A STUDY OF PHASE ANGLE ROOT LOCI TECHNIQUES

by

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1.1 Introduction

In the frequency domain study of feedback control systems, the Nyquist Criterion, the Bode Plot, and the Nichols Chart are very powerful tools of analysis. These techniques lead to the design of a feedback control systems from the frequency domain point of view with the system performance specifications given in terms of gain margin, phase margin, resonant peak, bandwith, etc.

However, by narrowing the region of analysis from the entire s-plane to the j -axis in the above mentioned techniques, the capability of maintaining control over both frequency and time domain responses is discarded.

During the last two decades, much work has been done on the development of design techniques which simultaneously control both frequency and time domain responses of linear feedback control systems. The most basic and important contributions were accomplished by:

1). Wiener (ref. 9), in his presentation of a statistical design method, considers the actual input signals to a system as described in terms of statistical average properties;

2). Guillemin (ref. 8), who applied the concepts of network synthesis to the synthesis of feedback control systems thus forcing the system to meet specifications in both the frequency and time domains; and

3). Evans (ref. 7), in his development of the root-locus method, adopted one of the basic viewpoints of the frequency domain in attempting to modify the open-loop system characteristics. Evans' work considers both the frequency and time domains.

The root locus method was first introduced by Evans in 1948, and has been greatly developed in the past two decades. In his technique, the Laplace transform and complex function theory are the basic tools and design is guided by the behavior of variation of the closed-loop transfer function poles with system parameters. The root-locus method can be used to adjust system gain, guide the design of compensation networks as need to satisfy a given set of specifications on transient or steady state performance of the system.

Although the conventional root-locus method seems to be a very good tool in analyzing feedback control systems, it is found that a more general and systematic technique of analysis can be derived from Evans' root-locus method. This is the Phase Angle Root Loci method of analyzing feedback control systems in the entire s-plane. By this technique, which combines the Phase Angle Root Loci and Constant Gain Root Loci, a more systematic and clear technique for designing a feedback control system and in reshaping the conventional root-locus to satisfy certain desirable specifications can be developed. This report is mainly dedicated to the study of this technique.

1.2 Phase Angle Root Loci (PARL) ----Definition

From the open loop transfer function G(s)H(s) of a given system, it is possible to obtain a family of loci by letting the phase angle of G(s)H(s) be equal to a constant angle. This family of loci on the s-plane is called the Phase Angle Root Loci (PARL) of the transfer function, G(s)H(s).

In a linear feedback control system without transportation lag, the open loop transfer function is a rational algebraic function of s which can be written

$$G(s)H(s) = K \cdot \frac{(s+z_1)(s+z_2)(s+z_3)\ \cdots\cdots\ (s+z_m)}{(s+p_1)(s+p_2)(s+p_3)\ \cdots\cdots\ (s+p_n)} \tag{1.1}$$

Consider the following expression where ϕ is treated as a parameter

$$G(s)H(s) = K \cdot \frac{(s+z_1)(s+z_2)(s+z_3)\ \cdots\cdots\ (s+z_m)}{(s+p_1)(s+p_2)(s+p_3)\ \cdots\cdots\ (s+p_n)} = e^{j\phi} \tag{1.2}$$

Those z_i's and p_i's are zeros and poles of the open-loop system.

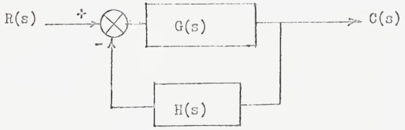


Fig. 1.1 A Feedback Control System

Therefore

$$\phi = \mathrm{Arg}\,(G(s)H(s)) = \mathrm{Arg}\left(K \cdot \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{k=1}^{n}(s+p_k)} \right)$$

$$= \mathrm{Arg} \prod_{i=1}^{m}(s+z_i) - \mathrm{Arg} \prod_{k=1}^{n}(s+p_k) \tag{1.3}$$

or

$$\phi = \sum_{i=1}^{m} \mathrm{Arg}\,(s+z_i) - \sum_{k=1}^{n} \mathrm{Arg}\,(s+p_k) \tag{1.4}$$

By definition, the PARL (Phase Angle Root Loci) are constructed by setting ϕ equal to a constant and varying the open-loop gain K in such a manner as to always satisfy Eq. (1.2).

Furthermore, it can be shown that the PARL of G(s)H(s) can be determined by superposition techniques. It is worthwhile to point out that the conventional root locus of Evans (ref. 7) is only one member of the family of loci that comprise the PARL. It is obtained by setting $\phi = \pm 180°$.

That is, the conventional root locus is derived from

$$G(s)H(s) = e^{\pm j\pi} = -1 \tag{1.5}$$

or

$$1 + G(s)H(s) = 0 \tag{1.6}$$

Equation (1.6) is the conventional characteristic equation for a feedback control system.

1.3 Constant Gain Root Loci (CGRL)

For any open-loop transfer function, G(s)H(s), Eq. (1.2) can be used to obtain a family of constant gain contours. For each member of this family the gain K is fixed and ϕ is allowed to vary.

The equation

$$G(s)H(s) = e^{j\phi}$$

implies

$$|G(s)H(s)| = 1$$

or, referring to Eq. (1.2) again, it is found that

$$K \cdot \frac{\prod_{i=1}^{m} |s+z_i|}{\prod_{k=1}^{n} |s+p_k|} = 1 \qquad \text{for} \quad 0 < K \leqslant \infty$$

or

$$K = \frac{\prod_{k=1}^{n} |s+p_k|}{\prod_{i=1}^{m} |s+z_i|} \tag{1.7}$$

It is interesting that the Constant Gain Root Loci in the s-plane depend only on the relative locations of poles and zeros of the open-loop transfer function and has nothing to do with the PARL. It can be shown that the PARL are orthogonal to the CGRL. (ref. 1,2,& 3). A few examples follows.

Example 1. $G(s)H(s) = K \cdot \frac{1}{s+2}$

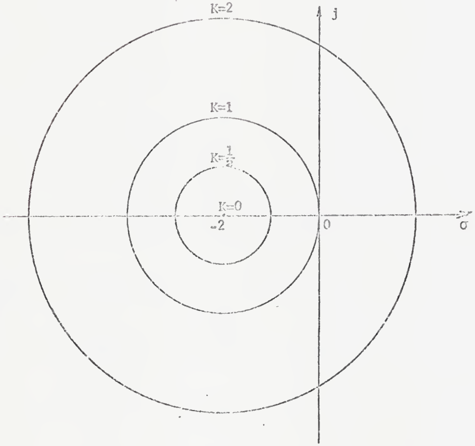


Fig. 1.2 CGRL for $G(s)H(s) = \frac{K}{s+2}$

Example 2. $G(s)H(s) = \frac{K}{s(s+1)(s+4)}$

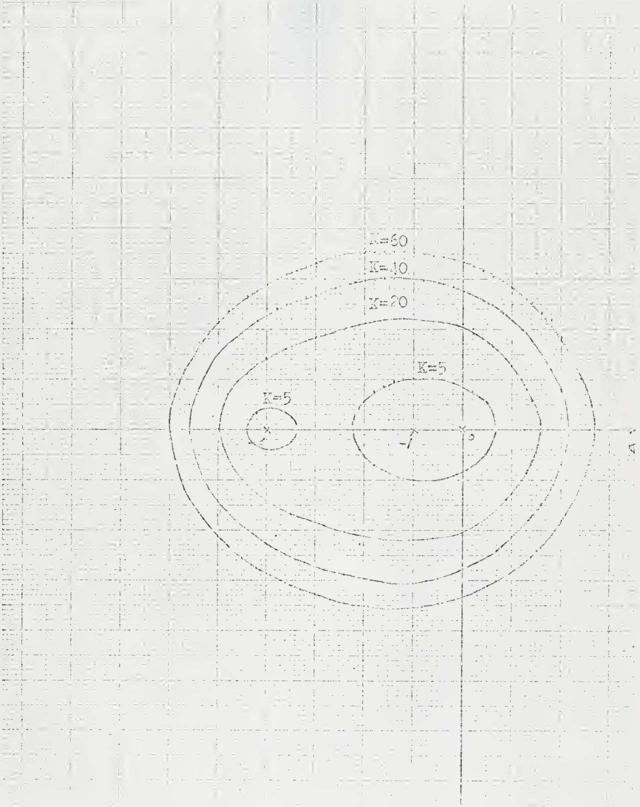


Fig. 1-5. [illegible] plot of $G(s) = \dfrac{K}{s(s+1)(s+4)}$.

