THE VOLUME ELEMENTS INTERCEPTED BY INTERSECTING CYLINDERS

by

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B. S., Kansas State College of Agriculture and Applied Science, 1934

A THESIS

submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

KANSAS STATE COLLEGE OF AGRICULTURE AND APPLIED SCIENCE

1935
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>THE INTERCEPTED VOLUME FORMED BY THE AXIAL INTERSECTION OF TWO UNEQUAL, RIGHT, CIRCULAR CYLINDERS</td>
<td>3</td>
</tr>
<tr>
<td>The Analytical Representation of the Problem</td>
<td>3</td>
</tr>
<tr>
<td>A Solution by Algebraic Methods</td>
<td>4</td>
</tr>
<tr>
<td>A Solution by Elliptic Functions</td>
<td>5</td>
</tr>
<tr>
<td>Elements of the Intercepted Volume</td>
<td>7</td>
</tr>
<tr>
<td>Special Cases</td>
<td>7</td>
</tr>
<tr>
<td>A Numerical Case</td>
<td>8</td>
</tr>
<tr>
<td>THE INTERCEPTED VOLUME FORMED BY A RANDOM, INTERNAL INTERSECTION OF TWO UNEQUAL, RIGHT, CIRCULAR CYLINDERS</td>
<td>10</td>
</tr>
<tr>
<td>Simplification of the Analytical Representation of the Problem</td>
<td>10</td>
</tr>
<tr>
<td>Reduction of the Volume Integral to Standard Forms</td>
<td>12</td>
</tr>
<tr>
<td>SUMMARY</td>
<td>22</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>23</td>
</tr>
</tbody>
</table>
INTRODUCTION

Two problems are to be considered. First is the determination of the intercepted volume formed by the axial intersection of two unequal, right, circular cylinders. Second is the determination of the intercepted volume formed by a random, internal intersection of two unequal, right, circular cylinders. The analytic expression of these volumes involves the three kinds of elliptic integrals.

An elliptic integral was first encountered in the problem of the rectification of the ellipse. From its association with the problem the integral received the appellation "elliptic". The first intensive study of integrals of this type was conducted by Adrien Marie Legendre (1752-1833), who showed that an integral depending upon the square root of a polynomial of fourth degree in \( x \) can be brought back to the three fundamental forms.

\[
\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \int_0^x \frac{x^2dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \text{and} \quad \int_0^x \frac{dx}{(x^2+a)\sqrt{(1-x^2)(1-k^2x^2)}}
\]

which are termed elliptic inte-
grals of the first, second, and third kinds, respectively. Numerical evaluation of the first and second kinds is conveniently effected by Landen's transformations.

The inverse functions defined by the elliptic integrals are termed elliptic functions. In 1825 Niels Henrik Abel did the pioneering work with elliptic functions. Carl Gustav Jacob Jacobi (1804-1851) discovered the theta-functions, which can be used in the numerical evaluation of the elliptic integral of the third kind. In the second problem treated below an elliptic integral of the third kind is encountered. However, a numerical evaluation will not be attempted, as the problem may be considered profitably without going into the extended application of the theta-functions.
THE INTERCEPTED VOLUME FORMED BY THE AXIAL INTERSECTION
OF TWO UNEQUAL, RIGHT, CIRCULAR CYLINDERS

The Analytical Representation of the Problem

A horizontal, circular cylinder of radius $a$ and a circular cylinder of radius $b$ ($a > b$) intersect centrally with an angle $\alpha$ between their axes. The cylinders $S_a$ and $S_b$ are mounted on the axes as shown in Fig. 1. Oblique coordinates are used, the YZ-plane being rotated about the Y-axis until it makes an angle $\alpha$ with the XY-plane. The equation of $S_a$ is its trace on the YZ-plane, which is $y^2/a^2 + z^2/(a^2\csc^2\alpha) = 1$. The equation of $S_b$ is its trace on the XY-plane, which is $x^2/(b^2\csc^2\alpha) + y^2/b^2 = 1$. The volume common to $S_a$ and $S_b$ is bounded on the sides by $S_b$ and topped at each end by $S_a$.

The element of volume stands upon the XY-plane and upon the ellipse represented by the equation of $S_b$, that is, the base of $S_b$. The slant height of the element is $Z_a$. Its volume is $Z_a \sin(\alpha) \, dxdy$. The total volume common to $S_a$ and
A Solution by Algebraic Methods

As \( Z_e = \csc(\alpha) \sqrt{a^2 - y^2} \), \( V = 6 \csc(\alpha) \int_{0}^{b} \sqrt{(a^2 - y^2)(b^2 - y^2)} \) dy.

Let \( y = bx \). The volume \( V = 8ab^2 \csc(\alpha) \int_{0}^{1} \sqrt{(1-x^2)(1-k^2x^2)} dx \),

where \( k^2 = \frac{b^2}{a^2} \). Let \( x = \sin(\phi) \). Let \( \Delta \phi = \sqrt{1-k^2 \sin^2 \phi} \).

