

PERIODICALLY REVERSE-SWITCHED CAPACITORS

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NOMENCLATURE

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$\bar{e} = L[e(t)]$	Laplace transform of $e(t)$	2
\dot{e}	Newton's notation for d/dt	2
e_0	$e(0)$ initial condition of function $e(t)$.	2
C	Capacitance	2
$R\bar{f}(\lambda)$	Region containing the poles of $\bar{f}(\lambda)$. . .	4
Z	Z-transform operator	4
Z_m	Z-transform operator with T replaced by mT	4
z	e^{-sT}	5
f_{nm}	$f(nmT)$, $f(t)$ at time $t = nmT$	5
R	Resistance	5
$n(t)$	The unit step function	6
P_n	Laplace transform on the n -th interval of length mT	6
v_0	$v(0)$ initial condition of function $v(t)$.	8
$\{f_n\} = \{f_0, f_1, \dots\}$	Notation for a sequence	10
V^{-1}	Matrix inverse of V	23
V'	Matrix transpose of V	23
z_k^*	Conjugate of z_k	24
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INTRODUCTION

The capacitor is a basic building block of electrical networks. Many new developments in capacitors, such as improvements in dielectric materials, occur. A new and different use of this basic unit is presented in this paper.

A capacitor may act as a digital feedback device by switching it in a special way. The charge stored on the capacitor during one period of operation may be used as the initial condition for the next period of operation. If these initial conditions are reversed by reversing the capacitor, a discrete or digital feedback is introduced since the action following is dependent upon the previous action.

The reverse-switched capacitor and its mathematical description are presented. Using these derivations, integrated and summed outputs of several integrators, whose capacitors are replaced with reverse-switched capacitors, are analyzed. The periodic quantizer is analyzed by Rademacher functions and then physically realized. A simple series resistance-capacitance network whose capacitor is replaced with a reverse-switched capacitor, is analyzed, and its limit cycle investigated. The computer programs and results are presented in each case.

In most cases, normalized impedances and admittances of 1 are used to facilitate calculations. This detracts in no way from generality of the methods used.

THE REVERSE-SWITCHED CAPACITOR

Operation

The role of initial conditions when a capacitor is set into operation has been thoroughly developed (LePage and Seely, 1). This thesis investigates the effects on a network containing a capacitor when not only initial conditions are present but when a periodic change of initial conditions takes place. These periodically changing conditions are effected by reversing the capacitor in the network at periodic intervals. The new device, a capacitor, with periodically reversed terminals will be called a reverse-switched capacitor and indicated by the symbol $\rightarrow\leftarrow$, which is the ordinary capacitor symbol with both "plate lines" curved outward. The reversing that takes place at the end of one switching period causes the final value of that switching period to be the initial value of the following period.

During the first period from time zero to the first switching time the reverse-switched capacitor will act the same as the unswitched device. The initial conditions will contribute the same as in the unswitched device. The differential equation may be written $i = c\dot{e}$ and Laplace transformed to read

$$\bar{i} = C s \bar{e} - C e_0 . \quad (1)$$

An analysis could be performed on any network by continuing in a piecewise linear manner; the solution would be obtained by ordinary means for the first period; the voltage on the capacitor could then be calculated; this reversed final voltage would then be used for the initial conditions of the next period. It can be seen, however, that this procedure is a slow and

tedious process. A Z-transform analysis overcomes this and other difficulties that arise. The procedure is exact and provides a clear and simple description of a reverse-switched capacitor.

The analysis used in this paper is based on the theory of a vanishingly small switching time. This is not realizable in the physical world; however, a close approximation can be made using modern technological advances. It is recognized that a somewhat different analysis could be based on finite switching times, such as an open circuit for a finite amount of time during switching. The reader is referred to Higgins and Howard (4) for work related to finite pulse lengths.

Physical Construction

Consider a capacitor whose leads are attached to a double-pole double-throw switch in an arrangement whereby the capacitor terminals are in one direction when the switch is in a given position. When the switch is thrown to the other position the capacitor's terminals are reversed in direction. The terminals of the reverse-switched capacitor are the common connections of the switch. The capacitor to be switched is connected to adjacent corners of the switch. Opposite corners are connected together to complete the buildup of the device shown in Fig. 1.

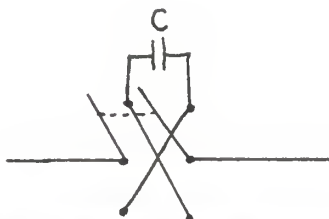


Fig. 1. Reverse-switched capacitor.

The reversing switch can be an electromechanical relay or a solid-state switch. The switch must be controlled by a periodic timing mechanism. The timer and switch combination should have as small a switching time as possible for the physical device to be properly described mathematically. If not, allowance must be made for the switching time duration.

Research on hardware is outside the scope of this paper; however, this does not hinder an introduction to it at the present time. The author has conducted an experiment using electromechanical equipment for the switch and found that with integrator circuits a small delay in the switching time can cause a noticeable deviation from the ideal output.

SUBSTITUTION ANALYSIS

The Z- and Z_m -Transform

The analysis makes use of the Z-transform and the special Z_m -transform. The Z_m -transform is the ordinary Z-transform where the sampling interval T is replaced with mT , the duration of the switching period.

Jury (6) presents Z-transform theory and some applications to sampled data systems. The Z-transform of a function is the complex convolution of its Laplace transform with the Laplace transform of a periodic Dirac delta function whose period is T seconds. This is represented as

$$\bar{z}f = \frac{1}{2\pi j} \oint_{R\bar{z}f(\lambda)} \bar{f}(\lambda) \frac{1}{1-e^{-(s-\lambda)T}} d\lambda \quad (2)$$

If the period T is replaced by mT we obtain

$$\bar{z}_m \bar{f} = \frac{1}{2\pi j} \oint_{R\bar{z}_m \bar{f}(\lambda)} \bar{f}(\lambda) \frac{1}{1-e^{-(s-\lambda)mT}} d\lambda \quad (3)$$

Representing $\frac{1}{1-e^{-(s-\lambda)T}}$ as an infinite sum and bringing the sum on the

outside gives

$$Z_m \bar{f} = \sum_{n=0}^{\infty} e^{-snmT} \frac{1}{2\pi j} \oint_{R\mathcal{C}\bar{f}(\lambda)} \bar{f}(\lambda) e^{\lambda nmT} d\lambda \quad (4)$$

$$Z_m \bar{f} = \sum_{n=0}^{\infty} e^{-snmT} f(nmT) = \sum_{n=0}^{\infty} Z^{nmT} f_{nm} \quad (5)$$

This is the Z_m -transform in infinite series form. The poles of $1/(1-e^{-\lambda T})$ may also be used to obtain

$$Z_m \bar{f} = \frac{1}{2\pi j} \oint_{R\mathcal{C}\bar{f}(s-\lambda)} \bar{f}(s-\lambda) \frac{1}{1-e^{-\lambda T}} d\lambda \quad (6)$$

Using Heavisides expansion where, $e^{-\lambda mT} = e^{-j2\pi n}$ or $\lambda = j2\pi n/mT$, results in a different form for $Z_m \bar{f}$.

$$Z_m \bar{f} = \sum_{n=-\infty}^{\infty} \bar{f}(s-(j2\pi n/mT)) \left[1/mT e^{-j2\pi n} \right] \quad (7)$$

$$Z_m \bar{f} = \frac{1}{mT} \sum_{n=-\infty}^{\infty} \bar{f}(s-(j2\pi n/mT)) \quad (8)$$

Theorems that apply to the ordinary Z-transform apply to the Z_m -transform also.

A Fundamental Development for Switched Integrators

An operational amplifier integrator was used in the early stages of development. The equation describing the integrator is simple and easy to work with. Referring to Smith and Wood (7), the equation for an integrator may be written

$$\frac{e}{R} + C\dot{v} = 0 \quad (9)$$

R and C are normalized to 1, yielding

$$e + \dot{v} = 0, \quad (10)$$

The P_n -transform is now defined as

$$P_n f \equiv \int_{nT}^{(n+1)T} f(t) e^{-st} dt. \quad (11)$$

Transforming by P_n yields

$$P_n e + P_n \dot{v} = 0. \quad (12)$$

A need for development of $P_n f$ and $P_n \dot{f}$ now exists. Consider first $P_n f$; then

$$P_n f = L \left[\left\{ u(t-nT) - u(t-(n+1)T) \right\} f(t) \right]$$

$$P_n f = \frac{1}{2\pi j} \oint_{\lambda=0} \frac{e^{-\lambda nT} - e^{-\lambda(n+1)T}}{\lambda} \bar{f}(s-\lambda) d\lambda \quad (13)$$

$$\sum_{n=0}^{\infty} P_n f = \frac{1}{2\pi j} \oint_{\lambda=0} \sum_{n=0}^{\infty} \frac{e^{-\lambda nT} - e^{-\lambda(n+1)T}}{\lambda} \bar{f}(s-\lambda) d\lambda. \quad (14)$$

The integrand sum will be of the collapsing type, yielding

$$\sum_{n=0}^{\infty} P_n f = \frac{1}{2\pi j} \oint_{\lambda=0} \frac{1 - e^{-mT\lambda}}{1 - e^{-\lambda T}} \bar{f}(s-\lambda) d\lambda \quad (15)$$

$$\sum_{n=0}^{\infty} P_n f = \frac{1}{2\pi j} \oint_{\lambda=0} \frac{\bar{f}(s-\lambda)}{\lambda} d\lambda = \bar{f}(s) = \bar{f}. \quad (16)$$

The same result may be obtained using the definition of $P_n f$ as follows:

$$P_n f = \int_{nT}^{(n+1)T} f(t) e^{-st} dt \quad (17)$$

$$\sum_{n=0}^{\infty} P_n f = \sum_{n=0}^{\infty} \int_{nmT}^{(n+1)mT} f(t) e^{-st} dt \quad (18)$$

$$= \int_0^{mT} f(t) e^{-st} dt + \int_{mT}^{2mT} f(t) e^{-st} dt + \dots \quad (19)$$

$$= \int_0^{\infty} f(t) e^{-st} dt \quad (20)$$

$$\sum_{n=0}^{\infty} P_n f = \bar{f} . \quad (21)$$

The next problem is evaluation of $P_n \dot{f}$. Integration by parts yields

$$P_n \dot{f} = \int_{nmT}^{(n+1)mT} \dot{f}(t) e^{-st} dt \quad (22)$$

$$= f(t) e^{-st} \Big|_{nmT}^{(n+1)mT} + s \int_{nmT}^{(n+1)mT} f(t) e^{-st} dt \quad (23)$$

$$P_n \dot{f} = f_{(n+1)m} e^{-s(n+1)mT} - f_{nm} e^{-snmT} + s P_n f . \quad (24)$$

By reversing the sign of the term $f_{(n+1)m} e^{-s(n+1)mT}$ the initial conditions will enter correctly into the formula and the reversal will also be entered properly. However, notation for $P_n \dot{f}$ will not be changed.

