POLYNOMIAL STRESSES IN PLANE ANISOTROPIC BEAMS

by

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NOTATION AND TERMINOLOGY

x_1, x_2, x_3	Rectangular coordinates; x_1, x_2 in the plane of the beam
$\phi\ (x_1, x_2)$	Airy stress function
σ_{ij}	stress tensor
ϵ_{ij}	strain tensor
S_{ijkl}	general elastic compliance tensor
C_{mn}	polynomial coefficients
2a	length of beam
2b	depth of beam
Plane anisotropic	the $x_1 - x_2$ plane is the only plane of elastic symmetry.
Orthotropic	three mutually perpendicular planes of elastic symmetry.

INTRODUCTION

In the classical two-dimensional theory of elasticity, plane stress problems of rectangular sections have been investigated for many years. The solution to a particular problem is often found by guess work or by other indirect approaches. Popular methods of solution are superposition, and the semi-inverse method. Unfortunately, these methods are directly dependent upon the skill and experience of the investigator.

Recently, due to work by Neou[3] and Hashin[1] a direct systematic method has been developed to solve the stress problem of a generally plane anisotropic rectangle. Some solutions to particular problems of this type have been given by Lekhnitskii.[9] Silverman[8] has solved several for plane orthotropic beams. However, the approaches by Lekhnitskii and Silverman lack the simplicity found in the method of Hashin.

In this report Hashin's method, referred to as the polynomial method, is discussed in detail. The method is then applied to several problems of generally plane anisotropic beams. In one particular case an interesting comparison is made between the polynomial solution and the least work solution. Finally, to fulfill the purpose of this report, anisotropic and isotropic beams are examined in each of the problems to determine the anisotropic effects on the stress distributions.

THE POLYNOMIAL METHOD OF SOLUTION

In plane stress problems of rectangular sections, the unit thickness of the plate is considered to be small in comparison to the depth (2b) and the length (2a) such that the assumption

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

is approximately satisfied everywhere. The equilibrium equations in this case are

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \qquad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \tag{1.1}$$

for no body force.

Equations (1.1) are identically satisfied by considering the Airy stress function Φ (x_1, x_2) which yields

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2{}^2} \qquad \sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1{}^2} \qquad \sigma_{12} = \frac{-\partial^2 \Phi}{\partial x_1 \partial x_2} \tag{1.2}$$

The compatability conditions for the two-dimensional problem are reduced to the single equation

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2{}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1{}^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \tag{1.3}$$

The compatibility equation (1.3) can be expressed in terms of stress by introducing the general anisotropic stress-strain relations

$$\epsilon_{ij} = S_{ijkl}\sigma_{kl} \tag{1.4}$$

where the S_{ijkl} are the general elastic compliances. The summation convention is used on repeated subscripts and the subscripts range from 1 to 2 for the two-dimensional case.

By substitution of (1.2) into (1.4) one obtains

$$\epsilon_{11} = S_{1111}\frac{\partial^2\phi}{\partial x_2^{\,2}} + S_{1122}\frac{\partial^2\phi}{\partial x_1^{\,2}} - 2S_{1112}\frac{\partial^2\phi}{\partial x_1\partial x_2} \tag{1.5a}$$

$$\epsilon_{22} = S_{2222}\frac{\partial^2\phi}{\partial x_1^{\,2}} + S_{2211}\frac{\partial^2\phi}{\partial x_2^{\,2}} - 2S_{2212}\frac{\partial^2\phi}{\partial x_1\partial x_2} \tag{1.5b}$$

$$\epsilon_{12} = -2S_{1212}\frac{\partial^2\phi}{\partial x_1\partial x_2} + S_{1222}\frac{\partial^2\phi}{\partial x_1^{\,2}} + 2S_{1211}\frac{\partial^2\phi}{\partial x_2^{\,2}} \tag{1.5c}$$

Inserting equations (1.5) into (1.3) yields the compatability condition in terms of $\Phi(x_1,x_2)$:

$$\beta_{22}\frac{\partial^4\phi}{\partial x_1^{\,4}} - \beta_{26}\frac{\partial^4\phi}{\partial x_1^{\,3}\partial x_2} + 2\gamma\frac{\partial^4\phi}{\partial x_1^{\,2}\partial x_2^{\,2}} - \beta_{16}\frac{\partial^4\phi}{\partial x_1\partial x_2^{\,3}}$$

$$+ \beta_{11}\frac{\partial^4\phi}{\partial x_2^{\,4}} = 0 \tag{1.6}$$

where

$$\beta_{22} = S_{2222} \qquad \beta_{26} = 4S_{2212} \qquad \gamma = \beta_{12} + \beta_{66} = S_{1122} + 2S_{1212}$$

$$\beta_{16} = 4S_{1112} \qquad \beta_{11} = S_{1111}$$

Equation (1.6) establishes the governing differential equation for $\Phi(x_1,x_2)$ in a plane anisotropic plate.

The boundary conditions are commonly given in terms of tractions. However, since the governing equation is expressed in terms of Φ, it is more convenient to describe the boundary conditions in terms of Φ. For a simply connected domain with no body force, the boundary conditions can be evaluated by the following method.[4]

Consider the region G, which is circumscribed by the boundary curve Γ, Fig. 1.

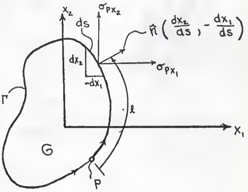

Fig. 1.

From the plane theory of elasticity, equilibrium of a boundary element produces the equations

$$\sigma_{px_1} = l\sigma_{11} + m\sigma_{12} \tag{1.7a}$$

$$\sigma_{px_2} = l\sigma_{12} + m\sigma_{22} \tag{1.7b}$$

where l and m are the direction cosines of the outward normal $\hat{n}$, and $(\sigma_{px_1}, \sigma_{px_2})$ are the components of the stress vector on the boundary. Substitution of the direction cosines for $\hat{n}$ (Fig. 1) into equations (1.7) yields

$$\sigma_{px_1} = \sigma_{11} \frac{dx_2}{ds} - \sigma_{12} \frac{dx_1}{ds} \tag{1.8a}$$

$$\sigma_{px_2} = \sigma_{12} \frac{dx_2}{ds} - \sigma_{22} \frac{dx_1}{ds} \tag{1.8b}$$

By integrating equations (1.8) counterclockwise from P it is found that

$$Rx_1 = \int_o^{\ell} \sigma_{px_1} ds = \int_o^{\ell} \{\sigma_{11} \frac{dx_2}{ds} - \sigma_{12} \frac{dx_1}{ds}\} ds \tag{1.9a}$$

$$Rx_2 = \int_o^{\ell} \sigma_{px_1} ds = \int_o^{\ell} \{\sigma_{12} \frac{dx_2}{ds} - \sigma_{22} \frac{dx_1}{ds}\} ds \tag{1.9b}$$

where Rx_1 and Rx_2 are the resultant boundary forces between o and in the x_1 and x_2 directions, respectively.

Substitution of equations (1.1) into equation (1.9a) produces

$$Rx_1 = \int_o^{\ell} [\frac{\partial^2 \phi}{\partial x_2{}^2} \frac{dx_2}{ds} + \frac{\partial^2 \phi}{\partial x_1 x_2} \frac{dx_1}{ds}] ds$$

$$= \int_o^{\ell} [\frac{\partial}{\partial x_2} (\frac{\partial \phi}{\partial x_2}) \frac{dx_2}{da} + \frac{\partial}{\partial x_1} (\frac{\partial \phi}{\partial x_2}) \frac{dx_1}{ds}] ds$$

$$= \int_o^{\ell} [\frac{\partial}{\partial s} (\frac{\partial \phi}{\partial x_2})] ds$$

$$= \frac{\partial \phi}{\partial x_2} \Big|_o^{\ell}$$

Similarly

$$R_{X_2} = -\frac{\partial\Phi}{\partial X_1}\Big|_o^{\ell}$$

It is convenient to choose Φ and its derivatives to be zero at P, which is permissible for zero body force.
Hence,

$$R_{X_1} = \frac{\partial\Phi}{\partial X_2} \tag{1.10a}$$

$$R_{X_2} = -\frac{\partial\Phi}{\partial X_1} \tag{1.10b}$$

where the derivatives are evaluated at $s = \ell$.

Integration of the total differential of Φ along with the aid of equations (1.8) yields

$$\Phi(\ell) = \int_o^{\ell} d\Phi = \int_o^{\ell}\left(\frac{\partial\Phi}{\partial X_1}dX_1 + \frac{\partial\Phi}{\partial X_2}dX_2\right)$$

$$= [-X_1 R_{X_2} + X_2 R_{X_1}]_o^{\ell} - \int_o^{\ell}(-X_1\sigma_{pX_2} + X_2\sigma_{pX_1})ds$$

$$= -X_1(\ell)R_{X_2} + X_2(\ell)\,R_{X_1} - \int_o^{\ell}(-X_1\sigma_{pX_2} + X_2\sigma_{pX_1})ds$$

$$= \int_o^{\ell}[X_2(\ell)-X_2]\sigma_{pX_1}ds - \int_o^{\ell}[X_1(\ell)-X_1]\sigma_{pX_2}ds$$

$$= M\,(\ell)$$

where the last expression is exactly equal to the counterclockwise moment about a point $s = \ell$ due to the tractions on the boundary between $s = o$ and $s = \ell$ and is denoted by $M(\ell)$.

In summary, the three boundary conditions for Fig. 1. are

$$R_{X_1} = \frac{\partial \phi}{\partial X_2} = \int_o^\ell \sigma_{pX_1} dS \tag{1.11a}$$

$$R_{X_2} = -\frac{\partial \phi}{\partial X_1} = \int_o^\ell \sigma_{pX_1} dS \tag{1.11b}$$

$$\phi(\ell) = M(\ell) = \int_o^\ell [X_2(\ell) - X_2]\sigma_{pX_1} dS - \int_o^\ell [X_1(\ell) - X_1]\sigma_{pX_2} dS \tag{1.11c}$$

and the governing equation is

$$\beta_{22}\frac{\partial^4 \phi}{\partial X_1^4} - \beta_{26}\frac{\partial^4 \phi}{\partial X_1^3 \partial X_2} + 2\gamma\frac{\partial^4 \phi}{\partial X_1^2 \partial X_2^2} - \beta_{16}\frac{\partial^4 \phi}{\partial X_1 \partial X_2^3}$$

$$+ \beta_{11}\frac{\partial^4 \phi}{\partial X_2^4} = 0 \tag{1.12}$$

For direct application in this report, consider the rectangular section of unit thickness in Fig. 2.

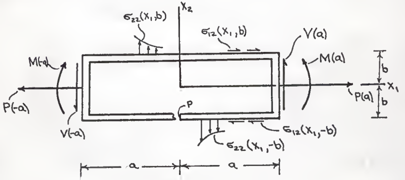

Fig. 2.

As indicated in Fig. 2., it is required that the stresses be specified on $\chi_2 = \pm b$, while only the resultant forces and moments are prescribed on $\chi_1 = \pm a$. For this reason the solution attempted is subject to Saint Venant's principle in the χ_1 direction; i.e., a>>b. Applying equations (1.11) to Fig. 2. leads to the following boundary conditions where the starting point, P, for counter clockwise integration is shown in Fig. 2.

