THE VOLUME ELEMENTS INTERCEPTED BY INTERSECTING CYLINDERS

by

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INTRODUCTION

Two problems are to be considered. First is the determination of the intercepted volume formed by the axial intersection of two unequal, right, circular cylinders. Second is the determination of the intercepted volume formed by a random, internal intersection of two unequal, right, circular cylinders. The analytic expression of these volumes involves the three kinds of elliptic integrals.

An elliptic integral was first encountered in the problem of the rectification of the ellipse. From its association with the problem the integral received the appellation "elliptic". The first intensive study of integrals of this type was conducted by Adrien Marie Legendre (1752-1833), who showed that an integral depending upon the square root of a polynomial of fourth degree in x can be brought back to the three fundamental forms.

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}, \qquad \int_0^x \frac{x^2dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}, \text{ and}$$

$$\int_0^x \frac{dx}{(x^2+a)\sqrt{(1-x^2)(1-k^2x^2)}},$$ which are termed elliptic inte-

grals of the first, second, and third kinds, respectively. Numerical evaluation of the first and second kinds is conveniently effected by Landen's transformations.

The inverse functions defined by the elliptic integrals are termed elliptic functions. In 1825 Niels Henrik Abel did the pioneering work with elliptic functions. Carl Gustav Jacob Jacobi (1804-1851) discovered the theta-functions, which can be used in the numerical evaluation of the elliptic integral of the third kind. In the second problem treated below an elliptic integral of the third kind is encountered. However, a numerical evaluation will not be attempted, as the problem may be considered profitably without going into the extended application of the theta-functions.

THE INTERCEPTED VOLUME FORMED BY THE AXIAL INTERSECTION OF TWO UNEQUAL, RIGHT, CIRCULAR CYLINDERS

The Analytical Representation of the Problem

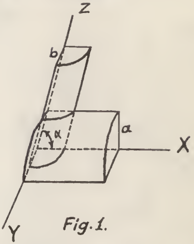

Fig. 1.

A horizontal, circular cylinder of radius a and a circular cylinder of radius b (a>b) intersect centrally with an angle α between their axes. The cylinders S_a and S_b are mounted on the axes as shown in Fig. 1. Oblique coordinates are used, the YZ-plane being rotated about the Y-axis until it makes an angle α with the XY-plane. The equation of S_a is its trace on the YZ-plane, which is $y^2/a^2+z^2/(a^2\csc^2\alpha)=1$. The equation of S_b is its trace on the XY-plane, which is $x^2/(b^2\csc^2\alpha)+y^2/b^2=1$. The volume common to S_a and S_b is bounded on the sides by S_b and topped at each end by S_a.

The element of volume stands upon the XY-plane and upon the ellipse represented by the equation of S_b, that is, the base of S_b. The slant height of the element is Z_a. Its volume is $Z_a\sin(\alpha)\,dxdy$. The total volume common to S_a and

S_b is $V=4\sin(\alpha)\int_0^b\int_{-\csc(\alpha)\sqrt{b^2-y^2}}^{\csc(\alpha)\sqrt{b^2-y^2}} Z_e\,dx\,dy$.

A Solution by Algebraic Methods

As $Z_e=\csc(\alpha)\sqrt{a^2-y^2}$, $V=8\csc(\alpha)\int_0^b\sqrt{(a^2-y^2)(b^2-y^2)}\,dy$.

Let $y=bx$. The volume $V=8ab^2\csc(\alpha)\int_0^1\sqrt{(1-x^2)(1-k^2x^2)}\,dx$,

where $k^2=b^2/a^2$. Let $x=\sin(\phi)$. Let $\Delta\phi=\sqrt{1-k^2\sin^2\phi}$.

Then $V=8ab^2\csc(\alpha)\int_0^{\pi/2}\left[1-(1+k^2)\sin^2\phi+k^2\sin^4\phi\right]\frac{d\phi}{\Delta\phi}$.--------(1)

Reduction of $\int_0^{\pi/2}\sin^2\phi\,\frac{d\phi}{\Delta\phi}$.

The above integral is identically equal to

$$(-1/k^2)\int_0^{\pi/2}(1-k^2\sin^2\phi)\frac{d\phi}{\Delta\phi}+1/k^2\int_0^{\pi/2}\frac{d\phi}{\Delta\phi}\equiv-1/k^2\int_0^{\pi/2}\Delta\phi\,d\phi+1/k^2\int^{\pi/2}\frac{d\phi}{\Delta\phi}$$

$=(1/k^2)\left[K-E\right]$, where K and E are complete elliptic integrals of the first and second kinds, respectively.

Reduction of $\int_0^{\pi/2}\sin^4\phi\,\frac{d\phi}{\Delta\phi}$.