II. THE CONSTRUCTION OF PHASE ANGLE ROOT LOCI

2.1 Rules of Construction of PARL

It was found that a set of rules for constructing PARL exists. These rules, in any case, should be regarded only as an aid to the construction of the PARL; they do not provided the exact plot. They are given as follows:

1). The starting points of the PARL. $(K = 0)$

The PARL start at the poles of $G(s)H(s)$.

Proof:

The PARL are considered to start at the points where the open-loop gain K is zero.

From equation (1.2) it can be found that

$$K \cdot \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{k=1}^{n}(s+p_k)} = e^{j\phi} \qquad (2.1)$$

Taking the absolute value of both sides of Eq. (2.1)

$$K \cdot \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{k=1}^{n}|s+p_k|} = |e^{j\phi}| \qquad (2.2)$$

where only positive values of K are considered, this last equation can be rewritten as

$$\frac{\prod_{i=1}^{n}|s+z_i|}{\prod_{k=1}^{n}|s+p_k|} = \frac{1}{K} \tag{2.3}$$

As K^{-1} approaches infinity which implies that s approaches the poles of G(s)H(s); that is, s approaches p_k.

2). The terminal points of the PARL. ($K = \infty$)

The PARL terminates at the zeros of G(s)H(s).

Proof:

The PARL are considered to end at the points where K becomes infinite. With reference to equation (2.3), as K approaches infinity, the value of the equation approaches zero, which requires that s must approaches to the zeros of G(s)H(s); that is: s approaches z_i.

3). Number of separate branches for each phase angle.

Let

N = Number of separate branches for a given phase angle.

Z = Number of finite zeros of G(s)H(s).

P = Number of finite poles of G(s)H(s).

then N = Max(Z, P).

Apparently there must be as many separate branches for each given phase angle, as the larger value of Z and P, since the loci must start at the poles and end at the zeros of G(s)H(s).

4). Mirror image property of the PARL with respect to the real axis in the s-plane.

The portion of the PARL corresponding to a phase angle of $-\phi$ and the portion of the [illegible] corresponding to a phase of $+\phi$ are mirror image of one another with respect to the real axis in the s-plane.

Proof:

From equation (1.4)

$$\phi = \sum_{i=1}^{m} \mathrm{Arg}(s+z_i) - \sum_{k=1}^{n} \mathrm{Arg}(s+p_k) = \mathrm{Arg}(\,G(s)H(s)\,) \qquad (2.4)$$

Now let

r = number of finite real zeros of $G(s)H(s)$.

v = number of pairs of complex conjugate zeros of $G(s)H(s)$.

t = number of finite real poles of $G(s)H(s)$.

q = number of pairs of complex conjugate poles of $G(s)H(s)$.

It is evidently true that $2v + r = m$ and $2q + t = n$.

With

$$s = \sigma + j\omega, \quad z_i = \alpha_i + j\beta_i, \quad p_k = \gamma_k + j\delta_k$$

equation (2.4) can be rewritten as

$$\begin{aligned}
\mathrm{Arg}(\,G(s)H(s)\,) = & \sum_{i=1}^{v} \mathrm{Arg}(\sigma + j\omega + \alpha_i + j\beta_i) \\
& + \sum_{i=1}^{v} \mathrm{Arg}(\sigma + j\omega + \alpha_i - j\beta_i) \\
& + \sum_{i=v+1}^{m} \mathrm{Arg}(\sigma + j\omega + \alpha_i) \\
& - \sum_{k=1}^{q} \mathrm{Arg}(\sigma + j\omega + \gamma_k + j\delta_k) \\
& - \sum_{k=1}^{q} \mathrm{Arg}(\sigma + j\omega + \gamma_k - j\delta_k) \\
& - \sum_{k=q+1}^{n} \mathrm{Arg}(\sigma + j\omega + \gamma_k).
\end{aligned}$$

or

$$\phi = \sum_{i=1}^{v} \tan^{-1}\left(\frac{\omega+\beta_i}{\sigma+\alpha_i}\right) + \sum_{i=1}^{v} \tan^{-1}\left(\frac{\omega-\beta_i}{\sigma+\alpha_i}\right) + \sum_{i=v+1}^{m} \tan^{-1}\left(\frac{\omega}{\sigma+\alpha_i}\right)$$

$$- \sum_{k=1}^{q} \tan^{-1}\left(\frac{\omega+\delta_k}{\sigma+\gamma_k}\right) - \sum_{k=1}^{q} \tan^{-1}\left(\frac{\omega-\delta_k}{\sigma+\gamma_k}\right) - \sum_{k=q+1}^{n} \tan^{-1}\left(\frac{\omega}{\sigma+\gamma_k}\right)$$

so

$$\phi = \sum_{i=1}^{v} \tan^{-1} \frac{\dfrac{\omega+\beta_i+\omega-\beta_i}{\sigma+\alpha_i}}{1 - \dfrac{\omega^2-\beta_i^2}{(\sigma+\alpha_i)^2}} + \sum_{i=v+1}^{m} \tan^{-1} \frac{\omega}{\sigma+\alpha_i}$$

$$- \sum_{k=1}^{q} \tan^{-1} \frac{\dfrac{\omega+\delta_k+\omega-\delta_k}{\sigma+\gamma_k}}{1 - \dfrac{\omega^2-\gamma_k^2}{(\sigma+\gamma_k)^2}} - \sum_{k=q-1}^{n} \tan^{-1} \frac{\omega}{\sigma+\gamma_k}$$

or

$$\phi = \sum_{i=1}^{v} \tan^{-1} \frac{\omega^2 \ (\sigma+\alpha_i)}{(\sigma+\alpha_i)^2 - (\omega^2-\beta_i^2)} + \sum_{i=v+1}^{m} \tan^{-1} \frac{\omega}{\sigma+\alpha_i}$$

$$- \sum_{k=1}^{q} \tan^{-1} \frac{\omega^2 \ (\sigma+\gamma_k)}{(\sigma+\gamma_k)^2 - (\omega^2-\delta_k^2)} - \sum_{k=q+1}^{n} \tan^{-1} \frac{\omega}{\sigma+\gamma_k} \qquad (2.5)$$

Since $\phi(\sigma,\omega)$ is an odd function in ω, as can be concluded from Eq. (2.5), $\phi(\sigma,\omega)$ will change sign when the PARL pass through the real axis. That is $\phi(\sigma,\omega) = -\phi(\sigma, -\omega)$, and the two portions of PARL under consideration are mirror images of one another with respect to the real axis.

5). Asymptotes of PARL.

For very large values of s, the PARL are asymptotic to straight lines with slope angles given by

$$\theta_k = \frac{2k\pi - \phi}{P - Z}$$

where $k = 0, 1, 2, 3, \ldots\ldots\ldots\ldots (P - Z)$.