On $\chi_2 = -b$

$$\Phi(\chi_1,-b) = \int_o^{\chi_1} (\chi_1 - s)\sigma_{22}(s,-b)\, ds \tag{1.13a}$$

$$\frac{\partial\Phi(\chi_1,-b)}{\partial\chi_2} = -\int_o^{\chi_1} \sigma_{12}(s,-b)\, ds \tag{1.13b}$$

and on $\chi_2 = +b$

$$\Phi(\chi_1,b) = \int_o^{a} (\chi_1 - s)\sigma_{22}(s,-b)\, ds$$

$$-2b \int_o^{a} \sigma_{12}(s,-b)\, ds + M(a) + V(a)(a-\chi_1)$$

$$+ P(a)b + \int_{\chi_1}^{a} (s-\chi_1)\sigma_{22}(s,b)\, ds \tag{1.14a}$$

$$\frac{\partial\Phi(\chi_1,b)}{\partial\chi_2} = -\int_o^{a} \sigma_{12}(s,-b)\, ds + P(a) + \int_{\chi_1}^{a} \sigma_{12}(s,b)\, ds \tag{1.14b}$$

In (1.14), M(a), V(a), and P(a) are the resultant moment and forces on the end $\chi_1 = a$ as shown in Fig. 2.

It is assumed that the stresaes on the boundaries $\chi_2 = \pm b$ are expressible as polynomials in χ_1;i.e.,

$$\sigma_{22}(\chi_1,-b) = \sum_{m=o}^{r} R_m \chi_1{}^m \tag{1.15a}$$

$$\sigma_{12}(\chi_1,-b) = \sum_{m=o}^{k} K_m \chi_1{}^m \tag{1.15b}$$

$$\sigma_{22}(\chi_1,b) = \sum_{m=o}^{t} T_m \chi_1{}^m \tag{1.15c}$$

$$\sigma_{12}(\chi_1,b) = \sum_{m=o}^{p} P_m \chi_1{}^m \tag{1.15d}$$

From the assumed conditions in (1.15), it follows directly that the boundary conditions (1.13-1.14) are also polynomiala in χ_1, and the aolution of (1.12) is chosen to be of the form

$$\Phi(\chi_1,\chi_2) = \sum_{m=o}^{M} \sum_{n=o}^{N} C_{mn} \chi_1{}^m \chi_2{}^n \tag{1.16}$$

where the C_{mn} are arbitrary constants.

Substitution of (1.16) into (1.12) and rearranging terms yields

$$\sum_{m=2}^{M} \sum_{n=2}^{N} \{\beta_{22}(m+2)(m+1)m(m-1)\ C_{m+2n-2}$$

$$-\beta_{26}(m+1)m(m-1)(n-1)C_{m+1n-1} + 2\gamma m(m-1)n(n-1)C_{mn}$$

$$-\beta_{16}(m-1)(n+1)n(n-1)C_{m-1m+1} + \beta_{11}(n+2)(n+1)n(n-1)C_{m-2n+2}\}$$

$$\chi_1^{m-2}\ \chi_2^{n-2} = 0$$

The fact that this equation must hold true for all x_1 and x_2 leads to the result

$$\beta_{22}(m+2)(m+1)m(m-1)C_{m+2n-2}$$

$$-\ \beta_{26}(m+1)m(m-1)(n-1)C_{m+1n-1}$$

$$+\ 2\gamma m(m-1)n(n-1)C_{mn}$$

$$-\beta_{16}(m-1)(n+1)n(n+1)\ C_{m-1n+1}$$

$$+\beta_{11}(n+2)(n+1)n(n-1)C_{m-2n+2} = 0 \qquad m\geq 2,\ n\geq 2 \qquad (1.17)$$

which is a set of recursion relations for the C_{mn}. It is convenient to formulate a coefficient matrix, Fig. 3.

$m \backslash n$	0	1	2	3	4	5	6	7	8
0	$\cancel{C_{00}}$	$\cancel{C_{01}}$	$\cancel{C_{02}}$	$\cancel{C_{03}}$	C_{04}	C_{05}	C_{06}	C_{07}	C_{08}
1	$\cancel{C_{10}}$	$\cancel{C_{11}}$	$\cancel{C_{12}}$	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}
2	$\cancel{C_{20}}$	$\cancel{C_{21}}$	C_{22}	C_{23}	C_{24}	C_{25}			
3	$\cancel{C_{30}}$			C_{33}					
4	C_{40}		C_{42}						
M= 5	C_{50}	C_{51}							
6	C_{60}								

Fig. 3.

Equation (1.17) relates five consecutive coefficients on a diagonal. One specific example is

$$4\beta_{22}C_{42}-3\beta_{26}C_{33}+8\gamma C_{24}-10\beta_{16}C_{15}+60\beta_{11}C_{06} = 0$$

which involves the five circled coefficients in Fig. 3. In equation (1.16) the largest power of x_1 is M. This indicates that all of the coefficients below the Mth row vanish, and for a given M the recursion relations show that all of the coefficients outside of the step domain on the extreme right (heavily marked in Fig. 3.) must also vanish. It is easily seen that, for all values of M, the domain of non-vanishing coefficients is formed by going four columns to the right on the Mth row and then proceeding up the diagonal steps until $m = 0$, and at this point $N = M+3$. So equation (1.16) takes the form

$$\Phi(x_1,x_2) = \sum_{m=o}^{M} \sum_{n=o}^{M+3} C_{mn}x_1^{\,m}x_2^{\,n} \tag{1.18}$$

where $m+n \leq M+3$ in accordance with the domain of non-vanishing coefficients in Fig. 3.

In order to completely determine the form of the stress function it is necessary that all of the C_{mn} in equation (1.18) be known. The number of unknown coefficients in the first four columns of Fig. 3. is

$$Z = 4(M+1)$$

and the number of coefficients in the upper diagonal square matrix is

$$W = (1/2)M(M+1)$$

Hence, the total numbers of unknown coefficients in (1.18) is given by

$$Q = Z + W = (1/2)(M+1)(M+8) \tag{1.19}$$

These coefficients must be determined from the recursion relations (1.17) and the boundary conditions as expressed in (1.13-1.14).

The recursion relations do not involve any of the C_{mn} in the shaded area of Fig. 3. Thus, each of the coefficients in the upper diagonal square matrix accounts for one recursion formula. It follows that the total number of recursion relations available is

$$W = (1/2)M(M+1) \tag{1.20}$$

For a complete solution

$$Q - W = Z = 4(M+1) \tag{1.21}$$

additional equations must be obtained from the boundary conditions.

After substitution of (1.15) into (1.13-1.14) the boundary conditions become

$$\phi(x_1,-b) = \sum_{m=o}^{M} H_m x_1^{\,m} \tag{1.22a}$$

$$\frac{\partial\Phi(\chi_1,-b)}{\partial\chi_2} = \sum_{m=o}^{M} I_m\chi_1^{\,m} \tag{1.22b}$$

$$\Phi(\chi_1,b) = \sum_{m=o}^{M} J_m\chi_1^{\,m} \tag{1.23a}$$

$$\frac{\partial\Phi(\chi_1,b)}{\partial\chi_2} = \sum_{m=o}^{M} L_m\chi_1^{\,m} \tag{1.23b}$$

where M is the largest power of χ_1 occurring in any one of the equations (1.22-1.23). The known coefficients H_m, I_m, J_m, and L_m take on appropriate values to account for those cases when the largest power of M of χ_1 does not occur in all four of the equations. Substitution of (1.18) into the left side of (1.22-1.23) yields

$$\sum_{m=o}^{M}\sum_{n=o}^{M+3} C_{mn}\chi_1^{\,m}(-b)^n = \sum_{m=o}^{M} H_m\chi_1^{\,m} \tag{1.24a}$$

$$\sum_{m=o}^{M}\sum_{n=1}^{M+3} C_{mn}\chi_1^{\,m}(-b)^{n-1} = \sum_{m=o}^{M} I_m\chi_1^{\,m} \tag{1.24b}$$

$$\sum_{m=o}^{M}\sum_{n=o}^{M+3} C_{mn}\chi_1^{\,m}(b)^n = \sum_{m=o}^{M} J_m\chi_1^{\,m} \tag{1.25a}$$

$$\sum_{m=o}^{M}\sum_{n=1}^{M+3} nC_{mn}\chi_1^{\,m}(b)^{n-1} = \sum_{m=o}^{M} L_m\chi_1^{\,m} \tag{1.25b}$$

By equating coefficients of equal powers of x_1, it is found that equations (1.24-1.25) produce $4(M+1)$ equations for the C_{mn}. This is exactly the number of equations required from the boundary conditions to completely determine the stress function in (1.18).

The genaral mathod of aolution can be outlined as follows: The stresses on the boundaries $x_2 = \pm b$ are expressed as polynomials in x_1. These stresses are substituted in equations (1.13-1.14) to form boundary conditions in terms of the polynomial stress function, $\Phi(x_1,x_2)$. The highest order M of x_1 in (1.18) is chosen to be the highest order of x_1 occurring in the boundary conditions. Then by using the equations expressed by the boundary conditions and the recursion relations, the system of Q equations and Q unknowns is solved directly for the coefficients C_{mn}. The polynomial method provides a systematic approach to solving the plane stress problem of generally anisotropic rectangles. It is obvious that this method cannot satisfy the exact stress distribution on all of the boundaries, since additional boundary conditions similar to (1.13-1.14) on the ends $x_1 = \pm a$ would cause the number of equations to exceed the numbers of unknowns.

EXAMPLE SOLUTIONS

To illustrate the application of the polynomial method, consider a beam loaded uniformly on two boundaries, Fig. 4.

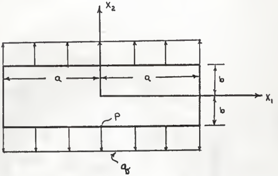

Fig. 4.

