# SOME APPLICATIONS OF BESSEL FUNCTIONS

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A. B., Bethel College, 1939

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A THESIS

submitted in partial fulfillment of the

requirements for the degree of

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE COLLEGE
OF AGRICULTURE AND APPLIED SCIENCE

1943

TABLE OF CONTENTS

## INTRODUCTION

Bessel Functions are named after a German mathematician and astronomer who lived from 1784 to 1846. A Bessel Function is the name given to a function that is a solution of Bessel's equation. He was not the first one to use these functions but he was the first to give a systematic development of their properties and some tables for the functions of lowest order. Functions of the zero order had been used as early as 1732 by Daniel Bernoulli and 1764 by L. Euler.

Bessel Functions are used to solve boundary value problems in heat, electricity, hydrodynamics, elasticity, and vibration. They are especially applicable to problems involving cylindrical coordinates.

It has been the purpose of this paper to solve a few particular problems involving Bessel Functions and to collect some solutions of problems previously solved.

# DERIVATION OF A BESSEL FUNCTION

Any solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0,$$

known as Bessel's equation, is called a Bessel function. It will be shown later how this equation arises in the process of obtaining solutions of certain differential equations written in cylindrical coordinates.

A particular solution of Bessel's equation can always be found in the form of a powers series multiplied by $x^p$, where p is not necessarily an integer, such as

$$y = x^p \sum_{j=o}^{\infty} a_j x^j, \qquad a_o \neq 0, \quad \text{or}$$

$$(1) \quad y = \sum_{j=o}^{\infty} a_j x^{p+j}.$$

Differentiate termwise twice

$$\frac{dy}{dx} = \sum_{j=o}^{\infty} a_j \,(p+j)\, x^{p+j-1},$$

$$\frac{d^2y}{dx^2} = \sum_{j=o}^{\infty} a_j \,(p+j)\,(p+j-1)\, x^{p+j-2}.$$

Substitute in equation (1)

$$\sum_{j=o}^{\infty} \left[x^2 a_j \,(p+j)\,(p+j-1)\, x^{p+j-2} + x a_j \,(p+j)\, x^{p+j-1} + (x^2 - n^2)\, a_j x^{p+j}\right] = 0,$$

which may be written

$$(2) \quad \sum_{j=o}^{\infty} \left[(p+j)\,(p+j-1) + (p+j) + (x^2 - n^2)\right] a_j x^{p+j} = 0.$$

Divide through by $x^p$ and expand the first two terms of the series

$$(p^2 - n^2)\, a_0 + a_0x^2 + \left[(p+1)^2 - n^2\right] a_1x + a_1x^3 + - - - = 0.$$

Collect like powers of x. Equation (2) then becomes

$$(3) \quad (p^2-n^2)a_0 + \left[(p+1)^2-n^2\right] a_1x + \sum_{j=2}^{\infty} \left\{\left[(p+j)^2-n^2\right] a_jx^j + a_{j-2}x^j\right\} = 0.$$

Since this is to be an identity in x, the coefficients of each power of x must vanish. The constant term vanishes if $p = \pm n$. The second term vanishes if $a_1 = 0$, and the coefficients of all the succeeding terms vanish if $\left[(p+j)^2 - n^2\right]a_j + a_{j-2} = 0$; that is, if $(p+j-n)\,(p+j+n)a_j = -a_{j-2}$. This is a recursion formula, giving each coefficient in terms of some preceding term.

Letting $p = n$, the formula becomes

$$(4) \quad j(2n + j)\, a_j = -a_{j-2}.$$

Since $a_1$ must be zero, it follows that

$$(5) \quad a_3 = a_5 = a_{2k-1} = 0 \quad \text{where } (k = 1, 2, 3, - - - ).$$

Replace j by 2j in (4)

$$a_{2j} = \frac{-1}{2^2 j\,(n+j)}\, a_{2j-2}\,.$$

Replace j by j-1

$$a_{2j-2} = \frac{-1}{2^2(j-1)\,(n+j-1)}\, a_{2j-4}\,,$$

so that

$$a_{2j} = \frac{(-1)^2}{2^4 j(j-1)\,(n+j)\,(n+j-1)}\, a_{2j-4}\,.$$

Continuing in this manner, it can be shown that

$$a_{2j} = \frac{(-1)^k\, a_{2j-2k}}{2^{2k} j(j-1) - - - (j-k+1)\,(n+j)\,(n+j-1) - - - (n+j-k+1)}$$

so that when $k = j$, we have the formula for $a_{2j}$ in terms of $a_0$.

