SELECTED TOPICS IN NON-EUCLIDEAN GEOMETRY

by

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INTRODUCTION TO NON-EUCLIDEAN GEOMETRY

from

Monographs on Modern Mathematics by J.W.A. Young

Non-Euclidean Geometry is a system of geometry which is built up without the use of the fifth postulate of Euclid. This postulate reads as follows:

"If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two lines, if produced indefinitely, meet on that side on which are the angles less than two right angles."

Many attempts have been made to establish such a system of geometry. That, in which Lobachevsky, a Russian, was the chief writer from 1833 to 1855, remained for a time the sole type of a non-Euclidean geometry. This was known as the Lobachevskian geometry. Riemann, however, in 1854, working from the standpoint of the differential calculus, discovered a new type to which the name of Riemannian geometry has been given. These two types of non-Euclidean geometry, it seems, have been the principal types recognized by mathematicians.

In the following paragraphs Euclid's fundamentals are assumed with the exception of the parallel postulate. A more general definition of parallels than that of Euclid will be given.
Let \( PQ \) (Fig. 1) be any straight line, and \( A \) a point not on \( PQ \). Through \( A \) there passes a set of lines intersecting \( PQ \), since any point on \( PQ \) may be joined to \( A \). Let \( AB \) be any line through \( A \) intersecting \( PQ \). The line \( AL \) is said to be parallel to \( PQ \) at the point \( A \), if (1) \( AL \) does not intersect \( PQ \) no matter how far produced and (2) any line in the angle opening \( BAL \), passing through \( A \), does intersect \( PQ \).

Let \( AH \) be perpendicular to \( PQ \). Then \( \angle HAK = \angle HAL \), for if \( \angle HAK > \angle HAL \), draw \( \angle HAC = \angle HAL \) and take \( C' \) on \( PQ \) so that \( HC' = HC \). Then the triangles \( HAC' \) and \( HAC \) are and \( \angle HAC' = \angle HAC = \angle HAL \). This is impossible since \( AL \) is parallel to \( HQ \); hence, \( \angle HAK > \angle HAL \). In like manner it can be shown that \( \angle HAL > \angle HAK \). Therefore \( \angle HAL = \angle HAK \).

\( \angle HAL \) is called the angle of parallelism for the distance \( AH \). The line \( AL \) is parallel to \( PQ \) at all of its points. Let \( AK \) be parallel to \( BQ \) at point \( A \) (Fig. 2).

Let \( A' \) be any point on \( AK \). To show that \( AK \) is parallel to \( BQ \) at point \( A' \), connect \( A' \) and \( B \) and through \( A' \) draw any line \( A'C \) in the angle open-
ing $BA'K$. Take D, any point on $A'C$ and connect D with A. Prolong AD until it meets BQ at $F$. (It will meet BQ, since $AK$ is parallel to BQ.) Also $A'C$ will meet BQ in a point between B and $F$; (Veblen, Th. 17 - "A line of a plane which contains one and only one point of a side of a triangle whose vertices are in the plane contains one other point of the triangle.) That is, any line through $A'$ in the angle opening $BA'K$ intersects BQ. But $A'K$ does not intersect BQ. Hence, it is parallel to BQ.

If a line is parallel to another line, the second is parallel to the first. Let LK (Fig. 3) be parallel to PQ.

Select a point B on PQ and erect a perpendicular to LK. This will meet LK at some point A since LK is parallel to PQ. Through B draw any line BC in the angle opening $QBA$. Construct the two angles $ABE$ and $ABD$ so that the

$\angle ABE = \angle ABD < \frac{1}{2} \angle QBC$. Then $BD = BE$; hence in $\angle BEK$ draw a line EF so that $\angle BEF = \angle BDK$, and EF meets BQ since LK is parallel to PQ. Now take $DG = EF$, and draw BG. Then the two triangles are equal and $\angle DBG = \angle EBF$. But $\angle DBE < \angle QBC$. Therefore, $\angle EBG > \angle EBC$. Hence the line BC meets LK at some point between $E$ and $G$; but BC is any line through B in the angle opening $QBA$, and LK and BQ do
not meet. Therefore DQ is parallel to LK.

If two lines are parallel to a third line, they are parallel to each other. Case 1: (Fig. 4). Let AK and DQ be parallel to ML. To prove: AK is parallel to DQ. Draw AC any line through A in \( \angle DAK \).

Then AC will meet ML in some point F since AK is parallel to ML. CF prolonged will also meet DQ, since ML is parallel to DQ. Hence any line through A in the \( \angle DAK \) meets DQ. Also AK cannot meet DQ, since it cannot meet ML. Therefore AK is parallel to DQ.

Case 2: (Fig. 5) Let AK and DQ be parallel to ML.

To prove: AK is parallel to DQ. Draw through A a line AK' parallel to DQ. Then by case 1, AK' is parallel to ML and coincides with AK.

The Euclidean Assumption

Replace Euclid No. 5 by "Through any point in a plane there goes one and only one line parallel to a given line."

Hence, (Fig. 6) \( \angle HAL = \angle HAK = \) a right angle. Take M the mid-point of AB, and draw MD perpendicular to PQ and intersecting AL in C. Then
\( \angle DCK \) is a right angle. The two right triangles \( \triangle AKC \) and \( \triangle BMD \) are \( \cong \), and \( \angle CAB = \angle ABD \). Therefore \( \angle QBA \) and \( \angle BAK \) together equal 2 right angles. By definition of parallels any line through \( A \) in the \( \angle BAK \) meets \( PQ \); hence the assumption is equivalent to Euclid No. 5.

**The Lobachevskian Assumption**

"Through any point in the plane there go two lines parallel to a given line." It follows that (Fig. 6) the \( \angle QBA + \angle BAK = 2 \) right angles, for if the sum is \( > 2 \) right angles, we could draw in \( \angle BAK \) a line not meeting \( BQ \). This is contrary to the assumption that \( AK \) and \( BQ \) are parallel. On the other hand, if the sum of the angles \( = 2 \) right angles, we have the Euclidean assumption. Therefore the angles must be acute.

**Theorem 1:** Let \( AB \) and \( CD \) be two parallel lines cut by a third line \( AC \), (Fig. 7) and let \( A'B' \) and \( C'D' \) be two other parallel lines cut by a third line \( A'C' \), and let \( \angle DCA = \angle D'C'A' \).

Then
1. If \( A'C' = AC \), \( \angle C'A'B' = \angle CAB \).
2. If \( A'C' < AC \), \( \angle C'A'B' > \angle CAB \).
3. If \( A'C' > AC \), \( \angle C'A'B' < \angle CAB \).
By law of converses, ("If the direct theorem is true, and the converse is true, then the opposite is true");

Let $\angle DCA = \angle D'C'A'$. Then it follows that:

Theorem 2:

(1) If $\angle C'A'B' = \angle CAB$, $A'C' = AC$.
(2) If $\angle C'A'B' > \angle CAB$, $A'C' < AC$.
(3) If $\angle C'A'B' < \angle CAB$, $A'C' > AC$.

Theorem 3: The angle of parallelism is fixed for a fixed distance, and decreases as the distance increases.

(Fig. 8) Where $p$ is fixed the angle of parallelism $\angle(p)$ is fixed; As $p$ increases $\angle(p)$ decreases. As $p$ decreases, $\angle(p)$ increases. $\angle(p) = 2 \tan \frac{\varphi}{2}$. ($K$ = space constant).