The boundary conditions in terms of loadings are

$$\sigma_{12}\ (x_1,b) = \sigma_{12}\ (x_1,-b) = 0 \qquad (2.1a)$$

$$\sigma_{22}\ (x_1,b) = \sigma_{22}\ (x_1,-b) = q \qquad (2.1b)$$

$$V(a) = V(-a) = 0 \qquad (2.2a)$$

$$M(a) = M(-a) = 0 \qquad (2.2b)$$

$$P(a) = P(-a) = 0 \qquad (2.2c)$$

Substitution of (2.1) (2.2) into (1.13-1.14) yields

$$\phi\ (x_1,-b) = \int_o^{x_1} (x_1 - a)\ q\,ds = q\int_0^{x_1} (x_1 s - \frac{s^2}{2})$$

$$= \frac{q x_1^{\,2}}{2} \qquad (2.3a)$$

$$\frac{\partial\phi(x_1,-b)}{\partial x_2} = 0 \qquad (2.3b)$$

$$\Phi\,(x_1,b) = \int_0^a (x_1-s)\ q\,ds + \int_{x_1}^a (s-x_1)\ q\,ds$$

$$= q x_1 a - \frac{qa^2}{2} + \frac{qa^2}{2} - q x_1 a + \frac{q x_1^{\,2}}{2} = \frac{q x_1^{\,2}}{2} \qquad (2.3c)$$

$$\frac{\partial\Phi\,(x_1,b)}{\partial x_2} = 0 \qquad (2.3d)$$

The highest power M of x_1 in (2.3) is 2. Hence, the stress function (1.18) becomes

$$\Phi(x_1,x_2) = \sum_{m=o}^{2}\ \sum_{n=o}^{5} C_{mn}\ x_1^{\ m}\ x_2^{\ n} \qquad m + n \leq 5$$

$$= C_{00} + C_{01}x_2 + C_{02}x_2^{\,2} + C_{03}x_2^{\,3} + C_{04}x_2^{\,4} + C_{05}x_2^{\,5}$$

$$+ C_{10}x_1 + C_{11}x_1x_2 + C_{12}x_1x_2^{\,2} + C_{13}x_1x_2^{\,3} + C_{14}x_1x_2^{\,4}$$

$$+ C_{20}x_1^{\,2} + C_{21}x_1^{\,2}x_2 + C_{22}x_1^{\,2}x_2^{\,2} + C_{23}x_1^{\,2}x_2^{\,3}$$

Inserting (2.4) into the left sides of (2.3) and equating coefficients of equal powers of x_1 produces the following set of equations:

From (2.3a)

$$C_{00} - C_{01}b + C_{02}b^2 - C_{03}b^3 + C_{04}b^4 - C_{05}b^5 = 0$$

$$C_{10} - C_{11}b + C_{12}b^2 - C_{13}b^3 + C_{14}b^4 = 0$$

$$C_{20} - C_{21}b + C_{22}b^2 - C_{23}b^3 = \frac{q}{2} \qquad (2.5a)$$

From (2.3b)

$$C_{01} - 2C_{02}b + 3C_{03}b^2 - 4C_{04}b^3 + 5C_{05}b^4 = 0$$

$$C_{11} - 2C_{12}b + 3C_{13}b^2 - 4C_{14}b^3 = 0 \qquad (2.5b)$$

$$C_{21} - 2C_{22}b + 3C_{23}b^2 = 0$$

From (2.3c)

$$C_{00} + C_{01}b + C_{02}b^2 + C_{03}b^3 + C_{04}b^4 + C_{05}b^5 = 0$$

$$C_{10} + C_{11}b + C_{12}b^2 + C_{13}b^3 + C_{14}b^4 = 0 \qquad (2.5c)$$

$$C_{20} + C_{21}b + C_{22}b^2 + C_{23}b^3 = \frac{q}{2}$$

From (2.3d)

$$C_{01} + 2C_{02}b + 3C_{03}b^2 + 4C_{04}b^3 + 5C_{05}b^4 = 0$$

$$C_{11} + 2C_{12}b + 3C_{13}b^2 + 4C_{14}b^3 = 0 \qquad (2.5d)$$

$$C_{21} + 2C_{22}b + 3C_{23}b^2 = 0$$

The available recursion relations are

$$4\gamma C_{22} - 3\beta_{16}C_{13} + 12\beta_{11}C_{04} = 0$$

$$\gamma C_{23} - \beta_{16}C_{14} + 5\beta_{11}C_{05} = 0 \qquad (2.6)$$

$$-\beta_{16}C_{23} + 2\beta_{11}C_{14} = 0$$

Thus, there are 15 equations and 15 unknowns.

The solution is simplified if corresponding pairs of equations (2.5) are added and subtracted. By solving equations (2.5) and (2.6) it is found that

$$C_{20} = \frac{q}{2} \quad \text{and} \quad C_{mn} = 0 \quad \text{for } m \neq 2,\ n \neq 0$$

The resulting stress function is obviously

$$\phi\,(x_1, x_2) = \frac{q}{2}\, x_1^{\,2}$$

from which it is found by (1.2) that

$$\sigma_{11} = 0$$

$$\sigma_{22} = q$$

$$\sigma_{12} = 0$$

For a second example consider the case of pure bending as illustrated in Fig. 5.

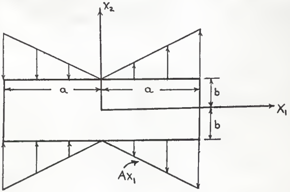

Fig. 5.

The boundary conditions are

$$\sigma_{12}\ (x_1,b) = \sigma_{12}\ (x_1,-b) = 0$$

$$\sigma_{22}\ (x_1,b) = \sigma_{22}\ (x_1,-b) = Ax_1$$

$$M(a) = M(-a) = 0$$

$$V(a) = V(-a) = 0$$

$$P(a) = P(-a) = 0$$

Substitution of the above values into (1.13-1.14) transforms the boundary conditions as follows:

$$\Phi\ (x_1,-b) = \frac{1}{6}Ax_1{}^3$$

$$\frac{\partial\Phi\ (x_1,-b)}{\partial x_2} = 0$$

$$\Phi\ (x_1,b) = \frac{1}{6}Ax_1{}^3$$

$$\frac{\partial\Phi\ (x_1,b)}{\partial x_2} = 0$$

Accordingly, M is 3 and the stress function becomes

$$\Phi\ (x_1,x_2) = \sum_{m=o}^{3}\ \sum_{n=o}^{7}\ C_{mn}\ x_1{}^m x_2{}^n \qquad m + n \leq 7$$

By expression (1.19) there are twenty-two total unknown coefficients. Expression (1.21) reveals that sixteen equations are obtained from the boundary conditions and by (1.20) six equations are available from the recursion formula. This set of equations is similar to that obtained in the previous example and is easily solved to yield

$$C_{30} = \frac{1}{6} A$$

$$C_{mn} = 0 \quad \text{for } m \neq 3,\ n \neq 0$$

The resulting stress function is

$$\Phi\ (x_1, x_2) = \frac{1}{6} A x_1^{\ 3}$$

and the stresses from (1.2) are

$$\sigma_{11} = 0 \qquad \sigma_{12} = 0 \qquad \sigma_{22} = A x_1$$

From the two previous examples it is seen that as the highest power, M, of x_1 increases on the boundaries it becomes necessary to solve an increasing number of equations. Consequently, if the polynomial expression of the loading has a very high power of x_1, the polynomial method becomes very impractical.

The results of the two examples satisfy all of the boundary conditions exactly; i.e. $\sigma_{11}\ (\pm a, x_2) = \sigma_{12}\ (\pm a, x_2) = 0$, and the application of St. Venant's principle is not necessary. In addition none of the anisotropic elastic compliances appear in either of the solutions. It can be concluded that the stresses in plane sections due to pure bending and pure compression or tension are not affected by anisotropy.

PARABOLIC TENSION

For further and more meaningful discussion of the polynomial method, consider the rectangular strip in Fig. 6.

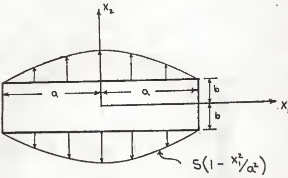

Fig. 6.

The prescribed loadings are

$$\sigma_{12}(x_1,-b) = \sigma_{12}(x_1,b) = 0$$

$$\sigma_{22}(x_1,-b) = \sigma_{22}(x_1,b) = S(1-x_1^2/a^2)$$

$$M(a) = M(-a) = 0$$

$$V(a) = V(-a) = 0$$

$$P(a) = P(-a) = 0$$

Directly, the boundary conditions in terms of $\Phi(x_1,x_2)$ are

$$\Phi(x_1,-b) = \frac{1}{2} S(1 - \frac{1}{6}\frac{x_1^2}{a^2})\, x_1^2$$

$$\frac{\partial \Phi\ (X_1,-b)}{\partial X_2} = 0$$

$$\Phi\ (X_1,b) = \frac{1}{2}\ S(1 - \frac{1}{6}\ \frac{X_1^{\ 2}}{a^2})X_1^{\ 2}$$

$$\frac{\partial \Phi\ (X_1,b)}{\partial X_2} = 0$$

Thus, M = 4 and the resulting stress function is

$$\Phi\ (X_1,X_2) = \sum_{m=o}^{4}\ \sum_{n=o}^{7} C_{mn}X_1^{\ m}X_2^{\ n} \qquad m + n \leq 7$$

Twenty equations from the boundary conditions and ten recursion relations are used to solve for the thirty unknown coefficients. Through a lengthy algebraic detail, the coefficients are found to be

$$C_{00} = \frac{1}{12}\ (\frac{\beta_{22}}{\beta_{11}})\ \frac{Sb^4}{a^2}$$

$$C_{02} = -\ \frac{1}{6}\ (\frac{\beta_{22}}{\beta_{11}})\ \frac{Sb^2}{a^2}$$

$$C_{04} = \frac{1}{12}\ (\frac{\beta_{22}}{\beta_{11}})\ \frac{S}{a^2}$$

$$C_{20} = \frac{1}{2}S$$

$$C_{40} = -\ \frac{1}{12}\ \frac{S}{a^2}$$

and all the remaining C_{mn} are zero.

The final form of the stress function is

$$\Phi\ (\chi_1,\chi_2) = \frac{1}{12}\ (\frac{\beta_{22}}{\beta_{11}})\ \frac{Sb^4}{a^2} - \frac{1}{6}\ (\frac{\beta_{22}}{\beta_{11}})\ \frac{Sb^2}{a^2}\ \chi_2^{\ 2}$$

$$+ \frac{1}{12}\ (\frac{\beta_{22}}{\beta_{11}})\ \frac{Sb}{a^2}\chi_2^{\ 4} + \frac{1}{2}\ S\ \chi_1^{\ 2}$$

$$- \frac{1}{12}\ \frac{S}{a^2}\ \chi_1^{\ 4}$$

from which

$$\sigma_{22} = S\ (1 - \frac{\chi_1^{\ 2}}{a^2}) \tag{2.7a}$$

$$\sigma_{12} = 0 \tag{2.7b}$$

$$\sigma_{11} = \frac{S}{a^2}\ (\frac{\beta_{22}}{\beta_{11}})\ (\chi_2^{\ 2} - \frac{b^2}{3}) \tag{2.7c}$$

From the result in (2.7c) it is seen that σ_{11} is not identically zero on either of the ends, $\chi_1 = \pm a$. In the actual problem (Fig. 6.) the ends are completely stress free. However, in either case the resultant forces and moments are zero.

$$M\ (\pm a) = \int_{-b}^{b} \chi_2\sigma_{11}d\chi_2 = 0$$

$$P\ (\pm a) = \int_{-b}^{b} \sigma_{11}d\chi_2 = 0$$

$$V\ (\pm a) = \int_{-b}^{b} \sigma_{12}d\chi_2 = 0$$

This indicates that the boundary conditions for $\chi_1 = \pm a$ are satisfied approximately, and the application of St. Venant's principle is necessary. In other words, if the length a is large in comparison to the depth b; i.e., a>>b, then the solution for the stresses is approximately valid at a sufficient distance from the ends.

In a plane orthotropic beam where the two planes of symmetry coincide with the x_1 and x_2 axes, it is found that

$$\beta_{22} = S_{2222} = 1/E_2$$

$$\beta_{11} = S_{1111} = 1/E_1$$

where E_1 and E_2 are the moduli of stiffness in the x_1 and x_2 directions, respectively. Thus,

$$\sigma_{11} = \frac{S}{a^2} (E_1/E_2) (x_2^2 - b^2/3) \tag{2.9}$$

The magnitude of σ_{11} is, from (2.9), directly dependent upon the dimensions of the beam, and the relative magnitudes of E_1 and E_2. For practical materials such as woods and composite materials, the magnitude of E_1/E_2 is generally less than 25. It has been mentioned that the results in (2.7) are subject to St. Venant's principle, which implies a>>b for the stresses to be valid at reasonable distances from the ends. In an attempt to find a basis for choosing dimensions of the beam which will yield reasonable results, consider the particular beam with an isotropic medium; i.e., $E_1 = E_2$.

