DERIVATION OF EXPECTED VALUES OF PERFORMANCE AND ACTIVITY FOR A MULTI-ITEM INVENTORY SYSTEM

by

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INTRODUCTION

This thesis is based on work done in the summer of 1965 at the Inventory Research Office at Frankford Arsenal in Philadelphia, Pennsylvania. Its purpose was to apply theoretical work done by the Ordnance Inventory Management Project in the late 1950's to a system that satisfied the assumptions of this earlier work. That system was the U. S. Army Logistical Center, Japan (USALCJ). The Commander-in-Chief, Pacific, had asked the Directorate of Inventory Control at USALCJ to review its techniques and to search for a better policy. As a result of a visit to USALCJ by Mr. Bernard B. Rosenman, Chief, of the Inventory Research Office (IRO), an informal study of USALCJ was begun by IRO but other high priority research projects impeded the research until June, 1965.

Except for the cost of a procurement action (CP) and the cost of holding a dollar's worth of a unit per year (CH), all the parameters that described the items of the system were estimated or determined. The Ordnance Inventory Management Project's research enabled one to specify an optimal policy that would minimize the total economic cost of a system with specified availability. The cost factors CP and CH must be known to do this. Management at USALCJ was primarily interested in finding what alternatives were possible in terms of changed investment, backorders, availability, and procurement activity through application of optimal policies. Management was secondarily interested in minimizing the total economic cost.

The purpose of this research was to forecast what would happen to the system under various optimal policies. An optimal policy was formulated when the availability and the CP/CH ratio were specified. Over eighty combinations of these policy specifying parameters were investigated.

An optimal policy was found that would reduce the investment in stock on hand yet maintain the levels of backorders and procurement. A policy was found that would reduce the dollar value of backorders while keeping investment and procurement at pre-implementation levels. Rather than take advantage of the maximum possible reduction in investment, management might have chosen to trade some of the possible reduction in investment for a partial reduction in the dollar value of backorders while holding procurement at the pre-implementation level of 14,240 per year.

Many other possibilities existed. For example, the capacity of the procurement group could have been decreased while holding investment and backorders at their old levels. Of course, and most likely, a combination of adjustments was possible.

A computer program was devised that would forecast the response of the system under an optimal policy determined by any selected availability and CP/CH ratio. A point of interest to the Inventory Research Office was to find how well the theoretical estimators for performance and activity under the (R,Q) policy in use at USALCJ would come to the figures actually observed in the system. Another purpose was to rederive the model and to present the original research.

The approach involved deriving the expected values of performance and activity for each item. Totals for the whole system were obtained by getting actual figures for a set of representative items and multiplying each figure for each representative item by the number of items in the system that were similar to it.

The notation used occasionally departs from the style commonly used by statisticians. All the acronyms used are at least similar to those used in the Army's research in logistics. For quick reference, a glossary of symbols has been placed at the end of the thesis.

EXPECTED VALUES FOR AN ITEM AND A SYSTEM

Managers of an inventory system are given resources--manpower, facilities, equipment, and money--to satisfy the needs of its customers. Management is responsible for the best possible employment of these resources. If the system is to be changed or when the demand of the customers changes, management must estimate the change in resource consumption that will result. Most aspects of these problems have been discussed by Churchman, Ackoff, and Arnoff (2). The primary purpose of this study was to help management at the U. S. Army Logistical Center, Japan, (USALCJ) find a better policy of the same type as that already in use. Estimators for performance and activity had to be derived to predict how the system would react under a proposed policy.

Policy is made when management chooses how it will invest its funds by setting a reorder point (R_j) and a replenishment quantity (Q_j) for each item. Operationally, a procurement group is made responsible for ordering a quantity of Q_j units of the jth item whenever its assets are less than or equal to R_j. Assets are the number of units in stock plus those on order minus those back-ordered. Any system governed by a policy like this will be referred to as being under an (R,Q) policy.

Inventory

Each item will have a long run average number of units in stock. Let the probability density of i_j units on hand of item j be $f_1(i_j)$. The expected level of inventory for the jth item is

$$E(I_j) = \sum_{i_j=1}^{\infty} i_j \cdot f_1(i_j) \tag{1}$$

The expected dollar value of investment in the item would be

$$DVI_j = UP_j \cdot E(I_j) \tag{2}$$

Since many items have about the same values for the parameters that describe them--such as unit price or average yearly demand--and are usually placed under the same (R,Q) policy, it is convenient if the various expected values are calculated for one representative item from each group of similar items. Let there be k classes and let the jth item represent the jth class. If there are n_j items in the jth class, the total, or aggregate, dollar value of stock on hand for the whole system would be expected to be

$$ADVI = \sum_{j=1}^{k} n_j \cdot DVI_j \tag{3}$$

ADVI is the expected average dollar value of stock on hand.

Requisitioning Objective

The maximum number of units that can be on hand at any time is the reorder point plus the replenishment quantity. The maximum investment in stock on hand, which has been called the requisitioning objective, is

$$RO_j = UP_j \cdot (R_j + Q_j) \tag{4}$$

The maximum operating capital that might be needed by the system under a proposed (R,Q) policy is called the aggregate requisitioning objective.

$$ARO = \sum_{j=1}^{k} n_j \cdot RO_j \tag{5}$$

Measures of Performance and Activity

Management needed to know how well the system would satisfy demand and how much work would be required under any proposed (R,Q) policy. Two indices of performance were estimated--the dollar value of backorders and the availability. One measure of work was derived--the number of procurement actions per year.

Backorders

One index of how poorly the system would perform was unsatisfied demand, the number of units backordered because requisitions were received for an item that was out of stock. Let the probability density of the number of units backordered be $f_2(b_j)$. The expected value of backorders is

$$E(B_j) = \sum_{b_j=1}^{\infty} b_j \cdot f_2(b_j) \qquad (6)$$

for the jth item. Backorders are considered a positive quantity even though they occur when there is "negative" stock, i.e. when the net stock level is below zero. The dollar value of backorders is

$$DVB_j = UP_j \cdot E(B_j) \qquad (7)$$

The aggregate expected dollar value of backorders over the whole catalog is

$$ADVB = \sum_{j=1}^{k} n_j \cdot DVB_j \qquad (8)$$

The loss due of unavailability of a unit is assumed to be directly proportional to its unit price. Therefore, ADVB is considered to be a better index of performance than the total number of units backordered. One disadvantage of the aggregate dollar value of backorders as an index of performance is that it does not take into account the demand placed on the system. That is, if demand increased markedly, then AIWB would probably increase regardless of how well the system coped with the new burden.

Availability

Availability is a measure of performance that relates the system's response to demand placed on it to the order of magnitude of that demand. Availability may be defined as the number of orders filled when first presented relative to the total number of orders received. Thus,

$$\text{Availability} = \frac{\text{Number of orders satisfied immediately}}{\text{Number of orders received}} \qquad (9)$$

Availability has the advantage of being easy to calculate for a given period. If as a matter or clerical procedure the number of units sent immediately to fill requisitions and the number of units requisitioned were accumulated, the resulting ratio would be the availability experienced during the period. A requisition for X units is considered to be X orders. This definition is necessary for two reasons. First, a requisition for 10,000 units that is filled immediately should be given more weight then a requisition for 10 units that is filled immediately. Second, it clarifies how to account for partially filled requisitions.

In terms of the net stock level the availability can be interpreted as the probability that net stock is above or at zero units. When availability is defined in this way it will be symbolized by α_j. Net stock consists of the stock on hand, I_j, minus the quantity backordered, B_j, so

$$N_j = \begin{cases} I_j & \text{for } I_j \geq 0 \\ -B_j & \text{for } B_j > 0 \end{cases} \qquad (10)$$

or

$$N_j = I_j - B_j \tag{11}$$

The expected availability of the jth item was found by summing the probability density function of N_j from zero to infinity.

Availability for the system was defined as the weighted average of the availabilities of the k representative items. An acronym, AVAIL, was used for this index of systemwide performance.

$$\text{AVAIL} = \frac{\sum_{j=1}^{k} n_j \cdot \alpha_j}{\sum_{j=1}^{k} n_j} \tag{12}$$

The systemwide availability might not be too meaningful if there were much variation among individual item availabilities. An optimal policy was required to provide nearly the same availability for all items.