If $P_n \dot{f}$ is summed after correction for reverse-switching, then

$$\sum_{n=0}^{\infty} P_n \dot{f} = - \sum_{n=0}^{\infty} f_{(n+1)m} e^{-s(n+1)mT} - \sum_{n=0}^{\infty} f_{nm} e^{-snmT} + s \sum_{n=0}^{\infty} P_n f , \quad (25)$$

$$= \sum_{n=1}^{\infty} f_{nm} e^{-snmT} - \sum_{n=0}^{\infty} f_{nm} e^{-snmT} + s \bar{f} , \quad (26)$$

$$\sum_{n=0}^{\infty} P_n \dot{f} = f_0 - 2 \sum_{n=0}^{\infty} f_{nm} e^{-snmT} + sf \quad (27)$$

The above equation can be written as

$$\sum_{n=0}^{\infty} P_n \dot{f} = sf - 2Z_m \bar{f} + f_0 \quad (28)$$

The forms are now available to substitute into equation (10) to analyze the switched integrator.

$$P_n \bar{e} + P_n \dot{\bar{v}} = 0 \quad (29)$$

$$\sum_{n=0}^{\infty} P_n \bar{e} + \sum_{n=0}^{\infty} P_n \dot{\bar{v}} = 0 \quad (30)$$

Substituting equations (20) and (27) into (30) yields the fundamental equation

$$\bar{e} + s\bar{v} = 2Z_m \bar{v} - v_0 \quad (31)$$

Here it is possible to check on the initial conditions. For \bar{v} in the first period the equation reads

$$\bar{e} + s\bar{v} = (2Z_m v_0 - v_0) \Big|_{z^m=0} \quad (32)$$

$$\bar{e} + s\bar{v} - v_0 = 0 \quad (33)$$

This checks with the unswitched capacitor shown in equation (1). R and C could be reintroduced at this point and then equation (31) would read

$$\bar{e}/R + Cs\bar{v} = 2CZ_m \bar{v} - Cv_0 \quad (34)$$

The analysis is carried on using equation (31) with $R = C = 1$. Solving for \bar{v} yields

$$\bar{v} = -\frac{\bar{e}}{s} + \frac{2}{s} Z_m \bar{v} - \frac{v_0}{s} \quad (35)$$

Taking the Z_m -transform

$$Z_m \bar{v} = -Z_m(\bar{e}/s) + \frac{2}{1-z^m} Z_m \bar{v} - \frac{v_0}{1-z^m} \quad (36)$$

and solving for $Z_m \bar{v}$ gives

$$Z_m \bar{v} = \frac{1-z^m}{1+z^m} Z_m(\bar{e}/s) + \frac{v_0}{1-z^m} \quad (37)$$

Substituting equation (37) into equation (35) yields

$$\bar{v} = -\bar{e}/s + \frac{2(1-z^m)}{s(1+z^m)} Z_m(\bar{e}/s) + \frac{v_0(1-z^m)}{s(1+z^m)}, \quad (38)$$

and taking the Z-transform yields

$$Z\bar{v} = -Z(\bar{e}/s) + \frac{2(1-z^m)}{(1-z)(1+z^m)} Z_m(\bar{e}/s) + \frac{v_0(1-z^m)}{(1-z)(1+z^m)} \quad (39)$$

It is desirable to use the simplest forcing function, the unit step function, to minimize calculation complexity. Making the substitution $\bar{e} = 1/s$, where $Z(\bar{e}/s) = Tz/(1-z)^2$ and $Z_m(\bar{e}/s) = mTz^m/(1-z^m)^2$, results in

$$Z\bar{v} = -\frac{Tz}{(1-z)^2} + \frac{2mTz^m}{(1-z)(1+z^m)(1-z^m)} + \frac{1-z^m}{(1-z)(1+z^m)} \quad (40)$$

Multiplying by the denominator and simplifying gives

$$Z\bar{v} \left[(1-z)^2(1-z^{2m}) \right] = -Tz(1-z^{2m}) + 2mTz^m(1-z) + v_0 \left[(1-z^m)^2(1-z) \right], \quad (41)$$

$$\begin{aligned} Z\bar{v}(1-2z+z^2-z^{2m+2}z^{2m+2}z^{2m+1}-z^{2m+2}) &= v_0 - (v_0+T)z + (2mT-2v_0)z^m \\ &+ (2v_0-mT)z^{m+1} + v_0z^{2m} + (T-v_0)z^{2m+1}, \end{aligned} \quad (42)$$

The recurrence relation used in digital computation may be written directly as

$$v_n = 2v_{n-1} - v_{n-2} + v_{n-2m} - 2v_{n-2m-1} + v_{n-2m-2} + x_n \quad (43)$$

where

$$\{X_n\} = \{v_0, -(v_0+T), 0, 0, \dots, 2(mT-v_0), 2(v_0-mT), 0, 0, \dots, v_0, (T-v_0), 0, 0, \dots\}.$$

It can now be seen why the sampled data approach was used since the result, in the form of a recurrence relation, is readily computed by a digital computer. Some solutions are given in Table 1. The computer program used to obtain these results is in Appendix C.

The Loop and Node Pair Equations

Using current as a driving source, a relation occurs between this current and the voltage across the reverse-switched capacitor alone. For this circuit we can write Newton's counterforce equation $i = C\dot{e}$. Using the relation for $C\dot{e}$ given by equation (28) gives

$$\bar{i} = C\bar{s}\bar{e} - 2CZ_m\bar{e} + Ce_0. \quad (44)$$

Equation (44) is called the node pair voltage equation.

From (44) the loop current equation can be derived by first solving for \bar{e} and Z_m transforming

$$\bar{i}/Cs = \bar{e} - (2/s)Z_m\bar{e} + (e_0/s) \quad (45)$$

$$Z_m(\bar{i}/Cs) = Z_m\bar{e} \left[1 - \frac{2}{1-z^m} \right] + \frac{e_0}{1-z^m}. \quad (46)$$

Solving for $Z_m\bar{e}$ and substituting into equation (45) gives the desired result

$$Z_m\bar{e} = -\frac{1-z^m}{1+z^m} Z_m(\bar{i}/Cs) + \frac{e_0}{1-z^m} \quad (47)$$

Table 1.

C C INTEGRATOR USING A REVERSE-SWITCHED CAPACITOR
 T=.1 V0=0. M=10

N	V(N)	N	V(N)	N	V(N)
	.00000	41	-.10000	81	-.10000
1	-.10000	42	-.20000	82	-.20000
2	-.20000	43	-.30000	83	-.30000
3	-.30000	44	-.40000	84	-.40000
4	-.40000	45	-.50000	85	-.50000
5	-.50000	46	-.60000	86	-.60000
6	-.60000	47	-.70000	87	-.70000
7	-.70000	48	-.80000	88	-.80000
8	-.80000	49	-.90000	89	-.90000
9	-.90000	50	1.00000	90	1.00000
10	1.00000	51	.90000	91	.90000
11	.90000	52	.80000	92	.80000
12	.80000	53	.70000	93	.70000
13	.70000	54	.60000	94	.60000
14	.60000	55	.50000	95	.50000
15	.50000	56	.40000	96	.40000
16	.40000	57	.30000	97	.30000
17	.30000	58	.20000	98	.20000
18	.20000	59	.10000	99	.10000
19	.10000	60	0.00000	100	0.00000
20	0.00000	61	-.10000	101	-.10000
21	-.10000	62	-.20000	102	-.20000
22	-.20000	63	-.30000	103	-.30000
23	-.30000	64	-.40000	104	-.40000
24	-.40000	65	-.50000	105	-.50000
25	-.50000	66	-.60000	106	-.60000
26	-.60000	67	-.70000	107	-.70000
27	-.70000	68	-.80000	108	-.80000
28	-.80000	69	-.90000	109	-.90000
29	-.90000	70	1.00000	110	1.00000
30	1.00000	71	.90000	111	.90000
31	.90000	72	.80000	112	.80000
32	.80000	73	.70000	113	.70000
33	.70000	74	.60000	114	.60000
34	.60000	75	.50000	115	.50000
35	.50000	76	.40000	116	.40000
36	.40000	77	.30000	117	.30000
37	.30000	78	.20000	118	.20000
38	.20000	79	.10000	119	.10000
39	.10000	80	0.00000	120	0.00000
40	0.00000	81	-.10000	121	-.10000

END OF PROGRAM AT STATEMENT 0007 + 00 LINES

$$\bar{i}/Cs = \bar{e} - \frac{2}{s} \left[-\frac{1-z^m}{1+z^m} Z_m(\bar{i}/Cs) + \frac{e_0}{1+z^m} \right] + (e_0/s) \quad (48)$$

$$\bar{e} = \bar{i}/Cs - \frac{2(1-z^m)}{s(1+z^m)} Z_m(\bar{i}/Cs) + e_0 \frac{(1-z^m)}{s(1+z^m)}, \quad (49)$$

Equation (49) is called the loop current equation.

Network Applications

Equations (44) and (49), developed in the preceding section, may be used to solve any network containing a reverse-switched capacitor. Both equations depend upon the sampled value at the instant of switching. The Z-transforms are, therefore, directly applicable and using them does not effect the exactness of solution.

The procedure for solving a network involving reverse-switched capacitors is as follows. The loop current or node pair voltage equation(s), whichever are desired, are written for the network. These are Laplace transformed if not already written in this form.

At the point(s) in the equation(s) where there is a voltage drop in the circuit due to a capacitor, the node pair voltage equation (49) is inserted. The initial conditions are taken care of in the new equation. The same procedure is followed for the loop current equation (44). The equation(s) obtained can then be manipulated much the same as those used in sampled-data feedback systems. The equation(s) are solved for the desired variable by Z_m transforming and substituting back into the equation not Z_m transformed. The desired variable may then be obtained, this equation Z transformed, and result set down in a recurrence relation. This recurrence relation may be easily programmed for a digital computer.