Then \( V = 8ab^2 \csc(\alpha) \int_{0}^{\Delta \phi} \left[ 1-(1+k^2)\sin^2 \phi + k^2 \sin^2 \phi \right] \frac{d\phi}{\Delta \phi} \). \(-\ldots-(1)\)

Reduction of \( \int_{0}^{\Delta \phi} \sin^4 \phi \frac{d\phi}{\Delta \phi} \).

The above integral is identically equal to

\[
(\frac{1}{k^2}) \int_{0}^{\Delta \phi} (1-k^2 \sin^2 \phi) \frac{d\phi}{\Delta \phi} + \frac{1}{k^2} \int_{0}^{\Delta \phi} \frac{d\phi}{\Delta \phi} = -1/k^2 \int_{0}^{\Delta \phi} \frac{d\phi}{\Delta \phi} + \frac{1}{k^2} \int_{0}^{\Delta \phi} \frac{d\phi}{\Delta \phi} = (1/k^2) \left( K - E \right),
\]

where \( K \) and \( E \) are complete elliptic integrals of the first and second kinds, respectively.

Reduction of \( \int_{0}^{\Delta \phi} \sin^4 \phi \frac{d\phi}{\Delta \phi} \).

Set up the following identity: \( \sin(\phi) \cos(\phi) \Delta(\phi) \)

\[
\int_{0}^{\phi} \frac{d}{\Delta \phi} = \int_{0}^{\phi} \left[ \cos^2 \phi \Delta \phi - \sin^2 \phi \Delta \phi - k^2 \sin^2 \phi \cos^2 \phi \right] \frac{d\phi}{\Delta \phi}
\]

\[
= \int_{0}^{\phi} \left[ 1-k^2 \sin^2 \phi - 2 \sin^2 \phi + 2k^2 \sin^4 \phi - k^2 \sin^2 \phi + k^2 \sin^4 \phi \right] \frac{d\phi}{\Delta \phi}
\]

\[
= \int_{0}^{\phi} \left[ \sin^2 \phi \Delta \phi + 3k^2 \sin^4 \phi \right] \frac{d\phi}{\Delta \phi} = \sin(\phi) \cos(\phi) \Delta(\phi).
\]
As \( \phi = \frac{\pi}{2} \),
\[
\int_0^{\pi/2} \frac{\sin^4 \phi}{4} \, d\phi = \frac{(2+2k^2)}{3k^2} \int_0^{\pi/2} \sin^2 \phi \, d\phi = \frac{K}{3k^2}.
\]

The integral on the right was reduced in the preceding paragraph. Hence,
\[
\int_0^{\pi/2} \sin^4 \phi \, d\phi = \frac{(2+2k^2)}{3k^2} (K-E) = \frac{K}{3k^2}
\]

\[
= \frac{(2+k^2)}{k^2} (K) - \frac{(2+2k^2)}{3k^2} E. \quad \text{Finally, (1) becomes}
\]

\[
V=2a \csc(\alpha) \left[ (a^2+b^2)E-(a^2-b^2)K \right], \quad \text{where } K \text{ and } E \text{ are elliptic integrals of the first and second kinds, respectively.}
\]

A Solution by Elliptic Functions

In the volume integral \( 8\csc(\alpha) \int_0^b \sqrt{(a^2-x^2)(b^2-x^2)} \, dx \)

let \( x = (b)\text{sn}(y,b/a) \), \( dx = (b)\text{cn}(y)\text{dn}(y) \, dy \). \quad \text{----------(1)}

\[
a^2-x^2 = a^2-b^2 \text{sn}^2(y) = a^2-b^2 \left(1-\text{cn}^2y\right), \quad \text{or}
\]

\[
a^2-x^2 = a^2\text{sn}^2(y), \quad \text{-------------------------(2)}
\]

\[
b^2-x^2 = b^2-b^2 \text{sn}^2(y) = b^2-b^2 \left(1-\text{cn}^2y\right), \quad \text{or}
\]

\[
b^2-x^2 = (b)^2\text{cn}^2(y). \quad \text{-------------------------(3)}
\]

From (1), (2), and (3) the volume becomes

\[
8\csc(\alpha) \cdot ab^2 \int_0^K \text{cn}^2 y \text{dn}^2 y \, dy. \quad \text{The limits in the last integral are } 0 \text{ and } K, \text{ as when } x=0, y=\text{sn}^{-1}(0)=0; \text{ when } x=b, y=\text{sn}^{-1}(1)=K. \quad \text{The last integral is equal to}
\]
\[ 8ab^2 \csc(\alpha) \int_0^K \frac{(1-\text{sn}^2y)(1-b^2\text{sn}^2y) \, dy}{a^2} \]

\[ = 8ab^2 \csc(\alpha) \int_0^K \left[ \frac{1-(a^2+b^2)\text{sn}^2y + b^2\text{sn}^4y}{a^2} \right] \, dy. \]  \hspace{1cm} \text{(4)}

**The Integration of** \( \int_0^y \text{sn}^2y \, dy \).

By definition, \( E(b/a, \phi) = \int_0^\phi \sqrt{1-(b^2/a^2)\sin^2 \phi} \, d\phi \).

