CONVERGENCE OF SOME STOCHASTIC MATRICES

by

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B. S., Kansas State University, 1961

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

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INTRODUCTION

In recent years, considerable use has been made of the theory involving steady-state Markov chains. Procedures have been developed to determine if a given stochastic process is of the type that will reach a steady-state i.e., the state which is independent of the initial condition. At the present time there is a definite lack of methods which give the rate at which a stochastic process approaches its steady-state.

Consider a finite Markov chain with states E_i $(i=1,2,\ldots,n)$ and the transition probability matrix $A=\left[p_{ij}\right]$ $i, j=1,2,\ldots,n$. Let the probability vector at time t be,

$$P(t) = (P_i(t)) \tag{1}$$

so that $P_i(t)$ is the probability that the process, defined by the above Markov chain, is in state E_i at time t. The matrix A and the vector (1) are related by,

$$P'(t+1) = P'(t)\, A. \tag{2}$$

We note that from (2)

$$P'(t) = P'(o)A^t. \tag{3}$$

Let us assume that the matrix A is such that the process reaches a stationary state so that,

$$\lim_{t\to\infty} P(t) = \pi = (\pi_i), \tag{4}$$

where π denotes the vector of stationary probabilities (Gantmacher, 1959).

Regular, finite Markov chains arise in theory of storage problems. Moran (1954) developed the stochastic matrix which characterizes a finite dam and found the stationary distribution for several cases. Gani defined the analogous problem for an inventory system (1955) and a queue system (1957). He recommended numerical methods for finding stationary probabilities. Prabhu (1958) contributed stationary probabilities for Moran's dam problem when capacity, X, is an integer multiple of output, M, and input is (i) geometric, (ii) negative binomial, (iii) Poisson. Chaddha (1960) generalized on previous work and defined M-policy as follows; a quanity M is added to an inventory of capacity X at regular time intervals, t, t+1,....., except when the content is greater than X-M, in which case the inventory is filled to capacity.

In previous work, the problem of determining rates of convergence to stationarity is not considered. Gani (1955) suggests that a method of escaping this problem is to "let the system run for awhile to overcome the initial effects." Chaddha (1960) points out that the rate of convergence is dependent on the characteristic roots (excluding unity) of the stochastic matrix.

In this thesis, numerical techniques will be used to find the stationary distributions of stochastic matrices for an M-policy where X and M take various different values. The demand distributions considered will be geometric and Poisson. The second largest characteristic root will be found. A method to predict the number of time periods the system must pass through to give a reliable estimate of stationarity using the second largest root will be presented. Finally application of the technique will be illustrated by numerical examples.

REGULAR MARKOV CHAINS AND STOCHASTIC MATRICES

Many of the results in the following sections depend upon the basic properties of Markov chains and stochastic matrices. Some results in the theory of Markov chains which will be of later interest are stated now, while keeping in mind that an exhaustive discussion is not intended. Feller (1950), Gantmachr (1959), and Kemeny and Snell (1960) treat the subject in greater detail.

A regular Markov chain has a transition matrix which is identifiable by the following property: A transition matrix, A, is regular if, and only if, for some t, A^t has no zero entries. (Kemeny and Snell 1960) The system of stochastic matrices which will be dealt with in the later sections of this thesis, will be seen to be of this regular type. This property also implies that it is possible for any state, E_j, to be reached from any other state, E_i, in t, time intervals.

The following theorem from Kemeny and Snell (1960) will be of interest in later sections.

THEOREM I. If A is a regular transition matrix then:

(i) The powers A^t approach a stochastic matrix A*.

(ii) Each row of A* is the same probability vector π'.

(iii) The components of π' are positive.

(iv) For any initial probability vector $P(o)'$, $P(o)' \cdot A^t$ approaches the vector π' as t tends to infinity.

(v) The vector π' is the unique probability vector such that $A'\pi = \pi$.

(vi) $A A^* = A^* A = A^*$.

The matrix A^* and the vector π, of equation (4), are referred to as the limiting matrix and the stationary probability vector for the Markov chain determined by A. Now, the equation (3) states that if the process is started in such a way that the initial states have a probability distribution P(o), then the probability distribution for the states after time t, is given by $P'(t) = P'(o)A^t$. Since the above theorem states that $\lim_{t \to \infty} P'(o)A^t = \pi'$ exists, and since π is dependent only on A we see that P(t) is approximately independent of the initial distribution, P(o), for a sufficiently large t.

A matrix with non-negative elements is defined as primitive if its largest characteristic root, λ_1, is real and positive, such that the inequality, $\lambda_1 > |\lambda_i|$, $(i=2,3,\ldots\ldots,n)$ holds. Gantmacher (1959), p. 80, 81 proves:

> THEOREM II. A matrix with non-negitive elements is primitive if, and only if, some power of the matrix has no zero elements.

It is at once apparent that this condition is fulfilled by any given regular transition matrix $A=\left[p_{ij}\right]$, since $p_{ij} \geqslant o$ for all i, j. In fact, the very property ($A^t > 0$; finite t) that insures the matrix A is regular, also insures that A is primitive, implying that the largest characteristic root of a regular transition matrix is positive and simple.

Grantmacher 1959, p. 63 states that for a primitive matrix with largest characteristic root λ_1, the following inequality holds

$$s \leq \lambda_1 \leq S, \tag{5}$$

where

$$s = \min_i \left(\sum_{j=1}^{n} p_{ij} \right)$$

$$S = \max_i \left(\sum_{j=1}^{n} p_{ij} \right) (j=1,2,\ldots\ldots,n). \tag{6}$$

But since we are dealing with a stochastic matrix which has every row sum equal to unity, the value of the largest characteristic root, λ_1, is unity.

Now, having shown that the value of λ_1 is unity, this fact may be used to determine the characteristic vector, V_1, corresponding to λ_1. Note that the vector equation

$$A\ V_i = \lambda_i\ V_i \quad , \tag{7}$$

for $i = 1$ reduces to

$$A\ V_1 = V_1 \tag{8}$$

and since

$$\sum_{j=1}^{n} p_{ij} = 1 \quad , \tag{9}$$

$V_1 = \left[1, 1, \ldots\ldots 1\right]'$ is obviously a solution. Also, for A', it is seen that

$$A'U_1 = \lambda_1\ U_1 \tag{10}$$

which is

$$A'U_1 = U_1 \tag{11}$$

But, this is seen to be the set of linear equations that define π, which implies that π is the characteristic vector of A' corresponding to $(\lambda_1 = 1)$. This is a quite useful property in that the numerical methods used to find characteristic vectors may be applied to the problem of determining the stationary distribution.

Now, it has been pointed out that a regular stochastic matrix, A, has a largest characteristic root that is simple, equal to unity, and corresponds to the characteristic vector $V_1 = (1, 1, \ldots\ldots, 1)'$. Also, the characteristic vector of A' corresponding to the characteristic root unity, was seen to be π and it was stated that $\lim_{t \to \infty} A^t = A^*$, where all rows of A^* are equal to π'. These properties can be illustrated by means of a numerical example. Consider the matrix

$$A = \begin{bmatrix} .9 & .1 & 0 \\ .81 & .09 & .1 \\ .729 & .081 & .19 \end{bmatrix} . \tag{12}$$

Since,

$$\sum_{j=1}^{3} p_{ij} = 1; \quad p_{ij} \geq 0 \ (i,\ j = 1,\ 2,\ 3)$$

holds for this example A is seen to be stochastic. For t = 2 we have

$$A^t = A^2 = \begin{bmatrix} .8910 & .0990 & .0100 \\ .8748 & .0972 & .0280 \\ .8602 & .0956 & .0442 \end{bmatrix} \tag{13}$$

which has all elements greater than zero, thus insuring that A is regular. (Note that A^2 is also stochastic.) The matrix equation,

$$\begin{bmatrix} .9 & .1 & 0 \\ .81 & .09 & .1 \\ .729 & .081 & .19 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \tag{14}$$

shows that $\lambda_1 = 1$ is a characteristic root of A and that $V_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$ is a characteristic vector of A. (Note that for a constant c, cV_1 is a characteristic vector.) Next, by solving the set of linear equations,

$$A'\pi = \pi \qquad \text{i.e.,}$$

$$\begin{bmatrix} .9 & .81 & .729 \\ .1 & .09 & .081 \\ 0 & .1 & .19 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} \tag{15}$$

we see that the stationary distribution,

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} .889 \\ .099 \\ .012 \end{bmatrix} \quad , \tag{16}$$

is the characteristic vector of A' corresponding to $\lambda_1 = 1$. To examine A^t as t increases, we point out that the case for t = 1 and t = 2 has been given and that for t = 3 we have,

$$A^3 = \begin{bmatrix} .889 & .099 & .012 \\ .886 & .098 & .015 \\ .884 & .098 & .018 \end{bmatrix} \quad , \tag{17}$$

and when t = 5,

$$A^5 = \begin{bmatrix} .889 & .099 & .012 \\ .889 & .099 & .012 \\ .889 & .099 & .012 \end{bmatrix} \quad , \tag{18}$$

which shows that $A^5 \sim A^*$. Also, using A^5 it is seen that an arbitrary probability vector, P(o), is transformed by A^* into π. For,

$$P(o) = \begin{bmatrix} .66 \\ .13 \\ .21 \end{bmatrix} \tag{19}$$

we see that,

$$P(o)'A^5 = \begin{bmatrix} .66 & .13 & .21 \end{bmatrix} \cdot \begin{bmatrix} .889 & .099 & .012 \\ .889 & .099 & .012 \\ .889 & .099 & .012 \end{bmatrix} = \begin{bmatrix} .889 & .099 & .012 \end{bmatrix} = \pi' \tag{20}$$

We see that for this example A^5 gives a good approximation to A*, which shows that the initial conditions of the Markov chain have little effect on the distribution after 5 time intervals. In general, the t required to insure a reasonable estimate of A* is not known. A discussion of this problem is presented in later sections.

M-POLICY

The random variable of the M-policy inventory process which we will consider is Z_t, the number of items in the inventory after replenishment at the end of the time interval t. Z_t is an integer having M as lower bound and X as upper bound such that (Chaddha, 1960),

$$M \leq Z_t \leq X \tag{21}$$

It is seen that Z_t can be in any one of $a + 1 = X - M + 1$ content states.

The demand distribution placed on the items in the inventory is defined as,

$$P = \{P_i\} \tag{22}$$

where

$$P_i = Pr \text{ (number of items demanded during a unit time interval, } I_t) = i \quad . \tag{23}$$

We see that the elements p_{ij}, $(i,\ j, = o,1,\ldots,a)$, of the stochastic matrix may be written as:

$$
\begin{aligned}
p_{ij} &= \sum_{k=M+i}^{\infty} P_k && (i=0,1,2,\ldots,a)\ (j=0) \\
&= P_{M+i-j} && (i=0,1,2,\ldots,a-M)\ (0<j\leq M+i) \\
&= 0 && (i=0,1,2,\ldots,a-M)\ (M+i<j) \\
&= P_{M+i-j} && (i=a,M+1.\ldots,a)\ (0<j<a) \\
&= \sum_{k=0}^{i-a+M} P_k && (i=a-M+1.\ldots,a)\ (j=a)
\end{aligned}
$$

or in matrix form (Chaddha 1960);

$$
\begin{array}{ll}
 & \begin{array}{lllll} Z_{t+1} = M & M+1\ldots\ldots 2M & 2M+1\ldots X-2M\ldots X-1 & & X \\ j = 0 & 1\ \ldots\ldots M & M+1\ \ldots a-M\ \ldots a-1 & & a \end{array} \\
\begin{array}{ll}
Z_t & i \\
M & 0 \\
M+1 & 1 \\
\vdots & \vdots \\
X-2M & a-M \\
X-2M+1 & a-M+1 \\
\vdots & \vdots \\
X & a
\end{array}
&
\left[\begin{array}{cccc}
\sum_{i=M}^{\infty} P_i & P_{M-1}\cdots P_0 & 0\ \cdots 0\ \cdots 0 & 0 \\
\sum_{i=M+1}^{\infty} P_i & P_M\ \cdots P_1 & P_0\ \cdots 0\ \cdots 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{i=a}^{\infty} P_i & P_{a-1}\cdots P_{a-M} & P_{a-M-1}\cdots P_M\cdots P_1 & P_0 \\
\sum_{i=a+1}^{\infty} P_i & P_a\ \cdots P_{a-M+1} & P_{a-M}\cdots P_{M+1}\cdots P_2 & \sum_{i=0}^{1} P_i \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{i=a+M}^{\infty} P_i & P_{a+M-1}\cdots P_a & P_{a-1}\cdots P_{2M}\cdots P_{M+1} & \sum_{i=0}^{M} P_i
\end{array}\right]
\end{array}
$$

(Z_t is content at time, t;Z_{t+1} is content at t+1)

By observing the matrix we note that the main diagonal and the two adjacent off diagonals are positive if P_{M+1}, P_M, and P_{M-1} are positive. This implies that for some t, A^t is positive which in turn implies A is regular.

In this thesis, two types of demand distributions will be considered in detail;

(i) geometric,

$$P_k = pq^k \quad (k = 0,1,2,\ldots.) \quad (p+q) = 1, 0<p<1 \tag{24}$$

(ii) Poisson,

$$P_k = e^{-m} \frac{m^k}{k!} \quad (k = 0,1,2,\ldots.) \tag{25}$$

The mean μ_G and variance Var (G) of the geometric distribution are

$$\mu_G = \frac{q}{p} \text{ and } \mathrm{Var}\ (G) = \frac{q}{p^2} \quad . \tag{26}$$

Since $0<p<1$, we note from (26) that

$$\mu_G < \mathrm{Var}\ (G) \quad . \tag{27}$$

For the Poisson distribution we have

$$\mu_P = \mathrm{Var}\ (P) = m \ . \tag{28}$$

It is pointed out that, for this type of inventory process, u is interpreted as the average demand on the system, i.e., the average number of items demanded during a unit time interval, I_t. For geometric demand distribution it is seen that,

$$P_0 = pq^0 = p = \Pr(\text{No items are sold during a unit interval}) \tag{29}$$

and that the average demand, μ_G, increases as p decreases. For Poisson distribution average demand increases as m increases.

A is obviously regular for geometric and Poisson demand distributions.

It is apparent by comparing the variances of the two demand distributions that for the case when $\mu_G = \mu_P$, the geometric distribution will always have greater variance than the Poisson distribution. The variance of the geometric distribution gets quite large for small values of p.

The characteristic roots of the M-policy stochastic matrix can be extracted from the characteristic equation of the matrix. Since the characteristic roots will be of much interest in following sections, a particular example will be studied and its roots found directly from the characteristic equation. Consider the case where maximum content, X, equals four, order size, M, equals one, and the demand distribution is geometric. We have,

$$A = \begin{bmatrix} q & p & 0 & 0 \\ q^2 & pq & p & 0 \\ q^3 & pq^2 & pq & p \\ q^4 & pq^3 & pq^2 & pq+p \end{bmatrix} . \tag{30}$$

The characteristic equation for this matrix is

$$C(\lambda) = \begin{vmatrix} q-\lambda & p & 0 & 0 \\ q^2 & pq-\lambda & p & 0 \\ q^3 & pq^2 & pq-\lambda & p \\ q^4 & pq^3 & pq^2 & pq+p-\lambda \end{vmatrix} = 0 \tag{31}$$

which after much algebra reduces to,

$$C(\lambda)=\lambda^4 - (1+3pq)\lambda^3+pq(3+pq)\lambda^2 - (pq)\lambda = 0$$

$$=\lambda(\lambda-1)\ (\lambda^2-3pq\lambda+(pq)^2) = 0 \tag{32}$$

therefore,

$$\lambda_1 = 1,\ \lambda_2 = \frac{3+\ 5}{2}\,pq,\ \lambda_3 = \frac{3-\ 5}{2}\,pq,\ \lambda_4 = 0 \quad .$$

For large matrices this method of obtaining characteristic roots has disadvantages which make it impractical for the present purposes. First, the involved algebra is quite time consuming and is of a nature that is not readily applicable to electronic computers; second, the characteristic vectors are not obtained without further work; and third, roots other than λ_2 will be of little interest for the present purpose. It becomes apparent that a more suitable method is needed to compute π and λ_2.

Before considering methods for finding π and λ_2, it should be pointed out that, although the M-policy, stochastic matrix is referred to in the present context as defining an inventory process; the same matrix defines a queueing process and defines a storage process for finite dams (Gani 1957). This, of course, means that any results obtained in inventory control can be applied to these other areas.

METHOD FOR FINDING CHARACTERISTIC ROOTS

In this thesis we are interested in finding the characteristic roots of A which determine its rate of convergence to A*. Since $\lambda_1 = 1$, λ_2 has the most important effect on the rate of convergence. This means that a method that would determine π, λ_2, and possibly λ_3, is in order.

Methods of handling the problem of finding characteristic roots and vectors fall into two classes, analytical and numerical. The analytical methods, in general, give ways of obtaining the characteristic equation of the matrix which then must be solved. Analytical methods have the advantage of giving the exact value of all roots. Most analytical methods also lead to ways of determining the characteristic vectors after the roots have been found. However, when the characteristic equation is of high degree, as indicated in the last section, analytical methods become very cumbersome and time consuming and we are forced to use numerical methods to find the solution of the equation.

Numerical iterative methods are, in general, simple to apply and will give the largest characteristic root and its corresponding characteristic vector. The iteration may be carried out to any degree of accuracy and having obtained λ_1 and V_1, A can be modified so that an iterative solution for λ_2 and V_2 is found. Iterative methods have the disadvantage of being less accurate for each succeeding root and, if two neighboring roots are almost the same size the convergence of the method is slow for the larger of the two roots. Let us assume that the roots of A are simple and distinct. When this is not the case, a modification is necessary. One very distinct advantage of iterative methods is that in most cases they are well adapted for use on an electronic computer.

For the present purpose, an iterative method is the most suitable and will be used for finding π and λ_2.

Faddeeva (1959) relates an iterative method which was used to handle the characteristic value problems encountered in this thesis. In order to have a better understanding of the way in which the results in later sections were obtained, this method is introduced now.

First, consider $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$, to be ordered with regard to absolute magnitude and for simplicity, consider each root to be real and distinct. Now, an arbitrary vector, Y_0, can be written as a linear function of the n characteristic vectors which determine the n dimensional space. Then

$$Y_0 = b_1V_1 + b_2V_2 + \ldots\ldots\ldots\ldots + b_nV_n. \tag{33}$$

Next form the vector sequence $\{Y_i\}$ $(i=1,2,\ldots\ldots,k)$ where

$$Y_1' = Y_0'A = b_1\lambda_1V_1' + b_2\lambda_2V_2' + \ldots\ldots\ldots\ldots + b_n\lambda_nV_n' \tag{34}$$

and

$$Y_k' = Y_0'A^k = b_1\lambda_1^kV_1' + b_2\lambda_2^kV_2' + \ldots\ldots\ldots\ldots + b_n\lambda_n^kV_n' \tag{35}$$

Now, let y_k be any component of the vector, Y_k, such that

$$y_k = C_1\lambda_1^k + C_2\lambda_2^k + \ldots\ldots\ldots + C_n\lambda_n^k \quad . \tag{36}$$

Now, it is seen that

$$\frac{y_{k+1}}{y_k} = \frac{C_1\lambda_1^{k+1} + C_2\lambda_2^{k+1} + \ldots\ldots\ldots + C_n\lambda_n^{k+1}}{C_1\lambda_1^k + C_2\lambda_2^k + \ldots\ldots\ldots + C_n\lambda_n^k} \tag{37}$$

$$= \frac{\lambda_1^{k+1}}{\lambda_1^k} \cdot \frac{1 + \frac{C_2}{C_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1} + \frac{C_3}{C_1}\left(\frac{\lambda_3}{\lambda_1}\right)^{k+1} + \ldots + \frac{C_n}{C_1}\left(\frac{\lambda_n}{\lambda_1}\right)^{k+1}}{1 + \frac{C_2}{C_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{k} + \frac{C_3}{C_1}\left(\frac{\lambda_3}{\lambda_1}\right)^{k} + \ldots + \frac{C_n}{C_1}\left(\frac{\lambda_n}{\lambda_1}\right)^{k}} \, . \tag{38}$$

From this it is evident that

$$\lim_{k\to\infty} \left(\frac{y_{k+1}}{y_k}\right) = \lambda_1 \qquad , \tag{39}$$

and for large k

$$\lambda_1 \approx \frac{y_{k+1}}{y_k} \qquad .$$

Also, for sufficently large k

$$Y_k'A = \lambda_1 Y_k' \qquad , \tag{40}$$

so that Y_k is the characteristic vector corresponding to λ_1.