$$(6)\quad a_{2j} = \frac{(-1)^j a_o}{2^{2j} j!\,(n+j)(n+j-1) \cdots (n+1)}\,, \quad (j = 1, 2, \cdots).$$

Since $a_o$ was left as an arbitrary constant, assign it the following value:

$$a_o = \frac{1}{2^n \Gamma(n+1)}\,.$$

Then (6) may be written

$$a_{2j} = \frac{(-1)^j}{2^{n+2j} j!\,\Gamma(n+j+1)}\,.$$

The function represented by equation (1) with the coefficients (5) and (6) is called a Bessel function of the first kind of order n.

$$(7)\quad J_n(x) = \sum_{j=o}^{\infty} \frac{(-1)^j}{j!\,\Gamma(n+j+1)} \left(\frac{x}{2}\right)^{n+2j}.$$

## GENERAL SOLUTIONS

Bessel's equation arises in the process of solving the following partial differential equations.

$$\text{I.}\quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0;$$

$$\text{II.}\quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t};$$

$$\text{III.}\quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k^2}\frac{\partial^2 u}{\partial t^2}.$$

## Case I

To solve Laplace's equation

$$(1) \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

let $u = R(r)\,\Phi(\phi)\,Z(z)$, where R is a function of r alone, $\Phi$ is a function of $\phi$ alone, and Z is a function of z alone. Then $\frac{\partial u}{\partial r} = R'\Phi Z$, $\frac{\partial u}{\partial z} = R\Phi Z'$, etc., where the prime denotes the ordinary derivative with respect to the only independent variable involved in the function. (1) can then be written

$$(2) \quad R''\Phi Z + \frac{1}{r} R'\Phi Z + \frac{1}{r^2} R\Phi'' Z + R\Phi Z'' = 0 .$$

Transposing the last term and dividing by $R\Phi Z$, (2) becomes

$$(3) \quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = -\frac{Z''}{Z} .$$

Since the member on the right is a function of z alone, it cannot vary with r and $\phi$, but it is equal to a function of r alone and $\phi$ alone. Therefore, it must be equal to a constant, say $-\lambda^2$, so that

$$(4) \quad Z'' - \lambda^2 Z = 0 \quad \text{and}$$

$$(5) \quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = -\lambda^2 .$$

A solution of (4) is

$$(6) \quad Z = A \cosh \lambda z + B \sinh \lambda z.$$

To solve (5) transpose the right member and the third term of the left member and multiply by $r^2$.

$$r^2 \frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = -\frac{\Phi''}{\Phi} .$$

Following the same line of reasoning as before, since the right member is a function of $\phi$ alone, it cannot vary with r, etc. Therefore, let it equal the constant $\mu^2$. Then

(7) $\Phi'' + \mu^2\Phi = 0;$

(8) $r\dfrac{R''}{R} + r\dfrac{R'}{R} + r^2\lambda^2 - \mu^2 = 0.$

A solution of (7) is

(9) $\Phi = C\cos\mu\phi + D\sin\mu\phi.$

Since u is a periodic function of $\Phi$ with a period of $2\pi$, let $\mu = n$ $(n = 0, 1, 2, - - - )$. Then (9) becomes

(9a) $\Phi = C\cos n\phi + D\sin n\phi.$

Multiply (8) by R: $r^2R'' + rR' + (r^2\lambda^2 - n^2)R = 0.$

This is Bessel's equation with the parameter $\lambda$ and therefore

(10) $R = J_n(\lambda r).$

Case II

To solve (1) $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t},$

let $u = R(r)\,\Phi(\phi)\,Z(z)\,T(t)$ where R is a function of r alone, etc. Then (1) can be written

(2) $R''\Phi ZT + \dfrac{1}{r}R'\Phi ZT + \dfrac{1}{r^2}R\Phi''ZT + R\Phi Z''T = \dfrac{1}{k}R\Phi ZT'.$

Divide through by $R\Phi ZT$ and equate to a constant, say $-\alpha^2$,

(3) $T' + k\alpha^2T = 0$ and

(4) $\dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} + \dfrac{1}{r^2}\dfrac{\Phi''}{\Phi} + \dfrac{Z''}{Z} = -\alpha^2.$

A solution for (3) is

(5) $T = Ae^{-k\alpha^2 t}$ .

