.REACTIVITY DETERMINATIONS IN
THE K.S.U. GRAPHITE SUB-
CRITICAL ASSEMBLY

by

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Nomenclature of Terms

a  Extrapolation dimension in the x direction, cm.
a_1  Physical radius of a control rod, cm.
a_0  Physical radius of the control rod, cm.
a_n  Orthogonality constants.
A  Amplitude coefficient for the mn mode.
b  Extrapolation dimension in the y direction, cm.
B^2  Buckling, cm^-2.
B_{g}^2  Geometric buckling, cm^-2.
B_{m}^2  Material buckling, cm^-2.
c  Extrapolated height of a reactor, cm.
C_E  End correction factor.
C_H  Harmonic and end correction factor.
d  Linear extrapolation distance, cm.
d_o  Linear extrapolation distance into the rod, cm.
D  Thermal diffusion coefficient.
f  Thermal utilization factor.
J  Neutron current, n/sec-cm^2 (n = neutrons).
J_{mn}  The mn modal current, n/sec-cm^2.
k_e  Effective multiplication factor.
k_{\infty}  Infinite multiplication factor
l^*  Thermal neutron lifetime, sec.
L  Thermal diffusion length, cm.
M^2  Migration area (= L^2 + \tau), cm^2.
M  Matrix operator.
\( n \) Neutron density, \( n/cm^3 \).
\( p \) Resonance escape probability.
\( P \) Poison matrix operator.
\( r_1 \) Outer radius of a control annulus, cm.
\( r_0 \) Inner radius of a control annulus, cm.
\( R_c \) Extrapolated critical reactor radius, cm.
\( R_0 \) Equivalent subcritical reactor radius, cm.
\( S \) Thermal neutron source term, \( n/sec\cdot cm^3 \).
\( S_{mn} \) \( m, n \) modal source constant.
\( v \) Average neutron velocity, \( cm/sec \).
\( \alpha^2 \) Radial component of the buckling (cylindrical geometry), \( cm^{-2} \).
\( \alpha_m^2 \) \( x \) component of the buckling, \( cm^{-2} \).
\( \beta \) Effective delay neutron fraction.
\( \beta_n^2 \) \( y \) component of the buckling, \( cm^{-2} \).
\( \gamma_{mn} \) \( m, n \) modal separation constant for the \( z \) direction, \( cm^{-1} \).
\( \epsilon \) Spacing between data points.
\( \eta \) Average number of fast fission neutrons emitted as a result of the capture of one thermal neutron in the fuel material.
\( \lambda \) Average decay constant for delay neutrons.
\( \lambda_n \) Transport mean free path.
\( \omega \) Reciprocal reactor period, \( sec^{-1} \).
\( \mu^2 \) Major buckling (two group), \( cm^{-2} \).
\( \nu^2 \) Minor buckling (two group), \( cm^{-2} \).
\( \zeta_a \) Macroscopic absorption cross section, \( cm^{-1} \).
\( \zeta_r \) Macroscopic slowing down cross section, \( cm^{-1} \).
\( \tau \) Fermi age, \( cm^2 \).
\( \phi \) Neutron flux, \( \text{n/sec-cm}^2 \).

\( \phi_f \) Fast group neutron flux, \( \text{n/sec-cm}^2 \).

\( \phi_t \) Thermal group neutron flux, \( \text{n/sec-cm}^2 \).

\( \nabla^2 \) Laplacian operator.

\( S_0^2 \) Error variance of the slope.

\( S_0^2 (Y_i) \) Error variance of the predicted least squares value of \( Y_i \).

\( \delta (x_i, y_i) \) Dirac delta function.

\( \Delta k_e \) Change in the effective multiplication factor.

\( \Phi_r \) Adjoint vector flux set.

\( \Phi_r' \) Perturbed vector flux set.

\( \mathcal{L} \) Non-leakage probability.
1.0 INTRODUCTION

The worth of a control rod in a given reactor system can be determined experimentally by two methods. These experimental methods are 1) a static method and 2) a kinetic method.

Several references (7, 9, 11, 14, 17, 23) discuss the theory of calculating the worth of a control rod for a given reactor system by the static method. The kinetic method is generally used to determine control rod worth in a critical reactor system. There are many references (2, 4, 7, 8, 10, 11, 13, 14, 16, 18, 23) which discuss the experimental and theoretical procedures used to find the worth of a control rod in a critical reactor system by the kinetic method.

Very little experimental work has been performed to determine control rod worth in a subcritical reactor system. It is the object of this paper to present a study of the various mathematical models which were used to calculate the control rod worth of a cadmium rod in the K.S.U. graphite subcritical reactor.

Methods used to calculate the reactivity worth of a centrally located poison rod were the following:

1. The effective multiplication factor, $k_e$, of the K.S.U. graphite pile was calculated from experimental data by the method presented by Kaiser (12). The effective multiplication factor of the pile with the poison rod was calculated by finding the theoretical increase in the geometric buckling due to the presence of the poison rod.

2. The method which Wood (25) presented was revised and extended to calculate the worth of a control rod in an
exponential assembly. This was done by experimentally measuring the inverse relaxation lengths with and without the poison rod present. The worth of the rod was calculated from this change in inverse relaxation lengths.

3. A method presented by Murray and Niestlie (18) was revised and extended to include a two group analysis for calculating the worth of a control rod in an exponential assembly. The values calculated for the infinite multiplication factor, $k_\infty$, by method one were used in these calculations.

4. One group and two group perturbation calculations were made to give an approximation to the reactivity worth of the control rod. In order to better approximate the reactivity worth of a control rod calculated by perturbation theory, the perturbed flux was estimated more accurately by a variational method.

5. The theoretical approximation equations presented by Glasstone and Edlund (7) were used to calculate the reactivity worth of the rod. These calculations were based on one group, corrected one group, and two group theories.

A kinetic study was attempted. Several hundred rod drops were performed. The data were much too random for any clear cut conclusions. A one delay group analog solution to the kinetics equations was attempted for this work. However, problems, such as defining a true source condition, arose which led this author to include the results for only illustrative purposes.
2.0 THEORY

2.1 One Group Diffusion Models Used to Calculate the Reactivity Worth of a Control Rod in a Subcritical Reactor System

2.1.1 A Model Using Experimental Data to Find the Effective Multiplication Factor of a Subcritical Reactor with No Control Rod. Thermal neutron diffusion theory has been studied by many investigators (4,5,6,7,12,14,17,21,23). The following paragraphs will serve as an outline to the basic theory of neutron diffusion in a multiplying medium with special emphasis on the assumptions involved in this theory.

The rate of change in neutron density in a volume element of a multiplying medium is the result of (a) the net flow of neutrons through the boundary of the element, (b) the number of neutrons absorbed by the medium per unit time, and (c) the production of neutrons by sources within the element. The neutron current, J, defined as the number of neutrons per second flowing through a unit area normal to the direction of flow, can be expressed, according to Fick's law, as

\[ J = -D(r) \nabla \phi (r,t). \]  \hspace{1cm} (1)

\(D(r)\) is the neutron diffusion coefficient and \(\phi (r)\) is the neutron flux which is equal to the average neutron velocity, \(v\), times the neutron density, \(n\). Then the net flow of neutrons into a volume element can be written in terms of the current as

\[- \text{div } J = \text{div } (D(r) \nabla \phi (r,t)) \text{.} \hspace{1cm} (2)\]

When the diffusion coefficient, \(D(r)\), is independent of position, it is also independent of the operator, div. The leakage term can then be written as \(Dv^2\phi (r,t)\) where \(v^2\phi\) is equal to \(\text{div } (\nabla \phi (r,t))\). The
neutron balance on the element of volume becomes
\[ D \nabla^2 \phi(r,t) - \frac{1}{\Sigma_a} \phi(r,t) + k_\infty \Sigma_a \phi(r,t) + S = \frac{\partial n(r,t)}{\partial t}. \] (3)

In general only steady state diffusion will be considered in this work. In this case Eq. (3) becomes
\[ D \nabla^2 \phi(r,t) - \frac{1}{\Sigma_a} \phi(r,t) + k_\infty \Sigma_a \phi(r,t) + S = 0. \] (4)

Eq. (4) will be solved subject to a point source boundary condition. Dividing Eq. (4) by \( D \) and defining \( \Sigma_m^2 \) as \( \Sigma_a (k_\infty - 1)/D \) results in the equation
\[ \nabla^2 \phi(r) + \Sigma_m^2 \phi(r) = 0. \] (5)

\( \Sigma_a \) is the averaged thermal neutron absorbance cross section and \( k_\infty \) is the infinite multiplication factor.

The validity of elementary diffusion theory depends on several important considerations which are listed below.

1. There are no collisions between neutrons, which means that each neutron diffuses independently of all other neutrons. The importance of this assumption results in a convenient mathematical simplification discussed by Weinberg and Wigner (23) and Hughes (11). This makes superposition of solutions possible.

2. The neutrons diffuse with a constant (average) energy and no energy is gained or lost (on the average) in a collision with a nucleus. This assumption is supported by Hughes (11) who states that experimental evidence indicates that the Maxwellian distribution is maintained reasonably well throughout the diffusion process, thus, allowing the diffusion equation to be energy independent.

3. The flux is a slowly varying function throughout the pile with no sharp dips or spikes. This assumption is essential to the validity
of Fick's law and means that diffusion theory is not applicable in close proximity to concentrated sources, absorbers, or the boundaries of the diffusing medium. It is also necessary that the medium be only slightly absorbing, since a high absorption cross-section would cause the neutron density to vary substantially within one mean-free path.

4. The neutron scattering is spherically symmetric or isotropic, allowing the velocity vector to be treated as a scalar in the neutron balance, Eq. (3). Since scattering is not isotropic near sources, boundaries, and absorbers, this assumption necessitates restrictions similar to those stated in condition (3).

5. The diffusion coefficient is independent of position. This allows D to be extracted from the term for divergence of the current in Eq. (2), leaving $D \nabla^2 \phi(r,t)$ for $\text{div}(D \text{grad} \phi(r,t))$. Eq. (6) is derived in Appendix (A). The result obtained for rectangular geometry is

$$
\phi(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos(m \pi x/a) \cos(n \pi y/b) \sinh \gamma_{mn}(c-z)
$$

The $m$ and $n$ are both odd, and $a$, $b$, and $c$ are the extrapolated dimensions of the pile in the $x$, $y$, and $z$ directions. The variables were assumed separable and the auxiliary separation constants equation was found to be

$$
P_m^2 = \alpha_m^2 + \beta_n^2 - \gamma_{mn}^2
$$

where

$$
\alpha_m = m \pi /a \quad \text{and} \quad \beta_n = n \pi /b
$$

In order to determine the quantity $A_{mn}$, a source boundary condition
is needed. The source boundary condition used in this work consisted of four point thermal sources, one located at each of the positions \( x = \pm x_0 \) and \( y = \pm y_0 \) in the source layer in the \( z \) dimension. The source condition was approximated by expanding in a series of cosine functions such that \( S \delta(x,y) \) at \( z=0 \) was written as

\[
S \delta(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S_{mn} \cos(m \pi x/a) \cos(n \pi y/b)
\]

where \( m \) and \( n \) were both odd.

The quantity \( S_{mn} \) was regarded as the sources for each \( mn \) mode of the flux. \( S_{mn} \) was determined by virtue of the orthogonality property of the cosines. Each member of Eq. (7) was multiplied by \( \cos(m \pi x/a) \) and \( \cos(n \pi y/b) \) and integrated over the interval of orthogonality, from \(-a/2\) to \( a/2\) and \(-b/2\) to \( b/2\). From the property of the Dirac delta function, the left member of Eq. (8) reduced to \( S \cos(m \pi x_1/a) \cos(n \pi y_1/b) \). By virtue of the orthogonal properties, the right member of Eq. (8) was found to be \( S_{mn} \) times one-half the range of orthogonality in each of the \( x \) and \( y \) directions. The result was determined to be

\[
S_{mn} = \frac{4}{ab} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} S \cos(m \pi x_1/a) \cos(n \pi y_1/b) dx_1 dy_1\]

(9)

The origin was located in the source plane so that only positive values in the \( z \) direction were considered. The current density in any one mode was assumed to be equal to only one half of the total source neutrons emitted in that mode. Using Fick's law and the expression for thermal flux which is given by Eq. (1), the \( mn \) modal current density was found to be

\[
J_{mn} = -D \left[ \frac{\partial \phi_{mn}}{\partial z} \right]_{z=0} = D \gamma_{mn} A_{mn} \cos(m \pi x/a) \cos(n \pi y/b) \cosh \gamma_{mn} \gamma_{mn} = \frac{1}{2} S_{mn}.
\]

(10)
Substituting from Eq. (9) and solving for \(A_{mn}\), yields Eq. (11).

\[
A_{mn} = \sum_{i=1}^{l_1} 2 S \cos(m \pi x_i / a) \cos(n \pi y_i / b) / ab \delta_{mn} \cosh \gamma_{mn} c. \tag{11}
\]

Substituting this result back into Eq. (8), the expression for total thermal neutron flux was found to be

\[
\phi(x,y,z) = \sum_{i=1}^{l_1} \sum_{m=1}^{l_2} \sum_{n=1}^{l_3} \left[ 2 S \cos(m \pi x_i / a) \cos(n \pi y_i / b) / ab \delta_{mn} \cos(m \pi x / a) \right]
\]

\[
\times \left[ \cos(m \pi x / a) \cos(n \pi y / b) \sinh \gamma_{mn}(c-z) / \cosh \gamma_{mn} c \right], \tag{12}
\]

where \(m\) and \(n\) are both odd.

Expressing the hyperbolic functions in terms of exponentials and rearranging Eq. (11) gives

\[
\phi(x,y,z) = \sum_{i=1}^{l_1} \sum_{m=1}^{l_2} \sum_{n=1}^{l_3} \left[ 2 S \cos(m \pi x_i / a) \cos(n \pi y_i / b) / ab \delta_{mn} \cos(m \pi x / a) \right]
\]

\[
\times \left[ \cos(n \pi y / b) \exp(-\gamma_{mn} z) \left[ 1 - \exp(-2 \gamma_{mn}(c-z)) / \left[ 1 + \exp(-2 \gamma_{mn} c) \right] \right] \right]. \tag{13}
\]

In order to determine \(\gamma_{11}\) easily, this equation was rearranged again in order to express the flux simply in terms of an exponential decay in the \(z\) direction.

\[
\phi(x,y,z) = C_E C_H \exp(-\gamma_{11} z) \tag{14}
\]

where

\[
C_E = \left[ 1 - \exp(-2 \gamma_{11}(c-z)) / \left[ 1 + \exp(-2 \gamma_{11} c) \right] \right]
\]

and

\[
C_H = 1 + \sum_{i=1}^{l_1} \sum_{m=1}^{l_2} \sum_{n=1}^{l_3} \gamma_{11} \cos(m \pi x / a) \cos(n \pi y / b) \cos(m \pi x_i / a) \cos(n \pi y_i / b)
\]

\[
\times \exp(-\gamma_{mn} z) \left[ 1 - \exp(-2 \gamma_{mn}(c-z)) \right] / C_E \gamma_{mn} \cos(\pi x / a) \cos(\pi y / b) \cos(\pi x_i / a)
\]

\[
\times \cos(\pi y_i / b) \exp(-\gamma_{11} z) \left[ 1 + \exp(-2 \gamma_{mn} c) \right].
\]
The \( m \) and \( n \) are odd but not simultaneously equal to unity. \( C_E \) is called the end correction factor and \( C_H \) is called the composite end and harmonic correction factor.

The iteration concept described by Kaiser (12) was used in an effort to calculate \( \gamma_{11} \), the inverse relaxation length. The following was the procedure used.

1. Assuming \( C_E C_H = 1.0 \), a least squares analysis was performed on the vertical traverse data taken at \( x = y = 0 \). The value thus obtained was used as an initial estimate for \( \gamma_{11} \).

2. The values of \( C_E \) and \( C_H \) were calculated using this initial estimate of \( \gamma_{11} \).

3. The modified count rates were then calculated by dividing the original count rates by \( C_E C_H \).

4. A least squares analysis was then performed on the modified count rates thus determining an improved estimate of \( \gamma_{11} \).

5. The new value for \( \gamma_{11} \) was compared with the previous value. When the change in the value of \( \gamma_{11} \) between successive iterations is considered negligible, the iteration process is complete. The final value of \( \gamma_{11} \) was used in the calculation of the material buckling and the effective multiplication factor. An IBM 1620 program designed to find \( \gamma_{11} \) by this method is discussed in Appendix (E).