Proof:

The general form of the open-loop transfer function can be written as

$$G(s)H(s) = K \cdot \frac{s^m + a_1 s^{m-1} + a_2 s^{m-2} + \ldots\ldots\ldots\ldots + a_m}{s^{m+p} + b_1 s^{m+p-1} + b_2 s^{m+p-2} + \ldots\ldots\ldots + b_{m+p}}$$

$$= K \cdot \frac{1}{\dfrac{s^{m+p} + b_1 s^{m+p-1} + b_2 s^{m+p-2} + \ldots\ldots\ldots + b_{m+p}}{s^m + a_1 s^{m-1} + a_2 s^{m-2} + \ldots\ldots\ldots\ldots + a_m}}$$

$$= K \cdot \frac{1}{s^p + (b_1 - a_1)s^{p-1} + \ldots\ldots\ldots\ldots + \dfrac{R(s)}{P(s)}}$$

where $p = P - Z$, and $R(s)$ is a polynomial in s with degree less than m, and

$$P(s) = s^m + a_1 s^{m-1} + \ldots\ldots\ldots\ldots\ldots\ldots + a_m$$

The PARL is constructed by setting

$$\frac{K}{s^p + (b_1 - a_1)s^{p-1} + \cdots\cdots\cdots\cdots + \frac{R(s)}{P(s)}} = e^{j\phi} \tag{2.6}$$

or

$$s^p + (b_1 - a_1)s^{p-1} + \cdots\cdots\cdots\cdots + \frac{R(s)}{P(s)} = Ke^{-j\phi} \tag{2.7}$$

As s becomes very large, the last term R(s)/P(s) becomes very small and relatively not significant when compared with the other terms. Only the first two terms of Equation (2.7) are considered significant. This approximation leads to

$$s^p + (b_1 - a_1)s^{p-1} = Ke^{-j\phi} \tag{2.8}$$

for very large values of s. And (2.8) can be written as

$$s\,(1 + \frac{b_1 - a_1}{s})^{\frac{1}{p}} = |K|^{\frac{1}{p}} \cdot e^{j(2k\pi - \phi)/p} \tag{2.9}$$

Using the binomial theorem the factor

$$(1 + \frac{b_1 - a_1}{s})^{\frac{1}{p}}$$

is expanded into an infinite series, with the result

$$s\,(1 + \frac{b_1 - a_1}{ps} + \cdots\cdots) = |K|^{\frac{1}{p}} \cdot e^{j(2k\pi - \phi)/p} \tag{2.10}$$

which for very large values of s becomes

$$s + \frac{b_1 - a_1}{p} = |K|^{\frac{1}{p}} \cdot e^{j(2k\pi - \phi)/p} \tag{2.11}$$

Substituting $s = \sigma + j\omega$ into Eq. (2.11) yields

$$\sigma + \frac{b_1 - a_1}{p} + j\omega = |K|^{\frac{1}{p}} \cdot \cos\frac{2k\pi - \phi}{p} + j\sin\frac{2k\pi - \phi}{p} \tag{2.12}$$

Equating the real and imaginary parts of Equation (2.12) yields

$$\sigma + \frac{b_1 - a_1}{p} = |K|^{\frac{1}{p}} \cdot \cos\frac{2k\pi - \phi}{p} \tag{2.13}$$

$$= |K|^{\frac{1}{p}} \cdot \sin\frac{2k\pi - \phi}{p} \tag{2.14}$$

with $k = 0, 1, 2, 3, \ldots\ldots\ldots, (P - Z)$.

Solving for $K^{\frac{1}{p}}$ from the last two equations leads to

$$|K|^{\frac{1}{p}} = \frac{\omega}{\sin\frac{2k\pi - \phi}{p}} = \frac{\sigma + \frac{b_1 - a_1}{p}}{\cos\frac{2k\pi - \phi}{p}} \tag{2.15}$$

and solving for ω yields

$$= \tan\left(\frac{2k\pi - \phi}{p}\right) \cdot \left[\sigma + \frac{b_1 - a_1}{p}\right] \tag{2.16}$$

Equation (2.16) represents a straight line in the s-plane. It is of the form

$$\omega = m(\sigma - \sigma_1)$$

where m is the slope and σ_1 is the intercept on the σ-axis. Thus

$$m = \tan\frac{2k\pi - \phi}{p} = \tan\frac{2k\pi - \phi}{P - Z} \tag{2.17}$$

and

$$\sigma_1 = -\frac{b_1 - a_1}{P - Z} \tag{2.18}$$

where $k = 0, 1, 2, 3, \ldots\ldots\ldots\ldots, (P - Z)$.

6). Intersection of PARL asymtotes on real axis.

a). The intersections of the asymptotes lie on the real axis.

b). The intersection of the asymptotes on the real axis is given by

$$\sigma_1 = \frac{\sum \text{Poles of } G(s)H(s) - \sum \text{Zeros of } G(s)H(s)}{P - Z} \tag{2.19}$$

Proof:

a). From the proven property that the $+\phi$ and the $-\phi$ loci are mirror images of one another with respect to the real axis and the phase angle of the loci changes sign after crossing the σ-axis, it is clear that the intersection of asymtotes must lie on the real axis.

b). This statement follows directly from Equation (2.18), since

$-b_1$ = sum of the roots of the denominator polynomial of $G(s)H(s)$, or sum of the poles of $G(s)H(s)$.

$-a_1$ = sum of the roots of the numerator polynomial of $G(s)H(s)$, or sum of the zeros of $G(s)H(s)$

therefore

$$\sigma_1 = -\left(\frac{b_1 - a_1}{P - Z}\right)$$

$$= \frac{\sum \text{Poles of } G(s)H(s) - \sum \text{Zeros of } G(s)H(s)}{P - Z}$$

7). PARL on real axis.

There exists no loci on the real axis except the loci with phase angle $\phi = \pm k\pi$, and k is a certain defined constant. On a given section of the real axis, PARL exists only for $\phi = \pm k\pi$, with k equal to the resultant of the number of zeros minus the number of poles on the real axis to the right of the given section.

A simple example will illustrate this clearly. A certain feedback control system has the open-loop transfer function:

$$G(s)H(s) = \frac{K(s^2 + 4s + 5)(s + 3)}{s(s+4)(s+5)(s+6)(s^2+2s+10)}$$

As was shown in Fig. 2.1, at any point s in the given section of real axis between poles $s = -4$ and $s = -5$, the phasors from the conjugate poles and zeros have phase angle of equal magnitude but opposite signs, therefore these poles and zeros contribute nothing to the phase angle of $G(s)H(s)$. It is quite evident that those real poles or zeros located to the left of the given region contribute zero phase to the phase angle of $G(s)H(s)$.

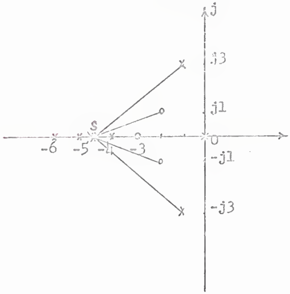


Fig. 2.1 PARL on the real axis.

If it is assumed

rp = number of poles to right of the given region

rz = number of zeros to right of the given region

then at any point of the given region of real axis

$$\mathrm{Arg}\,(\,G(s)H(s)\,) \;=\; \pm(rz - rp)\pi$$

Thus it can be concluded that any portion of the PARL which exists in the given region must have a constant phase angle $\pm(rz - rp)\pi$.

8). Breakaway points for PARL on real axis.

Breakaway points on real axis exist only for $\phi = \pm k\pi$, in this case the PARL must approach and leave a breakaway point on the real axis at an angle of $180^\circ/n$ apart, when n is the total number of loci approaching and leaving the point.

Proof: See ref. 6, pages 258-265.

9). Angles of departure (from poles) and angles of arrival (to zeros) of the PARL.

The angles of departure from poles and angles of arrival at zeros can be determined readily from the relation

$$\phi = \text{Arg}\,(\,G(s)H(s)\,) = \sum_{i=1}^{m} \text{Arg}\,(s + z_i) - \sum_{k=1}^{n} \text{Arg}\,(s + p_k)$$

for each fixed value of ϕ.