Adjusting the Replenishment Quantity

If performance is reasonably uniform for most items, that is, if the availability is about the same and if the dollar value of backorders is proportionate to the dollar value of annual demand, then overall measures of availability and backorders are good for evaluating how well the whole system is functioning. If, however, some items are causing more than their share of the aggregate dollar value of backorders by being understocked while others are continually in oversupply, then management normally adjusts the levels of assets at which replenishment orders are made. The reorder point of an understocked item is raised while that of an overstocked item is lowered. This eliminates unnecessary investment and shifts this money into item accounts that need a higher average stock on hand to reduce the average backorders.

For an item with steady demand the reorder point may be set so the stock is often about exhausted when the replenishment quantity arrives. An item with more variable demand may be given a higher reorder point and a smaller replenishment quantity so the system will respond to fluctuations in demand more readily. When the replenishment quantity is altered it changes the number of procurement actions that must be made to satisfy demand. If Q_j is decreased, the number of procurement actions per year must increase because demand continues unchanged and it takes more orders to purchase the same number of units as before the change. The opposite holds when Q_j is increased because it takes fewer procurement actions per year to get enough units to satisfy the annual demand. The expected number of procurement actions per year times the replenishment quantity should equal the average yearly demand so

$$E(NPY_j) = \frac{AYD_j}{Q_j} \tag{14}$$

and the total number of procurement actions expected per year would be

$$TNPY = \sum_{j=1}^{k} n_j \cdot E(NPY_j) \tag{15}$$

TNPY estimates the work required from the procurement group for a given (R,Q) policy.

Demand During the Procurement Lead Time

There is an interval of time between the placing of an order for Q_j units and the arrival of those units that is called the procurement lead time. If the demand during the procurement lead time, L_j, is a random variable Y_j, then

$$E(Y_j) = L_j \cdot AYD_j \tag{16}$$

where L_j is expressed in yearly units.

The various expected values above hold for any system under an (R,Q) policy. To find the expected values of investment, backorders, and availability, the probability density function of net stock for an item in a system like that of USALCJ will be derived in the next section.

DERIVATION OF EXPECTED VALUES OF PERFORMANCE AND ACTIVITY FOR AN ITEM

The expected values of performance and activity for an item in a system satisfying the following assumptions were derived as a part of the Ordnance Inventory Management Project. These results were presented by Christensen, Rosenman, and Calliher (1). The original derivation required that the random variable for the size of a requisition have a geometric density. It was implicit in the earlier research that the requisition size needed only a moment generating function for the same results to follow. This relaxation is explicitly stated here.

Assumptions

1. The number of requisitions received during a time period has a stationary Poisson distribution.

2. The size of a requisition is a random variable S whose distribution has a moment generating function.

3. The size of a requisition is statistically independent of the size of any other requisition.

4. All the moments of S are finite.

5. The procurement lead time is a fixed, known interval.

6. An item account is set in operation with assets at some level between the reorder point and the reorder point plus the replenishment quantity. It is equally likely that any of the points between R and R plus Q is selected. Assets are rectangularly distributed between R and R plus Q.

7. Assets are independent of demand.

Moment Generating Function of Demand

The number of requisitions received during the procurement lead time is a random variable X having a stationary Poisson density. If λ_j is the average number of requisitions received per unit time, the probability density function of X is

$$g_1(x_j;L) = \begin{cases} \dfrac{(\lambda_j L_j)^x e^{-\lambda_j L_j}}{x!} & X_j = 0, 1, 2, 3, \ldots \\ 0 & \text{Otherwise} \end{cases} \qquad (17)$$

The size of a requisition is a positive valued random variable S_i with probability density function $g_2(s_i)$ that has a moment generating function $m_s(t)$. The demand during the procurement lead time is a random variable Y that has a compound Poisson distribution (7). Let the probability density of demand be

$$h_1(y;L) = \begin{cases} 0 & \text{for } y < 0 \\ g_1(0;L) & \text{for } y = 0 \\ \sum_{x=1}^{\infty} g_2(s_1+s_2+s_3+\ldots+s_x = y \mid x) \cdot g_1(x;L) & \text{for } y > 0 \end{cases} \qquad (18)$$

The moment generating function of demand is

$$m_Y(t) = \sum e^{ty} \cdot h_1(y;L) \qquad (19)$$

$$= h_1(0;L) + \sum \sum \ldots \sum \sum_{x=1}^{\infty} \exp(t(s_1+s_2+\ldots+s_x))$$

$$\cdot g_2(s_1) \cdot g_2(s_2) \cdot \ldots \cdot g_2(s_x) \cdot g_1(x;L) \quad ds_1\, ds_2 \ldots ds_x$$

Since the S_i are positive, independent, and identically distributed,

$$m_Y(t) = \exp(-\lambda L) + \sum_{x=1}^{\infty} g_1(x;L) \quad \left(\sum_{s=0}^{\infty} \exp(ts) \cdot g_2(s)\right)^x \qquad (20)$$

$$= \exp(-\lambda L) + \sum_{x=1}^{\infty} \exp(-\lambda L) \cdot \frac{(\lambda L \cdot m_s(t))^x}{x!}$$

$$= \exp(-\lambda L) \sum_{x=0}^{\infty} \frac{(\lambda L\, m_s(t))^x}{x!}$$

The summation is the series expansion of $\exp(\lambda L\, m_s(t))$

$$m_Y(t) = \exp(-\lambda L) \cdot \exp(\lambda L \cdot m_s(t))$$

$$m_Y(t) = \exp(\lambda L(m_s(t) - 1)) \qquad (22)$$

Variance to Mean Ratio

The ratio of the variance of Y to the mean of Y can be found from

$$E(Y) = \lambda \cdot L \cdot E(S) \qquad (23)$$

$$E(Y^2) = (\lambda \cdot L \cdot E(S))^2 + \lambda \cdot L \cdot E(S^2) \qquad (24)$$

The variance of Y is

$$\sigma^2 = \lambda \cdot L \cdot E(S^2) \qquad (25)$$

so the variance to mean ratio is

$$VMR = E(S^2)/E(S) \qquad (26)$$

The VMR of an item was easily estimated and the standard deviation of Y was found from

$$\sigma = \sqrt{VMR\ L\ AYD} \qquad (27)$$

Distribution of Demand

Let Z be a random variable defined by the linear transformation

$$Z = \frac{Y - E(Y)}{\sigma} \tag{28}$$

The moment generating function of Z is

$$m_z(t) = \exp(-tE(Y)/\sigma) \cdot m_Y(t/\sigma) \tag{29}$$

From equation (22), repeated here with t/σ substituted for t and $\lambda LE(S)$ for $E(Y)$,

$$m_z(t) = \exp\left\{ -\lambda L \left[(tE(S)/\sigma) - m_s(t/\sigma) + 1 \right] \right\} \tag{30}$$

Now

$$m_s(t/\sigma) = E(\exp(tS/\sigma)) \tag{31}$$

which is equivalent to

$$m_s(t/\sigma) = E\left[\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{tS}{\sigma} \right)^k \right] \tag{32}$$

Substituting the right hand side of equation (32) for $m_s(t/\sigma)$ in $m_z(t)$ gives

$$m_z(t) = \exp\left[\lambda L \sum_{k=2}^{\infty} \frac{1}{k!} E\left(\frac{tS}{\sigma} \right)^k \right] \tag{33}$$

$$= \exp\left[\lambda L \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{t}{\sigma} \right)^k E(S^k) \right] \tag{34}$$