Now, given λ_1 and $V_1 = \left[V_{11}, V_{12}, \ldots . V_{1n}\right]'$, to find λ_2, we form the matrix

$$P = \begin{bmatrix} V_{11} & 0 & \cdots & & 0 \\ V_{12} & 1 & \cdots & & 0 \\ V_{13} & 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{1n} & 0 & \cdots & & 1 \end{bmatrix} \qquad , \tag{41}$$

and note that,

$$P^{-1} = \begin{bmatrix} \frac{1}{V_{11}} & 0 & \cdots & & 0 \\ -\frac{V_{12}}{V_{11}} & 1 & 0 & \cdots & 0 \\ -\frac{V_{13}}{V_{11}} & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ -\frac{V_{1n}}{V_{11}} & 0 & \cdots & & 1 \end{bmatrix} \qquad . \tag{42}$$

The matrix, $P^{-1}AP$, is similar to A, and both matrices have identical characteristic roots. Also,

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & b_{12} \cdots\cdots\cdots b_{1n} \\ 0 & \\ 0 & \\ \cdot & B \\ \cdot & \\ \cdot & \\ 0 & \end{bmatrix} , \tag{43}$$

so that,

$$\left|P^{-1}AP - \lambda I\right| = (\lambda_1 - \lambda) \quad \left|B - \lambda I\right| . \tag{44}$$

From this we see that the matrix B, of order n-1, has as its characteristic roots $\lambda_2, \lambda_3, \ldots\ldots\ldots, \lambda_n$. The root, λ_2, may now be determined by the iterative method for finding the dominant root.

The assumptions that were made above may not always hold but if $|\lambda_1| > |\lambda_2| > |\lambda_3|$ is true, the method should give the solutions for π and λ_2. The most undesirable property of this technique os the slow convergence to λ_1, when the ratio λ_1/λ_2 is near unity. In general, this iterative method has wide applicability to practical problems.

STATIONARY DISTRIBUTIONS

As stated previously, the stationary distribution, π, which satisfies the conditions, $A'\pi = \pi$, $\Sigma\ \pi_i = 1$, exists for any regular stochastic matrix, A. Also, for regular A, each element of π is non-zero. (Gantmacher, 1959).

The iterative method for finding π and λ_2 was used to obtain the stationary distributions for the M-policy matrix, A, for various different values of X and M. The cases when, X = 2(1)14, M= 1(1)13, p= .1(.1).9, are considered for geometric demand and X = 2(1)13, M = 1(1)12, m = 1(1)9 are considered for Poisson demand. The resulting stationary distributions are given in table 4 and table 5.

In discussing the stationary distributions it is helpful if we define,

$$D_G = (M - \mu_G) \tag{45}$$

and

$$D_P = (M - \mu_P). \tag{46}$$

Each M-policy matrix will have a D which is equal to the order size minus the average demand. It is seen that when $D > 0$, the order size is greater than the demand, and when $D < 0$, the converse is true.

In general, it was found that π was most evenly distributed and had its greatest variance when D = 0. As D increases in value, it is seen that the probability elements, π_i, corresponding to large Z_t increases in size and that the elements corresponding to small Z_t decrease in size. The inverse of this relationship is seen to hold, in that as D decreases in value, the probability elements, π_i, corresponding to small Z_t increase in size and elements corresponding to large Z_t decrease in size. This may be illustrated by taking an example from M-policy with geometric demand.

$X = 5 \quad M = 1$

Content	π_z	p=.8 $D_G=3/4$	p=.5 $D_G=0$	p=.2 $D_G=-3$
$Z_t = 1$	$\pi_1 =$	.001	.167	.750
2	$\pi_2 =$	.003	.167	.188
3	$\pi_3 =$	.012	.167	.047
4	$\pi_4 =$	.047	.167	.012
5	$\pi_5 =$	.938	.333	.004

This is the expected result in that the probability of an inventory system being full or near full should be greater when the demand during a time interval is less than the order size. Conversely when the order size is less than the demand an interal t, the inventory system is expected to be empty or near empty.

It may be seen by comparing table 4 to table 5, that, in general, when considering the stationary distribution of a given M-policy matrix, A, with Poisson demand and geometric demand, and with $D_G = D_P$; the variance of the random variable Z_t is greater for geometric demand. This implies π is more evenly distributed for geometric than for Poisson, for extreme values of D. It was mentioned previously that the variance of the geometric distribution is always greater than the variance of the Poisson distribution for $u_G = u_P$. It is believed that this fact is transferred from the demand distribution to the stationary distribution. For example when, $M = 2$, $X = 7$, $D_G = D_P = -2$, we have

$$\begin{aligned} \mathrm{Var}\left[Z_t(\text{geometric})\right] &= 1.8084 \\ \mathrm{Var}\left[Z_t(\text{Poisson})\right] &= .2295 \quad . \end{aligned} \tag{47}$$

Also,

$$E\left[Z_t(\text{geometric})\right] = 3.2424$$

$$E\left[Z_t(\text{Poisson})\right] = 2.1571 \quad . \tag{48}$$

The stationary distributions given in tables 4 and 5 are used to compute many parameters of M-policy inventory processes. In a later section these stationary distributions will be used to compute the average content of the system, the probability of placing an order at the end of an interval, the average demand not met during the interval, and an average cost function, thus illustrating the value of π in an applied situation.

SECOND LARGEST ROOT OF THE M-POLICY MATRIX

Before discussing the general properties of the second largest root of the M-policy matrix, a special property of the characteristic roots of the geometric M-policy matrix is pointed out.

THEOREM III. The M-policy stochastic matrix with geometric demand has,

(i) M characteristic roots equal to zero when

$$M \leq \frac{X}{2} - 1 \tag{49}$$

(ii) and has exactly two non-zero characteristic roots when

$$M > \frac{X}{2} - 1 \quad . \tag{50}$$

Proof. When (i) holds it is seen that the geometric M-policy matrix is

$$A = \begin{bmatrix} q^M & pq^{M-1} & pq^{M-2} & \ldots pq & p & 0 & 0 & 0 \\ q^{M+1} & pq^M & pq^{M-1} & \ldots pq^2 & pq^2 & p & 0 & 0 \\ q^{M+2} & pq^{M+1} & pq^M & \ldots pq^3 & pq^3 & pq & p & p \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \sum_{i=0}^{1} pq^i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q^X & pq^{X-1} & pq^{X-2} & \ldots pq^{X-M+1} & pq^{X-M} & pq^{X-M-1} & & \sum_{i=0}^{M} pq^i \end{bmatrix} \quad . \tag{51}$$

It is seen that a linear dependence exists in A, such that

$$\begin{aligned} \text{Column } 2 &= \frac{p}{q} \cdot (\text{column } 1) \\ \text{Column } 3 &= \frac{p}{q^2} \cdot (\text{column } 1) \\ &\vdots \\ \text{Column } M+1 &= \frac{p}{q^M} \cdot (\text{column } 1). \end{aligned} \tag{52}$$

Since the first M+1 columns are linearly dependent it follows that there will be M characteristic roots equal to zero and Theorem III is established when (i) holds. When (ii) holds the M-policy matrix is

$$A = \begin{bmatrix} q^M & pq^{M-1} \ldots\ldots\ldots pq^{2M-X+1} & \sum_{i=0}^{2M-X} pq^i \\ q^{M+1} & pq^M \ldots\ldots\ldots pq^{2M-X+2} & \sum_{i=0}^{2M-X+1} pq^i \\ q^{M+2} & pq^{M+1} \ldots\ldots\ldots pq^{2M-X+3} & \sum_{i=0}^{2M-X+2} pq^i \\ \vdots & \vdots \qquad\qquad \vdots & \vdots \\ q^X & pq^{X-1} \ldots\ldots\ldots pq^{M+1} & \sum_{i=0}^{M} pq^i \end{bmatrix} . \tag{53}$$

Now, it is seen that the first and last columns of A are linearly independent and all the interior columns are linear combinations of the first column. It follows that in this case the matrix A will have two non-zero characteristic roots and Theorem III is established when (ii) holds.

The characteristic equation of the geometric M-policy matrix when (50) holds, has been solved by Chaddha (unpublished result) and it was found that,

$$\lambda_2 = (X-M)\, pq^M \quad . \tag{54}$$

These properties of the characteristic roots of A for geometric demand are of interest since they affect the rate of convergence of A to A*.

The second largest characteristic roots for various M-policy matrices are given in table 6 (geometric demand) and in table 7 (Poisson demand). The iterative method discussed earlier was used to obtain these results.

When considering a given matrix, A, for both demands, λ_2 takes its largest value when D = 0 and decreases in value as $|D|$ increases. (See Fig. 1.). When M and μ are held constant and X is increased (which results in a larger matrix), λ_2 is also increased. (Fig. 2.).

When X and μ are held constant and M is increased, the situation is somewhat different, since D changes as M changes. If $|D|$ increases the, root, λ_2 decreases (Fig. 3.), but if $|D|$ decreases toward zero, then λ_2 increases and reaches a maximum at D = 0 (Fig. 3.) or in some cases, λ_2 reaches a maximum before $|D|$ reaches zero. (Fig. 4.). This last effect is due to the decreasing size of the matrix, A, as M increases.

In general, by inspecting table 6 and table 7, it is seen that for a given X, M, and D = 0; λ_2 is larger for Poisson than for geometric (Fig. 5.,6.). As $|D|$ increases, λ_2 for Poisson seems to decrease at a faster rate than λ_2 for geometric (Fig. 7.).

For a given X, M, and geometric demand,

$$\lambda_2 \approx C_{X,M,G}\, pq^M = C_{X,M,G}\, P_M \qquad M \le \frac{X}{2} - 1 \tag{55}$$

$$= (X-M)P_M \qquad M > \frac{X}{2} - 1 \quad . \tag{56}$$

The formula (56) is an exact result, and the formula (55) is seen to give a very satisfactory approximation of λ_2. Finite difference methods were applied to the data in table 6 to obtain this formula. In general the values given λ_2 by this formula should be accurate to three decimal places. The constant, $C_{X,M,G}$ is listed in table 1. To illustrate the use of table 1 consider the case when $X = 4$, $M = 1$. We have

$$C_{4,1,G} = 2.618 \qquad (57)$$

and

$$\lambda_2 \approx (2.618) \quad pq^1 \qquad (58)$$

which agrees with the previous answer for λ_2, (32).

For a given X, M, and Poisson demand, the formula

$$\lambda_2 = C_{X,M,P} \, e^{-m} \, \frac{m^M}{M!} = C_{X,M,P} \, P_M \qquad (59)$$

approximates λ_2 very well. This formula for λ_2 was obtained by observing that for a given X and M the ratio λ_2/P_M was a constant quantity. The constant quantity tabulated was obtained by using the largest of the λ_2 values. The values of the constant $C_{X,M,P}$, are given in table 2. Now consider an example for Poisson demand distribution with $m = 4$ and let $M = 3$, $X = 9$. We see that

$$C_{9,3,P} = 3.562 \qquad (60)$$

and

$$e^{-m} \frac{m^M}{M!} = .1954 \qquad (61)$$

so that

$$\lambda_2 \approx (3.562)\ (.1954) = .6960 \qquad (62)$$

which compares with the computed value, .6959.

The usefulness of being able to compute λ_2 directly from the demand distribution and the significance of the method used fall into several areas. First, by inspecting the approximating formules it is seen that they have a form similar to the form of the demand distribution. This indicates that possibly for other demand distributions a similar relationship with λ_2 exists. Second, given the values for $C_{X,M}$, estimates of the rate of convergence may be obtained without going through the time consuming process of computing λ_2. Third, this is a step toward the optimum solution of the problem which would be the ability to determine the rate of convergence using only the information given in the stochastic matrix.

The third largest characteristic root of A, λ_3, was computed in a few cases, but, in general, it was found that λ_3 could not be computed in a usable form. In some cases λ_3 was so close to zero that rounding errors destroyed its usefulness and, as shown, in numerous cases $\lambda_3 = 0$ (geometric, $M > \frac{X}{2} - 1$). The iterative method used for computing the roots has much less accuracy for the third root than for the second root. Since λ_2 is the most important root in determining rates of convergence the absence of λ_3 is not critical.

Table 1. $C_{X,M,G}$ (constants used in computing λ_2 for geometric distribution.)

X \ M	1	2	3	4	5	6	7	8	9	10
2	1.000									
3	2.000	1.000								
4	2.618	2.000	1.000							
5	3.000	3.000	2.000	1.000						
6	3.247	3.732	3.000	2.000	1.000					
7	3.414	4.303	4.000	3.000	2.000	1.000				
8	3.532	4.732	4.791	4.000	3.000	2.000	1.000			
9	3.618	5.064	5.449	5.000	4.000	3.000	2.000	1.000		
10	3.683	5.323	6.000	5.828	5.000	4.000	3.000	2.000	1.000	
11	3.732	5.529	6.449	6.541	6.000	5.000	4.000	3.000	2.000	1.000
12	3.771	5.694	6.823	7.162	6.854	6.000	5.000	4.000	3.000	2.000
13	3.802	5.828	7.136	7.702	7.606	7.000	6.000	5.000	4.000	2.000
14	3.827	5.939	7.398	8.162	8.275	7.873	7.000	6.000	5.000	5.000

M	11	12	13
12	1.000		
13	2.000	1.000	
14	3.000	2.000	1.000

Table 2. $C_{X,M,P}$ (constants used in computing λ_2 for Poisson distribution.)

X \ M	1	2	3	4	5	6	7	8	9	10
2	1.000									
3	1.707	1.000								
4	2.075	1.816	1.000							
5	2.278	2.379	1.866	1.000						
6	2.400	2.737	2.530	1.894	1.000					
7	2.478	2.973	3.000	2.620	1.913	1.000				
8	2.530	3.134	3.328	3.170	2.681	1.926	1.000			
9	2.573	3.247	3.562	3.574	3.290	2.726	1.935	1.000		
10	2.595	3.330	3.733	3.872	3.756	3.380	2.759	1.943	1.000	
11	2.615	3.392	3.861	4.095	4.110	3.896	3.449	2.785	1.949	1.000
12	2.633	3.440	3.958	4.266	4.380	4.298	4.007	3.504	2.806	1.953
13	2.647	3.477	4.039	4.398	4.591	4.612	4.452	4.241	3.549	2.823

M	11	12
12	1.000	
13	1.957	1.000

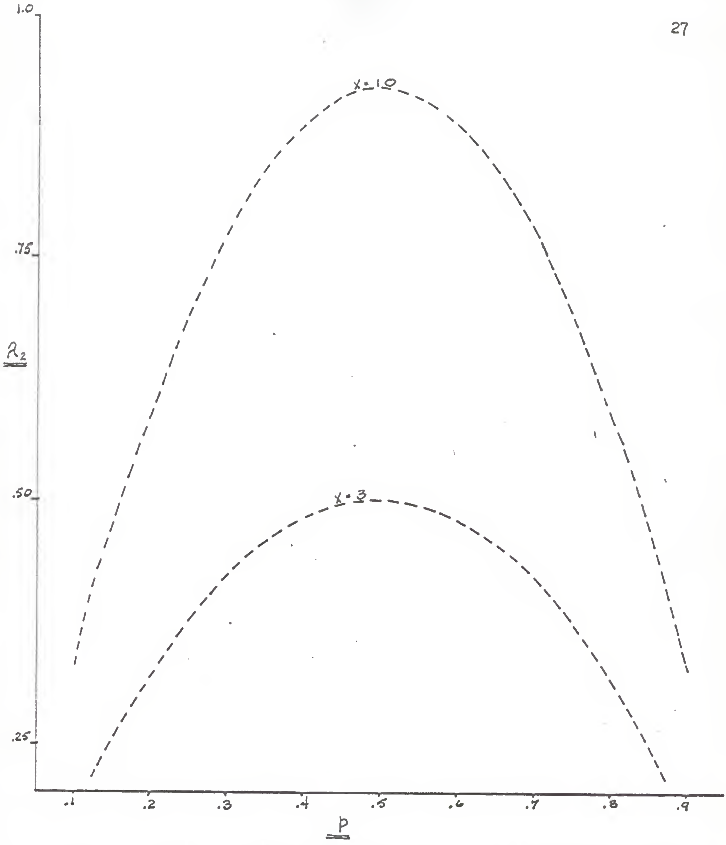

Fig. 1. Values of the second largest root, λ_2, when demand distribution is geometric with parameter p and order size M = 1.

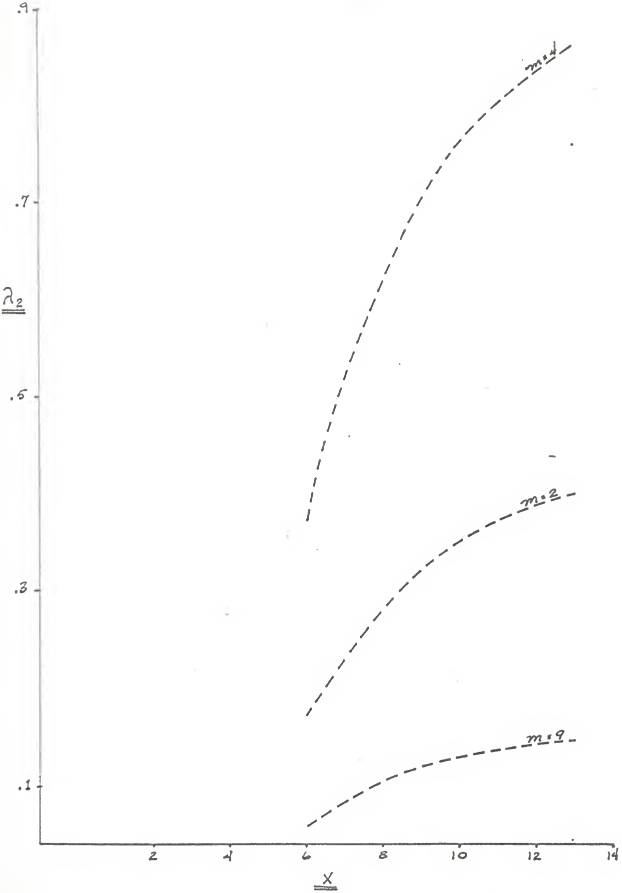

Fig. 2. Values of second largest root, λ_2, for inventory size X, Poisson demand distribution with parameter m, and order size M = 4.

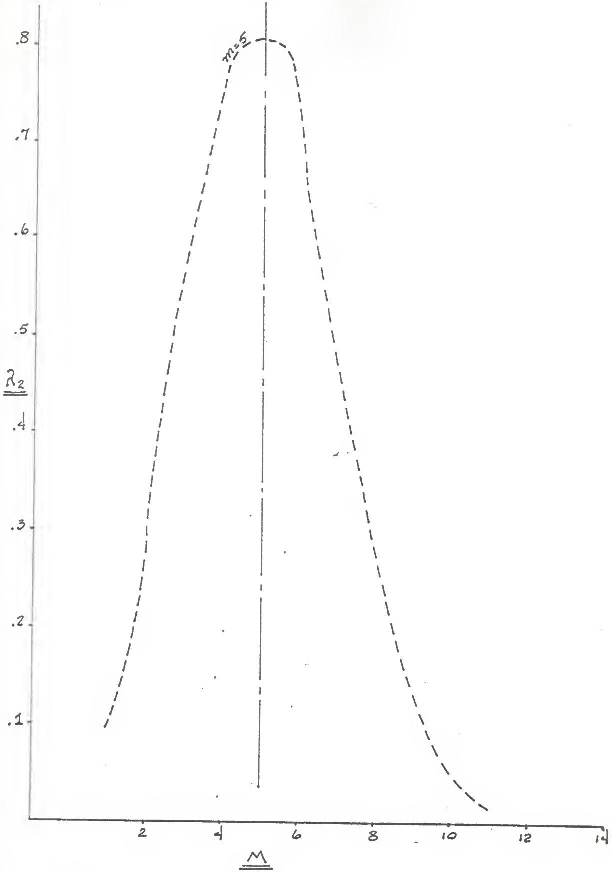

Fig. 3. Values of second largest root, λ_2, for order size M, Poisson demand with parameter m, and inventory size X = 13.