Set up the following identity: $\sin(\phi)\cos(\phi)\Delta(\phi)$

$$=\int_0^{\phi}\frac{d}{d\phi}(\sin\phi\cos\phi\Delta\phi)\,d\phi=\int_0^{\phi}\left[\cos^2\phi\Delta^2\phi-\sin^2\phi\Delta^2\phi-k^2\sin^2\phi\cos^2\phi\right]\frac{d\phi}{\Delta\phi}$$

$$=\int_0^{\phi}\left[1-k^2\sin^2\phi-2\sin^2\phi+2k^2\sin^4\phi\ -k^2\sin^2\phi+k^2\sin^4\phi\right]\frac{d\phi}{\Delta\phi}$$

$$=\int_0^{\phi}\frac{d\phi}{\Delta\phi}-(2+2k^2)\int_0^{\phi}\sin^2\phi\frac{d\phi}{\Delta\phi}+3k^2\int\sin^4\phi\frac{d\phi}{\Delta\phi}=\sin(\phi)\ \cos(\phi)\Delta(\phi).$$

As $\phi = \pi/2$, $\int_0^{\pi/2} \sin^4\phi \ \frac{d\phi}{\Delta\phi} = \frac{(2+2k^2)}{3k^2} \int_0^{\pi/2} \sin^2\phi \ \frac{d\phi}{\Delta\phi} - K/(3k^2)$.

The integral on the right was reduced in the preceding paragraph. Hence, $\int_0^{\pi/2} \sin^4\phi \ \frac{d\phi}{\Delta\phi} = \frac{(2+2k^2)}{3k^4} (K-E) - K/(3k^2)$

$= \frac{(2+k^2)}{3k^4} (K) - \frac{(2+2k^2)}{3k^4} E$. Finally, (1) becomes

$V = \frac{8}{3} a.\csc(\alpha) \left[(a^2+b^2)E - (a^2-b^2)K \right]$, where K and E are elliptic integrals of the first and second kinds, respectively.

A Solution by Elliptic Functions

In the volume integral $8\csc(\alpha) \int_0^b \sqrt{(a^2-x^2)(b^2-x^2)}\,dx$

let $x = (b)\operatorname{sn}(y, b/a)$. $dx = (b)\operatorname{cn}(y)\operatorname{dn}(y)dy$.-------------(1)

$a^2-x^2 = a^2-b^2\operatorname{sn}^2(y) = a^2-b^2 \left(\frac{1-\operatorname{dn}^2 y}{b^2/a^2}\right)$, or

$a^2-x^2 = a^2\operatorname{dn}^2(y)$.------------------------------------(2)

$b^2-x^2 = b^2-b^2\operatorname{sn}^2(y) = b^2-b^2 \ (1-\operatorname{cn}^2 y)$, or

$b^2-x^2 = (b)^2\operatorname{cn}^2(y)$. --------------------------------(3)

From (1), (2), and (3) the volume becomes

$8\csc(\alpha)\cdot ab^2 \int_0^K \operatorname{cn}^2 y \operatorname{dn}^2 y\,dy$. The limits in the last integral are 0 and K, as when $x = 0$, $y = \operatorname{sn}^{-1}(0) = 0$; when $x = b$, $y = \operatorname{sn}^{-1}(1) = K$. The last integral is equal to

$$8ab^2\csc(\alpha)\int_0^K (1-\mathrm{sn}^2y)(1-\frac{b^2}{a^2}\mathrm{sn}^2y)\,dy$$

$$= 8ab^2\csc(\alpha)\int_0^K \left[1-\left(\frac{a^2+b^2}{a^2}\right)\mathrm{sn}^2y+\frac{b^2}{a^2}\mathrm{sn}^4y\right]\,dy. \text{-----------}(4)$$

The Integration of $\int_0^y \mathrm{sn}^2y\,dy$.

By definition, $E(b/a,\phi)=\int_0^\phi \sqrt{1-(b^2/a^2)\sin^2\phi}\,\,d\phi$.

$d\phi = d(\mathrm{am}\ y) = \mathrm{dn}(y)dy$. By substitution, $E(b/a,\phi)=\int_0^y \mathrm{dn}^2y\,dy$,

as, when $\phi=0$, $y=0$. Hence, $E(b/a,\mathrm{am}\ y)=\int_0^y\left[1-(b^2/a^2)\mathrm{sn}^2y\right]dy$,

or $\int_0^y \mathrm{sn}^2y\,dy = (a^2/b^2)\left[y-E(b/a,\mathrm{am}\ y)\right]$.--------------(5)

The Integration of $\int_0^y \mathrm{sn}^4y\,dy$.