REVERSE-SWITCHED CAPACITOR IN A NETWORK

Application to Series RC Circuit

The loop current equation (49) may be used in the equation of the series RC Circuit, where the resistor is now in series with a reverse-switched capacitor and a voltage generator used as a source. The voltage equation of the circuit may be written directly in the Laplace transform notation,

$$\bar{e} = \bar{i}R + \bar{i}/Cs . \quad (50)$$

The voltage due to the capacitor is \bar{i}/Cs ; so in place of this may be written the loop current equation (49)

$$\bar{e} = \bar{i}R + \frac{\bar{i}}{Cs} - \left[\frac{2(1-z^m)}{s(1+z^m)} \right] Z_m(\bar{i}/Cs) + e_0 \left[\frac{(1-z^m)}{s(1+z^m)} \right] . \quad (51)$$

Normalizing $R = C = 1$ and dividing by $s+1$ yields

$$\frac{\bar{e}}{1+s} = \frac{\bar{i}}{s} - \left[\frac{2(1-z^m)}{s(s+1)(1+z^m)} \right] Z_m(\bar{i}/s) + \frac{e_0(1-z^m)}{(s+1)s(1+z^m)} \quad (52)$$

and Z_m transforming yields

$$\begin{aligned} Z_m \left[\frac{\bar{e}}{(s+1)} \right] &= Z_m(\bar{i}/s) \left[1 - Z_m \frac{2(1-z^m)}{s(s+1)(1+z^m)} \right] + \frac{e_0}{1+z^m} \\ &- \frac{e_0(1-z^m)}{(1-z^m e^{-mT})(1+z^m)} . \end{aligned} \quad (53)$$

Solving for $Z_m(\bar{i}/s)$ gives

$$Z_m(\bar{i}/s) = \left[\frac{(1+z^m)(1-z^m e^{-mT})}{(1-z^m)(1+z^m e^{-mT})} \right] Z_m \left[\frac{\bar{e}}{(s+1)} \right] - \frac{e_0 z^m (1-e^{-mT})}{(1-z^m)(1+z^m e^{-mT})} . \quad (54)$$

Substituting equation (54) into (52) and solving for \bar{i} results in

$$\bar{i} = \frac{s\bar{e}}{1+s} + \frac{2(1-z^m e^{-mT})}{(s+1)(1+z^m e^{-mT})} z_m \frac{\bar{e}}{s+1} - \frac{e_0 z^m (1-e^{-mT})}{(s+1)(1+z^m)(1+z^m e^{-mT})} - \frac{e_0(1-z^m)}{(s+1)(1+z^m)}. \quad (55)$$

Again for calculation purposes choose $\bar{e} = 1/s$. Doing this and combining the e_0 terms yields

$$\bar{i} = \frac{1}{s+1} + \frac{2z^m(1-e^{-mT})}{(s+1)(1-z^m)(1+z^m e^{-mT})} - \frac{e_0(1-z^m e^{-mT})}{(s+1)(1+z^m e^{-mT})}. \quad (56)$$

After Z transforming,

$$Z\bar{i} = \frac{1}{1-ze^{-T}} + \frac{2z^m(1-e^{-mT})}{(1-ze^{-T})(1-z^m)(1+z^m e^{-mT})} - \frac{e_0(1-z^m-z^m e^{-mT}+z^{2m} e^{-mT})}{(1-ze^{-T})(1-z^m)(1+z^m e^{-mT})} \quad (57)$$

multiplying through, and collecting terms the result is

$$\begin{aligned} Z\bar{i}(1-ze^{-T}-z^m(1-e^{-mT}) + z^{m+1}(e^{-T}-e^{-(m+1)T}) - z^{2m} e^{-mT} + z^{2m+1} e^{-(m+1)T}) \\ = 1-z^m+z^m e^{-mT}-z^{2m} e^{-mT}+2z^m-z^m e^{-mT}-e_0(1-z^m-z^m e^{-mT}+z^{2m} e^{-mT}). \end{aligned} \quad (58)$$

Equation (58) can be put into a recurrence relation as follows,

$$i_n = i_{n-1} e^{-T} + i_{n-m}(1-e^{-mT}) - i_{n-m-1}(e^{-T}-e^{-(m+1)T}) + i_{n-2m} e^{-mT} - i_{n-2m-1} e^{-(m+1)T} + x_n \quad (59)$$

where

$$\{x_n\} = \{(1-e_0), 0, \dots, 1-e^{-mT}+e_0(1+e^{-mT}), 0, \dots, -e^{-mT}(e_0+1), 0, \dots\}.$$

Equation (59) is directly applicable to digital computer programming, (Appendix C). The following data show the results of such a program. The data are plotted in Fig. 2.

TABLE 2. C C RC CIRCUIT WITH REVERSE-SWITCHED CAPACITOR
 $T=.1$ $E_0=0.$ $M=10$

N	I(N)	N	I(N)	N	I(N)	N	I(N)
1	1.000000	50	1.159989	100	1.114168	150	1.127316
2	.975310	51	1.131349	101	1.086660	151	1.099484
3	.951229	52	1.103417	102	1.059831	152	1.072338
4	.927744	53	1.076174	103	1.033664	153	1.045863
5	.904837	54	1.049603	104	1.008144	154	1.020042
6	.882497	55	1.023689	105	.983254	155	.994858
7	.860708	56	.998414	106	.958978	156	.970297
8	.839457	57	.973764	107	.935302	157	.946342
9	.818731	58	.949722	108	.912210	158	.922978
	.798516	59	.926274	109	.889689	159	.900191
10	1.221200	60	1.096617	110	1.132328	160	1.122118
11	1.191048	61	1.069542	111	1.104372	161	1.094414
12	1.161641	62	1.043135	112	1.077105	162	1.067393
13	1.132960	63	1.017380	113	1.050512	163	1.041040
14	1.104987	64	.992262	114	1.024576	164	1.015338
15	1.077705	65	.967764	115	.999280	165	.990271
16	1.051097	66	.943870	116	.974609	166	.965823
17	1.025145	67	.920566	117	.950546	167	.941978
18	.999834	68	.897838	118	.927079	168	.918722
19	.975149	69	.875671	119	.904190	169	.896041
20	1.048931	70	1.145976	120	1.118190	170	1.126173
21	1.023033	71	1.117683	121	1.090583	171	1.098369
22	.997774	72	1.090087	122	1.063657	172	1.071251
23	.973139	73	1.063174	123	1.037396	173	1.044803
24	.949113	74	1.036924	124	1.011784	174	1.019008
25	.925679	75	1.011323	125	.986804	175	.993850
26	.902824	76	.986354	126	.962441	176	.969314
27	.880534	77	.962002	127	.938679	177	.945383
28	.858794	78	.938250	128	.915504	178	.922043
29	.837590	79	.915086	129	.892902	179	.899279
30	1.183097	80	1.107540	130	1.129206	180	1.123021
31	1.153886	81	1.080196	131	1.101327	181	1.095295
32	1.125397	82	1.053526	132	1.074136	182	1.068253
33	1.097611	83	1.027515	133	1.047616	183	1.041879
34	1.070511	84	1.002146	134	1.021752	184	1.016156
35	1.044080	85	.977404	135	.996526	185	.991069
36	1.018302	86	.953273	136	.971923	186	.966601
37	.993161	87	.929737	137	.947927	187	.942737
38	.968640	88	.906783	138	.924524	188	.919463
39	.944725	89	.884395	139	.901699	189	.896763
40	1.078612	90	1.137480	140	1.120633	190	1.125482
41	1.051981	91	1.109396	141	1.092965	191	1.097695
42	1.026008	92	1.082005	142	1.065981	192	1.070594
43	1.000676	93	1.055291	143	1.039663	193	1.044162
44	.975970	94	1.029236	144	1.013994	194	1.018383
45	.951873	95	1.003825	145	.988960	195	.993241
46	.928372	96	.979041	146	.964544	196	.968720
47	.905451	97	.954870	147	.940731	197	.944804
48	.883096	98	.931295	148	.917505	198	.921479
49	.861293	99	.908302	149	.894854	199	.898729

TABLE 2. (CONT.)

N	I(N)	N	I(N)	N	I(N)	N	I(N)
200	1.123572	250	1.124667	300	1.124375	350	1.124481
201	1.095832	251	1.096900	301	1.096616	351	1.096720
202	1.068777	252	1.069819	302	1.069542	352	1.069643
203	1.042390	253	1.043406	303	1.043137	353	1.043236
204	1.016655	254	1.017646	304	1.017383	354	1.017480
205	.991555	255	.992523	305	.992267	355	.992362
206	.967076	256	.968020	306	.967772	356	.967865
207	.943201	257	.944122	307	.943880	357	.943971
208	.919915	258	.920815	308	.920579	358	.920668
209	.897204	259	.898082	309	.897853	359	.897941
210	1.125066	260	1.124243	310	1.124501	360	1.124449
211	1.097289	261	1.096487	311	1.096739	361	1.096689
212	1.070198	262	1.069416	312	1.069662	362	1.069613
213	1.043776	263	1.043013	313	1.043253	363	1.043206
214	1.018007	264	1.017263	314	1.017498	364	1.017452
215	.992874	265	.992149	315	.992379	365	.992335
216	.968362	266	.967656	316	.967880	366	.967838
217	.944455	267	.943767	317	.943986	367	.943946
218	.921139	268	.920468	318	.920682	368	.920643
219	.898398	269	.897744	319	.897954	369	.897916
220	1.123908	270	1.124579	320	1.124409	370	1.124480
221	1.096160	271	1.096815	321	1.096649	371	1.096719
222	1.069097	272	1.069736	322	1.069575	372	1.069643
223	1.042702	273	1.043326	323	1.043169	373	1.043235
224	1.016959	274	1.017568	324	1.017415	374	1.017480
225	.991853	275	.992447	325	.992298	375	.992362
226	.967366	276	.967946	326	.967802	376	.967865
227	.943484	277	.944050	327	.943910	377	.943972
228	.920192	278	.920744	328	.920608	378	.920669
229	.897474	279	.898014	329	.897881	379	.897942
230	1.124816	280	1.124324	330	1.124487	380	1.124463
231	1.097045	281	1.096566	331	1.096725	381	1.096702
232	1.069960	282	1.069493	332	1.069648	382	1.069626
233	1.043544	283	1.043088	333	1.043241	383	1.043219
234	1.017781	284	1.017336	334	1.017485	384	1.017464
235	.992654	285	.992221	335	.992367	385	.992347
236	.968148	286	.967726	336	.967869	386	.967850
237	.944246	287	.943836	337	.943975	387	.943957
238	.920935	288	.920535	338	.920672	388	.920655
239	.898200	289	.897810	339	.897944	389	.897928
240	1.124115	290	1.124529	340	1.124432	390	1.124483
241	1.096362	291	1.096766	341	1.096672	391	1.096721
242	1.069294	292	1.069688	342	1.069597	392	1.069645
243	1.042894	293	1.043279	343	1.043190	393	1.043238
244	1.017147	294	1.017523	344	1.017436	394	1.017483
245	.992036	295	.992403	345	.992319	395	.992365
246	.967545	296	.967904	346	.967823	396	.967868
247	.943659	297	.944009	347	.943930	397	.943975
248	.920362	298	.920704	348	.920628	398	.920672
249	.897641	299	.897975	349	.897901	399	.897945

TABLE 2. (CONT.)