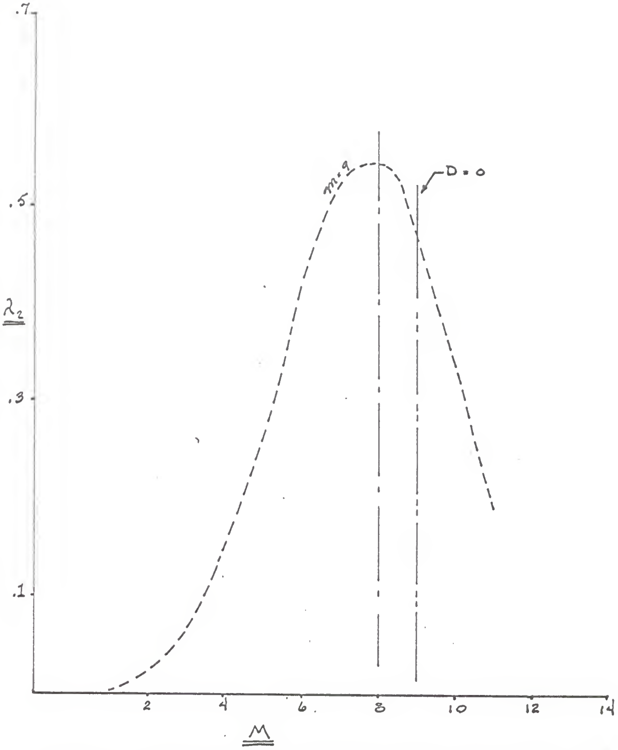

Fig. 4. Values of second largest root, λ_2, for order size M, Poisson demand with parameter m, and inventory size X = 13.

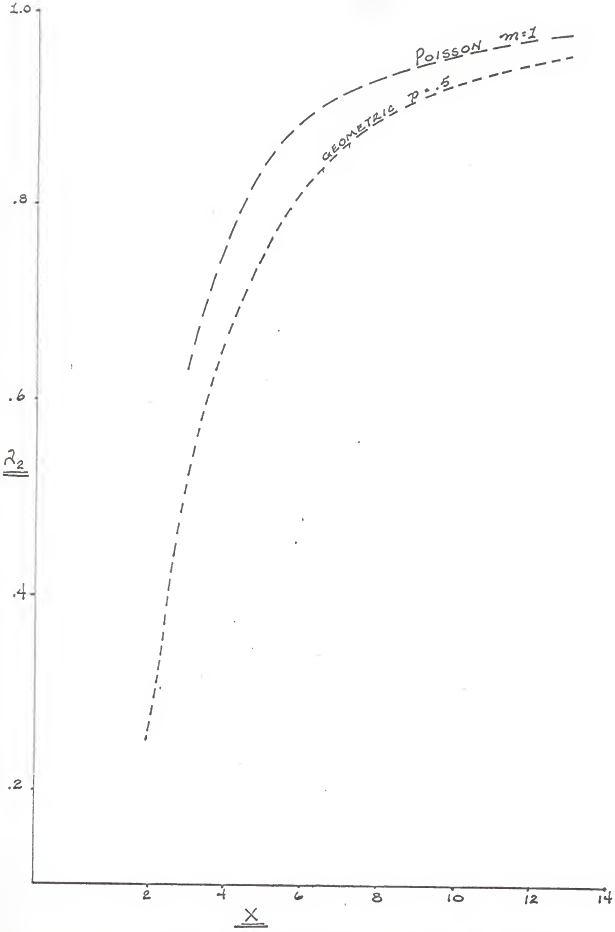

Fig. 5. Comparison of values of λ_2 for geometric and Poisson demand distributions with means $\mu = 1$ and order size $M = 1$.

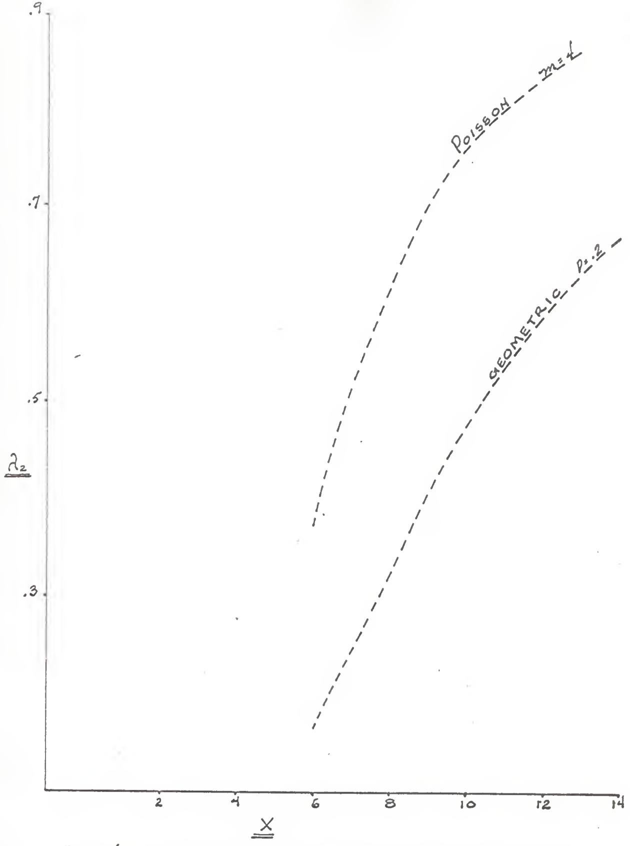

Fig. 6. Comparison of values of λ_2 for geometric and Poisson demand distributions with means $\mu = 4$ and order size $M = 4$.

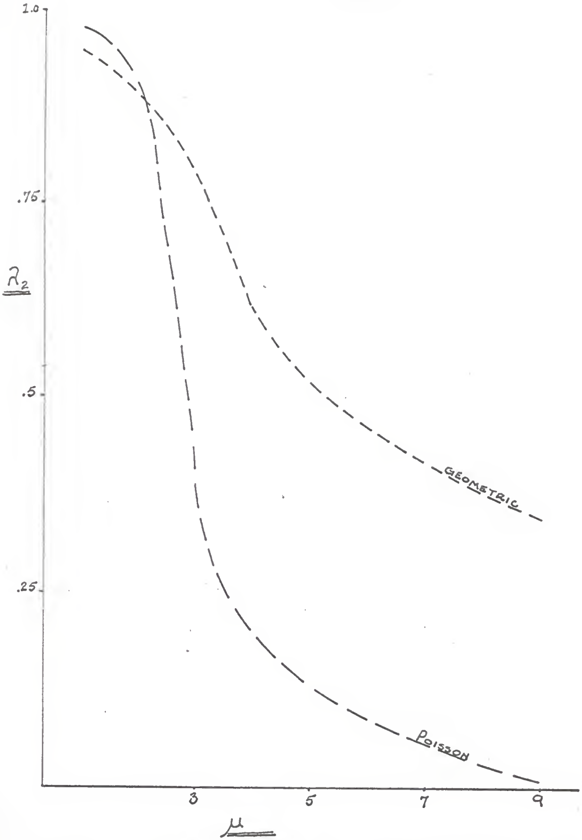

Fig. 7. Comparison of values of λ_2 for geometric and Poisson demand distributions with mean μ, inventory size X = 13, and order size M = 1.

METHOD TO APPROXIMATE THE RATE OF STATIONARITY

According to its bilinear resolution the matrix A^t maybe written (Faddeeva, 1959) as,

$$A^t = \lambda_1^t V_1 U_1' + \lambda_2^t V_2 U_2' + \dots\dots\dots + \lambda_n^t V_n U_n' \quad (63)$$

$$= A^* + \lambda_2^t V_2 U_2' + \dots\dots\dots + \lambda_n^t V_n U_n' \quad (64)$$

$$= A^* + f(\lambda_i^t) \quad (i = 2,3,\dots\dots n) \quad (65)$$

where V_i and U_i are characteristic vectors of A and A' respectively and $\lim_{t\ \infty} f(\lambda_i^t) = 0$. The equation (63) assumes the characteristic roots to be distinct but in (65) the only necessary assumption is that λ_1 is distinct (Faddeeva, 1959). In the present case, where A is a regular stochastic matrix, this last assumption holds since, $|\lambda_i| < 1$ $(i=2,3,\dots,n)$. It is apparent that λ_2 is the most important root in determining the rate of convergence to A*.

The problem is to find a value of t such that

$$|A^t - A^*| < N, \quad (66)$$

where N is a matrix of the same dimension as A and with every element equal to some small given value $d > o$.

It is noted that if the roots $\{\lambda_i\}$ are known then there is a value of t such that $|\lambda_i|^t$ is very small so that $f(\lambda_i^t)$ is very small and this value of t will suffice to satisfy (66). By considering only the effect of λ_2 on the rate of convergence, a value of t_1 needs to be found, so that

$$|\lambda_2|^{t_1} < \epsilon \quad (67)$$

where ϵ (> o) is some small given constant. From the inequality,

$$t_1 \quad \frac{\log \epsilon}{\log|\lambda_2|} \quad = \quad \frac{\log \epsilon}{\log C_{X,M} + \log P_M} \tag{68}$$

it should be noted that, by using t_1 as an estimate of the number of trials needed for convergence, the minimum number of trials needed will be obtained since in the calculation of t_1 we do not take into consideration the values of $\lambda_3, \lambda_4, \ldots, \lambda_n$.

In the extreme case when $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_n$, an estimate of the maximum number of trials needed can be obtained in much the same manner.

The inequality for this situation is given by,

$$(n-1) \qquad |\lambda_2|^{t_2} \quad < \epsilon \tag{69}$$

which reduces to

$$t_2 > \frac{\log \epsilon}{\log|\lambda_2|} \quad - \quad \frac{\log (n-1)}{\log \lambda_2} \tag{70}$$

$$t_2 > t_1 + \frac{\log (n-1)}{\log \frac{1}{C_{X,M}} + \log \frac{1}{P_M}} \tag{71}$$

Figures 8 and 9 illustrate the curve t_1 and figure 10 gives the graph of $t_2 - t_1$ for different n. Figure 8 is used to obtain a lower bound on the number of trials, and the number of non-zero roots (excluding unity) of the matrix determines which curve on figure 10 to use to obtain t_2.

It was determined that t should be such that,

$$|A^t - A^*| \quad < .001 \quad , \tag{72}$$

in order for A^t to give a reliable estimate of A^*. Knowing this,

ϵ = .0005 was chosen in order to offset any rounding errors and to have t_1 and t_2 slightly conservative.

To illustrate the use of t_1, t_2 and $C_{X,M}$ consider the M-policy with Poisson demand and X=6, M=2, m=1. We have

$$\lambda_2 = C_{6,2,P} \cdot P_2 = (2.737)\ (.1839) = .5033 \quad . \qquad (73)$$

From figure 8 it is seen that for λ_2 = .5033, t_1 = 12 (rounded up to integer). Figure 10 shows in this case that t_2 = 14. To check on this estimate, it is known from table 5 that

$$\pi' = \begin{bmatrix} .0027 & .0081 & .0276 & .0822 & .8794 \end{bmatrix}$$

and computing A^{14}, it is seen that

$$A^{14} = \begin{bmatrix} .0027 & .0082 & .0277 & .0823 & .8791 \\ .0027 & .0082 & .0276 & .0822 & .8792 \\ .0027 & .0082 & .0276 & .0822 & .8793 \\ .0027 & .0081 & .0276 & .0822 & .8794 \\ .0027 & .0081 & .0276 & .0822 & .8794 \end{bmatrix} . \qquad (74)$$

From this, it seen that A^{14} is a correct estimate of A* to three decimal places and is off by only 3 in the fourth decimal place. This degree of accuracy shows, in this case, that t_2 is, as stated, a good estimate of rate of convergence.

To better understand the behavior of t_1 and t_2, table 3 lists various M-policies and gives λ_2 and the estimates t_1, t_2 based on λ_2. For comparison a quantity t* is given to show the rate at which A^t approaches A*. t* is the smallest value of t such that

$$\left| A^t - A^{t-1} \right| < .001 \quad t = (2,3,.....). \qquad (75)$$

Table 3. Comparsion of t_1, t_2 with t*.

geometric demand

X	M	p	λ_2	t_1	t_2	t*
14	8	.1	.258	6	7	6
14	7	.1	.335	7	9	7
11	2	.1	.448	10	13	10
14	5	.2	.542	13	16	14
11	3	.2	.660	18	23	18
13	3	.3	.734	25	31	24
8	1	.6	.848	47	59	50

Poisson demand

X	M	m	λ_2	t_1	t_2	t*
9	7	4	.115	4	4	5
9	5	9	.200	6	7	7
8	3	6	.297	7	8	8
11	6	4	.406	9	10	10
9	5	4	.514	12	14	14
9	2	1	.597	15	19	20
11	3	2	.702	21	26	24

If the value of λ_3 was known, the estimate t_1 could be improved upon. If λ_3 is to be considered, the problem is to find the smallest value of t_{11} such that,

$$|\lambda_2|^{t_{11}} + |\lambda_3|^{t_{11}} < \epsilon \tag{76}$$

which reduces to

$$t_{11} > \frac{\log}{\log|\lambda_2|} + \frac{\log\left[1 + |\lambda_3/\lambda_2|^{t_{11}}\right]}{-\log|\lambda_2|},$$

$$t_{11} > t_1 + \frac{\log\left[1 + |\lambda_3/\lambda_2|^{t_{11}}\right]}{-\log|\lambda_2|} \quad . \tag{77}$$

If the extreme case $\lambda_3 = \lambda_4, \ldots\ldots\ldots\ldots, \lambda_n$ is considered, it is seen that

$$\left|\lambda_2\right|^{t_{12}} + (n-2)\ \left|\lambda_3\right|^{t_{12}} < \epsilon \qquad (78)$$

reduces to,

$$t_{12} > \frac{\log\ \epsilon}{\log\left|\lambda_2\right|} + \frac{\log\left[1 + (n-2)\left|\frac{\lambda_3}{\lambda_2}\right|^{t_{12}}\right]}{-\log\ \lambda_2} \ .$$

$$t_{12} > t_1 + \frac{\log\left[1 + (n-2)\left|\frac{\lambda_3}{\lambda_2}\right|^{t_{12}}\right]}{-\log\left|\lambda_2\right|} \ . \qquad (79)$$

Several facts become apparent by inspecting the equations which give t_{11} and t_{12}. First, the computation involved in finding t_{11} and t_{12} is greater than for t_1 and t_2. Second, when the ratio $|\lambda_3/\lambda_2|$ is small, t_{11} will be very close t_1 and the added effort of computing t_{11} is unnecessary. Third, when the ratio $|\lambda_3/\lambda_2|$ is near unity, t_{12} is very near t_2 in value, and it is doubtful that the improvement would be worth the added computation. The utility of λ_3 for estimating convergence seems to lie in its ability to lower the upper estimate when $|\lambda_3/\lambda_2|$ is small, and increase the lower estimate when $|\lambda_3/\lambda_2|$ is large. It is noted that when $\lambda_3 = 0$, $t_{12} = t_{11} = t_1$, and when $\lambda_3 = \lambda_2 =$, $t_{11} = t_2$ (with $n = 3$) and $t_{12} = t_2$.

In the example considered with Poisson demand, $X = 6$, $M = 2$, and $m = 2$, it was found that $\lambda_3 = .1963$. It is seen that in this case the ratio $\frac{\lambda_3}{\lambda_2}^{t_1}$ $(\approx |.4|^{12})$ is very small and that $t_{12} \approx t_1$.

In view of the above discussion it appears that λ_3 would be of value and would shorten the interval $(t_2 - t_1)$ when λ_2 is large $(>.85)$ but, otherwise the gain in accuracy does not justify the added computation.

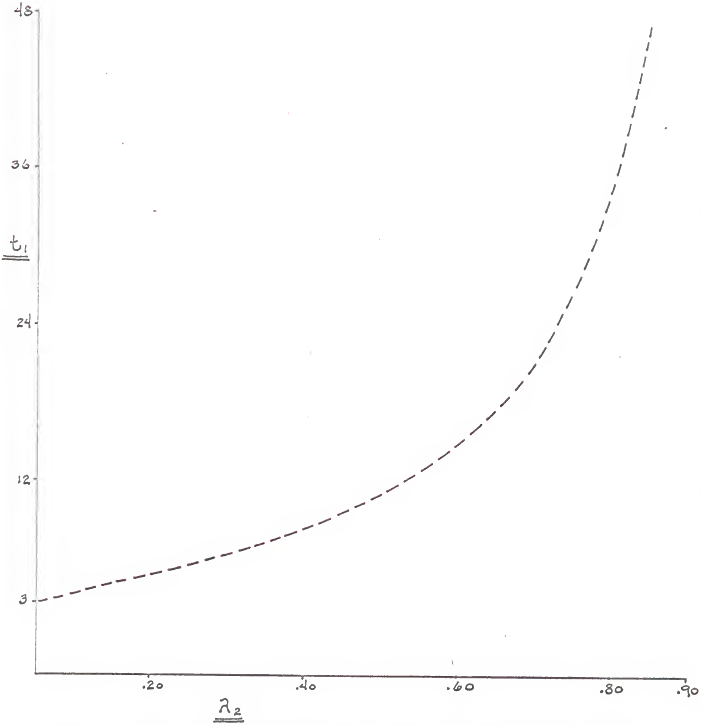

Fig. 8. Graph of t_1, the minimum number of intervals needed to reach the "near" steady state. λ_2 is the second largest characteristic root.

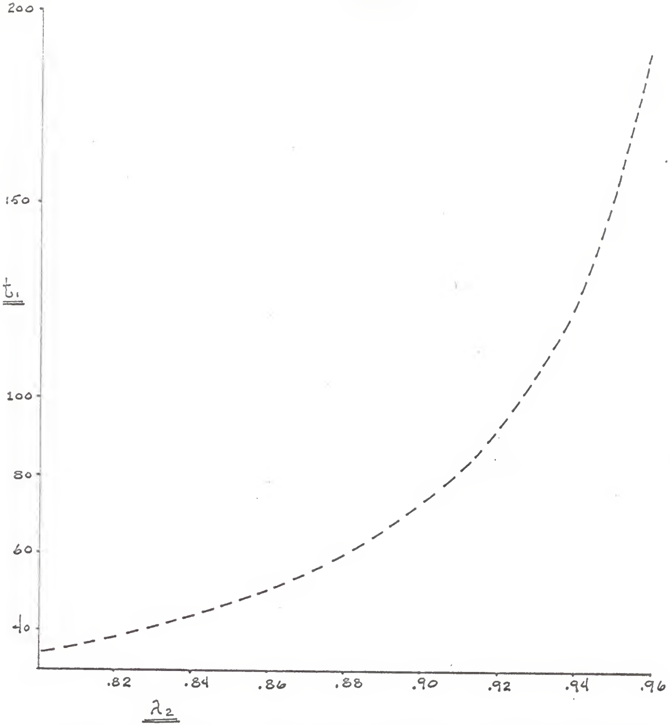

Fig. 9. Graph of t_1 the minimum number of intervals needed to reach the "near" steady state. λ_2 is the second largest characteristic root.

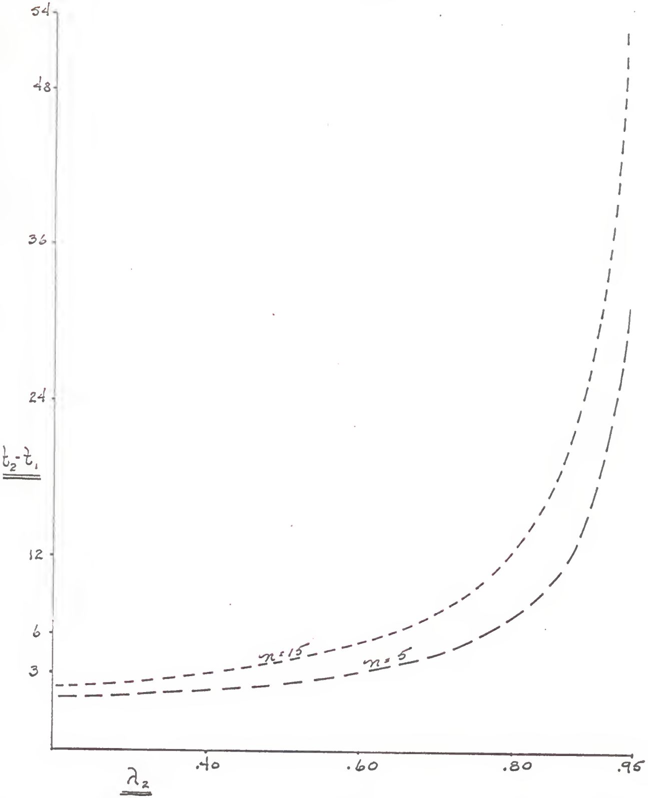

Fig. 10. Graph of $t_2 - t_1$, where t_2 is the maximum number of intervals needed to reach the "near" steady-state. λ_2 is the second largest characteristic root and n is the number of non-zero characteristic roots of the stochastic matrix.

