CORRELATION OF GAS FLOWS IN HORIZONTAL AND VERTICAL PIPES

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TABLE OF CONTENTS

NOMENCLATURE

A	flow area, ft^2
C	sound speed, ft/sec
C_p	constant pressure specific heat, ft lbf/slug ^oR
D	diameter of pipe, ft
F	impulse function, lbf
f	friction coefficient
g	acceleration of gravity, ft/sec^2
k	specific heat ratio
L	length of pipe, ft
M	Mach Number
P	pressure, lbf/ft^2 abs
Q	heat, ft lbf/slug
q_v	parameter defined on page 12
R	gas constant, ft lbf/slug ^oR
T	temperature, ^oR
v	specific volume, ft^3/slug
Z	elevation, ft
ρ	density, $slug/ft^3$
τ_w	wall shearing stress, lbf/ft^2
$(\ )^*$	condition at Mach Number unity
$(\ )^{*t}$	condition at Mach Number equal to $1/\sqrt{k}$
$(\ )_o$	stagnation condition
$(\ )_h$	horizontal property
$(\ )_v$	vertical property
f( )	functions defined on page 7

g( )   functions defined on page 13

h( )   functions defined on page 18 to 19

j( )   functions defined on pages 23 to 24

INTRODUCTION

From a one-dimensional point of view, the three most common factors tending to produce continuous changes in the state of a flowing stream for horizontal flow are: (1) changes in cross-sectional area (2) external heat exchange, and (3) wall friction. Furthermore, if the gas flows along a vertical pipe, the effects of change in elevation must be considered.

Four basic cases are considered in this report. The first case is isentropic flow. In horizontal isentropic flow, the area change is the main factor that causes the change in pressure, temperature and other properties. In vertical isentropic flow with constant cross-sectional area, the main factor that causes the change in properties is the change in elevation. These two flows have been correlated by a functional relation between area change for horizontal flow and elevation change for vertical flow. The other three cases studied are: (1) reversible diabatic flow with constant cross-sectional area for both the horizontal and vertical flows, (2) reversible isothermal flow with variable area in the case of horizontal flow and constant area in the case of vertical flow, (3) irreversible isothermal flow in which the area is constant in both the horizontal and vertical case.

The results of the correlations are presented on graphs in which the flow parameters that are used in the correlation are the independent variables and other quantities, such as Mach Number, pressure, etc., are dependent variables.

In all the cases studied the flow is subsonic, and in the vertical cases the flow is upward.

CORRELATION OF VERTICAL AND HORIZONTAL ISENTROPIC FLOWS

In reversible, isentropic, horizontal flow it is the change in cross-sectional area that physically causes the change in Mach Number, temperature, pressure and other properties. In reversible, isentropic, vertical flow, with constant cross-sectional area, it is the change in elevation that physically causes the change in Mach Number, temperature, etc,. As area change is the fundamental entity that causes property changes in horizontal flow, and elevation change is the fundamental entity that causes property changes in the vertical flow, these two fundamental entities have been chosen as the basis for the correlation of these two types of isentropic flow. As developed later this correlation is obtained through the equation

$$\left(\frac{A}{A^*}\right)_h = \left[1 + \frac{2(k-1)}{k+1}\frac{g}{C^{*2}}\Delta Z^*\right]^{\frac{k+1}{2(k-1)}} = \frac{1}{M}\left[\frac{2}{k+1}\left(1+\frac{k-1}{2}M^2\right)\right]^{\frac{k+1}{2(k-1)}}$$

Correlation of other properties are made in a similar manner. For example, the correlation between $\left(\frac{T}{T^*}\right)_h$ and $\left(\frac{T}{T^*}\right)_v$ is produced from the following relations

$$\left(\frac{T}{T^*}\right)_h = \left[M\left(\frac{A}{A^*}\right)\right]^{\frac{2(1-k)}{k+1}}$$

$$\left(\frac{T}{T^*}\right)_v = \left[1 + \frac{2(k-1)}{k+1}\frac{g}{C^{*2}}\Delta Z^*\right] / \left(\frac{2}{k+1}\right)\left(1 + \frac{k+1}{2}M^2\right)$$

$$\left(\frac{T}{T^*}\right)_v = \left[\frac{k+1}{k-1}\left(\frac{T^*}{T}\right)_h - \frac{2}{k-1}\right]^{\frac{1-k}{k+1}}$$

and

$$\left(\frac{T}{T^*}\right)_h = \frac{k+1}{k-1} / \left[\left(\frac{T}{T^*}\right)_v^{\frac{k+1}{1-k}} + \frac{2}{k+1}\right]$$

Isentropic Horizontal Flow

The relations for this type of flow are well established and may be found in any standard text on gas dynamics. The equations used in this report are from pages 83 to 87 of reference 1.

$$\left(\frac{A}{A^*}\right)_h = \frac{1}{M}\left[\frac{2}{k+1}\left(1 + \frac{k-1}{2}M^2\right)\right]^{\frac{k+1}{2(k-1)}} \tag{1-1}$$

$$\left(\frac{T}{T^*}\right)_h = \frac{k+1}{2 + (k-1)M^2} \tag{1-2}$$

$$\left(\frac{P}{P^*}\right)_h = \left[\frac{k+1}{2 + (k-1)M^2}\right]^{\frac{k}{k-1}} \tag{1-3}$$

$$\left(\frac{F}{F^*}\right)_h = \frac{1 + kM^2}{M\left[2(k+1)\left(1 + \frac{k-1}{2}M^2\right)\right]^{\frac{1}{2}}} \tag{1-4}$$

Isentropic Vertical Flow

<u>Basic Governing Equations</u> For isentropic flow in a vertical pipe with constant cross-sectional area, the basic governing equations can be written in the following forms:

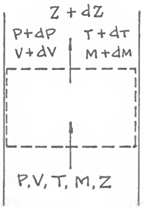


FIG. 1.

Energy Equation

$$C_p dT + VdV + gdZ = 0 \tag{1-5}$$

Momentum Equation

$$\frac{dP}{\rho} + VdV + gdZ = 0 \tag{1-6}$$

Continuity Equation

$$\frac{d\rho}{\rho} + \frac{1}{2}\frac{dV^2}{V^2} = 0 \tag{1-7}$$

Perfect Gas Relation

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{1-8}$$

Definition of Mach Number

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T} \tag{1-9}$$

Working Formulas From the above relations, expressions for pressure, temperature and other properties in terms of Mach Number (M) can be obtained. Combining (1-5) and (1-6)

$$C_p dT = \frac{dP}{\rho} = C^2 \frac{d\rho}{\rho} = kRT \frac{d\rho}{\rho}$$

where $C_p = kR/k-1$, and

$$\frac{dT}{T} = \frac{d\rho}{\rho} (k-1) \tag{1-10}$$

Substituting (1-6), (1-7) into (1-5) and eliminating $\frac{d\rho}{\rho}$,

$$\frac{dM^2}{M^2} = - (\frac{k+1}{k-1}) \frac{dT}{T} \tag{1-11}$$

Integrating between an arbitrary section and the critical section

$$\ln M^2 \Big|_M^1 = - (\frac{k+1}{k-1}) \ln T \Big|_T^{T^*}$$

$$(\frac{T}{T^*})_v = (\frac{1}{M^2})^{\frac{k+1}{k-1}} \tag{1-12}$$

From isentropic relations

$$(\frac{P}{P^*})_v = (\frac{T}{T^*})_v^{\frac{k}{k-1}} = (\frac{1}{M^2})^{\frac{k}{k+1}} \tag{1-13}$$

From the definition of the impulse function

$$(\frac{F}{F^*})_v = (\frac{P}{P^*})_v \frac{1+kM^2}{1+k} \tag{1-14}$$

Combining (1-14) and (1-13)

$$\left(\frac{F}{F^*}\right)_v = \left(\frac{1}{M}\right)^{\frac{2k}{k+1}} \left(\frac{1+kM^2}{1+k}\right) \tag{1-15}$$

Similarly

$$\left(\frac{T_o}{T_o^*}\right)_v = \left(\frac{T}{T^*}\right)_v \left(\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}}\right) \tag{1-16}$$

