr>
<td>6. Temperature drop of the distillate</td>
<td>°F</td>
<td>θ^n_1 &lt; x^n_1 - x^n_3</td>
</tr>
<tr>
<td>7. Temperature drop of the distillate in each stage in the second effect</td>
<td>°F</td>
<td>θ^n_2 &lt; x^n_1 - x^n_3</td>
</tr>
<tr>
<td>8. Temperature drop of the distillate in each stage in the third effect</td>
<td>°F</td>
<td>θ^n_3 &lt; x^n_1 - x^n_3</td>
</tr>
</tbody>
</table>
\[ \Delta t_1 = 9.00^\circ F, \]
\[ \theta_{1n} = 1.85^\circ F, \]
\[ \theta_{2n} = 2.17^\circ F, \]
\[ \theta_{3n} = 2.48^\circ F. \]

The computer flow diagram is given in Fig. 9, and the computer program symbols and the computer program are listed in Table 6. The results for both cases are tabulated in Table 7. Since it takes time to repeat the computation, the macro stage model was used for the subsequent simulation studies.

It should be pointed out that the determination of the recycle ratio for each effect is achieved by an iteration procedure. For a fixed range of flashing temperature and for a fixed distillate production, the recycle ratio becomes a dependent variable so that it can be evaluated by an iterative procedure for the conditions chosen. In this study, a value of the recycle ratio was assumed and the total distillate production for that effect was calculated until a distillate production of \( \frac{F}{6} \) was obtained.

4. FORMULATION OF THE DESIGN EQUATIONS ACCORDING TO THE MACRO STAGE MODEL.

In this section, we will consider the design equations for the macro-stage system. The only difference between the macro-stage and micro-stage system is that in the former we consider the performance of the heat recovery section as a whole unit, while in the latter we go into more detail by considering the performance at each stage within the heat recovery section. The basis of the formulation is a 50 million gallon per day plant. It is assumed that the salt water fed to the system contains 3.5 weight per cent salt and that the concentrated brine leaving the system (blowdown) contains 7 weight per cent salt. This information is presented together with other numerical
Fig. 9 Computer flow diagram (The micro stage model).
### TABLE 6

**A. EXPLANATION OF PROGRAMMING SYMBOLS**

*(THE MICROSTAGE MODEL)*

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>BHACT</td>
<td>fixed charge cost for the brine heater</td>
</tr>
<tr>
<td>CI</td>
<td>unit steam cost</td>
</tr>
<tr>
<td>C2</td>
<td>unit fixed charge cost for the brine heater</td>
</tr>
<tr>
<td>C3</td>
<td>unit fixed charge cost for the condenser tubes</td>
</tr>
<tr>
<td>C4</td>
<td>unit pumping cost</td>
</tr>
<tr>
<td>CPF1</td>
<td>average heat capacity of the flashing brine, the first effect</td>
</tr>
<tr>
<td>CPF2</td>
<td>average heat capacity of the flashing brine, the second effect</td>
</tr>
<tr>
<td>CPF3</td>
<td>average heat capacity of the flashing brine, the third effect</td>
</tr>
<tr>
<td>CPR1</td>
<td>heat capacity of recycle brine, the first effect</td>
</tr>
<tr>
<td>DCCT</td>
<td>pumping cost for the distillate and blowdown</td>
</tr>
<tr>
<td>DELHO</td>
<td>pumping head for the make-up feed</td>
</tr>
<tr>
<td>DELT1</td>
<td>temperature rise through the brine heater</td>
</tr>
<tr>
<td>FD</td>
<td>amount of make-up feed</td>
</tr>
<tr>
<td>FT</td>
<td>temperature of make-up feed</td>
</tr>
<tr>
<td>FBCT1</td>
<td>cost of recycle, the first effect</td>
</tr>
<tr>
<td>FBCT2</td>
<td>cost of recycle, the second effect</td>
</tr>
<tr>
<td>FBCT3</td>
<td>cost of recycle, the third effect</td>
</tr>
<tr>
<td>FPCT</td>
<td>pumping cost for the make-up feed</td>
</tr>
<tr>
<td>HFT</td>
<td>maximum brine temperature</td>
</tr>
<tr>
<td>HJCT1</td>
<td>fixed charge cost for the condenser tubes, heat rejection section, the second effect</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>-------------</td>
</tr>
<tr>
<td>HJCT2</td>
<td>fixed charge cost for the condenser tubes, heat rejection section, the second effect</td>
</tr>
<tr>
<td>HJCT3</td>
<td>fixed charge cost for the condenser tubes, heat rejection section, the third effect</td>
</tr>
<tr>
<td>P0</td>
<td>initial flashing pressure</td>
</tr>
<tr>
<td>P1</td>
<td>blowdown pressure, the first effect</td>
</tr>
<tr>
<td>P2</td>
<td>blowdown pressure, the second effect</td>
</tr>
<tr>
<td>P3</td>
<td>blowdown pressure, the third effect</td>
</tr>
<tr>
<td>R1</td>
<td>ratio of recycle brine in the first effect to the make-up feed</td>
</tr>
<tr>
<td>R2</td>
<td>ratio of recycle brine in the second effect to the make-up feed</td>
</tr>
<tr>
<td>R3</td>
<td>ratio of recycle brine in the third effect to the make-up feed</td>
</tr>
<tr>
<td>STMCT</td>
<td>steam cost</td>
</tr>
<tr>
<td>STRCT</td>
<td>structure cost</td>
</tr>
<tr>
<td>THIN</td>
<td>temperature drop of the distillate in each stage, heat recovery section, the first effect</td>
</tr>
<tr>
<td>TH2N</td>
<td>temperature drop of the distillate in each stage, heat recovery section, the second effect</td>
</tr>
<tr>
<td>TH3N</td>
<td>temperature drop of the distillate in each stage, heat recovery section, the third effect</td>
</tr>
<tr>
<td>TLWP1</td>
<td>total distillate production in the first effect</td>
</tr>
<tr>
<td>TLWP2</td>
<td>total distillate production in the second effect</td>
</tr>
<tr>
<td>TLWP3</td>
<td>total distillate production in the third effect</td>
</tr>
<tr>
<td>UB</td>
<td>overall heat transfer coefficient of the brine heater</td>
</tr>
<tr>
<td>UIAV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the first effect</td>
</tr>
<tr>
<td></td>
<td>Description</td>
</tr>
<tr>
<td>---</td>
<td>---------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>U2AV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the second effect</td>
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<td>U3AV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the third effect</td>
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<td>amount of distillate</td>
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<td>X4</td>
<td>unit production cost (lb basis)</td>
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<td>production cost due to the brine heater</td>
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B. SIMULATION OF THE MULTIFLASH EVAPORATION PROCESS (THE MICROSTAGE "ODEL"

DIMENSION X1(70),X2(70),X3(70),X4(70)
101 FORMAT(6X,6E10.4)
16 FORMAT(2X,7E10.3)
26 FORMAT(2X,7E10.4)
27 FORMAT(4E10.3,E10.4)
60 FORMAT(2E10.4)

READ 101,FD,CPF1,CPR1,U1AV,U2AV,U3AV
READ 101,C1,C2,C3,C4,CPF2,CPF3
READ 101,UB,RMS,DELHCPO,HFT,FT
READ 1U1,TH1N,TH2N,TH3N,DELT1
C
CALC,X1(I)
X1(I)=248.0
DC 11 I=2,21
11 X1(I)=X1(I-1)-TH1N
PUNCH 16,(X1(I),I=1,21)
C
CALC,X2(I),X3(I)
X2(I)=0.0
X3(I)=HFT-DELT1
R1=2.4
22 DC 21 I=2,21
  X2(I)=X2(I-1)+(FD*(1.0+R1)-X2(I-1))*CPF1*TH1N/(1098.3-611*X1(I))
21 X3(I)=X3(I-1)-(FD*(1.0+R1)*CPF1+X2(I-1)*(1.0-CPF1))*TH1N/(FD*(1.0+R1)
  1*CPR1)
  TLWP1=X2(21)+FD*(1.0+R1)*CPF1*(X1(21)+2.0-X3(21))/(1098.3-3055*(X1
  1(21)+X3(21)+2.0))
  IF (TLWP1-FD/6.) 31,32,32
31 R1=R1+3.01
GC TO 22
32 PUNCH 26,(X2(I),I=1,21)
   PUNCH 16,(X3(I),I=1,21)
PUNCH 27,R1, TLWP1
C
CALC,X4(I)
STMCT=2.*C1*(1.0+R1)*DELT1*CPR1/RMS
BHACT=(LOG(24.4+DELT1)-LOG(24.4))*(1.0+R1)*2.*C2*CPR1/UB
X4BH=STMCT+BHACT
X4(I)=X4BH