APPLICATIONS

To illustrate application of t_1 and t_2 consider an inventory process that follows M-policy. For geometric demand with $X = 5$, $M = 1$, $P = .5$ and starting with full inventory content (=5), consider estimates of stationarity after, say, five trials and consider estimates of stationarity after at least t_1 trials.

$$A = \begin{bmatrix} .5 & .5 & 0 & 0 & 0 \\ .25 & .25 & .5 & 0 & 0 \\ .125 & .125 & .25 & .5 & 0 \\ .0625 & .0625 & .125 & .25 & .5 \\ .03125 & .03125 & .0625 & .125 & .75 \end{bmatrix} \quad . \tag{80}$$

In this case it is seen that $C_{5,1,.5} = 3.00$ and $pq = .25$. The value of λ_2 is computed

$$\lambda_2 = (3.00)\ .25\ = .750 \quad . \tag{81}$$

From this it is seen that $t_1 = 27$ and $t_2 = 32$ (Fig. 8,10). The starting distribution is

$$P'(0) = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \end{vmatrix} \tag{82}$$

and from this

$$P(5) = \begin{bmatrix} .1222 \\ .1222 \\ .1370 \\ .1665 \\ .4521 \end{bmatrix} \qquad \begin{array}{c} P(30) = \\ \\ (t_1 < 30 < t_2) \end{array} \begin{bmatrix} .1666 \\ .1666 \\ .1666 \\ .1667 \\ .3334 \end{bmatrix} \quad . \tag{83}$$

Also, (table 4)

$$\pi' = \begin{bmatrix} .1667 & .1667 & .1667 & .1667 & .3333 \end{bmatrix}$$

which shows that P(30) is a very close estimate of π.

The average number of items in the inventory can be computed when the probabilities for each state are known. We see the average content is

(i) based on P(5)

$$\sum_{i=1}^{5} iP_i(5) = 3.8263 \tag{84}$$

(ii) based on π

$$\sum_{i=1}^{5} i\pi_i = 3.3333 \quad . \tag{85}$$

The average demand not met can also be computed if the probabilities are given. Average demand not met is the sum

$$\sum_{i=1}^{\infty} i(P_M(t)\cdot P_{M+i}+P_{M+1}(t)P_{M+1+i}+P_{M+2}(t)P_{M+2+i}+\ldots+P_X(t)P_{X+i}). \tag{86}$$

In our case it is seen that this reduces to

(i) based on P(5) $\frac{q}{p} \sum_{i=1}^{5} P_i(5)\, q^i = .2666$ (87)

(ii) based on π $\frac{q}{p} \sum_{i=1}^{5} \pi_i q^i = .1667$. (88)

The probability that an order is placed is,

(i) base on P(5)

$$1 - p_o P_5(5) = .7740 \tag{89}$$

(ii) based on π

$$1 - p_o \pi_5 = .8333 \quad . \tag{90}$$

To set up a hypothetical problem, assume that a government supplier, who is contracted to supply rockets at a test site, keeps five rockets on site ready to fire. Assume this inventory can be replenished with at most, one rocket per week, and five is the maximum number that can be kept on site. The rocket firings follow the geometric (p=.5) distribution and the contracter must pay a penalty cost, C_3, of \$50,000 each time the demand for a rocket is not met. Consider the weekly cost, C_1, of maintaining a rocket on site to be \$4,000 and the cost, C_2, of shipping a rocket to the test site to be \$4,000 also. This is clearly an example of an M-policy inventory system.

Total cost per week is found by using the equation:

Total cost per week = Average content x C_1

+ Pr [order is placed] x C_2+ Average demand

not met x C_3.

It is seen that for the example

(i) based on P(5)

$$\text{Total weekly cost} = (3.8263)C_1+(.7740)C_2+(.2666)C_3 \quad (91)$$
$$= \$31,731$$

(ii) based on π

$$\text{Total weekly cost} = (3.3333)C_1+(.8333)C_2+(.1666)C_3 \quad (92)$$
$$= \$25,000$$

The difference between these two results points out the danger involved in making estimates by using P(t) s, before a steady state is reached, and shows the utility of the estimators, t_1 and t_2.

For an example of the M-policy stochastic matrix with application in the field of queue theory, consider the following situation. Suppose that a clinic, with M staff physicians, has a waiting room with capacity X - M (=a). Assume that the service time for each patient is the same, I_t, and that patients are admitted from the waiting room only at the termination of the interval I_t. Also, assume the number of patients that arrive in the waiting room, during I_t, is a Poisson variable, and that any patient that arrives when the waiting room is full goes elsewhere for medical attention.

This queue system, with queue length the random variable, is seen to be analogous to the M-policy inventory system. The stochastic matrix for $M = 2$ and $a = 4$ is

No. of patients at time t+1

No. of patients in waiting room at time t

$$
\begin{array}{c}
\\ 4 \\ 3 \\ 2 \\ 1 \\ 0
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \end{array} \\
\left[\begin{array}{ccccc}
\sum_{i=2}^{\infty} p_i & p_1 & p_0 & 0 & 0 \\
\sum_{i=3}^{\infty} p_i & p_2 & p_1 & p_0 & 0 \\
\sum_{i=4}^{\infty} p_i & p_3 & p_2 & p_1 & p_0 \\
\sum_{i=5}^{\infty} p_i & p_4 & p_3 & p_2 & p_1+p_0 \\
\sum_{i=6}^{\infty} p_i & p_5 & p_4 & p_3 & \sum_{i=0}^{2} p_i
\end{array}\right]
\end{array}
$$

It is now apparent that table 5 and table 7 could be employed to determine the stationary distribution and rate of convergence for the problem. Knowing this, information concerning the clinic may be computed, i.e., average number of patients in the waiting room, average waiting time, average number of patients turned away, average idle time per physician, etc. From this. optimal levels could be obtained for M and a.

The stochastic matrix for M-policy can also be used to characterize the behavior of a finite dam under certain conditions. Suppose that a dam of capacity X, receives a random amount of water each year during the wet season. This amount is added to the water already in storage and the content of the dam will be less than X, or in the case of overflow will equal X. Say a quantity of water, M, is released each year, during the dry season, for irrigation and in the case when the dam does not contain, M, the entire amount in storage is released. If we consider the discrete analogue, where input is assumed to be discrete quantities of water, the stochastic matrix which describes the operation of the dam is the M-policy matrix. Thus, for a given rainfall distribution, the techniques developed in this thesis could be applied to determine the properties of the dam, i.e., average content, average ammount available for irrigation, etc.

From the variety of these examples it is evident that the M-policy is quite versatile in its applications. For this reason it is felt that the techniques developed, and similar extentions of these techniques, will find application in many areas concerned with Markov process theory.

CONCLUSIONS

From this thesis it becomes evident that the technique developed may be applied to any regular Markov chain for which the transition probability matrix is known. The M-policy model was followed to illustrate the technique with the hope that researchers and industrialists faced with similar problems, will find it useful.

The mathematics involved in determining π and λ_2 has been made feasible with the arrival of the high speed computer. With judicious programming, tables such as found in the Appendix can be readily obtained.

It is felt that the most significant development of this thesis is the technique developed to characterize the rate of convergence of an entire inventory process. It is hoped that the present effort will serve as a guide for future development of the technique to more complicated and more realistic systems. It is felt that the technique will have particular utility in queueing and storage theory where stochastic processes have extensive applicability.

ACKNOWLEDGMENT

This writer would like to express his gratitude to Dr. Roslan L. Chaddha, Kansas State University, for his untiring and encouraging attention during the preparation of this thesis and to the Computing Center, Kansas State University, for making available their I.B.M. 1620 computing facilities for the necessary numerical computation. This work was initiated during the summer of 1961 with the help of the National Science Foundation Undergraduate Research Participation Program.

APPENDIX

EXPLANATION OF TABLES

Table 4. (page 51) lists the stationary probability distribution,

$\pi = \quad \pi_i$

i = number of items in the inventory, $M \leq i \leq X$

π_i = probability of having i items in the inventory

for the M-policy stochastic matrix with geometric demand when $X = 2(1)14$, $M = 1(1)13$, and $p = .1(.1).9$.

Table 5. (page 70) is similar to Table 4. and lists the π_i for Poisson demand when $X = 2(1)13$, $M = 1(1)12$, and $m = 1(1)9$.

Table 6. (page 85) lists the second largest characteristic root, λ_2, of the M-policy stochastic matrix with geometric demand when $X = 2(1)14$, $M = 1(1)13$, and $p = .1(.1).9$.

Table 7. (page 88) lists λ_2 for Poisson demand when $X = 2(1)13$, $M = 1(1)12$, and $m = 1(1)9$.

Table 4.

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
	M=1, X=2								
1	.8901	.7619	.6203	.4737	.3333	.2105	.1139	.0476	.0110
2	.1099	.2381	.3797	.5263	.6667	.7895	.8861	.9524	.9890
	M=1, X=3								
1	.8890	.7529	.5914	.4154	.2500	.1231	.0466	.0118	.0012
2	.0988	.1882	.2534	.2769	.2500	.1846	.1086	.0471	.0110
3	.0122	.0588	.1552	.3077	.5000	.6923	.8448	.9411	.9878
	M=1, X=4								
1	.8889	.7507	.5798	.3839	.2000	.0758	.0196	.0029	.0001
2	.0988	.1876	.2485	.2559	.2000	.1137	.0456	.0117	.0012
3	.0108	.0469	.1065	.1706	.2000	.1706	.1065	.0469	.0110
4	.0014	.0147	.0652	.1896	.4000	.6398	.8283	.9384	.9877
	M=1, X=5								
1	.8889	.7502	.5750	.3654	.1667	.0481	.0083	.0007	
2	.0988	.1875	.2464	.2436	.1667	.0722	.0194	.0029	.0001
3	.0110	.0469	.1056	.1624	.1667	.1083	.1453	.0117	.0012
4	.0012	.0117	.0453	.1083	.1667	.1624	.1056	.0469	.0110
5	.0001	.0037	.0277	.1203	.3333	.6090	.8214	.9377	.9877
	M=1, X=6								
1	.8889	.7500	.5729	.3541	.1429	.0311	.0036	.0002	
2	.0988	.1875	.2455	.2360	.1429	.0466	.0082	.0007	
3	.0110	.0469	.1052	.1574	.1429	.0699	.0193	.0029	.0001
4	.0012	.0117	.0451	.1049	.1429	.1049	.0451	.0117	.0012
5	.0001	.0029	.0193	.0699	.1429	.1574	.1052	.0469	.0110
6		.0009	.0118	.0777	.2857	.5901	.8185	.9376	.9877
	M=1, X=7								
1	.8889	.7500	.5721	.3469	.1250	.0203	.0015		
2	.0988	.1875	.2452	.2312	.1250	.0305	.0035	.0002	
3	.0110	.0469	.1051	.1542	.1250	.0457	.0082	.0007	
4	.0012	.0117	.0450	.1028	.1250	.0685	.0193	.0029	.0001
5	.0001	.0029	.0193	.0685	.1250	.1028	.0450	.0117	.0012
6		.0007	.0082	.0457	.1250	.1542	.1051	.0469	.0110
7		.0002	.0051	.0508	.2500	.5781	.8173	.9375	.9877

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=1, X=8									
1	.8889	.7500	.5717	.3422	.1111	.0134	.0007		
2	.0988	.1875	.2450	.2281	.1111	.0201	.0015		
3	.0110	.0469	.1050	.1521	.1111	.0301	.0035	.0002	
4	.0012	.0117	.0450	.1014	.1111	.0451	.0083	.0007	
5	.0001	.0029	.0193	.0676	.1111	.0676	.0193	.0029	.0001
6		.0007	.0083	.0451	.1111	.1014	.0450	.0117	.0012
7		.0002	.0035	.0301	.1111	.1521	.1050	.0469	.0110
8		.0001	.0022	.0334	.2222	.5704	.8167	.9375	.9877
M=1, X=9									
1	.8889	.7500	.5715	.3392	.1000	.0088	.0003		
2	.0988	.1875	.2449	.2261	.1000	.0132	.0007		
3	.0110	.0469	.1050	.1508	.1000	.0199	.0015		
4	.0012	.0117	.0450	.1005	.1000	.0298	.0035	.0002	
5	.0001	.0029	.0193	.0670	.1000	.0441	.0083	.0007	
6		.0007	.0083	.0447	.1000	.0670	.0193	.0029	.0001
7		.0002	.0035	.0298	.1000	.1005	.0450	.0117	.0012
8			.0015	.0199	.1000	.1508	.1050	.0469	.0110
9			.0009	.0221	.2000	.5654	.8165	.9375	.9877
M=1, X=10									
1	.8889	.7500	.5715	.3372	.0909	.0058	.0001		
2	.0988	.1875	.2449	.2248	.0909	.0088	.0003		
3	.0110	.0469	.1050	.1499	.0909	.0132	.0007		
4	.0012	.0117	.0450	.0999	.0909	.0197	.0015		
5	.0001	.0029	.0193	.0666	.0909	.0296	.0035	.0002	
6		.0007	.0083	.0444	.0909	.0444	.0083	.0007	
7		.0002	.0035	.0296	.0909	.0666	.0193	.0029	.0001
8			.0015	.0197	.0909	.0999	.0450	.0117	.0012
9			.0007	.0132	.0909	.1499	.1050	.0469	.0110
10			.0004	.0146	.1818	.5621	.8164	.9375	.9878
M=1, X=11									
1	.8889	.7500	.5714	.3359	.0833	.0039	.0001		
2	.0988	.1875	.2449	.2239	.0833	.0058	.0001		
3	.0110	.0469	.1050	.1493	.0833	.0087	.0003		
4	.0012	.0117	.0450	.0995	.0833	.0131	.0007		
5	.0001	.0029	.0193	.0664	.0833	.0197	.0015		
6		.0007	.0083	.0442	.0833	.0295	.0035	.0002	
7		.0002	.0035	.0295	.0833	.0442	.0083	.0007	
8			.0015	.0197	.0833	.0664	.0193	.0029	.0001
9			.0007	.0131	.0833	.0995	.0450	.0117	.0012
10			.0003	.0087	.0833	.1493	.1050	.0469	.0110
11			.0002	.0097	.1667	.5599	.8164	.9375	.9877

Table 4.
(cont.)

M=1, X=12

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.8889	.7500	.5714	.3350	.0769	.0026			
2	.0988	.1875	.2449	.2237	.0769	.0039	.0001		
3	.0110	.0469	.1050	.1489	.0769	.0058	.0001		
4	.0012	.0117	.0450	.0993	.0769	.0087	.0003		
5	.0001	.0029	.0193	.0662	.0769	.0131	.0007		
6		.0007	.0083	.0441	.0769	.0196	.0015		
7		.0002	.0035	.0294	.0769	.0294	.0035	.0002	
8			.0015	.0196	.0769	.0441	.0083	.0007	
9			.0007	.0131	.0769	.0662	.0193	.0029	.0001
10			.0003	.0087	.0769	.0993	.0450	.0117	.0012
11			.0001	.0058	.0769	.1489	.1050	.0469	.0110
12			.0001	.0064	.1538	.5584	.8163	.9375	.9877

M=1, X=13

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.8889	.7500	.5714	.3345	.0714	.0017			
2	.0988	.1875	.2449	.2230	.0714	.0026			
3	.0110	.0469	.1050	.1487	.0714	.0039	.0001		
4	.0012	.0117	.0450	.0991	.0714	.0058	.0001		
5	.0001	.0029	.0193	.0661	.0714	.0087	.0003		
6		.0007	.0083	.0440	.0714	.0131	.0007		
7		.0002	.0035	.0295	.0714	.0196	.0015		
8			.0015	.0196	.0714	.0294	.0035	.0002	
9			.0007	.0131	.0714	.0440	.0083	.0007	
10			.0003	.0087	.0714	.0661	.0193	.0029	.0001
11			.0001	.0058	.0714	.0991	.0450	.0117	.0012
12			.0001	.0039	.0714	.1487	.1050	.0469	.0110
13				.0043	.1428	.5575	.8163	.9375	.9877

M=1, X=14

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.8889	.7500	.5714	.3340	.0667	.0011			
2	.0988	.1875	.2449	.2227	.0667	.0017			
3	.0110	.0469	.1050	.1485	.0667	.0026			
4	.0012	.0117	.0450	.0990	.0667	.0039	.0001		
5	.0001	.0029	.0193	.0660	.0667	.0058	.0001		
6		.0007	.0083	.0440	.0667	.0087	.0003		
7		.0001	.0035	.0293	.0667	.0130	.0007		
8			.0015	.0196	.0667	.0196	.0015		
9			.0007	.0130	.0667	.0293	.0035	.0002	
10			.0003	.0087	.0667	.0440	.0083	.0007	
11			.0001	.0058	.0667	.0660	.0193	.0029	.0001
12			.0001	.0039	.0667	.0990	.0450	.0117	.0012
13				.0026	.0667	.1485	.1050	.0469	.0110
14				.0029	.1333	.5568	.8163	.9375	.9877

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
	M=2, X=3								
2	.7932	.5872	.4021	.2523	.1429	.0708	.0288	.0083	.0010
3	.2067	.4128	.5979	.7477	.8571	.9292	.9712	.9917	.9989
	M=2, X=4								
2	.7829	.5505	.3401	.1820	.0833	.0317	.0093	.0017	.0001
3	.0870	.1376	.1458	.2113	.0833	.0475	.0216	.0068	.0009
4	.1301	.3118	.5142	.6966	.8333	.9208	.9691	.9914	.9990
	M=2, X=5								
2	.7800	.5319	.3007	.1369	.0500	.0144	.0030	.0004	
3	.0867	.1330	.1289	.9127	.0500	.0216	.0070	.0014	.0001
4	.0963	.1662	.1841	.1521	.1000	.0539	.0233	.0071	.0010
5	.0370	.1688	.3864	.6197	.8000	.9101	.9667	.9912	.9989
	M=2, X=6								
2	.7786	.5197	.2713	.1049	.0303	.0066	.0010	.0001	
3	.0865	.1299	.1163	.0699	.0303	.0098	.0023	.0003	
4	.0916	.1624	.1661	.1166	.0606	.0246	.0075	.0015	.0001
5	.0203	.0731	.1210	.1243	.0909	.0516	.0229	.0070	.0009
6	.0185	.1148	.3252	.5843	.7879	.9075	.9664	.9911	.9990
	M=2, X=7								
2	.7781	.5126	.2497	.0818	.0185	.0030	.0003		
3	.0865	.1281	.1070	.0545	.0185	.0045	.0007	.0001	
4	.0961	.1602	.1529	.0909	.0370	.0112	.0024	.0003	
5	.0203	.0721	.1114	.0970	.0555	.0236	.0074	.0015	.0001
6	.0129	.0581	.1132	.1252	.0926	.0522	.0230	.0070	.0009
7	.0062	.0690	.2658	.5506	.7778	.9056	.9661	.9911	.9989
	M=2, X=8								
2	.7779	.5079	.2328	.0645	.0114	.0014	.0001		
3	.0864	.1270	.0998	.0430	.0114	.0021	.0002		
4	.0960	.1587	.1425	.0717	.0227	.0051	.0008	.0001	
5	.0203	.0714	.1038	.0764	.0341	.0108	.0024	.0003	
6	.0129	.0575	.1056	.0987	.0568	.0238	.0074	.0015	.0001
7	.0037	.0322	.0897	.1168	.0909	.0519	.0229	.0070	.0009
8	.0027	.0451	.2258	.5289	.7727	.9049	.9661	.9911	.9990

Table 4.
(cont.)