From (1-12) and (1-16)

$$\left(\frac{T_o}{T_o^*}\right)_v = M^{\frac{2(1-k)}{1+k}} \left(\frac{2}{1+k} + \frac{k-1}{k+1} M^2\right) \tag{1-17}$$

$$\left(\frac{P_o}{P_o^*}\right)_v = \left(\frac{P}{P^*}\right)_v \left(\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}}\right)^{\frac{k}{k-1}} \tag{1-18}$$

From (1-13) and (1-18)

$$\left(\frac{P_o}{P_o^*}\right)_v = \left(\frac{1}{M}\right)^{\frac{2k}{k+1}} \left(\frac{2}{1+k} + \frac{k-1}{k+1} M^2\right)^{\frac{k}{k-1}} \tag{1-19}$$

Relation between Z and M Writing the energy equation between any section and the critical section

$$C_p(T^* - T) + \frac{V^{*2} - V^2}{2} + g\Delta Z^* = 0$$

Dividing through by C_pT^*

$$\left(1 - \frac{T}{T^*}\right) + \left(\frac{k-1}{2}\right) \frac{V^{*2} - V^2}{kRT^*} + \frac{k-1}{k} \frac{\Delta Z^*}{RT^*} g = 0$$

$$\frac{g}{C^{*2}} \Delta Z^* = \left(\frac{T}{T^*} - 1\right) \frac{1}{k-1} + \frac{1}{2} \frac{V^2 - V^{*2}}{C^{*2}} \tag{1-20}$$

$$\frac{V^2 - V^{*2}}{C^{*2}} = \frac{V^2}{V^{*2}} - 1 = \frac{M^2 kRT}{kRT^*} - 1 = M^2 \frac{T}{T^*} - 1 \tag{1-21}$$

From (1-20), (1-21) and (1-12)

$$\frac{g}{C^{*2}} \Delta Z^* = \frac{1}{k-1} \left[(M)^{\frac{2(1-k)}{1+k}} - 1 \right] + \frac{1}{2}(M)^{\frac{4}{k+1}} - \frac{1}{2} \tag{1-22}$$

Correlation of Vertical and Horizontal Flows

Rearranging (1-22)

$$1 + \frac{2(k-1)}{k+1} \frac{g}{C^{*2}} \Delta Z^* = (M)^{\frac{2(1-k)}{k+1}} \left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2} M^2\right) \tag{1-23}$$

$$\left[1 + \frac{2(k-1)}{k+1} \frac{g}{C^{*2}} \Delta Z^* \right]^{\frac{k+1}{2(k-1)}} = \frac{1}{M} \left[\frac{2}{k+1}\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k+1}{2(k-1)}} \tag{1-24}$$

The right side of (1-24) is equal to the area ratio $(\frac{A}{A^*})$ for horizontal isentropic flow, thus

$$\left[1 + \frac{2(k-1)}{k+1} \frac{g}{C^{*2}} \Delta Z^* \right]^{\frac{k+1}{2(k-1)}} = \left(\frac{A}{A^*}\right) \tag{1-25}$$

This relation is the equation that is used to correlate the vertical and horizontal isentropic flows. Similarly, from (1-15)

$$\left(\frac{F}{F^*}\right)_v = \left(\frac{1}{M}\right)^{\frac{2k}{k+1}} \left(\frac{1 + kM^2}{1+k}\right)$$

and, from (1-4)

$$\left(\frac{F}{F^*}\right)_h = \frac{1 + kM^2}{M \left[2(k+1)\left(1 + \frac{k-1}{2} M^2\right) \right]^{\frac{1}{2}}}$$

Rearranging

$$1 + kM^2 = \left(\frac{F}{F^*}\right)_h M \left[2(k+1)\left(1 + \frac{k-1}{2} M^2\right) \right]^{\frac{1}{2}} \tag{1-26}$$

From (1-15) and (1-26)

$$\left(\frac{F}{F^*}\right)_v = \left(\frac{F}{F^*}\right)_h \frac{\left[2 + (k-1)M^2\right]^{\frac{1}{2}}}{(k+1)^{\frac{1}{2}} (M)^{\frac{k-1}{k+1}}} \tag{1-27}$$

From (1-12) and (1-2)

$$\left(\frac{T}{T^*}\right)_v = \left[\frac{k+1}{k-1}\left(\frac{T^*}{T}\right)_h - \frac{2}{k+1}\right]^{\frac{1-k}{1+k}} \tag{1-28}$$

From (1-13) and (1-3)

$$\left(\frac{P}{P^*}\right)_v = \left[\frac{k+1}{k-1}\left(\frac{P^*}{P}\right)_h - \frac{2}{k+1}\right]^{\frac{-k}{k+1}} \tag{1-29}$$

From (1-2), (1-12) and (1-17)

$$\left(\frac{T_o}{T_o^*}\right)_v = \left(\frac{T}{T^*}\right)_v\left(\frac{T^*}{T}\right)_h \tag{1-30}$$

From (1-3), (1-13) and (1-19)

$$\left(\frac{P_o}{P_o^*}\right)_v = \left(\frac{P}{P^*}\right)_v\left(\frac{P^*}{P}\right)_h \tag{1-31}$$

From equations (1-1) and (1-22) it is seen that

$$M = f_1\left[\frac{A}{A^*}{}_h\right] = f_2\left[1 + \frac{2(k-1)}{k+1}\,\frac{g}{C^{*2}}\Delta Z^*\right]$$

Therefore

$$\left(\frac{F}{F^*}\right)_v = \left(\frac{F}{F^*}\right)_h f_3\left[\left(\frac{A}{A^*}\right)_h\right] = \left(\frac{F}{F^*}\right)_h f_4\left[1 + \frac{2(k-1)}{k+1}\,\frac{g}{C^{*2}}\Delta Z^*\right]$$

Likewise, from equations (1-2) and (1-22)

$$\left(\frac{T}{T^*}\right)_v = f_5\left[\left(\frac{A}{A^*}\right)_h\right] = f_6\left[1 + \frac{2(k-1)}{k+1}\,\frac{g}{C^{*2}}\Delta Z^*\right]$$

From equations (1-3) and (1-22) and T-P relation

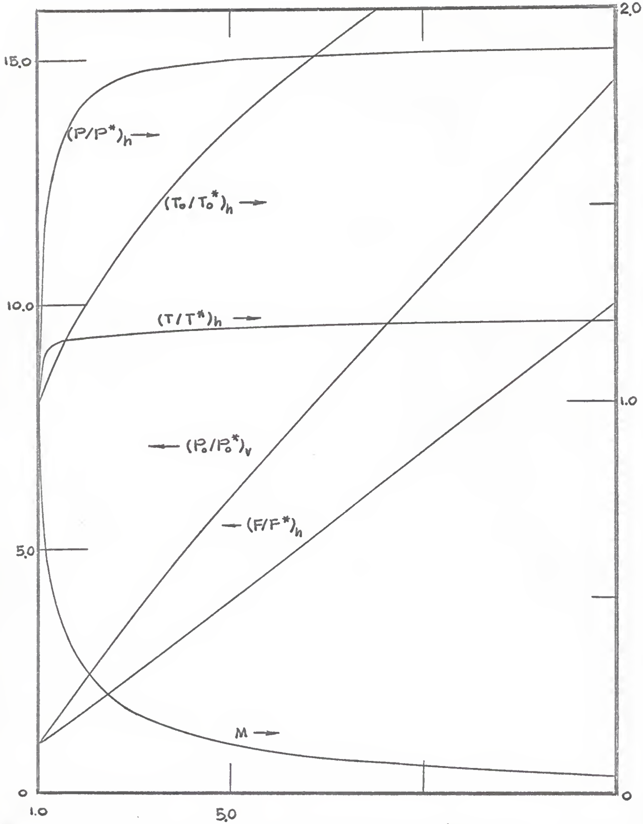
$$\left(\frac{P}{P^*}\right)_v = f_7\left[\left(\frac{A}{A^*}\right)_h\right] = f_8\left[1 + \frac{2(k-1)}{k+1}\,\frac{g}{C^{*2}}\Delta Z^*\right]$$