Theorem 4: If two lines have a common perpendicular, they neither intersect nor are parallel.

(a) (Converse of Th. 4) Two straight lines which neither intersect nor are parallel have a common perpendicular.

Let $LM$ and $EF$ (Fig. 9) be two straight lines which neither intersect nor are parallel. To show that they have a common perpendicular; Take $A$ and $B$, any two points on $LM$ and draw $AH$ and $BK$ perpendicular to $EF$. If $AH = BK$, it is evident that $LM$
and \( EF \) have a common perpendicular. Now suppose that \( BK < AH \). Draw \( KS \) parallel to \( LM \). Place the right angle \( PKB \) on the right angle \( FHA \) so that \( K \) falls on \( H \), \( KP \) takes the direction of \( HF \) and \( KB \) takes the direction of \( HA \). The point \( B \) falls at \( B' \) between \( H \) and \( A \), \( B'M' \) and \( BM \) the same position, and \( KS \) the position \( HS' \) parallel to \( B'M' \). Since \( \angle FKS = \angle FHS' \), a line parallel to \( KS \) and hence to \( LM \) drawn through \( H \) lies in the angle opening \( FHS' \). Hence \( HS' \) intersects \( LM \) and therefore \( B'M' \) intersects \( LM \) at some point \( P \).

Draw \( PR \) perpendicular to \( EF \). Place the right angle \( PHB' \) on the right angle \( PKB \). Then the line \( PR \) takes the position \( QT \), where \( QT \) is perpendicular to \( EF \) and \( \angle = \) to \( PR \). Now take \( W \) half way between \( R \) and \( T \) and draw \( WV \) perpendicular to \( EF \). Fold the figure \( TWV \) on \( WV \). Then \( T \) falls on \( R \), \( TQ \) coincides with \( RP \), and \( \angle WVQ \) coincides with \( \angle WVP \). Hence \( WV \) is the required perpendicular.

(b) Any angle is an angle of parallelism belonging to a certain distance.

Let \( LAE \) (Fig. 10) be any given angle \( \alpha \). To find a distance \( p \) for which \( \alpha \) is the angle of parallelism. Construct \( LAE = \alpha \), and on \( AK \) and \( AL \) take two points \( B \) and \( C \) so that \( AB = AC \). Connect \( B \) and \( C \) and draw \( BL' \) parallel to \( BL \), and \( CK' \) parallel
to CK. Draw also CF bisecting the \( \angle LCK' \), and BG bisecting the \( \angle KBL' \). The figure is then symmetric with respect to the line AE.

The lines CF and BG cannot intersect, for if they did intersect at a point T, we could draw TS parallel to AL and BL', and then, since \( \angle LCT = \angle L'BT \), and CT = BT, we should have \( \angle STC = \angle STB \) which is impossible. Also CF and BG cannot be parallel, for if they were, since \( \angle LCF = \angle L'BG \), and \( \angle CNL' = \angle BNF \), we should have CN = NB and therefore \( \angle NCB = \angle NBC = \angle K'CB \) which is impossible.

Since FC and BG neither intersect nor are parallel, they have a common perpendicular UV, which, by the symmetry of the figure, is also perpendicular to AE at H. UV is parallel to AK, for if UV is not parallel to AK, we could draw from each of the points U and V, a line parallel to AK and CK'. Since CU = BV, and \( \angle UCK' = \angle VBE \), these two parallels would make equal angles with UV, which is impossible. Hence the angle KAE is the angle of parallelism for the distance AH.

(c) Two parallel lines approach each other continually and their distance apart eventually becomes less than any assigned quantity.

Let LK and PQ (Fig. 11) be two parallel lines, and A and B two points on LK, the point B lying from A in the direction of parallelism. From A and B draw AH and BM
perpendicular to PQ. To prove: $BM \angle AH$. Take R half way between H and M, and draw RC perpendicular to PQ. Then the angle RCB is less than a right angle, since it is an angle of parallelism. Therefore the $\angle RCB < \angle RCA$. Hence if the quadrilateral RMBC is folded over on RC as an axis, the line MB takes the position HB', where MB = HB' < HA. Hence the lines LK and PQ continually approach each other.

To prove the second part of the theorem, let AK and HQ (Fig. 12) be any two parallel lines, and AH a perpendicular from A to HQ. Let $\xi$ be any assigned quantity, and lay off on AH the distance $HD < \xi$. Draw DL parallel to HQ and AK. Then the $\angle HDL < a$ right angle. Hence the line DE drawn from D perpendicular to AH will meet AK in some point C. From C draw CM perpendicular to HQ. Now $\angle MCD > \angle MCK$, for $\angle MCK$ is the angle of parallelism for the distance CM, and the line CD and MH neither intersect nor are parallel, since they have a common perpendicular. Hence if the quadrilateral MHDC is folded over on MC as an axis, it takes the position MH'D'C, where CK lies between
CD' and MQ. Then CK meets H'D' in some point K', where

\[ H'K' < H'D' = HD. \] Hence \( H'K' < \varepsilon. \)

(d) If two lines are not parallel, they will diverge if sufficiently far produced, and their distance apart will eventually become greater than any assigned distance.

Let AM and AN (Fig. 13) be two intersecting straight lines. Let B and D be two points on AM such that \( AD > AB, \) and let BC and DE be drawn perpendicular to AN. To prove: \( DE > BC. \)

Suppose, if possible, that \( DE = BC. \) Then a line drawn perpendicular to AN at the middle point of CE would be also perpendicular to AM, which is impossible, since AM and AN intersect. Suppose, if possible, that \( DE < BC. \) Take AF less than each of the distances DE and AB, and draw FG perpendicular to AN. Then FG is \( < AF < DE. \) But \( BC > DE, \) hence at some point K between G and C there is a perpendicular HK such that \( HK = DE. \) But this is impossible. Therefore \( DE > BC. \)

To show that there is no superior limit to the length of ED, take AH (Fig. 14) so that \( \angle MAN \) is the angle of parallelism for AH, and draw HL perpendicular to AM. Then AN and HL are parallel. Let "a" be any quantity, no matter how large, and
take Q on HL so that HQ = 2a. Connect Q and A, and at E, a point between A and H, draw a line perpendicular to AH, intersecting AQ in R. We can take E so near H that RE will differ from HQ by as little as we please, and certainly so that RE > a. But RE will intersect AN in a point D, since the angle of parallelism for AE is greater than the angle HAN. Then DE > RE > a. Since "A" is any positive number, there is no superior limit to the length of DE.

The Riemannian Assumption

"Through a point of the plane no line can be drawn parallel to a given line".

In other words, all lines of the pencil with its vertex at A (Fig. 1) intersect PQ. Here, the propositions of Euclid depending upon the assumption that two straight lines cannot enclose space are contradicted, except when applied to objective space in the domain of experience. We will then assume that the Euclidean assumptions, with the exception of the parallel postulate, are valid in a sufficiently restricted portion of space, that is, in a portion of space in which no straight line can be drawn of greater length than some fixed line of length M.

(a) All lines perpendicular to the same straight line meet in a point at a constant distance from the straight line. (Fig. 15) Let LK be any straight line and A and B
any two points upon it. By the Riemannian hypothesis $AO$ and $BO$, perpendicular to $LK$, meet in a point $O$. Since it is conceivable that the perpendiculars may meet more than once, we may assume that the two perpendiculars have no common point on the segment $AO$ or $BO$. It is assumed also that the triangle $ABO$ lies in the restricted portion of space mentioned above, so that in particular only one straight line can be drawn from $O$ to any point of the segment $AB$. Since the $\angle BAO = \angle ABO$, $BO = AO$. Construct $\angle BOM = \angle AOB$.