$$\frac{d}{dy}\left[\mathrm{sn}(y)\mathrm{cn}(y)\mathrm{dn}(y)\right] = \mathrm{cn}^2y\,\mathrm{dn}^2y-\mathrm{sn}^2y\,\mathrm{dn}^2y-(b^2/a^2)\mathrm{sn}^2y\,\mathrm{cn}^2y$$

$$= (1-\mathrm{sn}^2y)\left[1-(b^2/a^2)\mathrm{sn}^2y\right] -\mathrm{sn}^2y+(b^2/a^2)\mathrm{sn}^4y-\frac{b^2}{a^2}\mathrm{sn}^2y+\frac{b^2}{a^2}\mathrm{sn}^4y$$

$$= 1-\left(\frac{a^2+b^2}{a^2}\right)\mathrm{sn}^2y+(b^2/a^2)\mathrm{sn}^4y-\left(\frac{a^2+b^2}{a^2}\right)\mathrm{sn}^2y+(2b^2/a^2)\mathrm{sn}^4y$$

$$= 1-2\left(\frac{a^2+b^2}{a^2}\right)\mathrm{sn}^2y+(3b^2/a^2)\mathrm{sn}^4y.$$

Hence, $\int_0^y \mathrm{sn}^4y\,dy$

$$= -\frac{a^2y}{3b^2}+2\left(\frac{a^2+b^2}{3b^2}\right)\int_0^y \mathrm{sn}^2y\,dy+\frac{a^2}{3b^2}\,\mathrm{sn}(y)\mathrm{cn}(y)\mathrm{dn}(y). \text{-----}(6)$$

The original integral (4) becomes, by substitution from (5) and (6), $V=8ab^2K\csc(\alpha)-8a(a^2+b^2)\csc(\alpha)\left[K-E(b/a,1)\right]$ $+\frac{8b^4\csc(\alpha)}{a}\left[\frac{-a^2K}{3b^2}+\frac{2a^2(a^2+b^2)}{3b^4}\left\{K-E(b/a,1)\right\}\right]$, or

$V=\frac{8a\csc(\alpha)}{3}\left[(a^2+b^2)E-(a^2-b^2)K\right]$. This is the expression for V obtained on page 5.

Elements of the Intercepted Volume.

If a plane parallel to the XY-plane cuts the cylinders at the lowest point of the upper intersectional curve, it cuts the upper half of the common volume V into two parts--a cylinder (between the cutting plane and the XY-plane) and a cap bounded by the cutting plane and both surfaces S_a and S_b. The volume of this cap is evidently equal to half the common volume V minus the volume of the cylindrical section of S_b cut off by the cutting plane and the XY-plane. The lowest point on the intersectional curve of S_a and S_b is at the point on S_b where y is greatest, that is, $y=b$. There the vertical height (not the slant height) of the cylinder bounded by the cutting plane, the XY-plane, and S_b is $\sqrt{a^2-b^2}$. Hence, its volume is $\pi b^2\sqrt{a^2-b^2}\cdot\csc(\alpha)$. The volume of the cap (of which there are two) is

$$\frac{4a\csc(\alpha)}{3}\left[(a^2+b^2)E-(a^2-b^2)K\right]-\pi b^2\sqrt{a^2-b^2}\cdot\csc(\alpha).$$

Special Cases.

Observe that if $\alpha=\pi/2$, the volume V common to S_a and S_b is expressed by $V=\frac{8a}{3}\left[(a^2+b^2)E-(a^2-b^2)K\right]$. If $a=b$, the volume integral degenerates to $V = 8\csc(\alpha)\int_0^a (a^2-y^2)\,dy$

$= \frac{16a^3}{3}\csc(\alpha)$. Finally, if $\alpha = \pi/2$ and $a = b$, $V = \frac{16a^3}{3}$.

A Numerical Case

In a numerical evaluation of the general form of V, the elliptic integrals K and E are readily handled by means of Landen's transformations (Byerly, 1926), by which K, the complete integral of the first kind, is equal to $\frac{\pi}{2}(1+k_1)(1+k_2)(1+k_3)\ldots\ldots\ldots\ldots$, where $k_p = \dfrac{1-\sqrt{1-k_{p-1}^2}}{1+\sqrt{1-k_{p-1}^2}}$.

$$E(k,\tfrac{\pi}{2}) = K\left[1-\frac{k^2}{2}\left(1+\frac{k_1}{2}+\frac{k_1k_2}{2^2}+\frac{k_1k_2k_3}{2^3}+\ldots\ldots\right)\right]$$, where k_p is the same as above.

For a numerical example, let $a = 4$, $b = 1$, $\alpha = 60^\circ$. Then $V = \frac{32\csc(60^\circ)}{3}\left[17E-15K\right]$. By the use of five-place logarithms $k_1 = \dfrac{1-\sqrt{1-(1/16)}}{1+\sqrt{1-(1/16)}} = .016131$. $k_2 = \dfrac{1-\sqrt{1-.00026019}}{1+\sqrt{1-.00026019}}$ $= .000070006$. Neglecting the k's beyond k_2, we have $K = \frac{\pi}{2}(1+k_1)(1+k_2) = 1.570796(1.016131)(1.00007) = 1.5962$. As a check, this answer may be compared with the table value 1.59635. Greater accuracy may be obtained by taking more terms of the transformation.