Similarly

$$\left(\frac{P_o}{P_o^*}\right)_v = f_9\left[\left(\frac{A}{A^*}\right)_h\right] = f_{10}\left[1 + \frac{2(k-1)}{k+1}\,\frac{g}{C^{*2}}\Delta Z^*\right]$$

$$\left(\frac{T_o}{T_o^*}\right)_v = f_{11}\left[\left(\frac{A}{A^*}\right)_h\right] = f_{12}\left[1 + \frac{2(k-1)}{k+1}\,\frac{g}{C^{*2}}\Delta Z^*\right]$$

The results of the correlation have been plotted for subsonic upward flow and are shown in Fig. (2).



$$\left(\frac{A}{A^*}\right)_h = \left[1 + \frac{2(k-1)}{k+1}\frac{g}{C^{*2}}\Delta Z^*\right]^{\frac{k+1}{2(k-1)}}$$

FIG. 2 CORRELATION OF HORIZONTAL & VERTICAL ISENTROPIC FLOWS

CORRELATION OF VERTICAL AND HORIZONTAL REVERSIBLE DIABATIC FLOWS IN CONSTANT-AREA PIPE

Reversible Diabatic Horizontal Flow in Constant-Area Pipe

The relations for this type of flow are well established and may be found in standard texts on gas dynamics. The equations used here are from pages 194 to 196 of reference 1.

$$\left(\frac{T}{T^*}\right)_h = \frac{(k+1)^2 M^2}{(1+kM^2)^2} \tag{2-1}$$

$$\left(\frac{T_o}{T_o^*}\right)_h = \frac{2(k+1)M^2\left(1 + \frac{k-1}{2} M^2\right)}{(1 + kM^2)^2} \tag{2-2}$$

$$\left(\frac{P}{P^*}\right)_h = \frac{k + 1}{1 + kM^2} \tag{2-3}$$

$$\left(\frac{P_o}{P_o^*}\right)_h = \frac{k+1}{1+kM^2} \left[\frac{2}{k+1} + \frac{k-1}{k+1} M^2\right]^{\frac{k}{k-1}} \tag{2-4}$$

$$\left(\frac{V}{V^*}\right)_h = \frac{(k+1)M^2}{1 + kM^2} \tag{2-5}$$

In this case, it is the heat flow that causes the change in properties and thus it is the heat flow that is taken as the independent variable.

Reversible Diabatic Vertical Flow in Constant-Area Pipe

When comparing reversible, vertical flow in constant-area pipes for the two cases of diabatic flow and isentropic flow it is to be noted that the momentum and continuity equations are the same, while the energy equation differs by the heat flow term. In the isentropic case it is the change in elevation that causes changes in properties. In the vertical diabatic case it is the change in elevation and the heat flow that causes the change in properties.

In reversible, diabatic, horizontal flow (Rayleigh Line) the relation Pv^n equal to a constant is not the polytropic case in that n is not a constant.* In order to get a solution for the reversible, diabatic flow the author found it necessary to assume that n is a constant in Pv^n equal to a constant. Therefore the correlating factor in vertical flow is a function of heat flow, elevation change and the value of n.

Basic Governing Equations

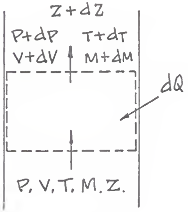


FIG. 3.

Energy Equation

$$dQ = C_p dT + VdV + gdZ \tag{2-6}$$

Momentum Equation

$$\frac{dP}{\rho} + VdV + gdZ = 0 \tag{2-7}$$

Continuity Equation

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \tag{2-8}$$

Perfect Gas Relation

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{2-9}$$

Definition of Mach Number

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T} \tag{2-10}$$

P - v Relation

$$Pv^n = \text{constant} \tag{2-11}$$

* It can be shown that n varies from zero to k as M goes from zero to unity.

Working Formulas From the above equations by assuming n a constant, the following useful formulas can be obtained.*

$$Q_v^* = C_pT^*(1 - \frac{T}{T^*}) - \frac{n}{n-1} RT^*(1 - \frac{T}{T^*}) \tag{2-14}$$

$$(\frac{T}{T^*})_v = 1 - \frac{Q_v^*}{C_pT^* - \frac{n}{n-1} RT^*} \tag{2-15}$$

$$(\frac{V}{V^*})_v = M(\frac{T}{T^*})_v^{\frac{1}{2}} \tag{2-16}$$

$$(\frac{P}{P^*})_v = \frac{V}{V^*}\frac{1}{M^2} \tag{2-17}$$

$$(\frac{T_o}{T_o^*})_v = (\frac{T}{T^*})_v(\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}})$$

$$= (\frac{T}{T^*})_v \left\lfloor \frac{2}{1+k} + \frac{(k-1)M^2}{k+1} \right\rfloor \tag{2-18}$$

$$(\frac{P_o}{P_o^*})_v = (\frac{P}{P^*}) \left[\frac{2}{1+k} + \frac{(k-1)M^2}{k+1}\right]^{\frac{k}{k-1}} \tag{2-19}$$

$$\frac{g}{C^{*2}} \Delta Z^* = \frac{n}{(n-1)k}\left[1 + (\frac{T}{T^*})_v\right] + \frac{1}{2} - \frac{1}{2}(\frac{V}{V^*})_v^2 \tag{2-20}$$

Correlation of Vertical and Horizontal Flows

Integrating (2-6) between an arbitrary section and the critical section

$$Q^* - g\Delta Z^* = C\,(T^* - T) + \frac{V^{*2}}{2}\left[1 - (\frac{V}{V^*})^2\right] \tag{2-22}$$

Dividing through by C^{*2}

$$(\frac{Q^* - g\Delta Z^*}{C^{*2}})_v = \frac{1}{k-1}\left[1 - (\frac{T}{T^*})_v\right] + \frac{1}{2}\left[1 - (\frac{V}{V^*})_v^2\right] \tag{2-23}$$

From (2-16)

$$(\frac{V}{V^*})^2 = M^2\,(\frac{T}{T^*}) \tag{2-24}$$

From (2-15)

* Derivations are in Appendix I.

$$(\frac{T}{T^*})_v = 1 - \frac{Q^*_v}{C_pT^* - \frac{n}{n-1} RT^*}$$

As $C_p = \frac{kR}{k-1}$ and $C^{*2} = kRT^*$, (2-15) becomes

$$(\frac{T}{T^*})_v = 1 - (\frac{Q^*}{C^{*2}})_v \left\lfloor \frac{k(k-1)(n-1)}{k(n-1) - n(k-1)} \right\rfloor$$

Let

$$N = \left\lfloor \frac{k(k-1)(n-1)}{k(n-1) - n(k-1)} \right\rfloor$$

$$(\frac{T}{T^*})_v = 1 - (\frac{Q^*}{C^{*2}})_v N \qquad (2\text{-}25)$$

Combining (2-24) and (2-25)

$$(\frac{V}{V^*})_v^2 = M^2 \left\lfloor 1 - (\frac{Q^*}{C^{*2}})_v N \right\rfloor \qquad (2\text{-}26)$$

Combining (2-25), (2-26) and (2-23)

$$(\frac{Q^* - g\Delta Z^*}{C^{*2}})_v = \frac{1}{k-1} \left\lfloor (\frac{Q^*}{C^{*2}})_v N \right\rfloor + \frac{1}{2} \left\lfloor 1 - M^2(1 - \frac{Q^*}{C^{*2}} N) \right\rfloor$$

$$= (\frac{1}{k-1} - \frac{M^2}{2})(\frac{Q^*}{C^{*2}})_v N + \frac{1 - M^2}{2}$$

$$\left\lfloor \frac{Q^*_v - g\Delta Z^*}{C^{*2}_v} + \frac{M^2 - 1}{2} \right\rfloor (\frac{C^{*2}}{Q^* N})_v = \frac{2 - (k-1)M^2}{2(k-1)} \qquad (2\text{-}27)$$

From the horizontal relation

$$(\frac{Q^*}{C^{*2}})_h = \frac{1}{k-1} \left\lfloor 1 - (\frac{T}{T^*})_h \right\rfloor + \frac{1}{2} \left\lfloor 1 - (\frac{V}{V^*})_h^2 \right\rfloor \qquad (2\text{-}28)$$