In (4) transpose the right member and the last term of the left member and equate to a constant $\mu^2$. Then,

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \alpha^2 = -\frac{Z''}{Z} = \mu^2. \text{ Then,}$$

(6) $Z'' + u^2 Z = 0$ and

(7) $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \alpha^2 - \mu^2 = 0.$$

A solution for (6) is

(8) $Z = B \cos \mu z + C \sin \mu z$.

In (7) transpose the third term, multiply by $r^2$, and equate to a constant $\beta^2$. Then,

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2(\alpha^2 - \mu^2) = -\frac{\Phi''}{\Phi} = \beta^2. \text{ Then,}$$

(9) $\Phi'' + \beta^2\Phi = 0$ and

(10) $$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2(\alpha^2 - \mu^2) - \beta^2 = 0.$$

A solution of (9) is

(11) $\Phi = D \cos \beta\phi + E \sin \beta\phi$, but since u is a periodic function of $\phi$ with a period of $2\pi$, let $\beta = n$ ($n = 1, 2, 3, - - -$). Then (11) becomes

(11a) $\Phi = D \cos n\phi + E \sin n\phi$.

In (10) let $(\alpha^2 - \mu^2) = \lambda^2$ and multiply by R. It then becomes

(12) $r^2R'' + rR' + (r^2\lambda^2 - n^2)R = 0$.

This is Bessel's equation and therefore the solution is

(13) $R = J_n(\lambda r)$.

## Case III

To solve

$$(1)\quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k^2}\frac{\partial^2 u}{\partial t^2}.$$

Let $u = R(r)\,\Phi(\phi)\,Z(z)\,T(t)$. Then (1) can be written

$$(2)\quad R''\Phi ZT + \frac{1}{r}R'\Phi ZT + \frac{1}{r^2}R\Phi''ZT + R\Phi Z''T = \frac{1}{k^2}R\Phi ZT''$$

Separate variables and equate to a constant as before.

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = \frac{1}{k^2}\frac{T''}{T} = -\alpha^2. \text{ Then}$$

$$(3)\quad T'' + k^2\alpha^2 T = 0 \quad \text{and}$$

$$(4)\quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} + \alpha^2 = 0.$$

A solution for (3) is

$$(5)\quad T = A\cos\alpha k\,t + B\sin\alpha k\,t.$$

Separate the variables in (4) and equate to a constant.

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \alpha^2 = -\frac{Z''}{Z} = \mu^2. \text{ Then,}$$

$$(6)\quad Z'' + \mu^2 Z = 0 \quad \text{and}$$

$$(7)\quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \alpha^2 - \mu^2 = 0.$$

A solution for (6) is

$$(8)\quad Z = C\cos\mu z + D\sin\mu z.$$

In (7) multiply by $r^2$, transpose the third term, and equate to a constant.

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + r^2(\alpha^2 - \mu^2) = - \frac{\Phi''}{\Phi} = \beta^2.$$

Then (9) $\Phi'' + \beta^2\Phi = 0$ and

(10) $r^2 \dfrac{R''}{R} + r \dfrac{R'}{R} + r^2(\alpha^2 - \mu^2) - \beta^2 = 0.$

A solution of (9) is

(11) $\Phi = E \cos \beta\phi + F \sin \beta\phi$. Since u is a periodic function of $\phi$ with a period of $2\pi$, we let $\beta = n$ ($n = 1, 2, 3, ---$). Then

(11a) $\Phi = E \cos n\phi + F \sin n\phi.$

In (10) multiply by R and let $(\alpha^2 - \mu^2) = \lambda^2$. Then

(12) $r^2R'' + rR' + (r^2\lambda^2 - n^2)R = 0.$

This is Bessel's equation and therefore the solution is

(13) $R = J_n(\lambda r).$

## PARTICULAR SOLUTIONS

### Case I

$\nabla^2 u = 0$ is the fundamental equation for: (1) Heat, steady state, (2) potential, (3) elasticity, and (4) hydrodynamics. Since the same differential equation applies to each of these fields, it follows that a solution of a particular problem in one field will also be a solution of a problem with analagous boundary conditions of any of the other fields.