If the rectangular parallelepiped reactor were critical, with no external source, the boundary conditions impressed on the solution for \( \phi(x,y,z) \) would be that the flux be zero at all extrapolated boundaries.
Thus, for a critical reactor at steady state, Eq. (6) becomes

$$\phi(x, y, z) = A \cos(\pi x/a) \cos(\pi y/b) \cos(\pi z/c)$$  \hspace{1cm} (15)

where a, b, and c are the extrapolated x, y, and z dimensions of the critical assembly. Only one mode was considered because (1) criticality can only be achieved in the first mode, and (2) after criticality has been achieved, the higher modes decay out exponentially. The auxiliary separation constants equation for the critical solution is

$$\mathbb{B}^2 = (\pi/a)^2 + (\pi/b)^2 + (\pi/c)^2 .$$  \hspace{1cm} (16)

Eq. (16) is the definition for geometric buckling in a rectangular parallelepiped reactor. Therefore, the condition for criticality is

$$\mathbb{B}^2 = \mathbb{B}_m = \mathbb{B}_g .$$  \hspace{1cm} (17)

The effective multiplication factor, $k_e$, is defined using the Fermi slowing down model by Glasstone and Edlund (7) as

$$k_e = k_\infty \exp(-\mathbb{B}_g T) / [1 + L^2 \mathbb{B}_g^2]$$  \hspace{1cm} (18)

where the condition for criticality is $k_e = 1$. Since criticality has been assumed, Eq. (18) can be rearranged to yield $k_\infty$, the infinite multiplication factor. Equating $k_e = 1$, $\mathbb{B}_g^2 = \mathbb{B}_m^2 = \mathbb{B}^2$ and rearranging Eq. (18) becomes

$$k_\infty = (1 + L^2 \mathbb{B}^2) \exp(\mathbb{B}^2 T) .$$  \hspace{1cm} (19)

The best estimate of $\gamma_{11}$, found by the previously described iterative technique, is used in Eq. (7) to find $\mathbb{B}_m^2$. This value is then used in
Eq. (19) to find \( k_\infty \). The infinite multiplication factor and the geometric buckling for the subcritical system under experimental consideration, found from Eq. (16), are used in Eq. (18) to find \( k_e \).

2.1.2 A Definition of the Extrapolation Distances of a Subcritical Rectangular Parallelepiped Reactor. Kaiser (12) described an experimental method to determine the effective pile size of a subcritical rectangular parallelepiped reactor system. It was found that the data were much too random for definite conclusions. For this reason, he assumed that the linear extrapolation distance (0.71 \( \lambda_n \)) could be used for determining the effective size in the \( x \), \( y \), and \( z \) directions. The quantity \( \lambda_n \) represents the transport mean free path of the reactor core medium.

A study of the extrapolation distance in the \( z \) direction was not made by Kaiser (12). Therefore, a method of improving on the estimate of the extrapolation distance in the \( z \) direction was sought and finally the following technique for determining this distance was used. This method involved a two parameter iterative procedure.

Three count rate measurements, equally spaced in the \( z \) direction and taken close to the top of the pile, were used. These points were given \( z \) positions of \( z \), \( z + \epsilon \), and \( z + 2 \epsilon \), where \( \epsilon \) was the spacing between each measured count rate. These count rates and the extrapolated height, \( c \), were related by Glasstone and Edlund (7) by the equation

\[
\tanh \gamma_{11}(c-z-\epsilon) = \frac{(\phi_1 + \phi_2)}{(\phi_1 - \phi_2)} \tanh \gamma_{11} \quad (20)
\]

Eq. (20) was incorporated with the iterative procedure for finding \( \gamma_{11} \). An initial guess for \( c \) was used in the iterative technique to
compute $\gamma_{11}$. This $\gamma_{11}$ and the newly generated flux values were used to calculate an improved value for $c$ by using Eq. (20). This $c$ was then used to compute a corrected $\gamma_{11}$ by the iterative procedure. This new value of $\gamma_{11}$ was used in Eq. (20). A new value for $c$ was determined. This value for $c$ was then used to find a new value for $\gamma_{11}$. When the change in the values for $\gamma_{11}$ and $c$ between successive iterations was considered negligible, the iteration process was stopped.

2.1.3 A One Group Calculational Model for Finding the Effective Multiplication Factor of a Subcritical Reactor with a Control Rod. To find the change in $k_e$ due to the insertion of a cylindrical control rod in a rectangular parallelepiped subcritical system, the rectangular parallelepiped subcritical system is approximated by a cylindrical system which has a $k_e$ equal to the $k_e$ for the rectangular parallelepiped system. Since $k_\infty$, $L^2$, and $C$ are equal for both subcritical systems, the geometric bucklings must be equal. In cylindrical geometry, the geometric buckling is given by a similar auxiliary separation constants equation as Eq. (16),

$$B_e^2 = \alpha^2 + (\pi/c)^2 \quad (21)$$

where $\alpha = (2.4048/R_o)$ for the cylindrical reactor with no control rod, and $R_o$ is the extrapolated radius of the reactor system. Equating the right members of Eq. (16) and (21), the equivalent radius for the cylindrical system can be found and is given as

$$R_o = (2.4048) \sqrt{(\pi/a)^2 + (\pi/b)^2} \quad (22)$$

In cylindrical geometry, the radial component of the solution to
Eq. (5) can be written as

\[ R(r) = A J_0(\alpha r) + B Y_0(\alpha r) \]  

(23)

where \( J_0 \) and \( Y_0 \) are Bessel functions of the first and second kind of order zero.

For a system with no control rod, Eq. (23) is subject to the following boundary conditions

1). the flux must be finite and non-negative
2). \( R(r = R_o) = 0 \)

Applying the first condition to Eq. (23), the constant, \( B \), is found to be equal to zero. The application of the second condition yields a definition for \( \alpha \). The quantity \( \alpha \) is equal to \( (2.4048/R_o) \). Therefore, the auxiliary separation constants equation for a subcritical system in cylindrical geometry can be written as

\[ \frac{B^2}{n^2} = \alpha^2 - \gamma^2 \]  

(24)

where \( \gamma \) is the inverse relaxation length.

However, if a control rod is inserted, the necessary boundary conditions are

1). \[ \frac{R(r = a)}{R(r = a)} = d \]
2). \( R(r = R_o) = 0 \),

where \( a_o \) is the physical boundary of the control rod and \( d_o \) is linear extrapolation distance into the rod. If these two boundary conditions are applied to Eq. (23) and the result rearranged, the following equation is found.
\[
\frac{J_0(\alpha_1 R_0)}{Y_0(\alpha_1 R_0)} = \frac{\left[ J_0(\alpha_1 R_0) + \alpha_1 \frac{d}{dR} J_1(\alpha_1 R_0) \right]}{\left[ Y_0(\alpha_1 R_0) + \alpha_1 \frac{d}{dR} Y_1(\alpha_1 R_0) \right]} \tag{25}
\]

where \( \alpha_1 \) is the new radial component of the buckling for the subcritical system with the control rod inserted.

Eq. (25) can be solved for \( \alpha_1 \). This value of \( \alpha_1 \) is then used in Eq. (21) to determine the new geometric buckling of the subcritical system with the control rod inserted. Using this geometric buckling and the one determined for the system without the control rod from Eq. (18), the change in \( k_e \) due to the control rod can be calculated.

2.1.4 A Model Using Experimental Data to Find the Effective Multiplication Factor of a Subcritical Reactor with a Control Rod. A method to predict the worth of a control rod in a critical reactor by using data from an exponential assembly was developed by Wood (25). The following method was based on a steady-state flux condition in a subcritical reactor system with an external source condition. A cylindrical exponential assembly with a central control rod was considered. For the assembly without the rod, the material buckling, \( \beta_m^2 \), was given by the separation constants equation

\[
\beta_m^2 = \alpha_0^2 - \gamma_0^2 \tag{28}
\]

where \( \alpha_0 \) is equal to \( 2.4048/R_0 \) for a cylindrical, steady-state reactor with no control rod, \( \gamma_0 \) is the inverse relaxation length, and \( R_0 \) is the extrapolated radius of the reactor.

The control rod was treated as an internal boundary when it was inserted, hence, the material buckling of the assembly was the same with
or without the rod. With the control rod inserted, the radial leakage was increased both inward toward the rod and outward through the boundary. Due to this increased leakage, the radial component of the geometric buckling was changed. Therefore, from Eq. (23), the relaxation length must also change. The material buckling of the assembly in terms of quantities defined with the control rod present becomes

\[ B^2_{m} = \alpha^2 - \gamma^2, \] (29)

where \( \alpha \) is the new radial component of the buckling and \( \gamma \) is the new inverse relaxation length. The linear extrapolation distance at the outer boundary is assumed to be unchanged when the control is inserted.

If \( \gamma_0 \) and \( \gamma \) are calculated by using the iterative technique described before, \( \alpha \) can be calculated by using Eq. (28) and (29).

The calculated value for \( \alpha \) can then be used to find the change in \( k_e \) in a manner as described below. In cylindrical geometry and with no control rod, the geometric buckling is given as

\[ B^2_{0} = \alpha_o^2 + (\pi/c)^2. \] (30)

However, if a control rod is inserted, the geometric buckling becomes

\[ B^2_{e2} = \alpha^2 + (\pi/c)^2 \] (31)

where \( \alpha \) is the same quantity which appears in Eq. (29). Employing the same definition for the effective multiplication factor, \( k_e \), as was used previously, the change in \( k_e \) due to the insertion of the rod is
\[ \Delta k_e = k_\infty \left[ \exp\left(-\frac{B_2^2}{E_2} \right) \frac{1}{\left( 1 + L^2 B_2^2 \right)} - \exp\left(-\frac{B_1^2}{E_1} \right) \frac{1}{\left( 1 + L^2 B_1^2 \right)} \right] \]  

(32)

where \( k_\infty \) is determined by the one group iterative method described previously.

In brief review, this method consists of the following main points.

1. Determine \( \gamma_0 \) and \( \gamma \) from the vertical traverse data taken without and with the control rod inserted.
2. The material buckling, \( B_m^2 \), was determined from Eq. (28).
3. The change in the radial component of the buckling, \( \alpha^2 \), was then determined from Eq. (29).
4. Eq's. (30) and (31) were then used to find the geometric bucklings of the subcritical reactor system without and with the control rod inserted.
5. These geometric bucklings, \( B_{g1}^2 \) and \( B_{g2}^2 \), were then used in Eq. (32) to find the change in the effective multiplication factor, \( k_e \), due to the presence of the control rod.

2.1.5 A Method of Extending the Results of Reactivity Measurements of a Control Rod in a Subcritical Reactor to a Critical Reactor of the Same Core Material. It was stated before that in cylindrical geometry the radial component of the general solution to Eq. (5) could be written as

\[ R(r) = A J_0(\alpha r) + B Y_0(\alpha r) . \]  

(33)

Also the boundary conditions, which are necessary when the control rod is inserted, were stated before as

\[ \left[ \frac{R(r)}{R'(r)} \right]_{r=a_1} = 0 \]
ii). \( R(r=R_o) = 0 \).

Eq. (20), however, states that the radial component of the buckling is \( B_c \). Applying the above boundary conditions to the above solution and rearranging slightly, an expression for \( d_o \) is derived.

\[
d_o = \frac{\left[ Y_0(\alpha R_o) J_0(\alpha a_1) - Y_0(\alpha a_1) J_0(\alpha R_o) \right]}{\alpha \left[ Y_1(\alpha a_1) J_0(\alpha R_o) - Y_0(\alpha R_o) J_1(\alpha a_1) \right]}
\]  

(34)

Assuming that the linear thermal extrapolation distance into the control rod is unchanged for a full-sized reactor, Eq. (34) can be applied to the full-sized reactor. A measure of the worth of a control rod in a critical reactor is then computed by finding the change in bucklings.

If an identical control rod were inserted in the critical reactor, the radius of the critical reactor, \( R_c \), should be inserted in boundary condition ii) for \( R_o \) and as a consequence, \( R_c \) can be substituted into Eq. (34). Therefore, Eq. (34) becomes

\[
d = \frac{\left[ Y_0(B R_c) J_0(B a_1) - Y_0(B a_1) J_0(B R_c) \right]}{B \left[ Y_1(B a_1) J_0(B R_c) - Y_0(B R_c) J_1(B a_1) \right]}
\]  

(35)

\( B^2 \) is the new radial component of the buckling. Using the value obtained for \( d \) from Eq. (34), Eq. (35) can be solved for \( B \). Thus, the change in the buckling due to the insertion of the control rod in a critical reactor is

\[
\Delta B^2 = B^2 - (2.405/R_c)^2
\]  

(36)

2.1.6 A One Group Approximation Method for Predicting the Reactivity Worth of a Control Rod (Nordheim – Scalottar). Glasstone and Edlund (7)
and Weinberg and Wigner (23) present a method attributed to Scalettar and Nordheim for predicting the approximate worth of a centrally located control rod in a large, cylindrical, bare, homogeneous reactor. Their solution is obtained by solving Eq. (5) in cylindrical geometry subject to the following boundary conditions:

i). \( \phi (r = R_o) = 0 \)

ii). \( \phi (r = a_o) = 0 \)

The assumptions made in the derivation of their final equation (Eq. (11.12.1), Ref. 7)

\[
\Delta k_e = 7.5 \frac{M^2}{R_o^2} \left[ 0.116 + \ln \left( \frac{R_o}{2.4a_o} \right) \right]^{-1}
\]

are as follows:

a. The control rod displaces an equivalent cylinder of the reactor core and does not leave a hole upon being withdrawn.

b. The modified one-group critical equation is valid, i.e.

\[
k_\infty = 1 + M^2 B^2
\]

where \( M^2 = L^2 + \zeta \) (migration area)

\[
B^2 = \text{Buckling}
\]

c. The flux goes to zero within the control rod at an extrapolated distance, \( d_o \).

d. The diameter of the rod is much smaller than the radius of the reactor. This allows for

\[
\alpha = \alpha_o + \Delta \alpha ,
\]

where \( \Delta \alpha \) is a small value. This is necessary to be able to use the asymptotic expressions for the Bessel functions.
2.2 Two Group Diffusion Models Used to Calculate the Reactivity Worth of a Control Rod in a Subcritical Reactor System

2.2.1 A Two Group Calculational Model for Finding the Change in Geometric Buckling Due to a Control Rod in a Subcritical Reactor.

Megheblian and Holmes (14) and Murray and Niestie (18) derived a two group method for calculating the worth of a centrally located control rod in a bare, critical, cylindrical reactor. This method with a few variations can be used to compute the worth of a control rod in a subcritical assembly.

A derivation of the solutions to the steady-state two group diffusion equations, Eq. (39) and (40), is presented in Appendix B.

\[ D_1 v^2 \Phi_1 - \Sigma_R \Phi_1 + k_o \Sigma_a \Phi_2 = 0 \]  \hspace{1cm} (39)

\[ D_2 v^2 \Phi_2 - \Sigma_a \Phi_2 + \Sigma_R \Phi_1 = 0 \]  \hspace{1cm} (40)

These solutions were found to be

\[ \Phi_1 = AX(r) + CY(r) \]  \hspace{1cm} (41)

\[ \Phi_2 = S_1 AX(r) + S_2 CY(r) \]  \hspace{1cm} (42)

The flux components \( X(r) \) and \( Y(r) \) has permissible solutions in cylindrical geometry of

\[ X(r) = A J_0(\alpha_1 r) + A Y_0(\alpha_1 r) \]  \hspace{1cm} (43)

\[ Y(r) = C J_0(\alpha_2 r) + C K_0(\alpha_2 r) \]  \hspace{1cm} (44)
For the cylindrical reactor with a centrally located control rod inserted, the boundary conditions are

i). \( \left\{ \frac{\partial}{\partial r} \phi_1 (r) \right\}_{r=a_1} = 0 \)

ii). \( \left\{ \phi_2 (r)/ \phi_2 ' (r) \right\}_{r=a_1} = 0 \)

iii). \( \phi_1 (r=R_c) = \phi_2 (r=R_c) = 0 \).

The quantity \( a_1 \) is the physical boundary of the control rod and \( R_c \) is the extrapolated critical radius of the reactor. Since \( \phi_1 (r) \), and \( \phi_2 (r) \) must vanish at the extrapolated radius of the reactor, \( X(r) \) and \( Y(r) \) must vanish individually at \( r = R_c \). Therefore, applying boundary condition iii) to Eq. (43) and (44), ratios of the constants \( A \), \( A_0 \), \( C \), and \( C_0 \) can be formed as

\[
A_o/A = - J_0(\alpha_1 R_c) / Y_0(\alpha_1 R_c) = - T_0(\alpha_1 R_c) \tag{45}
\]

\[
C_0/C = - I_0(\alpha_2 R_c) / K_0(\alpha_2 R_c) = - Q_0(\alpha_2 R_c). \tag{46}
\]

Boundary conditions i) and ii) were applied to Eq. (41) and (42). Using Eq. (45) and (46) and rearranging the result, was found to be

\[
\left[ 1 - C_1 T_0(\alpha_1 R_c) \right] / C_2 X_1 T_0(\alpha_1 R_c) = \frac{S_2 \left[ 1 - C_2 Q_0(\alpha_2 R_c) \right]}{S_1 \ C_4 \ X_2 \ Q_0(\alpha_2 R_c)} \tag{47}
\]

where \( C_1 = Y_0(\alpha_1 a_1) + \alpha_1 Y_1(\alpha_1 a_1) \)
\( C_2 = K_0(\alpha_2 a_1) + \alpha_2 K_1(\alpha_2 a_1) \)
\( C_3 = Y_1(\alpha_1 a_1) \)
\( C_4 = K_1(\alpha_2 a_1) \).
For a bare reactor with no control rod, the boundary conditions applied to Eq's. (43) and (44) are

1) The fluxes must be finite and non-negative

\[ \phi_1 (r=R_c) = \phi_2 (r=R_c) = 0. \]

Using the first condition, \( A_0 \) is zero by noting that \( Y_o (\alpha_1 r) \) goes to a negative infinity at the origin. If the second boundary condition is applied to Eq's. (43) and (44), the result is

\[ \alpha R_c = 2.4049. \] (48)

Only the first mode is used because (1) the reactor will go critical in only the first mode and (2) the higher modes will decay out exponentially.

In order to compute the worth of a centrally located control rod in a cylindrical, bare, subcritical reactor, the following procedure was used. In this procedure the above two group analysis was used only to find the increased buckling of the system containing the control rod. The change in the effective multiplication factor is found by using the Fermi slowing down model.