By the transformation given on page 8,

$$E = 1.5962\left[1-(1/32)(1+.008065+.0000028231)\right] = 1.5459.$$

The very small terms may be neglected if accuracy beyond four places is not desired. As a check, note the table value 1.54585.

The volume $V = \frac{32\csc(60^{\circ})}{3}\left[17E-15K\right] = \frac{32(2.3373)}{3(.86603)} = 28.788.$

THE INTERCEPTED VOLUME FORMED BY A RANDOM, INTERNAL INTERSECTION OF TWO UNEQUAL, RIGHT, CIRCULAR CYLINDERS

Simplification of the Analytical Representation of the Problem

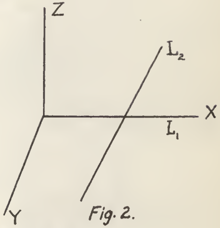

Fig. 2.

Let the axes of the two cylinders be represented by the random lines L_1 and L_2, which have no point in common. Without loss of generality, take the X-axis as L_1. Let L_2 be a random line whose equations are

$$S_1 \equiv A_1x + B_1y + C_1z + D_1 = 0.$$

$$S_2 \equiv A_2x + B_2y + C_2z + D_2 = 0.$$

The pencil of planes on L_2 is

$$S_1+kS_2 = (A_1 + kA_2)x + (B_1 + kB_2)y + (C_1 + kC_2)z + (D_1 + kD_2) = 0.$$

The direction cosines of L_1 are $\lambda = 1$, $\mu = 0$, $\nu = 0$. The angle between L_1 and the plane $S_1 + kS_2 = 0$ is given by

$$\sin(\theta) = \frac{\lambda(A_1+kA_2)+\mu(B_1+kB_2)+\nu(C_1+kC_2)}{\sqrt{(A_1+kA_2)^2+(B_1+kB_2)^2+(C_1+kC_2)^2}\sqrt{(\lambda^2+\mu^2+\nu^2)}}.$$

(Snyder and Sisam, 1914). If a plane of the pencil is parallel to L_1, then $\theta = 0$. The last equation reduces to $\lambda(A_1 + kA_2) = 0$, or $k = (-A_1/A_2)$, $A_2 \neq 0$. If $A_2 = 0$, $S_2 = 0$ is the desired plane parallel to L_1. Hence, the equation of a plane containing L_2 and parallel to L_1 is $(A_2B_1 - A_1B_2)y + (A_2C_1 - A_1C_2)z + (A_2D_1 - A_1D_2) = 0$. This proves that one plane containing L_2 may be constructed parallel to L_1.

In the general volume problem the axis of the cylinder S_a is the X-axis. The axis of the cylinder S_b is the random line L_2 in the above discussion. To simplify the analytic expression of the general problem, take a plane P_1 on the axis of S_b parallel to the axis of S_a by the process outlined above. Take a plane P_2 on the axis of S_a and parallel to P_1. Take P_2 as the new XZ-plane. The new XY-plane is perpendicular to P_2 and on the axis of S_a. The new YZ-plane is perpendicular to the other two planes and intersects the XY-plane in the same point with the axis of S_b. The volume problem with random internal intersection of the cylinders S_a and S_b $(a>b)$ is expressed analytically by the cylinder S_a on the X-axis and by the cylinder S_b, whose axis cuts the Y-axis and is parallel to the XZ-plane. A further simplification is accomplished by rotating the YZ-plane about the Y-axis until the axis of S_b lies in the new YZ-plane. (See Fig. 3 on page 12).

Reduction of the Volume Integral to Standard Forms

The axis of the cylinder S_b of radius b cuts the Y-axis at (0,L,0) and is parallel to the Z-axis, which makes an angle α with the X-axis. The axis of the cylinder S_a of radius a is the X-axis. $a > (L+b)$.

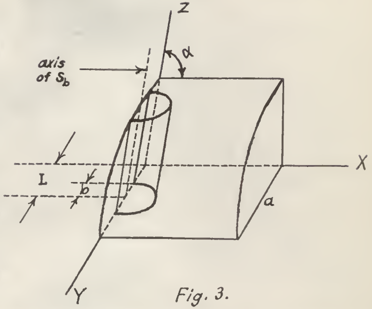

Fig. 3.