N	I(N)	N	I(N)	N	I(N)	N	I(N)
400	1.124473	450	1.124498	500	1.124514	550	1.124531
401	1.096712	451	1.096736	501	1.096752	551	1.096769
402	1.069636	452	1.069660	502	1.069676	552	1.069693
403	1.043229	453	1.043253	503	1.043268	553	1.043285
404	1.017474	454	1.017498	504	1.017513	554	1.017529
405	.992357	455	.992381	505	.992396	555	.992412
406	.967860	456	.967884	506	.967899	556	.967915
407	.943967	457	.943991	507	.944006	557	.944023
408	.920665	458	.920689	508	.920704	558	.920721
409	.897938	459	.897961	509	.897977	559	.897993
410	1.124487	460	1.124499	510	1.124518	560	1.124534
411	1.096725	461	1.096738	511	1.096756	561	1.096772
412	1.069649	462	1.069662	512	1.069680	562	1.069695
413	1.043242	463	1.043254	513	1.043272	563	1.043288
414	1.017487	464	1.017499	514	1.017517	564	1.017532
415	.992369	465	.992382	515	.992400	565	.992415
416	.967872	466	.967885	516	.967903	566	.967918
417	.943979	467	.943992	517	.944010	567	.944026
418	.920677	468	.920690	518	.920708	568	.920723
419	.897949	469	.897962	519	.897981	569	.897996
420	1.124482	470	1.124504	520	1.124521	570	1.124537
421	1.096721	471	1.096743	521	1.096759	571	1.096775
422	1.069645	472	1.069667	522	1.069683	572	1.069699
423	1.043238	473	1.043259	523	1.043275	573	1.043291
424	1.017483	474	1.017504	524	1.017520	574	1.017535
425	.992366	475	.992387	525	.992403	575	.992418
426	.967869	476	.967890	526	.967906	576	.967921
427	.943976	477	.943997	527	.944013	577	.944029
428	.920674	478	.920695	528	.920711	578	.920727
429	.897946	479	.897968	529	.897984	579	.898000
430	1.124492	480	1.124507	530	1.124524	580	1.124540
431	1.096730	481	1.096745	531	1.096763	581	1.096778
432	1.069654	482	1.069669	532	1.069686	582	1.069702
433	1.043247	483	1.043262	533	1.043279	583	1.043294
434	1.017492	484	1.017506	534	1.017523	584	1.017539
435	.992374	485	.992389	535	.992406	585	.992422
436	.967877	486	.967892	536	.967909	586	.967925
437	.943985	487	.943999	537	.944017	587	.944032
438	.920682	488	.920697	538	.920715	588	.920730
439	.897955	489	.897970	539	.897987	589	.898003
440	1.124491	490	1.124511	540	1.124527	590	1.124544
441	1.096730	491	1.096749	541	1.096766	591	1.096782
442	1.069654	492	1.069673	542	1.069689	592	1.069705
443	1.043246	493	1.043266	543	1.043282	593	1.043297
444	1.017491	494	1.017511	544	1.017526	594	1.017542
445	.992374	495	.992393	545	.992409	595	.992425
446	.967877	496	.967896	546	.967912	596	.967928
447	.943984	497	.944004	547	.944019	597	.944035
448	.920682	498	.920702	548	.920717	598	.920734
449	.897954	499	.897974	549	.897990	599	.898007

TABLE 2. (CONT.)

N	I(N)	N	I(N)	N	I(N)	N	I(N)
600	1.124547	650	1.124565	700	1.124582	750	1.124600
601	1.096785	651	1.096803	701	1.096820	751	1.096837
602	1.069709	652	1.069726	702	1.069743	752	1.069760
603	1.043301	653	1.043318	703	1.043335	753	1.043352
604	1.017545	654	1.017562	704	1.017579	754	1.017596
605	.992428	655	.992446	705	.992462	755	.992479
606	.967931	656	.967949	706	.967966	756	.967983
607	.944039	657	.944057	707	.944074	757	.944091
608	.920737	658	.920755	708	.920772	758	.920789
609	.898010	659	.898028	709	.898045	759	.898062
610	1.124551	660	1.124568	710	1.124586	760	1.124603
611	1.096789	661	1.096806	711	1.096824	761	1.096841
612	1.069712	662	1.069729	712	1.069747	762	1.069764
613	1.043304	663	1.043321	713	1.043339	763	1.043355
614	1.017549	664	1.017565	714	1.017583	764	1.017599
615	.992432	665	.992449	715	.992466	765	.992483
616	.967935	666	.967952	716	.967970	766	.967986
617	.944042	667	.944060	717	.944077	767	.944094
618	.920741	668	.920758	718	.920776	768	.920793
619	.898014	669	.898031	719	.898049	769	.898066
620	1.124554	670	1.124572	720	1.124589	770	1.124607
621	1.096792	671	1.096810	721	1.096827	771	1.096844
622	1.069716	672	1.069733	722	1.069750	772	1.069767
623	1.043308	673	1.043325	723	1.043342	773	1.043359
624	1.017552	674	1.017569	724	1.017586	774	1.017603
625	.992435	675	.992452	725	.992469	775	.992486
626	.967938	676	.967956	726	.967973	776	.967990
627	.944046	677	.944063	727	.944081	777	.944098
628	.920744	678	.920761	728	.920779	778	.920796
629	.898017	679	.898035	729	.898052	779	.898070
630	1.124558	680	1.124575	730	1.124593	780	1.124610
631	1.096796	681	1.096813	731	1.096830	781	1.096848
632	1.069719	682	1.069736	732	1.069754	782	1.069771
633	1.043311	683	1.043328	733	1.043345	783	1.043362
634	1.017556	684	1.017572	734	1.017589	784	1.017606
635	.992439	685	.992455	735	.992473	785	.992490
636	.967942	686	.967959	736	.967976	786	.967993
637	.944050	687	.944067	737	.944084	787	.944102
638	.920748	688	.920765	738	.920782	788	.920800
639	.898021	689	.898038	739	.898055	789	.898073
640	1.124561	690	1.124579	740	1.124596	790	1.124614
641	1.096799	691	1.096817	741	1.096834	791	1.096851
642	1.069723	692	1.069740	742	1.069757	792	1.069774
643	1.043315	693	1.043332	743	1.043348	793	1.043366
644	1.017559	694	1.017576	744	1.017592	794	1.017609
645	.992442	695	.992459	745	.992476	795	.992493
646	.967945	696	.967962	746	.967979	796	.967997
647	.944053	697	.944070	747	.944087	797	.944105
648	.920751	698	.920768	748	.920786	798	.920803
649	.898024	699	.898042	749	.898059	799	.898076

TABLE 2. (CONT.)

N	I(N)	N	I(N)	N	I(N)	N	I(N)
800	1.124617	850	1.124635	900	1.124653	950	1.124670
801	1.096855	851	1.096873	901	1.096890	951	1.096907
802	1.069777	852	1.069795	902	1.069813	952	1.069830
803	1.043369	853	1.043386	903	1.043404	953	1.043421
804	1.017613	854	1.017630	904	1.017647	954	1.017664
805	.992496	855	.992514	905	.992531	955	.992548
806	.968000	856	.968018	906	.968035	956	.968052
807	.944108	857	.944126	907	.944143	957	.944160
808	.920806	858	.920824	908	.920842	958	.920859
809	.898080	859	.898098	909	.898116	959	.898133
810	1.124621	860	1.124639	910	1.124657	960	1.124674
811	1.096858	861	1.096876	911	1.096894	961	1.096911
812	1.069781	862	1.069799	912	1.069816	962	1.069833
813	1.043372	863	1.043390	913	1.043407	963	1.043424
814	1.017616	864	1.017634	914	1.017651	964	1.017667
815	.992500	865	.992517	915	.992535	965	.992551
816	.968003	866	.968021	916	.968038	966	.968055
817	.944111	867	.944129	917	.944147	967	.944163
818	.920810	868	.920828	918	.920845	968	.920862
819	.898083	869	.898102	919	.898119	969	.898136
820	1.124624	870	1.124642	920	1.124660	970	1.124677
821	1.096862	871	1.096880	921	1.096897	971	1.096914
822	1.069784	872	1.069802	922	1.069820	972	1.069836
823	1.043376	873	1.043393	923	1.043410	973	1.043427
824	1.017620	874	1.017637	924	1.017654	974	1.017670
825	.992503	875	.992521	925	.992538	975	.992554
826	.968007	876	.968025	926	.968042	976	.968058
827	.944115	877	.944133	927	.944150	977	.944167
828	.920814	878	.920831	928	.920849		
829	.898087	879	.898105	929	.898123		
830	1.124628	880	1.124646	930	1.124664		
831	1.096865	881	1.096883	931	1.096901		
832	1.069788	882	1.069806	932	1.069823		
833	1.043379	883	1.043397	933	1.043414		
834	1.017623	884	1.017640	934	1.017657		
835	.992507	885	.992524	935	.992541		
836	.968010	886	.968028	936	.968045		
837	.944118	887	.944136	937	.944154		
838	.920817	888	.920835	938	.920852		
839	.898090	889	.898108	939	.898126		
840	1.124631	890	1.124649	940	1.124667		
841	1.096869	891	1.096887	941	1.096904		
842	1.069792	892	1.069809	942	1.069826		
843	1.043383	893	1.043400	943	1.043417		
844	1.017627	894	1.017644	944	1.017661		
845	.992510	895	.992528	945	.992544		
846	.968014	896	.968031	946	.968048		
847	.944122	897	.944140	947	.944157		
848	.920821	898	.920838	948	.920855		
849	.898094	899	.898112	949	.898129		