DC 41 I=2,21
41 X4(I)=X4(I-1)+2.*C3*(1.0+R1)*CPF1*(LOG(X1(I-1)-TH1N-X3(I))-LOG(X1(I
  I-1)-TH1N-X3(I-1)))/U1AV
PUNCH 46,(X4(I),I=1,21)
(CONTINUED)

```
C CALC(X1(I)) I=25,44
  X1(24)=X3(21)-2.
  DC 12 I=25,44
12 XI(I)=XI(I-1)-TH2N
  PUNCH 16,(X1(I), I=24,44)
C CALC*X2(I) »X3(I)
  X2(24)=0.0
  R2=2.2
  24 X3(24)=X3(21)-(1.+R1)*CPF1*( X1(21)+2.-X3(21))/(.965+R2*.9475)
  DC 23 I=25,44
    X2(I)= X2(I-1)+(FD*(5./6.+R2)- X2(I-1))*CPF2*TH2N/(1098.3-.611*XI
    1(I))
  23 X3(I)=X3(I-1)-(FD*(5./6.+R2)*CPF2+X2(I-1)*(1.-CPF2)+ FD/6.)*TH2N/
    (FD*(.965+R2*.9475))
    TLWP2=X2(44)+FD*(1.+R2)*CPF2*(X1(44)+2.-X3(44))/(1098.3-.3055*(X1
    1(44)+X3(44)+2.))
    IF(TLWP2-FD/6.) 33,34,34
  33 R2=R2+.01
  GC TC 24
34 PUNCH 26,(X2(I), I=24,44)
  PUNCH 16,(X3(I), I=24,44)
  PUNCH 27,R2,TLWP2
C CALC*X4(I)
  HJCT1=12.*C3*(1.+R1)*CPF1*(X1(21)+2.-X3(21))/(3.*(X3(21)-X3(24))+
  12.*(X1(21)-X3(21))-8.)/TUAV
  P1=EXP(14.07-9.*3660./(5.*((X3(21)-32.)+1989.))
  FBCT1=2.*C4*R1*144.*(PG-P1)/62.5/550./.85
  X4(24)=X4(21)+HJCT1+FBCT1
  DC 42 I=25,44
  42 X4(I)=X4(I-1)+2.*C3*(.965+R2*.9475)*((LCG(X1(I-1)-TH2N-X3(I)))-LCG
    1(X1(I-1)-TH2N-X3(I-1)))/TUAV
  PUNCH 26,(X4(I), I=24,44)
C CALC*X1(I) I=48,67
  X1(47)=X3(44)-2.
  DC 13 I=48,67
13 XI(I)=XI(I-1)-TH3N
  PUNCH 16,(X1(I), I=47,67)
C CALC*X2(I) »X3(I)
  X2(47)=0.0
  R3=2.2
  51 X3(47)=X3(44)-(1.+R2)*CPF2*(X1(44)+2.-X3(44))/(.965+R3*.930)
  DC 52 I=48,67
    X2(I)=X2(I-1)+(FD*(2./3.+R3)-X2(I-1))*CPF3*TH3N/(1098.3-.611*XI(I-
    1))
  52 X3(I)=X3(I-1)-(FD*(2./3.+R3)*CPF3+X2(I-1)*(1.-CPF3)+FD/3.)*TH3N/
    (FD*(.965+R3*.930))
    TLWP3=X2(67)+FD*(1.+R3)*CPF3*(X1(67)+2.-X3(67))/(1098.3-.3055*(X1
    1(67)+X3(67)+2.))
    IF(TLWP3-FD/6.) 53,54,54
  53 R3=R3+.01
  GC TC 51
```
(CONTINUED)

54 PUNCH  26,(X2(I),I=47,67)
PUNCH  16,(X3(I),I=47,67)
PUNCH  27,R3*TLWP3
C  CALC*X4(I),I=47,67
  HJCT2=12.0*C3*(1.+R2)*CPF2+(X1(44)+2.0-X3(44))/(3.0*(X3(44)-X3(47))+12.0*(X1(44)-X3(44))-8.0)/U2AV
  P2=EXP(14.07-9.0*3660.0/(5.0*(X3(44)-32.0)+1989.0))
  FBCT2=2.0*C4*R2*144./(P1-P2)/62.5/550./3600./.85
  X4(47)=X4(44)+HJCT2+FBCT2
DC  43 I=48,67
43  X4(I)=X4(I-1)+2.0*C3*(.965+R3*.93)*(LOG(X1(I-1)-TH3N-X3(I-1)))/U3AV
PUNCH  26,(X4(I),I=47,67)
  HJCT3=8.0*C3*(1.+R3)*CPF3*(X1(67)+2.0-X3(67))/(X1(67)+X3(67)-2.0*FT-16.0)/U3AV
  P3=EXP(14.07-9.0*3660.0/(5.0*(X3(67)-32.0)+1989.0))
  FBCT3=2.0*C4*R3*144./(P2-P3)/62.5/550./3600./.85
  DCCT=2.0*C4*144./(14.7-P3)/62.5/550./3600./.85
  X4(69)=X4(67)+HJCT3+FBCT3+DCCT
C  CALC*UPCT
  FPCT=2.0*C4*1.18*(DELHO+50.0)/550./3600.
  STRCT=43.2*7.48*24./(62.4*5.0*5.0)
  UPCT=(X4(69)+FPCT+STRCT)*1.5*62.4/7.48
PUNCH  60,X4(69),UPCT
END
DATA
  .3476E 08  .9590E 00  .9590E 00  .5150E 03  .5100E 03  .5050E 03
  .2500E-03  .3760E-04  .2397E-04  .6000E-02  .9485E 00  .9310E 00
  .5371E 03  .9289E 03  .3480E 02  .2982E 02  .2500E 03  .8500E 02
  .2000E 01  .2170E 01  .2380E 01  .8000E 01  .8000E 01
  .1850E 01  .2170E 01  .2480E 01  .9000E 01
## TABLE 7

RESULTS OF SIMULATION (THE MICROSTAGE MODEL)

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<td>Sea water make-up</td>
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Case I: $\Delta t_1 = 8.0 \degree F$, $\theta_1^n = 2.0 \degree F$, $\theta_2^n = 2.17 \degree F$, $\theta_3^n = 2.38 \degree F$

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<th>Blowdown Mlb/hr</th>
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Heat rejection section

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Heat rejection section

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Heat rejection section

| 67 | 95.374 | 93.374 | 85.000 | 5796.0 | 2.707 |
| 68 | 95.374 | 93.374 | 85.000 | 5796.0 | 2.707 |

unit production cost 0.2516 $/10^3$ gallon
TABLE 7 (Cont'd)

Case II: $\Delta t_1 = 9^\circ F$ $\theta_1^n = 1.85^\circ F$ $\theta_2^n = 2.17^\circ F$ $\theta_3^n = 2.48^\circ F$

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<th>Blowdown Brine Mlb/hr</th>
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Heat rejection section

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| 24 | 197.462 | 195.462 | 187.946 | 439.1  | 2.104 |
| 25 | 195.292 | 193.292 | 185.778 | 983.7  | 2.115 |
| 26 | 193.122 | 191.122 | 183.611 | 1227.0 | 2.126 |
| 27 | 190.952 | 188.952 | 181.443 | 1470.0 | 2.136 |
| 28 | 188.782 | 186.782 | 179.275 | 1712.0 | 2.147 |
| 29 | 186.612 | 184.612 | 177.107 | 1954.0 | 2.158 |
| 30 | 184.442 | 182.442 | 174.939 | 2194.0 | 2.169 |
| 31 | 182.272 | 180.272 | 172.770 | 2434.0 | 2.180 |
|-----------|-----------|-------------------------|---------------------|-------------------------------|-----------------------------|-------------------------------|
| 34        |           | 180.102                 | 178.102             | 170.601                       | 2673.0                      | 2.191                          |
| 35        |           | 177.932                 | 175.932             | 168.432                       | 2911.0                      | 2.201                          |
| 36        |           | 175.762                 | 173.762             | 166.263                       | 3148.0                      | 2.212                          |
| 37        |           | 173.592                 | 161.592             | 164.093                       | 3384.0                      | 2.223                          |
| 38        |           | 171.422                 | 169.422             | 161.923                       | 3620.0                      | 2.234                          |
| 39        |           | 169.252                 | 167.252             | 159.753                       | 3855.0                      | 2.245                          |
| 40        |           | 167.082                 | 165.082             | 157.583                       | 4089.0                      | 2.255                          |
| 41        |           | 164.912                 | 162.912             | 155.413                       | 4322.0                      | 2.266                          |
| 42        |           | 162.742                 | 160.742             | 153.242                       | 4554.0                      | 2.277                          |
| 43        |           | 160.572                 | 158.572             | 151.071                       | 4786.0                      | 2.288                          |

Heat rejection section

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Heat rejection section

|   |   |   |   |   |
|---|---|---|---|
| 67 |   |   |   |
| 68 | 91.887 | 89.887 | 85.00 |

**Unit Production cost:** 0.2539 $/10^3$ gallon
data in Table 7.

(A) Design Equations for the First Effect. In this section only the design equations for the brine heater (bht) and the heat recovery section are given. The design equations for the heat rejection section are included in part (B). An energy balance for the first effect (refer to Fig. 10) gives

\[ F(1+r_1)(C_{pl,av})x_0 = \frac{F}{6}(H_{vl,av}) + F(\frac{5}{6} + r_1)(C_{p,f1})x_1 \]  

(53)

where

- \( F \) = amount of make-up feed,
- \( x \) = temperature of flashing brine (°F)
- \( (H_{vl,av}) \) = average enthalpy of water vapor in the first effect relative to liquid at 0°F
- \( H_v = 1065 + 0.392x \) (BTU/lb) (an empirical formula obtained by curve fitting the steam table).
- \( (H_{vl,av}) = \frac{1065 + 0.392x_0 + 1065 + 0.392x_1}{2} \)
- \( r_1 = \) ratio of the recycle brine in the first effect to the make-up feed
- \( (C_{pl,av}) = \frac{(C_{p,F}) + r_1(C_{p,f1})}{1+r_1} \)
- \( (C_{p,F}) \) = heat capacity of the make-up feed,
- \( (C_{p,f1}) \) = heat capacity of the flashing brine at the exit of the first effect.

Simplifying and rewriting equation (53), we obtain

\[ \{ (C_{p,F}) + r_1(C_{p,f1}) \}x_0 = \frac{1}{6}[1065 + 0.196(x_0 + x_1)] + (\frac{5}{6} + r_1)(C_{p,f1})x_1 \]

Solving for \( r_1 \), we have
Fig. 10. Schematic flow diagram for the first effect.
\[ r_1 = \frac{1065 + 0.196(x_0 + x_1) + 5(C_p)^{f_{11}}x_1 - 6(C_p)^{F}x_0}{6(C_p)^{f_{11}}(x_0 - x_1)} \]  

(54)

For the steam that is used in the brine heater, the steam cost in dollars per pound of distillate product, \((x_4)_s\), is

\[ (x_4)_s = \frac{2C_1(1+r_1)(C_p)\Delta t_1}{\lambda_s} \]  

(55)

(See equation (17) in the micro-stage model.)

