SOME ELEMENTARY CONCEPTS IN MEASURE THEORY

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CHAPTER 1

INTRODUCTION

In order to gain an understanding of measure theory, a special case is discussed in the introduction of this report. Many of the properties of measure are introduced with this special case. Later the concept of measure is extended to more general settings.

In the space, the real line, let $\mathcal{P}$ be the collection of all semiclosed intervals of the form $\{x: a \leq x < b\}$, where a and b are finite real numbers. It is understood that $a \leq b$. These semiclosed intervals are denoted by $[a, b)$. The following results could also be obtained if closed or open intervals were considered.

The length of an interval is denoted by $l([a, b))$ and is defined by $l([a, b)) = b - a$. This is simply the length of the line segment with end points a and b.

<u>Theorem 1.1</u>[1] If $\{E_1, E_2, \cdots, E_n\}$ is a finite, disjoint class of sets in $\mathcal{P}$, each contained in a given set E_0 in $\mathcal{P}$, then

$$\sum_{i=1}^{n} l(E_i) \leq l(E_0).$$

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[1] This denotes the first theorem in Chapter 1. The notation, a.b, where a is the number of the chapter and b is the order of the theorem or definition in that chapter, is used throughout this report.

Proof: To show this, write $E_i = [a_i, b_i)$, $i = 0, 1, \cdots, n$ and assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. It follows that $a_0 \leq a_1 \leq b_1 \leq \cdots \leq a_n \leq b_n \leq b_0$. Therefore

$$\sum_{i=1}^{n} l(E_i) = \sum_{i=1}^{n} (b_i - a_i) \leq \sum_{i=1}^{n} (b_i - a_i) + \sum_{i=1}^{n} (a_{i+1} - b_i)$$

$$= b_n - a_1 \leq b_0 - a_0 = l(E_0).$$

Now consider the problem in which the union of a sequence of sets contains a given set. A result is obtained which is similar to Theorem 1.1. It is necessary, however, first to state a lemma that will be used in this proof.

Lemma 1.1. If a closed interval $F_0 = [a_0, b_0]$ is contained in the union of a finite number of bounded, open intervals, $U_1, U_2, \cdots, U_n$, $U_i = (a_i, b_i)$, $i = 1, 2, \cdots n$, then $b_0 - a_0 < \sum_{i=1}^{n} (b_i - a_i)$ (3, 34)[2].

Theorem 1.2. If $\{E_0, E_1, \cdots\}$ is a sequence of sets in $\mathcal{D}$ such that $E_0 \subset \bigcup_{i=1}^{\infty} E_i$, then $l(E_0) \leq \sum_{i=1}^{\infty} l(E_i)$.

Proof: Write $E_i = [a_i, b_i)$, $i = 0, 1, 2, \cdots$. If $a_0 = b_0$, then the result is easily seen. Otherwise, let ϵ be a positive number such that $\epsilon < b_0 - a_0$. For any positive number δ, $F_0 = [a_0, b_0 - \epsilon]$ and $U_i =$

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[2] Refers to page 34 of reference number 3 in the Bibliography. Similar notation is used throughout this report.

$(a_i - \frac{\delta}{2^i}, b_i)$, $i = 1, 2, \cdots$, it follows that $F_0 \subset \bigcup_{i=1}^{\infty} U_i$. Therefore, by the Heine-Borel theorem, there is a positive integer n such that $F_0 \subset \bigcup_{i=1}^{n} U_i$. From Lemma 1.1,

$$l(E_0) - \epsilon = (b_0 - a_0) - \epsilon < \sum_{i=1}^{n} (b_i - a_i + \frac{\delta}{2^i})$$

$$\leq \sum_{i=1}^{\infty} l(E_i) + \delta .$$

Since ϵ and δ are arbitrary, the conclusion follows.

Before the next theorem is stated, it is necessary to state two basic definitions. In defining the length of an interval, every set or interval was assigned a real number. This is a special case of a set function.

Definition 1.1. v is called a set function defined on $\mathfrak{F}$, where $\mathfrak{F}$ is a class of sets, if v assigns to every $A \in \mathfrak{F}$ a number, denoted by v(A), of the extended real number system.