Recalling equation (25)

$$= \exp\left[\tfrac{1}{2}t^2 + \lambda L \sum_{k=3}^{\infty} \frac{t^k}{k!} \frac{E(S^k)}{\sigma^k} \right]$$

$$= \exp\left[\tfrac{1}{2}t^2 + \lambda L \sum_{k=3}^{\infty} \frac{t^k}{k!} \frac{E(S^k)}{\lambda L\, E(S^2)\, \sigma^{k-2}} \right]$$

$$\exp\left[\tfrac{1}{2}t^2 + \sum_{k=3}^{\infty} \frac{t^k}{k!} \frac{E(S^k)}{E(S^2)} (\sigma^2)^{-\frac{k-2}{2}}\right]$$

$$\exp\left(\tfrac{1}{2}t^2\right) \cdot \exp\left[\sum_{k=3}^{\infty} \frac{t^k}{k!} \frac{E(S^k)}{E(S^2)} (\lambda L E(S^2))^{-\frac{k-2}{2}}\right]$$

$$m_z(t) = \exp(\tfrac{1}{2}t^2) \cdot \exp\left[\sum_{k=3}^{\infty} \frac{t^k}{k!} \frac{E(S^k)}{(E(S^2))^{k/2}} (\lambda L)^{-\frac{k-2}{2}}\right] \quad (35)$$

As the procurement lead time increases without bound, each term in the series goes to zero so

$$\lim_{L \to \infty} m_z(t) = \exp(\tfrac{1}{2}t^2) \quad (36)$$

This is the moment generating function of the standard normal distribution so Z is approximately a standard normal variate when the procurement lead time is long. Demand must be nearly normally distributed with mean E(Y) and variance σ^2.

Probability Density of Net Stock

Since all stock on order at time t-L is received by time t, the net stock at time t is

$$N_t = A_{t-L} - Y \quad (37)$$

The density of net stock is found from the joint density of assets and demand, $h_2(a_{t-L}, y)$. By assumption 6, assets have the rectangular density

$$f_3(a) = \begin{cases} 1/Q & \text{for } R < a \le R+Q \\ 0 & \text{elsewhere} \end{cases} \quad (38)$$

Assumption 7 states that assets are independent of demand.

$$h_2(a_{t-L},y) = f_3(a_{t-L}) \cdot h_1(y;L)$$

For each possible level of assets and time t-L, the demand during L must be exactly a_{t-L}-ns units for net stock (ns) to "arrive" at ns at time t. The density of ns is given by the convolution (7)

$$h_3(ns) = \sum_{a=R}^{R+Q} h_2(a_{t-L}, a_{t-L}-ns) \tag{39}$$

$$h_3(ns) = (1/Q) \sum_{a=R}^{R+Q} h_1(y=a_{t-L}-ns) \tag{40}$$

The density of ns for values less than or equal to R is found by making the substitution $y = a - ns$ (and dropping the subscript on a)

$$h_3(ns) = (1/Q) \sum_{y=R-NS}^{R+Q-NS} h_1(y) \tag{41}$$

$$h_3(ns) = (1/Q)\left[H_1(R\ Q-NS) - H_1(R-NS)\right] \tag{42}$$

for values of NS less than R.

For NS between R and R+Q, $h_1(y)$ in equation (40) is nonzero when the variable of integration takes on values such that a-ns is positive, i.e. for assets greater than or equal to net stock.

$$h_3(ns) = (1/Q) \sum_{a=ns}^{R+Q} h_1(y=a-ns) \tag{43}$$

Substituting $y = a - ns$ gives

$$h_3(ns) = (1/Q) \sum_{y=0}^{R+Q-NS} h_1(y) \tag{44}$$

so

$$h_3(ns) = (1/Q)\left[H_1(R\ Q-NS) - H_1(0)\right] \tag{45}$$

for values of NS between R and R+Q. Since the net stock cannot exceed the least upper bound for assets, R+Q, $h_3(ns)$ is zero for NS greater than R+Q. The probability density of net stock is

$$h_3(ns) = \begin{cases} (1/Q)\left[H_1(R+Q-ns) - H_1(R-ns)\right] & \text{for } ns<R \\ (1/Q)\left[H_1(R+Q-ns) - H_1(0)\right] & \text{for } R\le ns\le R+Q \\ 0 & \text{for } ns>R+Q \end{cases} \quad (46)$$

Availability

Availability has been defined (p. 6) as the probability that net stock is above or at zero units. In terms of the probability that an item will be out of stock, availability can be defined as

$$\alpha = 1 - H_3(0) \quad (47)$$

For a positive reorder point,

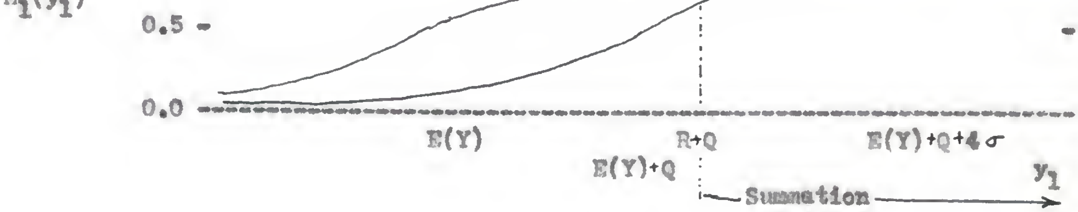
$$H_3(0) = (1/Q) \sum_{ns=-\infty}^{0} \left[H_1(R+Q-ns) - H_1(R-ns)\right]$$

Substitute y_1 for R+Q-ns and notice that y_1-Q equals R-ns.

$$H_3(0) = (1/Q) \sum_{R+Q}^{\infty} \left[H_1(y_1) - H_1(y_1-Q)\right] \quad (48)$$

The cross-hatched area in Fig. 1 is found by this summation.

Fig. 1: Geometric interpretation of equation (48)



If the maximum assets, R+Q, is near four standard deviations above the expected value of demand, E(Y), then $H_1(y)$ will be almost one over the interval of summation so

$$H_3(0) \leqq \sum_{R+Q}^{\infty} \left[1 - H_1(y_1-Q)\right] \tag{49}$$

Substitute y_2 for y_1-Q.

$$H_3(0) \leqq \sum_{R}^{\infty} \left[1 - H_1(y_2)\right] \tag{50}$$

When y_2 is above E(Y) plus four standard deviations the difference is practically zero so

$$H_3(0) \leqq \sum_{R}^{E(Y)+4\sigma} \left[1 - H_1(y_2)\right]$$

Dropping the subscript and using equation (47), the availability is

$$\alpha \cong 1 - \sum_{R}^{E(Y)+4\sigma} \left[1 \; H_1(y)\right] \tag{51}$$

Assets

The expected value of assets is

$$E(A) = R + \tfrac{1}{2}Q \tag{52}$$

Backorders

Backorders occur when net stock is below zero. It is customary to think of backorders as a positive variable so the following expected value was defined as backorders.

$$E(B) = \sum_{ns=0}^{\infty} ns \; h_3(-ns) \tag{53}$$

$$= \frac{1}{Q} \sum_{ns=0}^{\infty} ns\left[H_1(ns-R-Q) - H_1(ns-R)\right]$$

Using the same transformations on E(B) as were used to get the availability, the expected value of backorders was found to be

$$E(B) = (1/Q) \sum_{R}^{E(Y)+4\sigma} (y-R)\left[1 - H_1(y)\right] \qquad (54)$$

Stock on Hand--Inventory

Using the information in equations (11) and (37), the stock on hand was found to be related to the assets at time t-L, the demand during L, and backorders.

$$I = A_{t-L} - Y + B$$

so

$$E(I) = E(A) - E(Y) + E(B) \qquad (55)$$

The expected values in the right hand side of equation (55) are given in equations (52), (16), and (54).

A computer program was written which would use the equations derived in this section to calculate the expected values for representative items. These expected values were used in the equations of the previous section to get estimates of performance and activity for the whole system.

A DESCRIPTION OF THE U. S. ARMY LOGISTICAL CENTER, JAPAN

Representative Items

The range of unit prices was roughly divided into intervals on a logarithmic scale and the range of average yearly demands was also divided into similar intervals. Table 1 shows the frequencies of items in the various classifications.