The equation of S_a is $z^2/(a^2\csc^2\alpha)+y^2/a^2=1$. The equation of S_b is $x^2/(b^2\csc^2\alpha)+(y-L)^2/b^2=1$. The volume common to the cylinders is

$$2\sin(\alpha)\int_{L-b}^{L+b}\int_{-\sqrt{b^2-(y-L)^2}\csc(\alpha)}^{\sqrt{b^2-(y-L)^2}\csc(\alpha)} Z_a\,dx\,dy .$$

As $Z_a=\sqrt{(a^2-y^2)}\csc(\alpha)$, V becomes

$$4\csc(\alpha)\int_{L-b}^{L+b}\sqrt{(a^2-y^2)\left[b^2-(y-L)^2\right]}\,dy$$

Let $y=ax$. The last integral is, in indefinite form without the coefficient, $a^3\int\sqrt{(x^2-1)(x-\frac{L-b}{a})(x-\frac{L+b}{a})}\,dx$. Let $\frac{L-b}{a}=c$ and $\frac{L+b}{a}=f$. Drop the coefficient to get

$$\int\sqrt{(x^2-1)(x-c)(x-f)}\,dx.$$

Multiply numerator and denominator of the integrand by

$R_x = \sqrt{(x^2-1)(x-c)(x-f)}$ to get

$$\int \left[\frac{x^4-(c+f)x^3+(cf-1)x^2+(c+f)x-cf}{R_x}\right] dx \quad .$$

Hence the volume integral may be reduced to five of the type $\int \frac{x^n dx}{R_x}$, where $n=0, 1, 2, 3, 4.$-----------------(1)

Reduction of $\int \frac{x^4 dx}{R_x}$. $d(xR_x) = R_x dx +$

$\left[\frac{4x^3-3(c+f)x^2+2(cf-1)x+(c+f)}{2R_x}\right] xdx$, or

$xR_x = \frac{1}{2}\int \left[\frac{6x^4-5(c+f)x^3+4(cf-1)x^2+3(c+f)x-2cf}{R_x}\right] dx$. This equation may be solved for $\int \frac{x^4 dx}{R_x}$, which is then expressed in terms of xR_x and $\int \frac{x^n dx}{R_x}$, where $n=0, 1, 2, 3.$

Reduction of $\int \frac{x^3 dx}{R_x}$.

$d(R_x) = \frac{4x^3-3(c+f)x^2+2(cf-1)x+(c+f)}{2R_x} dx$, or

$R_x = \int \left[\frac{4x^3-3(c+f)x^2+2(cf-1)x+(c+f)}{2R_x}\right] dx$. This equation may be solved for $\int \frac{x^3 dx}{R_x}$, which is then given in terms of R_x and $\int \frac{x^n dx}{R_x}$, where $n=0, 1, 2.$

Simplification of R_x. To remove the x-terms from $Q=(x^2+2\lambda x+\mu)(x^2+2\lambda' x+\mu')$, proceed as in Edwards (1921). Let $x=(p+qz)/(1+z)$; then $x^2+2\lambda x+\mu =$

$$\frac{(p+qz)^2+2\lambda(p+qz)(1+z)+\mu(1+z)^2}{(1+z)^2}=\frac{H(z^2+2fz+g)}{(1+z)^2}$$, where

$H=q^2+2\lambda q+\mu$, and $1/H = \dfrac{f}{pq+\lambda(p+q)+\mu} = \dfrac{g}{p^2+2\lambda p+\mu}$.

Similarly, $x^2+2\lambda' x+\mu' = \dfrac{H'(z^2+2f'z+g')}{(1+z)^2}$, where H', f', g' are the same functions of p, q, λ', μ', as H, f, g, are of p, q, λ, μ. Hence $Q=\dfrac{HH'(z^2+2fz+g)(z^2+2f'z+g')}{(1+z)^4}$.

We shall be able to make f and f' zero by taking p and q so that $pq+\lambda(p+q)+\mu=0$ and $pq+\lambda'(p+q)+\mu'=0$, i.e.

$pq/(\lambda\mu'-\lambda'\mu)=(p+q)/(\mu-\mu')=1/(\lambda'-\lambda)=$

$= \dfrac{p-q}{\sqrt{(\mu-\mu')^2-4(\lambda'-\lambda)\;(\lambda\mu'-\lambda'\mu)}}$. Now $(\mu-\mu')^2-4(\lambda'-\lambda)\;(\lambda\mu'-\lambda'\mu)\equiv$

$\equiv\;(\mu+\mu'-2\lambda\lambda')^2-4(\mu-\lambda^2)\;(\mu'-\lambda'^2)=K^2$, say. So

$p+q=(\mu-\mu')/(\lambda'-\lambda)$ and $p-q=K/(\lambda'-\lambda)$, whence p and q are found."