Combining (2-1), (2-5) and (2-28)

$$(\frac{Q^*}{C^{*2}})_h = \frac{1}{k-1} \left\lfloor 1 - \frac{(k+1)^2 M^2}{(1+kM^2)^2} \right\rfloor + \frac{1}{2} \left\lfloor 1 - \frac{(k+1)^2 M^4}{(1+kM^2)^2} \right\rfloor \qquad (2\text{-}29)$$

Simplifying

$$(\frac{Q^*}{C^{*2}})_h = \frac{(k+1)(M^2-1)^2}{2(k-1)(1+kM^2)^2} \qquad (2\text{-}30)$$

Comparing (2-27) and (2-30)

$$\left(\frac{Q^*}{C^{*2}}\right)_h = \left[\frac{Q^*_v - g\Delta Z^*}{C^{*2}_v} + \frac{M^2-1}{2}\right]\left(\frac{C^{*2}}{Q^*N}\right)_v \frac{(k+1)(M^2-1)^2}{(1+kM^2)^2\left[2-(k-1)M^2\right]} \qquad (2\text{-}31)$$

From (2-30) and (2-31)

$$M = g_1(Q^*_h) = g_2(Q^*_v, \Delta Z^*)$$

From (2-15) and (2-31)

$$\left(\frac{T}{T^*}\right)_v = g_3(Q^*_h) = g_4(Q^*_v, \Delta Z^*)$$

From (2-15), (2-17) and (2-31)

$$\left(\frac{P}{P^*}\right)_v = g_5(Q^*_h) = g_6(Q^*_v, \Delta Z^*)$$

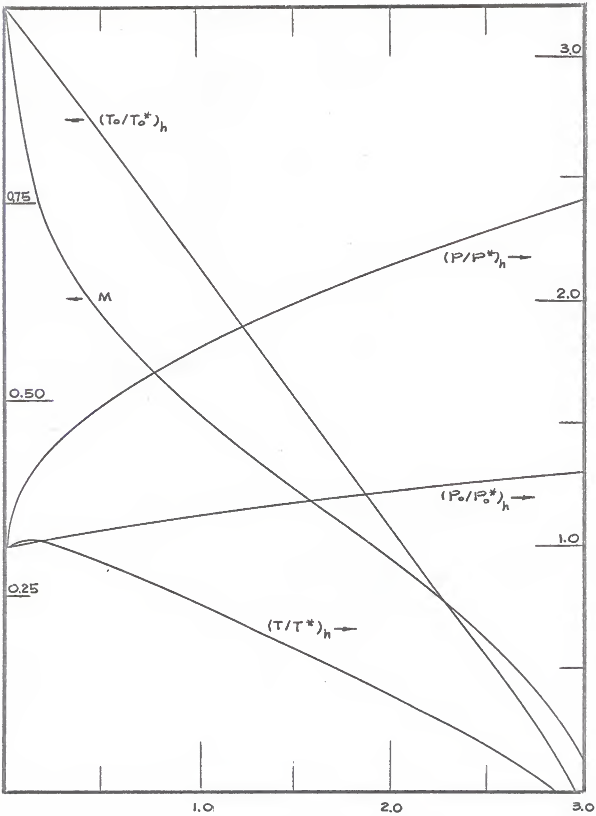
From (2-15), (2-18) and (2-31)

$$\left(\frac{T_o}{T^*_o}\right)_v = g_7(Q^*_h) = g_8(Q^*_v, \Delta Z^*)$$

From (2-17), (2-19) and (2-31)

$$\left(\frac{P_o}{P^*_o}\right)_v = g_9(Q^*_h) = g_{10}(Q^*_v, \Delta Z^*)$$

Since $(\frac{T}{T^*})_h$, $(\frac{P}{P^*})_h$, $(\frac{P_o}{P^*_o})_h$, $(\frac{T_o}{T^*_o})_h$ and M are functions of $(\frac{Q^*}{C^{*2}})_h$, and $(\frac{T}{T^*})_v$, $(\frac{P}{P^*})_v$, $(\frac{P_o}{P^*_o})_v$, $(\frac{T_o}{T^*_o})_v$ and M are functions of $(\frac{Q^*}{C^{*2}})_v$ and $(\frac{g}{C^{*2}}\Delta Z^*)$, the two flows are correlated through equation (2-31). The results of this correlation are shown on Fig. (4).



$$\left(\frac{Q^*}{C^{*2}}\right)_h = \left[\frac{Q_v^* - g\Delta Z^*}{C_v^{*2}} + \frac{M^2-1}{2}\right]\left(\frac{C^{*2}}{Q^*N}\right)_v \cdot \frac{(k+1)(M^2-1)^2}{(1+kM^2)^2[2-(k-1)M^2]}$$

FIG. 4 CORRELATION OF HORIZONTAL & VERTICAL REVERSIBLE DIABATIC FLOWS

CORRELATION OF HORIZONTAL AND VERTICAL REVERSIBLE ISOTHERMAL FLOWS

In this part, reversible isothermal cases in both horizontal and vertical flows are considered.

Reversible Isothermal Flow in Horizontal Pipe with Changing Cross-Sectional Area

The properties of gas flow in horizontal pipe with changing cross-sectional area are effected by the area ratio. For the isothermal case, heat transfer occurs, but the flow of heat, in order to maintain constant temperature, is dependent on the area change. In this case, the governing equations become

Basic Governing Equations

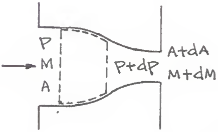


Fig. 5.

Energy Equation

$$dQ = VdV \tag{3-1}$$

Momentum Equation

$$\frac{dP}{\rho} + VdV = 0 \tag{3-2}$$

Continuity Equation

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \tag{3-3}$$

Perfect Gas Relation

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{3-4}$$

Definition of Mach Number

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} \tag{3-5}$$

Working Formulas From the above equations, useful working formulas for horizontal isothermal flow can be obtained. **

$$\left(\frac{V}{V^{*t}}\right)_h = \sqrt{k}\, M \tag{3-11}$$

$$\left(\frac{P}{P^{*t}}\right)_h = \left(\frac{\rho}{\rho^{*t}}\right)_h = e^{\left(\frac{1 - kM^2}{2}\right)} \tag{3-12}$$

$$\left(\frac{A}{A^{*t}}\right)_h = e^{\left(\frac{kM^2 - 1}{2}\right)} \left[\frac{1}{\sqrt{k}M}\right] \tag{3-13}$$

$$\left(\frac{P_o}{P^{*t}_o}\right)_h = e^{\frac{1 - kM^2}{2}} \left[\frac{2k}{3k-1}\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k}{k-1}} \tag{3-14}$$

$$\left(\frac{T_o}{T^{*t}_o}\right)_h = \frac{2k}{3k-1}\left(1 + \frac{k-1}{2} M^2\right) \tag{3-15}$$

$$Q^{*t}_h = \frac{RT}{2}\left(1 - kM^2\right) \tag{3-17}$$

$$\left(\frac{F}{F^{*t}}\right)_h = \frac{1 + kM^2}{2\sqrt{k}M} \tag{3-19}$$

Reversible Isothermal Flow in Vertical Pipe with Constant Cross-Sectional Area

In this case the main factor causing change in properties is the change of elevation. In order to maintain constant temperature the flow of heat must be adjusted to the change in elevation. Therefore it is the elevation change that is the correlating factor in this case.

Basic Governing Equations For a reversible isothermal vertical flow with constant cross-sectional area, the governing equations become

** Derivations are in Appendix II.

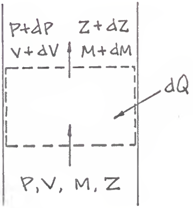


Fig. 6.