Table 1: Items received by the U. S. Army Logistical Center, Japan, from the Mutual Security Directorate, U. S. Army Terminal Agency, Atlantic

Class Number j	Unit Price UP_j	Estimated Average Yearly Demand AYD_j	Number of Items n_j
1	Less than $ .10	Less than 1,000	2,730
2	"	1,001 - 10,000	545
3	"	10,001 - 100,000	69
4	"	100,001 - 1,000,000	7
5	"	Over 1,000,000	0
6	$ .11 - $1.00	Less than 1,000	7,443
7	"	1,001 - 10,000	452
8	"	10,001 - 100,000	76
9	"	Over 100,000 (200,000)*	4
10	$1.01 - $10.00	Less than 100	7,925
11	"	101-1,000	1,149
12	"	1,001 - 10,000	312
13	"	10,001 - 100,000	29
14	"	Over 100,000	0
15	$10.01 - $100.00	Less than 10	2,132
16	"	11 - 100	1,240
17	"	101 - 1,000	323
18	"	1,001 - 10,000	47
19	"	Over 10,000 (50,000)*	6
20	$100.01 - $1,000.00	Less than 10	294
21	"	11-100	142
22	"	101 - 1,000	26
23	"	1,001 or more (2,000)*	1
24	$1,000 - $6,000	Less than 10	28
25	$1,000 - $20,000	11 - 100	21
26	$1,000 - $15,000		2
		Total	25,003

* The upper limit used in subsequent computations is shown in parentheses.

These are part of a larger system of 46,893 items handled by USALCJ. All of the items in this study were received from the Mutual Security Directorate, U. S. Army Terminal Agency, Atlantic, Brooklyn, N. Y.

The number of items in Table 1 varies inversely with unit price and average yearly demand. For this reason the representative items have been given unit prices and average yearly demands closer to the origin than the arithmetic average of the class limits. The following logarithmic average was used.

$$UP_j = \exp(\tfrac{1}{2}(\log_e UPL_j + \log_e UPU_j)) \tag{56}$$

$$AYD_j = \exp(\tfrac{1}{2}(\log_e AYDL_j + \log_e AYDU_j)) \tag{57}$$

where UPL_j was the lower limit for unit price in the jth class. UPU_j, $AYDL_j$, and $AYDU_j$ were similarly defined. Table 2 shows the unit prices and average yearly demands of the representative items. Also shown is the dollar value of yearly demand for each representative item.

$$DVYD_j = UP_j \cdot AYD_j \tag{58}$$

The unit price, average yearly demand, and dollar value of yearly demand are not affected by the (R,Q) policy so they are called invariant parameters.

Table 2: Representative unit prices, average yearly demands, and dollar values of yearly demand for items found from the logarithmic average of class limits

Class Number j	Representative Unit Price UP_j	Representative Average Yearly Demand AYD_j	Dollar Value of Yearly Demand $DVYD_j$
1	.0316	31.6227	1.0000
2	.0316	3163.8580	100.0500
3	.0316	31624.3270	1000.0492
4	.0316	316229.0000	10000.0410
6	.3316	31.6227	10.4880
7	.3316	3163.8580	1049.3331
8	.3316	31624.3270	10488.6040
9	.3316	141421.9000	46904.3460
10	3.1780	9.9999	31.7804
11	3.1780	317.8048	1009.9993
12	3.1780	3163.8580	10054.8960
13	3.1780	31624.3270	100503.6600
15	31.6385	3.1622	100.0499
16	31.6385	33.1662	1049.3326
17	31.6385	317.8048	10054.8930
18	31.6385	3163.8580	100099.9700
19	31.6385	22361.7890	707495.2500
20	316.2434	3.1622	1000.0495
21	316.2434	33.1662	10488.6070
22	316.2434	317.8048	100503.7000
23	316.2434	1414.9202	447459.3000
24	2449.5006	3.1622	7745.9996
25	4472.1562	33.1662	148324.6000
26	3873.0010	317.8048	1230858.4000

Source: Based on Table 1 and equations (56), (57), and (58)

Invariant Parameters

Any parameter that describes a representative item which is independent of the (R,Q) policy is called an invariant parameter. The unit price, average yearly demand, and the dollar value of yearly demand are invariant parameters that have been presented. The procurement lead time, the expected value of demand during the procurement lead time, the variance to mean ratio, and the standard deviation of demand during the procurement lead time are presented here.

The length of time needed for the next higher depot to process a requisition and for the goods to be transported to USALCJ remained the same regardless of the (R,Q) policy. Hence, the procurement lead time was an invariant parameter. The expected value of demand during the procurement lead time is the portion of the average yearly demand that was expected to occur during L.

Table 3: Procurement lead times and expected values of demand for representative items

Class Number j	Procurement Lead Time (In years) L_j	Expected Value of Demand in the Procurement Lead Time (In units) $E(Y_j)$
1	.75	23.7170
2	.75	2372.8935
3	.75	23718.2450
4	.21	66408.0900
6	.75	23.7170
7	.75	2372.8935
8	.21	6641.1086
9	.21	29698.5990
10	.75	7.4999
11	.75	238.3536
12	.21	664.4101
13	.21	6641.1086
15	.75	2.3716
16	.75	24.8746
17	.21	66.7390
18	.21	664.4101
19	.21	4695.9756
20	.75	2.3716
21	.21	6.9649
22	.21	66.7390
23	.21	297.1332
24	.21	.6440
25	.21	6.9649
26	.21	66.7390

Source: Procurement lead times, letter from USALCJ to IRO dated 26 January 1965; Expected values of demand based on L given here, AYD from Table 2, and equation (16)

The ratio of the variance of demand to expected demand during the procurement lead time is an invariant parameter. The variance to mean ratio (VMR_j) depends on the distribution of the requisition size (p. 12). An earlier study of customers in the continental United States estimated VMR_j from the average yearly demand and the unit price (1). Since AYD_j and UP_j were used to specify representative items, a similar relationship was sought.

A scatter diagram of 656 items showed that a good fit would be obtained with a linear regression equation involving logarithms of the three variables.

$$\log_e VMR_j - \overline{\log_e VMR} = \beta_{12.3}(\log_e AYD_j - \overline{\log_e AYD}) + \beta_{13.2}(\log_e UP_j - \overline{\log_e UP}) \tag{59}$$

$$\log_e VMR_j - 4.50360 = 0.93888(\log_e AYD_j - 5.74001) - 0.00961(\log_e UP_j - 1.05984) \tag{60}$$

The additional reduction in the error sum of squares obtained by use of this equation instead of one involving only AYD_j is the following

$$(R^2_{1.23} - r^2_{12}) \cdot \sum_{j=1}^{656} (\log_e VMR_j - \log_e VMR)^2 \tag{61}$$

The following statistic was used to test for a significant reduction (3).

$$F(1,n-3) = \frac{R^2_{1.23} - r^2_{12}}{(1 - R^2_{1.23})/(n-3)} \tag{62}$$

For $R^2_{1.23} = .94208$, $r^2_{12} = .94204$, $n = 656$, the observed F is 0.451, which is not significant. The additional benefit gained from the unit price was not significant so an equation depending only on AYD_j was used to estimate VMR_j.

$$\log_e VMR_j - 4.50 = 0.945\ (\log_e AYD_j - 5.74) \tag{63}$$

or

$$VMR_j = \exp(-.921 + 0.945 \log_e AYD_j) \tag{64}$$

which equals

$$VMR_j = .398 \cdot AYD_j^{.945} \tag{65}$$

Table 4: Variance to mean ratios and standard deviations of demand during the procurement lead time for the representative items

Class Number j	Variance to Mean Ratio VMR_j	Standard Deviation of Demand σ_j
1	10.4146	15.7163
2	808.8164	1385.3640
3	7123.0297	12997.9100
4	62754.6490	64555.5200
6	10.4146	15.7163
7	808.8164	1385.3640
8	7123.0297	6877.8490
9	29334.7740	29516.1200
10	3.5086	5.1297
11	92.1905	148.2360
12	808.8164	733.0660
13	7123.0297	6877.8490
15	1.1820	1.6743
16	10.8943	16.4618
17	92.1905	78.4391
18	808.8164	733.0660
19	5133.6744	4909.9490
20	1.1820	1.6743
21	10.8943	8.7107
22	92.1905	78.4391
23	378.0825	355.1728
24	1.1820	0.8859
25	10.8943	8.7107
26	92.1905	78.4391

Source: VMR from regression equation (65) with AYD from Table 2; standard deviation from equation (27) with L from Table 3

The (R,Q) Policy in Use

The policy for each item at USALCJ was determined by the dollar value of annual demand of that item. When demand brought assets down to a certain number of months of supply, a procurement requisition for a certain number of months of supply was sent to the next higher depot. A month of supply is the number of units needed to satisfy demand during an average month. Demand during an average month was estimated from the demand during the previous calendar year (1964). Table 5 gives the schedule used to set policy at USALCJ.