1. A critical reactor of the same \( k_{\infty} \) calculated by using the one group point thermal source model was assumed. This assumption allowed the quantities \( \mu^2 \) and \( \nu^2 \) to be calculated where,

\[
\mu^2 = \frac{1}{2} \left[ -\frac{1}{T} + 1/L^2 \right] \pm \sqrt{\left(1/T + 1/L^2\right)^2 + 4(k_{\infty} - 1)/T L^2}. \] (49)

Using these values of \( \mu^2 \) and \( \nu^2 \) to calculate \( \alpha_1 \) and \( \alpha_2 \) (see Appendix B), the critical radius, \( R_c \), can be calculated using Eq.(47).

2. With the critical radius of the reactor, having a control rod inserted, known, the radial component of the increased geometric buckling can be calculated. Since the subcritical reactor system
was unreflected, the minor two group buckling, \( \nu^2 \), was not applicable and the major two group buckling, \( \mu^2 \), was equal to \( \beta^2 \).

3. Using the term \( \alpha \) from above, the geometric buckling of the equivalent cylindrical subcritical system used in this work can be calculated. Use is made of the geometric buckling relation given as

\[
\frac{B^2_{\nu_2}}{B^2_{\mu_2}} = \alpha^2 + (\pi/c)^2.
\]  

The geometric buckling of the system without a control rod is given as

\[
\frac{B^2_{\nu_1}}{B^2_{\mu_1}} = \alpha^2_0 + (\pi/c)^2.
\]

4. The change in \( k_e \) is now calculated by using the Fermi slowing down definition for \( k_e \) (Eq. 18). This change is given by

\[
k_e = k_\infty \{ \exp(-B^2_{\nu_1} \tau) /[1 + L^2 B^2_{\nu_1}] - \exp(-B^2_{\mu_1} \tau) /[1 + L^2 B^2_{\mu_1}] \}.
\]

2.2.2 A Method for Extending the Results of Reactivity Measurements of a Control Rod in a Subcritical Reactor to a Critical Reactor of the Same Core Material. To treat the energy dependence of neutron absorption more accurately, a two group analysis was derived by Wood (25). In section 2.1 a method was described for calculating \( d_2 \), the thermal extrapolation distance into the control rod. If vertical traverses are taken in the multiplying medium, a fast extrapolation distance, \( d_1 \), can be found.

The solutions, Eq's. (43) and (44), to the two group diffusion equations, Eq's. (39) and (40), are considered valid. A separation of
variables technique is employed to solve the applicable wave equations. These wave equations are given in Appendix B as

\[ \nabla^2 x(r) + \mu^2 y(r) = 0 \]  
\[ \nabla^2 x(r) - \nu^2 y(r) = 0. \]

The auxiliary separation constants equations which apply to a subcritical assembly are given by

\[ \gamma_1^2 = \alpha_1^2 - \mu^2 \]  
\[ \gamma_2^2 = \alpha_2^2 + \nu^2. \]

The quantities \( \gamma_1 \) and \( \gamma_2 \) are inverse relaxation lengths and are equal if no internal boundaries exist in the reactor core medium. A relationship between \( \mu^2 \) and \( \nu^2 \) can be found by using Eq. (49). This relationship is

\[ \mu^2 = \nu^2 - \left(1/\tau + 1/L^2\right). \]

The radial flux components are of the same form as shown in Eq's. (43) and (44).

The two group boundary conditions which are assumed consist of the following:

i). \[ \phi_1 (r=R_o) = \phi_2 (r=R_o) = 0 \]

ii). \[ \left\{ \phi_1 (r) / \phi_1' (r) \right\}_{r=a_1} = d_1 \]

iii). \[ \left\{ \phi_2 (r) / \phi_2' (r) \right\}_{r=a_1} = d_2. \]

Applying boundary condition i) to Eq's. (43) and (44), equations identical
to Eq's. (45) and (46) are obtained. If boundary conditions ii) and iii) are applied to Eq's. (43) and (44), the following transcendental equation is obtained.

$$\left[ L_0(\alpha_{2a_1}, d_1) - Q_0(\alpha_{2R_0}) M_0(\alpha_{2a_1}, d_1) \right]/\left[ L_0(\alpha_{2a_1}, d_2) - Q_0(\alpha_{2R_0}) \right] \times M_0(\alpha_{2a_1}, d_2) = \left[ N_0(\alpha_{1a_1}, d_1) - T_0(\alpha_{2R_0}) O_0(\alpha_{1a_1}, d_1) \right]/\left[ N_0(\alpha_{1a_1}, d_2) - T_0(\alpha_{2R_0}) O_0(\alpha_{1a_1}, d_2) \right].$$

(59)

The quantities $L_0$, $M_0$, $N_0$, and $O_0$ are defined by

$$L_0(\alpha_{2a_1}, d_1) = I_0(\alpha_{2a_1}) - d_1 \left\{ \frac{d}{dr} I_0(\alpha_{2a_1}) \right\}_{r=a_1}, \quad i = 1, 2$$

$$M_0(\alpha_{2a_1}, d_1) = K_0(\alpha_{2a_1}) - d_1 \left\{ \frac{d}{dr} K_0(\alpha_{2a_1}) \right\}_{r=a_1}, \quad i = 1, 2$$

$$N_0(\alpha_{1a_1}, d_1) = J_0(\alpha_{1a_1}) - d_1 \left\{ \frac{d}{dr} J_0(\alpha_{1a_1}) \right\}_{r=a_1}, \quad i = 1, 2$$

$$O_0(\alpha_{1a_1}, d_1) = Y_0(\alpha_{1a_1}) - d_1 \left\{ \frac{d}{dr} Y_0(\alpha_{1a_1}) \right\}_{r=a_1}, \quad i = 1, 2.$$

Without the rod Eq. (55) becomes

$$\gamma_{10}^2 = \alpha_{10}^2 - \mu^2$$

(55')

where $\gamma_{10}$ is the inverse relaxation length and $\alpha_{10}$ is equal to $2.4048/R_0$. If the subcritical reactor is unreflected, Eq. (57) will not apply when the rod is withdrawn. However, if the rod is inserted, Eq. (55) and (56) are given as

$$\gamma_{11}^2 = \alpha_{11}^2 - \mu^2,$$

(55'')
\[ y_{21}^2 = \alpha_{21}^2 + \nu^2. \]  
\hspace{1cm} (56")

The quantity \( \mu^2 \) can be determined from Eq. (55') if the inverse relaxation length measured in the multiplying medium with no control rod present is used. With the control rod inserted, however, a new inverse relaxation length, \( y_{11} = y_{21} \), can be measured. This value is used in Eq. (56") to determine \( \alpha_{11}^2 \). Also, \( \alpha_{21}^2 \) can be calculated using \( \nu^2 \) and the measured value for \( y_{21} \).

The values of \( \alpha_{11}, \alpha_{21}, \) and \( d_2 \), determined in a previous section from measurements on the pure diffusing medium, are used in Eq. (59) to solve for \( d_1 \). Knowing \( d_1, d_2, \mu^2 \) and \( \nu^2 \), Eq. (59) can be used to solve for the bucklings \( \alpha_{13} \) and \( \alpha_{23} \) of a critical reactor of radius \( r_c \). The change in major bucklings is found to be

\[ \Delta \alpha_{11}^2 = \alpha_{13}^2 - (2.4043/r_c)^2. \]  
\hspace{1cm} (60)

This method can be extended to more groups by considering the appropriate boundary conditions for each group.

2.2.3 A Two Group Approximation Method for Predicting the Reactivity Worth of a Control Rod (Nordheim - Scalettar). To improve the value of \( \Delta k_e \) obtained, using modified one group theory, Glasstone and Edlund (7) present a method, based on a two group model, for predicting the worth of a centrally located rod in a large, cylindrical, bare, homogeneous reactor. Their solution is obtained by solving Eq's. (39) and (40) in cylindrical geometry subject to the following boundary conditions:

1). \( \phi_1 (r = R_o) = 0 \)

ia). \( \left\{ \left[ \frac{d}{dr} \phi_1 (r) \right] \right\}_{r=a_1} = 0 \)
ii). $\phi_2 (r = R_o) = 0$
iiia). $\phi_2 (r = a_o) = 0$.

The assumptions made in the derivation of their final equation, (Eq. 11.31.1) reference 7)

$$\Delta k_e = 7.5 \frac{L^2}{R_o^2} \left[ 0.116 \left( 1 + \frac{r}{L^2} \right) + \frac{r}{L^2} \ln \left( \frac{L}{\sqrt{L^2/M_{a_o}}} \right) + \ln \left( \frac{R_o}{2.4 a_o} \right) \right]^{-1},$$

are listed below.

a. The control rod displaces an equivalent cylinder of the reactor core and when the control rod is withdrawn, a void is not left.

b. The thermal flux goes to zero within the control rod at an extrapolated distance d. The first derivative of the fast flux is zero at the surface of the control rod.

c. The diameter of the control rod is much smaller than the radius of the reactor. This allows estimating $\Delta \mu$ to be small, where $\Delta \mu$ is written as

$$\Delta \mu = \mu - \mu_0.$$

This approximation is necessary to make the Bessel function approximations.

d. The value of k-1 is considered small. Therefore, the major and minor bucklings can be conveniently approximated.

The modified one group approximation is derived by multiplying Eq. (11.31.1) by $L^2/R^2$. The result is Eq. (11.33) from Glasstone and Edlund (7).

$$\Delta k_e = 7.5 \frac{L^2}{R^2} \left( 0.116 + \ln \frac{R}{2.4 a_o} \right)^{-1}$$
2.3 Perturbation Models for Estimating the Reactivity Worth of the Control Rod

2.3.1 A One Group Approximation. Perturbation theory can be used to estimate the worth of a control rod if the flux perturbation is small. The resultant change in reciprocal reactor period, $\Delta \omega$, derived from perturbation theory in Appendix C, becomes

$$\Delta \omega = \int_V \Phi_r^+ P \Phi_r' dV / \int_V \Phi_r^+ \Phi_r' dV .$$

(61)

The quantity $\Delta \omega$ is given as $(\omega' - \omega)$, where $\omega'$ is the reciprocal period of the perturbed reactor and $\omega$ is the reciprocal period of the unperturbed reactor. $P$ is the matrix operator representing the perturbation inserted into the reactor system. The quantities $\Phi_r^+$ and $\Phi_r'$ are spatially dependent vector flux sets representing the adjoint fluxes and perturbed fluxes, respectively.

The reactivity, $\Delta k_e/k_e$, change due to a perturbation in a reactor can be found by considering the time dependent diffusion equation in which the delayed neutron groups have been neglected. This equation was derived by Glasstone and Edlund (7), the result being

$$\left\{\Delta k_e/\lambda^*\right\} \phi' = d \phi'/dt .$$

(62)

The quantity $\lambda^*$ is the thermal neutron lifetime and is given as

$$\lambda^* = k_e/k_\infty \leq a_c v .$$

(63)

Substituting Eq. (63) into Eq. (62), the result is

$$\left\{k_\infty \leq a_c v \Delta k_e/k_e\right\} \phi' = d \phi'/dt .$$

(64)
If \( \phi' \) has a solution of the type

\[
\phi' = \phi_r^t \exp(\omega t)
\]  

(65)

where \( \phi' \) is dependent only on space, Eq. (64) can be rewritten as

\[
\left\{ k_\infty \Sigma_{a_c} v \Delta k_e/k_e \right\} \phi_r^t = \omega \phi_r^t.
\]  

(66)

Since the perturbation in the reactor is small, the unperturbed flux, \( \phi \), will be approximately equal to the perturbed flux, \( \phi' \). Making use of this approximation, multiplying Eq. (66) by \( \phi \), and integrating over the entire reactor volume, an equation relating \( \Delta k_e/k_e \) to the reciprocal reactor period, \( \omega \), is obtained as

\[
\frac{\Delta k_e}{k_e} = \omega \int_V \phi_r^2 \, dV / \int_V k_\infty \Sigma_{a_c} v \phi_r^2 \, dV.
\]  

(67)

Considering Eq. (61), if a one group calculation is to be performed, the matrix operator \( M \), is self adjoint, i.e. the flux functions \( \phi_r \), will form a complete orthogonal set. Therefore, the term \( \phi_r^+ \) is equal to \( \phi \). Also for small perturbations, \( \phi_r' \) is approximately equal to \( \phi \). Suppose further that a pure thermal absorption change is made in the reactor system. This change is due to the insertion of a centrally located control rod into the reactor system. This means that \( P \) can be given by

\[
P = - \Delta \left[ v \Sigma_a(r) \right].
\]  

(68)

Therefore, for the one group approximation with a pure absorption perturbation, Eq. (61) becomes

\[
\Delta \omega = \omega' - \omega = - \int_V \Delta \left[ v \Sigma_a(r) \right] \phi_r^2 \, dV / \int_V \phi_r^2 \, dV.
\]  

(69)
If a fictitious critical reactor, $\omega = 0$, is thought of as being a reference reactor, Eq. (69) will yield the $\Delta \omega$ for the equivalent cylindrical subcritical reactor system under consideration in this work. Therefore, $\Delta \omega_1$ is defined as the change in reciprocal reactor period due to the unperturbed reactor system being subcritical by an amount $k_{e1} - 1$ ($= k_{e1}$). The term $k_{e1}$ is the effective multiplication factor for the unperturbed, equivalent, cylindrical, subcritical reactor system.

With the control rod inserted, a new $\Delta \omega_2$ can be found which is thought of as the change in the reciprocal reactor period due to the reactor being subcritical and to the addition of the absorber. The change in the effective multiplication factor of the reactor caused by the insertion of the control rod is $k_{e2} - 1$ ($= \Delta k_{e2}$).

The quantity $\Delta \omega_1$ ($= \omega_1 - 0$) contains a fictitious $\Delta \left[ v \Sigma_a(r) \right]_r$, which can be thought of as the amount of absorber which makes the reference reactor subcritical by the amount $\Delta k_{e1}$. The term $\Delta \omega_2$ contains this same $\Delta \left[ v \Sigma_a(r) \right]_r$ plus the absorption term due to the presence of the control rod $\Delta \left[ v \Sigma_a(r) \right]_r$. Therefore, $\Delta \omega_1$ and $\Delta \omega_2$ can be written as

$$\Delta \omega = \omega_1 = - \int_V \Delta \left[ v \Sigma_a(r) \right]_r \phi_r^2 \, dV / \int_V \phi_r^2 \, dV$$  \hspace{1cm} (70)

$$\Delta \omega = \omega_2 = - \int_V \Delta \left[ v \Sigma_a(r) \right]_r \phi_r^2 \, dV - \int_V \Delta \left[ v \Sigma_a(r) \right]_r \phi_r^2 \, dV / \int_V \phi_r^2 \, dV.$$  \hspace{1cm} (71)

Substituting Eq. (70) and (71) into Eq. (67) and subtracting the quantity $(k_{e2} - 1) / k_{e2}$ is found to be
\[
\frac{(k_{e2} - 1)}{k_{e2}} = \frac{(k_{e1} - 1)}{k_{e1}} - \int_V \Delta \left[ v \Sigma_a(r) \right]_r \phi_r^2 \, dV / \int_V k_\infty \Sigma_{ac} \phi_r^2 \, dV .
\] (72)

In the one group model \(v\) is a constant representing the average neutron velocity and can be factored. Thus, Eq. (72) can be rewritten as

\[
\frac{(k_{e2} - 1)}{k_{e2}} = \frac{(k_{e1} - 1)}{k_{e1}} - \int_V \Delta \left[ \Sigma_a(r) \right]_r \phi_r^2 \, dV / \int_V \phi_r^2 \, dV .
\] (73)

For a hollow cylindrical annulus control rod, the value of \(\Delta \left[ \Sigma_a(r) \right]_r\) will be zero for \(r < r_0\) and \(r > r_1\), where \(r_0\) and \(r_1\) are the inner and outer radii, respectively. The quantity \(\Delta \left[ \Sigma_a(r) \right]_r\) will be equal to the value of the absorption cross section of the control rod material if the control rod is inserted in a hole initially containing only air. Therefore, Eq. (73) can be written as

\[
\frac{(k_{e2} - 1)}{k_{e2}} = \frac{(k_{e1} - 1)}{k_{e1}} - (\Delta \Sigma_a)_r \int_{r_0}^{r_1} \phi_r^2 \, r \, dr / k_\infty \Sigma_{ac} \times \int_V \phi_r^2 \, r \, dr .
\] (74)

Using an average value for \(k_{e1}\) obtained from the method explained in section 2.1, the quantity \((k_{e1} - 1) / k_{e1}\) can be calculated.

2.3.2 A Two Group Approximation. As in the one group model, assume the critical reference reactor to exist so that the equation for becomes

\[
\Delta \omega = \omega' = \int_V \Phi_r^+ \cdot P \Phi_r^- \, dV / \int_V \Phi_r^+ \Phi_r^- \, dV .
\] (75)

Here the quantities, \(\Phi_r^+\) and \(\Phi_r^-\), represent matrix operators. In order
to express Eq. (75) in terms of $\Delta k_e / k_e$, consider a perturbed reactor which has been perturbed by only a change in $\eta$, the average number of fast fission neutrons omitted as a result of the capture of one thermal neutron in fuel material, a factor in the infinite multiplication factor. Thus, the $P$ operator in Eq. (75) for this reactor becomes

$$P = \begin{pmatrix} 0 & v_1 f \lambda_{a_0} \Delta \eta \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} (76)

where $f$ is the thermal utilization factor and $\rho e$ has been assumed to be equal to unity. Substituting Eq. (76) into Eq. (75), and performing the necessary matrix multiplication, the following is the result,

$$\omega' = v_1 f \lambda_{a_0} \Delta \eta \int_{V} \phi_1^* \phi_2 \, dV / \int_{V} \phi_1^* \phi_2 \, dV.$$  \hspace{1cm} (77)

Here $\phi_1^*$ is the adjoint fast flux while $\phi_2$ is the thermal flux.