Then by the Riemannian hypothesis the line $OM$ meets $LK$ in a point $C$. The triangle $BOC$ has two angles and an included side congruent respectively to two angles and the included side of the triangle $AOB$.

Hence $\angle BCO = \angle ABO = \angle AOB = \angle BOM = \angle AOB$. By repeating this demonstration, we prove that if $P$ is a point on $LK$ such that $AP = m \cdot AB$, where $m$ is a positive integer, the line $OP$ is perpendicular to $LK$ at $P$, and $PO = AO$. But only one perpendicular can be drawn to $LK$ at $P$. Hence this perpendicular passes through $O$. Now take $D$, so that $AB = n \cdot AB$, where $n$ is a positive integer, and draw a line perpendicular to $LK$ at $D$. If this perpendicular should intersect either $BO$ or $AO$ at a point $O'$, in the segments $BO$
or AO, then BO and AO would also intersect at $O'$ which is contrary to the hypothesis.

Hence this perpendicular passes through 0, and $DO = AO$.

It follows that if $P$ is any point on $LK$ such that $AP = \frac{m}{n}AB$, where $m$ and $n$ are positive integers, the perpendicular to $LK$ at $P$ passes through $0$, and $PO = AO$. Also, since by hypothesis, only one straight line can be drawn from $P$ to $O$, the line $PO$ is perpendicular to $LK$.

Now let $P'$ be a point such that $AP' = AB$, where $\sqrt{2}$ is an irrational number. Take $P$ such that $AP = \frac{m}{n}AB$, draw $OP$ and $OP'$ and let $\frac{m}{n}$ pass through rational values approaching as a limit. $\angle AP'O = \lim \angle APO$; $P'O = \lim PO$. But $APO$ is always a right angle and $PO$ is always equal to $AO$. Hence $\angle AP'O$ is a right angle, and $P'O = AO$. The theorem is thus proved for the line $LK$. If $L'K'$ is any other line we may take $A'$ and $B'$, any two points on it, and draw the perpendiculars $A'O'$ and $B'O'$, intersecting at $O'$. Take $AB$ on $LK$ so that $AB = A'B'$. The two triangles $ABO$ and $A'B'O'$ are congruent, and $A'O' = AO$. The distance $AO$ is therefore independent of the line $LK$ or of the position of the point $A$ on the line. Place $AO = \angle A$. A corollary of the theorem is that all straight lines are of constant length. It is evident that, if $P$ is any point on $AB$, $\frac{AP}{AB} = \frac{\angle AOP}{\angle AOB}$.

Now if $\angle AOP = 2\pi$, the line $OA$ coincides with $OP$, and $AP$ becomes $L$, the total length of the line. Then the
length \( L = \frac{2 \angle AB}{\angle AOB} \).

(b) All lines which pass through a point 0 meet again in a point 0' such that the distance 0' is constant.

Let 0 (Fig. 15) be any point, and OA any line through 0. Take \( OA = \angle \), and draw LK perpendicular to AO. Let OB be any other line through 0, intersecting LK in B. Then OB is perpendicular to LK. Prolong AO to 0', so that \( AO' = AO \), and draw 0'B. The triangles AOB and A0'B are congruent since two sides and the included angle of one are equal respectively to two sides and the included angle of the other. Hence \( \angle ABO' = \angle ABO \) and they are right angles, and \( 0'B = OB = OA \). Therefore the line 0B0' is a straight line, and \( 00' = 2 \angle \).

Since all the lines are of finite length, any line through 0 returns through 0' to 0. Two cases are usually considered.

First, the point 0' may coincide with 0. The total length of a straight line is then \( 2 \angle \), and any two lines have only one point in common.

Secondly, the point 0' may be distinct from 0, but the lines 00' continued through 0' meet again in 0. The total length of a line is then \( 4 \angle \), and the two lines meet in two points. The Riemannian geometry, in this case, is the same as the geometry on the surface of a sphere.
The Sum of the Angles of a Triangle

Consider any triangle ABC (Fig. 16). Take E, the midpoint of AB, F, the mid-point of AC, and draw a straight line EF. From A, B, and C draw the lines AG, BK, and CL perpendicular to EF. In the right triangles AEG and EBK, $KA = EB$, and $\angle GEA = \angle BEK$.

Hence the two triangles are congruent, and $BK = AG$, and $\angle KBE = \angle GAE$. Similarly, the right triangles AGF and FLC are congruent, and $AG = CL$.

Also $\angle FCL = \angle GAF$. If we define equivalent figures as those which may be divided into parts which are congruent in pairs, it appears that the triangle ABC is equivalent to the quadrilateral BCLK. Also, the sum of the angles of the triangle ABC is equal to the sum of the angles KBC and LCB of the quadrilateral BCLK.

This quadrilateral BCLK has two right angles, L and K, and two equal sides, KB and LC, adjacent to the right angles and opposite to each other. Such a figure we shall call an isosceles-birectangular-quadrilateral. Thus the study of the angles and area of a triangle can be made by use of an equivalent isosceles-birectangular-quadrilateral, or an I.B.Q.
Let ABCD (Fig. 17) be an I.B.Q. with right angles at A and B. Let AB be called the base, CD the summit, and C and D the summit angles of the I.B.Q. Take L the mid-point of the base, and draw LK perpendicular to the base. Fold LBDK on LK as an axis. It is clear that the point D falls on C. Hence the summit angles of an I.B.Q. are equal.

Also LK is perpendicular to CD at its mid-point K, and the quadrilateral LBDK has three right angles. Through H, the mid-point of LK, draw EF perpendicular to LK. Fold HFDK on HF as an axis. The point D will fall at B', B, or B" according as KD is less than, equal to, or greater than LB. In these three cases the angle D is greater than, equal to, or less than the angle B, respectively. Hence each summit angle of an I.B.Q. is less than, equal to, or greater than, a right angle, according as the summit of the quadrilateral is greater than, equal to, or less than the base:

From the former discussion, it follows that:

1. In the Euclidean geometry each summit angle of an I.B.Q. is equal to a right angle.

2. In the Lobachevskian geometry each summit angle of an I.B.Q. is less than a right angle.

3. In the Riemannian geometry each summit angle
of an I.B.Q. is greater than a right angle.

NON-EUCLIDEAN TRIGONOMETRY

The definitions of the trigonometric functions based on Euclid are not available in the non-Euclidean geometries. So Lobachevsky constructed a "limit surface" or horisphere, on which the Euclidean geometry and trigonometry are valid at the same time that the Lobachevskian geometry is valid on the plane. But a more general method, which has also the advantage of operating entirely in the plane, must be used for the Riemannian geometry, and also applied to the Lobachevskian geometry.

Beginning with

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

the trigonometric functions are defined as follows:

\[ \sin x = \frac{e^{xi} - e^{-xi}}{2i} \]
\[ \cos x = \frac{e^{xi} + e^{-xi}}{2} \]
\[ \tan x = \frac{i}{e^{xi} + e^{-xi}} \]

where \( i = \sqrt{-1} \). These formulas obey all the formulas of trigonometry, and if \( x \) is real, they are real. If \( x \) is pure imaginary, the above formulas lead to the hyperbolic functions which are:
\[
-\text{isin } ix = \frac{e^x - e^{-x}}{2} = \sinh x
\]
\[
\cos ix = \frac{e^x + e^{-x}}{2} = \cosh x
\]
\[
-\text{itan } ix = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x
\]

If \( x \) is real, the hyperbolic functions are real. If the \( \cos x < 1 \), \( x \) is real; if the \( \cos x > 1 \), \( x \) is pure imaginary, except for the multiples of the period 2 which may always be added.