END OF PROGRAM AT STATEMENT 0014 + 00 LINES

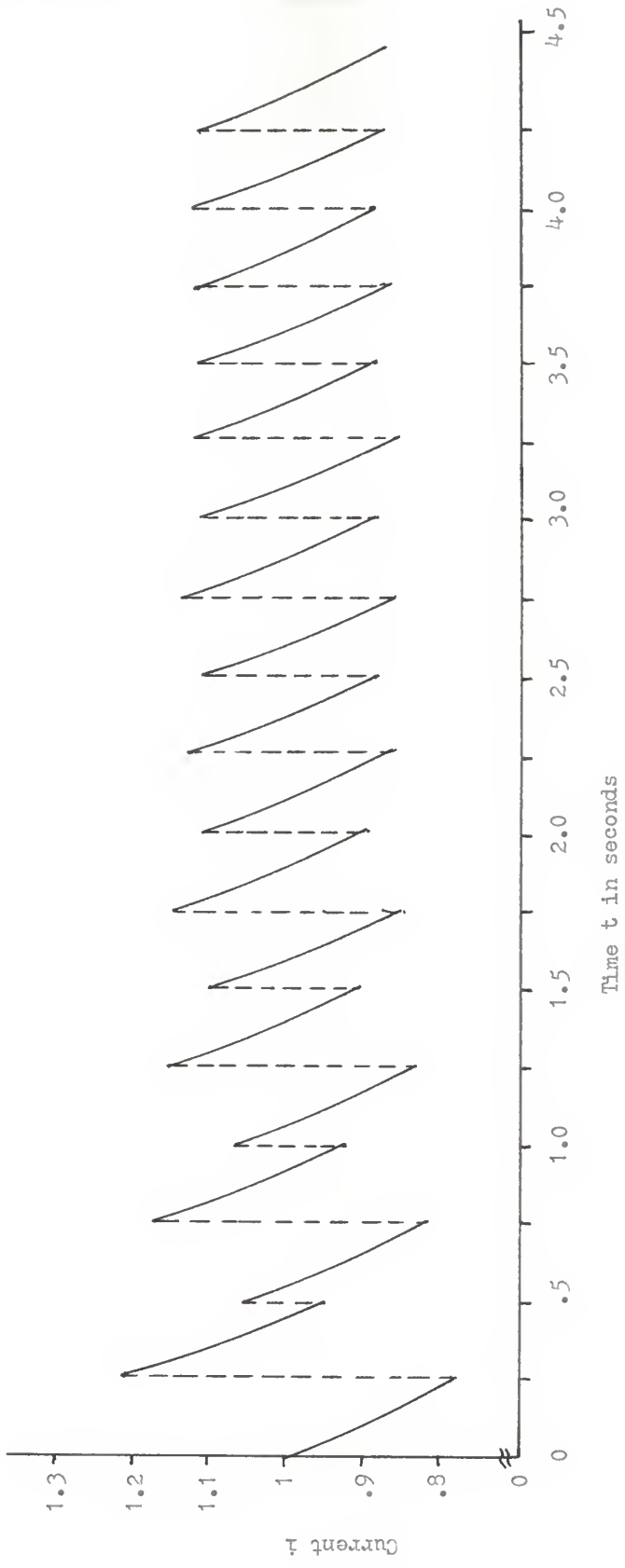


Fig. 2. Current in reverse-switched RC circuit with unit step input.

Limit Cycle of a Series RC Circuit

Results from the preceding section show that the output (current) takes the form of an ultimately periodic function. Looking at equation (57) it is seen that for zero initial conditions the second term contributes the ultimate periodicity. If initial conditions were present the term containing them would also enter into the ultimate periodicity, since the denominator is the same as the second term. The following methods could include this initial condition term if so desired. The term $1-z^m$ in the denominator is the only term that does not damp out.

The notation for a periodic sequence of period m is

$$\left\{ \overline{g_0, g_1, g_2, \dots, g_{m-1}} \right\}, \quad (60)$$

To gain insight to the problem the ultimately periodic sequence of f and g is investigated. The ultimately periodic sequence

$$\left\{ f_0, f_1, f_2, \dots, f_m, \overline{g_0, g_1, g_2, \dots, g_{m-1}} \right\} \quad (61)$$

may be expressed in Z-transform notation as shown below, where any of the g 's or f 's may be zero.

$$F(s) = f_0 + f_1 z + f_2 z^2 + \dots + f_m z^m + \frac{z^m (g_0 + g_1 z + g_2 z^2 + \dots + g_{m-1} z^{m-1})}{1-z^m} \quad (62)$$

This is expressed

$$\bar{F}(z) = \bar{f}(z) + \frac{z^m \bar{g}(z)}{1-z^m}, \quad (63)$$

where the degree of $g(z) < m$. Equation (57) may now be expressed in the form

$$\frac{z^m(1-e^{-mT})}{(1-ze^{-T})(1+z^me^{-mT})(1-z^m)} = \frac{A(z)}{(1-ze^{-T})(1+z^me^{-mT})} + \frac{B(z)}{1-z^m} \quad (64)$$

$B(z)$ must be of degree at least one less than its denominator. It is represented as follows:

$$B(z) = b_0 + b_1z + b_2z^2 + \dots + b_{m-1}z^{m-1} \quad (65)$$

The problem is solved whenever the b 's are found and this task is undertaken in the next section.

Determination of $B(z)$

A procedure will now be given for finding the coefficients of $B(z)$ in the limit cycle portion of the solution. Expressing equation (63) in the general form

$$\frac{P(z)}{(1-z^m)Q(z)} = \frac{A(z)}{Q(z)} + \frac{B(z)}{1-z^m} \quad (66)$$

and multiplying through by $(1-z^m)Q(z)$ yields

$$P(z) = (1-z^m)A(z) + Q(z)B(z) \quad (67)$$

The term $(1-z^m)A(z)$ will be zero for $z^m = 1$. The required m -th root of unity is

$$z_k = e^{j2\pi k/m} \text{ where } k = 1, 2, 3, \dots, m,$$

and equation (67) becomes,

$$P(z_k) = 0 + Q(z_k)B(z_k) \quad (68)$$

Dividing through we may now write

$$\frac{P(z_k)}{Q(z_k)} \equiv \beta_k = b_0 + b_1z_k + b_2(z_k)^2 + \dots + b_{m-1}(z_k)^{m-1} \quad (69)$$

For all k this result may be expressed as a matrix equation involving a transposed, with the elements unconjugated¹, Vandermonde matrix.

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 & \dots & z_1^{m-1} \\ 1 & z_2 & z_2^2 & z_2^3 & \dots & z_2^{m-1} \\ 1 & z_3 & z_3^2 & z_3^3 & \dots & z_3^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_m & z_m^2 & z_m^3 & \dots & z_m^{m-1} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{m-1} \end{pmatrix} \quad (70)$$

This matrix has an inverse due to the fact that

$$\sum_{k=0}^m z_k = 0. \quad (71)$$

Denoting the matrix by V the inverse may be written

$$V^{-1} = \frac{1}{m} V' \quad (72)$$

where V' denotes the transpose and involves the complex conjugation operation also. Some helpful simplification procedures can be derived from the definition of z_k . First note that

$$z_k = e^{j2\pi k/m} \quad (73)$$

therefore
$$z_k^n = e^{j2\pi k^n/m} \quad (74)$$

or
$$z_k^n = z_{kn}. \quad (75)$$

¹When this paper speaks of the transpose operation it will imply conjugation of the matrix elements also unless otherwise specified.

The kn obtained from equation (75) may contain numbers larger than m . Since there are only m components in the vector field the kn is computed modulo m . The subscript, therefore, always ranges from 0 to m .

By conjugating equation (73) we arrive at

$$z_k^* = e^{-j2\pi k/m} = 1 \cdot e^{-j2\pi k/m} \quad (76)$$

$$= e^{j2\pi m/m} \cdot e^{-j2\pi k/5} \quad (77)$$

$$= e^{j2\pi(m-k)/5} \quad (78)$$

or
$$z_k^* = z_{m-k} \quad (79)$$

Therefore, if z_k is a root then z_{m-k} is its complex conjugate. Thus the required complex conjugate pairs of roots exist in the given set of roots.

These operations can all be carried out at once to obtain the inverse of v . The inverse may now be used to solve equation (70).

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_{m-2} \\ b_{m-1} \end{pmatrix} = (1/m) \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ z_{m-1} & z_{m-2} & z_{m-3} & \dots & 1 \\ z_{m-2} & z_{m-4} & z_{m-6} & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_2 & z_4 & z_6 & \dots & 1 \\ z_1 & z_2 & z_3 & \dots & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \vdots \\ \beta_{m-1} \\ \beta_m \end{pmatrix} \quad (80)$$

The b 's may now be found by substituting in the β 's as obtained from

equation (69), and the limit cycle is

$$\frac{B(z)}{1-z^m} = \frac{b_0 + b_1 z + b_2 z^2 + \dots + b_{m-1} z^{m-1}}{1-z^m} \quad (81)$$

THE PERIODIC QUANTIZER $q(t)$

Rademacher Functions

The Rademacher functions are defined by

$$r_n(t) \equiv \text{sgn} [\sin(2^n \pi(t/a))] \quad (82)$$

where

$$\text{sgn } f(t) \equiv \begin{cases} +1 & \text{if } f(t) > 0 \\ 0 & \text{if } f(t) = 0 \\ -1 & \text{if } f(t) < 0 \end{cases}.$$



and so on up to $r_n(t)$.

Expansion of $q(t) = t$, $0 < t < 1$, by Rademacher Functions

The function $q(t)$ is called the periodic quantizing functions. $q(t) = t$ may be expanded on the interval, $0 < t < 1$, by means of the Rademacher functions.

For other applications of Rademacher functions the reader is referred to Kac (3).

$$q(t) = (1/2)r_1(t) - (1/4)r_2(t) - (1/8)r_3(t) - (1/16)r_4(t) \dots (1/2^n)r_n(t) \quad (83)$$

The derivation of this formula is now given.