Table 5: Reorder points and replenishment quantities in months of supply

Dollar Value of Yearly Demand	Reorder Point (Months)	Replenishment Quantity (Months)
\$ .01 - \$ 70.00	12	48
70.01 - 90.00	12	42
90.01 - 125.00	12	36
125.01 - 190.00	12	30
190.01 - 320.00	12	24
320.01 - 635.00	12	18
635.01 - 1,265.00	12	12
1,265.01 - 2,525.00	4.5	9
2,525.01 - 7,600.00	4.5	6
7,600.01 or more	4.5	3

Source: Letter from USALCJ to IRO dated 26 January 1965

The reorder point in months of supply and the reorder point in units are related by

$$R_j = M_j \frac{AYD_j}{12} \tag{66}$$

Similarly, the replenishment quantities in months of supply and in units are related by

$$Q_j = P_j \frac{AYD_j}{12} \tag{67}$$

Table 6: Reorder points and replenishment quantities for representative items

Class Number	Months of Supply Reorder Point	Months of Supply Replenishment Quantity	Units Reorder Point	Units Replenishment Quantity
j	M_j	P_j	R_j	Q_j
1	12	48	32	126.48
2	12	36	3,164	9,941.7
3	4.5	12	11,859	31,624.
4	4.5	3	118,586	79,057.
6	12	48	32	126.49
7	4.5	12	1,186	1,049.4
8	4.5	3	11,859	2,622.1
9	4.5	3	53,036	35,358.
10	12	48	10	40.
11	4.5	12	119	317.8
12	4.5	3	1,186	790.96
13	4.5	3	11,859	7,906.
15	12	3	3	9.49
16	4.5	12	12	33.16
17	4.5	3	119	79.45
18	4.5	3	1,186	790.96
19	4.5	3	8,386	5,590.5
20	12	12	3	3.16
21	4.5	3	12	8.29
22	4.5	3	119	79.45
23	4.5	3	531	353.73
24	4.5	3	2	.79
25	4.5	3	12	8.29
26	4.5	3	119	79.45

Source: The dollar value of yearly demand for each representative item, given in Table 2, was used to find the item's (M,P) policy in Table 5; the equivalent (R,Q) policy was determined with equations (66) and (67)

The reorder point has been thought of as the quantity such that there will be enough units in stock to satisfy demand during the procurement lead time plus a safety stock. The safety stock is a certain number (*a*) of standard deviations of demand.

$$R_j = E(Y_j) + \underline{a}_j \sigma_j \qquad (68)$$

Similarly, the replenishment quantity can be expressed as a certain number of standard deviations of demand.

$$Q_j = \underline{b}_j \sigma_j \qquad (69)$$

Since the expected value of demand and the standard deviation of demand are invariant parameters, an (R,Q) policy is really set by fixing the $\underline{a}_j$ and $\underline{b}_j$. Hence, an ($\underline{a}$,$\underline{b}$) policy for an item determines its (R,Q) policy.

Table 7: Coefficients $\underline{a}$ and $\underline{b}$ of the (R,Q) policies for the representative items

Class Number	$\underline{a}_j$	$\underline{b}_j$
1	.5270	8.0476
2	.5710	6.8514
3	-.9123	2.4330
4	.8082	1.2246
6	.5270	8.0483
7	-.8567	.7574
8	.7586	.3812
9	.7906	1.1979
10	.4873	7.7977
11	-.8051	2.1438
12	.7115	1.0789
13	.7586	1.1494
15	.3753	5.6680
16	-.7620	2.0143
17	.6662	1.0128
18	.7115	1.0789
19	.7515	1.1386
20	.3753	1.8873
21	.5780	.9517
22	.6662	1.0128
23	.6977	1.0553
24	1.5080	.8917
25	.5780	.9517
26	.6662	1.0128

Source: E(Y) from Table 3 and the standard deviation from Table 4 were used in equations (68) and (69) solved for $\underline{a}$ and $\underline{b}$, respectively

A computer program was written that would formulate optimal policies in terms of a and b coefficients for each representative item and then calculate the expected values of performance and activity for the whole system.

The model and the representative items were checked for validity by calculating the expected values for the system under the policy in use and comparing these figures with the observed indexes of performance and activity. The results are given in Table 8 when the arithmetic average of class limits was used to determine representative items. Information on these representative items is given in Appendix 1. These estimates were too far from the data so this set of representative items was rejected.

Table 8: Expected values and data on the U. S. Army Logistical Center, Japan; calculated for a set of representative items determined by arithmetic averages of class limits (Table 1) for average yearly demand and unit price

Performance Index	Expected Value[1]	Data[2]
Availability	0.9498	Unknown
Purchase Actions Per Year	20,209	13,100
Dollar Value of		
Stock on Hand (Inventory)	$38,260,000	$ 9,199,000
Backorders	1,556,000	1,604,000
Net Stock	36,710,000	7,595,000
Assets	61,270,000	22,180,000
Requisitioning Objective	80,137,000	23,570,000

Sources: (1) See Appendix 1 for supporting computations; (2) Letter and inclosures from USALCJ to IRO dated 26 January 1965

The representative items determined by logarithmic averages as presented in this section were then tried. Since the expected values came reasonably close to the data, the model and this set of representative items were accepted. Acceptance meant that the expected values calculated for any given (R,Q) policy would be considered to be a good prediction of how the system would actually operate if it were placed under that policy.

Table 9: Expected values and data on the U. S. Army Logistical Center, Japan; calculated for a set of representative items determined by logarithmic averages of class limits (Table 1) for average yearly demand and unit price

Performance Index	Expected Value[1]	Data[2]
Availability	0.9034	Unknown
Purchase Actions Per Year	14,240	13,100
Dollar Value of		
Stock on Hand (Inventory)	$11,330,000	$ 9,199,000
Backorders	1,203,000	1,604,000
Net Stock	10,130,000	7,595,000
Assets	19,280,000	22,180,000
Requisitioning Objective	25,480,000	23,570,000

Sources: (1) Computer program calculations; (2) Letter and inclosures from USALCJ to IRO dated 26 January 1965

OBTAINING OPTIMAL POLICIES AND FORECASTING SYSTEM PERFORMANCE

An optimal (a,b) policy minimizes the total economic cost of carrying each item in the system. The concept of total economic cost has been thoroughly discussed by Churchman, Ackoff, and Arnoff(2). In earlier work (1) it was assumed that the cost of a procurement action and the cost per year of holding a unit were known. These factors were not known for USALCJ. Instead, policies were formulated for various availabilities and cost factors to forecast what would happen to the system under various optimal policies. In this way, possible alternatives were found in terms of changed investment, backorders, availability, and procurement activity while minimizing the total economic cost.

The cost of a procurement action (CP) includes all costs associated with processing a requisition that are independent of the replenishment quantity. The cost of holding (CH) is the cost per dollar's worth of unit per year required to maintain, store, and handle an item.

$$CH = \frac{\text{Cost per year of keeping a unit in stock}}{\text{Unit Price}}$$

The cost of holding includes implicit costs such as interest.

Both cost factors were considered to be linear. That is, if there will be k times as many purchases under one policy relative to some base policy, the cost of procurement operations under the proposed policy would be k times the cost of procurement under the base policy. Similarly for the cost of holding--if inventory changes by a factor of k, so does the cost of holding this inventory. The same values of the cost factors were assumed to be appropriate for all items in the system.