\[ d\phi = d(\sin y) = \frac{dn(y)}{dy}. \]

By substitution, \( E(b/a, \phi) = \int_0^y \frac{dn^2y}{dy} \, dy \).

as, when \( \phi = 0, y = 0 \). Hence, \( E(b/a, \sin y) = \int_0^y \left[ 1-(b^2/a^2)\text{sn}^2y \right] \, dy \)

or \[ \int_0^y \text{sn}^2y \, dy = \left( \frac{a^2}{b^2} \right) \left[ y - E(b/a, \sin y) \right]. \]  \hspace{1cm} \text{(5)}

**The Integration of** \( \int_0^y \text{sn}^4y \, dy \).

\[ \frac{d}{dy} \left[ \text{sn}(y) \text{cn}(y) \text{dn}(y) \right] = \text{cn}^2y \text{dn}^2y - \text{sn}^2y \text{dn}^2y - (b^2/a^2)\text{sn}^2y \text{cn}^2y \]

\[ = (1-\text{sn}^2y) \left[ 1-(b^2/a^2)\text{sn}^2y \right] \text{sn}^2y + (b^2/a^2)\text{sn}^4y - \left( \frac{a^2+b^2}{a^2} \right)\text{sn}^2y + \left( \frac{2b^2}{a^2} \right)\text{sn}^4y \]

\[ = 1- \left( \frac{a^2+b^2}{a^2} \right)\text{sn}^2y + \left( \frac{b^2/a^2} \right)\text{sn}^4y - \left( \frac{a^2+b^2}{a^2} \right)\text{sn}^2y + \left( \frac{2b^2}{a^2} \right)\text{sn}^4y \]

\[ = 1 - 2 \left( \frac{a^2+b^2}{a^2} \right)\text{sn}^2y + \left( \frac{3b^2}{a^2} \right)\text{sn}^4y. \]

Hence, \[ \int_0^y \text{sn}^4y \, dy \]

\[ = -\frac{a^2}{2} + 2 \frac{a^2+b^2}{3b^2} \int_0^y \text{sn}^2y \, dy + \frac{a^2}{3b^2} \text{sn}(y) \text{cn}(y) \text{dn}(y). \]  \hspace{1cm} \text{(6)}

The original integral (4) becomes, by substitution from (5) and (6),

\[ V = 8ab^2 \csc(\alpha) - 8a(a^2+b^2) \csc(\alpha) \left[ K-E(b/a,1) \right] \]

\[ + \frac{8b^4 \csc(\alpha)}{a} \left[ \frac{a^2K+2a^2(a^2+b^2)}{3b^2} \left[ K-E(b/a,1) \right] \right], \]

or

\[ V = 8a \csc(\alpha) \left[ (a^2+b^2)E-(a^2-b^2)K \right]. \]  \hspace{1cm} \text{This is the expression}
Elements of the Intercepted Volume.

If a plane parallel to the XY-plane cuts the cylinders at the lowest point of the upper intersectional curve, it cuts the upper half of the common volume $V$ into two parts—a cylinder (between the cutting plane and the XY-plane) and a cap bounded by the cutting plane and both surfaces $S_a$ and $S_b$. The volume of this cap is evidently equal to half the common volume $V$ minus the volume of the cylindrical section of $S_b$ cut off by the cutting plane and the XY-plane. The lowest point on the intersectional curve of $S_a$ and $S_b$ is at the point on $S_b$ where $y$ is greatest, that is, $y = b$.

There the vertical height (not the slant height) of the cylinder bounded by the cutting plane, the XY-plane, and $S_b$ is $\sqrt{a^2 - b^2}$. Hence, its volume is $\pi b^2 \sqrt{a^2 - b^2} \cdot \csc(\alpha)$. The volume of the cap (of which there are two) is

$$\frac{4\pi \csc(\alpha)}{3} \left[ (a^2 + b^2)E - (a^2 - b^2)K \right] - \pi b^2 \sqrt{a^2 - b^2} \cdot \csc(\alpha).$$

Special Cases.

Observe that if $\alpha = \pi/2$, the volume $V$ common to $S_a$ and $S_b$ is expressed by $V = \frac{8\pi}{3} \left[ (a^2 + b^2)E - (a^2 - b^2)K \right]$. If $a = b,$
the volume integral degenerates to

$$V = 8 \csc(x) \int_0^a (a^2 - y^2) dy$$

$$= \frac{16a^3}{3} \csc(x).$$

Finally, if \(a = \frac{\pi}{2}\) and \(a = b\), \(V = \frac{16a^3}{3}\).

A Numerical Case

In a numerical evaluation of the general form of \(V\), the elliptic integrals \(K\) and \(E\) are readily handled by means of Landen's transformations (Byerly, 1926), by which \(K\), the complete integral of the first kind, is equal to

$$E\left(\frac{1+k_1}{2}\right)(1+k_2)(1+k_3) \ldots \ldots \ldots$$

where \(K_p = 1 - \sqrt{1-k_p^2}\)

$$K\left(\frac{1-k^2}{2} \frac{1-k^2}{2} \frac{1-k^2}{2} \right)$$

is the same as above.