If we place \( \cos mx = f(x) \), \( f(x) \) satisfies the functional equation \( f(x - y) - f(x-y) = 2f(x)f(y) \).

Conversely, if \( f(x) \) is a continuous function of \( x \), satisfying the above equation, then \( f(x) = \cos mx \), \( m \) being a constant, real or imaginary.

The sine and cosine of an acute angle may be defined as follows; (The extension to angles of any size is then made as in the ordinary trigonometry.)

Let \( A \) (Fig. 18) be an acute angle \( x \) in the right triangle \( ABC \), and \( BC \) the side opposite \( x \), and let \( AB \) become infinitesimal while the angle \( x \) remains constant.

Problem: To prove

1. That \( \lim_{AB \to 0} \frac{AC}{AB} \) is a definite number.
2. That \( \lim_{AB \to 0} \frac{AC}{AB} = f(x) \).
3. That \( f(x) \) is a continuous function.
4. That \( f(x) \) satisfies the equation
\[
f(x - y)' - f(x - y) = 2f(x)f(y).
\]
5. That \( f(x) = \cos mx \).
(1 and 2 may be assumed without demonstration.)

Proof:

3. Take \( B' \) on the extension of \( CB \) beyond \( B \), and let \( \Delta x \) be the measure of \( \angle BAB' \). If \( \Delta x \) is infinitesimal, then \( BB' \) is infinitesimal as compared with \( AB \), by (Th. 2, p. 49, Coolidge: If in a triangle whereof one angle is constant, a second angle may be made as small as desired, the side opposite this angle will be infinitesimal as compared to the other sides of the triangle).

The difference between two sides of a triangle (in plan geometry) is less than the third side; then \( AB' - AB = BB' \). But \( BB' \) is infinitesimal. Hence \( AB \) and \( AB' \) may differ by an infinitesimal amount, and \( \frac{AB'}{AC} - \frac{AB}{AC} \) will become and remain less than any assigned number. Therefore \( f(x) \) is continuous.

4. Suppose (Fig. 19) we have two angles \( \angle KAL \) and \( \angle LAX \) such that each is less than a right angle, and \( \angle LAX > \angle KAL \). Let \( \angle LAX = x \), and the \( \angle KAL = y \).

Take \( F \) on \( AL \), and find \( D \) so that \( AF = AD \), and the \( \angle PAL = \angle LAD \). AD is within
the $\angle x$. These points will certainly exist if $AF$ be very small. Join $P$ and $D$, and let $FD$ intersect $AL$ in point $B$. Through $F$, $B$, and $D$, draw perpendiculars to $AX$, meeting it in $G$, $C$, and $E$, respectively, which points are sure to exist if $AF$ be very small. Let us consider Fig. 20 a "close-up" of part of Fig. 19. Suppose there is a midpoint of $CE$ at $C'$. Erect a perpendicular to $CE$ at $C'$, intersecting $BD$ at $B'$. Lay off $CB'' = C'B'$. Then the figure $CC'B'B''$ is an I.B.Q. whose summit is $B'B''$. As $AD \to 0$ as a limit, $FD$ and $BB'$ each $\to 0$ as a limit. Hence $\angle BAB' \to 0$ as a limit, and $BB'$ is an infinitesimal of higher order than $AB$. But $AB$ and $CD$ are infinitesimals of the same order; hence $BB'$ is of higher order than $AD$. As $AD$, $AB$ and $DF \to 0$ as a limit, the $\angle B'B'B''$ is finite, and $\angle B'B''C$ approaches a right angle. Therefore $\angle BB'B''$ is finite and $BB'$ is an infinitesimal of the same order as $B'B''$, for the ratio $\frac{BB'}{B'B''}$ is a finite number. $B'B''$ and $CC'$ being the summit and base of an I.B.Q. are infinitesimals of the same order. Therefore $CC'$ is an infinitesimal of same order as $BB'$ and hence of higher order than $AD$. 
Then, \( \frac{AE}{AD} = f(x - y) + \epsilon_1 \)

\( \frac{AC}{AD} = \frac{AC}{AB} \cdot \frac{AB}{AD} = f(x) \cdot f(y) + \epsilon_2 \)

\( \frac{AG}{AF} = f(x + y) + \epsilon_3 \) but \( AF = AD \), \( \therefore \frac{AG}{AD} = f(x+y) + \epsilon_3 \).

\( \frac{AG+AE}{2} = AC + \epsilon_4 \) , \( (\epsilon_4 = CC') \)

\( \frac{AG + AE}{2} = 2AC + 2\epsilon_4 \)

\( \frac{AC + AE}{2} = 2 \frac{AC}{AD} + 2 \frac{\epsilon_4}{AD} \)

\( f(x + y) + \epsilon_3 + f(x - y) + \epsilon_1 = 2f(x)f(y) + 2\epsilon_2 + 2\epsilon_4 \frac{\epsilon_4}{AD} \).

As \( AD \to 0 \), \( \epsilon_1 \), \( \epsilon_2 \) and \( \epsilon_3 \) become 0, and since \( \epsilon_4 = CC' \), an infinitesimal of higher order than \( AD \), \( AD \) also becomes 0, and we have \( f(x + y) + f(x - y) = 2f(x)f(y) \).

5. \( f(x + y) = f(x) + \frac{f''(x)y}{2!} + \frac{f'''(x)y^2}{3!} + \frac{f''''(x)y^3}{4!} + \ldots \)

\( f(x - y) = f(x) - \frac{f'(x)y}{1!} + \frac{f''(x)y^2}{2!} - \frac{f'''(x)y^3}{3!} + \ldots \)

Adding,

\( f(x + y) + f(x - y) = 2f(x) + \frac{2f''''(x)y^2}{2} - \frac{2f''''(x)y^4}{4!} + \ldots \)

Dividing by \( 2f(x) \),

\( \frac{f(x + y) + f(x - y)}{2f(x)} = 1 + \frac{f''''(x)}{f(x)} \frac{y^2}{2!} - \frac{f''''(x)}{f(x)} \frac{y^4}{4!} + \ldots \)
\[
\frac{f(x + y) + f(x - y)}{2f(x)} = f(y),
\]

\[
\therefore f(y) = f'(x) \frac{f''(x)}{2} + f''(x) \frac{f'''(x)}{6} + \ldots
\]

\[
\therefore \frac{f'''(x)}{f(x)} = \text{a constant } k, \quad \frac{f''(x)}{f(x)} = \text{a constant also.}
\]

\[
f''(x) = kf(x) \quad \text{or} \quad \frac{\frac{d^2f(x)}{dx^2}}{f(x)} = kf(x) ; f'''(x) = kf'(x);
\]

\[
f''(x) = kf(x).
\]

\[
\frac{f''(x)}{f(x)} = k^2, \quad \text{and} \quad \frac{f'''(x)}{f(x)} = k^3.
\]

Then \( f(y) = 1 + \frac{k}{1!} + \frac{k^2}{2} \frac{y^2}{2!} + \frac{k^3}{3!} \frac{y^3}{3!} + \ldots \); Let \( k = -a^2 \)

Then \( f(x) = 1 - \frac{(ay)^1}{1!} + \frac{(ay)^2}{2!} - \frac{(ay)^3}{3!} + \ldots \).