The expected total economic cost of carrying an item under some (a,b) policy is a function of the cost of procurement, the cost of holding, and the availability. The total economic cost is

$$C(\underline{a},\underline{b};CP,CH) = CP \cdot NPY + CH \cdot UP \cdot E(I) \tag{70}$$

where it is understood that the cost is also a function of the invariant parameters (pp. 21-24) of the item. A Lagrange multiplier was introduced (5) so that the availability can be specified.

$$C(\underline{a},\underline{b};CP/CH,\alpha_s) = CP \cdot NPY + CH \cdot UP \cdot E(I) + \theta\,(\alpha - \alpha_s) \tag{71}$$

It will be shown that only the ratio of the cost factors influences the coefficients $\underline{a}$ and $\underline{b}$.

To minimize the total economic cost of the item, partial derivatives were taken with respect to $\underline{a}$, $\underline{b}$, and θ.

$$\frac{\partial C}{\partial \underline{a}} = CH \cdot UP \cdot \frac{\partial E(I)}{\partial \underline{a}} + \theta\,\frac{\partial \alpha}{\partial \underline{a}}$$

$$\frac{\partial C}{\partial \underline{b}} = -CP \cdot \frac{AYD}{\underline{b}^2\sigma} + CH \cdot UP \cdot \frac{\partial E(I)}{\partial \underline{b}} + \theta\,\frac{\partial \alpha}{\partial \underline{b}} \tag{72}$$

$$\frac{\partial C}{\partial \theta} = \alpha - \alpha_s$$

To find the partial derivatives of E(I), recall equation (55).

$$\frac{\partial E(I)}{\partial \underline{a}} = \frac{\partial}{\partial \underline{a}}\Big[E(A) - E(Y) + E(B)\Big] \tag{73}$$

$$\frac{\partial E(I)}{\partial \underline{a}} = \sigma + \frac{\partial E(B)}{\partial \underline{a}} \tag{74}$$

$$\frac{\partial E(I)}{\partial \underline{b}} = \tfrac{1}{2}\sigma + \frac{\partial E(B)}{\partial \underline{b}} \tag{75}$$

For a variable y and a constant k (8),

$$\frac{\partial}{\partial k}\int_{u(k)}^{v(k)} w(k,y)\ dy = w(k,v(k))\frac{\partial v(k)}{\partial k} - w(k,u(k))\frac{\partial u(k)}{\partial k} + \int_{u(k)}^{v(k)} \frac{\partial}{\partial k} w(k,y)\ dy \tag{76}$$

Applying equation (76) to the partial derivatives of the expected value of backorders,

$$\frac{\partial E(B)}{\partial \underline{a}} = \frac{1}{\underline{b}\sigma}\frac{\partial}{\partial \underline{a}}\int_{E(Y)+\underline{a}\sigma}^{E(Y)+4\sigma} (y-E(Y)-\underline{a}\sigma)(1 - G(y))\ dy \tag{77}$$

$$= \frac{-\sigma}{\underline{b}\sigma}\int_{E(Y)+\underline{a}\sigma}^{E(Y)+4\sigma} (1 - G(y))\ dy \tag{78}$$

$$\frac{\partial E(B)}{\partial \underline{a}} = -\sigma \alpha_s \tag{79}$$

$$\frac{\partial E(B)}{\partial \underline{b}} = -\frac{1}{\underline{b}} E(B) \tag{80}$$

Applying equation (76) to the partial derivatives of availability,

$$\frac{\partial \alpha}{\partial \underline{a}} = \frac{\partial}{\partial \underline{a}}\frac{1}{\underline{b}\sigma}\int_{E(Y)+\underline{a}\sigma}^{E(Y)+4\sigma} (1 - G(y))\ dy \tag{81}$$

$$\frac{\partial \alpha}{\partial \underline{a}} = -\frac{1}{\underline{b}}(1 - F(\underline{a})) \tag{82}$$

where $F(\underline{a})$ is the value of the standard normal cumulative distribution function.

$$\frac{\partial \alpha}{\partial \underline{b}} = -\frac{1}{\underline{b}} \alpha_s \tag{83}$$

Since Θ is a constant used to link availability to the cost function, it should not be a part of the final solution. The three equations (72) may be rewritten as two equations with Θ eliminated. The result is

$$Z = \frac{CP}{CH\ UP\ L\ VMR} = \frac{1}{2}\underline{b}^2 - \frac{\alpha_s (1-\alpha_s)\,\underline{b}^2}{1 - F(\underline{a})} - \frac{1}{\underline{b}}\int_{\underline{a}}^{4} (z-\underline{a})(1-F(z))\,dz \qquad (84)$$

$$\alpha_s = \frac{1}{\underline{b}}\int_{\underline{a}}^{4} (1 - F(z))\,dz \qquad (85)$$

CP and CH are the only variables in Z that are not invariant. It is their ratio, CP/CH, that really affects the values of $\underline{a}$ and $\underline{b}$. For this study the cost of holding was arbitrarily fixed at .15 and only CP was changed. The forecast of the system's response to various optimal policies formulated from many combinations of costs of procurement and availability were calculated.

These equations have been solved (1959) as a part of the Ordnance Inventory Management Project at the Massachusetts Institute of Technology and tables were provided on punched cards for this research. For many availabilities between .10 and .999 the values of $\underline{a}$ and $\underline{b}$ are given for values of Z between .002 and 10,000.

One other adjustment needed to be made. The management at USALCJ did not want to purchase less than one month's supply at a time so the computer program was written to choose the replenishment quantity according to equation (86).

$$Q_j = \mathrm{Max}(.0833\ AYD_j,\ \underline{b}_j\sigma) \qquad (86)$$

When Q was changed, the $(\underline{a},\underline{b})$ policy was no longer optimal. By adjusting the $\underline{a}$ coefficient, however, a policy was found which would minimize the cost subject to the constraint on Q. For most of the policies analyzed, Q was changed in three or fewer classes.

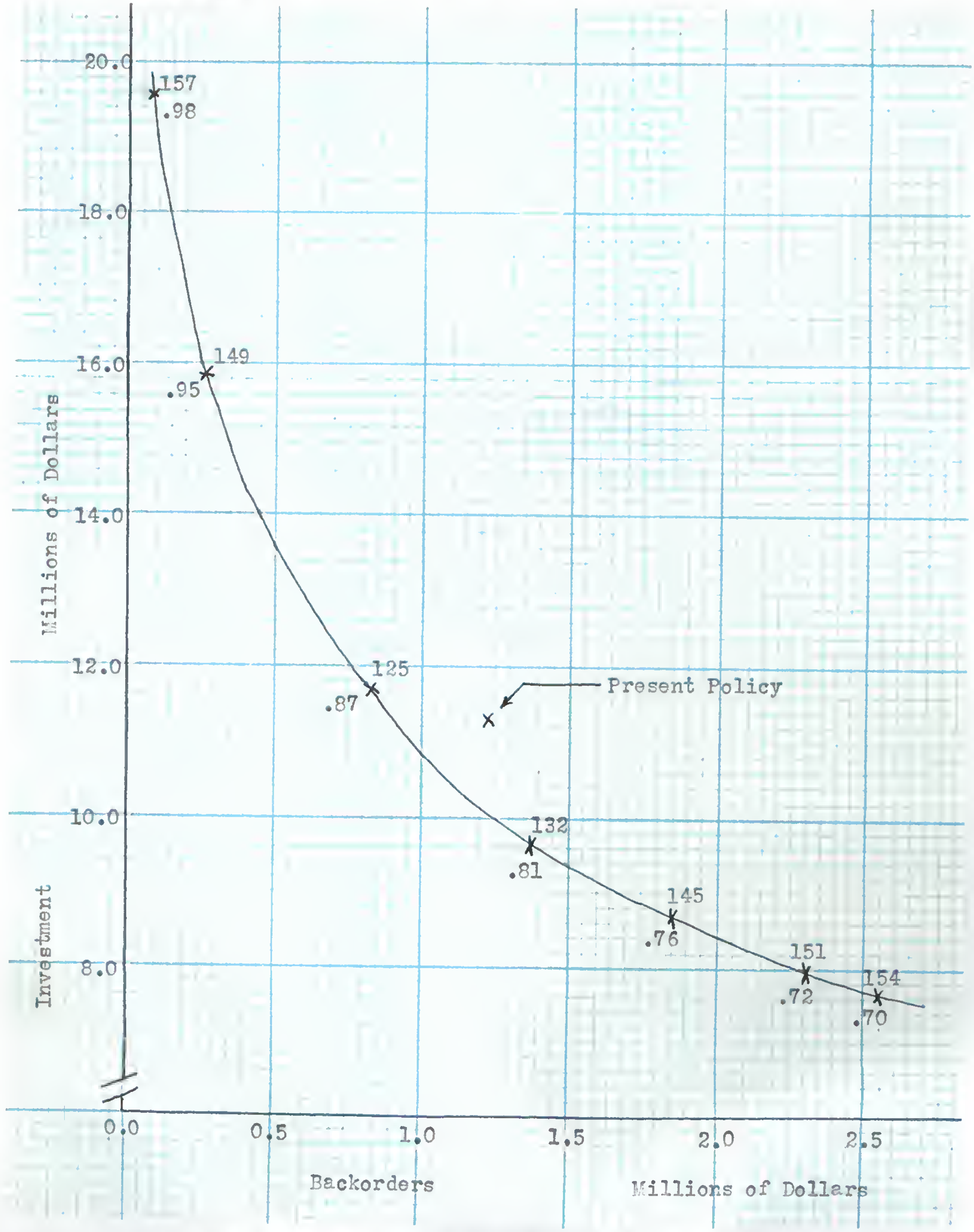
The computer program was written to find an optimal policy and calculate the expected values of performance and activity for the system. By adjusting the cost of procurement for each of several availabilities between .70 and .98, it was possible to find policies under which the total number of procurement actions per year would be expected to be close to 14,240, the present theoretical level. The iso-procurement action curve is shown in Fig. 2.