The stresses in (2.7) become

$$\sigma_{22} = S(1 - \frac{x_1^2}{a^2}) \tag{2.10a}$$

$$\sigma_{12} = 0 \tag{2.10b}$$

$$\sigma_{11} = \frac{S}{a^2}\left(x_2^2 - \frac{b^2}{3}\right) \tag{2.10c}$$

The polynomial solution (2.10) will be compared with Timoshenko's solution to the same problem by the method of least work.

Timoshenko shows that for an isotropic plate of unit thickness, the strain energy V is given by

$$V = \frac{1}{2E}\int_{-a}^{a}\int_{-b}^{b}\left[\sigma_{11}^2 + \sigma_{12}^2 + 2\sigma_{12}^2\right] dx_1 dx_2$$

By using equations (1.2) it is found that

$$V = \frac{1}{2E}\int_{-a}^{a}\int_{-b}^{b}\left[\left(\frac{\partial^2\Phi}{dx_1^2}\right)^2 + \left(\frac{\partial^2\Phi}{dx_1^2}\right)^2 + 2\left(\frac{\partial^2\Phi}{\partial x_1\partial x_2}\right)^2\right] dx_1 dx_2 \tag{2.11}$$

The principle of least work states that of all the stress functions which satisfy all the boundary conditions only the correct one will yield (2.11) an absolute minimum.

The stress function in series form is assumed to be

$$\Phi = \Phi_0 + \alpha_1\Phi_1 + \alpha_2\Phi_2 + \cdots + \alpha_n\Phi_n \tag{2.12}$$

where Φ satisfies all the boundary conditions and α_i's are arbitrary constants. Then by inserting (2.12) into (2.11), V becomes a function of the constants $\alpha_1, \alpha_2, \ldots, \alpha_n$. These constants can be calculated from the extremum conditions

$$\frac{\partial V}{\partial\alpha_1} = 0,\ \frac{\partial V}{\partial\alpha_3},\ \frac{\partial V}{\partial\alpha_2} = 0,\ \cdots,\ \frac{\partial V}{\partial\alpha_n} = 0$$

which yield a set of n linear equationa, in terms of the n unknown constants. The principle of least work requires that the values of the constants make (2.11) a minimum. If the assumption in (2.12) is near the correct stress function, then a fairly good approximation to the exact solution is obtained.

For the problem under consideration in Fig. 6. Timoshenko assumes*

$$\Phi\ (x_1,x_2) = \frac{1}{2}\ S\ x_1{}^2(1 - \frac{1}{6}\frac{x_1{}^2}{a^2})$$

$$+ (x_2{}^2 - b^2)^2(x_1{}^2 - a^2)^2\ (\alpha_1 + \alpha_2 x_2{}^2 + \alpha_3 x_1{}^2) \qquad (2.13)$$

Then by substituting (2.13) into (2.11) and requiring

$$\frac{\partial V}{\partial \alpha_1} = 0, \quad \frac{\partial V}{\partial \alpha_2} = 0\ , \quad \frac{\partial V}{\partial \alpha_3} = 0$$

a set of three linear equations in α_i is obtained. (see Timoshenko, p. 170, equations (g)). These equations can be solved for $\alpha_1,\alpha_2,\alpha_3$ in terms of S, a, and b. The three equations and tabulated values of $\alpha_1,\alpha_2,\alpha_3$ for various values of a and b are given in Appendix A.

From equation (2.13) the stresses are

$$\sigma_{11} = \frac{\partial^2\Phi}{\partial x_2{}^2} = 4(x_2{}^2 - b^2)(x_1{}^2 - a^2)^2(\alpha_1 + \alpha_2 x_2{}^2 + \alpha_3 x_1{}^2)$$

$$+ 8x_2{}^2\ (x_1{}^2 - a^2)^2(\alpha_1 + \alpha_2 x_2{}^2 + \alpha_3 x_1{}^2)$$

$$+ 16\alpha_2 x_2{}^2\ (x_2{}^2 - b^2)(x_1{}^2 - a^2)^2$$

*For the problem of Fig. 6. Timoshenko'a a and b must be interchanged and $y \to x_1, x \to x_2$

$$+ 2\alpha_2\ (x_2^{\ 2} - b^2)^2 (x_1^{\ 2} - a^2)^2 \tag{2.13a}$$

$$\sigma_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2} = -16 x_2 x_1\ (x_2^{\ 2} - b^2)\ (x_1^{\ 2} - a^2)(\alpha_1 + \alpha_2 x_2^{\ 2}$$

$$+ \alpha_3 x_1^{\ 2}) - 8\alpha_2 x_1 x_2\ (x_2^{\ 2} - b^2)^2 (x_1^{\ 2} - a^2)$$

$$- 8\alpha_3 x_1 x_2\ (x_2^{\ 2} - b^2)(x_1^{\ 2} - a^2)^2 \tag{2.13b}$$

$$\sigma_{22} = \frac{\partial^2\phi}{\partial x_1^{\ 2}} = S(1 - \frac{x_1^{\ 2}}{a^2})$$

$$+ 4\ (x_2^{\ 2} - b^2)^2 (x_1^{\ 2} - a^2)(\alpha_1 + \alpha_2 x_2^{\ 2} + \alpha_3 x_1^{\ 2})$$

$$+ 8 x_1^{\ 2}\ (x_2^{\ 2} - b^2)^2 (\alpha_1 + \alpha_2 x_2^{\ 2} + \alpha_3 x_1^{\ 2})$$

$$+ 16\alpha_3 x_1^{\ 2} (x_2^{\ 2} - b^2)(x_1^{\ 2} - a^2)$$

$$+ 2\alpha_3 (x_2^{\ 2} - b^2)^2 (x_1^{\ 2} - a^2)^2 \tag{2.13c}$$

The stresses in (2.13) satisfy all the boundary conditions exactly. This means that St. Venant's principle has not been used, and the results of the least work solution can be applied to any given dimensions of the beam or plate in Fig. 6. It is obvious that all of the stresses in (2.13) are directly dependent upon the values of $\alpha_1, \alpha_2, \alpha_3$. By the equations in Appendix A the α_i's are strongly influenced by the dimensions of the beam. Thus, the stresses in (2.13) are directly dependent upon the dimensions of the beam.

Any similarities between the stresses calculated by least work and the polynomial solution are not obvious because of the complex expressions in (2.13). The most radical appearing difference between (2.10) and (2.13) occurs in the shear stress. In order to present a meaningful comparison of the two solutions it is necessary to resort to direct calculations for both the results of (2.10) and (2.13). Then by graphical representations of the data it can be attempted to find a set of ratios a/b which will cause both the polynomial solution and the least work solution to yield similar results. The following comparison is presented in a systematic manner by examining the stress distributions for both methods of solution for progressing ratioa a/b.

Case I.

$$a/b = 1$$

In Fig. 7., $\frac{\sigma_{22}}{S}$ is plotted against $\frac{x_1}{a}$ across the section $x_2 = 0$ for both the polynomial solution and the least work solution.

From Fig. 7., the σ_{22} distribution across the section $x_2 = 0$ by the polynomial solution differs largely from the distribution found by least work. It can also be verified that large errors exist between the two solutions for σ_{11}. The largest shear stress by the least work solution is approximately 0.1S (See Appendix B) and this cannot be completely ignored in comparison to the zero shear stress found by the polynomial solution. These errora should be expected aince the approximate boundary conditions in the polynomial solution have a large effect when considering

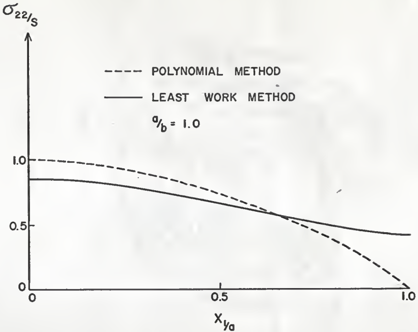

Fig. 7. σ_{22}/S across the section $x_2 = 0.0$.

a square plate; i.e., St. Venant's principle can not be applied to a square plate. It is concluded that the ratio $a/b = 1$ must be discarded for the polynomial solution.

Case II.

$a/b = 3$

The graph of $\frac{\sigma_{22}}{S}$ vs. $\frac{\chi_1}{a}$ for $\chi_2 = 0$ is shown in Fig. 8. The $\frac{\sigma_{22}}{S}$ distribution for both solutions is very nearly the same at the section $\chi_2 = 0$. It can be seen by examining (2.10a) and (2.13c) that the error between the two solutions for σ_{22} is maximum at $\chi_2 = 0$. This error decreases as $\frac{\chi_2}{b}$ increases until the same distribution is reached on the boundaries $\frac{\chi_2}{b} = \pm 1$.

The graph of $\frac{\sigma_{11}}{S}$ vs. χ_2/b is shown in Fig. 9. For the least work solution, $\frac{\sigma_{11}}{S}$ varies with χ_1/a. For the polynomial solution the $\frac{\sigma_{11}}{S}$ distribution remains constant for all values χ_1/a. Fig. 9. shows an remarkable similarity in the general shapes of the curves as long as χ_1/a is reasonably less than 1. When χ_1/a becomes greater than 0.7 the two solutions for σ_{11} begin to differ greatly. For this reason, the region for $\chi_1/a > 0.7$ cannot be considered in the polynomial solution. This is accounted for by the application of St. Venant's principle. For $\chi_1/a \leq 0.7$ the $\frac{\sigma_{11}}{S}$ distribution curves for least work oscillate about the fixed $\frac{\sigma_{11}}{S}$ distribution curve found by the polynomial method. At $\chi_1/a = 0.3$, the two curves nearly coincide. Since $\frac{\sigma_{11}}{S}$ is small in comparison to $\frac{\sigma_{22}}{S}$ for $a/b = 3$, the errors between the two solutions found in Fig. 9. become insignificant when the total effects of σ_{11} and σ_{22} are considered together. Also

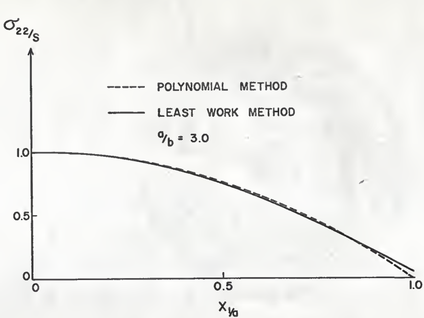

Fig. 8. σ_{22}/S across the section $x_2 = 0.0$.

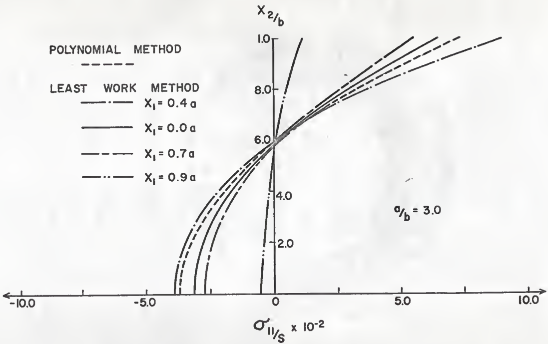

Fig. 9. σ_{11}/S across the sections X_1/a = constant.