For instance, consider the pole-zero configuration given in Fig. (2.2). It is desired to determined the angle at which the PARL with $\phi = \pm 60°$ leaves the pole at $s = -4 + j1$. A test point s_1 is selected such that s_1 is only slightly displaced from the pole. The angles contributed by all critical frequencies except the pole at $-4 + j1$ are determined approximately by the phasors from those poles and zeros to $-4 + j1$. The single angle contributed by the pole at $-4 + j1$ is then just sufficient to make the total phase angle at the test point equal to ϕ as shown in Fig (2.2).

In the example case under consideration the phase angle of the PARL is assumed to be 60°. So

$$-(\theta_{p1} - \theta_{p2}) = -60°$$

or

$$-90° - \theta_{p1} = -60°$$

so

$$\theta_{p1} = -30°$$

10). Calculation of K on PARL.

Referring to Equation (2.3), it can be found that

$$K = \frac{\prod_{k=1}^{n} \left| s + p_k \right|}{\prod_{i=1}^{m} \left| s + z_i \right|} \tag{2.21}$$

Thus K can be determined either graphically or analytically.

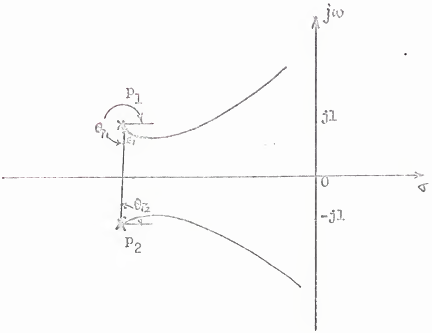


Fig. 2.2 Departure Angle at p_1.

2.2 Method of Construction of PARL.

Construction of PARL is a task much more elaborate than that of a conventional root locus even in the simplest cases. As was mentioned in section 1.2, it is possible to construct PARL by superimposing Phase Angle Root Loci for the basic form. The amount of labor can be reduced although it is still very complicated.

By the method of superposition, the first step is to factor the given open-loop transfer function into several simpler component transfer functions. Then by superposing two of the component transfer functions loci at a time, finally the required PARL of the given function can be obtained.

The amount of work required will depend on the degree of complexity of the given transfer function.

It is found that most of the open-loop transfer functions of interest can be constructed by superposing several basic forms of PARL. They are so called basic because of their simple geometrical forms. Each one of them is studied in the following:

1). Simple Pole. The transfer function considered has only one real pole in the left half of the s-plane with no finite zeros:

$$G(s)H(s) = \frac{K}{s+p} \tag{2.22}$$

The corresponding loci are shown in Fig. (2.3a). The loci of constant phase are radial lines emanating from the pole at $s = -p$.

2). Simple Zero. The transfer function considered has one real zero in the left half of the s-plane and no finite poles:

$$G(s)H(s) = K \cdot (s+z) \tag{2.23}$$

The corresponding loci are shown in Fig. (2.3b).

3). Simple Dipole. The transfer function to be considered is to have the following form:

$$G(s)H(s) = K \cdot \frac{s+z}{s+p} \tag{2.24}$$

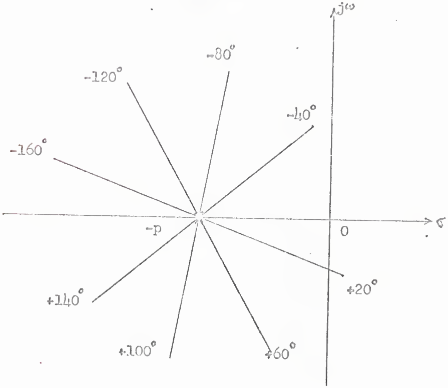


Fig. 2.3a PARL for a simple pole.

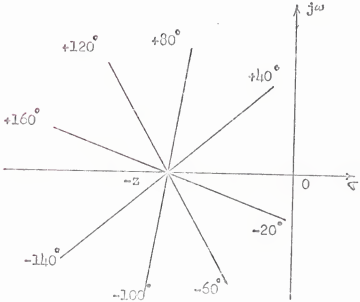


Fig. 2.3b PARL for a simple zero.

with both the pole and the zero on negative real axis.

Therefore

$$\operatorname{Arg}(G(s)H(s)) = \operatorname{Arg}(s+z) - \operatorname{Arg}(s+p) \tag{2.25}$$

let $s = \sigma + j\omega$, and $\phi = \operatorname{Arg}(G(s)H(s))$, so

$$\phi = \operatorname{Arg}(\sigma + j\omega + z) - \operatorname{Arg}(\sigma + j\omega + p)$$

$$= \tan^{-1}\frac{\omega}{\sigma+z} - \tan^{-1}\frac{\omega}{\sigma+p} = \tan^{-1}\frac{(p-z)}{\omega^2+\sigma^2+p^2+\sigma(p+z)} \tag{2.26}$$

For a given constant phase angle ϕ, this leads to

$$\tan\phi = \frac{(p-z)}{\omega^2+\sigma^2+p^2+\sigma(p+z)} \tag{2.27}$$

or

$$\sigma^2 + \sigma(p+z) + \omega^2 + p^2 = \left[(p-z)\cot\phi\right]\omega$$

so that

$$\sigma^2 + \sigma(p+z) + p^2 + \omega^2 - \left[(p-z)\cot\phi\right]\omega = 0$$

This last equation can be written as

$$\left[\omega - \frac{(p-z)}{2}\cot\phi\right]^2 + \left[\sigma + \frac{(p+z)}{2}\right]^2 = \frac{\csc^2\phi}{4}\left[(p+z)^2 - 2pz\cos^2\phi\right]$$

(2.27)

Evidently it can be concluded from Eq. (2.27) that the loci of the simple dipole for constant phase angles are portions of the circles passing through p and z, with center at ($\sigma = -\frac{p+z}{2}$, $\omega = \frac{p-z}{2}\cot\phi$) and radius

$$r = \frac{|\csc \phi|}{2} \left\{ (p+z)^2 - 2pz \cos^2 \phi \right\}^{\frac{1}{2}} \qquad (2.28)$$

The corresponding loci for phase lead and phase lag transfer functions are shown in Fig. (2.4) and Fig. (2.5) respectively.

It should be noted that the PARL of positive ϕ and the PARL of negative ϕ are mirror image of one another with respect to the real axis in each of the previous cases.

In order to understand the advantages of the superposition method, an example will be presented.

Example. Suppose that the open-loop transfer function is given as

$$G(s)H(s) = \frac{K}{s(s+1)(s+4)} \qquad (2.29)$$

By definition, the PARL of G(s)H(s) satisfy

$$\phi = \mathrm{Arg}\frac{1}{s} + \mathrm{Arg}\frac{1}{s+1} + \mathrm{Arg}\frac{1}{s+4} \qquad (2.30)$$

and naturally each point on the PARL satisfies Eq. (2.29).

First, write this equation as follows:

$$\mathrm{Arg}\frac{1}{s} + \mathrm{Arg}\frac{1}{s+1} = \phi_1 \qquad (2.31)$$

$$\mathrm{Arg}\frac{1}{s+4} + \phi_1 = \phi \qquad (2.32)$$

The PARL of Eq. (2.31) with ϕ_1 equal to a certain constant phase angle are easily constructed by first superposing the PARL of 1/s

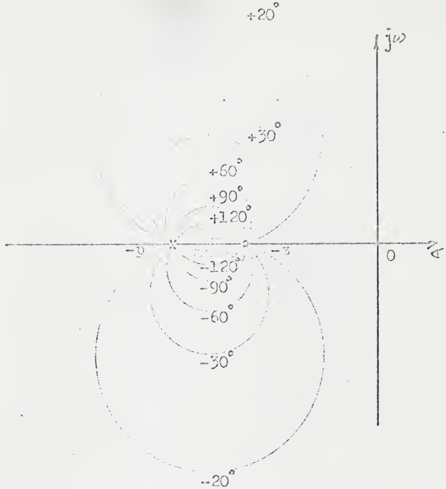


Fig. 2.4 PARL for a simple phase lead dipole.