For a numerical example, let \(a = 4\), \(b = 1\), \(a = 60^\circ\). Then

$$V = \frac{32 \csc(60^\circ)}{3} \left[ 17E - 15K \right].$$

By the use of five-place logarithms

\(k_1 = \sqrt{1-(1/16)} = .016131\).

\(k_2 = \frac{1-\sqrt{1-.00026019}}{1+\sqrt{1-.00026019}} = .000070006.\)

Neglecting the \(k\)'s beyond \(k_2\), we have

$$K = \frac{\pi}{2} (1+k_1)(1+k_2) = 1.570796(1.016131)(1.00007) = 1.5962.$$  

As a check, this answer may be compared with the table value 1.59635. Greater accuracy may be obtained by taking more terms of the transformation.
By the transformation given on page 9,
\[ E = 1.5962 \left[ 1 - \frac{1}{32} (1 + 0.008065 + 0.00000028231) \right] \approx 1.5459. \]
The very small terms may be neglected if accuracy beyond four places is not desired. As a check, note the table value 1.54565.

The volume \( V = \frac{32 \csc(60^\circ)}{3} \left[ 17R - 15K \right] = 32(2.3373) = 22.788. \)
THE INTERCEPTED VOLUME FORMED BY A RANDOM, INTERNAL INTERSECTION OF TWO UNEQUAL, RIGHT, CIRCULAR CYLINDERS

Simplification of the Analytical Representation of the Problem

Let the axes of the two cylinders be represented by the random lines $L_1$ and $L_2$, which have no point in common. Without loss of generality, take the $X$-axis as $L_1$. Let $L_2$ be a random line whose equations are

$$S_1 = A_1x + B_1y + C_1z + D_1 = 0.$$  
$$S_2 = A_2x + B_2y + C_2z + D_2 = 0.$$  

The pencil of planes on $L_2$ is

$$S_1 + kS_2 = (A_1 + kA_2)x + (B_1 + kB_2)y + (C_1 + kC_2)z + (D_1 + kD_2) = 0.$$  

The direction cosines of $L_1$ are $\lambda = 1$, $\mu = 0$, $\nu = 0$. The angle between $L_1$ and the plane $S_1 + kS_2 = 0$ is given by

$$\sin(\Theta) = \frac{\lambda(A_1 + kA_2) + \mu(B_1 + kB_2) + \nu(C_1 + kC_2)}{\sqrt{(A_1 + kA_2)^2 + (B_1 + kB_2)^2 + (C_1 + kC_2)^2 \lambda^2 + \mu^2 + \nu^2}}.$$  

( Snyder and Sisem, 1914 ). If a plane of the pencil
is parallel to \( L_1 \), then \( \theta = 0 \). The last equation reduces to 
\[ \lambda(A_1 + kA_2) = 0, \]
or 
\[ k = \frac{-A_1}{A_2}, \quad A_2 \neq 0. \]
If \( A_2 = 0 \), \( S_2 = 0 \) is the desired plane parallel to \( L_1 \). Hence, the equation of a plane containing \( L_2 \) and parallel to \( L_1 \) is 
\[ (A_2B_1 - A_1B_2)y + (A_2C_1 - A_1C_2)z + (A_2D_1 - A_1D_2) = 0. \]
This proves that one plane containing \( L_2 \) may be constructed parallel to \( L_1 \).

In the general volume problem the axis of the cylinder \( S_a \) is the \( X \)-axis. The axis of the cylinder \( S_b \) is the random line \( L_2 \) in the above discussion. To simplify the analytic expression of the general problem, take a plane \( P_1 \) on the axis of \( S_b \) parallel to the axis of \( S_a \) by the process outlined above. Take a plane \( P_2 \) on the axis of \( S_a \) and parallel to \( P_1 \). Take \( P_2 \) as the new \( XZ \)-plane. The new \( XY \)-plane is perpendicular to \( P_2 \) and on the axis of \( S_a \). The new \( YZ \)-plane is perpendicular to the other two planes and intersects the \( XY \)-plane in the same point with the axis of \( S_b \). The volume problem with random internal intersection of the cylinders \( S_a \) and \( S_b \) \((a > b)\) is expressed analytically by the cylinder \( S_a \) on the \( X \)-axis and by the cylinder \( S_b \), whose axis cuts the \( Y \)-axis and is parallel to the \( XZ \)-plane. A further simplification is accomplished by rotating the \( YZ \)-plane about the \( Y \)-axis until the axis of \( S_b \) lies in the new \( YZ \)-plane. (See Fig. 3 on page 12).
Reduction of the Volume Integral to Standard Forms

The axis of the cylinder $S_b$ of radius $b$ cuts the $Y$-axis at $(0,L,0)$ and is parallel to the $Z$-axis, which makes an angle $\alpha$ with the $X$-axis. The axis of the cylinder $S_a$ of radius $a$ is the $X$-axis. $a > (L+b)$.