The effective multiplication factor is given by

$$k_e = \eta \epsilon_p f \mathcal{L}.$$  \hspace{1cm} (78)

Where $\mathcal{L}$ is the non-leakage probability. Therefore, a uniform change in $\eta$ results in an expression such that

$$\Delta k_e / k_e = \Delta \eta / \eta.$$  \hspace{1cm} (79)

From the previous section, it is found that $\Delta k_e / k_e$ is equal to $\lambda^* \omega'$, so that the thermal neutron lifetime, $\lambda^*$, can be given by

$$\lambda^* = \Delta \eta / \eta \omega'.$$  \hspace{1cm} (80)
Substituting Eq. (77) for \( \omega^1 \) into Eq. (30) produces

\[
\lambda^* = \int_V \bar{\phi}_r^* \bar{\phi}_r \, dV / v_1 \eta f \xi_{ac} \int_V \phi_1^* \phi_2 \, dV.
\]  

(81)

Using Eq. (81), the reactivity, \( \Delta k_e / k_e \), can be expressed as

\[
\frac{\Delta k_e}{k_e} = \int_V \bar{\phi}_r^* P \bar{\phi}_r \, dV / v_1 \eta f \xi_{ac} \int_V \phi_1^* \phi_2 \, dV.
\]  

(82)

Consider now a perturbed reactor, in which the perturbation is a purely absorption change. Thus, the P operator in two group theory can be written as

\[
P = \begin{bmatrix}
0 & 0 \\
0 & -\Delta [v_2 \xi_{a}(r)]
\end{bmatrix}.
\]  

(83)

The reactivity associated with subcritical reactor is given by Eq. (82) where the P operator has been replaced by Eq. (83) and the necessary matrix multiplication has been performed.

\[
\frac{\Delta k_{e1}}{k_{e1}} = -\int_V \phi_2^* \Delta [v_2 \xi_{a}(r)] \phi_2 \, dV / v_1 \eta f \xi_{ac} \int_V \phi_1^* \phi_2 \, dV.
\]  

(84)

After the control rod is inserted, Eq. (82) becomes

\[
\frac{\Delta k_{e2}}{k_{e2}} = - \left\{ \int_V \phi_2^* \Delta [v_2 \xi_{a}(r)] \phi_2 \, dV - \int_V \phi_2^* \Delta [v_2 \xi_{a}(r)] \phi_2 \, dV \right\} / v_1 \eta f \xi_{ac} \int_V \phi_1^* \phi_2 \, dV.
\]  

(85)

Subtracting Eq. (84) from (85), produces the desired result of

\[
\frac{\Delta k_{e2}}{k_{e2}} = \frac{\Delta k_{e1}}{k_{e1}} - \int_V \phi_2^* \Delta [v_2 \xi_{a}(r)] \phi_2 \, dV / v_1 \eta f \xi_{ac} \int_V \phi_1^* \phi_2 \, dV.
\]  

(86)
The steady-state two group adjoint diffusion equations for a bare reactor can be written as

\[ \begin{align*}
\nabla^2 \psi_1^+ - v_1 \hat{\psi}_1^+ + v_2 \hat{\psi}_2^+ &= 0 \\
\nabla^2 \psi_2^+ - v_2 \hat{\psi}_2^+ + v_1 \eta f \hat{\psi}_1^+ &= 0
\end{align*} \]  

(87) \hspace{1cm} (88)

Eqs. (87) and (88) have solutions of the form

\[ \begin{align*}
\phi_1^+ &= S_1^+ A^+ X(r) + S_2^+ C^+ Y(r) \\
\phi_2^+ &= A^+ X(r) + C^+ Y(r)
\end{align*} \]  

(89) \hspace{1cm} (90)

where \( X(r) \) and \( Y(r) \) are given by Eq's. (43) and (44). By a similar method as was used in Appendix B to determine \( S_1 \) and \( S_2 \), the adjoint coupling coefficients, \( S_1^+ \) and \( S_2^+ \), can be determined to be

\[ \begin{align*}
S_1^+ &= \left( \frac{v_2}{v_1} \right) \left\{ 1 + \mu^2 \tau \right\} \\
S_2^+ &= \left( \frac{v_2}{v_1} \right) \left\{ 1 - \mu^2 \tau \right\}.
\end{align*} \]  

(91) \hspace{1cm} (92)

For a bare subcritical reactor, there will be no \( Y(r) \) component to the fluxes. Therefore, after substituting Eq's. (89), (90), (91), and (92) into Eq. (36), an expression for the reactivity of the centrally located control rod is given as

\[ \Delta k_{e2} / k_{e2} = \Delta k_{e1} / k_{e1} - \left( 1 + \mu_0^2 \right) \int_V \Delta \left[ \xi_a(r) \right] x^2(r) \, dV / \]

\[ \int_V \eta f \xi_{ac} \int_V x^2(r) \, dV. \]  

(93)

Since a two group model is being studied, the effective multiplication factor, \( k_e \), can be written as
The geometric buckling, $B_g^2$, for the one group reactor, is equal to the geometric buckling for the two group reactor, since the reactor core is bare. Therefore, Eq. (95) is identical with Eq. (74) if $p = 1$ is assumed to be unity. No new information is gained from a two group perturbation analysis on a bare reactor.

### 2.2.3 An Improvement for the Perturbation Models by a Variational Technique

Both one group and two group perturbation theory yield identical results for the reactivity worth of a control rod in a bare reactor. In order to improve these results, the assumption that the perturbed flux is equal to the unperturbed flux must be improved. The following method is proposed to make this improvement (3,14).

In section 2.0, the neutron diffusion equation for a multiplying medium was written as

$$-D_o \nabla^2 \phi_o(r) + \Sigma_a^o \phi_o(r) = \nabla \cdot \Sigma_f \phi_o(r)$$

where the superscript $o$ indicates an unperturbed system. Replacing $(\nabla \Sigma_f - \Sigma_a^o) / D_o$ by $B_o^2$, Eq. (95) becomes

$$\nabla^2 \phi_o(r) + B_o^2 \phi_o(r) = 0$$

where $\phi_o(r)$ is the flux corresponding to the lowest mode of the set of eigenfunctions $\phi_n(r)$. If all the eigenfunctions are considered, Eq. (96) can be written as

$$\nabla^2 \phi_n(r) + B_n^2 \phi_n(r) = 0$$
where the functions $\phi_n(r)$ will be assumed to be orthogonal functions.

For the system under consideration, the perturbation will occur in the thermal neutron absorption cross section. For the perturbed system, this cross section can be written as

$$\dot{\Sigma}_a(r) = \dot{\Sigma}_a^0 + \Delta \dot{\Sigma}_a(r)$$

(93)

where $\Delta \dot{\Sigma}_a(r)$ is the spatially dependent absorption cross section change. With this change in cross section, the reactor flux becomes

$$\phi(r) = \phi_0(r) + \Delta \phi(r)$$

(99)

where $\Delta \phi(r)$ is flux change due to the absorption cross section change. In order to maintain a steady-state operation, a hypothetical change in $\nu$ must be assumed.

$$\nu \rightarrow \nu + \Delta \nu$$

With these changes Eq. (95) becomes

$$-D_o \nabla^2 \Delta \phi(r) + \Delta \dot{\Sigma}_a(r) \phi_0(r) + \dot{\Sigma}_a^0 \Delta \phi(r) = \nu \dot{\Sigma}_f \Delta \phi(r) +$$

$$\Delta \nu \dot{\Sigma}_f \phi_0(r),$$

(100)

if the products of differentials are ignored and the terms contained in Eq. (95) are factored and replaced by zero.

Using all the eigenfunctions, $\phi(r)$ can be expanded such that

$$\phi(r) = \sum_{n=0}^{\infty} a_n \phi_n(r) = \phi_0(r) + \sum_{n=1}^{\infty} a_n \phi_n(r)$$

(101)

where the $a_0$ constant has been defined to be unity. Note that the second term of the right hand member of Eq. (101) is the $\Delta \phi(r)$
presented in Eq. (100).

To evaluate the constants $a_n$, Eq. (101) should be multiplied by $\phi_n(r)$ where $n \neq 0$, and the result integrated over the whole reactor volume, $V$. This results in an expression such as

$$
\int_V \phi_n(r) \phi(r) \, dV = \int_V \phi_n(r) \phi_o(r) \, dV + \int_V \phi_n(r) \Delta \phi(r) \, dV. \quad (102)
$$

Since the eigenfunctions $\phi_n(r)$ are orthogonal and $n \neq 0$ in Eq. (102), the first term of the right hand member is equal to zero. Also by virtue of orthogonality principals, the left hand member becomes

$$
\int_V \phi_n^2(r) \, dV.
$$

An expression for $\Delta \phi(r)$ in terms of $\phi_o(r)$ is obtained from Eq. (100)

$$
\Delta \phi(r) = \phi_o(r) \left[ \Delta \nu \xi_f - \Delta \xi_a \right] / -D_o \nu^2 + \left[ \xi_a^0 - \nu \xi_f \right]. \quad (103)
$$

In Eq. (103) the $\nabla^2$ can be replaced by $(-B_n^2)$ from Eq. (97) and $\left[ \xi_a^0 - \nu \xi_f \right]$ can be replaced by $(-D_o B_o^2)$. Therefore, Eq. (103) becomes

$$
\Delta \phi(r) = \phi_o(r) \left[ \Delta \nu \xi_f - \Delta \xi_a(r) \right] / D_o \left( E_n^2 - B_o^2 \right). \quad (104)
$$

Substituting for $\Delta \phi(r)$ in Eq. (104), the following is the result

$$
a_n = - \int_V \phi_n(r) \Delta \xi_a(r) \phi_o(r) \, dV / D_o \left( E_n^2 - B_o^2 \right) \int_V \phi_n^2(r) \, dV. \quad (105)
$$

The expression for the perturbed flux, found by using Eq's. (101) and (105), can now be used in the one group perturbation model, Eq. (74) for the perturbed flux and a better estimate of the worth of the rod is obtained (3).
3.0 EXPERIMENTAL FACILITIES

3.1 General Pile Description

The Kansas State University graphite pile, shown in Fig. (5), consisted of a rectangular parallelepiped, 68 in. square, and 100 in. high, resting on a concrete foundation. The pile was constructed of machined R-IHM Nuclear Grade Graphite blocks \(4 \pm 0.01\) inches in cross section and of specified lengths. In stacking, the long dimension of the blocks was alternated \(90^\circ\) from layer to layer. As seen in Fig. (5), holes were drilled through the entire length of certain graphite blocks to accommodate fuel slugs in an eight inch lattice formation. These holes were approximately 1.75 inch in diameter. The Nuclear Grade Graphite used for this pile has a thermal neutron absorption cross section between 3.7 and 4.5 millibarns.

For a portion of this work, each of the drilled fuel ports were filled with graphite cylinders. These graphite cylinders were made out of the same material as the graphite blocks and were 1.625 inches in diameter and 22.68 inches in length. Since the cylinders were of a smaller diameter than the fuel ports, small crescent shaped air gaps of 0.125 inches were left between the tops of the graphite cylinders and the tops of the fuel ports.

The blocks along the central vertical axis of the pile contained horizontal slots, shown in Fig. (5), with a cross section of 1.281 inches by 0.343 inches. These slots were used in this work to accommodate a BF\(_3\) probe. The density of the solid graphite blocks was 1.683 g/cc.
Fig. 1. Schematic Diagram of KSU pile.
Steichen (21) calculated the percentage air void to be approximately 0.3 per cent giving a corrected density for the pile of 1.676 g/cc.

The graphite pile contains a square machined hole with a side of 1.0625 inches extending from the top of the assembly to a distance of 20 inches above the concrete foundation. This hole, as can be seen by Fig. (1), was positioned 2.0 inches in the x dimension from the center of the pile. This hole was used to accommodate a 0.75 inch outside diameter aluminum tube filled with rolled sheet cadmium.

### 3.2 Non-Multiplying Configuration

**Table 1. Summary of Physical Specifications of Plutonium-Beryllium Neutron Sources**

<table>
<thead>
<tr>
<th>Source number</th>
<th>Grams Pu</th>
<th>Grams Be</th>
<th>Neutron Emission rate (n/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>365</td>
<td>7.87</td>
<td>15.99</td>
<td>((1.64\pm0.115)10^6)</td>
</tr>
<tr>
<td>366</td>
<td>7.87</td>
<td>16.01</td>
<td>((1.73\pm0.121)10^6)</td>
</tr>
<tr>
<td>367</td>
<td>7.86</td>
<td>15.89</td>
<td>((1.82\pm0.127)10^6)</td>
</tr>
<tr>
<td>368</td>
<td>7.86</td>
<td>15.88</td>
<td>((1.69\pm0.118)10^6)</td>
</tr>
<tr>
<td>369</td>
<td>7.86</td>
<td>16.09</td>
<td>((1.71\pm0.120)10^6)</td>
</tr>
</tbody>
</table>

For this portion of the work, experimentation was carried out in a diffusing medium with and without the poison rod inserted. A four-source configuration was used with the sources located in the source plane at positions 2, 4, 10, and 12 as shown in Fig. (2). In this configuration, each of the four sources was located 22 inches from the
ONE OF 13 POSSIBLE SOURCE LOCATIONS

Fig. 2. Source locations in KSU pile.
center of the pile. The sources were located in a plane six inches above the concrete foundation (see Fig. (1)).

The inner container of each source is 0.85 inch in diameter, 0.90 inch in height, and is made of tantalum. The outside container of each source is 1.02 inches in diameter, 1.30 inches in height, and is made of 18-8 stainless steel. All sources are sealed by welding.

### 3.3 Multiplying Configuration

For this portion of the work, the fuel ports contained aluminum tubes 66 inches long and 1.312 inches outside diameter. Each tube was supported in the fuel port by three aluminum rings, one fixed at the center, and two removable rings at the ends. Each aluminum tube accommodated eight Savannah River Type MK VIIIa fuel element rejects. Fig. (5) illustrates a portion of the loaded pile. In order that the poison rod would extend through the total length of the fuel region, the top three layers of the pile were left without fuel. The physical characteristics of the fuel slugs are shown in Figs. (3) and (4).

### 3.4 Neutron Detection System

Table (2) contains a listing of the various electronic components used throughout this work.

The Model NC-202 BF$_3$ probe's active volume, a length of 0.50 inches and a diameter of 0.1875 inches, contained $^{10}$BF$_3$ gas at a pressure of 70 cm Hg and an enrichment of 96 per cent. The model (RCL-10504) BF$_3$ probe's active volume with a length of 12 inches and a diameter of 1 inch, contained $^{10}$BF$_3$ gas at a pressure of 12 cm Hg and an
Fig. 3. The physical dimensions of the fuel slugs.
Fig. 4. Fuel elements for KSU pile
Fig. 5. KSU exponential pile.
FIG. 6 FLOW DIAGRAM FOR NEUTRON DETECTION SYSTEM
* LETTERS CORRESPOND TO LABELS ON FIG. 8
Table 2. A List of the Electronic Equipment Used

<table>
<thead>
<tr>
<th>Component</th>
<th>Type and Model</th>
<th>Nuclear Engr. Inventory No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>BF₃ Probe</td>
<td>Nuclear-Chicago Model NC-202</td>
<td>No. 97</td>
</tr>
<tr>
<td>BF₃ Probe</td>
<td>Radiation Counting Lab. RCL-1050⁴</td>
<td>No. 320</td>
</tr>
<tr>
<td>Pre Amplifier</td>
<td>Nuclear-Chicago Model 1062</td>
<td>No. 210</td>
</tr>
<tr>
<td>Count Rate Meter</td>
<td>B.J. Electronics Model DM1-D</td>
<td>No. 363</td>
</tr>
<tr>
<td>Scaler</td>
<td>Baird Atomics Model 132</td>
<td>No. 144</td>
</tr>
<tr>
<td>High Voltage Power Supply</td>
<td>John Fluke Model 400 BDA</td>
<td>No. 188</td>
</tr>
<tr>
<td>Timer</td>
<td>Baird Atomic Model 960</td>
<td>No. 147</td>
</tr>
<tr>
<td>Traversing Mechanism</td>
<td>K.S.U. Nuclear Engr.</td>
<td>No. 568</td>
</tr>
</tbody>
</table>

enrichment of 96 per cent. The Model NC-202 BF₃ probe with the associated counting system is shown in Fig. (7). Fig. (6) shows a block diagram of the neutron detection system.

The traversing mechanism, shown in Fig. (8), was capable of supporting the Model NC-202 BF₃ probe in a variety of positions in the pile. This mechanism was designed and built at Kansas State University and a detailed description of it is provided by L.R. Foulke (5).
Fig. 9. The physical dimensions of the poison rod.
3.5 Cadmium Poison Rod

Fig. (9) is an illustration of the poison rod used in this work. The 0.625 inch inside diameter aluminum tube contained a total of 2.4 pounds of 0.03 inch thick sheet cadmium. The sheet cadmium was shaped into a cylindrical annulus with a thickness of 0.09 inch. Thus, the actual poison element was a cylindrical annulus of dimensions 0.445 inch inside diameter, 0.625 inch outside diameter and 52 inches in length. Aluminum was used for the containing tube so that as little poison other than the actual cadmium was added to the reactor system as possible.