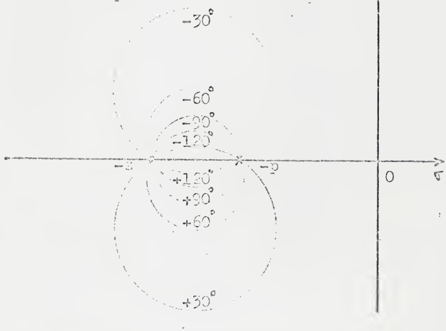


Fig. 2.5 PARL for a simple phase lag dipole.

and that of $1/(s+1)$ of the same phase angles individually as was shown in Fig. (2.6). Then by superposing the PARL of $1/(s+4)$ on those of Eq. (2.31) for the same phase angle individually one can obtain the required PARL for the transfer function given by Eq. (2.29).

Fig. (2.7) shows the result of this superposition.

2.3 Simple Geometrical Method of Construction of the PARL of Simple Dipole.

As has been mentioned previously, the PARL of a simple dipole is a very basic one. This is because of the ease with which the loci of roots of contant phase for such an open-loop transfer function can be obtained.

Shown in Fig. (2.8) is a phase lead dipole with the open-loop transfer function

$$G(s)H(s) = K \cdot \frac{s+z}{s+p} \qquad (2.33)$$

It was found by the author that the PARL for a given phase $\phi = \phi_1$ of $G(s)H(s)$ can be obtained by constructing a circle passing through the pole A and the zero B and containing an inscribed angle equal to the given phase angle ϕ_1. The portion of the circle in upper half of the s-plane is the required PARL. The portion below the real axis is the locus for $\phi = \phi_1 - 180°$ in this case of phase lead open-loop transfer function, otherwise the portion below is the locus for $\phi = \phi_1 + 180°$.

Since at any point P on the upper half of the s-plane it was found from Fig. (2.8) that

$$\phi = \text{Arg}\,(G(s)H(s)) = \text{Arg}\,(s+z) - \text{Arg}\,(s+p) = \phi_1 \qquad (2.34)$$

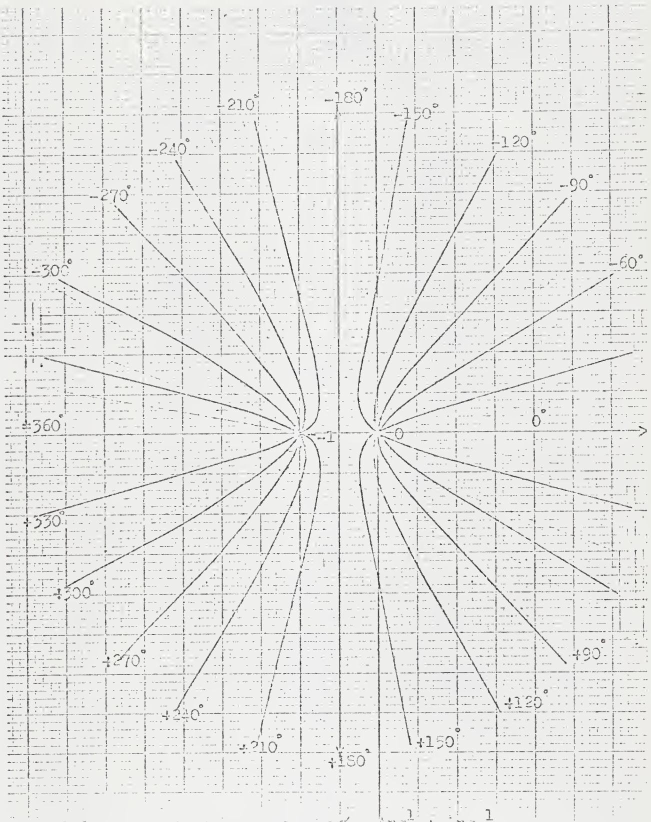


Fig. 2.6 PARL with phase angle of $\psi = \mathrm{Arg}\frac{1}{s} + \mathrm{Arg}\frac{1}{s+1}$

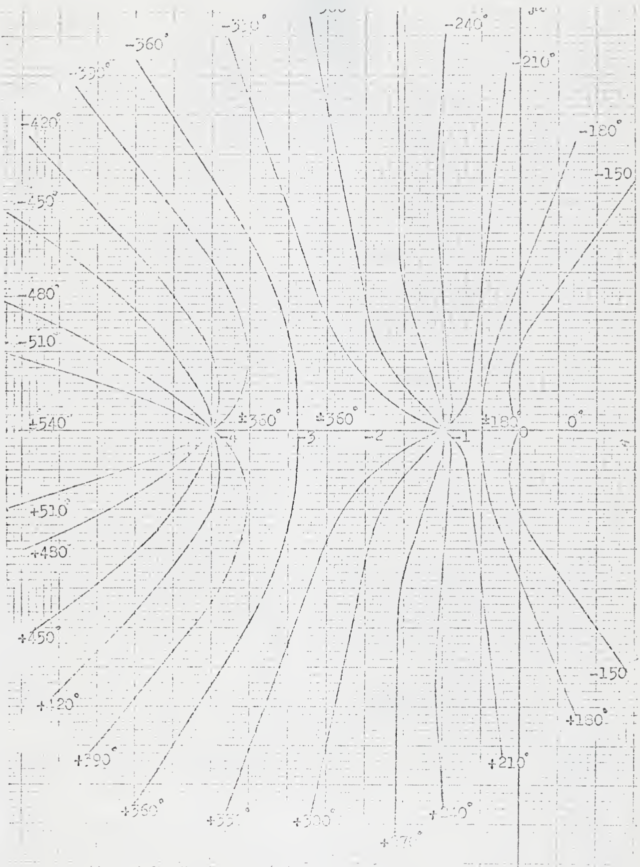


Fig. 2.7 PAM of $G(s) = \dfrac{K}{s(s+1)(s+2)}$

And as ϕ_1 is always the inscribed angle subtended by the chord $\overline{AB}$ formed by the open-loop pole and the open-loop zero, it must always be true that ϕ is equal to ϕ_1 or the constant phase angle.

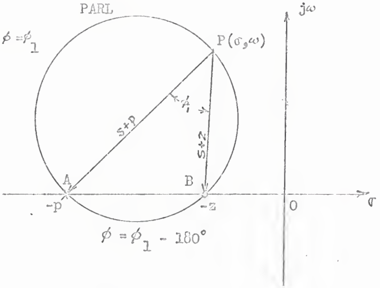


Fig. 2.8 Construction of PARL of a simple dipole.

III. COMPENSATION OF FEEDBACK CONTROL SYSTEMS FROM THE VIEWPOINT OF PARL AND CGRL

3.1 Introduction

Generally it is difficult or even impossible to design a feedback control system satisfying the performance specifications given for both the steady state response (steady state error) and the transient response (stability requirement) simultaneously. Usually the simple method of increasing the forward gain K can lead to an unsatisfactory transient response, although the system steady state error may be reduced.

Therefore it is usually necessary to insert some sort of compensation network or device into the system in addition to adjusting the forward gain in order to have the system satisfy both the steady-state and transient performance specifications. Although there has been a considerable amount of work devoted to this design problem, it is believed by the author that the compensation technique presented in this report represents another approach to the problem. It differs from the classical compensation techniques in that the PARL and the CGRL form the basis for the design decisions.

3.2 Application of PARL and CGRL to the Compensation Problem

It is assumed that the open-loop transfer function G(s)H(s) is a rational function of a complex variable defined in a certain region on the s-plane. At any point s belonging to the defined region on the s-plane, G(s)H(s) has a definite magnitude and phase. In other words, the open-loop transfer function G(s)H(s) at a given point on the s-plane will be completely determined whenever both the phase and the magnitude of G(s)H(s) are specified at that point.