The equation of $S_a$ is $z^2/(a^2\csc^2\alpha)+y^2/a^2=1$. The equation of $S_b$ is $x^2/(b^2\csc^2\alpha)+(y-L)^2/b^2=1$. The volume common to the cylinders is

$$V = 2\sin(\alpha) \int_{L-b}^{L+b} \int_{-\sqrt{b^2-(y-L)^2}}^{\sqrt{b^2-(y-L)^2}} z_a \, dx \, dy.$$

As $z_a = \sqrt{a^2-y^2} \csc(\alpha)$, $V$ becomes

$$V = 4\csc(\alpha) \int_{L-b}^{L+b} \sqrt{a^2-y^2} \left[ b^2(y-L)^2 \right]^{1/3} \, dx.$$

Let $y=ax$. The last integral is, in indefinite form without the coefficient,

$$a^3 \int \sqrt{\left(\frac{x^2-1}{a}\right)(x-L-b)(x-L+b)} \, dx.$$

Let $L-b = c$ and $L+b = f$. Drop the coefficient to get

$$\int \sqrt{\left(\frac{x^2-1}{a}\right)(x-c)(x-f)} \, dx.$$
Multiply numerator and denominator of the integrand by \( R_x = \sqrt{(x^2-1)(x-c)(x-f)} \) to get
\[
\int \frac{[x^4-(c+f)x^3+(cf-1)x^2+(c+f)x-cf]}{R_x} \, dx.
\]
Hence the volume integral may be reduced to five of the type \( \int \frac{x^n \, dx}{R_x} \), where \( n = 0, 1, 2, 3, 4 \).------------------(1)

**Reduction of** \( \int \frac{x^4 \, dx}{R_x} \)
\[
d(R_x) = \frac{d}{dx} [4x^3-3(c+f)x^2+2(cf-1)x+(c+f)] \, dx, \quad \text{or}
\]
\[
xR_x = \frac{1}{2} \int \frac{[6x^4-5(c+f)x^3+4(cf-1)x^2+3(c+f)x-2cf]}{R_x} \, dx.
\]
This equation may be solved for \( \int \frac{x^4 \, dx}{R_x} \), which is then expressed in terms of \( xR_x \) and \( \int \frac{x^n \, dx}{R_x} \), where \( n = 0, 1, 2, 3 \).

**Reduction of** \( \int \frac{x^3 \, dx}{R_x} \)
\[
d(R_x) = \frac{d}{dx} \frac{1}{2} \int [4x^3-3(c+f)x^2+2(cf-1)x+(c+f)] \, dx, \quad \text{or}
\]
\[
R_x = \int \frac{[4x^3-3(c+f)x^2+2(cf-1)x+(c+f)]}{R_x} \, dx.
\]
This equation may be solved for \( \int \frac{x^3 \, dx}{R_x} \), which is then given in terms of \( R_x \) and \( \int \frac{x^n \, dx}{R_x} \), where \( n = 0, 1, 2 \).

**Simplification of** \( R_x \). To remove the \( x \)-terms from
\[
Q = (x^2+2\lambda x+\mu)(x^2+2\lambda x+\mu),
\]
proceed as in Edwards (1921). Let \( x = (p+qz)/(1+z) \); then \( x^2+2\lambda x+\mu = \)
\[(p+qz)^2 + 2\lambda(p+qz)(1+z) + \mu(1+z)^2 = \frac{H(z^2 + 2fz + g)}{(1+z)^2}, \text{ where} \]

\[H = q^2 + 2\lambda q + \mu, \text{ and } \frac{1}{H} = \frac{f}{pq + \lambda(p+q) + \mu} = \frac{g}{p^2 + 2\lambda p + \mu} \]

Similarly, \[x^2 + 2\lambda x + \mu' = \frac{H'(z^2 + 2f'z + g')}{(1+z)^2}, \text{ where } H', f', \]

\[g' \text{ are the same functions of } p, q, \lambda, \mu', \text{ as } H, f, g, \text{ are of } p, q, \lambda, \mu. \text{ Hence } \frac{Q = \frac{H(H'(z^2 + 2fz + g))(z^2 + 2f'z + g')}{(1+z)^2}}{1/\lambda} \]

We shall be able to make \(f\) and \(f'\) zero by taking \(p\) and \(q\) so that \(pq + \lambda(p+q) + \mu = 0\) and \(pq + \lambda(p+q) + \mu' = 0, \text{ i.e.} \]

\[pq/(\mu - \lambda) = (p+q)/(\mu - \lambda) = 1/(\lambda - \lambda) = \]

\[\frac{p-q}{\sqrt{(\mu - \lambda)^2 - 4(\mu - \lambda) (\mu - \lambda)}}. \text{ Now } (\mu - \lambda)^2 - 4(\mu - \lambda) (\mu - \lambda) \equiv \]

\[(\mu + \mu' - 2\lambda \lambda')^2 - 4(\mu - \lambda) (\mu - \lambda) = K^2, \text{ say}. \text{ So} \]

\[p+q = (\mu - \lambda)/(\lambda - \lambda) \text{ and } p-q = K/(\lambda - \lambda), \text{ whence } p \text{ and } q \]

are found."