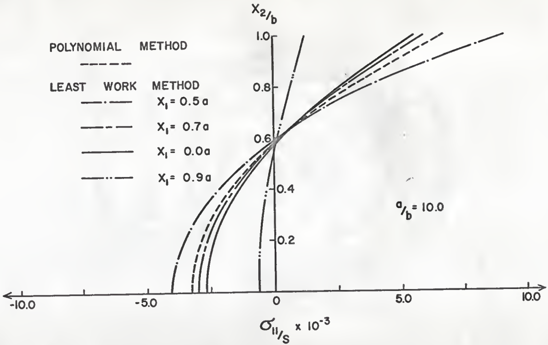

Fig. 10. σ_{11}/S across the sections x_1/a = constant.

these relatively small errors cannot be weighed heavily because of the totally different approaches of the two approximate methods.

In evaluating the shear stress for a comparison, no values can be considered for $X_1/a > 0.7$ bacause in this region the polynomial solution is invalid. The largest shear stress σ_{12} by least work (2.13b) found in the valid region is approximately 0.012S (See Appendix B). By the polynomial solution σ_{12} is zero everywhere. The largest error between the two solutions is obviously 0.012S. Reflecting back to Fig. 9. the largest error between the two solutions for σ_{11} is approximately 0.015S at $X_1/a = 0.4$. Hence, by the same argument presented for the errors in σ_{11} the error in σ_{12} between the two solutions also becomes relatively insignificant. For very large values of S the isolated errors between the two solutions for σ_{12} might appear significant. However, this difference is negligible when the total stress pattern is considered.

From the above discussion the similarities between the least work solution and the polynomial solution for the problem in Fig. 6. are quite evident.

Case III.

$$a/b > 3$$

The σ_{22} distribution found by the polynomial solution is nearly identical to the σ_{22} distribution found by least work for $a/b > 3$.

The graph of $\frac{\sigma_{11}}{S}$ vs. X_1/a for $a/b = 10$ is shown in Fig. 10. The only specific difference between Fig. 10. and Fig. 9. appears in the horizontal scale factor. The $\frac{\sigma_{11}}{S}$ distribution by both methods of solution is reduced by approximately one-tenth when a/b is increased from three to ten.

In addition $\frac{\sigma_{12}}{S}$ by least work is reduced when a/b is increased from three to ten, and the error between the two solutions for shear stress is directly reduced.

It can be shown that the similarities between the results produced by the polynomial and least work solutions become stronger as a/b increases above three, mainly because the error between the shear stresses decreases directly as a/b increases. (See Appendix B)

After examining cases I, II, and III, it can be concluded that the polynomial solution and the least work solution strongly support each other for ratios of $a/b \geq 3$. Consequently, ratios of $a/b \geq 3$ should yield reasonable approximate results when used in the stresses (2.10) derived by the polynomial method.

Since a range of ratios a/b has been determined for the polynomial solution, the orthotropic effects in (2.9) can be discussed in greater detail. Equation (2.9) is rewritten as

$$\sigma_{11} = S(b/a)^2(E_1/E_2)((s_2/b)^2-1/3) \tag{2.14}$$

and

$$(\sigma_{11})_{max} = \frac{2}{3} S \left(\frac{b}{a}\right)^2 \left(\frac{E_1}{E_2}\right) \tag{2.15}$$

which is the maximum σ_{11} found at $X_2 = \pm b$.

For an isotropic beam $E_1 = E_2$ and (2.15) becomes

$$\sigma_{11}{}^{1} = S(b/a)^2[(x_2/b)^2 - 1/3]$$

and

$$(\sigma_{11}{}^{1})_{max} = 2/3S(b/a)^2$$

According to S. G. Lekhnitskii, for pine wood

$$E_1/E_2 = 1/0.042 = 24$$

and from (2.14) and (2.15)

$$\sigma_{11}{}^{0} = 24S(b/a)^2[(x_2/b)^2-1/3]$$

$$(\sigma_{11}{}^{0})_{max} = 16S(b/a)^2$$

The effect of orthotropy is given by

$$\sigma_{11}{}^{0} = 24\sigma_{11}{}^{1}$$

where $\sigma_{11}{}^{1}$ and $\sigma_{11}{}^{0}$ are the isotropic and orthotropic stresses, respectively.

When $a/b = 10$

$$(\sigma_{11}{}^{1})_{max} = 0.00667S$$

and

$$(\sigma_{11}{}^{0})_{max} = 0.16S$$

In either case $(\sigma_{11})_{max}$ is small in comparison to

$$(\sigma_{22})_{max} = S$$

from (2.10). Although the orthotropic effects are significant in relation to the relative magnitudes of σ_{11}^{1} and σ_{11}^{0}, the effect upon the total stress field is negligible for $a/b = 10$.

When $a/b = 4$

$$(\sigma_{11}^{1})_{max} = 0.042S$$

and

$$(\sigma_{11}^{0})_{max} = S$$

Thus, the effects of orthotropy in the pine wood beam are quite significant for $a/b = 4$.

In the orthotropic pine wood beam the σ_{11}^{0} stress is found by multiplying the isotropic stress by twenty-four. This effect is large when the beam is short. However, when E_1/E_2 is changed from one to twenty-four the stress variation is also increased on the ends $x_1/a = \pm 1$. This increase in stress variation at the end boundaries may have a significant effect upon the stress distribution at the center section $x_1 = 0$ when the ratio a/b is small and although it has been verified that the polynomial solution yields reasonable results when $a/b = 3$ for the isotropic beam, this may not be completely true for the orthotropic pine wood beam.

The fact that the two solutions differ largely in σ_{11} when the beam is short leads to the conclusion that the orthotropic effects are significant when the beam is reduced in length. Obviously, the stress in (2.14) is also directly related to the degree of orthotropy.

COMBINED TENSION AND BENDING

In the example problems of pure bending and pure tension, the stress distribution was not affected by anisotropy. For the previous problems of parabolic tension, the stress distributions were found to be directly dependent upon the length of the beam and the degree of orthotropy. These problems seem to indicate that the effects of anisotropy are related to the manner in which the beam is loaded.

For further investigation consider the problem illustrated in Fig. 11.

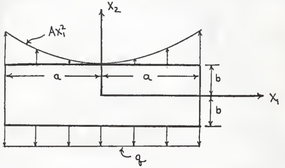

Fig. 11.

The loadings on the beam in Fig. 11. are

$$\sigma_{12}\ (x_1, \pm b) = 0 \tag{3.1a}$$

$$\sigma_{22}\ (x_1, \pm b) = Ax_1^{\ 2} \tag{3.1b}$$

$$\sigma_{22}\ (x_1, -b) = q \tag{3.1c}$$

$$V\ (\pm a) = 0$$

$$M(\pm a) = 0 \qquad P(\pm a) = 0 \tag{3.2}$$

Equilibrium of the beam demands that

$$\int_{-a}^{a} \sigma_{22}(x_1,+b)\,dx_1 - \int_{-a}^{a} \sigma_{22}(x_1,-b)\,dx_1 = Q \tag{3.3}$$

Substitution of (3.1b) and (3.1c) into (3.3) yields

$$A = \frac{3q}{a^2}$$

and (3.1b) becomes

$$\sigma_{22}(x_1,+b) = \frac{3q}{a^2} x_1^{\,2}$$

Directly, the boundary conditions for Φ are

$$\Phi(x_1,-b) = \frac{1}{2} q x_1^{\,2} \tag{3.4a}$$

$$\frac{\partial \Phi(x_1,-b)}{\partial x_2} = 0 \tag{3.4b}$$

$$\Phi(x_1,+b) = \frac{1}{4}\frac{q}{a^2} x_1^{\,4} + \frac{1}{4} q a^2 \tag{3.4c}$$

$$\frac{\partial \Phi(x_1,+b)}{\partial x_2} = 0 \tag{3.4d}$$

From (3.4c) $M = 4$, and the stress function in terms of the thirty unknown coefficients is

$$\Phi(x_1,x_2) = \sum_{m=o}^{4} \sum_{n=o}^{7} C_{mn}\, x_1^{\,m} x_2^{\,n} \qquad m + n \leq 7$$

The solution of this problem yields coefficients which are long and complex. A listing of these coefficients in given in Appendix C. The terms $\frac{\beta_{26}}{\beta_{11}}$ and $\frac{\beta_{16}}{\beta_{11}}$ appear frequently in the coefficients C_{mn}, and for this reason it is necessary to discuss the effect of these factors on the stress distribution.

A typical coefficient is

$$C_{16} = \frac{1}{80}\frac{\beta_{26}}{\beta_{11}}\frac{q}{a^2b^3} + \frac{1}{40}\frac{\gamma\beta_{16}}{\beta_{11}^{\ 2}}\frac{q}{a^2b^3} +$$

$$\frac{1}{40}\frac{\beta_{16}}{\beta_{11}}\frac{q}{a^2b^3}\left(\frac{\gamma}{\beta_{11}} - \frac{1}{2}\frac{\beta_{16}^{\ 2}}{\beta_{11}^{\ 2}}\right) \tag{3.5}$$

The elastic constants β_{26} and β_{16} are $4S_{2212}$ and $4S_{1112}$, respectively, and they relate normal strain to shear stress. The elastic constant β_{11} is S_{1111} and it relates normal strain to normal stress. In common engineering materials β_{26} and β_{16} are small in comparison to β_{11}, and for this reason $\frac{\beta_{16}}{\beta_{11}}$ and $\frac{\beta_{26}}{\beta_{11}}$ are small quantities. Hashin[1] has shown in an extreme case that the largest value of $\frac{\beta_{16}}{\beta_{11}}$ or $\frac{\beta_{26}}{\beta_{11}}$ is about $\pm$ 0.5.

In every coefficient where $\frac{\beta_{26}}{\beta_{11}}$ or $\frac{\beta_{16}}{\beta_{11}}$ appears, an accompanying factor

$$\frac{1}{L}\left(\frac{1}{a^i b^j}\right) \qquad \begin{matrix} i = 0,1,2 \\ j = -1,1,2,3 \end{matrix} \qquad \text{and} \qquad L >>> 1 \tag{3.6}$$

also appears. Since the polynomial solution is only valid for long slender beams; i.e., a>>b, (3.6) must be a small quantity. Thus, the total effect of $\frac{1}{L}\left(\frac{1}{a^ib^j}\right)\left(\frac{\beta_{26}}{\beta_{11}}\right)$ or $\frac{1}{L}\left(\frac{1}{a^ib^j}\right)\left(\frac{\beta_{26}}{\beta_{11}}\right)$ on the stress distributions is negligible.