For $n \geq 2$ the coefficients of $r_n(t)$ are $-1/2^n$.

$$C_n = \sum_{k=0}^{2^{n-1}-1} (-1)^k \int_{k/2^{n-1}}^{(k+1)/2^{n-1}} t dt = \sum_{k=0}^{2^{n-1}-1} \left((-1)^k t^2 / 2 \right) \Big|_{k/2^{n-1}}^{(k+1)/2^{n-1}} \quad (84)$$

$$= \sum_{k=0}^{2^{n-1}-1} \left((-1)^k / 2 \right) \left[\frac{(k+1)^2}{2^{2n-2}} - \frac{k^2}{2^{2n-2}} \right] \quad (85)$$

$$= (1/2^{n-1}) \sum_{k=1}^{2^{n-1}-1} (-1)^{k(1+2k)} = (1/2^{n-1}) \left[0 + 2 \sum_{k=0}^{2^{n-1}-1} (-1)^k k \right] \quad (86)$$

$$C_n = \frac{4}{2^{2n}} (-1) \frac{2^{n-1}}{2} = -1/2^n \quad (87)$$

For the case $n=1$, $C_1 = \int_0^1 t dt = 1/2$.

Physical Realization of $q(t)$ by Operational Amplifiers

It is desirable to construct the Rademacher functions $r_n(t)$ from a combination of switched integrator networks. Choosing $a=1$, $r_1(t)$ may be obtained by combining the following two functions:

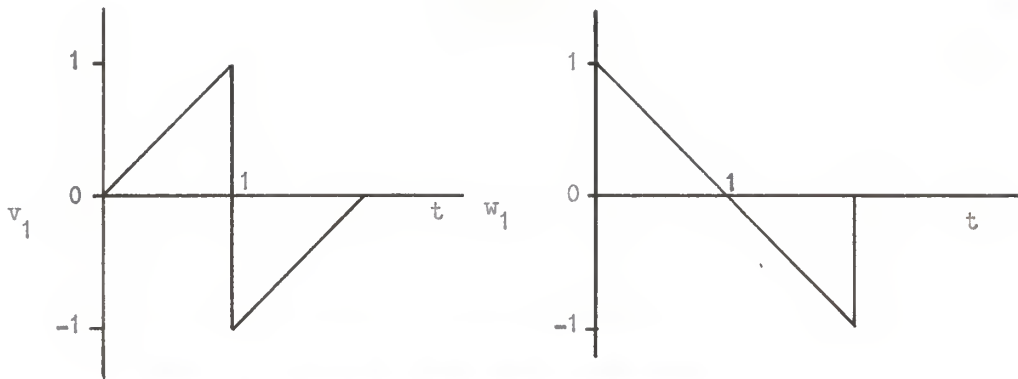


Fig. 3. Outputs of switched integrators.

These are produced by two switched integrator networks where one is started with an initial condition of 1 and the other 0. The one with initial conditions of 1 is reversed and both summed as shown in Fig. 4, to obtain $r_1(t)$.

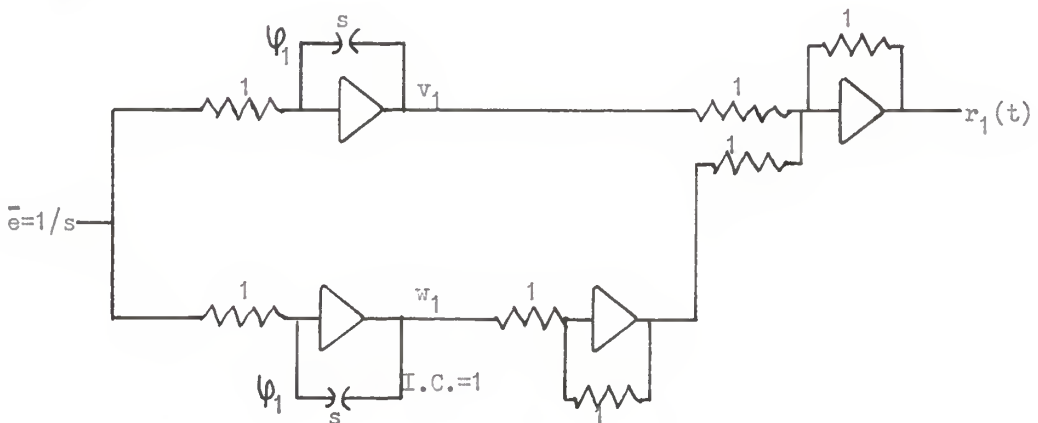


Fig. 4. Integrator network for $r_1(t)$.

Other $r_k(t)$ can be found in the same manner by replacing C with $C/2^{k-1}$ and φ_1 with $\varphi_{2^{k-1}}$; I.C. and \bar{e} remain the same. This gives a new capacitance value of $1/2^{k-1}$ farad.

Definition: φ_1 implies Z_m , φ_k implies $Z_{m/k}$ (88)

or m replaced by m/k . It is possible to normalize the Z_m to Z_{km} and $Z_{m/k}$ to Z_m after which the outputs must be normalized to 1. It seems best, however, to proceed using the first case.

The function $q(t)$ can be constructed by taking the $r_k(t)$ obtained from each of the $v_k + w_k$ and dividing by 2^k .

Replacements for Reduced Output

Reducing the output by 2^k , implies C being replaced with $2^k C$ and v_0 with $v_0/2^k$ in the integrator network.

$$\bar{e} + Cs\bar{v} = +2CZ_m\bar{v} - Cv_0$$

$$\bar{v} = -(\bar{e}/Cs) + (2/s)Z_m\bar{v} - v_0/s \quad (89)$$

$$\bar{v}/2^k = -(\bar{e}/C2^k s) + (2/2^k s)Z_m\bar{v} - (v_0/2^k s) \quad (90)$$

$$\bar{v}/2^k = -(\bar{e}/C2^k s) + (2/s)Z_m(\bar{v}/2^k) + (v_0/2^k s) \quad (91)$$

A comparison of equations (89) and (91) shows that the new network will have a new capacitance of $2^k C$ and new initial condition of $v_0/2^k$.

Performing the operations to get $r_k(t)$ and then the ones to get $r_k(t)/2^k$ we have C replaced with $C = C2^k/2^{k-1} = 2C$, v_0 with $v_0/2^k$ and φ_1 with $\varphi_{2^{k-1}}$. The new network can now correspond to any $r_k(t)/2^k$ when these substitutions are made as shown in Fig. 5.

To construct the negative we need only reverse the positions and initial conditions of the two integrator networks from that above as shown in Fig. 6. This is the same as changing the initial conditions from the bottom network to the top one and reversing them.

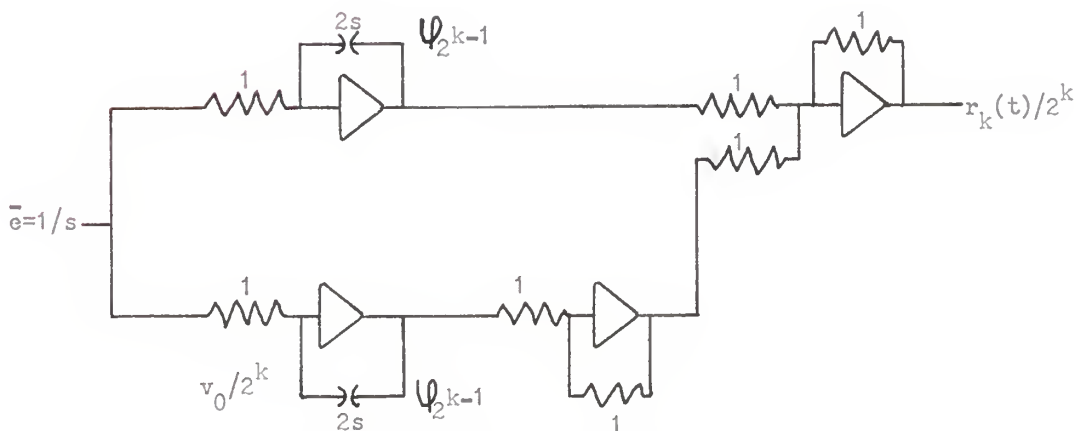


Fig. 5. Integrator network for $r_k(t)/2^k$, $v_0 = 1$.

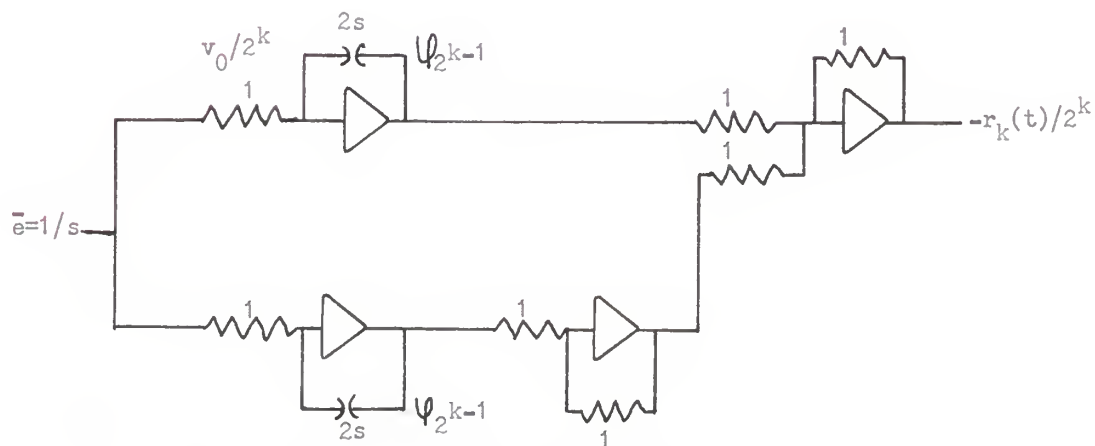


Fig. 6. Integrator network for $r_k(t)/2^k$, $v_0 = 1$.

We may construct $q(t)$ using the $r_1/2$ and $-r_k/2^k$ obtained, part of which is shown in Fig. 7.

Parallel Switched Integrators Equivalence

Two integrators in parallel switched at different rates may be replaced with a single network using only one amplifier. The two capacitors are placed across the amplifier and switched at their respective rates.

$$i_1 = C_1 \, de_1/dt \quad (92)$$

$$i_2 = C_2 \, de_2/dt \quad (93)$$

$$\bar{i}_1 = C_1 s \bar{e}_1 - 2Z_m \bar{e}_1 + e_1(0) \quad (94)$$

$$\bar{i}_2 = C_2 s \bar{e}_2 - 2Z_m \bar{e}_2 + e_2(0) \quad (95)$$

$$\bar{e}_1 = \bar{e}_2 = \bar{e} \quad (96)$$

$$i = i_1 + i_2 = (C_1 + C_2) s \bar{e} + 2(Z_m + Z_{2m}) \bar{e} - e(0) + e_2(0) \quad (97)$$

$$= (C_1 + C_2) s \bar{e} + 2(Z_m + Z_{2m}) \bar{e} - \bar{e}(0) \quad (98)$$

This can be generalized to any number of integrators,

$$i = (C_1 + C_2 + \dots + C_n) s \bar{e} + 2(Z_m + Z_{2m} + \dots + Z_{nm}) \bar{e} - \bar{e}(0), \quad (99)$$

where $e(0) = e_1(0) + e_2(0) + \dots + e_n(0)$,

which is the sum of the initial conditions of all the capacitors.