As an example, take \(Q = (x^2 - 1)\left(x^2 - (c+f)x + cf\right). \text{ Here} \]

\[\lambda = 0, \mu = -1, \lambda' = -(c+f)/2, \mu' = cf. \text{ Then } p+q = 2(1+cf) - (c+f) \]

Also \[p-q = -2 \frac{\sqrt{(1+cf)^2 - (c+f)^2}}{c+f} \]. \hspace{1cm} \text{-----------------------------------------} \hspace{1cm} (11)

Add (1) and (11) to get \[p = (1+cf) - \frac{\sqrt{(1+cf)^2 - (c+f)^2}}{c+f} \].

Subtract (11) from (1) to get \[q = (1+cf) + \frac{\sqrt{(1+cf)^2 - (c+f)^2}}{c+f} \].

Then \(Q\) becomes, by the substitution \(x = (p+q)/(1+y), \text{ where}\)
p and q are as above,

\[ q' = \left[ \frac{q^2-1}{y^2+p^2-1} \right] \left[ \frac{q^2-q(c+f)+cf}{(1+y)^4} \right] \]

\[ \equiv \frac{(Ay^2+B)(Cy^2+D) = (R_y)^2}{(1+y)^4} \] where A, B, C, and D

are as in the identity above.

**Reduction of \( \int \frac{x^2}{R_x} \).** Reduce this integral by the

\( x = (p+qy)/(1+y) \) given in the preceding paragraph, where

\( \frac{R_x^2}{R_x} = q \). Then \( dx = (q-p)dy \) and \( R_x = R_y/(1+y)^2 \). Hence

\[ \int \frac{x^2}{R_x} \frac{dx}{R_x} = (q-p) \int x^2 \frac{dy}{R_y} = (q-p) \int \frac{(p+qy)^2}{(1+y)^2} \frac{dy}{R_y} \]

\[ \equiv q^2(q-p) \int \frac{(y^2+2y+1)}{(y+1)^2} dy - 2q(q-p)^2 \int \frac{(1+y)dy}{(1+y)^2 R_y} + (q-p)^3 \int \frac{dy}{(1+y)^2 R_y} \]

Thus \( \int \frac{x^2}{R_x} \) depends upon \( \int \frac{dy}{R_y} \), \( \int \frac{dy}{(1+y)R_y} \), and

\[ \int \frac{dy}{(1+y)^2 R_y} \]. These three forms will now be reduced.

**Reduction of \( \int \frac{dy}{(1+y)^2 R_y} \).**

\[ q \left[ \frac{R_y}{(1+y)} \right] = \frac{(1+y)[2Ay(Cy+D) + 2Cy(Ay^2+B)] - 2R_x^2}{2R_y(1+y)^2} \]

\[ = \frac{AC(y^2-1)(y+1)^2 + (AB+BC+2AC)(y+1)-(A+B)(c+d) y}{R_y(1+y)^2} \]

\[ \frac{R_y}{1+y} = \frac{AC \int (y^2-1)dy + (AB+BC+2AC) \int \frac{dy}{(1+y)R_y} - (A+B)(c+d) \int \frac{dy}{(1+y)R_y}}{R_y(1+y)^2} \]
Then \[ \int \frac{dy}{(1+y)^2 R_y} \] depends upon \[ \int \frac{x^2 dy}{R_y}, \int \frac{dy}{R_y}, \int \frac{dy}{(1+y)R_y} \], and \( R_y/(1+y) \). Therefore \[ \int \frac{x^2 dx}{R_x} \] depends upon \[ \int \frac{y^2 dy}{R_y} \],

\[ \int \frac{dy}{R_y}, \text{ and } \int \frac{dy}{(1+y)R_y} \]

Reduction of \[ \int \frac{y^2 dy}{R_y} \]. This integral is identically equal to

\[ \frac{1}{A} \int \frac{(Ay+B)dy}{\sqrt{(Ay+B)(Cy+D)}} - B \int \frac{dy}{R_y} \].

In the first integral in the dexter let \( y=\sqrt{\frac{B(t^2-1)}{C}} \) to get

\[ \frac{1}{C} \int \sqrt{\frac{AD-BC-ADt^2}{l-t^2}} \text{ dt.} \]  \hspace{1cm} \text{(1)}

If \( AD-BC > 0 \), the integral \( (1) \) is \( \frac{1}{C} \sqrt{\frac{AD-BC}{l-k^2t^2}} \int \frac{dt}{l-t^2} \),

where \( k^2 = \left| \frac{AD}{AD-BC} \right|. \) In \[ \int \frac{1+k^2 t^2}{l-t^2} \text{ dt let } t=(1-z^2)^{1/2} \]