Of particular interest was the maximum possible reduction in investment when backorders and procurements were held at their theoretical levels of $1,203,000 and 14,240 per year, respectively. It was estimated that an optimal policy could reduce investment from $11.3 million to $10.1 million, a reduction of $1,200,000. If investment and procurements were held at their theoretical levels, an optimal policy was found that predicted a reduction in backorders to about $850,000, a change of $350,000, which is a 28 per cent reduction.

Of course, any other point on the iso-procurement action curve could be chosen. In particular, those points between the two points just mentioned would represent trades of a portion of the maximum possible reduction of investment or backorders for a partial reduction in the other.

If investment and backorders were held at their present theoretical levels, it was estimated that procurements could be reduced to 6,340 per year, a change of 55 per cent.

Fig. 2: Iso-procurement action curve for the U. S. Army Logistical Center, Japan, 14,240 per year. Availability appears below each policy analyzed.



Source: See Appendix 2 for complete information on each policy. The policy identification number is given for each point.

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Dr. Fryer is Head and Director of the Department of Statistics and the Statistical Laboratory. Dr. Sanger is Head of the Department of Mathematics.

Appendix 1: A description of the set of representative items defined by the arithmetic average of class limits in Table 1.

Let a set of representative items be defined by the arithmetic averages of the class limits for unit price and average yearly demand in Table 1. The invariant parameters describing each of these items were found and are presented below. The (R,Q) policy and the equivalent (a,b) policy are reported below in Tables 13 and 14 respectively. The expected values of performance and activity are presented in Table 8 (p. 28).

Table 10: Representative unit prices, average yearly demands, and dollar values of yearly demand for items found from the arithmetic average of class limits

Class Number j	Representative Unit Price UP_j	Representative Average Yearly Demand AYD_j	Dollar Value of Yearly Demand $DVYD_j$
1	.055	500.5	27.5275
2	.055	5500.5	302.5275
3	.055	55000.5	3025.0275
4	.055	550000.5	30250.0270
6	.555	500.5	277.7775
7	.555	5500.5	3052.7775
8	.555	55000.5	30525.2770
9	.555	150000.5	83250.2770
10	5.505	50.5	278.0025
11	5.505	550.5	3030.5025
12	5.505	5500.5	30280.2520
13	5.505	55000.5	302777.7500
15	55.005	5.5	302.5275
16	55.005	55.5	3052.7775
17	55.005	550.5	30280.2520
18	55.005	5500.5	302555.0000
19	55.005	30000.5	1650177.5000
20	550.005	5.5	3025.0275
21	550.005	55.5	30525.2770
22	550.005	550.5	302777.7500
23	550.005	1500.5	825282.5000
24	3500.005	5.5	19250.0270
25	10500.005	55.5	582750.2700
26	8000.005	550.5	4404002.7000

Source: Based on Table 1 and equation (58).

Table 11: Procurement lead times and expected values of demand for representative items

Class Number j	Procurement Lead Time (In years) L_j	Expected Value of Demand in the Procurement Lead Time (In units) $E(Y_j)$
1	.7500	375.3750
2	.7500	4125.3750
3	.2083	11456.6040
4	.2083	114565.1000
6	.7500	375.3750
7	.2083	1145.7541
8	.2083	11456.6040
9	.2083	31245.1040
10	.7500	37.8750
11	.2083	114.6691
12	.2083	1145.7541
13	.2083	11456.6040
15	.7500	4.1250
16	.2083	11.5606
17	.2083	114.6691
18	.2083	1145.7541
19	.2083	6249.1041
20	.2083	1.1456
21	.2083	11.5606
22	.2083	114.6691
23	.2083	312.5541
24	.2083	1.1456
25	.2083	11.5606
26	.2083	114.6691

Source: Procurement lead times, letter from USALCJ to IRO dated 26 January 1965; Expected values of demand based on L given here, AYD from Table 10, and equation (16)

Table 12: Variance to mean ratios and standard deviations of demand during the procurement lead time for the representative items

Class Number j	Variance to Mean Ratio VMR_j	Standard Deviation of Demand σ_j
1	141.6060	230.5544
2	1364.0337	2372.1610
3	12016.8670	11733.3900
4	105873.5700	110133.6000
6	141.6060	230.5544
7	1364.0337	1250.1380
8	12016.8670	11733.3900
9	31013.5950	31129.1300
10	16.2089	24.7772
11	154.9389	133.2918
12	1364.0337	1250.1380
13	12016.8670	11733.3900
15	1.9942	2.8681
16	17.7215	14.3133
17	154.9389	133.2918
18	1364.0337	1250.1380
19	6776.9033	6507.6550
20	1.9942	1.5115
21	17.7215	14.3133
22	154.9389	133.2918
23	399.6575	353.4326
24	1.9942	1.5115
25	17.7215	14.3133
26	154.9389	133.2918

Source: VMR from regression equation (65) with AYD from Table 10; standard deviation from equation (27) with L from Table 11

Table 13: Reorder points and replenishment quantities for representative items

Class Number j	Reorder Point (In units) R_j	Replenishment Quantity (In units) Q_j
1	500.5000	2002.000
2	5500.5000	11001.000
3	20617.0000	27500.300
4	206168.0000	137500.000
6	500.5000	1001.000
7	2061.9000	2750.250
8	20617.0000	13750.000
9	56228.0000	37500.100
10	50.5000	101.000
11	206.3500	275.250
12	2061.9000	1375.100
13	20617.0000	13750.100
15	5.5000	11.000
16	20.8040	27.750
17	206.3500	137.625
18	2061.9000	1375.125
19	11245.7000	7500.125
20	2.0617	2.750
21	20.8040	13.875
22	206.3500	137.625
23	562.4600	375.125
24	2.0617	1.375
25	20.8040	13.875
26	206.3500	137.625

Source: The dollar value of yearly demand for each representative item, given in Table 10, was used to find the item's (M,P) policy, then the equivalent (R,Q) policy was determined with equations (66) and (67)

Table 14: Coefficients a and b of the (R,Q) policies for the representative items

Class Number	a_j	b_j
1	.5427	8.6834
2	.5796	4.6375
3	.7807	2.3437
4	.8317	1.2484
6	.5427	4.3417
7	.7328	2.1999
8	.7807	1.1718
9	.8025	1.2046
10	.5095	4.0763
11	.6878	2.0650
12	.7328	1.0999
13	.7807	1.1718
14	.4794	3.8352
16	.6457	1.9387
17	.6878	1.0325
18	.7328	1.0999
19	.7678	1.1525
20	.6060	1.8193
21	.6457	.9693
22	.6878	1.0325
23	.7070	1.0613
24	.6060	.9096
25	.6457	.9693
26	.6878	1.0325

Source: E(Y) from Table 11 and the standard deviation from Table 12 were used in equations (68) and (69) solved for a and b respectively

Appendix 2: Expected values of performance and activity for certain availabilities and costs of procurement

From the more than eighty policies investigated, the following were used to establish the forecasts given in this thesis. Policies are listed in the order as one reads up the iso-procurement curve in Fig. 2 (p. 35).