For any further investigation only the caaes of iaotropy and orthotropy need to be considered. In either case both β_{16} and β_{26} are identically zero. For the case of an orthotropic beam in Fig. 11. the streases are found to be

$$\sigma_{11} = \frac{1}{2}\left(\frac{\beta_{22}}{\beta_{11}}\right)\left(\frac{b^2}{a^2}\right) q$$

$$+ \left[\frac{234}{560}\left(\frac{\beta_{22}}{\beta_{11}}\right)\left(\frac{b}{a^2}\right) - \frac{3}{8}\left(\frac{a^2}{b^3}\right) + \frac{3}{10}\left(\frac{1}{b}\right)\left(\frac{\gamma}{\beta_{11}}\right)\right.$$

$$\left. - \frac{162}{700}\left(\frac{\gamma}{\beta_{11}}\right)\left(\frac{b}{a^2}\right)\right] q x_2$$

$$- \frac{3}{2}\left(\frac{\beta_{22}}{\beta_{11}}\right)\left(\frac{1}{a^2}\right) q {x_2}^2$$

$$+ \left[\frac{3}{5}\left(\frac{\gamma}{\beta_{11}}\right)^2\left(\frac{1}{a^2 b}\right) - \frac{1}{2}\left(\frac{\gamma}{\beta_{11}}\right)\left(\frac{1}{b^3}\right) - \frac{3}{4}\left(\frac{\beta_{22}}{\beta_{11}}\right)\left(\frac{1}{a^2 b}\right)\right] q {x_2}^3$$

$$+ \left[\frac{42}{560}\left(\frac{\beta_{22}}{\beta_{11}}\right)\left(\frac{1}{a^2 b^3}\right) - \frac{42}{140}\left(\frac{\gamma}{\beta_{11}}\right)^2\left(\frac{1}{a^2 b^3}\right)\right] q {x_2}^5$$

$$+ \left[\frac{3}{4}\left(\frac{1}{b^3}\right) - \frac{9}{10}\left(\frac{1}{a^2 b}\right)\left(\frac{\gamma}{\beta_{11}}\right)\right] q {x_1}^2 x_2$$

$$+ \frac{3}{2}\left(\frac{\gamma}{\beta_{11}}\right)\left(\frac{1}{a^2 b^3}\right) q {x_1}^2 {x_2}^3 - \frac{3}{8}\left(\frac{1}{a^2 b^3}\right) q {x_1}^4 x_2 \qquad (3.7)$$

$$\sigma_{22} = \frac{1}{2} q + \left[\frac{3}{20}\left(\frac{b}{a^2}\right)\left(\frac{\gamma}{\beta_{11}}\right) - \frac{3}{4}\left(\frac{1}{b}\right)\right] q x_2$$

$$+ \left[\frac{1}{4} \left(\frac{1}{b^3}\right) - \frac{3}{10} \left(\frac{\gamma}{\beta_{11}}\right) \left(\frac{1}{a^2 b}\right) \right] x_2^{\,3} + \frac{3}{20} \left(\frac{\gamma}{\beta_{11}}\right) \left(\frac{1}{a^2 b^3}\right) q x_2^{\,5}$$

$$+ \frac{3}{2} \left(\frac{1}{a^2}\right) q x_1^{\,2} + \frac{9}{4} \left(\frac{1}{a^2 b}\right) q x_1^{\,2} x_2 - \frac{3}{4} \left(\frac{1}{a^2 b^3}\right) q x_1^{\,2} x_2^{\,3} \qquad (3.8)$$

$$\sigma_{12} = - \frac{3}{20} \left(\frac{\gamma}{\beta_{11}}\right) \left(\frac{b}{a^2}\right) q x_1 + \frac{3}{4} \left(\frac{1}{b}\right) q x_1$$

$$+ \frac{9}{10} \left(\frac{\gamma}{\beta_{11}}\right) \left(\frac{1}{a^2 b}\right) q x_1 x_2^{\,2} - \frac{3}{4} \left(\frac{1}{b^3}\right) q x_1 x_2^{\,2} \qquad (3.9)$$

$$- \frac{3}{4} \left(\frac{\gamma}{\beta_{11}}\right) \left(\frac{1}{a^2 b}\right) q x_1 x_2^{\,4} - \frac{3}{4} \left(\frac{1}{a^2 b}\right) q x_1^{\,3}$$

$$+ \frac{3}{4} \left(\frac{1}{a^2 b^3}\right) q x_1^{\,3} x_2^{\,2}$$

The two factors which are consistently found in the stresses are

$$\left(\frac{\gamma}{\beta_{11}}\right) \quad \text{and} \quad \left(\frac{\beta_{22}}{\beta_{11}}\right)$$

In terms of the general elastic compliances

$$\left(\frac{\gamma}{\beta_{11}}\right) = \frac{S_{1122} + 2S_{1212}}{S_{1111}} \qquad (3.10)$$

and

$$\left(\frac{\beta_{22}}{\beta_{11}}\right) = \frac{S_{2222}}{S_{1111}} \qquad (3.11)$$

For an orthotropic beam with the perpendicular planes of symmetry coinciding with x_1 and x_2

$$S_{1111} = \frac{1}{E_1} \qquad S_{2222} = \frac{1}{E_2}$$

$$S_{1122} = -\frac{\nu_{21}}{E_2} \qquad 2S_{1212} = \frac{1}{2G_{12}}$$

where

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

Directly

$$\left(\frac{\gamma}{\beta_{11}}\right) = -\nu_{12} + \frac{E_1}{2G_{12}} \tag{3.12a}$$

and

$$\left(\frac{\beta_{22}}{\beta_{11}}\right) = \frac{E_1}{E_2} \tag{3.12b}$$

For an isotropic beam

$$\left(\frac{\gamma}{\beta_{11}}\right) = 1 \tag{3.13a}$$

and

$$\left(\frac{\beta_{22}}{\beta_{11}}\right) = 1 \tag{3.13b}$$

For a Boron fiber-reinforced epoxy with the fibers parallel to X_1, Hashin[1] gives the following values of the elastic moduli.

$$E_1 = 24.3 \times 10^6 \text{ psi}$$

$$E_2 = 1.16 \times 10^6 \text{ psi}$$

$$G_{12} = 0.44 \times 10^{6} \text{ psi}$$

$$\nu_{12} = 0.252$$

Inserting these values in (3.12) yields

$$\left(\frac{\gamma}{\beta_{11}}\right) = 27.36 \tag{3.14a}$$

$$\left(\frac{\beta_{22}}{\beta_{11}}\right) = 20.95 \tag{3.14b}$$

Obviously, the Boron fiber-reinforced epoxy is extremely orthotropic.

Fig. 12. shows the σ_{11} stress across the section $\chi_1 = 0$ for $\frac{s}{b} = 10$. The dashed curve and the solid curve indicate the $\frac{\sigma_{11}}{q}$ distribution for the orthotropic epoxy and isotropic beams, respectively. The deviation between the orthotropic epoxy and isotropic beams is evident, but the significance of this deviation is questionable.

It is interesting to note that the curve for the isotropic σ_{11} stress is not exactly linear, but it still follows the flexure formula from basic mechanics of materials very closely. For a long thin beam the flexure formula is

$$\sigma_{11} = -\frac{M\chi_2}{I} \tag{3.15}$$

where I is the moment of inertia of the cross-section, M is the bending moment, and x_2 is the distance from the neutral axis. From (3.4c) the bending moment at $x_1 = 0$ is

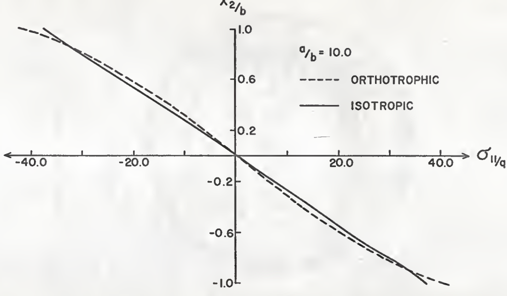

Fig. 12. σ_{11}/q across the section $x_1/a = 0.0$.

$$M = 1/4\ qa^2$$

For the beam of unit thickness in Fig. 11. it is easily verified that at $x_1 = 0$

$$\frac{\sigma_{11}}{q}\Big|_{x_2=\pm b} = \mp \frac{3}{8}\left(\frac{a}{b}\right)^2$$

With $a/b = 10$

$$\frac{\sigma_{11}}{q}\Big|_{x_2=\pm b} = \mp 37.5$$

From Fig. 12. by (3.7) for the isotropic beam

$$\frac{\sigma_{11}}{q}\Big|_{x_2=\pm b} = \mp 37.7$$

Since the solid curve is nearly linear, (3.15) produces good results when applied to the isotropic beam ($a/b=10$) in Fig. 11. at the section $x_1 = 0$. At $x_2/b=1.0$ and $x_1=0$ for the fiber-reinforced beam

$$\frac{\sigma_{11}}{q}\Big|_{x_2=b} = -42.7$$

This the maximum difference found between the orthotropic and isotropic stress when $a/b=10$, and from Fig. 12. the σ_{11} stress deviates moderately from formula (3.15).

Fig. 13., Fig. 14., and Fig. 15. show σ_{11}/q across the section $x_1=0$ for $a/b=5.0$, $a/b=3.5$, and $a/b=3.0$, respectively. As the beam becomes shorter σ_{11}/q is reduced for both the isotropic

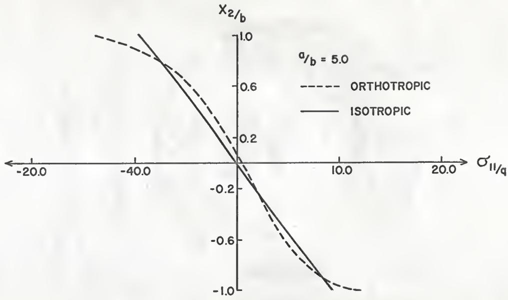

Fig. 13. σ_{11}/q across the section $\chi_1/a = 0.0$.

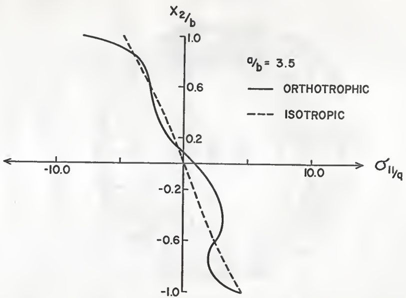

Fig. 14. σ_{11}/q across the section $\chi_1/a = 0.0$.

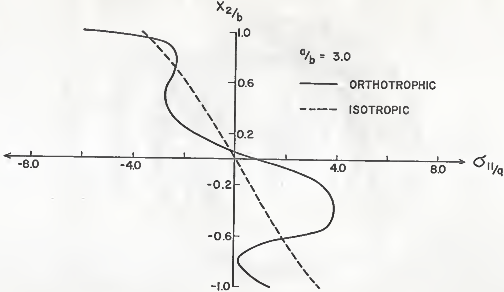

Fig. 15. σ_{11}/q across the section $x_1/a = 0.0$.

and orthotropic beams which is due to a reduction in the center bending moment. In all cases the curves for the isotropic σ_{11}/q are closely approximated by (3.15). This justifies the validity of the isotropic polynomial solution for the values of a/b given in the graphs, because for a/b as small as three the approximate conditions on the end boundaries do not affect the prominence of the center bending moment.

On the other hand, decreasing the length of the beam causes large changes in the orthotropic σ_{11}/q distribution at $x_1=0$. In Fig. 15. (a/b=3) for the fiber-reinforced beam

$$\frac{\sigma_{11}}{q}\bigg|_{x_2/b=-0.8} = 0.15$$

and for the isotropic beam

$$\frac{\sigma_{11}}{q}\bigg|_{x_2/b=-0.8} = 2.67$$

This difference is definitely significant. The variation between the isotropic and orthotropic σ_{11} stress increases gradually until a large difference is found when a/b=3. There is no question that the flexure formula is completely useless for the orthotropic stress shown in Fig. 15.