It becomes \[ \left[-(1+k^2)^{1/2}\right] \int \frac{1-k^2 z^2}{l-z^2} \text{ dz} \), \( 0 < k^2/(1+k^2) < 1 \),

the standard elliptic integral of the second kind. Take the other form of the above integral, namely

\[ \int \frac{1-k^2 t^2}{l-t^2} \text{ dt.} \] It is in standard form if \( k^2 < 1 \). If \( k^2 > 1 \),

let \( t=z/k \) to get

\[ \int \frac{k \cdot (1-z^2/k^2)dz}{\sqrt{(1-z^2)(1-z^2/k^2)}} \equiv \int \frac{dz}{\sqrt{(1-z^2)(1-z^2/k^2)}} \]

These are standard elliptic integrals of the second and first kinds, respectively, where \( 1/k^2 < 1 \).
Now return to (1) on page 16. If $BC - AD > 0$, (1) is of the form

$$\frac{(BC - AD)^{1/2}}{C} \int \frac{-1 \pm k^2t^2}{l - t^2} \, dt \quad \text{where } k^2 = \left| \frac{AD}{BC - AD} \right|.$$ 

First, take $\int \frac{1 - k^2t^2}{1 - t^2} \, dt$. Let $t = (1 - z^2)^{1/2}$ to get

$$-\int \frac{1 - k^2 + k^2z^2}{z^2 - 1} \, dz \quad \text{If } k^2 > 1, \text{ this is a form treated above. If } k^2 - 1 < 0, \text{ the integral is of the form}$$

$$-(1 - k^2)^{1/2} \int \frac{-1 - k_1^2z^2}{1 - z^2} \, dz \text{, the other possible form of the last integral. Here } k_1^2 = k^2/(1 - k^2) > 0. \text{ Rearrange this last form as}$$

$$\int (1 - k_1^2z^2) \, dz = (1 + k_1^2) \int \frac{dz}{\sqrt{(z^2 - 1)(1 + k_1^2z^2)}}$$

$$+ k_1^2 \int \frac{(z^2 - 1)dz}{\sqrt{(z^2 - 1)(1 + k_1^2z^2)}}. \text{ By letting } z = (1/k_1)(\phi^2 - 1)^{1/2}$$

transform the last to $k_1 \int \sqrt{1 + (1/k_1^2)(\phi^2 - 1)} \, d\phi$, which is a form treated above. Finally, in

$$\int \frac{dz}{\sqrt{(z^2 - 1)(1 + k_1^2z^2)}}$$

let $z = 1/(1 - t^2)^{1/2}$ to get $(1 + k_1^2)^{-1/2} \int \frac{dt}{\sqrt{(1 - t^2)(1 - k_2^2t^2)}}$, where $k_2^2 = (1 + k_1^2)^{-1} < 1$. The last is in standard first form. This completes the standardization of

$$\int \sqrt{\frac{y^2 + B}{Cy^2 + D}} \, dy.$$
Reduction of \( \int \frac{dy}{R_y} \). \( R_y^2 \) was the notation for \((Ay^2+B)(Cy^2+D)\). By the usual method of dividing out the constants from the radical and making a substitution \( y = nx \), where \( n \) is a judiciously chosen constant, the above integral is reduced to one of the following forms, depending on the signs of \( A, B, C, \) and \( D \). 

\[
\begin{align*}
(1) & \quad \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \\
(2) & \quad \int \frac{dx}{\sqrt{(1+x^2)(1-k^2x^2)}} \\
(3) & \quad \int \frac{dx}{\sqrt{(-1+x^2)(1-k^2x^2)}} \\
(4) & \quad \int \frac{dx}{\sqrt{(-1-x^2)(1-k^2x^2)}} \\
(5) & \quad \int \frac{dx}{\sqrt{(1-x^2)(1+k^2x^2)}} \\
(6) & \quad \int \frac{dx}{\sqrt{(1+x^2)(1+k^2x^2)}} \\
(7) & \quad \int \frac{dx}{\sqrt{(-1-x^2)(1+k^2x^2)}} \\
(8) & \quad \int \frac{dx}{\sqrt{(-1-x^2)(1+k^2x^2)}}
\end{align*}
\]

(1) is in standard first form if \( k^2 < 1 \). If not, the substitution \( x = z/k \) will standardize it.

For (2), the substitution \( x = (1/k)(1-z^2)^{1/2} \) will reduce it to (1).

In (3), the substitution \( x = (1-z^2)^{1/2} \) will yield (1) or (7), depending on the size of \( k^2 \).

The substitution \( x = (z^2-1)^{1/2} \) changes (4) to (1).

(5) is reduced to (2) by letting \( x = z/k \).
In (6) the substitution \( x = z(1-z^2)^{1/2} \) will yield (1) or (5), depending on the value of \( k^2 \).

(7) is reduced to (1) by letting \( x = (1-z^2)^{1/2} \).

In (8) let \( x = (z^2-1)^{1/2} \) to get (5) or (3), depending on the value of \( k^2 \).

Reduction of \[ \int \frac{dy}{(1+y)F_y} \]

This may be rewritten as \[ \int \frac{y \, dy}{(y^2-1)F_y} - \int \frac{dy}{(y^2-1)F_y} \]

The last integral in the dexter may be changed to one of the following forms, depending on the signs of A, B, C, and D, by dividing constants out of the radical and by making a judicious substitution \( y = nx^2 \).