Energy Equation

$$dQ = VdV + gdZ \tag{3-20}$$

Momentum Equation

$$\frac{dP}{\rho} + VdV + gdZ = 0 \tag{3-21}$$

Continuity Equation

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \tag{3-22}$$

Perfect Gas Relation

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{3-23}$$

Definition of Mach Number

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} \tag{3-24}$$

Working Formulas From the above equations, useful working formulas are obtained:*

$$\left(\frac{V}{V_*{}^t}\right)_v = \sqrt{k}\, M \tag{3-30}$$

$$\frac{g}{C_*{}^{t\,2}}\, \Delta Z_*{}^t = \frac{1}{k} \ln \left(\frac{1}{\sqrt{kM}}\right) - \frac{1}{2k} (1 - kM^2) \tag{3-31}$$

$$\left(\frac{P}{P_*{}^t}\right)_v = \frac{1}{\sqrt{k}\, M} \tag{3-32}$$

$$\left(\frac{P_o}{P_*{}^t}\right)_v = \frac{1}{\sqrt{k}\, M} \left[\frac{2k}{3k-1} \left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k}{k-1}} \tag{3-33}$$

$$\left(\frac{T_o}{T_o^*{}^t}\right)_v = \frac{2k}{3k-1} \left(1 + \frac{k-1}{2} M^2\right) \tag{3-34}$$

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* Derivations are in Appendix II.

Correlation of Vertical and Horizontal Flows

From (3-13)

$$\left(\frac{A}{A*^t}\right)_h = \frac{e^{\left(\frac{kM^2 - 1}{2}\right)}}{\sqrt{kM}}$$

$$\frac{1}{k\,M} = \left(\frac{A}{A*^t}\right)_h\, e^{\left(\frac{1 - kM^2}{2}\right)} \tag{3-35}$$

Substituting into (3-31)

$$\frac{g}{C*^{t^2}}\,\Delta Z*^t\,k = \ln\left(\frac{A}{A*^t}\right)_h + \frac{1 - kM^2}{2} - \frac{1 - kM^2}{2} = \ln\left(\frac{A}{A*^t}\right)_h \tag{3-36}$$

From (3-11) and (3-13)

$$\left(\frac{V}{V*^t}\right)_h = e^{\left(\frac{kM^2 - 1}{2}\right)}\left(\frac{A*^t}{A}\right) \tag{3-37}$$

From (3-13) and (3-12)

$$\left(\frac{P*^t}{P}\right)_h = \left(\frac{A}{A*^t}\right)_h \sqrt{k}\,M \tag{3-38}$$

From (3-13) and (3-14)

$$\left(\frac{P_o}{P*^t_o}\right)_h = \left(\frac{A*^t}{A}\right)_h \frac{1}{\sqrt{k}\,M}\left[\frac{2k}{3k-1}\left(1 + \frac{k-1}{2}M^2\right)\right]^{\frac{k}{k-1}} \tag{3-39}$$

From (3-13)

$$M = h_1\left[\left(\frac{A}{A*^t}\right)_h\right] \tag{3-40}$$

From (3-29)

$$\ln\left(\frac{V*^t}{V}\right)_v - \frac{1}{2}\left[1 - \left(\frac{V}{V*^t}\right)_v^2\right] = \frac{kg}{C*^{t^2}}\,\Delta Z*^t \tag{3-41}$$

From (3-31) and (3-32)

$$\left(\frac{P}{P_*^{\,t}}\right)_v = h_2 \left(\frac{kg}{C_*^{\,t^2}} \Delta Z_*^{\,t}\right) \tag{3-42}$$

Therefore, from (3-36) and (3-40)

$$M = h_1 \left[\left(\frac{A}{A_*^{\,t}}\right)_h\right] = h_3 \left(\frac{kg}{C_*^{\,t^2}} \Delta Z_*^{\,t}\right)$$

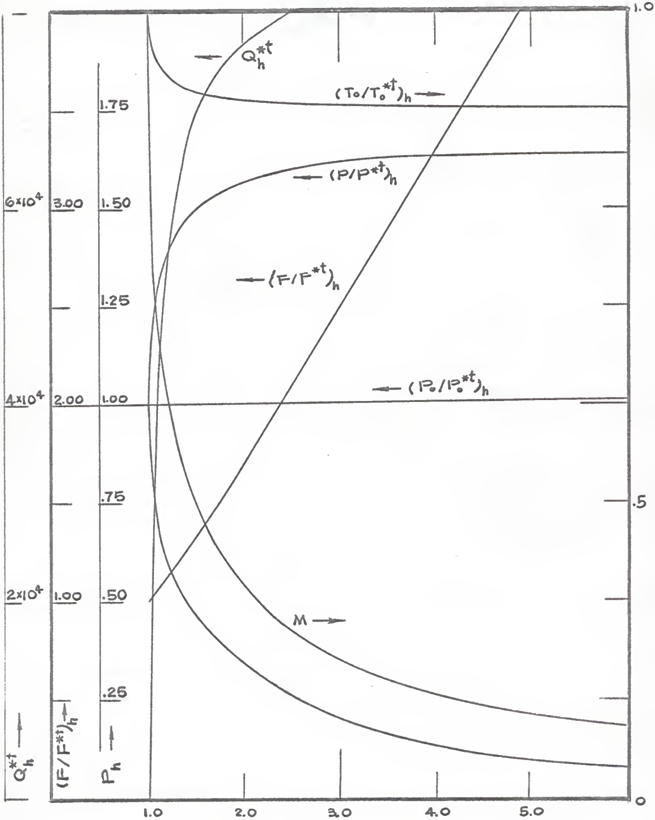
From (3-36) and (3-42)

$$\left(\frac{P}{P_*^{\,t}}\right)_v = h_2 \left(\frac{kg}{C_*^{\,t^2}} \Delta Z_*^{\,t}\right) = h_4 \left[\left(\frac{A}{A_*^{\,t}}\right)_h\right]$$

From (3-36), (3-33) and (3-32)

$$\left(\frac{P_o}{P_{*o}^{\,t}}\right)_v = h_5 \left(\frac{kg}{C_*^{\,t^2}} \Delta Z_*^{\,t}\right) = h_6 \left[\left(\frac{A}{A_*^{\,t}}\right)_h\right]$$

Since $\left(\frac{P}{P_*^{\,t}}\right)_h$, $\left(\frac{P_o}{P_{*o}^{\,t}}\right)_h$, $\left(\frac{T_o}{T_{*o}^{\,t}}\right)_h$, and M are functions of $\left(\frac{A}{A_*^{\,t}}\right)_h$, and $\left(\frac{P}{P_*^{\,t}}\right)_v$, $\left(\frac{P_o}{P_{*o}^{\,t}}\right)_v$, $\left(\frac{T_o}{T_{*o}^{\,t}}\right)_v$, and M are functions of $\left(\frac{g}{C_*^{2}}\, Z_*^{\,t}\right)$, the two flows are correlated through equation (3-36). The results of this correlation are shown on Fig. (7).



$$\left(\frac{A}{A^{*t}}\right)_h = e^{\frac{kg}{c^{*2}} \Delta z^{*t}}$$

FIG. 7 CORRELATION OF HORIZONTAL & VERTICAL REVERSIBLE ISOTHERMAL FLOW

CORRELATION OF VERTICAL AND HORIZONTAL IRREVERSIBLE ISOTHERMAL FLOWS

For irreversible case, wall friction is the main factor which causes properties change in both horizontal and vertical flows, the heat flow is dependent on the wall friction. Therefore, the correlation of these two flows is based on this fact.