M=2, X=9

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
2	.7778	.5051	.2194	.0513	.0070	.0006			
3	.0864	.1263	.0940	.0342	.0070	.0009	.0001		
4	.0960	.1578	.1343	.0570	.0140	.0023	.0003		
5	.0203	.0710	.0979	.0608	.0210	.0049	.0008	.0001	
6	.0129	.0572	.0995	.0786	.0350	.0109	.0024	.0003	
7	.0037	.0321	.0846	.0929	.0560	.0237	.0074	.0015	.0001
8	.0018	.0223	.0789	.1143	.0909	.0520	.0229	.0070	.0009
9	.0010	.0282	.1913	.5108	.7692	.9046	.9661	.9911	.9990

M=2, X=10

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
2	.7778	.5032	.2086	.0411	.0043	.0003			
3	.0864	.1258	.0894	.0274	.0043	.0004			
4	.0960	.1573	.1277	.0457	.0086	.0010	.0001		
5	.0203	.0708	.0930	.0487	.0129	.0023	.0003		
6	.0129	.0570	.0946	.0629	.0216	.0050	.0008	.0001	
7	.0037	.0319	.0804	.0744	.0345	.0109	.0024	.0003	
8	.0018	.0222	.0750	.0916	.0560	.0238	.0074	.0015	.0001
9	.0006	.0135	.0666	.1107	.0905	.0519	.0229	.0070	.0009
10	.0004	.0182	.1647	.4975	.7672	.9044	.9661	.9911	.9990

M=2, X=11

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
2	.7778	.5021	.1997	.3310	.0027	.0001			
3	.0864	.1255	.0856	.0221	.0027	.0002			
4	.0960	.1569	.1223	.0368	.0053	.0005			
5	.0203	.0706	.0891	.0392	.0080	.0010	.0001		
6	.0129	.0569	.0906	.0507	.0133	.0023	.0003		
7	.0037	.0319	.0770	.0599	.0213	.0050	.0008	.0001	
8	.0018	.0222	.0718	.0737	.0346	.0109	.0024	.0003	
9	.0006	.0135	.0638	.0891	.0559	.0238	.0074	.0015	.0001
10	.0003	.0089	.0581	.1086	.0904	.0519	.0229	.0070	.0009
11	.0002	.0116	.1422	.4867	.7660	.9043	.9661	.9911	.9990

M=2, X=12

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
2	.7778	.5013	.1923	.0268	.0016	.0001			
3	.0864	.1253	.0824	.0178	.0016	.0001			
4	.0960	.1567	.1177	.0297	.0033	.0002			
5	.0203	.0705	.0858	.0317	.0049	.0005			
6	.0129	.0570	.0872	.0410	.0082	.0010	.0001		
7	.0037	.0318	.0741	.0485	.0131	.0023	.0003		
8	.0018	.0222	.0691	.0596	.0213	.0050	.0008	.0001	
9	.0006	.0135	.0614	.0721	.0345	.0109	.0024	.0003	
10	.0003	.0089	.0559	.0878	.0558	.0238	.0074	.0015	.0001
11	.0001	.0056	.0503	.1066	.0903	.0519	.0229	.0070	.0009
12	.0001	.0074	.1237	.4784	.7652	.9043	.9661	.9911	.9990

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
	M=2, X=13								
2	.7778	.5008	.1860	.0217	.0010				
3	.0864	.1252	.0797	.0145	.0010				
4	.0960	.1565	.1139	.0241	.0020	.0001			
5	.0203	.0704	.0830	.0257	.0030	.0002			
6	.0129	.0567	.0844	.0332	.0051	.0005			
7	.0037	.0318	.0717	.0393	.0081	.0010	.0001		
8	.0018	.0221	.0669	.0484	.0132	.0023	.0003		
9	.0006	.0135	.0594	.0585	.0213	.0050	.0008	.0001	
10	.0003	.0089	.0541	.0712	.0345	.0109	.0024	.0003	
11	.0001	.0056	.0487	.0865	.0558	.0238	.0074	.0015	.0001
12		.0036	.0441	.1051	.0903	.0519	.0229	.0070	.0009
13		.0047	.1081	.4717	.7647	.9043	.9661	.9911	.9990
	M=2, X=14								
2	.7778	.5005	.1807	.0177	.0006				
3	.0864	.1251	.0775	.0118	.0006				
4	.0960	.1564	.1106	.0196	.0013				
5	.0203	.0704	.0806	.0209	.0019	.0001			
6	.0129	.0567	.0820	.0270	.0031	.0002			
7	.0037	.0318	.0697	.0320	.0050	.0005			
8	.0018	.0221	.0650	.0394	.0081	.0010	.0001		
9	.0006	.0135	.0577	.0476	.0132	.0023	.0003		
10	.0003	.0089	.0526	.0579	.0213	.0050	.0008	.0001	
11	.0001	.0056	.0473	.0703	.0345	.0109	.0024	.0003	
12		.0036	.0428	.0855	.0558	.0238	.0074	.0015	.0001
13		.0023	.0386	.1039	.0902	.0519	.0229	.0070	.0009
14		.0030	.0948	.4663	.7644	.9043	.9661	.9911	.9990
	M=3, X=4								
3	.7073	.4563	.2676	.1419	.0667	.0266	.0083	.0016	.0001
4	.2923	.5437	.7324	.8581	.9333	.9734	.9917	.9984	.9999
	M=3, X=5								
3	.6913	.4120	.2116	.0940	.0357	.0111	.0025	.0003	
4	.0768	.1030	.0907	.0627	.0357	.0166	.0059	.0013	.0001
5	.2319	.4849	.6977	.8433	.9286	.9723	.9916	.9984	.9999
	M=3, X=6								
3	.6802	.3784	.1702	.0630	.0192	.0046	.0008	.0001	
4	.0756	.0946	.0729	.0420	.0192	.0069	.0018	.0003	
5	.0840	.1182	.1042	.0700	.0385	.0174	.0060	.0013	.0001
6	.1602	.4088	.6527	.8250	.9231	.9711	.9914	.9984	.9999

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=3, X=7									
3	.6752	.3552	.1400	.0428	.0104	.0019	.0002		
4	.0750	.0888	.0600	.0285	.0104	.0029	.0006	.0001	
5	.0834	.1110	.0857	.0475	.0208	.0073	.0018	.0003	
6	.0926	.1388	.1224	.0792	.0417	.0181	.0061	.0013	.0001
7	.0738	.3062	.5919	.8020	.9167	.9698	.9912	.9984	.9999
M=3, X=8									
3	.6718	.3366	.1162	.0292	.0056	.0008	.0001		
4	.0746	.0841	.0498	.0195	.0056	.0012	.0002		
5	.0829	.1052	.0711	.0324	.0113	.0030	.0006	.0001	
6	.0921	.1315	.1016	.0541	.0226	.0076	.0019	.0003	
7	.0277	.0802	.0954	.0706	.0395	.0178	.0061	.0013	.0001
8	.0508	.2625	.5658	.7942	.9152	.9696	.9912	.9984	.9999
M=3, X=9									
3	.6696	.3218	.0974	.0200	.0031	.0003			
4	.0744	.0805	.0417	.0133	.0031	.0005	.0001		
5	.0827	.1006	.0596	.0222	.0061	.0013	.0002		
6	.0919	.1257	.0852	.0370	.0123	.0032	.0006	.0001	
7	.0277	.0767	.0799	.0484	.0215	.0074	.0019	.0003	
8	.0225	.0757	.0963	.0718	.0399	.0178	.0061	.0013	.0001
9	.0313	.2190	.5398	.7873	.9141	.9694	.9912	.9984	.9999
M=3, X=10									
3	.6685	.3102	.0823	.0137	.0017	.0001			
4	.0743	.0776	.0353	.0092	.0017	.0002			
5	.0825	.0969	.0504	.0153	.0033	.0005	.0001		
6	.0917	.1213	.0720	.0254	.0067	.0013	.0002		
7	.0276	.0739	.0676	.0333	.0117	.0031	.0006	.0001	
8	.0224	.0730	.0814	.0493	.0217	.0075	.0019	.0003	
9	.0157	.0670	.0947	.0720	.0400	.0178	.0061	.0013	.0001
10	.0172	.1801	.5164	.7818	.9133	.9694	.9912	.9984	.9999

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=3, X=11									
3	.6678	.3007	.0699	.0095	.0009	.0001			
4	.0742	.0752	.0300	.0063	.0009	.0001			
5	.0824	.0940	.0428	.0105	.0018	.0002			
6	.0916	.1175	.0612	.0175	.0036	.0006	.0001		
7	.0276	.0717	.0574	.0229	.0063	.0013	.0002		
8	.0224	.0708	.0692	.0340	.0118	.0031	.0006	.0001	
9	.0157	.0650	.0805	.0496	.0217	.0075	.0019	.0003	
10	.0073	.0518	.0887	.0709	.0399	.0178	.0061	.0013	.0001
11	.0110	.1534	.5002	.7788	.9130	.9694	.9912	.9984	.9999
M=3, X=12									
3	.6673	.2929	.0598	.0065	.0005				
4	.0741	.0732	.0256	.0043	.0005				
5	.0824	.0915	.0366	.0072	.0010	.0001			
6	.0915	.1144	.0523	.0121	.0020	.0002			
7	.0276	.0698	.0491	.0158	.0034	.0005	.0001		
8	.0224	.0689	.0591	.0234	.0064	.0013	.0002		
9	.0157	.0633	.0687	.0342	.0118	.0031	.0006	.0001	
10	.0073	.0505	.0758	.0489	.0217	.0075	.0019	.0003	
11	.0050	.0457	.0873	.0710	.0398	.0178	.0061	.0013	.0001
12	.0066	.1297	.4858	.7765	.9123	.9694	.9912	.9984	.9999
M=3, X=13									
3	.6671	.2865	.0513	.0045	.0003				
4	.0741	.0716	.0220	.0030	.0003				
5	.0824	.0895	.0314	.0050	.0005				
6	.0915	.1119	.0448	.0083	.0011	.0001			
7	.0276	.0683	.0421	.0109	.0019	.0002			
8	.0224	.0674	.0507	.0162	.0035	.0005	.0001		
9	.0157	.0619	.0590	.0236	.0064	.0013	.0002		
10	.0073	.0494	.0650	.0338	.0118	.0031	.0006	.0001	
11	.0050	.0447	.0749	.0490	.0217	.0075	.0019	.0003	
12	.0031	.0390	.0852	.0709	.0399	.0178	.0061	.0013	.0001
13	.0039	.1098	.4736	.7749	.9127	.9694	.9912	.9984	.9999

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=3, X=14									
3	.6669	.2811	.0441	.0031	.0001				
4	.0741	.0703	.0189	.0021	.0001				
5	.0823	.0878	.0270	.0035	.0003				
6	.0915	.1098	.0386	.0058	.0006				
7	.0275	.0670	.0362	.0075	.0010	.0001			
8	.0224	.0662	.0436	.0112	.0019	.0002			
9	.0157	.0607	.0508	.0163	.0035	.0005	.0001		
10	.0073	.0485	.0560	.0233	.0064	.0013	.0002		
11	.0050	.0438	.0645	.0338	.0118	.0031	.0006	.0001	
12	.0031	.0383	.0734	.0489	.0217	.0075	.0019	.0003	
13	.0017	.0326	.0831	.0707	.0399	.0178	.0061	.0013	.0001
14	.0024	.0939	.4638	.7739	.9127	.9694	.9912	.9984	.9999
M=4, X=5									
4	.6320	.3569	.1811	.0820	.0323	.0104	.0024	.0003	
5	.3638	.6431	.8189	.9180	.9677	.9896	.9976	.9997	1.
M=4, X=6									
4	.6117	.3135	.1375	.0521	.0167	.0042	.0007	.0001	
5	.0680	.0784	.0589	.0347	.0167	.0063	.0017	.0003	
6	.3203	.6081	.8036	.9132	.9667	.9894	.9975	.9997	1.
M=4, X=7									
4	.5955	.2780	.1051	.0331	.0086	.0017	.0002		
5	.0662	.0695	.0450	.0221	.0086	.0026	.0005	.0001	
6	.0735	.0869	.0643	.0368	.0172	.0064	.0017	.0003	
7	.2648	.5655	.7856	.9079	.9655	.9893	.9975	.9997	1.
M=4, X=8									
4	.5836	.2495	.0810	.0212	.0045	.0007	.0001		
5	.0648	.0624	.0347	.0141	.0045	.0010	.0002		
6	.0721	.0780	.0496	.0235	.0089	.0026	.0005	.0001	
7	.0801	.0975	.0708	.0392	.0179	.0065	.0017	.0003	
8	.1994	.5126	.7639	.9019	.9643	.9891	.9975	.9997	1.

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=4, X=9									
4	.5766	.2273	.0631	.0136	.0023	.0003			
5	.0641	.0568	.0270	.0091	.0023	.0004			
6	.0712	.0710	.0386	.0151	.0046	.0011	.0002		
7	.0791	.0888	.0552	.0252	.0093	.0027	.0005	.0001	
8	.0879	.1110	.0788	.0420	.0185	.0067	.0017	.0003	
9	.1212	.4450	.7373	.8950	.9630	.9889	.9975	.9997	1.
M=4, X=10									
4	.5710	.2084	.0493	.0087	.0012	.0001			
5	.0634	.0521	.0211	.0058	.0012	.0002			
6	.0705	.0651	.0302	.0097	.0024	.0004			
7	.0783	.0814	.0431	.0162	.0048	.0011	.0002		
8	.0870	.1018	.0616	.0270	.0096	.0027	.0005	.0001	
9	.0332	.0751	.0669	.0391	.0180	.0066	.0017	.0003	
10	.0965	.4161	.7278	.8934	.9628	.9889	.9975	.9997	1.
M=4, X=11									
4	.5668	.1923	.0387	.0056	.0006				
5	.0630	.0481	.0166	.0037	.0006	.0001			
6	.0700	.0601	.0237	.0062	.0012	.0002			
7	.0778	.0751	.0338	.0104	.0025	.0004			
8	.0864	.0939	.0483	.0174	.0050	.0011	.0002		
9	.0330	.0693	.0524	.0252	.0093	.0027	.0005	.0001	
10	.0297	.0746	.0678	.0395	.0181	.0066	.0017	.0003	
11	.0734	.3866	.7187	.8920	.9626	.9889	.9975	.9997	1.
M=4, X=12									
4	.5638	.1785	.0304	.0036	.0003				
5	.0626	.0446	.0130	.0024	.0003				
6	.0696	.0558	.0186	.0040	.0006	.0001			
7	.0773	.0697	.0266	.0067	.0013	.0002			
8	.0859	.0872	.0380	.0112	.0026	.0004			
9	.0328	.0643	.0413	.0162	.0048	.0011	.0002		
10	.0295	.0693	.0534	.0254	.0094	.0027	.0005	.0001	
11	.0251	.0726	.0683	.0397	.0181	.0066	.0017	.0003	
12	.0533	.3579	.7104	.8908	.9625	.9889	.9975	.9996	1.

Table 4.
(cont.)

M=4, X=13

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
4	.5617	.1667	.0240	.0023	.0002				
5	.0624	.0417	.0103	.0016	.0002				
6	.0693	.0521	.0147	.0026	.0003				
7	.0770	.0651	.0210	.0043	.0007	.0001			
8	.0856	.0814	.0300	.0072	.0013	.0002			
9	.0327	.0601	.0326	.0104	.0025	.0004			
10	.0294	.0647	.0421	.0164	.0049	.0011	.0001		
11	.0250	.0678	.0539	.0255	.0094	.0027	.0005	.0001	
12	.0192	.0685	.0679	.0397	.0181	.0066	.0017	.0002	
13	.0376	.3321	.7036	.8900	.9625	.9889	.9975	.9997	1.

M=4, X=14

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
4	.5601	.1563	.0190	.0015	.0001				
5	.0622	.0391	.0081	.0010	.0001				
6	.0691	.0488	.0116	.0017	.0002				
7	.0768	.0610	.0166	.0028	.0003				
8	.0854	.0763	.0237	.0046	.0007	.0001			
9	.0326	.0563	.0257	.0067	.0013	.0002			
10	.0293	.0606	.0333	.0105	.0025	.0004			
11	.0249	.0636	.0426	.0164	.0049	.0011	.0002		
12	.0193	.0642	.0537	.0256	.0094	.0027	.0005	.0001	
13	.0118	.0612	.0665	.0395	.0181	.0066	.0017	.0003	
14	.0286	.3127	.6992	.8896	.9624	.9881	.9975	.9997	1.

M=5, X=6

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
5	.5648	.2805	.1239	.0482	.0159	.0041	.0007	.0001	
6	.4352	.7195	.8761	.9518	.9841	.9959	.9993	.9999	1.

M=5, X=7

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
5	.5423	.2413	.0916	.0299	.0806	.0017	.0002		
6	.0603	.0603	.0393	.0199	.0806	.0025	.0005	.0001	
7	.3974	.6983	.8692	.9502	.9839	.9959	.9993	.9999	1.

M=5, X=8

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
5	.5231	.2088	.0679	.0185	.0041	.0007	.0001		
6	.0581	.0522	.0291	.0123	.0041	.0010	.0002		
7	.0646	.0653	.0416	.0206	.0082	.0025	.0005	.0001	
8	.3541	.6737	.8614	.9485	.9836	.9958	.9992	.9999	1.

Table 4.
(cont.)

M=5, X=9

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
5	.5072	.1819	.0505	.0115	.0021	.0003			
6	.0564	.0455	.0217	.0077	.0021	.0004			
7	.0626	.0568	.0309	.0128	.0042	.0010	.0002		
8	.0696	.0711	.0442	.0213	.0083	.0025	.0005	.0001	
9	.3042	.6447	.8526	.9467	.9833	.9958	.9993	.9999	1.

M=5, X=10

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
5	.4948	.1597	.0378	.0072	.0011	.0001			
6	.0550	.0399	.0162	.0048	.0011	.0002			
7	.0611	.0499	.0231	.0080	.0021	.0004			
8	.0679	.0624	.0330	.0133	.0042	.0010	.0002		
9	.0754	.0780	.0472	.0221	.0085	.0025	.0005	.0001	
10	.2459	.6101	.8427	.9448	.9831	.9958	.9993	.9999	1.

M=5, X=11

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
5	.4860	.1416	.0283	.0045	.0005				
6	.0540	.0354	.0121	.0030	.0005	.0001			
7	.0600	.0442	.0174	.0050	.0011	.0002			
8	.0667	.0553	.0248	.0083	.0022	.0004			
9	.0741	.0691	.0354	.0138	.0043	.0010	.0002		
10	.0823	.0864	.0506	.0229	.0086	.0026	.0005	.0001	
11	.1770	.5680	.8313	.9426	.9828	.9957	.9993	.9999	1.

M=5, X=12

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
5	.4786	.1260	.0213	.0028	.0003				
6	.0532	.0315	.0091	.0019	.0003				
7	.0591	.0394	.0130	.0031	.0005	.0001			
8	.0656	.0492	.0186	.0051	.0011	.0002			
9	.0729	.0615	.0266	.0086	.0022	.0004			
10	.0810	.0769	.0380	.0143	.0044	.0010	.0002		
11	.0369	.0646	.0452	.0220	.0085	.0025	.0005	.0001	
12	.1526	.5510	.8280	.9423	.9827	.9957	.9993	.9999	1.