(1) \[ \int \frac{dx}{(1+nx^2) \sqrt{(1-x^2)(1-k^2x^2)}} \]

(2) \[ \int \frac{dx}{(1+nx^2) \sqrt{(1+x^2)(1-k^2x^2)}} \]

(3) \[ \int \frac{dx}{(1+nx^2) \sqrt{(-1+x^2)(1-k^2x^2)}} \]

(4) \[ \int \frac{dx}{(1+nx^2) \sqrt{(-1-x^2)(1-k^2x^2)}} \]

(5) \[ \int \frac{dx}{(1+nx^2) \sqrt{(1-x^2)(1+k^2x^2)}} \]

(6) \[ \int \frac{dx}{(1+nx^2) \sqrt{(1+x^2)(1+k^2x^2)}} \]

(7) \[ \int \frac{dx}{(1+nx^2) \sqrt{(-1-x^2)(1+k^2x^2)}} \]

(8) \[ \int \frac{dx}{(1+nx^2) \sqrt{(-1+x^2)(1+k^2x^2)}} \]

(1) is a standard third form if \( k^2 < 1 \). If not, the substitution \( x = z/k \) will standardize it.

(2) becomes (1) by letting \( x = \frac{(1-z^2)}{k} \).
In (3) use \( x = (1-z^2)^{1/2} \) to yield (1) or (7), depending on the value of \( k^2 \).

(4) is changed to (1) by letting \( x = (z^2-1)^{1/2} \).

(5) is reduced to (2) by letting \( x = z/k \).

In (6) let \( x = z/(1-z^2)^{1/2} \) to reduce to

\[
\frac{\int (1-z^2)dz}{(1+n_1z^2)\sqrt{(1-z^2)(1+k_1^2z^2)}} = \frac{-1}{n_1} \int \frac{(1+n_1z^2)dz}{(1+n_1z^2)\sqrt{(1-z^2)(1+k_1^2z^2)}}
\]

\[+(1+n_1)/n_1 \int \frac{dz}{(1+n_1z^2)\sqrt{(1-z^2)(1+k_1^2z^2)}} \]  . The first in the dexter of the identity is an elliptic integral of the type treated in the reduction of \( \int \frac{dy}{F_y} \) above. The second in the dexter is of form (1) or (5) above, depending on the signs in the radical.

(7) is reduced to the first reduced form in (6) above (or a form that may be handled similarly) by the substitution \( x = (1-z^2)^{-1/2} \).

In (8) use \( x = (z^2-1)^{1/2} \) to reduce to (5) or (3), depending on the value of \( k^2 \).

Reduction of \( \int \frac{ydy}{(y^2-1)F_y} \) . Let \( y^2 = 1/t+1 \) to get

\[-\frac{1}{2} \int \frac{dt}{\sqrt{A+(1+B)t][C+(1+D)t]} \]  . This is an elementary form \( \int (ax^2+bx+c)^{-1/2}dx \), and is of varying forms
This completes the reduction of the integral \( \int \frac{x^2}{F_x} \) encountered on page 15.

**Reduction of \( \int \frac{x \, dx}{F_x} \).** By the substitution

\[ x = \frac{p+qy}{1+y}, \]

where \( p \) and \( q \) are as on page 14, the integral \( \int \frac{x \, dx}{\sqrt{(x^2-1)(x-c)(x-f)}} \) becomes

\[ (q-p) \int \frac{(p+qy) \, dy}{(1+y)R_y} \equiv p(q-p) \int \frac{dy}{(1+y)R_y} + q(q-p) \int \frac{ydy}{(1+y)R_y} \]

\[ = -(p-q)^2 \int \frac{dy}{(1+y)R_y} + q(q-p) \int \frac{dy}{R_y}. \]

Both of these have been reduced in the preceding discussion.

**Reduction of \( \int \frac{dx}{F_x} \).** The substitution \( x = 1/y+1 \)

yields

\[ -\int \frac{dy}{\sqrt{(1+2y)[1+(1-c)y][1+(1-f)y]}}, \]

Now let \( y+1/2 = z^2 \) to get

\[ -(2)^{1/2} \int \frac{dz}{\sqrt{[1+c+(1-c)z^2][1+f+(1-f)z^2]}}, \]

which was treated on page 17.

This completes the reduction of the volume integral

\( \int \sqrt{a^2-y^2} \left[ b^2-(y-L)^2 \right] \) dy to standard forms, which consist of the three types of elliptic integrals, several elementary integrals, and various algebraic expressions.
SUMMARY

The expression for the intercepted volume formed by the random, internal intersection of two unequal, right, circular cylinders involves, among other functions, the three kinds of elliptic integrals. If the intersection is made central, the elliptic integral of the third kind degenerates. Furthermore, if the radii of the cylinders are made equal, the elliptic integrals of the first and second kinds degenerate. Finally, if the axes of the cylinders intersect normally, the trigonometric factor becomes unity, leaving a simple algebraic expression.
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