Irreversible Isothermal Flow in Horizontal Pipe

For this case, relations are well established and can be found in any standard text of gas dynamics. The relations used here are from pages 180 to 181 of reference 1.

$$\left(\frac{V}{V^{*t}}\right)_h = \sqrt{k}\, M \tag{4-1}$$

$$\left(\frac{P}{P^{*t}}\right)_h = \frac{1}{\sqrt{k}\, M} \tag{4-2}$$

$$\left(\frac{P_o}{P_o^{*t}}\right)_h = \frac{1}{\sqrt{k}\, M} \left[\frac{2k}{3k-1}\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k}{k-1}} \tag{4-3}$$

$$\left(\frac{T_o}{T^{*t}}\right)_h = \frac{2k}{3k-1}\left(1 + \frac{k-1}{2} M^2\right) \tag{4-4}$$

$$\frac{4f}{D} L_{max} = \frac{1 - kM^2}{kM^2} + \ln kM^2 \tag{4-5}$$

$$Q_h^{*t} = \frac{RT}{2}\left(1 - kM^2\right) \tag{4-6}$$

Irreversible Isothermal Flow in Vertical Pipe

In case of combining effects of elevation change, heat transfer and friction, the governing equations become*

Basic Governing Equations

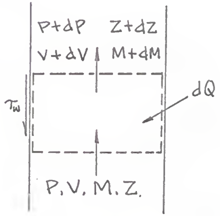


Fig. 8

Energy Equation

$$dQ = VdV + gdZ \tag{4-7}$$

Continuity Equation

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \tag{4-8}$$

Perfect Gas Relation

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{4-9}$$

Definition of Mach Number

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} \tag{4-10}$$

Momentum Equation

$$AdP + \rho AgdZ + \rho AVdV + \tau_w \pi DdZ = 0 \tag{4-11}$$

Working Formulas From the above equations, the expressions for various properties can be obtained.

$$\frac{4f}{D}\,\Delta Z*^t = \frac{2RTf + gD}{gD} \ln \frac{2g + \frac{4f}{D}C^2M^2}{2g + \frac{4f}{D}RT} - \frac{2RTf}{gD} \ln kM^2 \tag{4-19}$$

$$\frac{4f}{D}\,\Delta Z*^t = \frac{2RTf}{gD} \ln \left(\frac{V*^t}{V}\right)_v^2 + \left(\frac{2RTf}{gD} + 1\right) \ln \frac{2g + \frac{4f}{D}V^2}{2g + \frac{4f}{D}V*^{t2}} \tag{4-21}$$

$$\frac{4f}{D}\,\Delta Z*^t = \left(\frac{2RTf}{gD} + 1\right) \ln \frac{2g + \frac{4f}{D}RT\left(\frac{P*^t}{P}\right)^2}{2g + \frac{4f}{D}RT} - \frac{2RTf}{gD} \ln \left(\frac{P*^t}{P}\right)_v^2 \tag{4-23}$$

$$Q*_v^t = \frac{kRT}{2}\left(1 - kM^2\right) - g\Delta Z*^t \tag{4-24}$$

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* Derivations are in Appendix III.

Correlation of Vertical and Horizontal Flows

In this case, $\frac{4f}{D}$ L is the main factor in horizontal flow which causes the properties change. In vertical flow, it is $\frac{4f}{D}$ Z which causes the properties change. Through those two factors, the correlation of the vertical and horizontal flows are established.

From (4-19)

$$\ln kM^2 = \left(\frac{2RTf + gD}{gD}\right) \ln \frac{2g + \frac{4f}{D}C^2M^2}{2g + \frac{4f}{D}RT} - \frac{gD}{2RTf}\frac{4f}{D}\Delta Z{*}^t \tag{4-25}$$

From (4-5)

$$\ln kM^2 = \frac{4f}{D} L_{max} - \frac{1 - kM^2}{kM^2} \tag{4-26}$$

Combining (4-25) and (4-26)

$$\frac{4f}{D} L_{max} = \frac{1 - kM^2}{kM^2} + \left(1 + \frac{gD}{2RTf}\right) \ln \frac{2g + \frac{4f}{D}C^2M^2}{2g + \frac{4f}{D}RT} - \frac{gD}{2RTf}\frac{4f}{D}\Delta Z{*}^t \tag{4-27}$$

From (4-2) and (4-5)

$$\left(\frac{P}{P{*}^t}\right)_h = j_1\left(\frac{4f}{D} L\right)$$

From (4-5)

$$M = j_2\left(\frac{4f}{D} L\right)$$

From (4-3) and (4-5)

$$\left(\frac{P_o}{P{*}_o^t}\right)_h = j_3\left(\frac{4f}{D} L\right)$$

From (4-4) and (4-5)

$$\left(\frac{T_o}{T{*}_o^t}\right)_h = j_4\left(\frac{4f}{D} L\right)$$

From (4-6) and (4-5)

$$Q_h^{*t} = j_5 \left(\frac{4f}{D} L\right)$$

From (4-19)

$$M = j_6 \left(\frac{4f}{D} Z\right)$$

From (4-21)

$$\left(\frac{V}{V^{*t}}\right)_v = j_7 \left(\frac{4f}{D} Z\right)$$

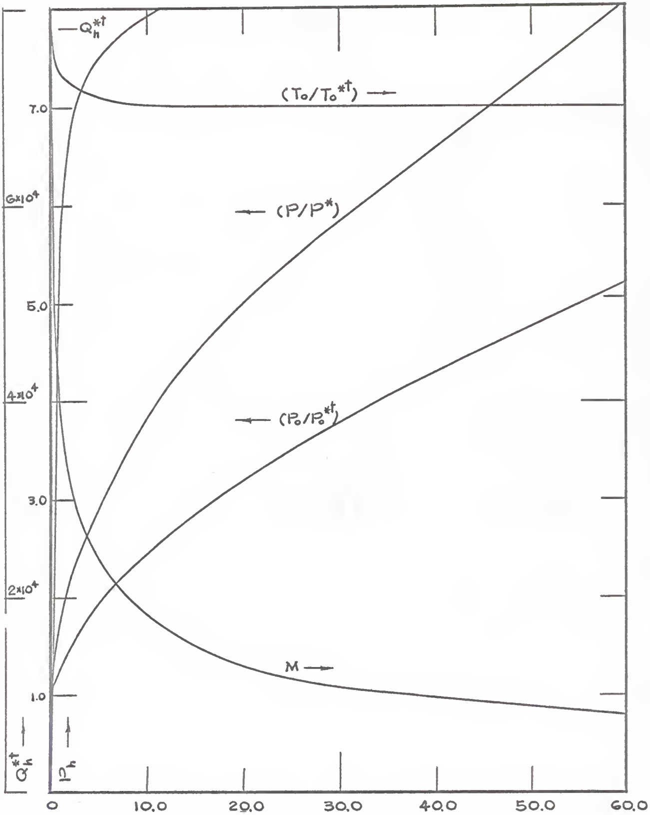
From (4-23)

$$\left(\frac{P}{P^{*t}}\right)_v = j_8 \left(\frac{4f}{D} Z\right)$$

From (4-24)

$$Q_v^{*t} = j_9 \left(\frac{4f}{D} Z\right)$$

Since $\left(\frac{P}{P^{*t}}\right)_h$, M, $\left(\frac{P_o}{P_o^{*t}}\right)_h$, $\left(\frac{T_o}{T_o^{*t}}\right)_h$, Q are functions of $\frac{4f}{D}$ L, and $\left(\frac{P}{P^{*t}}\right)_v$, M, $\left(\frac{P_o}{P_o^{*t}}\right)_v$, $\left(\frac{T_o}{T_o^{*t}}\right)_v$, Q are functions of $\frac{4f}{D}$ Z, this correlation between these two flows is established through equation (4-27). The results of this correlation are shown on Fig. (9).



$$\frac{4f}{D} L_{max} = \frac{1-kM^2}{kM^2} + \left(1 + \frac{gD}{2RTf}\right) \cdot \ln \frac{2g + (4f/D)c^2M^2}{2g + (4f/D)RT} - \frac{gD}{2RTf}\frac{4f}{D}\Delta Z^{*\dagger}$$

FIG. 9 CORRELATION OF HORIZONTAL & VERTICAL IRREVERSIBLE, ISOTHERMAL FLOWS

APPENDICES

APPENDIX I

DERIVATION OF WORKING FORMULAS FOR REVERSIBLE, DIABATIC VERTICAL FLOW

In this case, by assuming n** in Pv^n = constant is a constant, we are able to integrate the term $\frac{dP}{\rho}$ in the momentum equation.