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
	M=5, X=13								
5	.4724	.1125	.0160	.0017	.0001				
6	.0525	.0281	.0069	.0013	.0001				
7	.0583	.0352	.0098	.0019	.0003				
8	.0648	.0440	.0140	.0032	.0006	.0001			
9	.0720	.0549	.0200	.0053	.0011	.0002			
10	.0800	.0687	.0286	.0089	.0022	.0004			
11	.0364	.0577	.0340	.0137	.0043	.0010	.0002		
12	.0346	.0651	.0456	.0221	.0085	.0025	.0005	.0001	
13	.1290	.5338	.8249	.9420	.9827	.9957	.9993	.9999	1.
	M=5, X=14								
5	.4674	.1009	.0121	.0011	.0001				
6	.0519	.0252	.0052	.0007	.0001				
7	.0577	.0315	.0074	.0012	.0001				
8	.0641	.0394	.0106	.0020	.0003				
9	.0712	.0493	.0151	.0033	.0006	.0001			
10	.0791	.0616	.0216	.0056	.0011	.0002			
11	.0360	.0517	.0256	.0085	.0022	.0004			
12	.0342	.0584	.0344	.0137	.0043	.0010	.0002		
13	.0316	.0651	.0460	.0221	.0085	.0025	.0005	.0001	
14	.1066	.5169	.8222	.9417	.9827	.9957	.9993	.9999	1.
	M=6, X=7								
6	.5051	.2213	.0854	.0285	.0079	.0016	.0002		
7	.4949	.7787	.9146	.9715	.9921	.9984	.9998	1.	1.
	M=6, X=8								
6	.4817	.1874	.0620	.0174	.0040	.0007	.0001		
7	.0535	.0469	.0266	.0116	.0040	.0010	.0002		
8	.4648	.7657	.9114	.9709	.9921	.9984	.9998	1.	1.
	M=6, X=9								
6	.4609	.1593	.0451	.0107	.0020	.0003			
7	.0512	.0398	.0193	.0071	.0020	.0004			
8	.0569	.0498	.0276	.0119	.0040	.0010	.0002		
9	.4310	.7511	.9079	.9703	.9920	.9983	.9998	1.	1.

Table 4.
(cont.)

i	v= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=6, X=10									
6	.4428	.1359	.0329	.0065	.0010	.0001			
7	.0492	.0340	.0141	.0044	.0010	.0002			
8	.0547	.0425	.0201	.0073	.0020	.0004			
9	.0607	.0531	.0288	.0121	.0040	.0010	.0002		
10	.3926	.7346	.9041	.9697	.9919	.9983	.9998	1.	1.
M=6, X=11									
6	.4274	.1164	.0240	.0040	.0005				
7	.0475	.0291	.0103	.0027	.0005	.0001			
8	.0528	.0364	.0147	.0044	.0010	.0002			
9	.0586	.0455	.0210	.0074	.0020	.0004			
10	.0651	.0568	.0300	.0123	.0040	.0010	.0002		
11	.3486	.7158	.9000	.9691	.9919	.9983	.9998	1.	1.
M=6, X=12									
6	.4146	.1003	.0176	.0025	.0003				
7	.0461	.0251	.0075	.0016	.0003				
8	.0512	.0313	.0108	.0027	.0005	.0001			
9	.0569	.0392	.0154	.0045	.0010	.0002			
10	.0632	.0490	.0219	.0076	.0020	.0004			
11	.0702	.0612	.0313	.0126	.0041	.0010	.0002		
12	.2978	.6940	.8955	.9685	.9918	.9983	.9998	1.	1.
M=6, X=13									
6	.4048	.0868	.0129	.0015	.0001				
7	.0450	.0217	.0055	.0010	.0001				
8	.0500	.0271	.0079	.0017	.0003				
9	.0555	.0339	.0113	.0028	.0005	.0001			
10	.0617	.0424	.0161	.0046	.0010	.0002			
11	.0685	.0530	.0230	.0077	.0021	.0004			
12	.0762	.0663	.0328	.0129	.0041	.0010	.0002		
13	.2384	.6687	.8906	.9678	.9917	.9983	.9998	1.	1.

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
	M=6, X=14								
6	.3960	.0754	.0094	.0009	.0001				
7	.0440	.0188	.0040	.0006	.0001				
8	.0489	.0236	.0058	.0010	.0001				
9	.0543	.0295	.0083	.0017	.0003				
10	.0604	.0348	.0118	.0028	.0005	.0001			
11	.0671	.0460	.0168	.0047	.0010	.0002			
12	.0745	.0575	.0241	.0079	.0021	.0004			
13	.0388	.0531	.0303	.0125	.0041	.0010	.0002		
14	.2160	.6593	.8895	.9677	.9917	.9983	.9998	1.	1.
	M=7, X=8								
7	.4521	.1715	.0591	.0170	.0039	.0007	.0001		
8	.5479	.8249	.9409	.9830	.9961	.9993	.9999	1.	1.
	M=7, X=9								
7	.4284	.1465	.0425	.0103	.0020	.0003			
8	.0476	.0366	.0182	.0069	.0020	.0004			
9	.5240	.8169	.9394	.9828	.9961	.9993	.9999	1.	1.
	M=7, X=10								
7	.4071	.1228	.0305	.0063	.0010	.0001			
8	.0452	.0307	.0131	.0042	.0010	.0002			
9	.0503	.0384	.0187	.0070	.0020	.0004			
10	.4974	.8081	.9377	.9826	.9960	.9993	.9999	1.	1.
	M=7, X=11								
7	.3881	.1032	.0219	.0038	.0005				
8	.0431	.0258	.0094	.0025	.0005	.0001			
9	.0479	.0323	.0134	.0042	.0010	.0002			
10	.0532	.0403	.0192	.0070	.0020	.0004			
11	.4677	.7984	.9360	.9824	.9960	.9993	1.	1.	1.
	M=7, X=12								
7	.3712	.0870	.0158	.0023	.0002				
8	.0412	.0217	.0068	.0015	.0002				
9	.0458	.0272	.0097	.0026	.0005	.0001			
10	.0509	.0400	.0138	.0043	.0010	.0002			
11	.0566	.0425	.0197	.0071	.0020	.0004			
12	.4342	.7877	.9342	.9822	.9960	.9993	1.	1.	1.

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=7, X=13									
7	.3565	.0735	.0114	.0014	.0001				
8	.0396	.0184	.0049	.0009	.0001				
9	.0440	.0230	.0070	.0016	.0002				
10	.0489	.0287	.0099	.0026	.0005	.0001			
11	.0543	.0359	.0142	.0043	.0010	.0002			
12	.0604	.0448	.0203	.0072	.0020	.0004			
13	.6037	.7758	.9323	.9820	.9960	.9993	1.	1.	1.
M=7, X=14									
7	.3439	.0623	.0082	.0009	.0001				
8	.0382	.0156	.0035	.0006	.0001				
9	.0425	.0195	.0050	.0009	.0001				
10	.0472	.0243	.0072	.0016	.0003				
11	.0524	.0304	.0102	.0026	.0005	.0001			
12	.0582	.0380	.0146	.0044	.0010	.0002			
13	.0647	.0475	.0209	.0073	.0020	.0004			
14	.3529	.7625	.9303	.9818	.9960	.9993	.9999	1.	1.
M=8, X=9									
8	.4048	.1389	.0411	.0101	.0020	.0003			
9	.5952	.8611	.9589	.9899	.9980	.9997	1.	1.	1.
M=8, X=10									
8	.3815	.1151	.0293	.0061	.0010	.0001			
9	.0424	.0288	.0125	.0041	.0010	.0002			
10	.5761	.8561	.9582	.9898	.9980	.9997	1.	1.	1.
M=8, X=11									
8	.3603	.0955	.0209	.0037	.0005				
9	.0400	.0239	.0089	.0025	.0005	.0001			
10	.0445	.0298	.0128	.0041	.0010	.0002			
11	.5551	.8508	.9574	.9897	.9980	.9997	1.	1.	1.
M=8, X=12									
8	.3412	.0794	.0149	.0022	.0002				
9	.0379	.0198	.0064	.0015	.0002				
10	.0421	.0248	.0091	.0025	.0005	.0001			
11	.0468	.0310	.0130	.0041	.0010	.0002			
12	.5320	.8450	.9566	.9896	.9980	.9997	1.	1.	1.

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=8, X=13									
8	.3239	.0661	.0106	.0014	.0001				
9	.0360	.0165	.0045	.0009	.0001				
10	.0400	.0206	.0065	.0015	.0002				
11	.0444	.0258	.0093	.0025	.0005	.0001			
12	.0494	.0323	.0133	.0042	.0010	.0002			
13	.5063	.8387	.9558	.9896	.9980	.9997	1.	1.	1.
M=8, X=14									
8	.3084	.0551	.0076	.0008	.0001				
9	.0343	.0138	.0032	.0005	.0001				
10	.0381	.0172	.0046	.0009	.0001				
11	.0423	.0215	.0066	.0015	.0002				
12	.0470	.0269	.0095	.0025	.0005	.0001			
13	.0522	.0336	.0135	.0042	.0010	.0002			
14	.4777	.8319	.9550	.9895	.9980	.9997	1.	1.	1.
M=9, X=10									
9	.3627	.1103	.0286	.0061	.0010				
10	.6373	.8897	.9714	.9939	.9990	1.	1.	1.	1.
M=9, X=11									
9	.3402	.0908	.0203	.0037	.0005				
10	.0378	.0227	.0087	.0024	.0005	.0001			
11	.6220	.8865	.9711	.9939	.9990	.9999	1.	1.	1.
M=9, X=12									
9	.3196	.0747	.0144	.0022	.0002				
10	.0355	.0187	.0062	.0015	.0002				
11	.0395	.0234	.0088	.0024	.0005	.0001			
12	.6055	.8832	.9707	.9939	.9990	.9999	1.	1.	1.
M=9, X=13									
9	.3008	.0616	.0102	.0013	.0001				
10	.0334	.0154	.0044	.0009	.0001				
11	.0371	.0192	.0062	.0015	.0002				
12	.0413	.0241	.0089	.0025	.0005	.0001			
13	.5874	.8797	.9703	.9939	.9990	.9999	1.	1.	1.

Table 4.
(cont.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=9, X=14									
9	.2837	.0508	.0072	.0008	.0001				
10	.0315	.0127	.0031	.0005	.0001				
11	.0350	.0159	.0044	.0009	.0001				
12	.0389	.0198	.0063	.0015	.0002				
13	.0432	.0248	.0090	.0025	.0005	.0001			
14	.5676	.8760	.9699	.9938	.9990	.9999	1.	1.	1.
M=10, X=11									
10	.3251	.0878	.0199	.0036	.0005				
11	.6749	.9122	.9801	.9964	.9995	1.	1.	1.	1.
M=10, X=12									
10	.3036	.0718	.0141	.0022	.0002				
11	.0337	.0180	.0060	.0015	.0002				
12	.6627	.9102	.9799	.9964	.9995	1.	1.	1.	1.
M=10, X=13									
10	.2839	.0588	.0099	.0013	.0001				
11	.0315	.0147	.0043	.0009	.0001				
12	.0350	.0184	.0061	.0015	.0002				
13	.6495	.9082	.9797	.9963	.9995	1.	1.	1.	1.
M=10, X=14									
10	.2658	.0481	.0088	.0008	.0001				
11	.0295	.0120	.0013	.0005	.0001				
12	.0328	.0150	.0043	.0009	.0001				
13	.0365	.0188	.0061	.0015	.0002				
14	.6353	.9060	.9795	.9963	.9995	1.	1.	1.	1.
M=11, X=12									
11	.2916	.0699	.0139	.0022	.0002				
12	.7084	.9301	.9861	.9978	.9998	1.	1.	1.	1.
M=11, X=13									
11	.2712	.0569	.0098	.0013	.0001				
12	.0301	.0142	.0042	.0009	.0001				
13	.6987	.9288	.9860	.9978	.9998	1.	1.	1.	1.

Table 4.
(concl.)

i	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
	M=11, X=14								
11	.2525	.0464	.0069	.0008	.0001				
12	.0281	.0116	.0030	.0005	.0001				
13	.0312	.0145	.0042	.0009	.0001				
14	.6882	.9275	.9859	.9978	.9998	1.	1.	1.	1.
	M=12, X=13								
12	.2616	.0557	.0097	.0013	.0001				
13	.7384	.9443	.9903	.9987	.9999	1.	1.	1.	1.
	M=12, X=14								
12	.2418	.0462	.0068	.0008	.0001				
13	.0290	.0110	.0029	.0005	.0001				
14	.7292	.9428	.9902	.9987	.9999	1.	1.	1.	1.
	M=13, X=14								
13	.2347	.0445	.0068	.0008	.0001				
14	.7653	.9556	.9932	.9992	.9999	1.	1.	1.	1.

Table 5.

i	m= 1	2	3	4	5	6	7	8	9
	M=1, X=2								
1	.4180	.8144	.9415	.9802	.9930	.9975	.9991	.9997	.9999
2	.5820	.1856	.0585	.0198	.0070	.0025	.0009	.0003	.0001
	M=1, X=3								
1	.2994	.8007	.9406	.9802	.9930	.9975	.9991	.9997	.9999
2	.2929	.1623	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.4077	.0370	.0035	.0004					
	M=1, X=4								
1	.2308	.7976	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.2303	.1620	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.2253	.0328	.0033	.0004					
4	.3136	.0075	.0002						
	M=1, X=5								
1	.1875	.7970	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.1875	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.1871	.0329	.0033	.0004					
4	.1830	.0067	.0002						
5	.2548	.0015							
	M=1, X=6								
1	.1579	.7968	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.1579	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.1579	.0329	.0033	.0004					
4	.1576	.0067	.0002						
5	.1541	.0014							
6	.2146	.0003							
	M=1, X=7								
1	.1364	.7968	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.1364	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.1364	.0329	.0033	.0004					
4	.1364	.0067	.0002						
5	.1361	.0014							
6	.1331	.0003							
7	.1853	.0001							

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
	M=1, X=8								
1	.1200	.7968	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.1200	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.1200	.0329	.0033	.0004					
4	.1200	.0067	.0002						
5	.1200	.0014							
6	.1197	.0003							
7	.1171	.0001							
8	.1631								
	M=1, X=9								
1	.1100	.7968	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.1098	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.1093	.0329	.0033	.0004					
4	.1087	.0067	.0002						
5	.1077	.0014							
6	.1066	.0003							
7	.1052	.0001							
8	.1019								
9	.1406								
	M=1, X=10								
1	.0968	.7968	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.0968	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.0968	.0329	.0033	.0004					
4	.0968	.0067	.0002						
5	.0968	.0014							
6	.0968	.0003							
7	.0968	.0001							
8	.0966								
9	.0945								
10	.1315								
	M=1, X=11								
1	.0882	.7968	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.0882	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.0882	.0329	.0033	.0004					
4	.0882	.0067	.0002						
5	.0082	.0014							
6	.0882	.0003							
7	.0882	.0001							
8	.0882								
9	.0880								
10	.0861								
11	.1199								

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
M=1, X=12									
1	.0827	.7968	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.0827	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.0825	.0329	.0033	.0004					
4	.0823	.0067	.0002						
5	.0820	.0014							
6	.0816	.0003							
7	.0811	.0001							
8	.0806								
9	.0801								
10	.0795								
11	.0774								
12	.1074								
M=1, X=13									
1	.0769	.7968	.9405	.9802	.9930	.9975	.9991	.9997	.9999
2	.0769	.1619	.0560	.0194	.0069	.0025	.0009	.0003	.0001
3	.0769	.0329	.0033	.0004	.0001				
4	.0769	.0067	.0002						
5	.0769	.0014							
6	.0769	.0003							
7	.0769	.0001							
8	.0769								
9	.0766								
10	.0760								
11	.0742								
12	.0694								
13	.0885								
M=2, X=3									
2	.0984	.4433	.7434	.8927	.9559	.9818	.9925	.9969	.9988
3	.9016	.5567	.2566	.1073	.0441	.0182	.0075	.0031	.0012
M=2, X=4									
2	.0317	.3365	.7086	.8855	.9545	.9816	.9925	.9969	.9988
3	.0810	.2317	.1711	.0832	.0365	.0156	.0065	.0027	.0011
4	.8873	.4318	.1202	.0313	.0090	.0029	.0010	.0003	.0001
M=2, X=5									
2	.0095	.2680	.6941	.8840	.9543	.9816	.9925	.9969	.9988
3	.0270	.1916	.1696	.0834	.0365	.0156	.0065	.0027	.0011
4	.0826	.2041	.0904	.0273	.0085	.0028	.0010	.0003	.0001
5	.8809	.3364	.0459	.0053	.0007	.0001			

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
	M=2, X=6								
2	.0027	.2216	.6877	.8836	.9543	.9116	.9925	.9969	.9988
3	.0081	.1598	.1684	.0834	.0365	.0156	.0065	.0027	.0011
4	.0276	.1755	.0905	.0274	.0085	.0028	.0010	.0003	.0001
5	.0822	.1639	.0339	.0044	.0006	.0001			
6	.8793	.2792	.0195	.0012	.0001				
	M=2, X=7								
2	.0007	.1889	.6851	.8836	.9543	.9816	.9925	.9969	.9988
3	.0023	.1363	.1678	.0833	.0365	.0156	.0065	.0027	.0011
4	.0083	.1508	.0903	.0274	.0085	.0028	.0010	.0003	.0001
5	.0275	.1455	.0341	.0044	.0006	.0001			
6	.0822	.1408	.0147	.0010	.0001				
7	.8789	.2377	.0080	.0002					
	M=2, X=8								
2	.0002	.1646	.6840	.8836	.9543	.9816	.9925	.9969	.9988
3	.0007	.1187	.1675	.0833	.0365	.0156	.0065	.0027	.0011
4	.0024	.1315	.0902	.0274	.0085	.0028	.0010	.0003	.0001
5	.0083	.1278	.0341	.0044	.0006	.0001			
6	.0275	.1277	.0148	.0010	.0001				
7	.0822	.1224	.0060	.0002					
8	.8788	.2072	.0034						
	M=2, X=9								
2	.0001	.1458	.6836	.8836	.9543	.9816	.9925	.9969	.9988
3	.0002	.1052	.1674	.0833	.0365	.0156	.0065	.0027	.0011
4	.0007	.1165	.0901	.0274	.0085	.0028	.0010	.0003	.0001
5	.0024	.1134	.0341	.0044	.0006	.0001			
6	.0083	.1141	.0148	.0010	.0001				
7	.0275	.1129	.0061	.0002					
8	.0822	.1085	.0025						
9	.8787	.1835	.0014						
	M=2, X=10								
2		.1309	.6834	.8836	.9543	.9816	.9925	.9969	.9988
3	.0001	.0944	.1673	.0833	.0365	.0156	.0065	.0027	.0011
4	.0002	.1046	.0901	.0274	.0085	.0028	.0010	.0003	.0001
5	.0007	.1017	.0341	.0044	.0006	.0001			
6	.0024	.1026	.0148	.0010	.0001				
7	.0083	.1022	.0061	.0002					
8	.0275	.1014	.0026						
9	.0822	.0974	.0010						
10	.8787	.1647	.0006						

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
	M=2, X=11								
2		.1187	.6833	.8836	.9543	.9816	.9925	.9969	.9988
3		.0857	.1673	.0833	.0365	.0156	.0065	.0027	.0011
4	.0001	.0949	.0901	.0274	.0085	.0028	.0010	.0003	.0001
5	.0002	.0923	.0341	.0044	.0006	.0001			
6	.0007	.0930	.0148	.0010	.0001				
7	.0024	.0928	.0061	.0002					
8	.0083	.0928	.0026						
9	.0275	.0920	.0011						
10	.0822	.0883	.0004						
11	.8787	.1494	.0002						
	M=2, X=12								
2		.1086	.6833	.8836	.9543	.9816	.9925	.9969	.9988
3		.0784	.1673	.0833	.0365	.0156	.0065	.0027	.0011
4		.0868	.0901	.0274	.0085	.0028	.0010	.0003	.0001
5	.0001	.0845	.0341	.0044	.0006	.0001			
6	.0002	.0851	.0148	.0010	.0001				
7	.0007	.0849	.0061	.0002					
8	.0024	.0850	.0026						
9	.0083	.0849	.0011						
10	.0275	.0842	.0004						
11	.0827	.0808	.0002						
12	.8787	.1367	.0001						
	M=2, X=13								
2		.1001	.6832	.8836	.9543	.9816	.9925	.9969	.9988
3		.0722	.1673	.0833	.0365	.0156	.0065	.0027	.0011
4		.0800	.0901	.0274	.0085	.0028	.0010	.0003	.0001
5		.0778	.0341	.0044	.0006	.0001			
6	.0001	.0784	.0148	.0010	.0001				
7	.0002	.0783	.0061	.0002					
8	.0007	.0783	.0026						
9	.0024	.0783	.0011						
10	.0083	.0783	.0004						
11	.0275	.0776	.0002						
12	.0822	.0745	.0001						
13	.8787	.1260							
	M=3, X=4								
3	.0202	.1743	.4546	.7041	.8550	.9320	.9687	.9858	.9937
4	.9798	.8257	.5454	.2959	.1450	.0680	.0313	.0142	.0063