Table I. shows the isotropic and orthotropic stress distribution on the end $x_1/a=1$ for a/b=3. The variation of the isotropic σ_{11} stress is very small in comparison to the variation of the orthotropic σ_{11} stress. In both cases it can be shown that the resultant force and moment created by the end

TABLE I.

$\frac{\sigma_{11}}{q}$ at the end $\frac{x_1}{a} = 1.0$ by the Polynomial method when $\frac{a}{b} = 3.0$

Orthotropic

$\frac{x_1}{a}$	$\frac{x_2}{b}$	$\frac{\sigma_{11}}{q}$
1.000	1.000	13.723
1.000	0.800	1.722
1.000	0.600	-6.524
1.000	0.400	-8.693
1.000	0.200	-5.319
	0.000	1.164
	-0.200	7.367
	-0.400	9.903
	-0.600	6.337
	-0.800	-3.864
1.000	-1.000	-18.378

Isotropic

$\frac{x_1}{a}$	$\frac{x_2}{b}$	$\frac{\sigma_{11}}{q}$
1.000	1.000	0.268
1.000	0.800	-0.019
	0.600	-0.142
	0.400	-0.140
	0.200	-0.059
	0.000	0.056
	-0.200	0.157
	-0.400	0.198
	-0.600	0.133
	-0.800	-0.083
1.000	-1.000	-0.490

stresses are zero. For a short beam a large variation of end stress obviously has a greater effect on the stress at $x_1=0$ than a small variation of end stress. Thus, when a/b=3 the orthotropic σ_{11} stress is not as accurate as the isotropic stress.

The difference found between the isotropic and orthotropic shear stress, σ_{12}, shown in Fig. 16. for a/b=5 is of no real consequence since the shear stress is small in comparison to σ_{11}. The difference between the isotropic and orthotropic σ_{22} was found to be less than that for σ_{12}, and for this reason neither σ_{12} nor σ_{22} have been investigated in detail.

For the problem in Fig. 11. it is concluded that the σ_{11} stress distribution is directly dependent upon the interrelation of the ratio a/b and the degree of orthotropy. If the beam is long ($a/b \geq 10$) the stress is not affected greatly, even by a highly orthotropic material. If the beam is short the orthotropic solution differs largely from the isotropic solution. From Fig. 15. it is seen that the σ_{11} stress distribution would still be affected by a lesser orthotropic reinforcement. It is difficult to determine the actual effects of orthotropy when the beam is short because of the large boundary stress variation. Since the isotropic solution yields good results when the ratio a/b=3 and the corresponding orthotropic solution differs largely from the isotropic solution, there appears to be large orthotropic effects in a short beam although an accurate measure of the effects cannot be obtained with the polynomial solution.

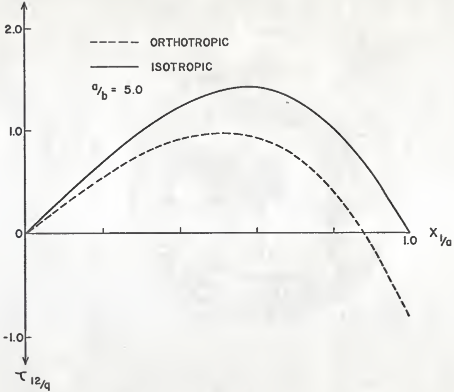

Fig. 16. σ_{12}/q across the section $x_2/b = 0.0$.

CONCLUSION

The polynomial method provides a direct and systematic approach to the solution of generally plane anisotropic beam problems with a>>b.

Problems solved by Hashin[1] along with the last problem of combined tension and bending solved in this report reveal that the anisotropic factors $\frac{\beta_{16}}{\beta_{11}}$ and $\frac{\beta_{26}}{\beta_{11}}$ are not large enough to cause any significant deviation from the stresses found when considering an isotropic beam. Since $\frac{\beta_{16}}{\beta_{11}}$ and $\frac{\beta_{26}}{\beta_{11}}$ are negligible factors, it is only necessary to compare the stress distributions in orthotropic and isotropic beams. In either type of beam β_{16} and β_{26} are identically zero.

In the example problems of pure tension and pure bending the stress diatribution is completely independent of the elastic properties of the beam.

In the problem of parabolic tension the solution for the σ_{11} stress is directly affected by the degree of orthotropy. This effect is aignificant if the beam is relatively short. If $\frac{a}{b}$ is large, the σ_{11} stress becomes negligible and any orthotropic effects on such a small stress is of little consequence.

In the problem of combined tension and bending it is quite obvious that the solution is directly related to the length of the beam and the degree of orthotropy.

For the problems investigated in this report, the effects of anisotropy are directly dependent upon (1) the degree of orthotropy, (2) the dimensions of the beam, and (3) the type of loading which the beam undertakes. If the beam is long ($\frac{a}{b} > 10$), the isotropic solution yields reasonable results irregardless of the beam material. However, in relatively short beams the orthotropic properties may play an important role in the solution.

In general it is concluded that any significant anisotropic influence upon the stresses must take place in short beams or square plates. The polynomial method cannot be used to determine the anisotropic or orthotropic stresses accurately in square plates or short beams because of the approximation of the end conditions. Continued investigations should deal with beams or plates ($a/b \leq 3$) by methods which can adequately fulfill all the boundary conditions.

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APPENDIX A

The three equations for α_1, α_2, α_3 found by Timoshenko's least work solution of the problem in Fig. 6. are

$$(1)\ \alpha_1 \left(\frac{64}{7} + \frac{256}{49}\frac{a^2}{b^2} + \frac{64}{7}\frac{a^4}{b^4}\right) + \alpha_2 b^2 \left(\frac{64}{77} + \frac{64}{49}\frac{a^4}{b^4}\right)$$

$$+ \alpha_3 b^2 \left(\frac{64}{49}\frac{a^2}{b^2} + \frac{64}{77}\frac{a^6}{b^6}\right) = \frac{S}{b^4 a^2}$$

$$(2)\ \alpha_1 \left(\frac{64}{11} + \frac{64}{7}\frac{a^4}{b^4}\right) + \alpha_2 b^2 \left(\frac{192}{143} + \frac{256}{77}\frac{a^2}{b^2} + \frac{192}{7}\frac{a^4}{b^4}\right)$$

$$+ \alpha_3 b^2 \left(\frac{64}{77}\frac{a^2}{b^2} + \frac{64}{77}\frac{a^6}{b^6}\right) = \frac{S}{b^4 a^2}$$

$$(3)\ \alpha_1 \left(\frac{64}{7} + \frac{64}{11}\frac{a^4}{b^4}\right) + \alpha_2 b^2 \left(\frac{64}{77} + \frac{64}{77}\frac{a^4}{b^4}\right)$$

$$+ \alpha_3 b^2 \left(\frac{192}{7}\frac{a^2}{b^2} + \frac{256}{77}\frac{a^4}{b^4} + \frac{192}{143}\frac{a^6}{b^3}\right) = \frac{S}{b^4 a^2}$$

The values of α_1, α_2, α_3 for various ratios of $\frac{a}{b}$ are

$\frac{a}{b}$	α_1	α_2	α_3
1	$0.04040\ S/b^6$	$0.01172\ S/b^8$	$0.01172\ S/b^8$
2	$0.00120\ S/b^6$	$0.00007\ S/b^8$	$0.00064\ S/b^8$
3	$0.00115\times10^{-4} S/b^6$	$0.24273\times10^{-5}\ S/b^8$	$0.51440\times10^{-4}\ S/b^8$
4	$0.16836\times10^{-4} S/b^6$	$0.22414\times10^{-6}\ S/b^8$	$0.62646\times10^{-5}\ S/b^8$
5	$0.43528\times10^{-5}\ S/b^6$	$0.36332\times10^{-7}\ S/b^8$	$0.11128\times10^{-5}\ S/b^8$
8	$0.25905\times10^{-6}\ S/b^6$	$0.84000\times10^{-9}\ S/b^8$	$0.26635\times10^{-7}\ S/b^8$
10	$0.68210\times10^{-7}\ S/b^6$	$0.13793\times10^{-9}\ S/b^8$	$0.44817\times10^{-8}\ S/b^8$

APPENDIX B

Shear stress in the beam shown in Fig. 6.

by the method of least work.

a/b=1.0

$x_{1/b}$	$x_{2/b}$	$\sigma_{12/S}$
.0000	.2000	-0.00000
.1000	.2000	-.00899
.2000	.2000	-.01775
.3000	.2000	-.02598
.4000	.2000	-.03324
.5000	.2000	-.03892
.6000	.2000	-.04213
.7000	.2000	-.04168
.8000	.2000	-.03595
.9000	.2000	-.02292
1.0000	.2000	-0.00000
.0000	.4000	-0.00000
.1000	.4000	-.01686
.2000	.4000	-.03324
.3000	.4000	-.04855
.4000	.4000	-.06198
.5000	.4000	-.07236
.6000	.4000	-.07809
.7000	.4000	-.07698
.8000	.4000	-.06619
.9000	.4000	-.04204
1.0000	.4000	-0.00000
.0000	.6000	-0.00000
.1000	.6000	-.02141
.2000	.6000	-.04213
.3000	.6000	-.06138
.4000	.6000	-.07809
.5000	.6000	-.09080
.6000	.6000	-.09754
.7000	.6000	-.09569
.8000	.6000	-.08186
.9000	.6000	-.05174
1.0000	.6000	-0.00000
.0000	.8000	-0.00000
.1000	.8000	-.01830
.2000	.8000	-.03595
.3000	.8000	-.05223
.4000	.8000	-.06619
.5000	.8000	-.07660
.6000	.8000	-.08186
.7000	.8000	-.07986
.8000	.8000	-.06793
.9000	.8000	-.04268
1.0000	.8000	-0.00000

a/b=3.0

	$x_{1/b}$	$x_{2/b}$	$\sigma_{12/S}$
	.0000	.2000	-0.00000
	.3000	.2000	.00104
	.6000	.2000	.00168
	.9000	.2000	.00161
	1.2000	.2000	.00062
	1.5000	.2000	-.00125
	1.8000	.2000	-.00371
	2.1000	.2000	-.00612
	2.4000	.2000	-.00743
	2.7000	.2000	-.00609
	3.0000	.2000	-0.00000
	.0000	.4000	-0.00000
	.3000	.4000	.00181
	.6000	.4000	.00293
	.9000	.4000	.00279
	1.2000	.4000	.00107
	1.5000	.4000	-.00221
	1.8000	.4000	-.00652
	2.1000	.4000	-.01074
	2.4000	.4000	-.01302
	2.7000	.4000	-.01066
	3.0000	.4000	-0.00000
	.0000	.6000	-0.00000
	.3000	.6000	.00206
	.6000	.6000	.00333
	.9000	.6000	.00316
	1.2000	.6000	.00118
	1.5000	.6000	-.00257
	1.8000	.6000	-.00750
	2.1000	.6000	-.01232
	2.4000	.6000	-.01491
	2.7000	.6000	-.01220
	3.0000	.6000	-0.00000
	.0000	.8000	-0.00000
	.3000	.8000	.00153
	.6000	.8000	.00247
	.9000	.8000	.00234
	1.2000	.8000	.00084
	1.5000	.8000	-.00198
	1.8000	.8000	-.00567
	2.1000	.8000	-.00928
	2.4000	.8000	-.01122
	2.7000	.8000	-.00917
	3.0000	.8000	-0.00000