But this is \( \cos (ay) ; \)

\[
\therefore f = \text{cosine function.}
\]

Hence we have \( \lim_{AB} \frac{AC}{AB} = \cos mA ; \) since \( AC < AB, \) \( m \) is real, and if we take the system of measurement of angle by which a right angle = \( \frac{\pi}{2} \), \( m = 1 \); hence \( \lim_{AB} \frac{AC}{AB} = \cos A. \) Also \( \frac{BC}{AB} = \sin A. \)

It may be shown also that if \( CD \) is the summit and \( AB \) the base of an I.B.Q., \( \frac{CD}{AB} \) approaches a limit as \( AB \rightarrow 0 \), and that the limit is a continuous function of "a", satisfying the above functional equation.

Hence, \( \lim_{AB} \frac{CD}{AB} = \cos mA. \)
In the Lobachevskian geometry, CD > AB, and m is pure imaginary. In this case we place \( m = \frac{1}{k} \), where \( k \) is real, and \( \lim \frac{CD}{AB} = \cos \frac{\alpha}{k} = \frac{a}{k} \).

In the Riemannian geometry, CD < AB, and m is real. In this case we place \( m = \frac{1}{k} \), and have \( \lim \frac{CD}{AB} = \cos \frac{a}{k} \).

The constant \( k \) depends upon the unit of distance used.

Without details of proof, we shall now assume the general formulas in Young's Monographs as follows: (\( ABC \) being a right triangle, \( C \) a right angle, \( AB = c \), \( AC = b \), \( BC = a \))

1. \( \cos mc = \cos ma \cdot \cos mb \)
2. \( \sin ma = \sin mc \cdot \sin A \)
3. \( \tan ma = \tan mc \cdot \cos B \)
4. \( \cos A = \cos ma \cdot \sin B \)
5. \( \sin mb = \sin mc \cdot \sin B \)
6. \( \tan mb = \tan mc \cdot \cos A \)
7. \( \cos B = \cos mb \cdot \sin A \)

In which, in the Lobachevskian geometry, we replace \( m \) by \( \frac{1}{k} \); and in the Riemannian geometry, by \( \frac{1}{k} \).

NON EUCLIDEAN ANALYTIC GEOMETRY

If \( x \) and \( y \) are assumed arbitrarily to be the coordinates of a point \( P \), using the coordinate axes \( OX \) and \( OY \), there is not necessarily a corresponding point \( P' \) in the
Lobachevskian geometry, since perpendiculars at points on the axes may be parallel or non-intersecting.

If we take the polar coordinates of \( P \) with the rectangular coordinates, between the two sets there exist, in either the Riemannian or the Lobachevskian geometry, the relations:

\[
\tan mx = \tan mr \cos \Theta, \tag{1}
\]
\[
\tan my = \tan mr \sin \Theta \tag{2}
\]

whence \( \tan^2 mx + \tan^2 my = \tan^2 mr \).

The equation of a straight line becomes

\[
\tan mx \cos \alpha + \tan my \sin \alpha = \tan mp \tag{3}
\]

The distance between two points \( P \) and \( P \) becomes

\[
\cos m PP = \frac{1 + \tan mx \tan mx + \tan my \tan my}{\sqrt{1 + \tan^2 mx + \tan^2 my + \sqrt{1 + \tan^2 mx + \tan^2 my}}} \tag{4}
\]

The angle between two lines becomes

\[
\phi = \cos^{-1} \frac{\cos \alpha \cos \alpha + \sin \alpha \sin \alpha + \tan mp \tan mp}{\sqrt{1 + \tan^2 mp}} \tag{5}
\]

In the Riemannian geometry, instead of \( x \) and \( y \), we will use the new coordinates \( \xi \) and \( \eta \), where

\[
\xi = k \tan \frac{x}{k}, \quad \eta = k \tan \frac{y}{k} \tag{6}
\]

and in the Lobachevskian geometry,

\[
\xi = -ik \tan \frac{i x}{k} = k \tanh \frac{x}{k}, \quad \eta = -ik \tan \frac{i y}{k} = k \tanh \frac{y}{k}. \tag{7}
\]

(1) To obtain the equation of the straight line in the Lobachevskian geometry, substitute (7) in (3) above, and the equation becomes

\[
\xi \cos \alpha + \eta \sin \alpha = k \tanh \frac{P}{k} \tag{8}
\]

This may be written \( A\xi + B\eta + C = 0 \), where

\[
A = \cos \alpha, \quad B = \sin \alpha, \quad C = k \tanh \frac{P}{k}. \tag{9}
\]
\[
\cos \alpha = \frac{A}{\sqrt{A^2 + B^2}} \quad \sin \alpha = \frac{B}{\sqrt{A^2 + B^2}} \quad k \tan \frac{\theta}{k} = \frac{-C}{\sqrt{A^2 + B^2}} \quad (10)
\]

(since \(\sin^2 \alpha + \cos^2 \alpha = 1\), and in a right triangle whose sides are \(A\) and \(B\), and whose hypotenuse is \(C\), the above relations exist).

Now if \(p\) is real, \(\tan \frac{\theta}{k} < 1\), and \(\tanh \frac{\theta}{k} < 1\), and from

\[
C^2 < k(A^2 + B^2) \quad (9)
\]

Conversely, equation (9) represents a straight line provided \(C < k(A^2 + B^2)\), for then \(\alpha = \cos^{-1} \frac{A}{\sqrt{A^2 + B^2}}\) and

\[
p = k \tan^{-1} \frac{-C}{\sqrt{A^2 + B^2}}.
\]

2. To obtain the formula for the distance between points \(P_1\) and \(P_2\) in the Lobachevskian geometry, substitute \(m = \frac{1}{k}\) in (4) above, and we have

\[
\cos \frac{1P_1P_2}{k} = \frac{1 + \tan \frac{i \varepsilon_{12}}{k} + \tan \frac{i \varepsilon_{21}}{k} + \tan \frac{i \eta_{12}}{k} \tan \frac{i \eta_{21}}{k}}{\sqrt{1 + \tan^2 \frac{i \varepsilon_{12}}{k} + \tan^2 \frac{i \varepsilon_{21}}{k} + \tan^2 \frac{i \eta_{12}}{k} + \tan^2 \frac{i \eta_{21}}{k}}}
\]

\[
cosh \frac{P_1P_2}{k} = \frac{-k^2 + \varepsilon_1^2 + \eta_1 \eta_2}{\sqrt{-k^2 + \varepsilon_1^2 + \eta_1 \eta_2}}
\]

\[
cosh \frac{P_1P_2}{k} = \frac{i (k^2 - \varepsilon_1 \varepsilon_2 - \eta_1 \eta_2)}{\sqrt{i (k^2 - \varepsilon_1^2 - \eta_1^2) \sqrt{i (k^2 - \varepsilon_2^2 - \eta_2^2)}}}
\]

\[
cosh \frac{P_1P_2}{k} = \frac{\eta - \eta_z \eta_2 - \eta_1 \eta_2}{\sqrt{\eta - \eta_z \eta_2 - \eta_1 \eta_2}}
\]

(11)