As an example, take $Q=(x^2-1)\left[x^2-(c+f)x+cf\right]$. Here $\lambda=0$, $\mu=-1$, $\lambda'=-(c+f)/2$, $\mu'=cf$. Then $p+q=\dfrac{2(1+cf)}{(c+f)}$----(I)

Also $p-q=\dfrac{-2\sqrt{(1+cf)^2-(c+f)^2}}{c+f}$. ----------------- (II)

Add (I) and (II) to get

$$p=\frac{(1+cf)-\sqrt{(1+cf)^2-(c+f)^2}}{c+f}.$$

Subtract (II) from (I) to get

$$q=\frac{(1+cf)+\sqrt{(1+cf)^2-(c+f)^2}}{c+f}.$$

Then Q becomes, by the substitution $x=(p+qy)/(1+y)$, where p and q are as above,

$$Q' = \frac{\left[(q^2-1)y^2+p^2-1\right]\left[\left\{q^2-q(c+f)+cf\right\}y^2+p^2-p(c+f)+cf\right]}{(1+y)^4}$$

$$\equiv \frac{(Ay^2+B)\ (Cy^2+D)}{(1+y)^4} \equiv (R_y)^2/(1+y)^4$$, where A, B, C, and D are as in the identity above.

<u>Reduction of</u> $\int \frac{x^2dx}{R_x}$. Reduce this integral by the $x=(p+qy)/(1+y)$ given in the preceding paragraph, where $R_x^2 = Q$. Then $dx = \frac{(q-p)dy}{(1+y)^2}$ and $R_x = R_y/(1+y)^2$. Hence

$$\int \frac{x^2dx}{R_x} = (q-p)\int\frac{x^2\ dy}{R_y} = (q-p)\int\frac{(p+qy)^2dy}{(1+y)^2\ R_y} \equiv$$

$$\equiv q^2(q-p)\int\frac{(y^2+2y+1)dy}{(y+1)^2R_y} - 2q(q-p)^2\int\frac{(1+y)dy}{(1+y)^2R_y} + (q-p)^3\int\frac{dy}{(1+y)^2R_y}$$

$$\equiv q^2(q-p)\int\frac{dy}{R_y} - 2q(q-p)^2\int\frac{dy}{(1+y)R_y} + (q-p)^3\int\frac{dy}{(1+y)^2\ R_y}\ .$$

Thus $\int\frac{x^2dx}{R_x}$ depends upon $\int\frac{dy}{R_y}$, $\int\frac{dy}{(1+y)R_y}$, and $\int\frac{dy}{(1+y)^2R_y}$. These three forms will now be reduced.

<u>Reduction of</u> $\int\frac{dy}{(1+y)^2\ R_y}$.

$$d\left[\frac{R_y}{(1+y)}\right] = \frac{(1+y)\left[2Ay(Cy^2+D)+2Cy(Ay^2+B)\right]\ -2R_y^2}{2R_y(1+y)^2}\ dy$$

$$= \frac{AC(y^2-1)(y+1)^2+(AD+BC+2AC)(y+1)-(A+B)\ (C+D)}{R_y(1+y)^2}\ dy\text{ , or}$$

$$\frac{R_y}{1+y} = AC\int\frac{(y^2-1)dy}{R_y} + (AD+BC+2AC)\int\frac{dy}{(1+y)R_y} - (A+B)\ (C+D)\int\frac{dy}{(1+y)^2R_y}.$$

Then $\int \frac{dy}{(1+y)^2 R_y}$ depends upon $\int \frac{y^2 dy}{R_y}$, $\int \frac{dy}{R_y}$, $\int \frac{dy}{(1+y)R_y}$,

and $R_y/(1+y)$. Therefore $\int \frac{x^2 dx}{R_x}$ depends upon $\int \frac{y^2 dy}{R_y}$,

$\int \frac{dy}{R_y}$, and $\int \frac{dy}{(1+y)R_y}$.

<u>Reduction of</u> $\int \frac{y^2 dy}{R_y}$. This integral is identically equal to

$$\frac{1}{A}\int \frac{(Ay^2+B)\,dy}{\sqrt{(Ay^2+B)(Cy^2+D)}} - \frac{B}{A}\int \frac{dy}{R_y}.$$

In the first integral in the dexter let $y = \sqrt{\frac{D}{C}(t^2-1)}$ to get

$$\frac{1}{C}\int \sqrt{\frac{AD-BC-ADt^2}{1-t^2}}\,dt. \text{ ------------------------------ } (1)$$

If $AD-BC > 0$, the integral (1) is $\frac{1}{C}\sqrt{AD-BC}\int \sqrt{\frac{1 \pm k^2t^2}{1-t^2}}\,dt$,

where $k^2 = \left|\frac{AD}{AD-BC}\right|$. In $\int \sqrt{\frac{1+k^2t^2}{1-t^2}}\,dt$ let $t = (1-z^2)^{+1/2}$

It becomes $\left[-(1+k^2)^{+1/2}\right]\int \sqrt{\frac{1-k^2z^2/(1+k^2)}{1-z^2}}\,dz$, $0 < k^2/(1+k^2) < 1$, the standard elliptic integral of the second kind. Take the other form of the above integral, namely

$\int \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt$. It is in standard form if $k^2 < 1$. If $k^2 > 1$, let $t = z/k$ to get

$$\equiv \int \frac{k\cdot(1-z^2/k^2)\,dz}{\sqrt{(1-z^2)(1-\frac{z^2}{k^2})}} \equiv \frac{1}{k}\int \frac{(1-z^2)\,dz}{\sqrt{(1-z^2)(1-\frac{z^2}{k^2})}} - \frac{k^2-1}{k^2}\int \frac{dz}{\sqrt{(1-z^2)(1-z^2/k^2)}}.$$

These are standard elliptic integrals of the second and first kinds, respectively, where $1/k^2 < 1$.