In a particular problem in heat a finite cylinder is given, with one base kept at $0^\circ$, the convex surface is insulated and the temperature of the other base is a function of the distance from the axis. The conditions that must be fulfilled are:

(1) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0;$

(2) $u(r,0) = 0, \qquad 0 \leqq r \leqq a;$

(3) $\left.\frac{\partial u}{\partial r}\right]_{r=a} = 0, \qquad 0 \leqq z \leqq b;$

(4) $u(r,b) = f(r).$

Since the temperature is symmetric with respect to the z axis, $\frac{\partial^2 u}{\partial \phi^2} = 0.$

The general solution of (1) was previously shown to be $u = R(r)\,Z(z)$ where

(5) $Z = A \cosh \lambda z + B \sinh \lambda z$

and $R = J_n(\lambda r)$, but since $\frac{\partial^2 u}{\partial \phi^2} = 0$, $n = 0$. Then

(6) $R = J_o(\lambda r).$

Applying condition (2) to equation (5) it becomes $A \cosh 0 + B \sinh 0 = 0$. Since $\sinh 0 = 0$ it leaves $A \cosh 0 = 0$. To satisfy this, A must equal 0. Then

(7) $Z = B \sinh \lambda z.$

Applying condition (3) to equation (6) it becomes

(8) $J'_o(\lambda a) = 0.$

The only functions $R(r)$ which will satisfy equation (3) are $J'_o(\lambda_j r)$, where $\lambda_j$ are the positive roots of equation (8).

The only particular solutions of $u = R(r)\,Z(z)$ that will satisfy conditions (1), (2), and (3) are

(9) $u(r,z) = \sum_{j=1}^{\infty} A_j J_o(\lambda_j r) \sinh \lambda_j z.$

To satisfy condition (4) the coefficients A must be so chosen that

$$(10) \quad f(r) = \sum_{j=1}^{\infty} A_j J_o(\lambda_j r) \sinh \lambda_j b.$$

According to the Fourier - Bessel expansion, this is true if

$$(11) \quad A_j = \frac{2\lambda_j^2}{\sinh(\lambda_j b)(\lambda_j^2 a^2 + h^2 - n^2)\left[J_n(\lambda_j a)\right]^2} \int_o^a r f(r) J_n(\lambda_j r)\, dr,$$

where $\lambda_j$, ($j = 1, 2, 3, ---$), are the positive roots of

$$(12) \quad \lambda a J'_n(\lambda a) + h J_n(\lambda a) = 0.$$

The conditions on $\lambda_j$ were given in equation (8). This is a special case of equation (12) where $h = n = 0$. Therefore,

$$(13) \quad A_j = \frac{2}{\sinh(\lambda_j b)\, a^2 \left[J_o(\lambda_j a)\right]^2} \int_o^a r\, f(r)\, J_o(\lambda_j r)\, dr.$$

However, in the special case where $h = n = 0$, $\lambda_1$ is to be taken as zero and the first term of the series is

$$A_1 = \frac{2}{a^2 \sinh(\lambda_1 b)} \int_o^a r\, f(r)\, dr.$$

The solution can then be written

$$(14) \quad u(r,z) = \frac{2}{a^2 \sinh(\lambda_1 b)} \int_o^a r\, f(r)\, dr +$$

$$\sum_{j=2}^{\infty} \frac{2 \sinh(\lambda_j z)}{a^2 J_o(\lambda_j a)^2 \sinh(\lambda_j b)} \int_o^a r\, f(r)\, J_o(\lambda_j r)\, dr.$$

The convex surface and one base of a cylinder of radius a and length b are kept at constant temperature zero; the temperature of each point of the other base is a given function of the distance of the point from the center of the base. The solution of the following differential equation with the boundary conditions will give the temperature of any point of the cylinder after permanent temperatures have been established.

$$(1)\quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0;$$

$$(2)\quad u\,(r + 0) = 0, \qquad 0 \leqq r \leqq a;$$

$$(3)\quad u\,(a,\ z) = 0, \qquad 0 \leqq z < b;$$

$$(4)\quad u\,(r,\ b) = f(r), \qquad 0 < r \leqq a.$$

The solution is

$$u\,(r,\ z) = \sum_{j=1}^{\infty} A_j \frac{\sinh\,(\lambda_j z)}{\sinh\,(\lambda_j b)}\ J_o(\lambda_j r),$$

where $A_j = \dfrac{2}{a^2\left[J_1(\lambda_j a)\right]^2}\displaystyle\int_o^c r\,f(r)\,J_o(\lambda_j r)\,dr.$

Let the potential on the surface of a hollow cylindrical ring at, $r = a$, and at both bases, $z = o$ and $z = c$, be kept at zero and on the inside at $r = b$ let it be a function of the height $z$ only. The solution of the following differential equation with the boundary conditions will give the potential inside the hollow cylindrical ring.