4.0 EXPERIMENTAL PROCEDURE

Vertical Traverse Measurements

All data were collected using the Model NC-202 BF₃ probe and associated counting equipment which was previously described. The counter was operated at 1400 volts with a pulse height sensitivity of 0.8.

Two separate vertical traverses were made, one without the poison rod and one with the poison rod. For the traverse without the poison rod, the BF₃ probe was positioned horizontally with its active volume centered on the vertical or z axis. The vertical traverse was made with the BF₃ probe in this position in order to obtain a maximum count rate. Also, a mathematical simplification in the iterative analysis procedure resulted from data obtained in this position.

For a mathematical simplification, the poison rod was assumed to be located at the center of the reactor system. All the models used in
the analysis of these data assumed that the poison rod was located in this position. However, in the physical system, the poison rod was centered on a point 2.5 inches in the y direction from the vertical axis. This can be seen from Fig. (1). In order to obtain as close as possible to a maximum count rate and for the mathematical simplification, the data collected with the poison rod inserted were taken at a position 2.5 inches from the center of the pile in the x direction.

Each count rate measurement was taken so that approximately 10,000 counts were collected at each z position. This was done so that each data point would have approximately the same variance. The z positions used for collecting data are shown in Fig. (1).

5.0 DATA PRESENTATION AND ANALYSIS

5.1 Presentation of Raw Data

The data taken for each of the vertical traverses were corrected for background counts. The data for each traverse, both with the rod and without the rod, are presented in Tables (3a) and (3b). The deviation reported is the standard deviation of each measurement.

5.2 Analysis of the Vertical Traverse Data

The data, presented in the previous section, were analysed so that the best estimate of $\beta_{11}$ was obtained by a least squares fit of the modified count rates. The IBM 1620 code, presented in Appendix E, was used to calculate these slope values.

Error variances of these slopes were calculated using a method given by Mickley, et al. (15). The error variance of the slope is given by
Table 3a.  The Actual Data (Corrected for Background) Collected With No Poison Rod

<table>
<thead>
<tr>
<th>Distance From Source</th>
<th>Set No. 1</th>
<th>Set No. 2</th>
<th>Set No. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inches</td>
<td>C/m.</td>
<td>C/m.</td>
<td>C/m.</td>
</tr>
<tr>
<td>42.156</td>
<td>638.5±6.3</td>
<td>641.1±6.3</td>
<td>643.7±5.7</td>
</tr>
<tr>
<td>50.156</td>
<td>401.2±3.7</td>
<td>394.4±3.6</td>
<td>393.0±3.6</td>
</tr>
<tr>
<td>58.156</td>
<td>250.1±2.0</td>
<td>252.6±2.1</td>
<td>250.7±2.1</td>
</tr>
<tr>
<td>66.156</td>
<td>101.0±1.6</td>
<td>161.1±1.6</td>
<td>156.7±1.6</td>
</tr>
<tr>
<td>74.156</td>
<td>103.2±0.9</td>
<td>101.0±0.39</td>
<td>100.5±0.37</td>
</tr>
</tbody>
</table>

Table 3a (continued)

<table>
<thead>
<tr>
<th>Distance From Source</th>
<th>Set No. 4</th>
<th>Set No. 5</th>
<th>Set No. 6</th>
</tr>
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<tbody>
<tr>
<td>Inches</td>
<td>C/m.</td>
<td>C/m.</td>
<td>C/m.</td>
</tr>
<tr>
<td>42.156</td>
<td>638.7±5.6</td>
<td>640.3±5.7</td>
<td>634.6±6.5</td>
</tr>
<tr>
<td>50.156</td>
<td>397.8±3.8</td>
<td>394.9±3.8</td>
<td>342.4±3.8</td>
</tr>
<tr>
<td>58.156</td>
<td>249.3±2.2</td>
<td>249.9±2.0</td>
<td>247.5±2.3</td>
</tr>
<tr>
<td>66.156</td>
<td>159.8±1.5</td>
<td>160.9±1.1</td>
<td>156.6±1.2</td>
</tr>
<tr>
<td>74.156</td>
<td>98.8±0.78</td>
<td>99.0±0.69</td>
<td>100.2±0.84</td>
</tr>
</tbody>
</table>
Table 3b. The Actual Data (Corrected for Background) Collected With the Poison Rod Inserted

<table>
<thead>
<tr>
<th>Distance From Source</th>
<th>Set No. 1</th>
<th>Set No. 2</th>
<th>Set No. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inches</td>
<td>C/m.</td>
<td>C/m.</td>
<td>C/m.</td>
</tr>
<tr>
<td>42.156</td>
<td>500.3±4.1</td>
<td>491.3±4.9</td>
<td>503.6±3.3</td>
</tr>
<tr>
<td>50.156</td>
<td>291.5±2.5</td>
<td>291.7±2.7</td>
<td>294.4±2.7</td>
</tr>
<tr>
<td>53.156</td>
<td>173.5±1.7</td>
<td>175.1±1.8</td>
<td>175.6±1.7</td>
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<tr>
<td>66.156</td>
<td>114.4±0.96</td>
<td>107.7±0.93</td>
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<td>74.156</td>
<td>68.7±0.68</td>
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</table>

Table 3b (continued)

<table>
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<th>Distance From Source</th>
<th>Set No. 4</th>
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<tbody>
<tr>
<td>Inches</td>
<td>C/m.</td>
<td>C/m.</td>
<td>C/m.</td>
</tr>
<tr>
<td>42.156</td>
<td>530.4±5.3</td>
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<td>486.6±2.9</td>
</tr>
<tr>
<td>50.156</td>
<td>301.0±2.9</td>
<td>290.0±2.8</td>
<td>304.4±2.3</td>
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<tr>
<td>53.156</td>
<td>183.7±1.9</td>
<td>171.6±1.7</td>
<td>176.8±1.6</td>
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<tr>
<td>66.156</td>
<td>113.7±1.1</td>
<td>105.7±0.81</td>
<td>107.9±1.0</td>
</tr>
<tr>
<td>74.156</td>
<td>69.2±0.33</td>
<td>63.3±0.66</td>
<td>67.3±0.77</td>
</tr>
</tbody>
</table>
\[ \text{se} \left( b \right) = \frac{1}{n_1 - 2} \sum_{i=1}^{n_1} (Y_i - \bar{y}_i)^2. \]  

This equation was formulated on the basis that the data could be fit by a linear plot. In the present problem, the quantity \( y_i \) is the logarithm of the modified count-rate corresponding to position \( z_i \).

\( y_i \) is calculated by

\[ Y_i = a + b z_i \]  

where \( a \) and \( b \) are the least squares intercept and slope, respectively.

Mickley, et al. (15) also present a method for finding the confidence limit envelope for a set of data. The estimate of the error variance of \( Y_i \) caused by a variance of \( y_i \) is given by

\[ S_o^2 (Y_i) = \left\{ \frac{1}{n_i} \left( Y_i - y_i \right)^2 / (n_1 - 2) \right\} \cdot \left\{ 1/n_i + z_i / \sum_{i=1}^{n_i} z_i^2 \right\}. \]

Using the t test, the confidence limits of \( Y_i \) are given as

\[ \text{Confidence limits of } Y_i = (\pm t) \text{ se}(Y_i). \]

The term \( (n_1 - 2) \) represents the number of degrees of freedom associated with \( S_o^2 (Y_i) \). The number of degrees of freedom in this work was 3, giving a value of \( t = 3.182 \) corresponding to a 95 per cent confidence limit.

All the data, both with the rod and without the rod, are plotted in Figs. (10-17). The values given for \( Y \) were calculated from the least squares technique; error variances of the \( Y \) values and the 95 per cent confidence limits are presented on these plots.
Fig. 10. A plot of the data presented in Tables 3a and 3b.
Fig. 11. A plot of the data presented in Tables 3a and 3b.
Fig. 13. A plot of the data presented in Tables 3a and 3b.
Fig. 13. A plot of the data presented in Tables 3a and 3b.
Fig. 14. A plot of the data presented in Tables 3a and 3b.
Fig. 15. A plot of the data presented in Tables 3a and 3b.
Fig. 16. A plot of the data presented in Tables 3a and 3b.
6.0 RESULTS AND CONCLUSIONS

6.1 Defining the Effective Pile Size and Equivalent Cylindrical Pile Radius

In most reactor calculations a convenient boundary condition and one that is often used is that the neutron flux vanishes at an extrapolation distance past the physical boundary of the reactor. This assumes that the negative neutron current, $J_x^-$, equals zero at the physical boundary of the reactor. For the case of a semi-infinite slab, the flux shape in the $x$ direction is a cosine function (see Fig. (18)). The flux is also a cosine function in the $x$ and $y$ and $r$ directions for the rectangular parallelepiped and cylindrical geometries. However, in the $z$ direction for a cylindrical or rectangular parallelepiped subcritical reactor, the flux shape is an exponential...
A question then arises as to the validity of using $0.71 \lambda_n$ for the extrapolation distance in the $x$ direction.

Kaiser (12) described an experimental method for determining the extrapolation distance in the $x$ and $y$ dimensions. This method consisted of taking data near the boundary and fitting the data by a least squares technique to a cosine function. Then the cosine curve could actually be extrapolated to zero. Kaiser (12) concluded that his data were much too random for any definite conclusions. His calculations showed that the pile looked like an inverted pyramid. Therefore, he used $0.71 \lambda_n$ for the extrapolation distance in both the $x$ and $y$ dimensions.

Based on Kaiser's conclusions, $0.71 \lambda_n$ was used for the extrapolation distance, $d$, in the $x$ and $y$ dimensions in this work. The value used for $\lambda_n$ was $2.53 \pm 0.09$ cm. This was the same value which Foulke (5) and Kaiser (12) used. Thus, the effective $x$ and $y$ dimensions of the rectangular pile are 69.375 inches by 69.375 inches.

A study of the extrapolated height, $c$, was not made by Kaiser (12). Therefore, a technique to make this study was developed. This technique for determining the extrapolated height, $c$, consisted of a two parameter, $\gamma_{11}$ and $c$, iterative procedure. This iterative procedure consisted of using Eq. (20) to calculate the extrapolated height, $c$, and the IBM 1620 iterative technique (12) to find the $\gamma_{11}$ parameter. Using this method, an extrapolated height of 107.36 inches was determined.

An analysis was performed to show that for the external source positioning used in this work the technique described above for finding...
the parameter, \( \gamma_{11} \), will yield a better estimate of this parameter than the value obtained when using 0.71 \( \lambda_n \) plus the physical boundary of the pile for the extrapolated height, c. Table (4) shows the results of this analysis.

The source positions used in this work were one source located at each of the following four positions in the source layer:

1. \( x = \pm 22 \) inches, \( y = 0 \) inches
2. \( x = 0 \) inches, \( y = \pm 22 \) inches.

These positions were approximately one half the distance between the pile's origin and outside boundaries. For this reason, the harmonic content of the parameter, \( \gamma_{11} \), should be negligible and consequently both the \( C_E \) and \( C_H \) factors should be very nearly unity.

Table 4. The Results of the Extrapolated Height Analysis

<table>
<thead>
<tr>
<th>( z ), in.</th>
<th>Count Rates</th>
<th>Modified Count Rates</th>
<th>( C_E )</th>
<th>( C_H )</th>
<th>Modified Count Rates</th>
<th>( C_E )</th>
<th>( C_H )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Extrap. Height</td>
<td></td>
<td></td>
<td>Extrapol. Height</td>
<td></td>
<td></td>
</tr>
<tr>
<td>42.156</td>
<td>638.5</td>
<td>615.4</td>
<td>0.9992</td>
<td>1.038</td>
<td>619.8</td>
<td>0.9960</td>
<td>1.034</td>
</tr>
<tr>
<td>50.156</td>
<td>401.2</td>
<td>394.4</td>
<td>0.9982</td>
<td>1.019</td>
<td>398.2</td>
<td>0.9907</td>
<td>1.017</td>
</tr>
<tr>
<td>58.156</td>
<td>250.1</td>
<td>248.8</td>
<td>0.9957</td>
<td>1.009</td>
<td>253.4</td>
<td>0.9785</td>
<td>1.008</td>
</tr>
<tr>
<td>66.156</td>
<td>161.0</td>
<td>161.9</td>
<td>0.9894</td>
<td>1.005</td>
<td>168.7</td>
<td>0.9503</td>
<td>1.004</td>
</tr>
<tr>
<td>74.156</td>
<td>103.2</td>
<td>105.7</td>
<td>0.9744</td>
<td>1.002</td>
<td>116.4</td>
<td>1.8848</td>
<td>1.002</td>
</tr>
</tbody>
</table>
It can be seen from Table (4) that when 0.71 \( \lambda_u \) is used for the extrapolation distance in the z direction, the \( C_Z \) factors are quite different from unity near the top of the pile. When the iterative procedure described previously for finding \( \gamma_{11} \) and Eq. (20) is used, the calculated curve of \( \phi \) versus z position is nearly linear and the original data points are positioned very close to the calculated curve. Also, simple least squares fits of linear curves to the data points on semi-logarithmic plots yield slopes very close to the values obtained for \( \gamma_{11} \) from the iterative technique. However, when 0.71 \( \lambda_u \) is used for the extrapolation distance, the calculated curve is quite different from the linear plot expected because of the source positioning.

As described in section 2.1.3, the equivalent cylindrical radius was calculated from Eq. (22). The value found for this radius was 37.55 inches.

6.2 Calculation of the Control Rod Worth of a Centrally Located Control Rod in a Subcritical Reactor System by One Group Models

6.2.1 The Results of the Semi-Experimental and Calculational Method for Finding the Change in the Effective Multiplication Factor.

The results of using the IBM 1620 program, which are presented in Appendix E, to calculate the effective multiplication factor of the K.S.U. graphite subcritical assembly are presented in Table (5). The effective multiplication factor of the graphite pile with a centrally located control rod was calculated by using Eq's. (18) and (24).

The values of \( \Delta k_e \) presented in Table (5) were calculated from six different sets of count rate versus z position data. As can be seen from
Table 5. One Group Effective Multiplication Factors

<table>
<thead>
<tr>
<th>Set Number</th>
<th>k_{e0} No. Rod</th>
<th>k_{e1} With Rod</th>
<th>- Δk_e</th>
<th>% Dev. From the Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5324</td>
<td>0.5377</td>
<td>0.0447</td>
<td>+1.36</td>
</tr>
<tr>
<td>2</td>
<td>0.5787</td>
<td>0.5344</td>
<td>0.0443</td>
<td>-0.45</td>
</tr>
<tr>
<td>3</td>
<td>0.5740</td>
<td>0.5302</td>
<td>0.0436</td>
<td>-0.68</td>
</tr>
<tr>
<td>4</td>
<td>0.5729</td>
<td>0.5293</td>
<td>0.0436</td>
<td>-1.13</td>
</tr>
<tr>
<td>5</td>
<td>0.5757</td>
<td>0.5316</td>
<td>0.0441</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0.5767</td>
<td>0.5325</td>
<td>0.0442</td>
<td>+0.23</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td></td>
<td>0.0441</td>
<td></td>
</tr>
</tbody>
</table>

Tables (3a and 3b), section 5.1, the lowest data point was taken at a z position of 42.156 inches from the source and the highest data point was taken at a z position of 74.156 inches.

6.2.1. The Results of Calculating the Change in the Effective Multiplication Factor by Experimentally Measuring the Inverse Relaxation Lengths. The results of finding the change in the effective multiplication factor, Δk_e, by measuring the change in relaxation length due to the presence of the control rod, are presented in Table (6). Six sets of count rate versus z position both with and without the control rod were analyzed to find the Δk_e.

The first portion of the IBM 1620 program presented in Appendix E was used to analyze each set of data to find the parameter, \( \gamma_{11} \).
Table 6. Results of the Difference in Relaxation Lengths

<table>
<thead>
<tr>
<th>Set</th>
<th>$c_r$ $^{-1}$</th>
<th>$c_s$ $^{-1}$</th>
<th>$k_{c0}$</th>
<th>$k_{c1}$</th>
<th>$-\Delta k_e$</th>
<th>Per Cent Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.021753</td>
<td>0.02345</td>
<td>0.5324</td>
<td>0.5506</td>
<td>0.0318</td>
<td>-6.74</td>
</tr>
<tr>
<td>2</td>
<td>0.021890</td>
<td>0.02562</td>
<td>0.5737</td>
<td>0.5464</td>
<td>0.0323</td>
<td>-5.28</td>
</tr>
<tr>
<td>3</td>
<td>0.022110</td>
<td>0.02401</td>
<td>0.5740</td>
<td>0.5327</td>
<td>0.0353</td>
<td>+3.52</td>
</tr>
<tr>
<td>4</td>
<td>0.022160</td>
<td>0.02410</td>
<td>0.5729</td>
<td>0.5367</td>
<td>0.0332</td>
<td>+6.16</td>
</tr>
<tr>
<td>5</td>
<td>0.022030</td>
<td>0.02335</td>
<td>0.5757</td>
<td>0.5417</td>
<td>0.0340</td>
<td>-0.29</td>
</tr>
<tr>
<td>6</td>
<td>0.021980</td>
<td>0.02335</td>
<td>0.5767</td>
<td>0.5418</td>
<td>0.0349</td>
<td>2.35</td>
</tr>
</tbody>
</table>

Average: 0.0341

6.2.3 The Results of the One Group Approximation Method for Finding the Change in the Effective Multiplication Factor. The change in the effective multiplication factor was calculated using an approximate method presented by Glasstone and Edlund (7). The results of these one group calculations are presented in Table (7). These values are also plotted in Fig. (20).