Now return to (1) on page 16. If BC-AD >0, (1) is of the form

$$\frac{(BC-AD)^{1/2}}{C}\int\sqrt{\frac{-1\pm k^2t^2}{1-t^2}}\,dt \quad , \text{ where } k^2=\left|\frac{AD}{BC-AD}\right| \ .$$

First, take $\int\sqrt{\frac{1-k^2t^2}{t^2-1}}\,dt$. Let $t=(1-z^2)^{1/2}$ to get

$-\int\sqrt{\frac{1-k^2+k^2z^2}{z^2-1}}\,dz$. If $k^2>1$, this is a form treated above. If $k^2-1<0$, the integral is of the form

$-(1-k^2)^{1/2}\int\sqrt{\frac{-1-k_1{}^2z^2}{1-z^2}}\,dz$, the other possible form of the last integral. Here $k_1^2=k^2/(1-k^2)>0$. Rearrange this last form as

$$\int\frac{(1-k_1^2z^2)dz}{\sqrt{(z^2-1)(1+k_1^2z^2)}}\equiv(1+k_1^2)\int\frac{dz}{\sqrt{(z^2-1)(1+k_1^2z^2)}}$$

$$+k_1^2\int\frac{(z^2-1)dz}{\sqrt{(z^2-1)(1+k_1^2z^2)}}.$$ By letting $z=(1/k_1)(\phi^2-1)^{1/2}$ transform the last to $k_1\int\sqrt{\frac{1+(1/k_1^2)-(\phi^2/k_1^2)}{1-\phi^2}}\,d\phi$, which is a form treated above. Finally, in $\int\frac{dz}{\sqrt{(z^2-1)(1+k_1^2z^2)}}$ let $z=1/(1-t^2)^{1/2}$ to get $(1+k_1^2)^{-1/2}\int\frac{dt}{\sqrt{(1-t^2)(1-k_2^2t^2)}}$,

where $k_2^2=(1+k_1^2)^{-1}<1$. The last is in standard first form. This completes the standardization of

$$\int\sqrt{\frac{Ay^2+B}{Cy^2+D}}\,dy.$$

Reduction of $\int \frac{dy}{R_y}$. R_y^2 was the notation for $(Ay^2+B)(Cy^2+D)$. By the usual method of dividing out the constants from the radical and making a substitution $y = nx$, where n is a judiciously chosen constant, the above integral is reduced to one of the following forms, depending on the signs of A, B, C, and D.

(1) $\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, (2) $\int \frac{dx}{\sqrt{(1+x^2)(1-k^2x^2)}}$,

(3) $\int \frac{dx}{\sqrt{(-1+x^2)(1-k^2x^2)}}$, (4) $\int \frac{dx}{\sqrt{(-1-x^2)(1-k^2x^2)}}$,

(5) $\int \frac{dx}{\sqrt{(1-x^2)(1+k^2x^2)}}$, (6) $\int \frac{dx}{\sqrt{(1+x^2)(1+k^2x^2)}}$,

(7) $\int \frac{dx}{\sqrt{(-1+x^2)(1+k^2x^2)}}$, (8) $\int \frac{dx}{\sqrt{(-1-x^2)(1+k^2x^2)}}$.

(1) is in standard first form if $k^2 < 1$. If not, the substitution $x = z/k$ will standardize it.

For (2), the substitution $x = (1/k)(1-z^2)^{1/2}$ will reduce it to (1).

In (3) the substitution $x = (1-z^2)^{1/2}$ will yield (1) or (7), depending on the size of k^2.

The substitution $x = (z^2-1)^{1/2}$ changes (4) to (1).

(5) is reduced to (2) by letting $x = z/k$.

In (6) the substitution $x = z(1-z^2)^{-1/2}$ will yield (1) or (5), depending on the value of k^2.

(7) is reduced to (1) by letting $x = (1-z^2)^{-1/2}$.

In (8) let $x = (z^2-1)^{1/2}$ to get (5) or (3), depending on the value of k^2.

Reduction of $\int \frac{dy}{(1+y)R_y}$.

This may be rewritten as $\int \frac{y\,dy}{(y^2-1)R_y} - \int \frac{dy}{(y^2-1)R_y}$.