The $q(t)$ network can be simplified by using the parallel network equivalence just shown. Only one amplifier is then needed for each side of the network as shown in Fig. 8.

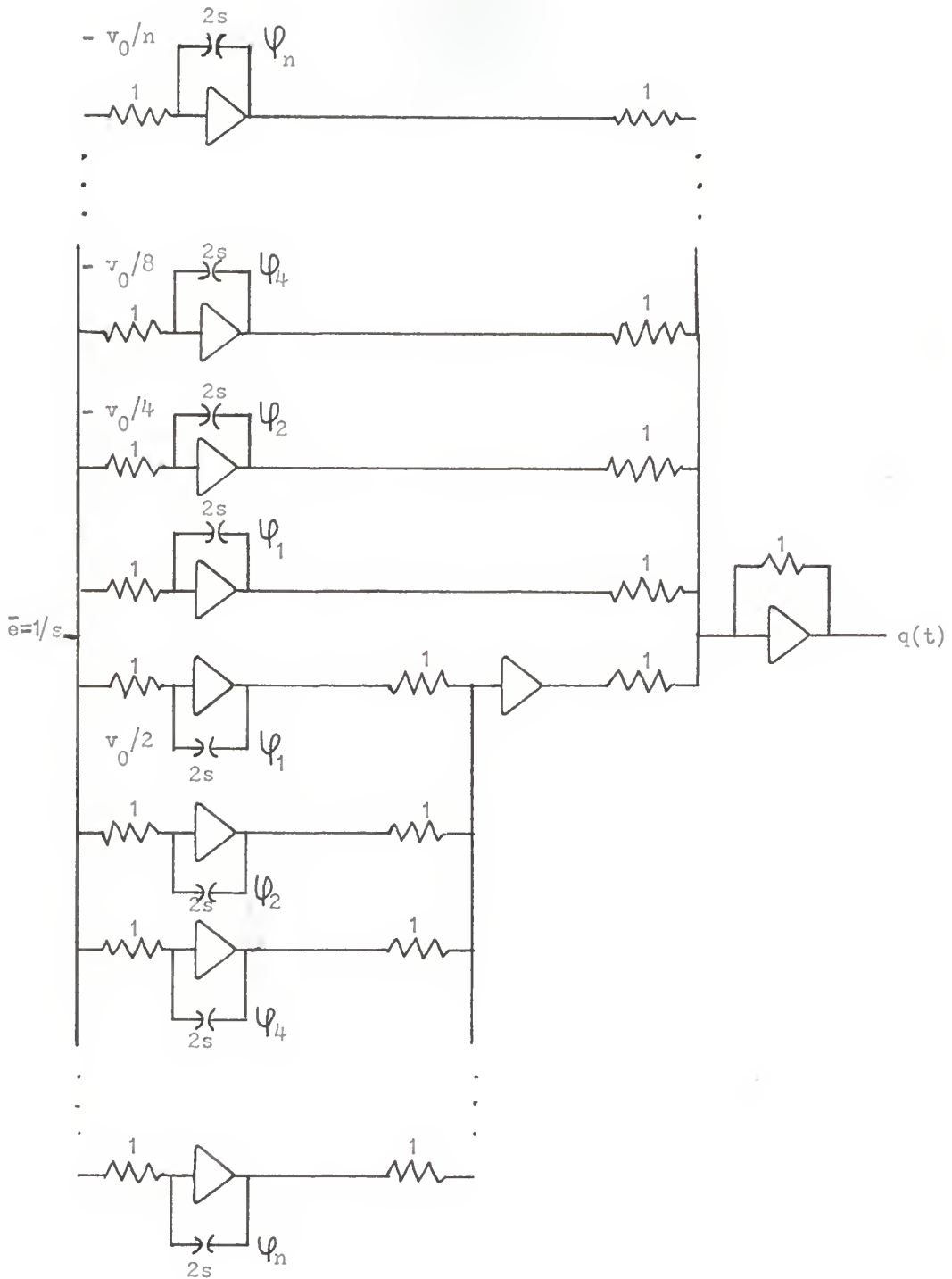


Fig. 7. Integrator network for $q(t)$, $v_0 = 1$.

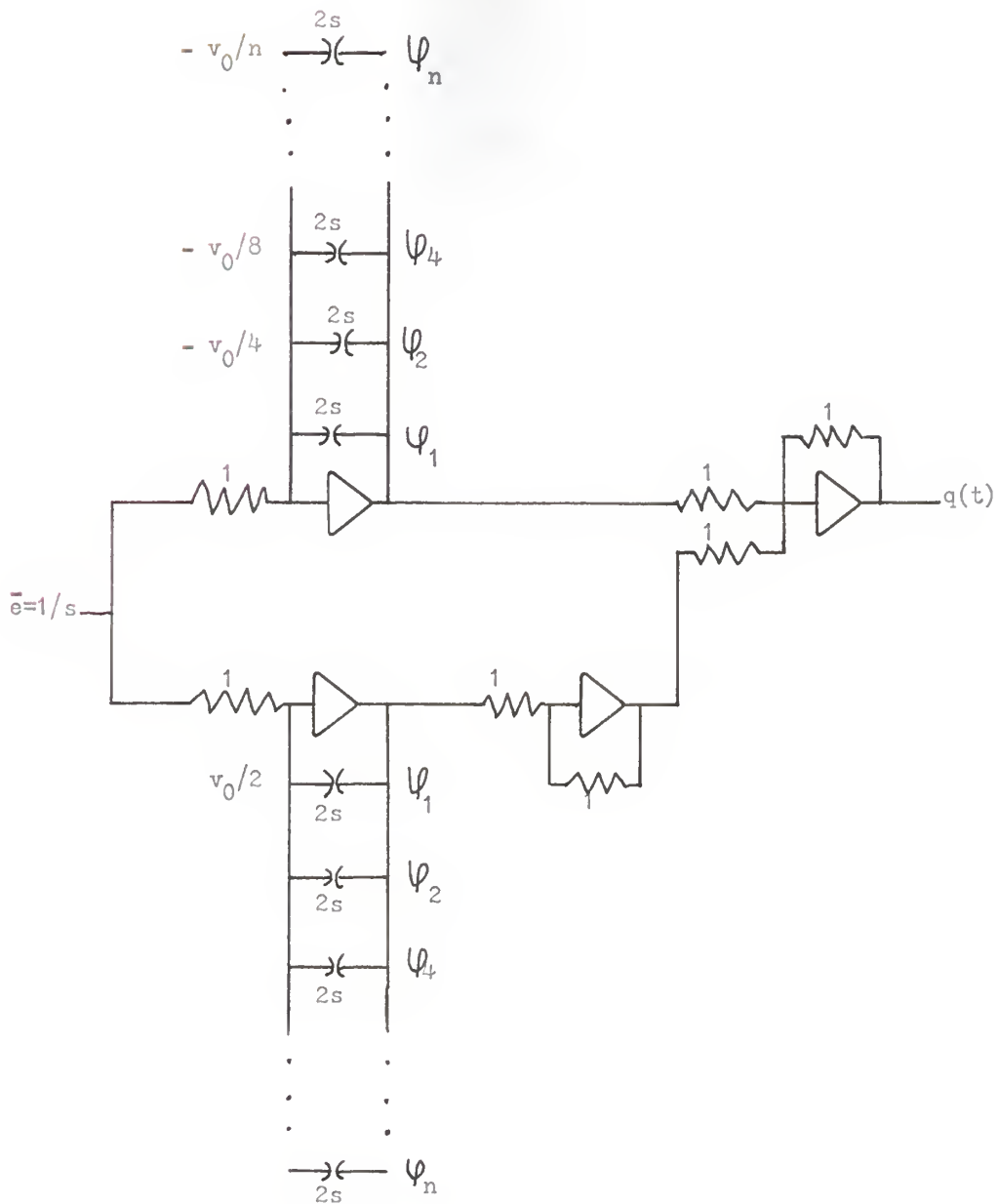


Fig. 8. Integrator network for $q(t)$, $v_0 = 1$.

Calculation of $q(t)$ from the Network

The total network output for the $(1/2)r_1(t)$ is the difference between the output from the integrator with initial conditions and the one without.

$$(1/2)\bar{r}_1 = \bar{v}_1 - \bar{w}_1 \quad (100)$$

$$\bar{v}_1 = -\frac{\bar{e}}{Cs} + \frac{2(1-z^n)}{s(1+z^n)} \frac{\bar{e}}{nCs} + \frac{v_0}{1+z^n} - \frac{v_0}{s} \quad (101)$$

$$\bar{w}_1 = -\frac{\bar{e}}{Cs} + \frac{2(1-z^n)}{s(1+z^n)} \frac{\bar{e}}{nCs} \quad (102)$$

$$C = 2, v_0 = 1/2 \quad (103)$$

$$(1/2)\bar{r}_1 = (1/2s) \frac{1-z^n}{1+z^n} \quad (104)$$

The same may be done for the other $\bar{r}_k/2^k$ with $C = 2$, $v_0 = 1/2^k$, and $n = n/2^{k-1}$. This gives

$$\frac{1}{2^k} \bar{r}_k = \frac{1}{2^k s} \left(\frac{1-z^{n/2^{k-1}}}{1+z^{n/2^{k-1}}} \right) \quad (105)$$

We may now compute \bar{q} by substituting into equation (87).

$$\bar{q} = \frac{1}{2s} \left(\frac{1-z^n}{1+z^n} \right) - \frac{1}{s} \left[\frac{1}{4} \left(\frac{1-z^{n/2}}{1+z^{n/2}} \right) + \frac{1}{8} \left(\frac{1-z^{n/4}}{1+z^{n/4}} \right) + \dots + \frac{1}{n} \left(\frac{1-z^{2n/n}}{1+z^{2n/n}} \right) \right] \quad (106)$$