It can be seen from Table (7) that this approximate method yields quite high values for $\Delta k_e$ compared to the results of the other models used here. These results show that erroneous approximations were made which make this a poor calculational model.
Table 7. The Calculated Results of the Approximate Methods

<table>
<thead>
<tr>
<th>$a_0/R_0 \times 10^{-1}$</th>
<th>$-\Delta k_e$ One Group</th>
<th>$-\Delta k_o$ Two Group</th>
<th>$-\Delta k_e$ One Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.1350</td>
<td>0.0830</td>
<td>0.08365</td>
</tr>
<tr>
<td>10</td>
<td>0.1503</td>
<td>0.0985</td>
<td>0.09308</td>
</tr>
<tr>
<td>20</td>
<td>0.1095</td>
<td>0.1119</td>
<td>0.1049</td>
</tr>
<tr>
<td>30</td>
<td>0.1831</td>
<td>0.1215</td>
<td>0.1133</td>
</tr>
<tr>
<td>40</td>
<td>0.1941</td>
<td>0.1294</td>
<td>0.1202</td>
</tr>
<tr>
<td>50</td>
<td>0.2037</td>
<td>0.1363</td>
<td>0.1261</td>
</tr>
<tr>
<td>60</td>
<td>0.2122</td>
<td>0.1425</td>
<td>0.1314</td>
</tr>
<tr>
<td>70</td>
<td>0.2200</td>
<td>0.1482</td>
<td>0.1362</td>
</tr>
<tr>
<td>80</td>
<td>0.2272</td>
<td>0.1525</td>
<td>0.1407</td>
</tr>
<tr>
<td>85</td>
<td>0.2304</td>
<td>0.1560</td>
<td>0.1423</td>
</tr>
<tr>
<td>90</td>
<td>0.2340</td>
<td>0.1535</td>
<td>0.1449</td>
</tr>
<tr>
<td>100</td>
<td>0.2404</td>
<td>0.1633</td>
<td>0.1483</td>
</tr>
</tbody>
</table>

6.3 Calculation of Control Rod Worth of a Centrally Located Control Rod in a Subcritical Reactor System by Using the Two Group Models

6.3.1 The Results of the Two Group Calculational Method for Finding the Change in the Effective Multiplication Factor. The change in the effective multiplication factor was calculated by first finding the increase in the radial buckling due to the control rod. This increased buckling was then used in Eq. (33) to give $\Delta k_e$. The results of these calculations are presented in Table (8).
Table 3. The Results of the Two Group Method

<table>
<thead>
<tr>
<th>Set Number</th>
<th>$k_{o1}$ No Rod</th>
<th>$k_{o2}$ With Rod</th>
<th>$-\Delta k_o$</th>
<th>% Dev. From the Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5324</td>
<td>0.5525</td>
<td>0.0299</td>
<td>-0.33</td>
</tr>
<tr>
<td>2</td>
<td>0.5737</td>
<td>0.5435</td>
<td>0.0302</td>
<td>+0.67</td>
</tr>
<tr>
<td>3</td>
<td>0.5740</td>
<td>0.5442</td>
<td>0.0288</td>
<td>-0.67</td>
</tr>
<tr>
<td>4</td>
<td>0.5729</td>
<td>0.5432</td>
<td>0.0297</td>
<td>-1.00</td>
</tr>
<tr>
<td>5</td>
<td>0.5757</td>
<td>0.5458</td>
<td>0.0299</td>
<td>-0.33</td>
</tr>
<tr>
<td>6</td>
<td>0.5767</td>
<td>0.5463</td>
<td>0.0304</td>
<td>+1.33</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td></td>
<td><strong>0.0300</strong></td>
<td></td>
</tr>
</tbody>
</table>

The values of $k_{o2}$ were found by using the $k_o$'s calculated by the method of section 2.1. These values of $k_o$ were used to calculate the major and minor two group bucklings, $\mu^2$ and $\nu^2$ respectively. Eq. (47) was then used to calculate the radius of a fictitious critical reactor. This radius was used to calculate the increase in the radial component of the buckling. The geometric buckling of the system with the control rod was calculated using these results. Eq. (52) was used to find $\Delta k_o$. 
6.3.2 The Results of the Two Group Approximation Method for finding the Change in the Effective Multiplication Factor. The two group approximate and corrected one group approximate equations derived and presented by Glassstone and B'luin (7) were employed to calculate $\Delta k_e$, theoretically. These results are presented in Table (7). Also a comparative plot of the one group, two group, and corrected one group

![Graph showing results of approximate methods]

**Fig. 20.** The results of the approximate methods.

values for $-\Delta k_e$ versus the ratio of the control rod radius to reactor radius is presented in Fig. (20).

The results show the values for $\Delta k_e$ to be quite high. This would seem to indicate that erroneous assumptions had been made in the derivation.
6.4 Perturbation Calculations

The change in the effective multiplication factor due to the insertion of the cadmium annulus was calculated by using Eq. (74), section 2.3. As was explained in section 2.3, a one group or multigroup calculation yields identical results for the case of a bare reactor. Since the cadmium annulus was in excess of 0.03 inch thick, the inner radius, \( r_o \), in Eq. (74) was varied for calculational purposes. The average value of \( k_{e1} \) calculated by the method of section 2.1 was also used. Therefore, a plot of \( \Delta k_e (= k_{e2} - k_{e1}) \) versus \( \Delta r (= r_1 - r_o) \) was made. This plot is presented in Fig. (21).

It can be seen that from the first order perturbation assumption (the perturbed and unperturbed fluxes are approximately equal) \( \Delta k_e \) is linear with respect to the cadmium thickness. This means that the last 1 mil increment of cadmium is just as effective as the first 1 mil increment. However, most references (7, 14, and 17) state that 0.03 inch is "black" or acts as an absolute sink to thermal neutrons. This means that the \( \Delta k_e \) versus \( \Delta r \) plot should reach a constant value for a \( \Delta r \) of 0.03 inch. Because the first order perturbation calculations don't yield these results, another method of analysis was sought and is presented below.

In section 2.3 there is presented a method for evaluating the constants of a flux function synthesized from orthogonal functions. This method will yield a better approximation to the perturbed flux. Therefore, a better value for \( k_{e2} \) can be calculated. Eq. (105) was used to evaluate the orthogonality constants. The orthogonal functions used were zero-order Bessel functions of the first kind with arguments such as \( \alpha_n r \).
Fig. 21. The Results of the Improved Perturbation Calculations
where $\alpha$ is $j_n/R_0$. The $j_n$ is the $n$-th zero of the $J_0$ Bessel function and $R_0$ is extrapolated equivalent radius. These new flux functions were then used as the perturbed fluxes in Eq. (74). The results of the $\Delta k_e$ versus $\Delta r$ calculations are presented in Fig. (21).

It can be seen that the two term flux approximation yields results which are in keeping with theoretical expectations. It can also be seen that the cadmium in excess of one transport mean free path ($0.007$ cm.) is not effective in decreasing the effective multiplication factor. Fig. (21) shows that for the three and four term flux approximations the maximum $\Delta k_e$ for the cadmium poison is decreased. Furthermore, the $\Delta r$ after which further addition of cadmium is ineffective, is also decreased. Actually, the results of using the three and four term flux approximations showed the $\Delta k_e$ values reached a maximum at $\Delta r$ values of approximately $0.005$ cm. and $0.0035$ cm., respectively.

If Fig. (21) is carefully noted, it can be seen that the maximum values of $\Delta k_e$ is less when more terms are assumed for the flux function. However, the difference in maximum values becomes less as the number of terms increases. If an infinite number of terms is assumed for the flux function, an exact representation of the perturbed flux is obtained. This flux function will yield an exact curve for $\Delta k_e$ versus $\Delta r$. 
6.5 General Comparisons

Table (9) is a comparison of the $\Delta k_e$ values of the cadmium annulus in the K.S.U. graphite, calculated by using experimental data from three models.

Table 9. Comparison of Three Semi-Experimental Methods

<table>
<thead>
<tr>
<th>Set Number</th>
<th>$-\Delta k_e$ One Group Sec. 2.1.3</th>
<th>$-\Delta k_e$ Relaxation Length Sec. 2.1.4</th>
<th>$-\Delta k_e$ Two Group Sec. 2.2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0447</td>
<td>0.0318</td>
<td>0.0299</td>
</tr>
<tr>
<td>2</td>
<td>0.0443</td>
<td>0.0323</td>
<td>0.0302</td>
</tr>
<tr>
<td>3</td>
<td>0.0438</td>
<td>0.0353</td>
<td>0.0298</td>
</tr>
<tr>
<td>4</td>
<td>0.0436</td>
<td>0.0362</td>
<td>0.0297</td>
</tr>
<tr>
<td>5</td>
<td>0.0441</td>
<td>0.0340</td>
<td>0.0299</td>
</tr>
<tr>
<td>6</td>
<td>0.0442</td>
<td>0.0349</td>
<td>0.0304</td>
</tr>
<tr>
<td>Average</td>
<td>0.0444</td>
<td>0.0341</td>
<td>0.0300</td>
</tr>
</tbody>
</table>

It can be seen from Table (9) that the one group model yields higher values than the relaxation length model. The two group model gives $\Delta k_e$ values lower than both the one group and relaxation length models.

Since the one group and relaxation length models assume all neutrons are of thermal energy, these models should yield higher values for $\Delta k_e$ than the two group model. However, the two group model accounts for the
neutron energy spectrum better, consequently, the $\Delta k_0$ values calculated by this model are more accurate. The one group model uses a theoretical method to find the increase in radial buckling and hence, the $\Delta k_0$ value. The relaxation length model makes use of an experimentally determined change in relaxation length to calculate the $\Delta k_0$ value of the cadmium poison in the K.S.U. graphite pile. Therefore, the relaxation model yields a more accurate value for $\Delta k_0$ than the one group model.

As was explained in the perturbation section, an exact solution for $\Delta k_0$ could be calculated if the perturbed flux could be represented exactly. It can be seen from Fig. (21) that the series representation of the perturbed flux yields a $\Delta k_0$ value which is less than the two group $\Delta k_0$ value. If an infinite series were used to represent the perturbed flux, an exact value for $\Delta k_0$ could be calculated based on the improved perturbation model. By comparing Fig. (21) and Table (9), it can be seen that the two group model and the improved perturbation model yields similar values for the $\Delta k_0$ of the cadmium poison in the Kansas State University graphite subcritical reactor system.
This work has served as a good introduction to the theory and experiments involving control rod statics and kinetics in subcritical assemblies. All of this work can possibly be extended to a different reactor system, such as the water moderated, natural uranium system available at Kansas State University.

In addition to the static methods presented in this work, further refinements could be developed if transport theory calculations were used to determine $\Delta k_e$. Different functions could be used for the series representation of the perturbed flux and $\Delta k_e$ could be calculated using the series representation in the perturbation calculations.

The two kinetic methods presented by Hogan (10) and the method developed by Kimel, et. al. (13) should be investigated further for possible extension to the subcritical reactor. Absorbers ranging from "gray" to "black" could be inserted into the reactor. The methods presented above could be applied to these absorbing rods.
ACKNOWLEDGEMENT

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   Perturbation Integrals for Two Group Calculations and

2. Drake, M. K.
   Theoretical Reactor Kinetic Models and Experimental Verification.

3. Doggett, W. O.
   Private Communication.

4. Etherington, H.

5. Foulko, L. R.
   K.S.U. Pile Standardization and Study of Slowing Down and Diffusion
   Models. Kansas Engineering Experiment Station. Manhattan, Kansas
   (1961).

6. Glasstone, S.

7. Glasstone, S. and Edlund, M. E.

8. Henry, A. F.
   The Application of Reactor Kinetics to the Analysis of Experiments.

9. Horz, J. B.

10. Hogan, W. S.
    Negative Reactivity Measurements, Nuc. Sci. and Eng., 8(6) 518-22
    (1960).

11. Hughes, D. J.
    Pile Neutron Research. Addison-Wesley (1953).

12. Kaiser, R. E.
    Experimental and Theoretical Investigation of the Determination
    of the Diffusion Length of Thermal Neutrons in Graphite. Kansas
    Engineering Experiment Station. Manhattan, Kansas (1961).

    Determination of Time Behavior of Neutron Density and of
    Reactivity on the Argonaut Reactor. Nuc. Sci. and Eng., 6(3) 233-7
    (1959).

15. Hickley, K. S., Sherwood, C. K. and Reed, C. E. 

A Generalized Variational Method for Reactor Analysis, A Ph.D. 

17. Murray, R. L. 

18. Murray, R. L. and Westlie, J. W. 

19. Price, W. J. 

20. Snedecor, G. W. 

21. Steichen, C. U. 
The Experimental Determination and Analytic Verification of the 
Age of Pu-Be Source Neutrons in Graphite. M.S. Thesis. Kansas 
State University (1960).

22. Van Antclamation, J. W. 
Notes on Electronic Analog Computers: Scaling and Mechanization. 
The Boeing Airplane Company: Wichita Division Analog Computer 
Laboratory (1959).

23. Weinberg, A. H. and Hine, E. P. 
The Physical Theory of Neutron Chain Reactors. Chicago: University 

24. Wayland, Harold. 
Van Nostrand (1957).

25. Wood, E. B. 
Hanford Progress Report, HM-64966.
APPENDIX A

Solution to the Thermal Diffusion Equation

The recovery for the solution to the thermal diffusion equation is shown in Fig. (A-1). The equation to be solved in a multiplying medium is

\[ \nabla^2 \phi + \beta_n \phi = 0. \]  \hspace{1cm} (A-1)

![Diagram of a rectangular parallelepiped geometry for the thermal diffusion equation.](image)

Fig. A-1. The rectangular parallelepiped geometry used in this work.

The solution is subject to the boundary conditions that the flux, \( \phi \), be zero at all extrapolated boundaries. These boundary conditions may be stated mathematically as

\[ \phi(\pm a/2, y, z) = \phi(x, \pm b/2, z) = \phi(x, y, c) = 0 \]

where \( a, b, \) and \( c \) are the extrapolated \( x, y, \) and \( z \) dimensions of the pile.

Eq. (A-1) may be written in the form
\[
(\partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2) + b^2 = 0. \tag{A-2}
\]

Assuming the variables are separable, a solution of Eq. (A-2) should exist such that

\[
\phi(x, y, z) = X(x) Y(y) Z(z). \tag{A-3}
\]

Substituting this solution into Eq. (A-2), the diffusion equation becomes

\[
YZX'' + XZY'' + XYZ'' + b^2 XYZ = 0. \tag{A-4}
\]

Dividing both sides of this equation by \(XYZ\) gives

\[
X''/X + Y''/Y + Z''/Z + b^2 = 0. \tag{A-5}
\]

In order to meet the boundary conditions let each term of Eq. (A-5) be equal to a constant as follows

\[
X''/X = -\alpha_m^2 \tag{A-6}
\]

\[
Y''/Y = -\beta_m^2 \tag{A-7}
\]

\[
Z''/Z = \gamma_{mn}^2. \tag{A-8}
\]

Rewriting Eq. (A-6) gives

\[
X'' + \alpha_m^2 X = 0. \tag{A-9}
\]

A solution of this equation is

\[
X(x) = A_m \cos \alpha_m x. \tag{A-10}
\]

To meet the boundary condition that \(X(a/2) = 0\), \(\alpha_m\) must be given by
\[ \alpha_n = \frac{n \pi}{a}, \, n = 1, 3, 5, \ldots \]  
(A-11)

so that the solution for \( u(x) \) is

\[ u(x) = A_n \cos \frac{n \pi x}{a}, \, n = 1, 3, 5, \ldots \]  
(A-12)

Similarly for \( u(y) \)

\[ u(y) = A_n \cos \frac{n \pi y}{a}, \, n = 1, 3, 5, \ldots \]  
(A-13)

Rewriting Eq. (A-3) gives

\[ z^n - \xi_{mn} z = 0 . \]  
(A-14)

Eq. (A-14) has a solution given by

\[ z(z) = C_{mn} \sinh \xi_{mn} z . \]  
(A-15)

This function may be made to fit the boundary condition by altering the argument to give

\[ z(z) = C_{mn} \sinh \xi_{mn} (c-z) . \]  
(A-16)

Thus, the solution for the flux, \( \phi \), is found by combining the solutions for \( u(x) \), \( u(y) \), and \( z(z) \) as a product solution.

\[ \phi(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{n \pi x}{a} \cos \frac{n \pi y}{a} \sinh \xi_{mn} (c-z) \]  
(A-17)

The quantities \( m \) and \( n \) are summed over odd integers. Substituting the constants into Eq. (A-5), the auxiliary separation constant equation is obtained and can be written as

\[ -\alpha_n^2 - \gamma_n^2 + \beta_n^2 = -\mu_n^2 . \]
Then \( n \pi /a \) and \( m \pi /b \) can be substituted for \( \beta_n \) and \( \alpha_m \). The result is

\[
E_m^2 + \gamma_{mn}^2 - (n \pi /a)^2 - (m \pi /b)^2 = 0.
\] (A-12)

\( E_m^2 \) is the material buckling of the pile.
APPENDIX B
Solution to the Two Group Diffusion Equations

The equations to be solved, using cylindrical geometry, are

\[ D_1 \nabla^2 \phi_1 - E_R \phi_1 + k_\infty E_a \phi_2 = 0 \]  \hspace{1cm} (B-1)

\[ D_2 \nabla^2 \phi_2 - E_a \phi_2 + E_R \phi_1 = 0 \]  \hspace{1cm} (B-2)

where \( \phi_2 \) is the thermal flux

\( \phi_1 \) is \( p \) times the fast flux without resonance capture

\( D_1, D_2 \) are the fast and thermal diffusion coefficients

\( E_R \) is a slowing down cross section for the fast neutrons

\( E_a \) is the thermal neutron absorption cross section

\( k_\infty \) is the infinite multiplication factor.