The fixed charge cost for the brine heater, \((x_4)_B\), is (see equation (18) in the micro-stage model)

\[ (x_4)_B = \frac{2C_2(1+r_1)(C_p)_{pl}}{U_B} \frac{\Delta t_1}{\lambda_s} \{\ln(24.4 + \Delta t_1) - \ln(24.4)\} \]

(56)

The rate of heat transfer for the condenser tubes of the heat recovery section of the first effect is

\[ q_{1r} = F(1+r_1)(C_{pl})_{av}(x_0 - x_1 - \Delta t_1) \]

The heat transfer area, \(A_{1r}\), is given by

\[ A_{1r} = \frac{q_{1r}}{(U_{1r})_{av}(\Delta t_{1r})_{av}} \]

where
\[
(\Delta t_{1r})^{\text{av}} = \frac{(\Delta t_1 - 2) + (\Delta t_1 - 2 - \frac{x_0-x_1-\Delta t_1}{20})}{2}
\]
\[
= \frac{41 \Delta t_1 - x_0+x_1 - 80}{40}
\]

Hence, the fixed charge cost, \((x_4)_{c1}\), is

\[
(x_4)_{c1} = \frac{C_3 F \{C_p F + r_1(C_p f_1)\}(x_0-x_1-\Delta t_1)}{\frac{F}{2}(U_1)^{\text{av}}(41\Delta t_1 - x_0+x_1 - 80)}
\]

\[
(x_4)_{c1} = \frac{80C_3 \{C_p F + r_1(C_p f_1)\}(x_0-x_1-\Delta t_1)}{(U_1)^{\text{av}}(41\Delta t_1 - x_0+x_1 - 80)}
\]  \( (57) \)

The recycle cost for the first effect, \((x_4)_{rl}\), is (see equation (20) for the micro-stage model)

\[
(x_4)_{rl} = \frac{2C_4 r_1 x 144(P_0-P_1)}{62.5 \times 550 \times 3600 \times 0.85}
\]  \( (58) \)

where

\[
P_1 = \exp \left\{14.07 - \frac{9 \times 3660}{5(x_1 - 32) + 1989}\right\}
\]

(B) **Design Equations for the Second Effect.** Making an energy balance for the second effect (refer to Fig. 11), we have

\[
F(\frac{5}{6} + r_2)(C_p^2)_{av} x_1 = \frac{F}{6}(H v_2)_{av} + F(\frac{2}{3} + r_2)(C_p)f_2 x_2
\]  \( (59) \)

Since
Fig. 11. Schematic flow diagram for the second effect.
\[
(C_{p2})_{av} = \frac{\frac{5}{6} (C_{p})_{f1} + r_2 (C_{p})_{f2}}{\frac{5}{6} + r_2},
\]
\[
(H_{v2})_{av} = 1065 + 0.196(x_1 + x_2).
\]

Equation (59) may be rewritten as
\[
\frac{5}{6} (C_{p})_{f1} + r_2 (C_{p})_{f2} x_1 = \frac{1065 + 0.196(x_1+x_2)}{6} + \frac{\frac{2}{3} + r_2 (C_{p})_{f2} x_2}{(C_{p})_{f2} x_2 - x_1}.
\]

Hence \( r_2 \) is obtained as
\[
r_2 = \frac{1065 + 0.196(x_1+x_2) + 4(C_{p})_{f2} x_2 - 5(C_{p})_{f1} x_1}{6(C_{p})_{f2} (x_2 - x_1)}.
\] (60)

The value of \( \Delta t_2 \) is obtained by making an energy balance for the heat rejection section of the first effect. We can write
\[
q_{1j} = F(1+r_1)(C_{p1})_{av} \Delta t_1
\]
\[
= F((C_{p})_{f} + r_1 (C_{p})_{f1}) \Delta t_1.
\]

We also have
\[
q_{1j} = F((C_{p})_{f} + r_2 (C_{p})_{f2}) \Delta t_2.
\]

Hence
\[
\Delta t_2 = \frac{(C_{p})_{f} + r_1 (C_{p})_{f1}}{(C_{p})_{f} + r_2 (C_{p})_{f2}} \Delta t_1.
\] (61)

The heat transfer area for the condenser tubes in the heat rejection section of the first effect, \( A_{1j} \), is given by
\[ A_{lj} = \frac{q_{lj}}{(U_{l})_{av}(\Delta t)_{avljl}} \]

From equation (25) in the micro-stage model, we note that \( \Delta t_{avljl} \), the average temperature difference for heat transfer, is

\[ \Delta t_{avljl} = \frac{3(x_{3}^{20} - x_{3}^{23}) + 2(x_{1}^{20} - x_{3}^{20}) - 8}{6} \]

\[ = \frac{3\Delta t_{2} + 2(\Delta t_{1} - 2) - 8}{6} \]

\[ = \frac{3\Delta t_{2} + 2\Delta t_{1} - 12}{6} \]

The fixed charge cost for the condenser tubes of the heat rejection section of the first effect, \( (x_{4})_{lj} \), can thus be written as

\[ (x_{4})_{lj} = \frac{12C_{3}(C_{p}F + r_{1}(C_{p}F_{1})\Delta t_{1}}{(U_{1})_{av}(3\Delta t_{2} + 2\Delta t_{1} - 12))} \quad (62) \]

The rate of heat transfer for the condenser tubes of the heat recovery section of the second effect is

\[ q_{2r} = F[\left(\frac{5}{6} + r_{2}\right)(C_{p2})_{av} + \frac{1}{6})(x_{1} - x_{2} - \Delta t_{2}) \]

\[ = F[\frac{5}{6}(C_{p}F_{1} + r_{2}(C_{p}F_{2}) + \frac{1}{6})(x_{1} - x_{2} - \Delta t_{2})] \quad . \]

For the condenser tubes the fixed charge cost \( (x_{4})_{c2} \) is

\[ (x_{4})_{c2} = \frac{C_{3}[F\frac{5}{6}(C_{p}F_{1} + r_{2}(C_{p}F_{2}) + \frac{1}{6})(x_{1} - x_{2} - \Delta t_{2})]}{2(U_{2})_{av}\frac{1}{41\Delta t_{2} - x_{1} + x_{2} - 80}} \]

\[ = \frac{80C_{3}[\frac{5}{6}(C_{p}F_{1} + r_{2}(C_{p}F_{2}) + \frac{1}{6})(x_{1} - x_{2} - \Delta t_{2})]}{(U_{2})_{av}(41\Delta t_{2} - x_{1} + x_{2} - 80)} \quad (63) \]
The recycle cost for the second effect, \((x_4)_{r_2}\) is (see equation (32) in the micro-stage model)

\[
(x_4)_{r_2} = \frac{2C_4 r_2 \times 144(P_1 - P_2)}{62.5 \times 550 \times 3600 \times 0.85}
\]  
(64)

where

\[
P_2 = \exp \left\{ 14.07 - \frac{9 \times 3660}{5(x_2 - 32) + 1989} \right\}
\]

(C) Design Equations for the Third Effect. An energy balance for the third effect (refer to Fig. 12) gives

\[
F(\frac{2}{3} + r_3)(C_{p3})_{av}x_2 = \frac{F}{6} (H_{v3})_{av} + F(\frac{1}{2} + r_3)(C_{p})_{f3}x_3
\]

(65)

Since

\[
(C_{p3})_{av} = \frac{\frac{2}{3}(C_{p})_{f2} + r_3(C_{p})_{f3}}{\frac{2}{3} + r_3}
\]

and

\[
(H_{v3})_{av} = 1065 + 0.196(x_2 + x_3),
\]

equation (65) can be written as

\[
\left(\frac{2}{3}(C_{p})_{f2} + r_3(C_{p})_{f3}\right)x_2 = \frac{1065 + 0.196(x_2 + x_3)}{6} + (\frac{1}{2} + r_3)(C_{p})_{f3}x_3
\]

Solving for \(r_3\), one obtains

\[
r_3 = \frac{1065 + 0.196(x_2 + x_3) + 3(C_{p})_{f3}x_3 - 4(C_{p})_{f2}x_2}{6(C_{p})_{f3}(x_2 - x_3)}
\]

(66)
Fig. 12. Schematic flow diagram for the third effect.
The value of $\Delta t_3$ is obtained by making an energy balance around the heat rejection section of the second effect. One can write two equations

\begin{align*}
q_{2j} &= F \left[ \frac{5}{6} (C_p f_1) + r_2 (C_p f_2) + \frac{1}{6} \right] \Delta t_2 \\
q_{2j} &= F \left[ (C_p f) + r_3 (C_p f_3) \right] \Delta t_3
\end{align*}

which when combined gives

\[ \Delta t_3 = \frac{\frac{5}{6} (C_p f_1) + r_2 (C_p f_2) + \frac{1}{6}}{(C_p f_3 + r_3 (C_p f_3))} \Delta t_2 \quad (67) \]

The fixed charge cost for the condenser tubes of the heat rejection section of the second effect, $(x_4)_{2j}$, is

\[ (x_4)_{2j} = \frac{12C \left( \frac{5}{6} (C_p f_1) + r_2 (C_p f_2) + \frac{1}{6} \right) \Delta t_2}{(U_2)_{av} (3\Delta t_3 + 2\Delta t_2 - 12)} \quad (68) \]