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
	M=3, X=5								
3	.0042	.0866	.3590	.6615	.8422	.9286	.9679	.9856	.9936
4	.0168	.1163	.1993	.1631	.0953	.0488	.0236	.0111	.0051
5	.9789	.7971	.4417	.1755	.0625	.0225	.0085	.0033	.0013
	M=3, X=6								
3	.0008	.0412	.2918	.6369	.8370	.9276	.9677	.9856	.9936
4	.0036	.0610	.1670	.1606	.0957	.0490	.0237	.0111	.0051
5	.0170	.1195	.1823	.1097	.0464	.0183	.0072	.0029	.0012
6	.9786	.7783	.3559	.0927	.0208	.0050	.0014	.0004	.0001
	M=3, X=7								
3	.0001	.0193	.2447	.6239	.8353	.9274	.9677	.9856	.9936
4	.0007	.0298	.1447	.1582	.0957	.0490	.0237	.0111	.0051
5	.0037	.0631	.1607	.1096	.0467	.0183	.0072	.0029	.0012
6	.0170	.1186	.1540	.0602	.0161	.0043	.0013	.0004	.0001
7	.9785	.7692	.2959	.0481	.0062	.0008	.0001		
	M=3, X=8								
3		.0090	.2103	.6167	.8347	.9274	.9677	.9856	.9936
4	.0001	.0141	.1248	.1565	.0957	.0490	.0237	.0111	.0051
5	.0007	.0310	.1402	.1088	.0467	.0183	.0072	.0029	.0012
6	.0037	.0630	.1394	.0607	.0162	.0044	.0013	.0004	.0001
7	.0170	.1176	.1304	.0309	.0046	.0007	.0001		
8	.9785	.7654	.2549	.0263	.0021	.0002			
	M=3, X=9								
3		.0042	.1844	.6129	.8345	.9274	.9677	.9856	.9936
4		.0066	.1094	.1555	.0956	.0490	.0237	.0111	.0051
5	.0001	.0146	.1233	.1082	.0467	.0183	.0072	.0029	.0012
6	.0007	.0310	.1240	.0606	.0163	.0044	.0013	.0004	.0001
7	.0037	.0627	.1205	.0314	.0046	.0007	.0001		
8	.0170	.1174	.1150	.0171	.0016	.0002			
9	.9785	.7636	.2234	.0142	.0007				

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
	M=3, X=10								
3		.0019	.1642	.6109	.8344	.9274	.9677	.9856	.9936
4		.0031	.0974	.1550	.0956	.0490	.0237	.0111	.0051
5		.0068	.1099	.1078	.0467	.0183	.0072	.0029	.0012
6	.0001	.0147	.1108	.0604	.0162	.0044	.0013	.0004	.0001
7	.0007	.0309	.1089	.0314	.0046	.0007	.0001		
8	.0037	.0627	.1078	.0174	.0016	.0001			
9	.0170	.1173	.1023	.0093	.0005				
10	.9785	.7627	.1989	.0075	.0002				
	M=3, X=11								
3		.0009	.1479	.6098	.8344	.9274	.9677	.9856	.9936
4		.0014	.0878	.1547	.0956	.0490	.0237	.0111	.0051
5		.0032	.0989	.1076	.0467	.0183	.0072	.0029	.0012
6		.0068	.0998	.0603	.0163	.0044	.0013	.0004	.0001
7	.0001	.0146	.0984	.0314	.0046	.0007	.0001		
8	.0007	.0309	.0986	.0175	.0016	.0002			
9	.0037	.0626	.0971	.0095	.0005				
10	.0170	.1172	.0921	.0051	.0002				
11	.9785	.7623	.1793	.0042	.0001				
	M=3, X=12								
3		.0004	.1346	.6091	.8344	.9274	.9677	.9856	.9936
4		.0007	.0799	.1545	.0956	.0490	.0237	.0111	.0051
5		.0015	.0900	.1075	.0467	.0183	.0072	.0029	.0012
6		.0032	.0908	.0603	.0163	.0044	.0013	.0004	.0001
7		.0068	.0896	.0313	.0046	.0007	.0001		
8	.0001	.0146	.0900	.0175	.0016	.0002			
9	.0007	.0309	.0897	.0095	.0005				
10	.0037	.0626	.0833	.0051	.0002				
11	.0170	.1172	.0839	.0028	.0001				
12	.9785	.7621	.1631	.0023					

Table 5.
(cont.)

M=3, X=13

i	m= 1	2	3	4	5	6	7	8	9
3		.0002	.1248	.6088	.8344	.9274	.9677	.9856	.9936
4		.0003	.0740	.1545	.0956	.0490	.0237	.0111	.0051
5		.0007	.0833	.1075	.0467	.0183	.0072	.0029	.0012
6		.0015	.0839	.0602	.0163	.0044	.0013	.0004	.0001
7		.0032	.0826	.0313	.0046	.0007	.0001		
8		.0068	.0827	.0175	.0016	.0002			
9	.0001	.0146	.0824	.0095	.0005				
10	.0007	.0308	.0819	.0052	.0002				
11	.0037	.0626	.0804	.0028	.0001				
12	.0170	.1172	.0762	.0015					
13	.9785	.7620	.1478	.0013					

M=4, X=5

i	m= 1	2	3	4	5	6	7	8	9
4	.0037	.0579	.2220	.4613	.6786	.8254	.9100	.9550	.9780
5	.9963	.9421	.7780	.5387	.3214	.1746	.0900	.0450	.0220

M=4, X=6

i	m= 1	2	3	4	5	6	7	8	9
4	.0006	.0207	.1321	.3743	.6331	.8085	.9047	.9535	.9776
5	.0031	.0413	.1255	.1779	.1532	.1002	.0572	.0305	.0156
6	.9967	.9380	.7424	.4478	.2136	.0913	.0381	.0160	.0068

M=4, X=7

i	m= 1	2	3	4	5	6	7	8	9
4	.0001	.0067	.0764	.3094	.6027	.7994	.9023	.9529	.9774
5	.0005	.0154	.0795	.1552	.1505	.1009	.0577	.0307	.0156
6	.0032	.0422	.1289	.1658	.1152	.0591	.0273	.0123	.0054
7	.9962	.9357	.7154	.3693	.1316	.0406	.0127	.0042	.0015

M=4, X=8

i	m= 1	2	3	4	5	6	7	8	9
4		.0020	.0437	.2621	.5841	.7954	.9016	.9528	.9774
5	.0001	.0051	.0476	.1342	.1472	.1008	.0577	.0307	.0157
6	.0005	.0157	.0824	.1486	.1147	.0596	.0274	.0122	.0054
7	.0032	.0423	.1275	.1447	.0752	.0280	.0098	.0035	.0013
8	.9962	.9349	.6987	.1304	.0788	.0163	.0036	.0008	.0002

M=4, X=9

i	m= 1	2	3	4	5	6	7	8	9
4		.0006	.0250	.2269	.5727	.7937	.9013	.9527	.9774
5		.0016	.0277	.1169	.1446	.1006	.0577	.0307	.0157
6	.0001	.0052	.0497	.1314	.1334	.0596	.0274	.0122	.0054
7	.0005	.0157	.0823	.1327	.0757	.0283	.0099	.0035	.0013
8	.0032	.0423	.1256	.1240	.0453	.0112	.0027	.0007	.0002
9	.9962	.9346	.6897	.2681	.0483	.0066	.0009	.0001	

Table 5.
(cont.)

M=4, X=10

i	m= 1	2	3	4	5	6	7	8	9
4		.0002	.0143	.1999	.5657	.7929	.9013	.9527	.9774
5		.0005	.0159	.1031	.1429	.1006	.0577	.0307	.0157
6		.0016	.0290	.1164	.1122	.0596	.0275	.0112	.0054
7	.0001	.0052	.0499	.1193	.0753	.0283	.0099	.0035	.0013
8	.0005	.0157	.0816	.1159	.0460	.0113	.0027	.0007	.0002
9	.0032	.0423	.1245	.1088	.0277	.0045	.0007	.0001	
10	.9962	.9346	.6847	.2365	.0302	.0028	.0003		

M=4, X=11

i	m= 1	2	3	4	5	6	7	8	9
4			.0082	.1787	.5614	.7926	.9013	.9527	.9774
5		.0001	.0091	.0921	.1418	.1005	.0577	.0307	.0157
6		.0005	.0167	.1042	.1114	.0596	.0246	.0122	.0054
7		.0016	.0292	.1073	.0749	.0283	.0099	.0035	.0013
8	.0001	.0052	.0496	.1058	.0460	.0114	.0027	.0007	.0002
9	.0005	.0157	.0811	.1031	.0282	.0045	.0007	.0001	
10	.0032	.0423	.1241	.0975	.0175	.0019	.0002		
11	.9962	.9345	.6819	.2114	.0188	.0012	.0001		

M=4, X=12

i	m= 1	2	3	4	5	6	7	8	9
4			.0047	.1615	.5588	.7925	.9013	.9527	.9774
5			.0053	.0833	.1411	.1005	.0577	.0307	.0157
6		.0001	.0096	.0941	.1108	.0596	.0274	.0122	.0054
7		.0005	.0169	.0971	.0745	.0283	.0099	.0035	.0013
8		.0016	.0291	.0962	.0459	.0114	.0027	.0007	.0002
9	.0001	.0052	.0494	.0952	.0283	.0045	.0007	.0001	
10	.0005	.0157	.0810	.0935	.0179	.0019	.0002		
11	.0032	.0423	.1238	.0881	.0109	.0008	.0001		
12	.9962	.9345	.6802	.1911	.0118	.0005			

M=4, X=13

i	m= 1	2	3	4	5	6	7	8	9
4			.0027	.1474	.5571	.7924	.9013	.9527	.9774
5			.0030	.0760	.1407	.1005	.0577	.0307	.0157
6			.0055	.0859	.1105	.0596	.0274	.0122	.0054
7		.0001	.0097	.0885	.0743	.0283	.0099	.0035	.0013
8		.0005	.0168	.0878	.0457	.0114	.0027	.0007	.0002
9		.0016	.0290	.0873	.0283	.0045	.0007	.0001	
10	.0001	.0052	.0494	.0871	.0180	.0020	.0002		
11	.0005	.0157	.0809	.0852	.0112	.0008	.0001		
12	.0032	.0423	.1236	.0804	.0068	.0003			
13	.9962	.9345	.6793	.1743	.0074	.0002			

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
	M=5, X=6								
5	.0006	.0172	.0933	.2547	.4658	.6604	.8017	.8903	.9415
6	.9994	.9828	.9067	.7453	.5342	.3396	.1983	.1097	.0585
	M=5, X=7								
5	.0001	.0049	.0429	.1674	.3856	.6141	.7818	.8831	.9391
6	.0005	.0127	.0584	.1260	.1622	.1442	.1017	.0629	.0361
7	.9994	.9824	.8986	.7066	.4522	.2417	.1166	.0540	.0248
	M=5, X=8								
5		.0013	.0184	.1079	.3232	.5804	.7693	.8791	.9379
6	.0001	.0038	.0283	.0885	.1440	.1413	.1024	.0635	.0364
7	.0005	.0128	.0600	.1288	.1530	.1154	.0671	.0347	.0171
8	.9994	.9821	.8934	.6749	.3798	.1628	.0612	.0226	.0085
	M=5, X=9								
5		.0003	.0075	.0689	.2761	.5579	.7627	.8775	.9375
6		.0010	.0125	.0593	.1262	.1377	.1022	.0637	.0365
7	.0001	.0038	.0290	.0915	.1390	.1147	.0678	.0350	.0172
8	.0005	.0129	.0603	.1275	.1364	.0830	.0378	.0157	.0064
9	.9994	.9820	.8907	.6528	.3223	.1068	.0296	.0081	.0024
	M=5, X=10								
5		.0001	.0030	.0441	.2405	.5430	.7594	.8769	.9374
6		.0002	.0052	.0387	.1109	.1345	.1019	.0637	.0365
7		.0010	.0128	.0618	.1241	.1129	.0678	.0351	.0172
8	.0001	.0038	.0292	.0915	.1263	.0834	.0382	.0158	.0064
9	.0005	.0129	.0603	.1246	.1187	.0556	.0186	.0058	.0018
10	.9994	.9820	.8895	.6390	.2795	.0706	.0104	.0027	.0006
	M=5, X=11								
5			.0012	.0283	.2128	.5332	.7578	.8767	.9374
6		.0001	.0021	.0250	.0983	.1322	.1018	.0637	.0365
7		.0002	.0053	.0406	.1108	.1113	.0678	.0351	.0172
8		.0010	.0130	.0622	.1147	.0728	.0383	.0158	.0064
9	.0001	.0038	.0292	.0903	.1118	.0563	.0189	.0059	.0019
10	.0005	.0129	.0602	.1230	.1043	.0367	.0087	.0019	.0004
11	.9994	.9820	.8892	.6306	.2474	.0476	.0068	.0010	.0001

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
M=5, X=12									
5			.0004	.0182	.1908	.5266	.7570	.8766	.9374
6			.0008	.0161	.0881	.1305	.1017	.0637	.0365
7		.0001	.0021	.0264	.0995	.1100	.0677	.0351	.0172
8		.0002	.0053	.0410	.1037	.0820	.0383	.0158	.0064
9		.0010	.0129	.0617	.1029	.0563	.0189	.0059	.0019
10	.0001	.0038	.0292	.0893	.0996	.0375	.0088	.0019	.0004
11	.0005	.0129	.0602	.1220	.0934	.0248	.0042	.0007	.0001
12	.9994	.9820	.8890	.6253	.2219	.0323	.0034	.0004	
M=5, X=13									
5			.0002	.0118	.1729	.5222	.7566	.8766	.9374
6			.0003	.0104	.0799	.1294	.1016	.0637	.0365
7			.0008	.0170	.0902	.1090	.0677	.0351	.0172
8		.0001	.0021	.0267	.0942	.0814	.0383	.0158	.0064
9		.0002	.0053	.0408	.0940	.0559	.0189	.0059	.0019
10		.0010	.0129	.0612	.0926	.0376	.0089	.0019	.0004
11	.0001	.0038	.0292	.0888	.0902	.0254	.0043	.0007	.0001
12	.0005	.0129	.0602	.1213	.0848	.0170	.0021	.0002	
13	.9994	.9820	.8889	.6218	.2011	.0220	.0016	.0001	
M=6, X=7									
6	.0001	.0046	.0353	.1235	.2785	.4690	.6466	.7822	.8727
7	.9999	.9954	.9647	.8765	.7215	.5310	.3534	.2178	.1273
M=6, X=8									
6		.0011	.0134	.0661	.1952	.3943	.6006	.7601	.8638
7	.0001	.0035	.0233	.0690	.1235	.1502	.1362	.1013	.0667
8	.9999	.9954	.9633	.8649	.6813	.4555	.2632	.1385	.0696
M=6, X=9									
6		.0002	.0046	.0334	.1345	.3343	.5651	.7450	.8583
7		.0009	.0092	.0390	.0925	.1351	.1335	.1021	.0675
8	.0001	.0035	.0237	.0710	.1258	.1428	.1135	.0721	.0607
9	.9999	.9953	.9624	.8565	.6472	.3878	.1880	.0809	.0336
M=6, X=10									
6			.0015	.0163	.0922	.2878	.5397	.7359	.8555
7		.0002	.0033	.0205	.0665	.1197	.1297	.1018	.0676
8		.0009	.0094	.0402	.0954	.1309	.1126	.0728	.0411
9	.0001	.0035	.0238	.0716	.1245	.1292	.0868	.0453	.0212
10	.9999	.9953	.9620	.8515	.6214	.3324	.1312	.0442	.0145

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
	M=6, X=11								
6			.0005	.0077	.0633	.2519	.5221	.7308	.8543
7			.0011	.0102	.0467	.1060	.1261	.1014	.0676
8		.0002	.0033	.0211	.0692	.1170	.1106	.0729	.0412
9		.0009	.0094	.0405	.0953	.1206	.0871	.0458	.0214
10	.0001	.0035	.0238	.0716	.1216	.1140	.0624	.0255	.0095
11	.9999	.9953	.9619	.8489	.6039	.2895	.0916	.0235	.0058
	M=6, X=12								
6			.0001	.0036	.0436	.2237	.5099	.7280	.8538
7			.0003	.0049	.0325	.0945	.1234	.1011	.0676
8			.0011	.0105	.0490	.1060	.1086	.0728	.0412
9		.0002	.0034	.0213	.0397	.1103	.0863	.0460	.0215
10		.0009	.0094	.0406	.0939	.1080	.0632	.0259	.0096
11	.0001	.0035	.0238	.0714	.1191	.1006	.0438	.0135	.0038
12	.9999	.9953	.9618	.8478	.5923	.2568	.0649	.0126	.0024
	M=6, X=13								
6				.0016	.0302	.2012	.5013	.7625	.8536
7			.0001	.0023	.0226	.0850	.1213	.1009	.0676
8			.0003	.0050	.0342	.0957	.1069	.0727	.0412
9			.0011	.0106	.0496	.1003	.0852	.0460	.0215
10		.0002	.0034	.0213	.0690	.1001	.0630	.0260	.0096
11		.0009	.0094	.0405	.0925	.0966	.0447	.0138	.0039
12	.0001	.0035	.0238	.0714	.1174	.0902	.0319	.0072	.0015
13	.9999	.9953	.9618	.8472	.5846	.2310	.0466	.0069	.0010
	M=7, X=8								
7		.0011	.0122	.0544	.1489	.2969	.4715	.6358	.7658
8	1.	.9989	.9878	.9456	.8511	.7031	.5285	.3642	.2342
	M=7, X=9								
7		.0002	.0040	.0245	.0879	.2176	.4014	.5903	.7423
8		.0009	.0084	.0326	.0753	.1199	.1405	.1293	.1001
9	1.	.9989	.9876	.9429	.8367	.6626	.4582	.2804	.1576
	M=7, X=10								
7			.0012	.0103	.0497	.1572	.3436	.5539	.7250
8		.0002	.0029	.0154	.0472	.0939	.1277	.1267	.1008
9		.0009	.0084	.0332	.0776	.1219	.1343	.1108	.0751
10	1.	.9989	.9875	.9411	.8255	.6271	.3944	.2086	.0991

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
M=7, X=11									
7			.0003	.0041	.0272	.1131	.2978	.5267	.7137
8			.0009	.0067	.0277	.0709	.1142	.1229	.1005
9		.0002	.0029	.0157	.0487	.0965	.1242	.1098	.0759
10		.0009	.0085	.0335	.0783	.1205	.1230	.0884	.0508
11	1.	.9989	.9874	.9401	.8181	.5990	.3408	.1522	.0591
M=7, X=12									
7			.0001	.0015	.0146	.0815	.2618	.5070	.7069
8			.0002	.0027	.0155	.0524	.1019	.1192	.0999
9			.0009	.0068	.0286	.0736	.1128	.1077	.0759
10		.0002	.0029	.0158	.0492	.0964	.1155	.0885	.0515
11		.0009	.0085	.0335	.0783	.1175	.1098	.0668	.0315
12	1.	.9989	.9874	.9396	.8138	.5787	.2983	.1108	.0343
M=7, X=13									
7				.0006	.0077	.0589	.2333	.4928	.7028
8			.0001	.0010	.0084	.0384	.0912	.1161	.0995
9			.0003	.0028	.0170	.0548	.1019	.1054	.0757
10			.0009	.0069	.0290	.0740	.1063	.0876	.0517
11		.0002	.0029	.0158	.0493	.0948	.1045	.0675	.0320
12		.0009	.0085	.0335	.0781	.1146	.0975	.0492	.0184
13	1.	.9989	.9874	.9394	.8115	.5645	.2652	.0815	.0199
M=8, X=9									
8		.0002	.0038	.0220	.0728	.1704	.3115	.4736	.6270
9	1.	.9998	.9962	.9780	.9272	.8296	.6885	.5264	.3730
M=8, X=10									
8			.0011	.0087	.0370	.1079	.2360	.4072	.5823
9		.0002	.0027	.0139	.0399	.0790	.1160	.1324	.1232
10	1.	.9998	.9961	.9774	.9230	.8131	.6480	.4604	.2945
M=8, X=11									
8			.0003	.0032	.0177	.0659	.1765	.3514	.5454
9			.0008	.0057	.0214	.0532	.0939	.1215	.1208
10		.0002	.0027	.0140	.0409	.0813	.1177	.1273	.1078
11	1.	.9998	.9961	.9771	.9200	.7996	.6119	.3998	.2260