The homogeneous parts of Eq's. (B-1) and (B-2) will satisfy equations of the type

\[ \nabla^2 \phi_1 + B_2^2 \phi_1 = 0 \]  \hspace{1cm} (B-3)

\[ \nabla^2 \phi_2 + b_2^2 \phi_2 = 0 \]  \hspace{1cm} (B-4)

The quantity \( B_2^2 \) used in both Eq's. (B-3) and (B-4) is the same term.

To illustrate the validity of this statement, Eq. (B-2) may be solved for \( \phi_1 \) in terms of \( \phi_2 \) and this result substituted into Eq. (B-2). The first of these manipulations results in an expression involving only \( \phi_2 \) and the second equation is one involving only \( \phi_1 \). In both equations, the operator parts are identical.

From Eq's. (B-3) and (B-4), the quantities \(-b_2^2 \phi_1\) and \(-B_2^2 \phi_2\) may be inserted in Eq's. (B-1) and (B-2) for \( \nabla^2 \phi_1 \) and \( \nabla^2 \phi_2 \), respectively.

The results after rearranging are
\[-(D_1 \varepsilon_2^2 + \varepsilon_2) \phi_1 + k \varepsilon_2 \phi_2 = 0 \quad (B-5)\]

\[\varepsilon_2 \phi_1 - (D_2 \varepsilon_2^2 + \varepsilon_2) \phi_2 = 0. \quad (B-6)\]

Since Eq's. (B-5) and (B-6) are simultaneous equations, both equal to zero, the determinant of their coefficients must be equal to zero, if a non-trivial solution exists. Setting this determinant equal to zero, expanding the determinant, and rearranging the result, the following quadratic equation in \(B^2\) is obtained.

\[
\left[ \frac{B^2}{2} + \frac{1}{\varepsilon} + \frac{1}{L^2} \right] B^2 - [k_\infty - 1/\varepsilon L^2] = 0 \quad (B-7)
\]

The quantities \(L^2\) and \(\varepsilon\) are the thermal diffusion length \((D_2/\varepsilon_a)\) and Fermi age \((D_1/\varepsilon_R)\), respectively.

Using the quadratic formula, Eq. (B-7) can be solved for the two roots of \(B^2\). These two roots of \(B^2\) are given by the following expression

\[
B^2 = \left\{ \frac{\mu^2}{\nu^2} \right\} = \frac{1}{2} \left[ -(1/\varepsilon + 1/L^2) \pm \sqrt{(1/\varepsilon + 1/L^2)^2 + 4(k_\infty - 1)/\varepsilon L^2} \right] \quad (B-3)
\]

Using these quantities for \(B^2\) in Eq's. (B-3) and (B-4) and defining \(X(r)\) and \(Y(r)\) to be the flux components corresponding to \(\mu^2\) and \(\nu^2\), respectively, these equations become

\[
\nabla^2 X(r) + \mu^2 X(r) = 0 \quad (B-9)
\]

\[
\nabla^2 Y(r) - \nu^2 Y(r) = 0 \quad (B-10)
\]

The solutions to Eq's. (B-1) and (B-2) are linear combinations of \(X(r)\) and \(Y(r)\). The permissible solutions for \(X(r)\) and \(Y(r)\) in cylindrical geometry are
\[ X(r) = A \left[ J_0(\alpha_1 r) + n Y_0(\alpha_1 r) \right] \]  
\[ Y(r) = C \left[ J_0(\alpha_2 r) + n Y_0(\alpha_2 r) \right] \]  
\[ (B-11) \]
\[ (B-12) \]

For a critical cylindrical actor of finite height, the auxiliary separation constants equations define \( \alpha_1 \) and \( \alpha_2 \) as

\[ \alpha_1^2 = \nu^2 - (\pi/c)^2 \]  
\[ (B-13) \]
\[ \alpha_2^2 = \nu^2 + (\pi/c)^2 \]  
\[ (B-14) \]

Since \( \phi_1 \) and \( \phi_2 \) are linear combinations of \( X(r) \) and \( Y(r) \), the expressions for \( \phi_1 \) and \( \phi_2 \) can be written as

\[ \phi_1 = A X(r) + C Y(r) \]  
\[ (B-15) \]
\[ \phi_2 = A' X(r) + C' Y(r) \]  
\[ (B-16) \]

Eq's. (B-15) and (B-16) represent the most general solutions to Eq's. (B-1) and (B-2). However, permissible solutions exist such that \( \phi_1 = A X(r) \) and \( \phi_2 = A' X(r) \). If these solutions are substituted into either Eq. (B-1) or (B-2), a relationship between \( A \) and \( A' \) can be shown. Using only Eq. (B-1) and rearranging the result, the ratio \( A'/A \), the coupling coefficient \( S_1 \), is given as

\[ S_1 = A'/A = \frac{\xi_2}{\xi_2 D_2 \nu^2 + \xi_1} \]  
\[ (B-17) \]

Now if only \( \phi_1 = CY \) and \( \phi_2 = C' Y \) are substituted into Eq. (B-1), a second coupling coefficient, \( S_2 \), is given by

\[ S_2 = C'/C = \frac{\xi_2}{\xi_2 D_2 \nu^2} \]  
\[ (B-18) \]
Therefore, the general solutions of Eq's. (E-1) and (E-2) are

\[
\phi_1 = A J_0(\alpha_1 r) + A_0 Y_0(\alpha_1 r) + C I_0(\alpha_2 r) + C_0 K_0(\alpha_2 r) \quad (E-19)
\]

\[
\phi_2 = S_1 A J_0(\alpha_1 r) + A_0 Y_0(\alpha_1 r) + S_2 \left[ (I_0(\alpha_2 r) + C_0 K_0(\alpha_2 r)) \right]. \quad (E-20)
\]
APPENDIX C

Derivation of the Perturbation Equations

Good discussions of perturbation theory can be found in many references (1, 7, 14, 16, 17, 23). Glasstone and Edlund (7) present a derivation of the perturbation equations which is general enough to include both one group and two group perturbation theory. Therefore, this derivation will be followed quite closely in the following discussion.

In general, the diffusion equations can be written, using the compact matrix notation, as

$$\mathbf{X} \mathbf{\Phi} = \mathbf{C} / \partial t$$

where $\mathbf{X}$ is the matrix operator and $\mathbf{\Phi}$ is a vector set of fluxes. If only one group of neutrons is considered, $\mathbf{X}$ and $\mathbf{\Phi}$ become

$$\mathbf{X} = v \left[ \nabla^2 + \frac{\mathbf{C} \mathbf{\Phi}}{\mathbf{D} \mathbf{\Phi}} (\mathbf{v} - 1) \right]$$

$$\mathbf{\Phi} = \phi (r,t)$$

whereas, if two groups of neutrons are considered, these quantities are written as

$$\mathbf{X} = \begin{bmatrix} v_1 (D_1 \nabla^2 - \Sigma_p) & v_1 k \Sigma_a \\ v_2 \leq R & v_1 (D_2 \nabla^2 - \Sigma_a) \end{bmatrix}$$

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1 (r,t) \\ \phi_2 (r,t) \end{bmatrix}$$
Consider a solution of Eq. (C-1) of the type

$$\hat{\Phi} = \exp(\omega t) \hat{\phi}. \tag{C-6}$$

where $\hat{\phi}$ is a vector flux set depending only on space and $\omega$ is the same for every group since the groups are coupled by the fission chain reaction. Using Eq. (C-6), Eq. (C-1) can be written as

$$M \frac{\partial \hat{\phi}}{\partial t} = \omega \hat{\phi}. \tag{C-7}$$

The quantity $\omega$ is actually the reciprocal of the reactor period. Eq. (C-7) will yield a series of eigenvalues, $\omega_k$, and a set of corresponding eigenfunctions, $\hat{\phi}_k(r)$. The eigenvalues will have only one positive value corresponding to the stable reactor period, and all the others will be negative. All the negative eigenvalues represent transients which decay out with time. Therefore, only the positive eigenvalue is of interest here.

If the reactor is perturbed a small amount, Eq. (C-7) can be written as

$$(X + P) \frac{\partial \hat{\phi}'}{\partial t} = \omega' \hat{\phi}' \tag{C-8}$$

where $P$ is the matrix operator due to the change in the reactor system. Due to this change, $\hat{\phi}$ becomes $\hat{\phi}'$ and $\omega$ becomes $\omega'$. Based on these considerations, a general equation for $\omega' - \omega$ can now be derived. This equation can then be used to find the worth of a control rod in a particular reactor system.

Consider two particular eigenfunctions, $\hat{\phi}_r^k$ and $\hat{\phi}_r^l$ so that Eq. (C-7) can be written as
\[ X \Phi_r^k = \omega_k \Phi_r^k \quad (C-9) \]
\[ X \Phi_r^1 = \omega_1 \Phi_r^1 . \quad (C-10) \]

Multiply Eq. (C-9) by \( \Phi_r^1 \) and Eq. (C-10) by \( \Phi_r^k \), integrate both equations over the reactor volume, and subtract the result. The final result will be in a form of the type
\[ \int_V \Phi_r^1 \cdot M \cdot \Phi_r^k \, dV - \int_V \Phi_r^1 \cdot X \cdot \Phi_r^k \, dV = (\omega_k - \omega_1) \int_V \Phi_r^1 \cdot \Phi_r^k \, dV. \quad (C-11) \]
If the eigenfunctions, \( \Phi_r^k \), are orthogonal, i.e.
\[ \int_V \Phi_r^k \cdot \Phi_r^1 \, dV = \delta_k^1 \quad \text{if } k = 1 \]
\[ = 0 \quad \text{if } k \neq 1 \]
then it follows that
\[ \int_V \Phi_r^1 \cdot X \cdot \Phi_r^k \, dV = \int_V \Phi_r^1 \cdot M \cdot \Phi_r^k \, dV. \quad (C-12) \]

In this case the matrix operator \( X \) is said to self-adjoint.

Consider now the case when the eigenfunctions, \( \Phi_r^k \), do not form an orthogonal set. For this case an adjoint operator \( X^* \) can be defined such that
\[ \int_V \Phi_r^1 \cdot X^* \Phi_r^k \, dV = \int_V \Phi_r^1 \cdot X \Phi_r^k \, dV. \quad (C-13) \]
The \( \Phi_r^k \)'s are called the adjoint functions and are the eigenfunctions of the adjoint matrix equation
\[ X^* \Phi_r^+ = \lambda^* \Phi_r^+ \quad (C-14) \]
where \( \lambda^* \) is the complex conjugate of \( \lambda \). The adjoint matrix operator is, in general, formed by replacing each element of the matrix operator
by its complex conjugate and interchanging rows and columns. Since all the elements of \( M \) are real for a reactor system, \( M^+ \) is formed merely by interchanging the rows and columns of \( M \).

A similar proof can be performed to illustrate the validity of Eq. (C-13) as was performed to show Eq. (C-12). Consider now Eq's. (C-8) and (C-14). Multiply Eq. (C-8) by \( \Phi_r^+ \) and Eq. (C-14) by \( \Phi_r' \), integrate the resulting equations over the whole reactor volume, and subtract the results. The final result will be in a form such as

\[
\int_V \Phi_r' \times \Phi_r' \, dV - \int_V \Phi_r' \times \Phi_r' \, dV + \int_V \Gamma_r' \times \Phi_r' \, dV =
\]

\[
(\omega' - \omega) \int_V \Phi_r' \times \Phi_r' \, dV \quad \text{(C-15)}
\]

where \( \omega' \) is equal to \( \omega \). Using Eq. (C-13), Eq. (C-15) can be put into a form such as

\[
\Delta \omega = \int_V \Phi_r' \times \Phi_r' \, dV / \int_V \Phi_r' \times \Phi_r' \, dV \quad \text{(C-16)}
\]

where \( \Delta \omega \) is equal to \( \omega' - \omega \). Eq. (C-16) is then used in the particular group model to find the worth of a change in the reactor system, denoted by the \( P \) matrix operator.
APPENDIX D

An IBM 1620 Program for Finding the Effective Multiplication Factor in the K.O.U. Graphite Supercritical Pile

This program was written to determine the material buckling, \( B_m^2 \), and the effective multiplication factor, \( k_{\text{eff}} \), in the K.O.U. exponential graphite pile. The program was written in Fortran I. The Fortran program print-out and logic diagram are included in this section.

This program is based on a point thermal source solution to the thermal diffusion equation. The best estimate of \( \gamma_{11} \) is determined by a harmonic and end correction iterative analysis which was outlined in section 2.1. \( B_m^2 \) is calculated from the equation

\[
B_m^2 = \left( \frac{\pi}{a} \right)^2 + \left( \frac{\pi}{b} \right)^2 - \gamma_{11}^2 .
\]

The program then calculates the infinite multiplication factor from

\[
k_{\infty} = \exp(2B_m^2) \cdot (1 + \frac{B_m^2}{B_m^2}) .
\]

The effective multiplication factor can then be calculated from

\[
k_{\text{eff}} = k_{\infty} \exp(-B_m^2) / \left[ 1 + \frac{B_m^2}{B_m^2} \right] .
\]

where \( B_m^2 \) is the geometric buckling and is formed from

\[
B_m^2 = \left( \frac{\pi}{a} \right)^2 + \left( \frac{\pi}{b} \right)^2 + \left( \frac{\pi}{c} \right)^2 .
\]

This program will analyze a maximum of six sets of count rate versus z position data. More data sets can be analyzed by changing a few cards in the source deck and recompiling. Or the program can be restarted each time a new series of six sets of data is to be analyzed.
Table D-1. Data Loading Sequence

<table>
<thead>
<tr>
<th>Card Number</th>
<th>Symbol</th>
<th>Data</th>
<th>Position</th>
<th>Format</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>B</td>
<td>Extrapolated x dimension</td>
<td>1</td>
<td>E10.4</td>
</tr>
<tr>
<td>1</td>
<td>H</td>
<td>Extrapolated z dimension</td>
<td>2</td>
<td>E10.4</td>
</tr>
<tr>
<td>1</td>
<td>W</td>
<td>Effective Diffusion Length</td>
<td>3</td>
<td>E10.4</td>
</tr>
<tr>
<td>1</td>
<td>TAU</td>
<td>Fermi Age</td>
<td>4</td>
<td>E10.4</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
<td>Source Position in x direction</td>
<td>5</td>
<td>E10.4</td>
</tr>
<tr>
<td>1</td>
<td>WY</td>
<td>Effective Fuel Height</td>
<td>6</td>
<td>E10.4</td>
</tr>
<tr>
<td>2-6</td>
<td>Z(I)</td>
<td>Z position</td>
<td>1</td>
<td>E10.4</td>
</tr>
<tr>
<td>7-11</td>
<td>C(I)</td>
<td>Count rate, Corresponding to Z position</td>
<td>1</td>
<td>E10.4</td>
</tr>
</tbody>
</table>

The input data were loaded in the following manner after the object and subroutine decks had been loaded.