The rate of heat transfer for the condenser tubes of the heat recovery section of the third effect, $q_{3r}$, is

\begin{align*}
q_{3r} &= F \left[ \frac{2}{3} + r_3 (C_p f_3) + \frac{1}{3} \right] (x_2 - x_3 - \Delta t_3) \\
&= F \left( \frac{2}{3} (C_p f_2) + r_3 (C_p f_3) + \frac{1}{3} \right) (x_2 - x_3 - \Delta t_3)
\end{align*}

The fixed charge cost, $(x_4)_{c3}$, is

\begin{align*}
(x_4)_{c3} &= \frac{C_F \left( \frac{2}{3} (C_p f_2) + r_3 (C_p f_3) + \frac{1}{3} \right) (x_2 - x_3 - \Delta t_3)}{\frac{F}{2} (U_3)_{av} (41\Delta t_3 - x_2^2 + x_3^2 - 80)} \\
&= \frac{80C \left( \frac{2}{3} (C_p f_2) + r_3 (C_p f_3) + \frac{1}{3} \right) (x_2 - x_3 - \Delta t_3)}{(U_3)_{av} (41\Delta t_3 - x_2 + x_3 - 80)} \quad (69)
\end{align*}
The fixed charge cost for the condenser tubes of the heat rejection section of the third effect, \((x_4)_{3j}\), is

\[
(x_4)_{3j} = \frac{8C_3 \left[ \frac{2}{3} (C_p)_{f2} + r_3 (C_p)_{f3} + \frac{1}{3} \right] \Delta t_3}{(U_3)_{av} (\Delta t_3 + 2(x_3 - T_F) - 8)}
\]  

(70)

where from equation (47) in the microstage model, we see that

\[
x_1^{66} + x_3^{66} - 2T_F - 6 = (x_1^{66} + 2 - x_3^{66}) + 2(x_3^{66} - T_F) - 6
\]

\[
= \Delta t_3 + 2(x_3 - T_F) - 8.
\]

Note that \(x_3\) is the blowdown temperature of the third effect.

The recycle cost for the third effect, \((x_4)_{r3}\), is \[see equation (44)\]

\[
(x_4)_{r3} = \frac{2C_4 r_3 \times 144(P_3 - P_3)}{62.5 \times 550 \times 3600 \times 0.85}
\]  

(71)

where

\[
P_3 = \exp \left\{ 14.07 - \frac{9 \times 3660}{5(x_3 - 32) + 1989} \right\}
\]

The dumping cost for the distillate and spent brine, \((x_4)_{DC}\), is \[see equation (48) for the microstage model\]

\[
(x_4)_{DC} = \frac{2C_4 \times 144(14.7 - P_3)}{52.5 \times 550 \times 3600 \times 0.85}
\]  

(72)

The structure cost, \((x_4)_{Str}\), is \[see equation (51) for the microstage model\]

\[
(x_4)_{Str} = \frac{43.2 \times 7.48 \times 24}{62.4 \times 5 \times 107}
\]  

(73)

The pumping cost for the make-up feed, \((x_4)_{FP}\), is \[see equation (50) for the
micro-stage model

\[ (x_4)_{FP} = \frac{2C_4 \times 1.18 \left\{ (P_0 - 14.7) \frac{144}{62.5} + 50 \right\}}{550 \times 3600} \] (74)

In summary, the total cost is the sum of the individual costs, which are

1. the steam cost, equation (55)
2. the fixed charge cost for the brine heater, equation (56)
3. the fixed charge cost for the condenser tubes of each effect, equations (57), (62), (63), (68), (69), and (70).
4. the recycle cost for each effect, equations (58), (64), and (71)
5. the pumping cost for the make-up feed, equation (74)
6. the pumping cost for the distillate and blowdown, equation (72)
7. the structure cost, equation (73).

In addition to these costs, the general and administrative costs listed in Table 1 must be considered.

5. SIMULATION ACCORDING TO THE MACRO-STAGE MODEL

The simulation was carried out on the IBM 1620 computer. The design equations are listed in Table 8. The ranges of variables are shown in Table 9. The computer program symbols and computer program are put in Tables 10 and 11. The results are shown in Tables 12, 13 and 14.

6. DISCUSSION OF RESULTS

Comparison of results by both models are shown in Table 13. From this, we see that there is good agreement. The minor differences might be attributed to the temperature differences for heat transfer used for each case. It is also seen that the recycle ratios of each effect based on the amount of
<table>
<thead>
<tr>
<th>Brine heater</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(1) Steam Cost</strong></td>
</tr>
<tr>
<td>$2C_1 \left( \frac{(C_p F) + r_1 (C_p f_1)}{\lambda_s} \right) \Delta t_1$</td>
</tr>
<tr>
<td>$r_1$ is calculated from</td>
</tr>
<tr>
<td>$r_1 = \frac{1065 - 0.196(x_0 x_1) + 5(C_p F) x_1 - 6(C_p F) x_0}{6(C_p f_1)(x_0 - x_1)}$</td>
</tr>
<tr>
<td>$C_1 =$ given</td>
</tr>
<tr>
<td>$\lambda_s =$ given</td>
</tr>
<tr>
<td>$(C_p F), (C_p f_1)$ given</td>
</tr>
<tr>
<td>$\Delta t_1, x_0, x_1$ assumed</td>
</tr>
</tbody>
</table>

| **(2) Fixed charge cost for the brine heater,** |
| $2C_2 \left( \frac{(C_p F) + r_1 (C_p f_1)}{\lambda_s} \right) \left\{ \ln(24.4 + \Delta t_1) - \ln(24.4) \right\}$ |
| $C_2 =$ given |
| $U_B =$ given |
### TABLE 8 (Cont'd)

**The first effect**

**heat recovery section**

\[
80C_3 \left\{ \frac{(C_p)_F + r_1(C_p)_{F1}}{(U_1)_{av.}} \right\} (x_0 - x_1 - \Delta t_1) \quad (57)
\]

\[
(U_1)_{av.} (41\Delta t_{1} - x_0 + x_1 - 80)
\]

\[C_3 : \text{given}\]

\[(U_1)_{av.} : \text{given}\]

\[
\frac{288 C_4 r_1 (P_0 - P)}{62.5 \times 550 \times 3600 \times 0.85}
\]

\[62.5 \times 550 \times 3600 = 0.85 \quad (58)\]

\[C_4 : \text{given}\]

\[P_0 : \text{given}\]

\[P_1 \text{ is calculated from}\]

\[P_1 = \exp \left( 14.07 - \frac{9 \times 3660}{5(x_1 - 32) + 1989} \right),\]

**heat rejection section**

\[
12C_3 \left\{ \frac{(C_p)_F + r_1(C_p)_{F1}}{(U_1)_{av.}} \right\} \Delta t_1 \quad (62)
\]

\[(U_1)_{av.} (3\Delta t_2 + 2\Delta t_1 - 12)\]

\[\Delta t_2 \text{ is calculated from}\]

\[
r_2 = \frac{1065 + 0.196(x_1 + x_2) + 4(C_p)_{F2} x_2 - 5(C_p)_{F1} x_1}{6(C_p)_{F2} (x_1 - x_2)}\]

\[6(C_p)_{F2} (x_1 - x_2) \quad (60)\]
TABLE 8 (Cont'd)

\[ \Delta t_2 = \frac{(C_p)_{F1} + r_1(C_p)_{F1}}{(C_p)_{F1} + r_2(C_p)_{F2}} \Delta t_1 \]  

(61)

\[ (C_p)_{F2} \text{ is given,} \]

\[ x_2 = \text{assumed,} \]

The second effect

heat recovery section

(1) fixed charge cost for condenser tube

\[ 80C_3 \frac{\frac{5}{6}(C_p)_{F1} + r_2(C_p)_{F2} + \frac{1}{6}}{(U_2)_{av.} \cdot (41\Delta t_2 - x_1 + x_2 - 80)} \]  

(63)

\[ (U_2)_{av.} = \text{given} \]

(2) recycling cost

\[ \frac{288C_4 r_2 (P_1 - P_2)}{62.5 \times 550 \times 3600 \times 0.85} \]  

(64)

\[ P_2 = \exp \left( 14.07 - \frac{9 \times 3660}{5(x_2 - 32) + 1989} \right) \]

The second effect

heat rejection section

(1) fixed charge cost for the condenser tube

\[ \frac{12C_3 \frac{\frac{5}{6}(C_p)_{F1} + r_2(C_p)_{F2} + \frac{1}{6}}{(U_2)_{av.} \cdot (3\Delta t_3 + 2\Delta t_2 - 12)} \Delta t_2}{\Delta t_3} \]  

(68)

\[ \Delta t_3 \text{ is calculated from} \]

\[ r_3 = \frac{1065 + 0.196 (x_2 + x_3) + 3(C_p)_{F3} x_3 - 4(C_p)_{F2} x_2}{6(C_p)_{F3} (x_2 - x_3)} \]  

(66)
TABLE 8 (Cont'd)

\[
\Delta t_3 = \frac{\frac{5}{6} (\frac{C}{p} f_1) + r_2 (\frac{C}{p} f_2) + \frac{1}{3}}{(\frac{C}{p} f_2 + r_3 (\frac{C}{p} f_3) + \frac{1}{3}) \Delta t_2}
\]

\[
(C_p f_3) = \text{given}
\]

\[
x_3 = \text{assumed}
\]

The third effect

(1) fixed charge cost for the condenser tube

\[
80C_3 \left( \frac{2}{3} (\frac{C}{p} f_2) + r_3 (\frac{C}{p} f_3) + \frac{1}{3} \right) \left( x_2 - x_3 - \Delta t_3 \right)
\]

\[
\left( \frac{U_3}{U_3} \right)_{\text{av.}} \frac{(41 \Delta t_3 - x_2 + x_3 - 80)}{(U_3)_{\text{av.}}}
\]

\[
(U_3)_{\text{av.}} = \text{given}
\]