From this point of view, it is found that compensation of a feedback control system can be achieved by first developing a technique of finding a network which will successfully compensate the phase at a certain point in the s-plane where the transient specifications are satisfied. Phase compensation is achieved by forcing the PARL with an phase angle of ± 180° of the compensated system through the desired operating point. An adjustment of the open-loop forward gain will be necessary in order to satisfy the steady state specifications. In other words, the compensation problem in the s-plane is composed of two steps, one dealing with phase compensation and one dealing with the forward gain adjustment.

Consider a linear feedback control system with the open-loop transfer function

$$G(s)H(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = K \cdot F(s) \qquad (3.1)$$

It is assumed that the given system is operated with a forward gain K_1 which satisfies the steady state requirements of the system. The open-loop transfer function becomes

$$G(s)H(s) = K_1 \cdot F(s) \qquad (3.2)$$

A portion of the PARL and CGRL of the uncompensated system is shown in Fig. (3.1). The notation of Fig. (3.1) is defined as

$(PARL)_u$ = PARL of the uncompensated system with open-loop transfer function given by Eq. (3.1).

$(CGRL)_u$ = CGRL of the uncompensated system with open-loop transfer function given by Eq. (3.1).

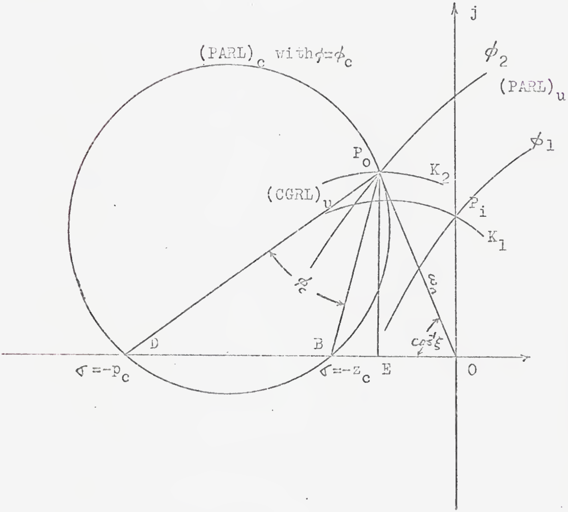


Fig. 3.1 Application of PARL and CGRL to the compensation problem.

$(PARL)_c$ = PARL of the compensation network transfer function.

$P_i(0,\omega_i)$ = One of the dominant poles of the uncompensated closed-loop system with open-loop forward gain $K = K_1$.

$P_o(\sigma_o,\omega_o)$ = One of the desired dominant pole of the compensated closed-loop system.

ξ = Desired damping coefficient of the dominant poles of the compensated system.

ω_n = Desired undamped natural frequency of the dominant poles of the compensated system.

$\sigma_o = \xi\,\omega_n$ = Damping constant (actual damping).

$\omega_o = \sqrt{1-\xi^2}$ = Conditional frequency.

As shown in Fig. (3.1), the uncompensated system has a dominant closed-loop pole located at the intersection of the (PARL) with $\phi = \phi_1 \pm 180°$ and the $(CGRL)_u$ with $K = K_1$. In such a case, the transient response would be completely undesirable although the steady state response satisfies the system specifications as mentioned previously. Suppose that it is found that both the transient and the steady state response would satisfy the system requirements if the dominant closed-loop poles were moved to point P_o. P_o is the intersection point of the $(PARL)_u$ with $\phi = \phi_2$ and the $(CGRL)_u$ for $K = K_2$. Clearly both phase and gain compensation are needed to achieve this relocation of the operating point.

The first step in the compensation procedure is to determine the amount of phase shift that the compensation network must supply at P_o. This can be found by inspection of the phase angle at P_o from the $(PARL)_u$ diagram. In particular, the compensating network must supply sufficient phase shift, ϕ_c, at P_o to insure that the PARL of the compensated system has a branch corresponding to a phase of ± 180 degrees at P_o. That is

$$\phi_c + \phi_2 = \pm 180° \tag{3.3}$$

or,

$$\phi_c = -(\phi_2 \pm 180°) \quad . \tag{3.4}$$

In these last two equations -180 degrees should be used if the conventional root locus of the $(PARL)_u$ under consideration was a lagging phase locus, otherwise $+180$ degrees should be used in Eq. (3.3). Evidently phase lead compensation is needed when

$$(\phi_2 \pm 180°) \leq 0 \quad , \tag{3.5}$$

or phase lag compensation is needed when

$$(\phi_2 \pm 180°) \geq 0 \quad . \tag{3.6}$$

Now suppose that phase lead compensation is needed for the current consideration. Due to the simplicity and convenience of the geometry in the construction of both the PARL and CGRL for simple dipoles, the suggested compensation network has the form

$$G_c = K_c \cdot \frac{s+z_c}{s+p_c} \tag{3.7}$$

A few branches of the PARL of the dipole are plotted in Figure 3.2.

It is assumed that

$$p_c = K' \cdot z_c \tag{3.8}$$

where K' is an arbitrary constant. The factor K' is greater than 1 for a phase lead compensation network and less than 1 for a phase lag compensation network. (See paragraph 2.2, for phase lead and phase lag dipole transfer function PARL).

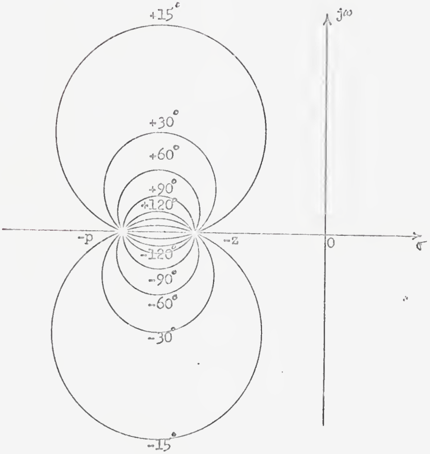


Fig. 3.2 PARL of $G_c(s) = K_c \cdot \frac{s+z_c}{s+p_c}$

A forward gain K_i is inserted so that the desired steady state error coefficient can be obtained. This will lead to a compensated open-loop transfer function of the form

$$(G(s)H(s))_c = K_1 \cdot (G(s)H(s)) \cdot K_c \cdot \frac{s+z_c}{s+p_c} \tag{3.9}$$

$$= K_1 K_2 K_c \cdot F(s) \cdot \frac{s+z_c}{s+p_c} \tag{3.10}$$

since the open-loop transfer function corresponding to the branch of the $(PARL)_u$ passing through P_o is

$$G(s)H(s) = K_2 \cdot F(s) \tag{3.11}$$

Suppose that the steady state error specifications require a positional error constant of K_p. Equation (3.2) implies

$$K_p = \lim_{s\to 0} (K_1 \cdot F(s)) = K_1 \cdot \lim_{s\to 0} F(s) \tag{3.12}$$

and for the compensated system the positional error constant K_p can be found from

$$K_p = \lim_{s\to 0} (G(s)H(s))_c = \lim_{s\to 0} (K_1 \cdot K_2 \cdot K_c \cdot F(s) \cdot \frac{s+z_c}{s+p_c})$$

$$= K_1 K_2 K_c \cdot \frac{z_c}{p_c} \cdot \lim_{s\to 0} F(s) \tag{3.13}$$

Since the steady state error must be maintained within the specifications, the value of K_p must be the same before and after the compensation network is inserted. Referring to Equations (3.12), (3.13), and (3.8) reveals that

$$K_p = K_1 \cdot \lim_{s\to 0} F(s) = K_i K_2 K_c \frac{1}{K'} \cdot \lim_{s\to 0} F(s)$$

or when it is solved for K_i,

$$K_i = \frac{K_1 K'}{K_2 K_c} \tag{3.14}$$

where K_c is the gain of the compensation network at P_o.