$$(1)\quad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial z^2} = 0;$$

$$(2)\quad v\,(a,\ \phi,\ z) = f(z);$$

$$(3)\quad v\,(b,\ \phi,\ z) = 0, \qquad\qquad a > b;$$

$$(4)\quad v\,(r,\ \phi,\ 0) = 0;$$

$$(5)\quad v\,(r,\ \phi,\ c) = 0.$$

The solution is

$$V = \sum_{k=1}^{\infty} \frac{A_k \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{k\pi z}{c}\right)\left[\frac{I_o\left(\frac{k\pi r}{c}\right)}{I_o\left(\frac{k\pi a}{c}\right)} - \frac{K_o\left(\frac{k\pi r}{c}\right)}{K_o\left(\frac{k\pi b}{c}\right)}\right]}{\frac{I_o\left(\frac{k\pi a}{c}\right)}{I_o\left(\frac{k\pi b}{c}\right)} - \frac{K_o\left(\frac{k\pi a}{c}\right)}{K_o\left(\frac{k\pi b}{c}\right)}},$$

where $A_j = \frac{2}{c}\int_o^c f(z) \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{k\pi z}{c}\right)dz.$

Case II

$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t}$ is the fundamental variable state heat equation.

Let the convex surface of a finite cylinder with insulated bases be kept at temperature zero and the initial temperature a function of the distance from the axis only. Since the function is independent of $\phi$ and z, the heat equation and the boundary equations are:

(1) $\frac{\partial u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{k}\frac{\partial u}{\partial t}$, $(0 < r < c,\ t > 0)$;

(2) $u\,(c,t) = 0$, $(t > 0)$;

(3) $u\,(r,0) = f(r)$, $(0 < r < c)$.

The general solution of (1) was previously shown to be $u = R(r)\,T(t)$ where

(4) $R = J_n(\ r)$ and

(5) $T = A\,e^{-k\lambda^2 t}$.

Since $\frac{\partial^2 u}{\partial \phi^2} = 0$, $n = 0$. Therefore (4) becomes

(4a) $R = J_o(\lambda r)$.

Applying condition (2) on (4a) it becomes

(6) $J_o(\lambda c) = 0.$

This will satisfy the condition if in $J_o(\lambda_j r)$, $\lambda_j$ are the positive roots of equation (6).

Equations (1) and (2) will be satisfied if

$$(7)\quad u(r,t) = \sum_{j=1}^{\infty} A_j J_o(\lambda_j r)\, e^{-k\lambda_j^2 t}.$$

To satisfy condition (3) the coefficients $A_j$ must be determined so that

$$(8)\quad f(r) = \sum_{j=1}^{\infty} A_j J_o(\lambda_j r).$$

According to the Fourier-Bessel expansion, this is true if

$$(9)\quad A_j = \frac{2}{c^2 \left[J_1(\lambda_j c)\right]^2} \int_o^c r\, f(r)\, J_o(\lambda_j r)\, dr.$$

The solution can then be written

$$(10)\quad u\,(r,t) = \frac{2}{c^2} \sum_{j=1}^{\infty} \frac{J_o(\lambda_j r)}{\left[J_1(\lambda_j c)\right]^2}\, e^{-k\lambda_j^2 t} \int_o^c r\, f(r)\, J_o(\lambda_j r)\, dr.$$

Let the surface of an infinite cylinder of radius c undergo heat transfer into surroundings kept at temperature zero, according to Newton's law. The solution of the following differential equation with the boundary conditions will give the temperature at a given point in the cylinder at a given time.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad (0 < r < c,\ t > 0);$$

$$c\,\frac{\partial u\,(c,t)}{\partial r} = -\,hu\,(c,t), \qquad (t > 0);$$

$$u\,(r,0) = f(r), \qquad (0 < r < c).$$

The solution is

$$u\ (r,t) = \sum_{j=1}^{\infty} A_j J_o(\lambda_j r)\ e^{-k\lambda_j^2 t},$$

where $A_j = \dfrac{2\ \lambda_j^2}{(\lambda_j^2 c^2+h^2)\left[J_o(\lambda_j c)\right]^2} \displaystyle\int_o^c r\ J_o(\lambda_j r)\ f(r)\ dr.$

Case III

$\nabla^2 u = \dfrac{1}{k^2}\dfrac{\partial^2 u}{\partial t^2}$ is the fundamental vibration equation.