The output will be written out by the typewriter and all output will be labeled. Sense switches 1 and 2 in the on position will give all of the correction factor values used to correct the count rate data.
DIMENSION AND FORMAT
READ CONSTANTS E, H, W, Td, A, WY
READ Z(I)
MN=1
READ C(I)
I=1
SET SUMLN=0, SUMZ2=0, SMLN2=0, SUMZL=0, SUMZ=0.
CALCULATE SUMLN=ΣlnC, SUMZ2=ΣZ², SMLN2=Σ(lnC)², SUMZL=ΣZ(lnC), SUMZ=ΣZ
I=I+1
NO
I=5
YES
CALCULATE SLOPE = -\( \gamma _{11} \)
\( \gamma _{11} = \text{FIRS} \)

LOOP 3
CALC. CORR. COEFF. R
PRINT \( \gamma _{11} \), R
CALCULATE \( E_m \gamma _{11}^2 = \gamma _{11}^2 \)
I=1
CALCULATE CE = \( 1 - \frac{\gamma _{11}(C-Z)}{1 + e^{2\gamma _{11} C}} \)
LOOP 5
CH=0
CALCULATE CONST = \( \frac{\beta _{11} \gamma _{11} Z}{CE \cos \frac{\alpha x}{b} \cos \frac{\pi y}{b}} \)
M=3
N=2
LOOP 1
CALCULATE \( T_2 = \cos \frac{n\pi x}{b} \)
LOOP 2
CALCULATE \( T_3 = \cos \frac{m\pi x}{a} \)
CALCULATE CC(I) = \( \frac{C(I)}{(CE)x(CH)} \)
PRINT CC(I), CE, CH

LOOP 2
DOES m=7
YES
CALC. CH = CH + (T1 \times T2 \times T3)

LOOP 1
DOES n=7
YES
CALC. CH = 1 + CH \times CONST

LOOP 1
NO
m=m+1
YES
n=n+1
NO
m=m+1

PRINT A
PROGRAM LISTING - APPENDIX C

DIMENSION C(6),Z(6),CC(6),RST(50)
1 FORMAT(10,1E10.4,1E10.4,1E10.4,1E10.4,1E10.4,1E10.4)
2 FORMAT(15.8,15.8,15.8,15.8)
3 FORMAT(15.8,15.8,15.8,15.8,1)
113 FORMAT(15.8,9H= RST /)
161 FORMAT(15.8,9H= EME /)
119 FORMAT(15.8,6H= EME /)
162 FORMAT(15.8,9H= EME /)
163 FORMAT(15.8,7H= AKEM /)
166 FORMAT(15.8,7H= AKEM /)
219 FORMAT(15.8,4H= R /)
READ 1, B, H, W, TAU, A, WY
PRINT 1, B, H, W, TAU, A, WY
DO 113 I = 1,5,1
113 READ 1, Z(I)
114 DO 5 I = 1,5,1
5 READ 1, C(I)
PRINT 235
235 FORMAT(16H C(I) C/X, 17H Z(I) CM /)
DO 235 I = 1,5,1
231 PRINT 3, C(I),Z(I)
I = 1
IN = 1
SUMN = 0.
SUMN2 = 0.
SUMZ = 0.
SUMZ2 = 0.
SUMZL = 0.
10 TM2 = LOG(C(I))
TM22 = TM2**2.
SUMN = SUMN + TM2
SUMN2 = SUMN2 + TM22
SUMZ = SUMZ + Z(I)
TM3 = Z(I)**2.
SUMZ2 = SUMZ2 + TM3
TM4 = Z(I)*TM2
SUMZL = SUMZL + TM4
TEST = I - 5
IF(TEST) 15,20,20
15 I = I + 1
GO TO 10
20 BET(IN) = (((SUMZ*SUMN)-(5.*SUMZL))/((SUMZ**2.)-(5.*SUMZ2))
BET = SUMN*SUMZ2
SQR = SQR(TST)
AR = SUMZL/SQR
PRINT 319, AR
PRINT 113, BET(IN)
IF(BET(IN)) 100,105,105
100 BET(IN) = - BET(IN)
105 I = 1
FIR = BET(IN)
TRMA = -2.*FIRS*(H-Z(I))
TRMB = -2.*FIRS*H
TRMC = EXP(TRMA)
TRMD = EXP(TRMA)
TRM2 = 1. - TRMC
TRMT = 1. - TRMD
CE = TRM2/TRXF
TRMG = FIRS*Z(I)
TRMH = EXP(TRMG)
TRMI = TRMH/CE
TERA = TRM2*FIRS
ARGA = 3.1416*A/B
TERA = COS(ARGA)
TERC = TERS2
AX = 3.
AN = 1.
BCX = 0.
ARGB = AX*3.1416*A/B
TRD = COS(ARGB)
IF(TRD) 50, 51, 52
TRD = -TRD
51 ARCC = AN*3.1416*A/B
TERE = COS(ARGC)
IF(TERE) 50, 51, 52
TERE = -TERE
TERF = TRD + TERE
BETM2 = (AX*3.1416/B)**2. + (AN*3.1416/B)**2. - BM2
IF(BETM2) 90, 150, 150
90 BETM2 = -BETM2
GO TO 150
150 TRMI = SQRT(BETM2)
TX = TRMI*Z(I)
TRX = EXP(TX)
TRN = 1./TRX
TRNO = -2.*TRNO*(H-Z(I))
TRYP = EXP(TRNO)
TRTQ = 1. - TRYP
TRQP = -2.*TRP*H
TRMQ = EXP(TRQP)
TRMT = 1. + TRMQ
F = TRMQ/TRTQ
BCH = BCH+TERA*TRMC*TERF*F/(TRMC*TERC)
TESI = AM-7.
IF(TESI) 50, 51, 52
50 AM = AM+2.
GO TO 30
51 TESJ = AM-7.
IF(TESJ) 50, 51, 52
52 AM = AM+2.
GO TO 30
53  CH = 1.*CH
   CC(I) = C(I)/(CH*CE)
   TM2 = -I
   IF(RES2) 59,53,53
56  IF(SENSE SWITCH 1) 60,200
60  PRINT 2, Z(I),C(I),CC(I),CH
200  I = I+1
   CC TO 27
59  IF(SENSE SWITCH 2) 61,201
61  PRINT 2, Z(I),C(I),CH,CE
201  SMLA = 0.
   SMZ2 = 0.
   SUMZ = 0.
   SML2 = 0.
   ZSLA = 0.
   I = 1
64  TM2 = LOG(CC(I))
   SM2 = SML*I*2.
   SMLA = SMLA+TM2
   SMLA2 = SMLA2+TM2
   SUMZ = SUMZ + Z(I)
   TMIC = Z(I)*2.
   SM2 = SMZ2+TMIC
   TMIC2 = 2*(I)*TMIC
   ZSLA = ZSLA+TM2
   TEST = I-5
   IF(TEST) 65,66,66
65  I = I+1
   CO TO 64
66  EN = EN+1
   BET(I,') = (SUMZ*SMLA-5.*ZSLA)/(SUMZ**2.-5.*SMZ2)
   SRT1 = SMLA2*SML2
   SRT2 = SRT(SRT1)
   AM1 = ZSLA/SRT1
   IF(BET(I,')) 400,410,410
400  BET(IN) = -BET(IN)
410  DIFF = BET(IN)-FIRS
   IF(DIFF) 900,910,910
900  DIFF = -DIFF
910  TEST = 0.000C1-DIFF
   IF(TEST) 105,62,62
62  IN2M = 2.*(3.1416/2)**2.-BET(IN)**2.
   TN1 = E10N*TAN
   ENN = EXP(DBR)
   AKINF = DBR*(1.+DBR**2.)**2.
   SM2 = 2.*(3.1416/2)**2.+(3.1416/2)**2.
   TN2 = -BET*TM1
   TM2 = EXP(TN1)
   AKINF = AKINF/((1.+TM2**2.)*C2)
   IF(SENSE SWITCH 3) 1000,1001
1000  PRINT 113, BET(IN)
   PRINT 119, ENN
PRINT 161, D62
PRINT 162, D62
1001 PRINT 162, AKINT
PRINT 166, AKEFF
PRINT 910, AM
TESTS = IX-6
IF (TESTS) 117, 173, 173
117 XI = IX+1
GO 20 114
173 END
Appendix B

1. Analytic Solution to the One Delay Neutron Group Reactor Kinetic Equations

In order to obtain an estimate of the dynamic characteristics of a subcritical reactor for a step change in reactivity, a one-group kinetic model was set up on the analog computer. The standard neutron kinetic equations for a critical reactor were derived by Drake (2). These equations were modified to give

\[ \frac{dn}{dt} = \left( k_e (1 - \beta) / \lambda^* \right) n(t) - n(t) / \lambda^* + \lambda c(t) + S \]  \hspace{1cm} \text{(E-1)}

\[ \frac{dc}{dt} = \left( k_e \beta / \lambda^* \right) n(t) - \lambda c(t) \]  \hspace{1cm} \text{(E-2)}

\( n(t) = \) neutron density

\( c(t) = \) precursor density

\( k_e = \) effective multiplication constant

\( \beta = \) effective delay neutron fraction

\( \lambda^* = \) thermal neutron lifetime

\( \lambda = \) averaged delay neutron decay constant

\( S = \) source term

The only modification needed to obtain Eq.'s. (E-1) and (E-2) from the critical kinetic equations is the addition of the source term, \( S \), in Eq. (E-1). For a near critical system, the source neutrons become negligible and the source term is dropped from Eq. (E-1). For the reactor system studied in this work, \( k_e \) is quite low, (0.5), and the source term is needed to sustain a neutron flux. In the physical problem, the source term was not space independent. However, Eq.'s.
\( (2-1) \) and \( (2-2) \) are spatially independent. Therefore, a spatial independent source term must be approximated in order that a solution may be found for Eq's. \( (2-1) \) and \( (2-2) \).

The approximation made was to consider an idealized average unit volume and writing a neutron balance on this unit volume. In the physical system, the source neutrons enter the system from the lower boundary. Thus, the source term actually is spatial dependent. The unit volume on which the neutron balance was written was assumed to have a number of source neutrons equal to the total number of source neutrons entering the system divided by the total reactor core volume.

Eq's. \( (2-1) \) and \( (2-2) \) can be applied to a critical reactor system in which a step insertion of reactivity was made. For a critical system \( \Delta k_e \) was defined to be

\[
\Delta k_e = k_e - 1
\]  

(2-3)

where \( k_e \) is the value of the effective multiplication factor for the system after the step change in reactivity. The effective multiplication factor for the initially critical system is defined to be unity. If the physical system which was considered in this work, was thought of as a critical system containing control elements of enough reactivity value to give the system an initial effective multiplication factor of \( k_{e1} \); the actual system under consideration was one in which a negative reactivity was "built-in". After dropping the poison rod into this system, the effective multiplication factor was lowered still further to a value of \( k_{e2} \). Three \( \Delta k_e \)'s can now be defined,

\[
\Delta k_{e1} = k_{e1} - 1
\]  

(2-4)
\[ \Delta k_{e_2} = k_{e_2} - k_{e_1} \]

\[ \Delta k_e = \Delta k_{e_1} + \Delta k_{e_2} = k - 1. \]

In view of the above separation of activities, Eqs. (E-1) and (E-2) can now be rewritten as

\[ \frac{dn}{dt} = \left\{ \left[ (1 - \beta) k_{e_1} - \frac{1}{\lambda^*} \right] \right\} n(t) + \lambda c(t) + \left\{ k_{e_2} \frac{(1 - \beta)}{\lambda^*} \right\} n(t) + s \] (E-7)

and

\[ \frac{dc}{dt} = \left\{ k_{e_1} \frac{\beta}{\lambda^*} \right\} n(t) + \left\{ \Delta k_{e_2} \frac{\beta}{\lambda^*} \right\} n(t) - \lambda c(t). \] (E-8)

To derive the machine equations which were used for the analog solution, maximum values of all the variables were estimated. Since the neutron density was operated between two relative levels, an arbitrary value could be assigned to be its maximum value. With this value chosen, the maximum values of all the other quantities were calculated through the use of the following boundary conditions.

a. \( t < (0^-) \)

\[ \frac{dn}{dt} = \frac{dc}{dt} = k_{e_2} = 0 \]

\[ n(t) = n_o \quad c(t) = c_o \]

b. \( t \rightarrow \infty \)

\[ \frac{dn}{dt} = \frac{dc}{dt} = 0 \]

\[ n(t) = n_{\infty} \quad c(t) = c_{\infty} \]

Applying these boundary conditions to Eqs. (E-7) and (E-8), the following relations were obtained,

\[ S = -\left\{ \Delta k_{e_1} / \lambda^* \right\} n_o \] (E-9)

\[ c_o = \left\{ \beta k_{e_1} / \lambda \lambda^* \right\} n_o \] (E-10)

All the nuclear parameters \((\beta, \lambda^*, \lambda)\) were known. The value for \(k_{e_1}\), calculated by using the vertical traverse data, was assumed to be
correct. The one group approximations for $\Delta k_{o2}$ due to the poison rod give an upper bound for $\Delta k_{o2}$. A series of curves of the flux versus time with $\Delta k_{o2}$ as a parameter was found.

The problem of parameter and time scaling is discussed in the next section.

2. Derivation of the Machine Equations for the Analog Solution

As was previously derived, the following are the modified spatially independent reactor kinetic equations which apply to a subcritical reactor system with a homogenous source, $S$:

$$\frac{dn}{dt} = \left\{ \frac{\Delta k_{o2}}{\lambda^p} \right\} n + \left\{ (1 - \beta) \frac{\Delta k_{o2}}{\lambda^p} \right\} n + \lambda c + S \quad (E-11)$$

$$\frac{dc}{dt} = \left\{ k_{o1} \beta / \lambda^p \right\} n + \left\{ \Delta k_{o2} \beta / \lambda^p \right\} n - \lambda c \quad (E-12)$$

When the boundary condition at $(t = 0-)$ was applied, the following two defining equations resulted:

$$c_o = \left\{ k_{o1} \beta / \lambda^p \right\} n_o \quad (E-13)$$

$$S = - \left\{ \Delta k_{o1} / \lambda^p \right\} n_o. \quad (E-14)$$

Table (E-1) contains the nuclear parameters used in deriving the machine equations. Hughes' data for delayed neutron fractions were used. The value taken for thermal neutron lifetime is a typical value for a graphite moderated thermal reactor.
Table E-1. Analog Constants

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>0.0602</td>
<td>sec(^{-1} )</td>
</tr>
<tr>
<td>( \beta^* )</td>
<td>10(^{-3} )</td>
<td>sec</td>
</tr>
<tr>
<td>( \beta )</td>
<td>7.55\times10(^{-3} )</td>
<td></td>
</tr>
<tr>
<td>1-( \beta )</td>
<td>0.99245</td>
<td></td>
</tr>
<tr>
<td>( n_0 )</td>
<td>0.91</td>
<td>r/cc</td>
</tr>
<tr>
<td>( k_{e1} )</td>
<td>0.5757</td>
<td></td>
</tr>
<tr>
<td>( k_{e2} )</td>
<td>0.4757</td>
<td></td>
</tr>
<tr>
<td>( \Delta k_e )</td>
<td>-0.4243</td>
<td></td>
</tr>
<tr>
<td>( \Delta k_{e2} )</td>
<td>-0.10</td>
<td></td>
</tr>
</tbody>
</table>

Using these constants, the following two equations were calculated.

\[
(\text{d}n/\text{d}t)_{\text{max.}} = -0.099245 \times 10^3 \ n_0 \quad (E-15)
\]

\[
(\text{d}c/\text{d}t)_{\text{max.}} = -0.755 \ n_0 \quad (E-16)
\]

with these equations and the constants of Table (E-1), the method of
scaling presented by Van Arsdalen (22) can be followed to derive the
following machine equations

\[
[2.5] = -(0.4236)(10) [250c] - (0.99245) [250r] + (0.0501) [4c] + (0.4243)[10] \quad (E-17)
\]
\[
\begin{align*}
\left[ 200^{\frac{\alpha}{25}} \right] &= (0.5216)(10) \left[ 250^{\frac{\alpha}{25}} \right] - (0.906) \left[ 250^{\frac{\alpha}{25}} \right] - (0.601) \\
&\times (10) \left[ 4^{\alpha} \right] \\
\left[ 250^{\frac{\alpha}{25}} \right] &= \int \left[ 2.5^{\frac{\alpha}{25}} \right] \, dt + (1) \left[ \cdot \right] \\
\left[ 4^{\alpha} \right] &= (1.33 \times 10^{-9}) \int \left[ 200^{\frac{\alpha}{25}} \right] \, dt + (0.8672) \left[ 10 \right]
\end{align*}
\]

where \( t = 10^{-2} \tau \) and \( \tau \) is the machine time. A possible program for the PACE TR-10 analog computer is presented in Fig. (22). In Table (E-2), a listing of the potentiometer settings is presented.

**Table E-2. Analog Potentiometer Settings**

<table>
<thead>
<tr>
<th>(-\Delta k_e)</th>
<th>Potentiometer No. 2</th>
<th>Potentiometer No. 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.002</td>
<td>0.006</td>
</tr>
<tr>
<td>0.03</td>
<td>0.204</td>
<td>0.725</td>
</tr>
<tr>
<td>0.055</td>
<td>0.345</td>
<td>0.499</td>
</tr>
<tr>
<td>0.015</td>
<td>0.149</td>
<td>0.136</td>
</tr>
</tbody>
</table>

3. Results of the Analog Solution

The one delayed neutron group analog solution was worked on the PACE TR-10 analog computer. Values for control rod worth, \( \Delta k_e \), were varied from -0.1 to -0.015. A series of transient response curves to the step insertion of these \( \Delta k_e \) values were obtained. These curves are presented in Fig. (E-2).
This analog solution was obtained in order that a comparison between it and an experimentally determined curve could be made. The calcium poison rod, described in section 3.5, was dropped several hundred times. The data were accumulated in a 256 channel analyzer with a multi-scaler plug-in logic. Definite transient response curves were obtained but the data were not reproducible. No conclusions could be reached from these data.
Fig. E-2. The results of the one delay group analog solution.
Fig. E-1. Analog solution flow diagram
REACTIVITY DETERMINATIONS IN
THE K.S.U. GRAPHITE SUB-
CRITICAL ASSEMBLY

by

NORMAN DEAN ECKHOFF

B.S., Kansas State University, 1961

AN ABSTRACT OF
A MASTER'S THESIS

submitted in partial fulfillment
of the requirements for the degree

MASTER OF SCIENCE

Department of Nuclear Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas
1963
The objectives of this thesis were to make reactivity determinations and control rod worth calculations for a neutron poison, a hollow cadmium cylinder, in the Kansas State University graphite subcritical assembly.

To carry out these objectives, several mathematical models were used. The reactivity determinations were made by using the following three methods to analyze the data taken in the graphite subcritical assembly. These methods were: 1. a one group analysis, 2. a relaxation length analysis, and 3. a two group analysis. The theoretical control rod worth calculations were made through the use of the following three methods: 1. an approximate one and two group technique, 2. a perturbation technique, and 3. an improved perturbation technique. These methods were all developed for a subcritical reactor system.

A comparison of the results of the calculations showed that the two group analysis and the improved perturbation technique yielded similar values for the reactivity of the control rod.