If in equation (11) we place \(\varepsilon_z = \varepsilon\) \(\eta_z = \eta\) \(\varepsilon_2 = \varepsilon + \eta_2\) \(\eta_2 = \eta + \eta_2\), and \(P_1P_2 = ds\), it becomes, as far as infinitesimals of the second order are concerned:

\[
cosh \frac{P_1P_2}{k} \text{ becomes } \cosh \frac{ds}{k} = 1 + \frac{1}{2} \left( \frac{ds}{k} \right)^2 + (\text{higher order}).
\]
Then \[
\frac{K - \xi, \eta, \xi - \eta, \eta^2}{\sqrt{K^2 - \xi^2 - \eta^2}} = \frac{K^2 - \xi (\xi + d\xi) - \eta (\eta + d\eta)}{\sqrt{K^2 - \xi^2 - \eta^2}} - \frac{\zeta^2 - \xi^2 - \eta^2}{\sqrt{K^2 - \xi^2 - \eta^2}} - \frac{\eta^2}{\sqrt{K^2 - \xi^2 - \eta^2}}
\]

Then \[
1 + \frac{1}{2} \frac{(ds)^2}{K^2} + \ldots = \frac{(K^2 - \xi^2 - \eta^2) - (\xi d\xi + nd\eta)}{\sqrt{K^2 - \xi^2 - \eta^2}} - (d\xi^2 + d\eta^2)
\]

For convenience, let \((K^2 - \xi^2 - \eta^2) = u, (\xi d\xi + nd\eta) = v,\)

\((d\xi^2 + d\eta^2) = w.\) Then

\[
1 + \frac{1}{2} \frac{(ds)^2}{K^2} + \ldots = \frac{u - v}{\sqrt{u - 2v - w}} + \ldots + \frac{(u - v)^2 - \frac{v^2 + uw}{u^2 - 2uv - uw}}{\sqrt{u - 2v - w}} + \ldots
\]

But \(v^2\) is an infinitesimal of second order; \(uw\) is also an infinitesimal of second order; \(u^2\) is finite. Therefore the fraction is an infinitesimal of second order and is \(< 1,\) so we may expand the expression by the binomial theorem. Thus

\[
1 + \frac{1}{2} \frac{(ds)^2}{K^2} + \ldots = 1 + \frac{1}{2} \frac{(v^2 + uw)}{u^2 - 2uv - uw} + \ldots
\]

Again, the expression \((u^2 - 2uv - uw)\) may be expanded, (since \(u^2\) is finite, and \((2uv + uw)\) is an infinitesimal of second order, and \(< u^3,\) giving \((u^2)^{-1}(-1)(u^2)(2uv + uw) + \ldots\))

Thence \[
1 + \frac{1}{2} \frac{(ds)^2}{K^2} = 1 + \frac{1}{2} (v^2 + uw) \left( \frac{1}{u^2} + \frac{2uv - uw}{u^3} \right) - \ldots
\]

\[
\frac{(ds)^2}{K^2} = \frac{v^2 + uw}{u^2} = \frac{(\xi d\xi + nd\eta)^2 + (K^2 - \xi^2 - \eta^2)(d\xi^2 + d\eta^2)}{(K^2 - \xi^2 - \eta^2)^2}
\]

\[
= \frac{\xi d\xi^2 + 2\xi d\xi \cdot nd\eta + n^2 d\eta^2 + K^2 (d\xi^2 + d\eta^2) - \xi^2 d\xi^2 - \eta^2 d\eta^2 - n^2 d\xi^2 - n^2 d\eta^2}{(K^2 - \xi^2 - \eta^2)^2}
\]

\[
= \frac{K^2 (d\xi^2 + d\eta^2) - (n^2 d^2 - 2n d\xi \cdot d\eta + \xi d\eta + \eta d\xi)}{(K^2 - \xi^2 - \eta^2)^2}
\]
In terms of polar coordinates, \( \xi = k \cdot \tanh\frac{P}{k} \cos \theta; \)

\( \eta = k \tanh\frac{P}{k} \sin \theta. \)

\[
\begin{align*}
\frac{\text{d} \xi}{\text{d} \theta} &= k \cdot \text{sech}\frac{P}{k} \cdot \frac{1}{k^2} \text{d}P \cdot \cos \theta - k \cdot \tanh\frac{P}{k} \cdot \sin \theta \cdot \text{d}\theta \\
\frac{\text{d} \eta}{\text{d} \theta} &= k \cdot \text{sech}\frac{P}{k} \cdot \frac{1}{k^2} \text{d}P \cdot \sin \theta - k \cdot \tanh\frac{P}{k} \cdot \cos \theta \cdot \text{d}\theta \\
\frac{\text{d} \xi}{\text{d} \eta} &= \frac{\text{sech}\left(\frac{P}{k}\right) \text{d}P \cdot \cos^{2}\theta - 2k \cdot \text{sech}\left(\frac{P}{k}\right) \text{d}P \cdot \tanh\frac{P}{k} \cdot \sin \theta \cdot \cos \theta \cdot \text{d}\theta + k^2 \tanh^2\left(\frac{P}{k}\right) \sin^2 \theta \cdot \text{d}\theta^2}{\text{sech}\left(\frac{P}{k}\right) \text{d}P \cdot \sin^2 \theta + 2k \cdot \text{sech}\left(\frac{P}{k}\right) \text{d}P \cdot \tanh\frac{P}{k} \cdot \sin \theta \cdot \cos \theta \cdot \text{d}\theta + k^2 \tanh^2\left(\frac{P}{k}\right) \cos^2 \theta \cdot \text{d}\theta^2}
\end{align*}
\]

Add; \( \frac{\text{d} \xi}{\text{d} \eta} + \frac{\text{d} \eta}{\text{d} \xi} = \text{sech}\left(\frac{P}{k}\right) \text{d}P \cdot \cos^2 \theta + k^2 \tanh^2\left(\frac{P}{k}\right) \text{d}\theta \)

\[
\frac{\text{d} \xi}{\text{d} \eta} = k \cdot \tanh\left(\frac{P}{k}\right) \cdot \text{sech}\left(\frac{P}{k}\right) \text{d}P \cdot \sin \theta \cdot \cos \theta - k^2 \tanh^2\left(\frac{P}{k}\right) \sin^2 \theta \cdot \text{d}\theta
\]

Subtract; \( \eta \frac{\text{d} \xi}{\text{d} \eta} - \frac{\text{d} \xi}{\text{d} \eta} = -k^2 \tanh^2\left(\frac{P}{k}\right) \text{d}\theta \)

\[
\begin{align*}
\eta \frac{\text{d} \xi}{\text{d} \eta} - \frac{\text{d} \xi}{\text{d} \eta} &= k^4 \tanh^2\left(\frac{P}{k}\right) \text{d}\theta^2 , \quad \frac{\text{d} \xi}{\text{d} \eta} = k^2 \tanh^2\left(\frac{P}{k}\right) \cos^2 \theta , \\
\eta^2 &= k^2 \tanh^2\left(\frac{P}{k}\right) \sin^2 \theta; \quad (\xi^2 + \eta^2) = k^2 \tanh^2\left(\frac{P}{k}\right), \quad k^2 \xi^2 - \eta^2 = k^2 - k^2 \tanh^2\left(\frac{P}{k}\right) = k^2 \text{sech}^2\left(\frac{P}{k}\right)
\end{align*}
\]

\[
ds = \frac{k \sqrt{k^2 \left( \text{sech}^2\left(\frac{P}{k}\right) \text{d}P^2 + k^2 \tanh^2\left(\frac{P}{k}\right) \text{d}\theta^2 \right)^2 - K^4 \tanh^4\left(\frac{P}{k}\right) \text{d}\theta^2}}{k^2 \text{sech}^2\left(\frac{P}{k}\right)}
\]

\[
ds = \frac{k^2 \left( \text{sech}^2\left(\frac{P}{k}\right) \text{d}P^2 + k^2 \tanh^2\left(\frac{P}{k}\right) \text{d}\theta \right)^2}{k^2 \text{sech}^2\left(\frac{P}{k}\right)}
\]
\[
= \sqrt{\text{sech}^2 \frac{p^2}{k} \cdot \text{sech}^2 \frac{p}{k} \cdot \text{sech}^2 \frac{p^2}{k} \cdot \text{sech}^2 \frac{p}{k}}
\]

whence the circumference of the circle \( p = a \), since \( da^2 = 0 \), is
\[
C = \int \text{d} \theta = \int_{0}^{2\pi} k \sinh \frac{a}{k} \text{d} \theta
\]

but sinh \( \frac{a}{k} = \frac{e^a - e^{-a}}{2} \).