Table 15: Expected dollar values. (Thousands of Dollars)

Policy Identification Number	Backorders	Stock on Hand (Investment)	Assets	Stockage Objective
167	3,997	6,437	11,591	16,147
154	2,554	7,597	14,193	18,744
151	2,302	7,905	14,753	19,302
145	1,848	8,632	15,934	20,483
132	1,338	9,756	17,569	22,107
125	807	11,586	19,929	24,449
149	245	15,890	24,795	29,254
157	83	19,540	28,607	33,024
165	37	22,031	31,144	35,537
182	1,227	11,349	19,273	28,911

Table 16: Purchases per year, actual availability, parameters determining policy, (cost of holding was always .15).

Policy Identification Number	Purchases Per Year	Actual Availability	Table Availability	Cost of Procurement Action
167	14,240	.6122	.6000	6.50
154	14,260	.7075	.7000	8.75
151	14,260	.7270	.7200	9.25
145	14,240	.7644	.7600	10.37
132	14,240	.8113	.8100	11.83
125	14,250	.8680	.8700	13.75
149	14,250	.9495	.9500	17.37
157	14,235	.9797	.9800	19.75
165	14,260	.9899	.9900	21.00
182	6,340	.8281	.8100	95.00

GLOSSARY OF SYMBOLS

Each symbol used in this thesis is followed by the first page on which it appears and/or the page on which it is defined and explained. The subscript j, which is used to identify the item, is omitted here.

Greek Letters

α	7	Availability
$\beta_{1.23}$	23	Multiple regression coefficient
$\beta_{1.32}$	23	Multiple regression coefficient
θ	31	Lagrange multiplier
λ	11	Mean number of requisitions received per unit time
σ	12	Standard deviation of demand during the procurement lead time

Latin Letters

$\underline{a}$	26	Coefficient determining the safety stock
A	14	Assets, a random variable
ADVB	5	Aggregate dollar value of backorders
ADVI	4	Aggregate dollar value of inventory
ARO	5	Aggregate requisitioning objective
AVAIL	7	Systemwide availability
AYD	8	Average yearly demand
AYDL	20	Lower limit of the average yearly demand for a class
AYDU	20	Upper limit of the average yearly demand for a class
$\underline{b}$	27	Coefficient determining the replenishment quantity
B	5	Backorders, a random variable
CH	30	Cost of holding a dollar's worth of a unit per year

CP	30		Cost of a single procurement action
DVB	5		Dollar value of backorders
DVI	4		Dollar value of inventory
DVYD	20		Dollar value of yearly demand
E(A)	17		Expected value of assets
E(B)	5	18	Expected value of backorders
E(I)	4	18	Expected value of inventory
E(Y)	9		Expected value of demand during the procurement lead time
$f_1(i_j)$	3		Density function for inventory
$f_2(b_j)$	5		Density function for backorders
$f_3(a_j)$	14		Density function for assets
$g_1(x;L)$	11		Density function for the number of requisitions received during the procurement lead time
$h_1(y;L)$	11		Density function for demand during the procurement lead time
$h_2(a,y)$	14		Joint density function for assets and demand during the procurement lead time
$h_3(ns_j)$	15	16	Density function for net stock
I	3		Inventory, a random variable
IRO	1		Inventory Research Office, Frankford Arsenal, Philadelphia, Pa.
j	3		Representative item identification number
L	9		Procurement lead time in years
M	25		Reorder point in months of supply
$m_s(t)$	11		Moment generating function of the requisition size random variable
$m_Y(t)$	11		Moment generating function of demand
$m_z(t)$	13		Moment generating function of a standard normal random variable
n_j	4		Number of items in the jth class

ns	15	A particular net stock level
NPY	8	Number of procurement actions per year
NS	15	Net stock, a random variable
P	25	Replenishment quantity in months of supply
Q	3	Replenishment quantity in units
r_{12}	23	Simple correlation coefficient
R	3	Reorder point in units
$R_{1.23}$	23	Multiple correlation coefficient
RO	4	Requisitioning objective
(R,Q) policy		Defined on page 3
S	10	Requisition size, a random variable
TNPY	8	Total number of procurement actions per year
UP	4	Unit price
UPL	20	Lower limit of the unit price for a class
UPU	20	Upper limit of the unit price for a class
USALCJ	1	U. S. Army Logistical Center, Japan
VMR	12	Variance to mean ratio
X	11	Number of requisitions received during the procurement lead time, a random variable
Y	11	Demand during the procurement lead time
Z	13	Standard normal random variable

DERIVATION OF EXPECTED VALUES OF PERFORMANCE AND ACTIVITY

FOR A MULTI-ITEM INVENTORY SYSTEM

by

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ABSTRACT

The expected values of performance and activity for a multi-item inventory system can be estimated from the expected values for the items in the system. The expected values of interest were the investment, dollar value of backorders, the availability, and the number of procurement actions necessary to replenish stock. Demand could not be lost in the military inventory system being studied, the U. S. Army Logistical Center, Japan. The expected values for each item were found to be functions of the invariant parameters of the item and the policy used to govern replenishment of the stock of the item. Possible alternatives in terms of changed investment, backorders, procurement activity, and availability were found by applying and evaluating various policies.

The dollar value of backorders was considered to be a better index of performance than the total number of units backordered because the loss due to the unavailability of a unit was assumed to be proportional to its unit price. The availability is the probability that net stock is above or at zero units, i.e. it is the probability that there are no backorders.

Under the assumption that demand had a compound Poisson distribution, the expected values for a typical item were derived. It was found that demand was approximately normally distributed for long procurement lead times. The mean and variance of demand during the procurement lead time are invariant parameters of each item.

The 25,003 items were classified into groups such that all items in each group had about the same values for their invariant parameters. The expected values for each representative item were calculated by a computer for a given policy. Totals for the whole system were found by adding the weighted expected values for each representative item.

All policies evaluated were of the type presently in use. When assets are equal to or below a certain number of units, the reorder point, an order is sent to the appropriate depot in the United States for a certain number of units of the item, called the replenishment quantity. The assets are the number of units on hand plus the number on order minus the number backordered.

A set of representative items were found such that their expected values of performance and activity under the present policy were reasonably close to those experienced at the Logistical Center. The usefulness of the conclusions is based on the assumption that the system will respond to any given policy in a manner similar to the change in the expected values for the representative items relative to the theoretical expected values calculated from the representative items under the present policy.

Policies were made by using the results of the Ordnance Inventory Management Project. For a given availability and cost factor ratio, this earlier research minimized the total economic cost of carrying an item and, therefore, minimized the cost of the system. By trying various combinations of cost factor ratios and availabilities, possible changes in the Logistical Center were explored.

For each availability from .60 to .99, the cost factor ratio was adjusted to get the same number of procurement actions as are presently experienced (14,240) while investment and backorders were determined. One of these policies could theoretically change investment to $10.1 million (from the present $11.3 million), a reduction of $1.2 million. Another policy was found that forecast a reduction of $350,000 in the dollar value of backorders, a change of 28%.

A policy was found that would change the number of procurements per year to 6,340, a reduction of 55%, while keeping investment and backorders at their present levels.