Let a membrane be stretched over a fixed circular frame $r = c$ in the plane $z = 0$, with an initial displacement of $z = f(r,\phi)$. The displacement of the membrane will be found as the continuous solution of the following differential equation with the given boundary conditions.

(1) $\dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \phi^2} = \dfrac{1}{k^2}\dfrac{\partial^2 z}{\partial t^2};$

(2) $\left.\dfrac{\partial z}{\partial t}\right]_{t=0} = 0,$ $\qquad (0 \leqq r \leqq c,\ -\pi < \phi \leqq \pi);$

(3) $z(c,\phi,t) = 0,$ $\qquad (-\pi < \phi \leqq \pi,\ t > 0);$

(4) $z(r,\phi,0) = f(r,\phi),$ $\qquad (0 \leqq r \leqq c,\ -\pi < \phi \leqq \pi).$

The general solution to equation (1) was previously shown to be $z = R(r)\ \Phi(\phi)\ T(t)$ where

(5) $R = J_n(\lambda r),$

(6) $\Phi = A \cos n\phi + B \sin n\phi$ and

(7) $T = C \cos \lambda kt + D \sin \lambda kt.$

Differentiating (7) and applying condition (2), it becomes

$\frac{\partial T}{\partial t} = -\lambda kC \sin \lambda kt + \lambda kD \cos \lambda kt$. Letting $t = 0$,

$0 = -\lambda kC \sin \lambda k0 + \lambda kD \cos \lambda k0$. To satisfy this, D must be zero. Therefore

(8) $T = C \cos \lambda kt$.

Applying condition (3) to (5) determines that the roots of (5) will have to be any of the positive roots $\lambda_{nj}$ of the equation

(9) $J_n(\lambda c) = 0$.

All the conditions except (4) will then be satisfied by the following equation:

$$(10) \quad z(r,\phi,t) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} J_n(\lambda_{nj} r)\,(A_{nj} \cos n\phi + B_{nj} \sin n\phi) \cos \lambda_{nj} kt.$$

This last condition will be satisfied provided $A_{nj}$ and $B_{nj}$ are such that

$$f(r,\phi) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} J_n(\lambda_{nj} r)\,(A_{nj} \cos n\phi + B_{nj} \sin n\phi).$$

According to the Fourier-Bessel expansion, these are:

$$(11) \quad A_{nj} = \frac{2}{c^2 \left[J_{n+1}(\lambda_{nj} c)\right]^2} \int_0^c r\, J_n(\lambda_{nj} r)\,dr \int_{-\pi}^{\pi} f(r,\phi) \cos n\phi \, d\phi,$$

$$(n = 1, 2, - - - ),$$

$$(12) \quad A_{oj} = \frac{1}{c^2 \left[J_1(\lambda_{oj} c)\right]^2} \int_0^c r J_o(\lambda_{oj} r)\,dr \int_{-\pi}^{\pi} f(r,\phi)\,d\phi,$$

$$(13) \quad B_{nj} = \frac{2}{c^2 \left[J_{n+1}(\lambda_{nj} c)\right]^2} \int_0^c r J_n(\lambda_{nj} r)\,dr \int_{-\pi}^{\pi} f(r,\phi) \sin (n\phi)\, d\phi.$$

The required solution is equation (1) with coefficients (11), (12), and (13).

## CONCLUSION

The main results of this paper are the solution of particular problems given in Case I, Case II, and Case III. By the proper substitutions in these equations the solution of certain types of problems in vibration, elasticity, heat, electricity, and hydrodynamics may be found.

## ACKNOWLEDGMENT

Indebtedness is acknowledged to Dr. G. C. Munro and Dr. D. T. Sigley, major instructors, for directing this study.

## REFERENCES

Byerley, W. E.
Fourier's series. New York. Gin. 287 p. 1893.

Cajori, F.
A history of mathematics. New York. Macmillan. 516 p. 1929.

Churchill, R. V.
Fourier's series and boundary value problems. New York. McGraw-Hill. 206 p. 1941.

Filon, L. N. G.
Elastic equilibrium of strained circular cylinders. Phil. Trans. Roy. Soc. London. Series A, 198: 147-233. 1902.

Gray, A. and Mathews, G. B.
A treatise on Bessel Functions. London. Macmillan. 327 p. 1931.

Smythe, W. R.
Static and dynamic electricity. New York. McGraw-Hill. 559 p. 1939.

Timoshenko, S.
Theory of elastic stability. New York. McGraw-Hill. 517 p. 1936.