a/b=4.0	x_1/b	x_2/b	σ_{12}/S
	.0000	.2000	-0.00000
	.4000	.2000	.00062
	.8000	.2000	.00103
	1.2000	.2000	.00107
	1.6000	.2000	.00062
	2.0000	.2000	-.00031
	2.4000	.2000	-.00157
	2.8000	.2000	-.00283
	3.2000	.2000	-.00356
	3.6000	.2000	-.00297
	4.0000	.2000	-0.00000
	.0000	.4000	-0.00000
	.4000	.4000	.00108
	.8000	.4000	.00181
	1.2000	.4000	.00186
	1.6000	.4000	.00107
	2.0000	.4000	-.00055
	2.4000	.4000	-.00275
	2.8000	.4000	-.00495
	3.2000	.4000	-.00623
	3.6000	.4000	-.00520
	4.0000	.4000	-0.00000
	.0000	.6000	-0.00000
	.4000	.6000	.00124
	.8000	.6000	.00206
	1.2000	.6000	.00212
	1.6000	.6000	.00122
	2.0000	.6000	-.00064
	2.4000	.6000	-.00315
	2.8000	.6000	-.00567
	3.2000	.6000	-.00713
	3.6000	.6000	-.00595
	4.0000	.6000	-0.00000
	.0000	.8000	-0.00000
	.4000	.8000	.00092
	.8000	.8000	.00154
	1.2000	.8000	.00158
	1.6000	.8000	.00090
	2.0000	.8000	-.00049
	2.4000	.8000	-.00237
	2.8000	.8000	-.00426
	3.2000	.8000	-.00535
	3.6000	.8000	-.00447
	4.0000	.8000	-0.00000

a/b=10.0

x_1/b	x_2/b	σ_{11}/S
1.0000	.2000	.00005
2.0000	.2000	.00008
3.0000	.2000	.00008
4.0000	.2000	.00005
5.0000	.2000	-.00001
6.0000	.2000	-.00010
7.0000	.2000	-.00019
8.0000	.2000	-.00024
9.0000	.2000	-.00020
10.0000	.2000	-0.00000
.0000	.4000	-0.00000
1.0000	.4000	.00008
2.0000	.4000	.00013
3.0000	.4000	.00014
4.0000	.4000	.00009
5.0000	.4000	-.00002
6.0000	.4000	-.00018
7.0000	.4000	-.00033
8.0000	.4000	-.00042
9.0000	.4000	-.00036
10.0000	.4000	-0.00000
.0000	.6000	-0.00000
1.0000	.6000	.00009
2.0000	.6000	.00015
3.0000	.6000	.00016
4.0000	.6000	.00010
5.0000	.6000	-.00003
6.0000	.6000	-.00020
7.0000	.6000	-.00038
8.0000	.6000	-.00049
9.0000	.6000	-.00041
10.0000	.6000	-0.00000
.0000	.8000	-0.00000
1.0000	.8000	.00007
2.0000	.8000	.00011
3.0000	.8000	.00012
4.0000	.8000	.00007
5.0000	.8000	-.00002
6.0000	.8000	-.00015
7.0000	.8000	-.00029
8.0000	.8000	-.00036
9.0000	.8000	-.00031
10.0000	.8000	-0.00000

APPENDIX C

Coefficients C_{mn} of

$$\Phi\,(x_1,x_2) = \sum_{m=o}^{4}\ \sum_{n=o}^{7} C_{mn}\, x_1^{\,m} x_2^{\,n} \qquad\qquad m + n \le 7$$

for the problem in Fig. 11.

C_{00}, C_{01}, C_{10} have not been calculated since they do not affect the stresses.

$$C_{02} = \frac{1}{4}\left(\frac{\beta_{22}}{\beta_{11}}\right)\left(\frac{b}{a}\right)^2 q$$

$$\begin{aligned} C_{03} = {} & \frac{39}{560}\left(\frac{\beta_{22}}{\beta_{11}}\right)\left(\frac{b}{a^2}\right) q - \frac{191}{2240}\,\frac{\beta_{16}\beta_{26}}{\beta_{11}^{\,2}}\left(\frac{b}{a^2}\right) q \\ & - \frac{1}{16}\,\frac{a^2}{b^3}\, q + \frac{5}{160}\,\frac{\beta_{16}^{\,2}\gamma}{\beta_{11}^{\,3}}\left(\frac{b}{a^2}\right) q \\ & + \frac{1}{20}\left(\frac{\gamma}{\beta_{11}} - \frac{1}{2}\,\frac{\beta_{16}^{\,2}}{\beta_{11}^{\,2}}\right)\left(\frac{1}{b}\right) q \\ & + \frac{9}{800}\,\frac{\beta_{16}^{\,2}}{\beta_{11}^{\,2}}\left(\frac{\gamma}{\beta_{11}} - \frac{1}{2}\,\frac{\beta_{16}^{\,2}}{\beta_{11}^{\,2}}\right)\left(\frac{b}{a^2}\right) q \\ & - \frac{27}{700}\,\frac{\gamma}{\beta_{11}}\left(\frac{\gamma}{\beta_{11}} - \frac{1}{2}\,\frac{\beta_{16}^{\,2}}{\beta_{11}^{\,2}}\right)\left(\frac{b}{a^2}\right) q \end{aligned}$$

$$C_{04} = -\frac{1}{8}\,\frac{\beta_{22}}{\beta_{11}}\left(\frac{1}{a^2}\right) q$$

$$C_{05} = -\frac{3}{80}\left(\frac{\beta_{22}}{\beta_{11}}\right)\left(\frac{1}{a^2 b}\right) q + \frac{1}{20}\,\frac{\beta_{16}\beta_{26}}{\beta_{11}^{\,2}}\left(\frac{1}{a^2 b}\right) q$$

$$- \frac{1}{40} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^2}{\beta_{11}^2}\right) \left(\frac{1}{b^3}\right) \quad q$$

$$+ \frac{3}{100} \frac{\gamma}{\beta_{11}} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{16^2}{\beta_{11}^2}\right) \left(\frac{1}{a^2 b}\right) q$$

$$- \frac{3}{200} \frac{\beta_{16}^2}{\beta_{11}^2} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^2}{\beta_{11}^2}\right) \left(\frac{1}{a^2 b}\right) q$$

$$- \frac{1}{40} \frac{\beta_{16}^2 \gamma}{\beta_{11}^3} \left(\frac{1}{a^2 b}\right) q$$

$$C_{06} = 0$$

$$C_{07} = \frac{1}{560} \frac{\beta_{22}}{\beta_{11}} \left(\frac{1}{a^2 b^3}\right) q - \frac{11}{2240} \frac{\beta_{16}\beta_{26}}{\beta_{11}^2} \left(\frac{1}{a^2 b^3}\right) q$$

$$+ \frac{1}{160} \frac{\beta_{16}^2}{\beta_{11}^3} \left(\frac{1}{a^2 b^3}\right) q + \frac{1}{160} \frac{\beta_{16}^2}{\beta_{11}^2} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^2}{\beta_{11}^2}\right) \left(\frac{1}{a^2 b^3}\right) q$$

$$- \frac{1}{140} \frac{\gamma}{\beta_{11}} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^2}{\beta_{11}^2}\right) \left(\frac{1}{a^2 b^3}\right) q$$

$$C_{11} = 0$$

$$C_{12} = - \frac{27}{80} \frac{\beta_{26}}{\beta_{11}} \left(\frac{b}{a^2}\right) q + \frac{7}{40} \frac{\gamma\beta_{16}}{\beta_{11}^2} \left(\frac{b}{a^2}\right) q$$

$$- \frac{1}{8} \frac{\beta_{16}}{\beta_{11}} \left(\frac{1}{b}\right) q + \frac{3}{40} \frac{\beta_{16}}{\beta_{11}} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^2}{\beta_{11}^2}\right) \left(\frac{b}{a^2}\right) q$$

$$C_{13} = 0$$

$$C_{14} = \frac{3}{16} \frac{\beta_{26}}{\beta_{11}} \left(\frac{1}{a^2 b}\right) q - \frac{1}{8} \frac{\beta_{16}\gamma}{\beta_{11}^2} \left(\frac{1}{a^2 b}\right) q$$

$$- \frac{3}{40} \frac{\beta_{16}}{\beta_{11}} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^{2}}{\beta_{11}^{2}}\right) \left(\frac{1}{a^{2}b}\right) q + \frac{1}{16} \frac{\beta_{16}}{\beta_{11}} \left(\frac{1}{b^{3}}\right) q$$

$$C_{15} = 0$$

$$C_{16} = - \frac{1}{80} \frac{\beta_{26}}{\beta_{11}} \left(\frac{1}{a^{2}b^{3}}\right) q + \frac{1}{40} \frac{\gamma\beta_{16}}{\beta_{11}^{2}} \left(\frac{1}{a^{2}b^{3}}\right) q$$

$$+ \frac{1}{40} \frac{\beta_{16}}{\beta_{11}} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^{2}}{\beta_{11}^{2}}\right) \left(\frac{1}{a^{2}b^{3}}\right) q$$

$$C_{20} = \frac{1}{4} q$$

$$C_{21} = \frac{3}{40} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^{2}}{\beta_{11}^{2}}\right) \left(\frac{b}{a^{2}}\right) q - \frac{3}{8} \left(\frac{1}{b}\right) q$$

$$C_{22} = 0$$

$$C_{23} = - \frac{3}{20} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^{2}}{\beta_{11}^{2}}\right) \left(\frac{1}{a^{2}b}\right) q + \frac{1}{8} \left(\frac{1}{b^{3}}\right) q$$

$$C_{24} = 0$$

$$C_{25} = \frac{3}{40} \left(\frac{\gamma}{\beta_{11}} - \frac{1}{2} \frac{\beta_{16}^{2}}{\beta_{11}^{2}}\right) \left(\frac{1}{a^{2}b^{3}}\right) q$$

$$C_{30} = - \frac{1}{16} \frac{\beta_{16}}{\beta_{11}} \left(\frac{b}{a^{2}}\right) q$$

$$C_{31} = 0$$

$$C_{32} = \frac{1}{8} \frac{\beta_{16}}{\beta_{11}} \left(\frac{1}{a^{2}b}\right) q$$

$$c_{33} = 0$$

$$c_{34} = -\frac{1}{16}\frac{\beta_{16}}{\beta_{11}}\left(\frac{1}{a^2b^3}\right) q$$

$$c_{40} = \frac{1}{8}\left(\frac{1}{a^2}\right) q$$

$$c_{41} = \frac{3}{16}\left(\frac{1}{a^2b}\right) q$$

$$c_{42} = 0$$

$$c_{43} = -\frac{1}{16}\left(\frac{1}{a^2b^3}\right) q$$

POLYNOMIAL STRESSES
IN PLANE ANISOTROPIC BEAMS

by

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B. S., Cornell College, 1965

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A systematic solution to plane stress problems of generally anisotropic beams is discussed in detail. The method of solution, which is subject to Saint Venant's principle, is employed to solve several plane problems. It is found that anisotropy has very little influence on the stress fields in long thin beams. The largest anisotropic effects are found when a beam is relatively short and the degree of orthotropy is large. The findings of this report demand further investigations of the anisotropic stresses in short beams and square plates by methods which are not dependent upon Saint Venant's principle.