From (2-6) and (2-7)

$$dQ = C_p dT - \frac{dP}{\rho} \tag{2-12}$$

$$Pv^n = \text{const} \qquad v = \left(\frac{C}{P}\right)^{\frac{1}{n}} \tag{2-13}$$

Combining (2-12) and (2-13)

$$dQ = C_p dT - \left(\frac{C}{P}\right)^{\frac{1}{n}} dP$$

Integrating between an arbitrary section and the critical section

$$Q_v^* = C_p(T^* - T) - C^{\frac{1}{n}} \frac{n}{n-1}\left(P^{*\frac{n-1}{n}} - P^{\frac{n-1}{n}}\right)$$

$$= C_p(T^* - T) - \frac{n}{n-1} R(T^* - T)$$

$$= C_p T^*\left(1 - \frac{T}{T^*}\right) - \frac{n}{n-1} RT^*\left(1 - \frac{T}{T^*}\right) \tag{2-14}$$

From (2-14)

$$\left(\frac{T}{T^*}\right)_v = 1 - \frac{Q_v^*}{C_p T^* - \frac{n}{n-1} RT^*} \tag{2-15}$$

From definition of Mach Number

$$\left(\frac{T}{T^*}\right)_v = \left(\frac{V}{V^*}\right)_v^2 \frac{1}{M^2}$$

** The value of n must be of such magnitude that there is heat flow into the flowing gas. This assumption is made so that the direction of heat flow is the same in both the horizontal and vertical cases.

$$(\frac{V}{V^*})_v = M\left[1 - \frac{Q_v^*}{C_pT^* - \frac{n}{n-1}RT^*}\right]^{\frac{1}{2}} \tag{2-16}$$

$$(\frac{P}{P^*})_v = (\frac{\rho}{\rho^*})_v(\frac{T}{T^*}) = (\frac{V}{V^*})\frac{1}{M^2}$$

$$= \frac{1}{M}\left[1 - \frac{Q_v^*}{C_pT^* - \frac{n}{n-1}RT^*}\right]^{\frac{1}{2}} \tag{2-17}$$

$$(\frac{T_o}{T_o^*})_v = (\frac{T}{T^*})_v\left(\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}}\right)$$

$$= (1 - \frac{Q_v^*}{C_pT^* - \frac{n}{n-1}RT^*})\left[\frac{2}{1+k} + \frac{(k-1)}{k+1}M^2\right] \tag{2-18}$$

$$(\frac{P_o}{P_o^*}) = (\frac{P}{P^*})_v\left(\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}}\right)^{\frac{k}{k-1}}$$

$$= \frac{1}{M}(1 - \frac{Q_v^*}{C_pT^* - \frac{n}{n-1}RT^*})^{\frac{1}{2}}\left[\frac{2}{1+k} + \frac{(k-1)}{k+1}M^2\right]^{\frac{k}{k+1}} \tag{2-19}$$

From (2-7)

$$\frac{dP}{\rho} + VdV + gdZ = 0$$

Integrating

$$\frac{n}{n-1}R(T^* - T) + \frac{V^{*2} - V^2}{2} = g\Delta Z^*$$

$$\frac{n}{n-1}RT^*(1 - \frac{T}{T^*}) + \frac{V^{*2}}{2}\left[1 - (\frac{V}{V^*})^2\right] = g\Delta Z^*$$

Dividing through by C^{*2}

$$\frac{n}{(n-1)k}(1 - \frac{T}{T^*}) + \frac{1}{2}\left[1 - (\frac{V}{V^*})_v^2\right] = \frac{g}{C^{*2}}\Delta Z^* \tag{2-20}$$

APPENDIX II

DERIVATION OF WORKING FORMULAS FOR HORIZONTAL REVERSIBLE ISOTHERMAL FLOW

In this case, $dQ = TdS$ is applicable, and

$$dS = C_p \frac{dT}{T} - R\frac{dP}{P}$$

$$dT = 0$$

$$dS = -R\frac{dP}{P}$$

$$dQ = -RT\frac{dP}{P} \tag{3-6}$$

From the governing equations and (3-6), expressions for pressure, temperature velocity, etc. can be derived in terms of M.

Combining (3-1) and (3-6)

$$-RT\frac{dP}{P} = VdV \tag{3-7}$$

Integrating between an arbitrary section and the critical section**

$$-RT\ln\left(\frac{P^{*t}}{P}\right)_h = \frac{V^{*t^2} - V^2}{2}$$

$$RT\ln\left(\frac{P}{P^{*t}}\right)_h = \frac{V^{*t^2}}{2}\left[1 - \left(\frac{V}{V^{*t}}\right)_h^2\right] \tag{3-8}$$

From the definition of Mach Number

$$V^{*t} = M^{*t}\sqrt{kRT} \tag{3-9}$$

$$M^{*t} = 1/\sqrt{k}$$

$$V^{*t} = \sqrt{RT}$$

** The critical section for reversible isothermal flow occurs at the minimum cross-sectional area. At this section $M = 1/\sqrt{k}$, (see page 180 and problem 6-10 on page 188 of reference 1).

$$V = M\sqrt{kRT} \tag{3-10}$$

$$\left(\frac{V}{V*^t}\right)_h = \sqrt{k}\, M \tag{3-11}$$

Combining (3-8) and (3-11)

$$2 \ln \left(\frac{P}{P*^t}\right)_h = 1 - kM^2$$

$$\left(\frac{P}{P*^t}\right)_h = \left(\frac{\rho}{\rho*^t}\right)_h = e^{\left(\frac{1-kM^2}{2}\right)} \tag{3-12}$$

From the continuity equation

$$\frac{A}{A*^t} = \frac{\rho*^t}{\rho}\frac{V*^t}{V} = \frac{e^{\left(\frac{kM^2-1}{2}\right)}}{kM} \tag{3-13}$$

From the definition of stagnation pressure

$$\left(\frac{P_o}{P*^t_o}\right)_h = \left(\frac{P}{P*^t}\right)_h \left(\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}\frac{1}{k}}\right)^{\frac{k}{k-1}}$$

$$= e^{\left(\frac{1-kM^2}{2}\right)} \left[\frac{2k}{3k-1}\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k}{k-1}} \tag{3-14}$$

From the definition of stagnation temperature

$$\left(\frac{T_o}{T*^t_o}\right)_h = \left(\frac{T}{T*^t}\right)_h \left(\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}\frac{1}{k}}\right)$$

$$= \frac{2k}{3k-1}\left(1 + \frac{k-1}{2} M^2\right) \tag{3-15}$$

From (3-1)

$$dQ = \frac{1}{2}\, dV^2$$

Integrating

$$Q*^t_h = \frac{V*^{t2}}{2}\left[1 - \left(\frac{V}{V*^t}\right)^2\right] \tag{3-16}$$

From (3-16) and (3-11)

$$Q_{h}^{*t} = \frac{RT}{2} (1 - kM^2) \tag{3-17}$$

From the definition of the impulse function

$$\left(\frac{F}{F^{*t}}\right)_h = \left(\frac{P}{P^{*t}}\right)_h \left(\frac{A}{A^{*t}}\right)_h \frac{1 + kM^2}{2} \tag{3-18}$$

From (3-12), (3-13) and (3-18)

$$\left(\frac{F}{F^{*t}}\right)_h = \frac{1 + kM^2}{kM(2)} \tag{3-19}$$

DERIVATION OF WORKING FORMULAS FOR REVERSIBLE ISOTHERMAL VERTICAL FLOW

From (3-20) and (3-21)

$$dQ = - \frac{dP}{\rho} = - \frac{dP}{P} \frac{P}{\rho} = - RT \frac{dP}{P} \tag{3-25}$$

From (3-25) and (3-20)

$$- RT \frac{dP}{P} = VdV + gdZ \tag{3-26}$$

From (3-22) and (3-23)

$$\frac{dP}{P} = \frac{d\rho}{\rho} = - \frac{dV}{V}$$

$$RT \frac{dV}{V} = VdV + gdZ$$

$$RT \frac{dV}{V} - VdV = gdZ$$

$$RT \frac{dV}{V} - \frac{1}{2} dV^2 = gdZ \tag{3-27}$$

Integrating between an arbitrary section and the critical section

$$RT \ln V \Big|_{V}^{V^{*t}} - \frac{1}{2}(V^2) \Big|_{V}^{V^{*t}} = g(Z^{*t} - Z)$$

$$RT \ln \left(\frac{V^{*t}}{V}\right)_v - \frac{1}{2} (V^{*t^2} - V) = g\Delta Z^{*t}$$

$$RT \ln \left(\frac{V^{*t}}{V}\right)_v - \frac{1}{2} V^{*t2}\left(1 - \frac{V^2}{V^{*t2}}\right) = g\Delta Z^{*t} \tag{3-28}$$