Thence \( C = k \pi (e^a - e^{-a}) \).

3. To apply the formula for the angle \( \phi \) between two straight lines to the Lobachevskian geometry, in (5) substitute
\[
\cos \alpha = \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin \alpha = \frac{B}{\sqrt{A^2 + B^2}}, \quad m = \frac{1}{k};
\]

Thence
\[
\cos \phi = \frac{A_1 A_2}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} + \frac{B_1 B_2}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} + \tan \frac{p}{k} \tan \frac{p}{k}
\]

Multiply both numerator and denominator by \((-ik)\),

\[
\cos \phi = \frac{A_1 A_2}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} + \frac{B_1 B_2}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} + (-ik \tan \frac{p}{k} \chi - ik \tan \frac{p}{k})
\]

But \((-ik \cdot \tan \frac{p}{k}) = k \tanh \frac{p}{k} = \frac{-c}{\sqrt{A^2 + B^2}}\),

so
\[
\cos \phi = \frac{-k^2(A_1 A_2 + B_1 B_2)}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} + \frac{C_1 C_2}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} + \frac{C_1 C_2}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}}
\]

\[
= \frac{1}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} \left( k^2(A_1^2 + B_1^2) - C_1^2 \right) \left( k^2(A_2^2 + B_2^2) - C_2^2 \right)
\]

\[
\therefore \cos \phi = \frac{k^2(A_1 A_2 + B_1 B_2) + C_1 C_2}{k^2(A_1^2 + B_1^2) - C_1^2} \frac{k^2(A_2^2 + B_2^2) - C_2^2}{k^2(A_1^2 + B_1^2) - C_1^2} (14)
\]
In the Riemannian geometry, the equation for the straight line becomes \( \cos - \sin = k \tan \frac{D}{k} \); \( (15) \)

for distance between points

\[
\cos \frac{PP}{K} = \frac{K^2 + z_1 z_2 + \eta_1 \eta_2}{\sqrt{K^2 + z_1^2 + \eta_1^2} \sqrt{K^2 + z_2^2 + \eta_2^2}} \quad (16)
\]

and for the angle \( \phi \), \( \cos \phi = \frac{k^2 (A_1 A_2 + B_1 B_2 + C_1 C_2)}{\sqrt{k^2 (A_1^2 + B_1^2)} + C_1^2 \sqrt{k^2 (A_2^2 + B_2^2)} + C_2^2} \) \( (17) \)

**SUMMARY OF FORMULAS**

1. In an I.B.Q. whose summit is CD, and base, AB, and whose side AC is a given length "a", \( \lim_{CD \to AB} \frac{CD}{AB} = \cos \frac{ma}{k} \), hence in L.G. where \( m = \frac{1}{k} \), \( \lim_{CD \to AB} \frac{CD}{AB} = \cos \frac{1a}{k} = \cosh \frac{a}{k} \) \( (1) \)

2. If ABC be a triangle with a right angle at C, and \( AB = c, AC = b, BC = a \), for the L.G. we have

\[
\cosh \frac{c}{k} = \cosh \frac{a}{k} \cosh \frac{b}{k} \quad (2)
\]

\[
\sinh \frac{a}{k} = \sinh \frac{c}{k} \sin A \quad (3)
\]

\[
\tanh \frac{a}{k} = \tanh \frac{c}{k} \cos B \quad (4)
\]

\[
\cos A = \cosh \frac{a}{k} \sin B \quad (5)
\]

\[
\sinh \frac{b}{k} = \sinh \frac{c}{k} \sin B \quad (6)
\]

\[
\tanh \frac{b}{k} = \tanh \frac{c}{k} \cos A \quad (7)
\]

\[
\cos B = \cosh \frac{b}{k} \sin A \quad (8)
\]

3. If ABC be any triangle with vertices A, B, and C, and the opposite sides \( a, b, \) and \( c \), respectively,
\[
\cos ma = \cos mc \cdot \cos mb - \sin mb \cdot \sin mc \cdot \cot A,
\]
hence in the L.G. where \(m = \frac{1}{k}\),
\[
\cosh \frac{a}{k} = \cosh \frac{c}{k} \cdot \cosh \frac{b}{k} - \sinh \frac{c}{k} \cdot \cot A. \quad (9)
\]

4. If we use new coordinates \(\xi, \eta\), in place of \(x, y\), where \(\xi = -ik \tan \frac{1x}{k} = k \tanh \frac{x}{k}\), and \(\eta = -ik \tan \frac{1y}{k}\) = \(k \tanh \frac{y}{k}\), the equation of the straight line in L.B.

becomes
\[\xi \cos \alpha + \eta \sin \alpha = k \tanh \frac{p}{k}\]

or \(a \xi + b \eta + c = 0\) \(\quad (10)\)

where \(\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}\), \(\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}\) and
\[
k \tanh \frac{p}{k} = \frac{-c}{\sqrt{a^2 + b^2}}. \quad (12)
\]

5. Distance between points \(P_1: (\xi_1, \eta_1)\) and \(P_2: (\xi_2, \eta_2)\)
is measured by \(\cosh \frac{P_1P_2}{k} = \frac{K^2 - \xi_1 \xi_2 - \eta_1 \eta_2}{\sqrt{K^2 - \xi_1^2 - \eta_1^2 \cdot \sqrt{K^2 - \xi_2^2 - \eta_2^2}}} \quad (13)\)

6. The angle \(\phi\) between two lines is measured by
\[
\cos \phi = \frac{K^2(a_1 a_2 + b_1 b_2) - c_1 c_2}{\sqrt{K^2(a_1^2 + b_1^2) - c_1^2} \cdot \sqrt{K^2(a_2^2 + b_2^2) - c_2^2}} \quad (14)
\]

7. Let \(P(\xi, \eta)\) be any point on a Lobachevskian plane, \((r, \theta)\) its polar coordinates, where \(r\) is always positive.

Then
\[
\begin{aligned}
\xi &= k \tanh \frac{r}{k} \cos \theta \\
\eta &= k \tanh \frac{r}{k} \sin \theta \\
\xi^2 + \eta^2 &= k^2 \tanh^2 \frac{r}{k} < k^2
\end{aligned}
\]
\( \xi \) and \( \eta \) may be taken as ordinary Cartesian coordinates upon a Euclidean plane, i.e.; a plane on which the Euclidean geometry is assumed to hold. Then to \( P \) on the Lobachevskian plane corresponds a point \( P' \) on the Euclidean plane, and \( P' \) lies inside the circle \( \xi^2 + \eta^2 = k \), called the fundamental circle.