(2) recycle cost

\[
288C_4 r_3 (P_2 - P_3)
\]

\[
62.5 \times 550 \times 3600 \times 0.85
\]

\[
P_3 = \text{calculated from}
\]

\[
P_3 = \exp \left( 14.07 - \frac{9 \times 3660}{5(x_3 - 32) + 1989} \right)
\]

The third effect

heat recovery section

heat rejection section

(1) fixed charge cost for the condenser tubes,

\[
8C_3 \left( \frac{2}{3} (\frac{C}{p} f_2) + r_3 (\frac{C}{p} f_3) + \frac{1}{3} \right) \Delta t_3
\]

\[
\left( \frac{U_3}{U_3} \right)_{\text{av.}} \left( \Delta t_3 + 2(x_3 - T_F) - 8 \right)
\]

\[
T_F = \text{given}
\]
<table>
<thead>
<tr>
<th>TABLE 8 (Cont'd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2) damping cost for the distillate and blowdown</td>
</tr>
<tr>
<td>(3) pumping cost for the make-up feed</td>
</tr>
<tr>
<td>(4) construction cost</td>
</tr>
</tbody>
</table>

<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2 \times C_4 \times \frac{144(14.7 - P_3)}{62.5 \times 550 \times 3600} )</td>
</tr>
<tr>
<td>( P_3 = \text{given (from steam table)} )</td>
</tr>
<tr>
<td>( \frac{2 \times C_4 \times 1.18 {P_0 - 14.7} \times 144}{550 \times 3600} )</td>
</tr>
<tr>
<td>( \frac{43.2 \times 7.48 \times 24}{62.4 \times 5 \times 107} )</td>
</tr>
</tbody>
</table>
TABLE 9
RANGE OF VARIABLES

1. Maximum temperature of flashing brine
   
2. Minimum temperature of flashing
   
3. Intermediate blowdown temperature
   
4. Total temperature difference between the flashing brine and recycle brine

\( (T_f)_{\text{max}} < Ts \) (steam temperature)

\( (T_f)_{\text{min}} > T_F \) (temperature of make-up feed)

\( (T_f)_{\text{max}} > x_1, x_2 > (T_f)_{\text{min}} \)

\( 2 < \Delta t_n < 15, \ n=1, 2, 3 \)

In this section, we fix \( (T_f)_{\text{max}} \) at 250\(^\circ\)F and \( (T_f)_{\text{min}} \) at 95.2 \(-91.2\)\(^\circ\)F. The intermediate temperatures are chosen such that the temperature interval is greater the previous one each time.
Fig. 13. Computer flow diagram (the macro stage model).
**TABLE 10**  
**EXPLANATION OF PROGRAM SYMBOLS**  
**(THE MACROSTAGE MODEL)**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>BHACT</td>
<td>fixed charge cost for the brine heater</td>
</tr>
<tr>
<td>C1</td>
<td>unit steam cost</td>
</tr>
<tr>
<td>C2</td>
<td>unit fixed charge cost for the brine heater</td>
</tr>
<tr>
<td>C3</td>
<td>unit fixed charge cost for the condenser tubes</td>
</tr>
<tr>
<td>C4</td>
<td>unit pumping cost</td>
</tr>
<tr>
<td>CPF</td>
<td>heat capacity of the make-up feed</td>
</tr>
<tr>
<td>CPF1</td>
<td>heat capacity of the flashing brine, the first effect</td>
</tr>
<tr>
<td>CPF2</td>
<td>heat capacity of the flashing brine, the second effect</td>
</tr>
<tr>
<td>CPF3</td>
<td>heat capacity of the flashing brine, the third effect</td>
</tr>
<tr>
<td>DCCT</td>
<td>pumping cost for the distillate and blowdown</td>
</tr>
<tr>
<td>DELT1</td>
<td>temperature rise through the brine heater</td>
</tr>
<tr>
<td>DELT2</td>
<td>temperature difference between the flashing and recycle brine, the second effect</td>
</tr>
<tr>
<td>DELT3</td>
<td>temperature difference between the flashing and recycle brine, the third effect</td>
</tr>
<tr>
<td>F1X0</td>
<td>unit production cost due to the first effect</td>
</tr>
<tr>
<td>F2X1</td>
<td>unit production cost due to the second and third effect</td>
</tr>
<tr>
<td>FBCT1</td>
<td>recycling cost, the first effect</td>
</tr>
<tr>
<td>FBCT2</td>
<td>recycling cost, the second effect</td>
</tr>
<tr>
<td>FBCT3</td>
<td>recycling cost, the third effect</td>
</tr>
<tr>
<td>FPCT</td>
<td>pumping cost for the make-up feed</td>
</tr>
<tr>
<td>Variable</td>
<td>Description</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
</tr>
<tr>
<td>FT</td>
<td>temperature of make-up feed</td>
</tr>
<tr>
<td>P0</td>
<td>initial flashing pressure</td>
</tr>
<tr>
<td>P1</td>
<td>blowdown pressure, the first effect</td>
</tr>
<tr>
<td>P2</td>
<td>blowdown pressure, the second effect</td>
</tr>
<tr>
<td>P3</td>
<td>blowdown pressure, the third effect</td>
</tr>
<tr>
<td>R1</td>
<td>ratio of recycle brine in the first effect to the make-up feed</td>
</tr>
<tr>
<td>R2</td>
<td>ratio of recycle brine in the second effect to the make-up feed</td>
</tr>
<tr>
<td>R3</td>
<td>ratio of recycle brine in the third effect to the make-up feed</td>
</tr>
<tr>
<td>RMS</td>
<td>latent heat of steam</td>
</tr>
<tr>
<td>STMCT</td>
<td>steam cost</td>
</tr>
<tr>
<td>STRCT</td>
<td>structure cost</td>
</tr>
<tr>
<td>TAC1R</td>
<td>fixed charge cost for the condenser tubes, heat recovery section, the first effect</td>
</tr>
<tr>
<td>TAC1J</td>
<td>fixed charge cost for the condenser tubes, heat rejection section, the first effect</td>
</tr>
<tr>
<td>TAC2R</td>
<td>fixed charge cost for the condenser tubes, heat recovery section, the second effect</td>
</tr>
<tr>
<td>TAC2J</td>
<td>fixed charge cost for the condenser tubes, heat rejection section, the second effect</td>
</tr>
<tr>
<td>TAC3R</td>
<td>fixed charge cost for the condenser tubes, heat recovery section, the third effect</td>
</tr>
<tr>
<td>TAC3J</td>
<td>fixed charge cost for the condenser tubes, heat rejection section, the third effect</td>
</tr>
<tr>
<td>U1AV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the first effect</td>
</tr>
<tr>
<td>Variable</td>
<td>Description</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
</tr>
<tr>
<td>U2AV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the second effect</td>
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<td>U3AV</td>
<td>average overall heat transfer coefficient of the condenser tubes, the third effect</td>
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<tr>
<td>UB</td>
<td>overall heat transfer coefficient of the brine heater</td>
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<td>unit production cost</td>
</tr>
<tr>
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<td>maximum brine temperature</td>
</tr>
<tr>
<td>X1</td>
<td>blowdown temperature, the first effect</td>
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<td>blowdown temperature, the second effect</td>
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<td>blowdown temperature, the third effect</td>
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TABLE 11

SIMULATION OF THE MULTIFLASH EVAPORATION PROCESS (THE MACROSTAGE MODEL)

21 FORMAT(6X,6E10.4)
51 FORMAT(1H 3F10.3,2X,E11.5)
52 FORMAT(1H 5F10.3,2X,E10.4)
53 FORMAT(1H E11.4)
READ 21, CPF, CPF1, UB, U1AV, U2AV, U3AV
READ 21, X0, C1, C2, C3, C4
READ 21, RMS, CPF2,CPF3, FT, P0
READ 21, DELT1, X1, X2, X3
PRINT 21, CPF, CPF1, UB, U1AV, U2AV, U3AV
PRINT 21, X0, C1, C2, C3, C4
PRINT 21, RMS, CPF2,CPF3, FT, P0
PRINT 21, DELT1, X1, X2, X3
C
CALC.F1X0

12  R1= (1065. + .196*(X0+X1)+5. *CPF1*X1-6. *CPF*X0)/(6. *CPF1*(X0-X1))
STMCT=2.*C1*(CPF+CPF1*R1)*DELT1/RMS
BHACT=2.*C2*(CPF+CPF1*R1)*LOG(24.4+DELT1)-LOG(24.4))/UB
TAC1R=80.*C3*(CPF+CPF1*R1)*(X0-X1-DELT1)/U1AV/(41.*DELT1-X0+X1-180.)
P1=EXP(14.07-9.*3660.)/(5.*(X1-32.)+1989.)
FBCT1=2.*C4*R1*144.*(P0-P1)/62.5/550./3600./.85
F1XU=STMCT+BHACT+TAC1R+FBCT1
PUNCH 51, X1, DELT1, R1, F1X0
C
CALC.F2X1

12  R2= (1065. + .196*(X1+X2)+4.*CPF2*X2-5. *CPF1*X1)/(6. *CPF2*(X1-X2))
DEL12=(CPF+CPF1*R1)*DELT1/(CPF+CPF2*R2)
R3= (1065. + .196*(X2+X3)+3.*CPF3*X3-4.*CPF2*X2)/(6. *CPF3*(X2-X3))
DELT3= (5.*CPF1/6.+CPF2*R2+1./6.)*DELT2/(CPF+CPF3*R3)
TAC1J=12.*C3*(CPF+CPF1*R1)*DELT1/U1AV/(3.*DELT2+2.*DELT1-12.)
TAC2R=80.*C3*(5.*CPF1/6.+CPF2*R2+1./6.)*(X1-X2-DELT2)/U2AV/(41.*
1DELT2-X1+X2-80.)
TAC2J= 12.*C3*(5.*CPF1/6.+CPF2*R2+1./6.)*DELT2/U2AV/(3.*DELT3+
12.*DELT2-12.)
TAC3R=80.*C3*(2.*CPF2/3.+CPF3*R3+1./3.)*(X2-X3-DELT3)/U3AV/(41.*
1*DELT3-X2+X3-80.)
TAC3J=80.*C3*(2.*CPF2/3.+CPF3*R3+1./3.)*DELT3/U3AV/(DELT3+2.*(X3
1-FT)-8.)
P2 = \exp(14.07 - 9.03660 / (5.0 (X2 - 32.0) + 1989))