With Equations (3.8) and (3.14) substituted into Eq. (3.10), the compensated open-loop transfer function turns out to be

$$(G(s)H(s))_c = \frac{K_1 K'}{K_2 K_c} K_2 \cdot F(s) \cdot \frac{s+z_c}{s+p_c} \cdot K_c$$

$$= K_1 K' \cdot F(s) \cdot \frac{s+z_c}{s+p_c} \tag{3.15}$$

Futhermore, it is found by the author that z_c or p_c, K', σ_o, ω_o, ω_n, and ϕ_c are related to one another by the following formula

$$K' z_c^2 - \sigma_o (K'+1) z_c + \omega_n^2 - \sqrt{\left[\omega_o^2 + (z_c - \sigma_o)^2\right]\left[\omega_o^2 + (K' z_c - \sigma_o)^2\right]} \cdot \cos\phi_c = 0 \tag{3.16}$$

In other words, the suggested procedure of compensating a feedback control system can be summarized as follows:

1). Specify σ_o, ω_o, ω_n for the dominant closed-loop poles. This determines P_o in the s-plane.

2). Determine ϕ_c from Eq. (3.4).

3). Determine whether phase lead or phase lag compensation should be employed from Equations (3.5) and (3.6).

4). Specify K' or z_c.

5). Obtain either z_c or K' from Eq. (3.16) or construct the branch of the PARL of the simple dipole transfer function of Eq. (3.7) that passes through P_o and z_c as mentioned in section 2.2 and determine p_c by inspection. (See Fig. 3.1).

6). Determine the value of K_c at point P_o on the branch of PARL of the simple dipole that passes through P_o.

7). Subtitute the resulting values into Eq. (3.15) and obtain the desired open-loop transfer function.

An example illustrating the results derived in this section follows.

Suppose that a certain feedback control system has the open-loop transfer function

$$G(s)H(s) = \frac{259(s^2 + s + 1.25)}{(s+4)(s+5)(s^2 + 0.4 s + 9.04)} \tag{3.23}$$

It is desired to compensate the system in such a way that the transient response has

$$\xi = 0.56$$
$$\omega_n = 1.80$$

without changing the steady state positional error constant.

In this example the PARL and CARL for the uncompensated system are constructed. On the s-plane it is found that for the desired transient [illegible] the closed-loop dominant pole in the upper half s-plane must be relocated to P_o. The [illegible] and [illegible] show that

$$\phi_2 = -150^\circ$$

$$K_2 = 61.7$$

For the purpose of phase compensation, the compensation network must have a phase angle ϕ_c such that

$$\phi_c + \phi_2 = \phi_1 = -180^\circ,$$

or

$$\begin{aligned} \phi_c &= -(\phi_2 - \phi_1) = -(-150^\circ - (-180^\circ)) \\ &= -30^\circ \end{aligned} \tag{3.24}$$

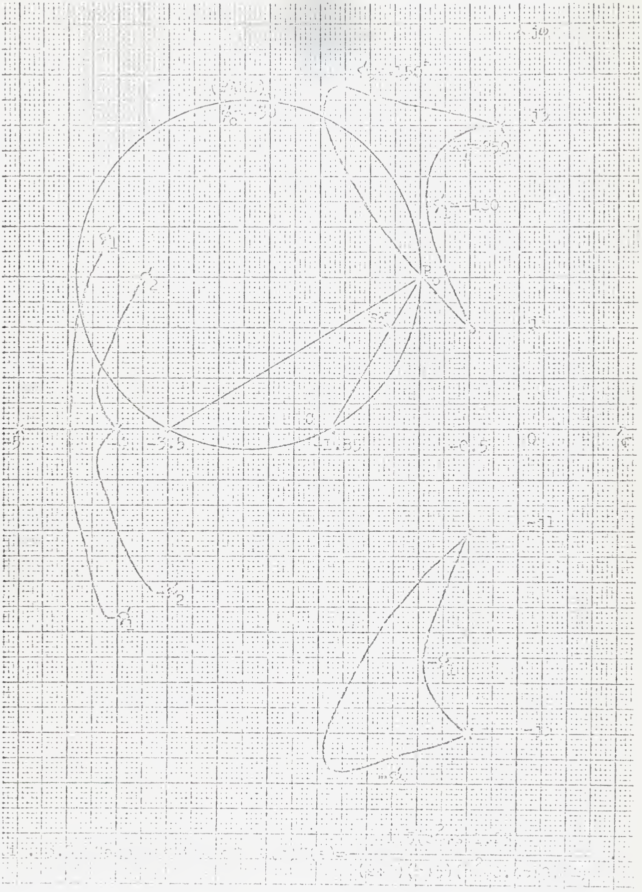
Thus phase lag compensation is required. Therefore according to the previously assumed criterion, K' must be chosen less than 1 for

$$K' = \frac{p_c}{z_c} \leqslant 1 \tag{3.25}$$

Now, assume that z_c is chosen with a value of

$$z_c = 3.5$$

as shown in Fig. (3.3). Connect points P_o and the zero at $-z_c$ with straight line AP_o. Construct an angle with line AP_o as one edge and P_o the vertex. The angle AP_oC must be equal to ϕ_c, where C is the intercept of the other edge with the real axis. Point C must be to the right of point A in this [illegible], otherwise it should be to the left of point A. [illegible]


jω
(PAIR)
-5
-3.5
-1.50
-0.5
0
σ
-j1
-j2

$$p_c = 1.85 \tag{3.26}$$

so that

$$K' = \frac{1.85}{3.5} = 0.53 \leqslant 1$$

as required.

And it was found that the gain at P_o on the branch of (PARL) with a phase angle of ϕ_c is

$$K_c = 1.72$$

According to Eq. (3.7), the compensation network transfer function would be

$$G_c(s) = \frac{1.72(s+3.5)}{(s+1.85)} \tag{3.27}$$

and according to Eq. (3.15) the compensated system open-loop transfer function would be

$$\begin{aligned}(G(s)H(s))_c &= (259)\cdot(0.53)\cdot\frac{s+3.5}{s+1.85}\cdot\frac{(s^2+s+1.25)}{(s+4)(s+5)(s^2+0.4s+9.04)} \\ &= \frac{137\,(s+3.5)\cdot(s^2+s+1.25)}{(s+1.85)\cdot(s+4)\cdot(s+5)\cdot(s^2+0.4\,s+9.04)}\end{aligned} \tag{3.28}$$

as required.

A calculation for the purpose of demonstrating that the previous procedures retain the correct steady state error shows that for the uncompensated system

$$K_p = \frac{(259)(1.25)}{4.5(9.04)} = 1.8$$

and for the compensated system

$$K = \frac{(137)(3.5)(1.25)}{4.5(1.85)(9.04)} = 1.8$$

The reshaped conventional root locus diagram is shown in Fig. 3.4.

3.3 Derivation of Equation (3.16)

The unique relation between variables z_c, p_c, σ_o, ω_o, ω_n, K' and $\emptyset_c$ is so important in the compensation procedure that a detailed derivation of the equation is necessary.