Conversely, if \((\xi', \eta')\) be the coordinates of any point on the Euclidean plane,

\[
\begin{align*}
\cos \Theta &= \frac{\xi}{\sqrt{\xi^2 + \eta^2}} \\
\sin \Theta &= \frac{\eta}{\sqrt{\xi^2 + \eta^2}} \\
r &= \frac{k}{2} \log \frac{k + \sqrt{k^2 - \xi^2 - \eta^2}}{k - \sqrt{k^2 - \xi^2 - \eta^2}}
\end{align*}
\]

\( 16 \)

**SOME APPLICATIONS OF THE LOBACHEVSKIAN GEOMETRY**

1. **Conic Sections:** The path of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed line is called a conic section. The fixed point is called the focus of the conic, the fixed line the directrix, and the constant ratio the eccentricity, \( e \).

If \( e < 1 \), the conic is an ellipse; if \( e = 1 \), the conic is a parabola, and if \( e > 1 \), the conic is a hyperbola.
General equation; Let the focus of a conic be \((1,j)\) and the directrix \((\lambda)\) the line \(x \cos \beta + y \sin \beta = p\) (Fig. 21). The distance of the moving point \((x,y)\) from the line \((\lambda)\) is \(x \cos \beta + y \sin \beta - p\). Hence the equation of the conic is \(\sqrt{(x - 1)^2 + (y - j)^2} = e(x \cos \beta + y \sin \beta - p)\) or \((x - 1)^2 + (y - j)^2 = e^2(x \cos \beta + y \sin \beta - p)^2\).

If in this equation we substitute for \(x, y,\) and \(p\) the values \(\xi, \eta,\) and \(k \tanh \frac{p}{k}\) respectively the equation becomes, in the L.G.

\[(\xi - 1)^2 + (\eta - j)^2 = e(\xi \cos \beta + \eta \sin \beta - k \tanh \frac{p}{k})^2\]  

(1)

Polar equation; To find the equation of a conic in polar coordinates, let us take the focus as pole, and the line through the focus perpendicular to the directrix as polar axis. (Fig. 21) Denote by \(2\alpha\) the length of \(Q_1Q_2\) (the latus rectum); then the distance \(OB\) from the focus to the directrix \(\lambda\) is \(\frac{\alpha}{e}\), for the point \(Q_1\) is a point on the curve. Hence by the definition

\[OQ_1 = e \cdot Q_1 M\]

so that \[OB = Q_1 M = \frac{OQ_1}{e} = \frac{\alpha}{e}\].

Now assume a point \(P\) \((r, \theta)\) in a general position on the curve, and drop a perpendicular \(PL\) to the directrix.
Since \( OP = e \cdot PL \), \( PL = \frac{r}{e} \).

Evidently \( ON + PL = OB \),

or \( r \cos \theta + \frac{r}{e} = \frac{d}{e} \), or \( r = \frac{d}{1 + e \cos \theta} \).

Making substitutions in the above from formula (16) of the summary, we have in L.G. the polar equation of the conic:

\[
\frac{k}{2} \log \frac{r + \sqrt{r^2 + \eta^2}}{r - \sqrt{r^2 + \eta^2}} = \frac{d}{1 + \frac{\sqrt{r^2 + \eta^2}}{\sqrt{E + \eta^2}}}.
\]

or \( \frac{k}{2} \log \frac{r + \sqrt{r^2 + \eta^2}}{r - \sqrt{r^2 + \eta^2}} = \frac{d}{\sqrt{E + \eta^2}} \) (2)

2. The ellipse: The ellipse has been defined as the conic section for which \( e < 1 \). To obtain the cartesian equation of the curve, let us take as focus (Fig. 22) the point \((ae, 0)\), and as directrix the line \( x = \frac{a}{e} \) (the point \( F_1 \) and line of Fig. 22) where \( a \) is a constant.

With these assumptions the defining equation \( p = ed \) becomes

\[
\sqrt{(x - ae)^2 + y^2} = e\left(\frac{a}{e} - x\right),
\]

or \( x - 2aex + a^2 - y^2 = a^2 - 2ae x + e^2 x^2 \),

\[
x^2(1 - e^2) + y^2 = a^2(1 - e^2),
\]

or \( \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \). Let \( b^2 = a^2(1 - e^2) \).
and the equation becomes \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). (Since \( e < 1 \), \( b^2 \) is positive, and \( b \) is real).

To obtain the equation of the ellipse in L.G., substitute in the above, \( \xi \) and \( \eta \) respectively for \( x \) and \( y \), where \( \xi = k \tanh \frac{x}{k} \) and \( \eta = k \tanh \frac{y}{k} \), and the equation becomes

\[
\frac{k^2 \tanh^2 \frac{x}{k}}{a^2} + \frac{k^2 \tanh^2 \frac{y}{k}}{b^2} = 1
\]

or \( k^2 (b^2 \tanh^2 \frac{x}{k} + a^2 \tanh^2 \frac{y}{k}) = a^2 b^2 \) \hspace{1cm} (3)

Center at a point \((m,n)\); The equation

\[
\frac{(x - m)^2}{a^2} + \frac{(y - n)^2}{b^2} = 1
\]

becomes in L.G. \( \frac{(k \tanh \frac{x}{k} - m)^2}{a^2} + \frac{(k \tanh \frac{y}{k} - n)^2}{b^2} = 1 \). \hspace{1cm} (4)

Polar form; Where \( \mathbf{P} = \{x = r \cos \Theta, y = r \sin \Theta\} \)

\( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) becomes \( \frac{r^2 \cos^2 \Theta}{a^2} + \frac{r^2 \sin^2 \Theta}{b^2} = 1 \).

If \((\xi, \eta)\) be the coordinates of the point \( P \) on the Euclidean plane, by formulas (16) the equation becomes

\[
\left[ \frac{k}{\log \frac{k + \sqrt{\xi^2 + \eta^2}}{k - \sqrt{\xi^2 + \eta^2}}} \right]^2 \cdot \frac{\xi^2}{\xi^2 + \eta^2} = \left[ \frac{k}{\log \frac{k - \sqrt{\xi^2 + \eta^2}}{k + \sqrt{\xi^2 + \eta^2}}} \right]^2 \cdot \frac{\xi^2}{\xi^2 + \eta^2} = 1
\]

or \( \left[ \frac{k}{\log \frac{k - \sqrt{\xi^2 + \eta^2}}{k + \sqrt{\xi^2 + \eta^2}}} \right]^2 \cdot \frac{\xi^2}{\xi^2 + \eta^2} = 1 \) \hspace{1cm} (5)
Tangent to the ellipse at a given point of contact:

In the Euclidean geometry, if the given point is \((x_1, y_1)\) the tangent at this point on the curve is found by the formula \(m = -\frac{b^2 x_1}{a^2 y_1}\). In L.G. the corresponding formula becomes \(m = -\frac{b^2 \xi_1}{a^2 \eta_1}\), and the equation of the tangent is
\[
\frac{\xi}{a^2} + \frac{\eta}{b^2} = 1
\]  

Normal to the ellipse at a given point: Since the slope of the normal at any point is the negative reciprocal of the slope of the tangent, for the normal, at the point \((x_1, y_1)\) the slope \(m_1 = \frac{a^2 y_1}{b^2 x_1}\), and the equation of the normal becomes in L.G. \(\eta - \eta_1 = \frac{a^2 \eta_1}{b^2 \xi_1} (\xi - \xi_1)\).
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