FBCT2 = 2.0C4*R2*144.0*(P1-P2) / 62.5 / 550.0 / 3600.0 / 85

P3 = \exp(14.07 - 9.03660 / (5.0 (X3 - 32.0) + 1989))

FBCT3 = 2.0C4*R3*144.0*(P2-P3) / 62.5 / 550.0 / 3600.0 / 85

DCCT = 2.0C4*144.0*(14.7 - P3) / 62.5 / 550.0 / 3600.0 / 85

FPCT = 2.0C4*1.18*(P0-14.7)*144.0 / 62.5 / 550.0 / 3600.0

STRCT = 43.2*7.48*24.0 / (62.4*5.0E7)

F2X1 = TAC1J + TAC2R + TAC2J + TAC3R + TAC3J + FBCT2 + FBCT3 + DCCT + FPCT + STRCT

PUNCH 52, X2, R2, DELT2, R3, DELT3, F2X1

UPCT = (F1X0 + F2X1) * 62.4 * 1.3E3 / 7.48

PUNCH 53, UPCT

DELT1 = DELT1 + 0.5

IF (DELT1 = 8.5) 12, 12, 15

STOP

END

DATA

(Continued)

9650E 00 9580E 00 5371E 03 5150E 03 5100E 03 5050E 03
2500E 03 2500E-03 3760E-04 2397E-04 6000E-02
9289E 03 9475E 00 9300E 00 8500E 02 2982E 02
6000E 01 2060E 03 1530E 03 9400E 02
7000E 01 2040E 03 1520E 03 9200E 02
8000E 01 2020E 03 1506E 03 9520E 02
9000E 01 2040E 03 1520E 03 9160E 02
1000E 02 1980E 03 1450E 03 9000E 02
1200E 02 1960E 03 1460E 03 9200E 02
1300E 02 2040E 03 1440E 03 9250E 02
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<th>( r_2 )</th>
<th>( r_3 )</th>
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</tr>
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<td>( ^\circ F )</td>
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<td>( lb/lb )</td>
<td>( lb/lb )</td>
<td>$/10^3 gallon</td>
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<td>2.523</td>
<td>2.613</td>
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<td>152.0</td>
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TABLE 13

COMPARISON OF RESULTS

THE MACROSTAGE MODEL

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<th>$x_2$</th>
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<td>lb/lb</td>
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<tr>
<td>8</td>
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THE MICROSTAGE MODEL

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<th>$(T_f)_{46}$</th>
<th>$(T_f)_{68}$</th>
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<th>$r_2$</th>
<th>$r_3$</th>
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<tbody>
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<td>$^oF$</td>
<td>$^oF$</td>
<td>lb/lb</td>
<td>lb/lb</td>
<td>lb/lb</td>
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make-up feed according to the macrostage model are lower than those by the microstage model. The macrostage model also gave lower unit production costs.

As we mentioned in Section 1, the cost figures calculated are only the sum of the in-plant operation and capital charge costs. Additional cost items listed in Table 1 should be incorporated to make up the unit production cost, and thus the unit production cost becomes $0.42 \sim 0.43$ per 1000 gallons. According to Ref. (3) the best unit production cost reported for a plant of 50 MM gallon per day with a single effect multistage model is $0.449$ per 1000 gallons. The cost difference might be due to the thermodynamic advantage of the multieffect multistage model; however, we did not calculate the cost due to friction losses in the condenser tubes and channels. These are known to be of minor importance in the evaluation of cost functions.

The simulation results by varying $\Delta t_1$ in the range of $7.0 \sim 8.0^\circ F$ with fixed values for $x_1 = 204^\circ F$, $x_2 = 152^\circ F$, $x_3 = 92^\circ F$ according to the macrostage model are shown in Table 14.
TABLE 14
CALCULATION RESULTS BY THE MACROSTAGE MODEL

\[ x_1 = 204^\circ F \quad x_2 = 152^\circ F \quad x_3 = 92^\circ F \]

<table>
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<tr>
<th>( \Delta t_1 )</th>
<th>( r_1 )</th>
<th>( f_1(x_0) )</th>
<th>( r_2 )</th>
<th>( \Delta t_2 )</th>
<th>( r_3 )</th>
<th>( \Delta t_3 )</th>
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<th>Unit Cost</th>
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<td>lb/1b</td>
<td>( \circ F )</td>
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APPENDIX II

OPTIMIZATION OF THE MULTI-EFFECT, MULTI-STAGE EVAPORATION PROCESS BY THE MAXIMUM PRINCIPLE

We shall employ the design equations obtained by the macro-stage model in Appendix I to carry out the optimization of the multiflash evaporation process by means of a discrete form of the maximum principle [1]. The reason we use this model is that it is simpler in its form than the micro-stage model. Since we assumed equal temperature drops for the flashing brine in each stage and also that the temperature differences between the flashing brine and the recycle brine are equal all the way through the heat recovery section of each effect, we can treat the process as a stage-wise one-dimensional decision process and apply the discrete maximum principle, to optimize it. In the formulation of the discrete maximum principle we choose the temperatures of the flashing brine at the exit of each effect as state variables and the ratios of the recycle brine in each effect to the make up feed as decision variables. The optimization problem is defined so that the first stage is the combination of the brine heater and the heat recovery section of the first effect, the second stage is just the heat recovery section of the second effect, and the third stage is the third effect itself. The reason why the heat rejection sections of the first two effects are excluded is that they involve two different $\Delta t$'s of the adjacent effects. This leads to a "process with decisions in memory" [1] in evaluating the cost equations. However, by excluding these two sections from our optimization study, the problem is reduced to the standard type without memory.
1. **STATEMENT OF THE PROBLEM**

It should be noted that equal distillate production is assumed for each effect in this formulation. Then it follows that the blowdown concentrations are

\[
(C_f)_{1b} = 0.042, \\
(C_f)_{2b} = 0.0525, \\
(C_f)_{3b} = 0.07.
\]

The heat capacities of the flashing brine corresponding to these concentrations are

\[
(C_p)_{f1} = 0.958, \\
(C_p)_{f2} = 0.9475, \\
(C_p)_{f3} = 0.93.
\]

An energy balance around the brine heater and first effect (see Fig. 1) gives

\[
q + F(C_p)_{f1} tr_2 + Fr_2(C_p)_{r2} tr_2 = \frac{F}{6}(x_1 - \Delta t_{BPE}) + \frac{5}{6} r_2 F(C_p)_{p2} av x_1
\]

where \( x_1 = t_f \)

\[
q + F(1+r_2)(C_p)_{p2} av tr_2 = F(1+r_2)(C_p)_{p2} av x_1
\]

Solving for \( \Delta t_2 = x_1 - tr_2 \), we obtain

\[
\Delta t_2 = x_1 - tr_2 = \frac{q}{F(1+r_2)(C_p)_{p2} av}
\]

In the same manner it can be shown that

\[
\Delta t_3 = \frac{q}{F(1+r_3)(C_p)_{p3} av}
\]
Fig. 1. A block diagram for the first effect.
and that

$$\Delta t_1 = \frac{q}{F(1+r_1)(C_p)_{av}}$$  \hspace{1cm} (4)$$

Since, the magnitudes of the recycle ratios are known from the previous simulation, we can assign approximate values to the average heat capacities without knowing the exact values of the recycle ratio

$$(C_p)_{av} = 0.96,$$

$$(C_p_2)_{av} = 0.952,$$

$$(C_p_3)_{av} = 0.94.$$  

Note that equations (2), (3), and (4) all contain the factor $\frac{q}{F}$, and for this reason we will choose $\frac{q}{F}$ as a parameter in our optimization study.

2. OPTIMAL SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

According to the algorithm of the discrete maximum principle [1], we define the temperature of the flashing brine as the state variable $x_1$ with a transformation equation at each stage of the form

$$x_1^n = T^n (x_1^{n-1}; \theta^n)^*$$

where $\theta^n$ is the recycle ratio which is the decision to be made at each stage.

Using the notation introduced here, equation (54) in Appendix I becomes

$$\theta^1 = \frac{1065 + 0.196(x_1^0 + x_1^1) + 5 \times 0.958x_1^1 - 6 \times 0.955x_1^0}{6 \times 0.958 (x_1^0 - x_1^1)}$$  \hspace{1cm} (5)$$

*All superscripts stand for the stage number.*
Solving this equation for $x_1^1$ gives

$$x_1^1 = T^1(x_1^0, \theta^1) = \frac{(5.748 \theta^1 + 5.594)x_1^0 - 1065}{5.748 \theta^1 + 4.986}$$

(6)

Similarly, equation (60) becomes

$$\theta^2 = \frac{1065 + 0.196(x_1^1 + x_1^2) + 4 \times 0.9475x_1^2 - 5 \times 0.958x_1^1}{6 \times 0.9475(x_1^1 - x_1^2)}$$

(7)

which may be rewritten as

$$x_1^2 = T^2(x_1^1; \theta^2) = \frac{(5.685 \theta^2 + 4.594)x_1^1 - 1065}{5.685 \theta^2 + 3.986}$$

(8)

From equation (66), we can obtain

$$\theta^3 = \frac{1065 + 0.196(x_1^2 + x_1^3) + 3 \times 0.93x_1^3 - 4 \times 0.9475x_1^2}{6 \times 0.93(x_1^2 - x_1^3)}$$

(9)

by introducing the new notation. Solving equation (9) for $x_1^3$ gives

$$x_1^3 = T^3(x_1^2; \theta^3) = \frac{(5.58 \theta^3 + 3.594)x_1^2 - 1065}{5.58 \theta^3 + 2.986}$$

(10)

We shall introduce a new state variable $x_2$ for the accumulated unit production cost. We shall show that the accumulated unit production cost can be written in the form

$$x_2^n = x_2^{n-1} + G^n(x_1^{n-1}; \theta^n), x_2^0 = 0 \quad n = 1, 2, 3.$$  

(11)
It is obvious that \[ \sum_{n=1}^{3} G^n(x_{1}^{n-1}; \theta^n) = x_2^3. \] Thus the problem is transformed into the standard form in which a sequence of \( \theta^n \), \( n = 1, 2, 3 \) is to be chosen so as to minimize the objective function \( x_2^3 \) for a process described by equations (6), (8), and (10).