The magnitude of the phasors P_oD, P_oE and DE in Fig. (3.1) are related as follows:

$$P_oD = \sqrt{P_oE^2 + DE^2} = \sqrt{\omega_o^2 + (p_c - \sigma_o)^2} \qquad (3.17)$$

$$P_oB = \sqrt{\omega_o^2 + (z_c - \sigma_o)^2} \qquad (3.18)$$

$$DB = p_c - z_c = (K' - 1)z_c \qquad (3.19)$$

From an elementary trigonometric formula, it is found that

$$\cos\emptyset_c = \frac{P_oD^2 + P_oB^2 - DB^2}{2 \cdot P_oD \cdot P_oB} \qquad (3.20)$$

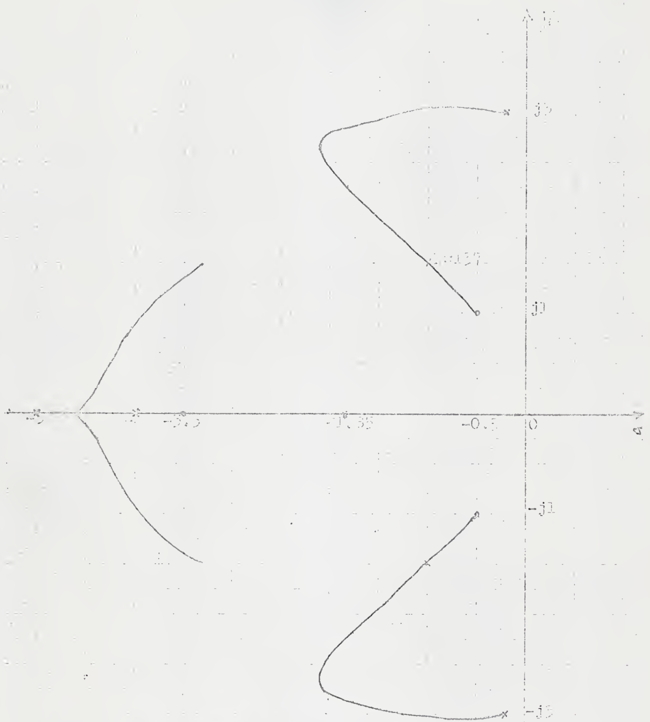


Fig. 3.4 Root locus diagram for

$$(G(s)H(s)) = \frac{137(s+2.5)(s^2+s+1.[illegible])}{(s+1.[illegible])(s+[illegible])(s)(s^2+[illegible]+3.[illegible])}$$

and substitution of the results of Equations (3.17), (3.18) and (3.19) into Eq. (3.20) leads to

$$\cos\phi_c = \frac{(\omega_o^2 + (p_c-\sigma_o)^2) + (\omega_o^2 + (z_c-\sigma_o)^2) - (p_c-z_c)^2}{2((\omega_o^2 + (z_c-\sigma_o)^2)\cdot(\omega_o^2 + (p_c-\sigma_o)^2))^{\frac{1}{2}}}$$

$$= \frac{2(\omega_o^2+\sigma_o^2) + p_c^2 + z_c^2 - 2\sigma_o(p_c+z_c) - (K'^2-2K'+1)z_c^2}{2((\omega_o^2 + (z_c-\sigma_o)^2)\ (\omega_o^2 + (p_c-\sigma_o)^2))^{\frac{1}{2}}}$$

$$= \frac{2\omega_n^2 + (K'^2+1)z_c - 2\sigma_o z_c(K'+1) - K'^2 z_c^2 + 2K'z_c^2 - z_c^2}{2((\omega_o^2 + (z_c-\sigma_o)^2)\ (\omega_o^2 + (p_c-\sigma_o)^2))^{\frac{1}{2}}}$$

$$= \frac{K'z_c^2 - \sigma_o(K'+1)z_c + \omega_n^2}{2((\omega_o^2 + (z_c-\sigma_o)^2)\ (\omega_o^2 + (K'z_c-\sigma_o)^2))^{\frac{1}{2}}}$$

(3.21)

This last equation can be rearranged to yield

$$K'z_c^2-\sigma_o(K'+1)z_c+\omega_n^2-((\omega_o^2+((z_c-\sigma_o)^2)(\omega_o^2+(K'z_c-\sigma_o)^2))^{\frac{1}{2}}\cos\phi_c = 0$$

(3.22)

IV. SUMMARY

The conventional characteristic equation

$$1 + G(s)H(s) = 0 \qquad (4.1)$$

of a feedback control system makes the studying of the characteristics of a feedback control system on the s-plane possible. The problem of reshaping the conventional root locus on the s-plane in such a way as to meet a certain set of requirements can be accomplished by using conventional frequency response design techniques and then interpretating the results in the s-plane. The principle difficulty with this technique is that the designer can not maintain adequate control of the system transient response. The generalized root contours of Kuo (ref. 6) represent an effort to overcome this difficulty. In this report it is shown that the phase angle root locus and constant gain root locus can be used to compensate a linear feedback control system in such a way as to give the designer simultaneous control of both the transient and steady state performance.

From a phase angle point of view, the conventional characteristic equation can be generalized into

$$G(s)H(s) - e^{j\phi} = 0 \qquad (4.2)$$

where ϕ is the phase parameter of the open-loop transfer function. When ϕ is set equal to 180°, Eq. (4.2) reduces to the conventional closed-loop characteristic equation. It is noted that according to Eq. (4.2) each point in the s-plane has a unique phase angle ϕ associated with it. In other words, there is defined a phase plane corresponding to a given open-loop transfer function. It is possible to measure the difference in phase between any two points in s-plane. The phase angle root loci as defined in this report, depicts this phase difference in such a way as to aid in the compensation problem. Furthermore, each point in the s-plane corresponds to a unique open-loop forward gain K. The constant gain root locus depicts this gain relationship in such a way as to be useful in the compensation problem.

A procedure that utlizes phase angle root locus and constant gain root locus diagrams as tools for determining the pole and zero location of simple dipole compensation network is presented. The procedure is illustrated by an example. Although the example is a simple one in terms of the number of open-loop poles and zeros and in terms of the complexity of the compensation network, it is clear that the computational problem is a difficult one. However, it appears that the problem is amenable to digital computer implementation.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. John G. Truxal, Control System Synthesis. McGraw-Hill Book Company, Inc., 1955.

2. Otto J. M. Smith, Feedback Control Systems. McGraw-Hill Book Company, Inc., 1958.

3. C. J. Savant, Jr., Basic Feedback Control System Design. McGraw-Hill Book Company, Inc., 1958.

4. Yaohan Chu, "Synthesis of Feedback Control System by Phase-Angle Loci," AIEE Trans. on Applications and Industry, vol. 71, pp. 330-339, 1952.

5. George J. Thaler, Servomechanism Analysis. McGraw-Hill Book Company, Inc., 1953.

6. Benjamin C. Kuo, Automatic Control Systems. Prentice-Hall Inc., 1965.

7. W. R. Evans, "Graphical Analysis of Control Systems," Transactions AIEE, vol. 67, pp. 547-551, 1948;

"Control System Synthesis by Root Locus Method," Transaction AIEE, vol. 69, pp. 66-69, 1950.

8. Guillemin, E. A., Communication Networks, vol. II. John Wiley & Sons, Inc., New York, 1942;

"A Summary of Modern Method of Network Synthesis," Advances in Electronics, vol. III, Academic Press Inc., New York, 1951.

9. Wiener, Norbert, Nonlinear Problems in Random Theory. Cambridge Technology Press of M.I.T., 1958.

Interpolation, and Smoothing of Stationary Time Series with Engineering Applications. Cambridge Technology Press of M.I.T., 1950.

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A STUDY OF PHASE ANGLE ROOT LOCI TECHNIQUES

by

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B. S., National Taiwan University, Taiwan, China, 1963

AN ABSTRACT OF A MASTER'S REPORT

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Manhattan, Kansas

1968

ABSTRACT

Loci in the s-plane of constant phase of the open-loop transfer function of a linear feedback control system are called phase angle root loci (PARL). Loci in the s-plane of constant open-plane gain are termed constant gain root loci (CGRL). Various characteristics of these loci are derived and discussed.

A procedure that utlizes phase angle root loci and constant gain root loci diagrams as tools for determining the pole and zero location of simple dipole compensation network is presented. The procedure is illustrated by an example. Although the example is a simple one in terms of the number of poles and zeros and in terms of the complexity of the compensation network, it is clear that the computational problem is a difficult one. However, it appears that the problem is amenable to digital computer implementation.