A reduction is now made on the expression for \bar{q} . Substituting $z = e^{-snT}$ then $snT = u$ into equation (106) results in

$$s\bar{q} = \frac{1}{2} \left(\frac{1-e^{-u}}{1+e^{-u}} \right) - \left[\frac{1}{4} \left(\frac{1-e^{-u/2}}{1+e^{-u/2}} \right) + \frac{1}{8} \left(\frac{1-e^{-u/4}}{1+e^{-u/4}} \right) + \dots + \frac{1}{n} \left(\frac{1-e^{-2u/n}}{1+e^{-2u/n}} \right) \right] \quad (107)$$

Making the substitution $(1-e^{-u})/(1+e^{-u}) = \tanh(u/2)$, and manipulating to get a tanh series yields

$$\bar{s}_q = (1/2)\tanh(u/2) - [\tanh(u) + (1/2)\tanh(u/2) + (1/4)\tanh(u/4) + (1/8)\tanh(u/8) + \dots] + \tanh(u) + (1/2)\tanh(u/2), \quad (108)$$

which after summing becomes

$$\bar{s}_q = \tanh(u) + \tanh(u/2) - \sum_{k=0}^n \frac{1}{2^k} \tanh(u/2^k). \quad (109)$$

The tanh series may be summed and the reader is referred to Appendix A for a proof. Making the substitution for the tanh series and collecting terms yields

$$\bar{s}_q = \coth(u) - \coth(u/2) + (1/2^n)\coth(u/2^n). \quad (110)$$

Now $\coth(a) - \coth(b) = -\sinh(a-b)/(\sinh(a)\sinh(b))$. Making this substitution into equation (110), substituting $u = smT$, and reducing gives

$$\bar{q} = \frac{1}{s} [2^{-n} \coth(smT2^{-n}) - \csch(smT)]. \quad (111)$$

SUMMARY

It was desirable in the investigation of a new idea to establish a firm basis for further investigation rather than to come up with extensive results. Some applications of this basic research, however, are applied and the results presented.

It is seen that, although the physical construction of the reverse-switched capacitor is rather simple, a somewhat complex mathematical description evolves in the analysis, but the Z-transform simplifies this description. The results of a Z-transform analysis are exact at the sampling points; however, no information is provided between sampling points.

The results from the switched integrator can be seen to be an inverted type of sawtooth waveform. Whenever the switching period, sampling period, or initial conditions are varied the results still take the form of an ordinary integrator with the output displaced by some amount during the switching period as in the particular case given.

The results, from the application of a reverse-switched capacitor to an RC circuit, show that the current stays around the unity level whereas in an ordinary RC circuit the current will eventually become zero when a unit step voltage is applied. A transient is evident which diminishes to zero in the limit cycle. The limit cycle can be readily calculated for a particular circuit.

The orthogonal Rademacher functions describe the periodic quantizer $q(t)$ and physical realizations of $q(t)$ are readily available because of this.

Using this thesis as a basis, a great number of extensions could be made of the reverse-switched capacitor to other RC, RL and RLC networks with the object of a practical application.

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APPENDICES

Appendix A

The hyperbolic tangent series sums to two terms consisting of the hyperbolic cotangent. Using two identities for hyperbolic functions the procedure is as follows:

$$\frac{\cosh^2(x) + \sinh^2(x)}{2\sinh(x)\cosh(x)} = \frac{\cosh(2x)}{\sinh(2x)} = \coth(2x) \quad (112)$$

$$\coth(x) + \tanh(x) = 2 \coth(2x) \quad (113)$$

$$\tanh(x) = -\coth(x) + 2 \coth(2x) \quad (114)$$

$$\tanh(x/2) = -\coth(x/2) + 2 \coth(x) \quad (115)$$

$$(1/2)\tanh(x/2) = -(1/2)\coth(x/2) + \coth(x) \quad (116)$$

Equations (114) + (116) yields

$$\tanh(x) + (1/2)\tanh(x/2) = -(1/2)\coth(x/2) + 2 \coth(2x). \quad (117)$$

$$\tanh(x/4) = -\coth(x/4) + 2 \coth(x/2) \quad (118)$$

$$(1/4)\tanh(x/4) = -(1/4)\coth(x/4) + (1/2)\coth(x/2) \quad (119)$$

Equations (117) and (119) yields

$$\begin{aligned} \tanh(x) + (1/2)\tanh(x/2) + (1/4)\tanh(x/4) &= -(1/4)\coth(x/4) \\ &+ 2 \coth(2x). \end{aligned} \quad (120)$$

An induction proof begins.

$$\text{Assume } \sum_{k=0}^n (1/2^k)\tanh(x/2^k) = -(1/2^n)\coth(x/2^n) + 2 \coth(2x) \quad (121)$$

$$\tanh(x/2^{n+1}) = -\coth(x/2^{n+1}) + 2 \coth(x/2^n) \quad (122)$$

$$(1/2^{n+1})\tanh(x/2^{n+1}) = -(1/2^{n+1})\coth(x/2^{n+1}) + (1/2^n)\coth(x/2^n) \quad (123)$$

Equations (121) and (123) yield

$$\sum_{k=0}^{n+1} (1/2^k)\tanh(x/2^k) = -(1/2^{n+1})\coth(x/2^{n+1}) + 2 \coth(2x) \quad (124)$$

End of induction proof!

Appendix B

```
C C INTEGRATOR USING A REVERSE-SWITCHED CAPACITOR
C SAMPLING TIME T, SWITCHING TIME MT, AND INITIAL CONDITIONS E0
  DIMENSION X(50), V(200)
  READ, T, V0
  PUNCH 5
5  FORMAT (//3(2X1HN,5X,4HV(N),5X)//)
  DO 1 N=1,22
  X(N+22)=0.
1  V(N)=0.
  X(22)=V0
  X(23)=-(V0+T)
  X(32)=2.*(10.*T-V0)
  X(33)=2.*(V0-10.*T)
  X(42)=V0
  X(43)=T-V0
  DO 6 N=23, 150
  V(N)=2.*V(N-1)-V(N-2)+V(N-20)-2.*V(N-21)+V(N-22)
  IF(N-44) 4,4,6
4  V(N)=V(N)+X(N-1)
6  CONTINUE
  DO 3 I=23, 63
  J=I-23
  K=I+18
  L=I+58
3  PUNCH 7,J,V(I),K,V(I+41),L,V(I+81)
7  FORMAT (3(I4,2X,F7.5,4X))
  END
```



```

C C RC CIRCUIT WITH REVERSE-SWITCHED CAPACITOR
C SAMPLING TIME T, SWITCHING TIME MT, AND INITIAL CONDITIONS E0
  DIMENSION X(50), A(999)
  READ, T, EC
  PUNCH 2
2  FORMAT(//4(2X1HN,5X,4HI(N),5X,)//)
  DC 4 I = 1, 21
  X(I+20) = 0.
4  A(I) = 0.
  C1 = EXP(-T)
  C2 = EXP(-10.*T)
  C3 = EXP(-11.*T)
  C4 = 1.-C2
  X(21) = 1.-E0
  X(31) = C4+E0*(1.+C2)
  X(41) = (-C2*(E0+1.))
  DC 6 N = 22, 999
  A(N)=C1*A(N-1)+C4*A(N-10)-(C1-C3)*A(N-11)+C2*A(N-20)-C3*A(N-21)
  IF (N-42) 6, 6, 5
6  A(N) = A(N) + X(N-1)
  DC 8 I = 22, 71
  J = I - 22
  K = I + 28
  L = I + 78
  M = I + 128
8  PUNCH 9,J,A(I),K,A(I+50),L,A(I+100),M,A(I+150)
9  FORMAT (4(I4, 2X, F8.6, 3X))
  DC 10 I = 22, 71
  J = I + 178
  K = I + 228
  L = I + 278
  M = I + 328
10 PUNCH 9,J,A(I+200),K,A(I+250),L,A(I+300),M,A(I+350)
  DC 11 I = 22, 71
  J = I + 378
  K = I + 428
  L = I + 478
  M = I + 528
11 PUNCH 9,J,A(I+400),K,A(I+450),L,A(I+500),M,A(I+550)
  DC 12 I = 22, 71
  J = I + 578
  K = I + 628
  L = I + 678
  M = I + 728
12 PUNCH 9,J,A(I+600),K,A(I+650),L,A(I+700),M,A(I+750)
  DC 15 I = 22, 71
  J = I + 778
  K = I + 828
  L = I + 878
  M = I + 928
  IF (I-49) 13,13,14
13 PUNCH 9,J,A(I+800),K,A(I+850),L,A(I+900),M,A(I+950)
  GO TO 15
14 PUNCH 9,J,A(I+800),K,A(I+850),L,A(I+900)
15 CONTINUE
  END

```

PERIODICALLY REVERSE-SWITCHED CAPACITORS

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When a capacitor is periodically reversed in a circuit, the final voltage on the capacitor acts as an initial condition for the next period. The analysis of the device could be carried out in this piecewise linear fashion but would be slow and tedious. The device may be constructed, using a double-pole double-throw switch that is periodically operated, to reverse the capacitor in the circuit.

The Z-transform analysis effects easy and straightforward solutions to problems involving the reverse-switched capacitor. These give recurrence relations which adapt easily to digital computer solutions.

The operational amplifier integrator provides a simple network that is the starting point for analysis. The developments are used in obtaining two important equations, the loop current and node pair voltage equations. These may be used for application to any other network by proper expression of the network equations in the Laplace transform domain. The solution can then be carried on in the Z-transform domain in much the same manner as in sampled-data feedback systems.

A simple application of the loop current equation is to the series RC circuit. With a step input of voltage the current, which is called the output, may be solved. After the solution, it is seen that there is a limit cycle added to a transient. The solution for the limit cycle was investigated. This solution can be accomplished with a knowledge of complex conjugate root pairs and Vandermonde matrices.

The periodic quantizer $q(t) = t$ is a special function evolved from a network of integrators which have outputs $r_k(t)/2^k$, where $r_k(t)$ is a Rademacher function. The integrator network is a construction developed from the set of $r_k(t)$. There are replacements for elements in the integrator

to obtain the reduced output $r_k(t)/2^k$. A set of parallel switched integrators may be replaced by one integrator with the switched capacitors in parallel. The tanh series involved in calculating $q(t)$ is summable to a compact form in terms of the coth and csch functions.

This thesis describes the reverse-switched capacitor and provides clues for new applications.