The terms which comprise the cost functions, \( G^n(x_1^{n-1}; \theta^n), n = 1, 2, 3, \) have been derived in Appendix I. In this optimization study, \( G^1(x_1^0; \theta^1) \) is comprised of the steam cost, the fixed charge cost for the brine heater, and the fixed charge cost for the condenser tubes of the heat recovery section of the first effect. The steam cost is obtained from equation (55) of Appendix I as

\[
(x_4)_s = \frac{2C_q \rho}{F \lambda_s} \quad \text{(12)}
\]

The fixed charge cost for the brine heater is obtained by rewriting equation (56) as

\[
(x_4)_B = \frac{1.92C_2(1+\theta^1)}{U_B} \left\{ \ln(24.4 + \frac{q}{0.96F(1+\theta^1)}) - \ln(24.4) \right\} \quad \text{(13)}
\]

The fixed charge cost for the condenser tubes of the heat recovery section of the first effect is obtained by substituting equation (6) into equation (57). This gives

\[
(x_4)_{c1} = \frac{80C_3(1+\theta^1)0.96\left\{ x_1^0 - \frac{(5.748\theta^1 + 5.594)x_1^0 - 1065}{5.748\theta^1 + 4.986} - \frac{q}{F(1+\theta^1)0.96} \right\}}{41q - x_1^0 + \frac{(5.748\theta^1 + 5.594)x_1^0 - 1065}{5.748\theta^1 + 4.986} - 80}
\]

\[
(U_1)_{av} = \frac{41q}{F(1+\theta^1)0.96} - x_1^0 + \frac{(5.748\theta^1 + 5.594)x_1^0 - 1065}{5.748\theta^1 + 4.986} - 80
\]
We will not include the recycle cost in this optimization study because the total recycle cost itself is small compared to the total production cost. In addition to this, the pressure

\[
\Delta P_{\text{total}} = (P_0 - P_1) + (P_1 - P_2) + (P_2 - P_3)
\]

\[
= \Delta P_1 + \Delta P_2 + \Delta P_3
\]

is almost constant since the total temperature range of the flashing process can only be varied a few degrees Fahrenheit. Thus, an increase in one \( \Delta P \) will decrease the other two \( \Delta P \)s. It is also intuitively seen that the recycle cost for each effect itself cannot vary much.

As mentioned earlier, the production cost due to the fixed charge costs for the condenser tubes in the heat rejection sections of the first and the second effects are excluded from our consideration. Since there are only 6 of the 68 stages in these sections the optimal unit production cost will not be influenced very much in case they are not taken into account. Since they involve decisions in memory as will be shown below, they complicate the optimization problem.

Rewriting equation (62) in Appendix I for the fixed charge cost for the condenser tubes in the heat rejection section of the first effect we have
after substituting equations (2) and (4) into equation (62)

\[(x_4)_1 j = \left( \frac{12C_3 G}{F} \right) \left( U_1 \right)_{av} \left\{ \frac{3q}{F(1+\theta^2)0.952} + \frac{2q}{F(1+\theta^1)0.96} - 12 \right\} \]

\[(x_4)_1 j = \left( \frac{12C_3}{U_1} \right)_{av} \left\{ \frac{3}{0.952(1+\theta^2)} + \frac{2}{0.96(1+\theta^1)} - \frac{12F}{q} \right\} \]

A similar expression can be obtained for the heat rejection section of the second effect by substituting equations (2) and (3) into equation (68) of Appendix I. One obtains

\[(x_4)_2 j = \left( \frac{12C_3}{U_2} \right)_{av} \left\{ \frac{3}{0.94(1+\theta^3)} + \frac{2}{0.952(1+\theta^2)} - \frac{12F}{q} \right\} \]

Equations (15) and (16) each contain two decision variables and because of this they can't be included in cost functions of the form \( G_n(x_1^{n-1}; \theta^n) \). In order to include equations (15) and (16) a new state variable would have to be introduced (1). However, in this study equations (15) and (16) will be added in after the optimal solution has been found.

From equations (12), (13), and (14) we thus obtain

\[ G^1(x_1^0; \theta^1) = (x_4)_s + (x_4)_D + (x_4)_c_1 \]

In this optimization study, \( G^2(x_1^1; \theta^2) \) is the fixed charge cost for the condenser tubes in the heat recovery section of the second effect. Substituting equations (2) and (8) into equation (63) in Appendix I we obtain
\[ G^2(x_1; \theta^2) = \frac{80C_3(1+\theta^2)0.952 \left\{ x_1 - \frac{(5.685 \theta^2 + 4.594) x_1^1 - 1065}{5.685 \theta^2 + 3.986} \right\}}{(U_2)_{av} \left\{ \frac{q}{F(1+\theta^2)0.952} - x_1^1 + \frac{(5.685 \theta^2 + 4.594)x_1^1 - 1065}{5.685 \theta^2 + 3.986} \right\}} \]

which can be simplified to give

\[ G^2(x_1; \theta^2) = \frac{76.16C_3(1+\theta^2) \left\{ 0.952(1+\theta^2)(1065-0.608x_1^1)\frac{q(5.685\theta^2+3.986)}{F} \right\}}{(U_2)_{av} \left\{ \frac{41q(F(5.685\theta^2+3.986)-0.952(1+\theta^2)(1383.9-0.608x_1^1+454.8\theta^2)}{F} \right\}} \]

\[ G^3(x_1^2; \theta^3) \]

represents the fixed charge cost for the condenser tubes in both the heat recovery and heat rejection sections of the third effect. From equation (69) in Appendix I, the fixed charge cost for the condenser tubes in the heat recovery section of third effect is obtained by substituting equations (3) and (10) for \( x_1^3 \) and \( \Delta t_3 \)

\[ (x_4^c)_{c3} = \frac{80C_3(1+\theta^3)0.94 \left\{ x_1^2 - \frac{5.58 \theta^3 + 3.594}{} \right\}}{(U_3)_{av} \left\{ \frac{41q(F(5.685\theta^3+3.986)-0.94(1+\theta^3)(1383.9-0.608x_1^1+446.4\theta^3)}{F} \right\}} \]

On simplification, it becomes

\[ (x_4^c)_{c3} = \frac{75.2C_3(1+\theta^3) \left\{ 0.94(1+\theta^3)(1065-0.608x_1^2) - \frac{q(F(5.58\theta^3+2.986))}{F} \right\}}{(U_3)_{av} \left\{ \frac{41q(F(5.58\theta^3+2.986)-0.94(1+\theta^3)(1303.9-0.608x_1^1+446.4\theta^3))}{F} \right\}} \]
\[
(x_4)_{3j} = \frac{8C_3 (1 + \theta^3)^{0.94}}{(U_3)^{av} \left( F(1 + \theta^3)^{0.94} + \frac{\mathcal{F}(l + \theta^3) q}{5.58 \theta^3 + 2.986} \right)} \\
= \frac{8C_3}{(U_3)^{av} \left( \frac{1}{0.94(1 + \theta^3)} + \frac{2(15.8 \theta^3 + 3.594) x_1^2 - 1065}{q(5.58 \theta^3 + 2.986)} \right)}
\]

On further simplification, it becomes
\[
(\text{U}_3)^{av} \left[ (5.58 \theta^3 + 2.986) \frac{3F}{q} (5.58 \theta^3 + 3.594) x_1^2 - 1065 \right]^{0.94(1 + \theta^3)} - 0.94(1 + \theta^3) \frac{F}{q} (2T_F + 8)(5.58 \theta^3 - 2.986)
\]

Thus, we can write
\[
G^3(x_1^2; \theta^3) = (x_4)_{c3} + (x_4)_{3j}
\]

Now we observe that our cost equations have the standard form of the discrete maximum principle.

\[
x_0^1 = 0
\]
\[
x_1^1 = x_2^0 + G^1(x_1^0; \theta^1),
\]
\[
x_2^1 = x_2^1 + G^2(x_1^1; \theta^2),
\]
\[
x_2^2 = x_2^2 + G^3(x_1^2; \theta^3),
\]

and

\[
\operatorname{Min.} S = \operatorname{Min.} \sum_{n=1}^{3} G^n(x_1^{n-1}; \theta^n) = \operatorname{Min.} x_2^3
\]
A block diagram for the process is shown in Fig. 2.