Dividing through by C^{*t2}

$$\frac{RT}{C^{*t2}} \ln \left(\frac{V^{*t}}{V}\right) - \frac{1}{2} \frac{V^{*t2}}{C^{*t2}} \left(1 - \frac{V^2}{V^{*t2}}\right) = \frac{g}{C^{*t2}} \Delta Z^{*t}$$

$$\frac{1}{k} \ln \left(\frac{V^{*t}}{V}\right) - \frac{1}{2k} \left(1 - \frac{V^2}{V^{*t2}}\right) = \frac{g}{C^{*t2}} \Delta Z^{*t} \tag{3-29}$$

$$\frac{V}{V^{*t}} = \sqrt{k}M \tag{3-30}$$

From (3-29) and (3-30)

$$\frac{1}{k} \ln \left(\frac{1}{\sqrt{k}\, M}\right) - \frac{1}{2k} \left(1 - kM^2\right) = \frac{g}{C^{*t2}} \Delta Z^{*t} \tag{3-31}$$

From the continuity and the perfect gas relation

$$\left(\frac{P}{P^{*t}}\right)_v = \frac{1}{\sqrt{k}\, M} \tag{3-32}$$

From the definition of stagnation pressure

$$\left(\frac{P_o}{P_o^{*t}}\right)_v = \left(\frac{P}{P^{*t}}\right)_v \left(\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}\frac{1}{k}}\right)^{\frac{k}{k-1}}$$

$$= \frac{1}{\sqrt{k}\, M} \left[\left(\frac{2k}{3k-1}\right)\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k}{k-1}} \tag{3-33}$$

Similarly, the stagnation temperature is

$$\left(\frac{T_o}{T_o^{*t}}\right)_v = \left(\frac{T}{T^{*t}}\right)_v \left(\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}\frac{1}{k}}\right)$$

$$= \frac{2k}{3k-1} \left(1 + \frac{k-1}{2} M^2\right) \tag{3-34}$$

APPENDIX III

DERIVATION OF WORKING FORMULAS FOR VERTICAL IRREVERSIBLE ISOTHERMAL FLOW

Dividing through by ρA, and introducing $\tau_w = \frac{\rho V^2}{2} f$, equation (4-11) becomes

$$\frac{dP}{\rho} + gdZ + VdV + \frac{4f}{D}\left(\frac{V^2}{2}\right)dZ = 0 \tag{4-12}$$

$$\frac{dP}{P}\,\frac{P}{\rho} + gdZ + VdV + \frac{4f}{D}\left(\frac{V^2}{2}\right)dZ = 0 \tag{4-13}$$

From the perfect gas relation

$$\frac{P}{\rho} = RT$$

(4-12) becomes

$$RT\,\frac{dP}{P} + gdZ + \frac{1}{2}\,dV^2 + \frac{4f}{D}\,\frac{V^2}{2}\,dZ = 0$$

$$dZ\left(g + \frac{4f}{D}\,\frac{V^2}{2}\right) + RT\,\frac{dP}{P} + \frac{1}{2}\,dV^2 = 0 \tag{4-14}$$

Rearranging

$$dZ = \frac{-\left(RT\,\frac{dP}{P} + \frac{1}{2}\,dV^2\right)}{\left(g + \frac{4f}{D}\,\frac{V^2}{2}\right)} \tag{4-15}$$

From (4-7), (4-8) and (4-9)

$$\frac{dP}{P} = \frac{d\rho}{\rho} = \frac{1}{2}\,\frac{dV^2}{V^2} = -\,\frac{1}{2}\,\frac{dM^2}{M^2} \tag{4-16}$$

Combining (4-15) and (4-16)

$$dZ = \frac{\frac{RT}{2}\,\frac{dM^2}{M^2} - \frac{C^2}{2}\,dM^2}{\left(g + \frac{4f}{D}\,\frac{C^2M^2}{2}\right)}$$

$$dZ = \frac{RT}{2g + \frac{4f}{D} C^2M^2} \frac{dM^2}{M^2} - \frac{C^2}{2g + \frac{4f}{D} C^2M^2} dM^2 \tag{4-17}$$

Integrating between an arbitrary section and the critical section

$$Z^{*t} - Z = -\frac{RT}{2g} \ln \left[\frac{2g + \frac{4f}{D} C^2M^2}{M^2}\right]\Bigg|_M^{1/\sqrt{k}} - \frac{D}{4f} \ln \left(2g + \frac{4f}{D} C^2M^2\right)\Bigg|_M^{1/\sqrt{k}}$$

$$\Delta Z^{*t} = \frac{RT}{2g} \ln \frac{2g + \frac{4f}{D} C^2M^2}{kM^2 \left(2g + \frac{4f}{D} RT\right)} + \frac{D}{4f} \ln \frac{2g + \frac{4f}{D} C^2M^2}{2g + \frac{4f}{D} RT} \tag{4-18}$$

$$\frac{4f}{D} \Delta Z^{*t} = \frac{RT}{2g} \frac{4f}{D} \ln \frac{2g + \frac{4f}{D} C^2M^2}{kM^2 \left(2g + \frac{4f}{D} RT\right)} + \ln \frac{2g + \frac{4f}{D} C^2M^2}{2g + \frac{4f}{D} RT}$$

$$= \frac{2RTf + gD}{gD} \ln \frac{2g + \frac{4f}{D} C^2M^2}{2g + \frac{4f}{D} RT} - \frac{2RTf}{gD} \ln kM^2 \tag{4-19}$$

From (4-17) and (4-10)

$$dZ = \frac{RT}{2g + \frac{4f}{D} V^2} \frac{dV^2}{V^2} - \frac{1}{2g + \frac{4f}{D} V^2} dV^2 \tag{4-20}$$

Integrating

$$\Delta Z^{*t} = \frac{RT}{2g} \ln \left(\frac{2g + \frac{4f}{D} V^2}{V^2}\right)\left(\frac{V^{*t2}}{2g + \frac{4f}{D} V^{*t2}}\right)$$

$$+ \frac{D}{4f} \ln \left(\frac{2g + \frac{4f}{D} V^2}{2g + \frac{4f}{D} V^{*t2}}\right)$$

$$\frac{4f}{D} \Delta Z^{*t} = \frac{2RTf}{gD} \ln \left(\frac{V^{*t2}}{V^2}\right)_V + \left(\frac{2RTf}{gD} + 1\right) \ln \frac{2g + \frac{4f}{D} V^2}{2g + \frac{4f}{D} V^{*t2}} \tag{4-21}$$

From (4-16)

$$\left(\frac{P}{P^{*t}}\right)_V = \left(\frac{1}{kM^2}\right)^{\frac{1}{2}}$$

$$kM^2 = \left(\frac{P^{*t}}{P}\right)_V^2 \tag{4-22}$$

Combining (4-22) and (4-19), eliminating M,

$$\frac{4f}{D}\,\Delta Z^{*t} = \frac{2RTf + gD}{gD}\ln\frac{2g + \frac{4f}{D}RT\left(\frac{P^{*t}}{P}\right)^2}{2g + \frac{4f}{D}RT} - \frac{2RTf}{gD}\ln\left(\frac{P^{*t}}{P}\right)_V^2 \tag{4-23}$$

From (4-7)

$$dQ = \frac{1}{2}\,dV^2 + g\,dZ$$

Integrating

$$Q_V^{*t} = \frac{kRT}{2}\left(1 - kM^2\right) + g\Delta Z^{*t} \tag{4-24}$$

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CORRELATION OF GAS FLOWS IN HORIZONTAL
AND VERTICAL PIPES

by

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AN ABSTRACT OF A MASTER'S REPORT

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The main effort of this report is to find out the relations between gas flows in horizontal and vertical pipes.

The horizontal relations which are well established and can be found in any standard gas dynamics text. The vertical relations are developed in this report.

Four cases are considered, namely: (1) isentropic flow (2) reversible diabatic flow (3) reversible isothermal flow and (4) irreversible isothermal flow.

In the first case, change in cross-sectional area for horizontal flow is the main factor that causes the change in pressure, temperature and other properties. In isentropic, constant-area vertical flow, change in elevation is the main factor that causes the change in properties. Through these two factors, relations between these two flows are established.

Likewise, the other three cases are correlated in a similar manner.