The last integral in the dexter may be changed to one of the following forms, depending on the signs of A, B, C, and D, by dividing constants out of the radical and by making a judicious substitution $y = n'x$.

$$(1) \int \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}, \quad (2) \int \frac{dx}{(1+nx^2)\sqrt{(1+x^2)(1-k^2x^2)}},$$

$$(3) \int \frac{dx}{(1+nx^2)\sqrt{(-1+x^2)(1-k^2x^2)}}, \quad (4) \int \frac{dx}{(1+nx^2)\sqrt{(-1-x^2)(1-k^2x^2)}},$$

$$(5) \int \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1+k^2x^2)}}, \quad (6) \int \frac{dx}{(1+nx^2)\sqrt{(1+x^2)(1+k^2x^2)}},$$

$$(7) \int \frac{dx}{(1+nx^2)\sqrt{(-1+x^2)(1+k^2x^2)}}, \quad (8) \int \frac{dx}{(1+nx^2)\sqrt{(-1-x^2)(1+k^2x^2)}}.$$

(1) is a standard third form if $k^2 < 1$. If not, the substitution $x = z/k$ will standardize it.

(2) becomes (1) by letting $x = \frac{(1-z^2)^{1/2}}{k}$.

In (3) use $x = (1-z^2)^{1/2}$ to yield (1) or (7), depending on the value of k^2.

(4) is changed to (1) by letting $x = (z^2-1)^{1/2}$.

(5) is reduced to (2) by letting $x = z/k$.

In (6) let $x = z/(1-z^2)^{1/2}$ to reduce to

$$\int \frac{(1-z^2)dz}{(1+n_1z^2)\sqrt{(1-z^2)(1\pm k_1^2z^2)}} \equiv \frac{-1}{n_1}\int \frac{(1+n_1z^2)dz}{(1+n_1z^2)\sqrt{(1-z^2)(1\pm k^2z^2)}}$$

$$+(1+n_1)/n_1 \int \frac{dz}{(1+n_1z^2)\sqrt{(1-z^2)(1\pm k^2z^2)}}$$

. The first in the dexter of the identity is an elliptic integral of the type treated in the reduction of $\int \frac{dy}{R_y}$ above. The second in the dexter is of form (1) or (5) above, depending on the signs in the radical.

(7) is reduced to the first reduced form in (6) above (or a form that may be handled similarly) by the substitution $x = (1-z^2)^{-1/2}$.

In (8) use $x = (z^2-1)^{1/2}$ to reduce to (5) or (3), depending on the value of k^2.

Reduction of $\int \frac{ydy}{(y^2-1)R_y}$. Let $y^2 = 1/t+1$ to get

$$\frac{-1}{2}\int \frac{dt}{\sqrt{[A+(1+B)t][C+(1+D)t]}}$$

. This is an elementary form $\int (ax^2+bx+c)^{-1/2}dx$, and is of varying forms according to the signs of the constants.

This completes the reduction of the integral $\int \frac{x^2 dx}{R_x}$ encountered on page 15.

<u>Reduction of</u> $\int \frac{x\,dx}{R_x}$. By the substitution $x = (p+qy) / (1+y)$, where p and q are as on page 14, the integral $\int \frac{x\,dx}{\sqrt{(x^2-1)(x-c)(x-f)}}$ becomes

$$(q-p)\int \frac{(p+qy)dy}{(1+y)R_y} \equiv p(q-p)\int \frac{dy}{(1+y)R_y} + q(q-p)\int \frac{y\,dy}{(1+y)R_y}$$

$$\equiv -(p-q)^2\int \frac{dy}{(1+y)R_y} + q(q-p)\int \frac{dy}{R_y}.$$ Both of these have been reduced in the preceding discussion.

<u>Reduction of</u> $\int \frac{dx}{R_x}$. The substitution $x = 1/y+1$ yields $-\int \frac{dy}{\sqrt{(1+2y)[1+(1-c)y][1+(1-f)y]}}$. Now let $y+1/2 = z^2$ to get $-(2)^{1/2}\int \frac{dz}{\sqrt{\left[\frac{(1+c)}{2}+(1-c)z^2\right]\left[\frac{(1+f)}{2}+(1-f)z^2\right]}}$, which was treated on page 17.

This completes the reduction of the volume integral $\int \sqrt{(a^2-y^2)\,[b^2-(y-L)^2]}\;dy$ to standard forms, which consist of the three types of elliptic integrals, several elementary integrals, and various algebraic expressions.

SUMMARY

The expression for the intercepted volume formed by the random, internal intersection of two unequal, right, circular cylinders involves, among other functions, the three kinds of elliptic integrals. If the intersection is made central, the elliptic integral of the third kind degenerates. Furthermore, if the radii of the cylinders are made equal, the elliptic integrals of the first and second kinds degenerate. Finally, if the axes of the cylinders intersect normally, the trigonometric factor becomes unity, leaving a simple algebraic expression.

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