The procedure for solving such an optimization problem by the discrete maximum principle is to introduce adjoint vectors $z^n$ and Hamiltonian functions $H^n$ such that

$$H^n = \sum_{i=1}^{2} z^n_i x^n_i \quad n = 1, 2, 3,$$

$$z^{n-1}_i = \frac{\partial H^n}{\partial x^n_{i-1}}, \quad n = 1, 2, 3; \quad i = 1, 2.$$

Then,

$$H^1 = z^1_1 x^1_1 + z^1_2 x^1_2 = z^1_1 x^1_1 + z^1_2 G^1(x^0_1; \theta^1)$$

$$H^2 = z^2_1 x^2_1 + z^2_2 x^2_2 = z^2_1 x^2_1 + z^2_2(x^1_1 + G^2(x^1_1; \theta^2))$$

$$H^3 = z^3_1 x^3_1 + z^3_2 x^3_2 = z^3_1 x^3_1 + z^3_2(x^2_1 + G^3(x^2_1; \theta^3))$$

It is obvious that

$$z^1_2 = \frac{\partial H^2}{\partial x^1_2} = z^2_2 = \frac{\partial H^3}{\partial x^2_2} = z^3_2 = 1,$$

$$z^3_1 = 0,$$

Hence, we can write
Fig. 2. A block diagram for the multiflash evaporation process.
\[ H^1 = z_1 \frac{x_1^2 + x_2}{x_1^2 + x_2} = z_1 \frac{(5.748 \theta^1 + 5.594)x_1^0}{5.748 \theta^1 + 4.986} + \frac{2C_1q}{F \lambda_s} \]
\[ + \frac{1.92C_2(1 + \theta^1)}{U_B} \left\{ \ln(24.4) + \frac{q}{0.96(1 + \theta^1)} - \ln(24.4) \right\} \]
\[ + \frac{76.8C_3(1+\theta^1)\left\{0.96(1+\theta^1)(1065-0.608x_1^0) - \frac{q}{F}(5.748\theta^1+4.986)\right\}}{(U_1)_{av}\left\{41q(5.748\theta^1+4.986) - 0.96(1+\theta^1)(1463.9-0.608x_1^0+459.8\theta^1)\right\}} \]  
\[ (21) \]

\[ H^2 = z_2 \frac{x_1^2 + x_2}{x_1^2 + x_2} = z_2 \frac{(5.685 \theta^2 + 4.594)x_1^0}{5.685 \theta^2 + 3.986} \]
\[ + \frac{x_2^2}{x_2^2 + 2} = z_2 \frac{(5.685 \theta^2 + 4.594)x_1^0}{5.685 \theta^2 + 3.986} \]
\[ + \frac{76.6C_3(1+\theta^2)\left\{0.952(1+\theta^2)(1065-0.608x_1^0) - \frac{q}{F}(5.685\theta^2+3.986)\right\}}{(U_2)_{av}\left\{41q(5.685\theta^2+3.986)-0.952(1+\theta^2)(1383.9-0.608x_1^0+454.8\theta^2)\right\}} \]  
\[ (22) \]

\[ H^3 = z_3 \frac{x_1^3 + x_2}{x_1^3 + x_2} = z_3 \frac{(5.58 \theta^3 + 3.594)x_1^0}{5.58 \theta^3 + 2.986} \]
\[ + \frac{x_2^3}{x_2^3 + 2} = z_3 \frac{(5.58 \theta^3 + 3.594)x_1^0}{5.58 \theta^3 + 2.986} \]
\[ + \frac{75.2C_3(1+\theta^3)\left\{0.94(1+\theta^3)(1065-0.608x_1^0) - \frac{q}{F}(5.58\theta^3+2.986)\right\}}{(U_3)_{av}\left\{41q(5.58\theta^3+2.986)-0.94(1+\theta^3)(1303.9-0.608x_1^0+446.4\theta^3)\right\}} \]
\[ + \frac{7.52C_3(1 + \theta^3)(5.58 \theta^3 + 2.986)}{(U_3)_{av}\left\{(5.58\theta^3+2.986) + \frac{2F}{q}\left\{(5.58\theta^3+3.594)x_1^0-1065\right\}0.94(1+\theta^3) - \frac{0.94 F}{q} (1 + \theta^3)2T_F + 8)(5.58 \theta^3 - 2.986)\right\}} \]  
\[ (23) \]
3. COMPUTATIONAL SCHEME

A computational procedure which might be used to obtain the optimal values for the process, if the decision variable is not constrained or the optimal value is in the interior of the admissible range of the decision variable, is as follows:

1. Assume a value for \( x_1^3 \)

2. Differentiate equation (23) to obtain \( \frac{\partial H^3}{\partial \theta^3} \). If the decision variable is not constrained, we must have

\[
\frac{\partial H^3}{\partial \theta^3} = 0 \tag{24}
\]

3. Determine \( x_1^2 \) and \( \theta^3 \) by solving equations (10) and (24) simultaneously. This might be done by assuming an initial value for \( \theta^3 \), finding \( x_1^2 \) from equation (10), evaluating \( \frac{\partial H^3}{\partial \theta^3} \), and then using the method of steepest descent in the form

\[
\theta^{\text{new}} = \theta^{\text{old}} - k \frac{\partial H^3}{\partial \theta^3} \theta^{\text{old}}
\]

to find a better value of \( \theta^3 \).

4. Evaluate \( z_1^2 = \frac{\partial H^3}{\partial x_1^2} \) using the obtained values of \( x_1^2 \) and \( \theta^3 \) and substitute it into equation (22).

5. Use the \( \frac{\partial H^2}{\partial \theta^2} = 0 \), relation

\[
\frac{\partial H^2}{\partial \theta^2} = 0 \tag{25}
\]

and equation (8) to determine \( x_1^1 \) and \( \theta^2 \) by solving these equations simultaneously as in step 3.

6. Evaluate \( z_1^1 = \frac{\partial H^2}{\partial x_1^1} \) for the obtained values of \( x_1^1 \) and \( \theta^2 \); and substitute
it into equation (21),

7. Use equation (6) and the relation \( \frac{\partial H_1}{\partial \theta^1} = 0 \) (26) to determine \( x_1^0 \) and \( \theta^1 \) by solving these equations simultaneously as in step (3),

8. Compare the computed value of \( x_1^0 \) with the given value of \( x_1^0 \). The above procedure is repeated until the calculated value of \( x_1^0 \) is approximately equal to the given value.

It is recalled that the plant we are considering has the capacity of 50 MM gallons of distillate per day so that every cost item should be based on this capacity.

In Ref. (2) and Ref. (3) the values \( q/F = 20.6 \) and 33.7, respectively were used. The system under investigation here is closer to the case given in Ref. (2). Based on these previous studies a reasonable range of the parameter, \( q/F \), and the decision variables to be used in the computation is

1) \( q/F = 22-28 \)

2) \( \theta^n = 2.0-2.6 \).

The range of the decision variables is taken intuitively from the simulation results and review of Ref. (2).

In principle, smaller values of \( \theta^n \) are desired.

Because

\[ \Delta t^n = \frac{q}{F(1 + \theta^n)(C_{pn av})} \]

If \( q/F \) is given, we can see that the smaller the values of \( \theta^n \), the larger the value of \( \Delta t^n \), and thus less condenser tube area is needed for each effect so that the production cost is reduced. But \( \theta^n \) can't be too
small because a certain amount of recycle brine is needed for each effect in order to increase the heat capacity of the flashing brine stream so that for a given production of distillate the total temperature drop of the flashing brine satisfies the conditions

3) \( x^0_1 = 250^\circ F \)

\( x^3_1 = 90\sim95^\circ F \)

The maximum brine temperature \( x^0_1 \) is fixed at about \( 250^\circ F \) from the point of view of scale-formation. The blowdown temperature cannot be too high or too low. In case it is too high, heat waste becomes appreciable and in case it is too low, the condenser tube area in the heat rejection section and the pumping cost for the coolant increase. In view of this the blowdown temperature \( x^3_1 \) should be taken as \( 90\sim95^\circ F \) in this study.

It is also noted that the computational procedure stated earlier is only valid for the case in which the optimal decision is interior to the set of admissible decisions. If the optimal decision lies on the boundary of the set the necessary condition for \( x^3_2 \) to be a minimum is

\[ H^n = \text{minimum} \]

where \( n \) denotes only those stages where the optimal decision lies on the boundary of the set.
REFERENCES


APPLICATIONS OF DYNAMIC PROGRAMMING TO PROCESS ENGINEERING

by

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B. S., National Taiwan University, 1955

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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1967
This work shows how the method of dynamic programming can be used to optimize chemical processing systems by treating several examples.

The principle of optimality and the functional equation which constitutes the basic algorithm of the method of dynamic programming are presented. The computational scheme for the functional equation and the table entry procedure are given in detail followed by some simple examples. The optimal allocation of back-mix reactor volumes for a two phase, second order reaction system is considered for both countercurrent and co-current flow. The results show that the countercurrent flow scheme is more economical than the co-current flow scheme especially at higher conversions.

The optimization of a multieffect, multistage evaporator system with a nominal capacity of 20 MM gallons per day of distillate is considered. Using the cost estimating procedures recommended by the Office of Saline Water, an optimal unit production cost of 40¢ per 1000 gallons was obtained. This particular study serves to illustrate the applicability of dynamic programming to the optimization of a multistage decision process whose first appearance seems to be complex.

The refinement of solutions and the reduction in dimensionality are discussed. Since the discretization errors are inherent with the method of dynamic programming, one or another method of interpolation or extrapolation should be used to refine the solutions. Furthermore, the methods that can be used to obtain a reduction in dimensionality are described fully because of the exponential increase in computer memory that occurs with an increase in the number of state variables. Often the successful application of the dynamic programming technique is obstructed by the dimensionality problem. The k-th best policy is also given to illustrate how to find suboptimal
policies which are the best alternatives to the optimal policy. The application of the k-th best method is illustrated by several simple examples and the optimization of a multistage decision process with parallel redundancy. The optimization procedure for treating complex multistage decision processes is also given and some examples are treated in order to illustrate the technique for optimizing nonsequential, geometrically complex processes.

The transformation and cost equations for both a microstage and a macro-stage model of a multieffect, multistage flash evaporation system are developed and compared. Due to the simplicity of the macrostage model, the design equations for the macrostage model are used for the dynamic programming optimization. Results of simulation studies which were made using both the microstage and the macrostage models are presented.