Table 5.
(cont.)

i	m= 1	2	3	4	5	6	7	8	9
M=8, X=12									
8			.0001	.0011	.0081	.0392	.1315	.3064	.5170
9			.0002	.0022	.0106	.0339	.0735	.1095	.1171
10			.0008	.0058	.0219	.0549	.0962	.1184	.1068
11		.0002	.0027	.0141	.0412	.0821	.1163	.1176	.0887
12	1.	.9998	.9961	.9769	.9182	.7899	.5824	.3481	.1704
M=8, X=13									
8				.0003	.0035	.0229	.0982	.2705	.4957
9			.0001	.0008	.0050	.0207	.0564	.0983	.1136
10			.0002	.0022	.0109	.0350	.0760	.1082	.1046
11			.0008	.0058	.0221	.0556	.0960	.1109	.0887
12		.0002	.0028	.0141	.0413	.0821	.1134	.1060	.0696
13	1.	.9998	.9961	.9768	.9172	.7838	.5560	.3061	.1281
M=9, X=10									
9			.0011	.0082	.0330	.0901	.1886	.3235	.4752
10	1.	1.	.9989	.9918	.9670	.9099	.8114	.6765	.5248
M=9, X=11									
9			.0003	.0029	.0149	.0500	.1260	.2514	.4121
10			.0008	.0054	.0192	.0466	.0809	.1122	.1257
11	1.	1.	.9989	.9917	.9660	.9043	.7931	.6364	.4623
M=9, X=12									
9			.0001	.0010	.0062	.0264	.0815	.1932	.3582
10			.0002	.0020	.0090	.0268	.0575	.0931	.1161
11			.0008	.0054	.0195	.0468	.0833	.1136	.1212
12	1.	1.	.9989	.9916	.9653	.9001	.7777	.6001	.4045
M=9, X=13									
9				.0003	.0025	.0133	.0516	.1479	.3140
10			.0001	.0007	.0039	.0147	.0389	.0749	.1054
11			.0002	.0020	.0092	.0274	.0593	.0952	.1134
12			.0008	.0055	.0196	.0473	.0841	.1123	.1128
13	1.	1.	.9989	.9916	.9648	.8973	.7661	.5697	.3544

Table 5.
(concl.)

i	m= 1	2	3	4	5	6	7	8	9
M=10, X=11									
10			.0003	.0029	.0139	.0445	.1060	.2044	.3336
11	1.	1.	.9997	.9971	.9861	.9555	.8940	.7956	.6664
M=10, X=12									
10				.0009	.0057	.0220	.0629	.1422	.2646
11			.0002	.0019	.0085	.0240	.0500	.0818	.1086
12	1.	1.	.9997	.9971	.9858	.9540	.8771	.7760	.6269
M=10, X=13									
10				.0003	.0022	.0103	.0357	.0963	.2077
11			.0001	.0007	.0036	.0125	.0314	.0606	.0919
12			.0002	.0019	.0086	.0244	.0515	.0841	.1098
13	1.	1.	.9997	.9971	.9857	.9528	.8816	.7591	.5906
M=11, X=12									
11			.0001	.0009	.0055	.0206	.0559	.1206	.2182
12	1.	1.	.9999	.9991	.9945	.9794	.9441	.8794	.7818
M=11, X=13									
11				.0003	.0021	.0093	.0299	.0754	.1568
12			.0001	.0006	.0035	.0117	.0282	.0533	.0819
13	1.	1.	.9999	.9991	.9945	.9791	.9419	.8713	.7612
M=12, X=13									
12				.0003	.0020	.0089	.0277	.0670	.1340
13	1.	1.	1.	.9997	.9980	.9911	.9723	.9330	.8660

Table 6.

X	p=	.1	.2	.3	.4	.5	.6	.7	.8	.9
	M=1									
2		.0900	.1600	.2100	.2400	.2500	.2400	.2100	.1600	.0900
3		.1800	.3200	.4200	.4800	.5000	.4800	.4200	.3200	.1800
4		.2356	.4189	.5498	.6283	.6545	.6283	.5498	.4189	.2356
5		.2700	.4800	.6300	.7200	.7500	.7200	.6300	.4800	.2700
6		.2922	.5195	.6819	.7793	.8117	.7793	.6819	.5195	.2922
7		.3073	.5463	.7170	.8194	.8536	.8194	.7170	.5463	.3073
8		.3179	.5651	.7417	.8477	.8830	.8477	.7417	.5651	.3179
9		.3256	.5789	.7598	.8683	.9045	.8683	.7598	.5789	.3256
10		.3314	.5892	.7733	.8838	.9206	.8838	.7733	.5892	.3314
11		.3359	.5971	.7837	.8957	.9330	.8957	.7837	.5971	.3359
12		.3394	.6033	.7919	.9050	.9427	.9050	.7919	.6033	.3394
13		.3422	.6083	.7984	.9125	.9505	.9125	.7984	.6083	.3422
14		.3444	.6123	.8037	.9185	.9568	.9185	.8037	.6123	.3444
	M=2									
3		.0810	.1280	.1470	.1440	.1250	.0960	.0630	.0320	.0090
4		.1620	.2560	.2940	.2880	.2500	.1920	.1260	.0640	.0180
5		.2430	.3840	.4410	.4320	.3750	.2880	.1890	.0960	.0270
6		.3023	.4777	.5486	.5374	.4665	.3583	.2351	.1194	.0336
7		.3485	.5508	.6325	.6196	.5378	.4131	.2711	.1377	.0387
8		.3833	.6057	.6956	.6814	.5915	.4543	.2981	.1514	.0426
9		.4102	.6482	.7445	.7293	.6331	.4862	.3194	.1621	.0456
10		.4312	.6814	.7825	.7666	.6654	.5110	.3354	.1703	.0479
11		.4478	.7077	.8127	.7962	.6911	.5308	.3483	.1769	.0498
12		.4612	.7288	.8370	.8199	.7117	.5466	.3587	.1822	.0512
13		.4721	.7460	.8568	.8393	.7286	.5595	.3672	.1865	.0525
14		.4811	.7602	.8731	.8552	.7424	.5702	.3742	.1900	.0535
	M=3									
4		.0729	.1024	.1029	.0864	.0625	.0384	.0189	.0064	.0009
5		.1458	.2048	.2058	.1728	.1250	.0768	.0378	.0128	.0018
6		.2187	.3072	.3087	.2592	.1875	.1152	.0567	.0192	.0027
7		.2916	.4096	.4116	.3456	.2500	.1536	.0756	.0256	.0036
8		.3493	.4906	.4930	.4140	.2995	.1840	.0906	.0307	.0043
9		.3973	.5580	.5608	.4708	.3406	.2093	.1030	.0349	.0049
10		.4374	.6144	.6174	.5184	.3750	.2304	.1134	.0384	.0054
11		.4702	.6604	.6637	.5572	.4031	.2477	.1219	.0413	.0058
12		.4974	.6987	.7021	.5895	.4264	.2620	.1290	.0439	.0061
13		.5202	.7307	.7342	.6165	.4460	.2740	.1349	.0457	.0064
14		.5393	.7575	.7612	.6392	.4624	.2841	.1398	.0473	.0067

Table 6.
(cont.)

X	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=4									
5	.0656	.0828	.0720	.0518	.0312	.0154	.0057	.0013	.0001
6	.1312	.1638	.1440	.1037	.0625	.0307	.0113	.0026	.0002
7	.1968	.2458	.2161	.1555	.0938	.0461	.0170	.0038	.0003
8	.2624	.3277	.2881	.2074	.0125	.0614	.0227	.0051	.0004
9	.3281	.4096	.3602	.2592	.1562	.0768	.0283	.0064	.0004
10	.3824	.4775	.4198	.3021	.1821	.0895	.0330	.0075	.0005
11	.4292	.5359	.4712	.3391	.2044	.1005	.0371	.0084	.0006
12	.4699	.5867	.5159	.3713	.2238	.1100	.0406	.0092	.0006
13	.5053	.6309	.5547	.3992	.2407	.1183	.0437	.0099	.0007
14	.5355	.6687	.5879	.4231	.2551	.1254	.0463	.0104	.0007
M=5									
6	.0591	.0655	.0504	.0311	.0156	.0061	.0017	.0003	
7	.1181	.1311	.1008	.0622	.0313	.0123	.0034	.0005	
8	.1771	.1966	.1513	.0933	.0469	.0184	.0051	.0008	
9	.2362	.2621	.2017	.1244	.0625	.0246	.0068	.0010	
10	.2952	.3277	.2521	.1555	.0781	.0307	.0085	.0013	
11	.3543	.3932	.3025	.1866	.0937	.0369	.0102	.0015	.0001
12	.4047	.4492	.3456	.2132	.1071	.0421	.0117	.0018	.0001
13	.4491	.4984	.3835	.2366	.1188	.0467	.0129	.0019	.0001
14	.4886	.5423	.4172	.2574	.1293	.0508	.0141	.0021	.0001
M=6									
7	.0531	.0524	.0353	.0187	.0078	.0025	.0005	.0001	
8	.1063	.1049	.0706	.0373	.0156	.0049	.0010	.0001	
9	.1594	.1573	.1059	.0560	.0234	.0074	.0015	.0002	
10	.2126	.2097	.1412	.7465	.0312	.0098	.0020	.0002	
11	.2657	.2621	.1765	.0933	.0391	.0123	.0026	.0003	
12	.3189	.3146	.2118	.1120	.0469	.0147	.0031	.0003	
13	.3720	.3670	.2471	.1306	.0547	.0172	.0036	.0004	
14	.4184	.4128	.2779	.1469	.0615	.0193	.0040	.0004	
M=7									
8	.0478	.0419	.0247	.0112	.0039	.0010	.0002		
9	.0957	.0839	.0494	.0224	.0078	.0020	.0003		
10	.1435	.1258	.0741	.0336	.0117	.0029	.0005		
11	.1913	.1678	.0988	.0448	.0156	.0039	.0006		
12	.2391	.2097	.1235	.0560	.0195	.0049	.0008	.0001	
13	.2870	.2517	.1482	.0672	.0234	.0059	.0009	.0001	
14	.3348	.2936	.1729	.0784	.0273	.0069	.0010	.0001	

Table 6.
(concl.)

X	p= .1	.2	.3	.4	.5	.6	.7	.8	.9
M=8									
9	.0431	.0356	.0173	.0067	.0020	.0004			
10	.0861	.0671	.0346	.0134	.0040	.0008	.0001		
11	.1291	.1007	.0519	.0202	.0059	.0012	.0001		
12	.1722	.1342	.0692	.0269	.0078	.0016	.0002		
13	.2152	.1678	.0865	.0336	.0098	.0020	.0002		
14	.2583	.2013	.1038	.0403	.0117	.0024	.0003		
M=9									
10	.0387	.0269	.0121	.0040	.0010	.0002			
11	.0775	.0537	.0242	.0081	.0020	.0003			
12	.1162	.0805	.0363	.0121	.0029	.0005			
13	.1550	.1074	.0484	.0161	.0039	.0006	.0001		
14	.1937	.1342	.0605	.0202	.0049	.0008	.0001		
M=10									
11	.0349	.0215	.0085	.0024	.0005	.0001			
12	.0697	.0429	.0169	.0048	.0010	.0001			
13	.1046	.0644	.0254	.0073	.0015	.0002			
14	.1395	.0859	.0339	.0096	.0020	.0003			
M=11									
12	.0314	.0172	.0059	.0015	.0002				
13	.0628	.0344	.0119	.0029	.0005	.0001			
14	.0941	.0515	.0178	.0044	.0007	.0001			
M=12									
13	.0282	.0137	.0042	.0009	.0001				
14	.0564	.0274	.0083	.0017	.0002				
M=13									
14	.0254	.0110	.0029	.0005	.0001				

Table 7.

X	m= 1	2	3	4	5	6	7	8	9
M=1									
2	.3679	.2707	.1494	.0733	.0337	.0149	.0064	.0027	.0011
3	.6280	.4621	.2550	.1251	.0575	.0254	.0109	.0046	.0019
4	.7633	.5616	.3099	.1520	.0699	.0309	.0132	.0056	.0023
5	.8380	.6166	.3402	.1669	.0767	.0339	.0145	.0061	.0025
6	.8828	.6495	.3584	.1758	.0808	.0356	.0153	.0064	.0027
7	.9115	.6706	.3701	.1815	.0835	.0369	.0158	.0067	.0028
8	.9309	.6849	.3780	.1854	.0853	.0378	.0163	.0068	.0030
9	.9461	.6950	.3835	.1881	.0866	.0387	.0169	.0069	.0033
10	.9546	.7024	.3876	.1902	.0880	.0403	.0179	.0072	.0038
11	.9621	.7079	.3907	.1919	.0892	.0427	.0195	.0076	.0044
12	.9686	.7122	.3931	.1936	.0929	.0459	.0215	.0088	.0052
13	.9737	.7156	.3949	.1957	.0960	.0496	.0237	.0097	.0061
M=2									
3	.1839	.2707	.6240	.1465	.0842	.0446	.0224	.0107	.0050
4	.3341	.4917	.4070	.2662	.1530	.0810	.0406	.0195	.0091
5	.4377	.6440	.5331	.3486	.2004	.1062	.0532	.0255	.0119
6	.5035	.7409	.6133	.4011	.2306	.1221	.0612	.0294	.0137
7	.5469	.8047	.6661	.4356	.2504	.1327	.0664	.0319	.0149
8	.5764	.8482	.7021	.4592	.2639	.1398	.0700	.0336	.0157
9	.5973	.8789	.7275	.4758	.2735	.1449	.0725	.0349	.0162
10	.6125	.9013	.7460	.4879	.2805	.1486	.0744	.0358	.0167
11	.6239	.9181	.7599	.4970	.2857	.1514	.0758	.0365	.0170
12	.6327	.9310	.7706	.5040	.2897	.1535	.0769	.0371	.0174
13	.6396	.9411	.7790	.5095	.2929	.1552	.0778	.0378	.0179
M=3									
4	.0613	.1804	.2240	.1954	.1404	.0892	.0521	.0286	.0150
5	.1144	.3367	.4181	.3646	.2620	.1665	.0973	.0534	.0280
6	.1551	.4564	.5667	.4942	.3551	.2257	.1319	.0724	.0379
7	.1839	.5413	.6721	.5861	.4211	.2680	.1564	.0859	.0450
8	.2040	.6005	.7456	.6502	.4672	.2970	.1735	.0953	.0499
9	.2184	.6428	.7980	.6959	.5000	.3179	.1857	.1020	.0534
10	.2289	.6736	.8363	.7293	.5240	.3331	.1946	.1069	.0560
11	.2367	.6966	.8650	.7542	.5419	.3445	.2013	.1105	.0579
12	.2425	.7143	.8868	.7733	.5556	.3572	.2063	.1133	.0594
13	.2474	.7280	.9049	.7882	.5563	.3600	.2103	.1155	.0605

Table 7.
(cont.)

X	m=	1	2	3	4	5	6	7	8	9
	M=4									
5		.0153	.0902	.1680	.1954	.1755	.1339	.0912	.0573	.0337
6		.0290	.1709	.3183	.3701	.3324	.2536	.1728	.1085	.0639
7		.0402	.2364	.4403	.5119	.4598	.3507	.2390	.1500	.0884
8		.0486	.2860	.5327	.6194	.5563	.4243	.2892	.1815	.1070
9		.0548	.3225	.6006	.6983	.6972	.4784	.3261	.2046	.1206
10		.0594	.3494	.6507	.7565	.6794	.5183	.3532	.2217	.1306
11		.0628	.3695	.6882	.8001	.7186	.5482	.3736	.2345	.1382
12		.0654	.3849	.7168	.8334	.7485	.5710	.3892	.2442	.1439
13		.0674	.3968	.7390	.8592	.7717	.5887	.4012	.2518	.1484
	M=5									
6		.0031	.0361	.1008	.1563	.1755	.1606	.1277	.0916	.0607
7		.0059	.0690	.1929	.2990	.3356	.3073	.2443	.1752	.1162
8		.0082	.0968	.2703	.4191	.4705	.4307	.3425	.2456	.1628
9		.0101	.1187	.3317	.5142	.5773	.5285	.4202	.3014	.1998
10		.0115	.1356	.3787	.5871	.6591	.6033	.4797	.3441	.2281
11		.0126	.1483	.4143	.6423	.7211	.6601	.5249	.3765	.2496
12		.0134	.1581	.4416	.6846	.7686	.7036	.5595	.4013	.2660
13		.0141	.1657	.4628	.7175	.8055	.7374	.5863	.4205	.2788
	M=6									
7		.0005	.0120	.0504	.1042	.1462	.1606	.1490	.1221	.0911
8		.0010	.0232	.0971	.2007	.2816	.3093	.2870	.2352	.1754
9		.0014	.0328	.1374	.2840	.3985	.4378	.4061	.3329	.2483
10		.0017	.0407	.1704	.3521	.4942	.5428	.5036	.4128	.3078
11		.0020	.0469	.1964	.4060	.5697	.6258	.5805	.4759	.3549
12		.0022	.0517	.2167	.4479	.6285	.6904	.6405	.5250	.3915
13		.0024	.0555	.2325	.4806	.6744	.7408	.6872	.5633	.4201
	M=7									
8		.0001	.0034	.0236	.0595	.1045	.1377	.1490	.1396	.1171
9		.0001	.0067	.0418	.1152	.2021	.2665	.2884	.2702	.2267
10		.0002	.0095	.0596	.1643	.2881	.3798	.4110	.3851	.3231
11		.0003	.0119	.0745	.2053	.3602	.4738	.5139	.4814	.4039
12		.0003	.0138	.0866	.2386	.4185	.5517	.5971	.5594	.4693
13		.0003	.0153	.0962	.2651	.4650	.6129	.6634	.6214	.5214

Table 7.
(concl.)

X	m= 1	2	3	4	5	6	7	8	9
	M=8								
9		.0009	.0081	.0298	.0653	.1032	.1304	.1395	.1318
10		.0017	.0157	.0578	.1268	.2006	.2533	.2712	.2560
11		.0024	.0226	.0829	.1818	.2876	.3631	.3887	.3669
12		.0030	.0284	.1043	.2287	.3618	.4568	.4891	.4617
13		.0035	.0332	.1220	.2675	.4231	.5343	.5720	.5399
	M=9								
10		.0002	.0027	.0133	.0363	.0689	.1014	.1124	.1132
11		.0004	.0053	.0258	.0707	.1341	.1976	.2418	.2568
12		.0005	.0076	.0371	.1018	.1931	.2845	.3481	.3697
13		.0007	.0096	.0470	.1287	.2443	.3599	.4403	.4676
	M=10								
11			.0008	.0053	.0181	.0413	.0710	.0993	.1186
12		.0001	.0016	.0103	.0354	.0807	.1387	.1939	.2316
13		.0001	.0023	.0149	.0512	.1166	.2004	.2802	.3348
	M=11								
12			.0002	.0019	.0083	.0225	.0452	.0722	.0970
13			.0004	.0038	.0161	.0441	.0884	.1413	.1899
	M=12								
13			.0001	.0006	.0034	.0113	.0264	.0481	.0728

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CONVERGENCE OF SOME STOCHASTIC MATRICES

by

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B. S., Kansas State University, 1961

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

This thesis provides an introduction to the problem of determining the rate at which a regular finite Markov process approaches it's steady-state. Methods of determining the convergence of a stochastic process to a steady-state are in existence. A procedure to determine the length of time which must elapse before the process can be said to have reached the "near" steady-state is a logical extension. For a regular Markov process, a method utilizing the characteristic roots of the involved stochastic matrix is developed to predict the number of time intervals the process must pass through in order to insure that a "near" steady-state has been reached.

Regular Markov chains and stochastic matrices are discussed. The numerical method to find the dominant characteristic root and the corresponding characteristic vector is introduced. The utility of applying characteristic root methods to Markov processes is pointed out.

The particular Markov process dealt with is an inventory process (M-policy) considered with two types of consumer demand (geometric & Poisson). The M-policy stochastic matrix is given and it's properties for these types of demand is noted. The stationary distributions and the second largest characteristic roots are tabled for the M-policy with various different sizes of inventory, replenishment, and average demand.

The second largest characteristic root is used to develop a method to predict the time required for the process to reach the "near" steady-state. Finally, examples of the application of the method in inventory theory, queue theory and dam theory are given.