Definition 1.2. A set function v defined on a class of sets $\mathfrak{F}$ is countably additive if, for a disjoint sequence of sets $\{E_i\} \subset \mathfrak{F}$ such that $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{F}$,

$$v\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} v(E_i).$$

The length l of the intervals in $\mathfrak{D}$ is a set function. The following theorem states that l is countably additive.

Theorem 1.3. The length l, defined on $\mathcal{B}$ is countably additive.

Proof: If $\{E_i\}$ is a disjoint sequence of sets in $\mathcal{B}$ whose union is also in $\mathcal{B}$, then, $\sum_{i=1}^{n} l(E_i) \leq l(E)$, where $E = \bigcup_{i=1}^{\infty} E_i$. Hence, $\sum_{i=1}^{\infty} l(E_i) \leq l(E)$. By Theorem 1.2, the conclusion follows.

The idea of a set function that is countably additive is now extended from semiclosed intervals to a more general class of sets. The space is the plane, Euclidean 2-space, and $\mathcal{B}$ is the collection of all rectangles. Then the set function defined on $\mathcal{B}$ is the area of a rectangle. If one were to continue in this manner the space could be extended to Euclidean n-space. The set function, "length" will be denoted by $\bar{l}$. A list of desirable properties that $\bar{l}$ might be expected to satisfy has been proposed by Thielman (5, 132).

These properties are:

(1) $\bar{l}$ is defined for every set $A \subseteq E_n$. (E_n denotes Euclidean n-space.)

(2) $\bar{l}(A) \geq 0$ for all A in E_n.

(3) $\bar{l}$ is countably additive.

(4) If $A \subseteq E_n$, then $\bar{l}$ is invariant under translation or rotation in E_n.

(5) $\bar{l}(A)$ reduces to s^n if the set A is a "cube" of "edge" s in E_n.

It is shown in the following example that a "length" satisfying these five properties cannot exist if one accepts Zermelo's Axiom of Choice (2, 166).

Let a be an irrational number. In terms of a, a subdivision of the interval $A = \{x : 0 \leq x < 1\}$ into a denumerable number of sets can be obtained. Each of these sets is a translation of every other one. In this example every real number is identified with the number of its residue class modulo 1 which is in A. Associate with every real number x the denumerable set

$$S_x = \{\cdot \cdot \cdot, x - 2a, x - a, x, x + a, x + 2a, \cdot \cdot \cdot\}.$$

Now it is proven that for every x and y, S_x and S_y are either identical or disjoint. Assume S_x and S_y are not disjoint. Then there is a $z \in S_x \cap S_y$. Let $w \in S_x$. Then $z = x + na$, $z = y + ma$, and $w = x + pa$, where n, m, and p are integers. It follows that

$$w = x + pa = z + (p - n) a = y + (p - n + m)a,$$

so that $w \in S_y$. Therefore $\{S_x\}$ is a decomposition of A into disjoint denumerable sets.

Now define a set S consisting of one and only one element from each of the sets S_x. Also for every integer m, let

$$S(m) = S + ma = \{x + ma : x \quad S\}.$$

The sets S(m) are a denumerable number of translations of S.

Next it is proven that every z is in an S(m) and that it is in only one S(m). Now $z \in S_x$ for one and only one $x \in S$. Hence $z = x + ma$ for some m so that $z \in S(m)$. Suppose $z \in S_n$. Then $z = y + na$, $y \in S$. It follows that $x + ma = y + na$. Hence $x - y = (n-m)a$. This means that $y \in S_x$, so that $S_y = S_x$. Since S has only one element from each S_x, $y = x$. Then $(n-m)a = 0$, so that $n = m$.

It has now been proven that the sets $S(m) = S_m$ are disjoint translations of S such that

$$A = \bigcup_{m=-\infty}^{\infty} S_m.$$

Suppose $\overline{l}(T)$ is a set function satisfying properties (1) - (5). What can the "length" $\overline{l}(S(0))$ of $S = S(0)$ be?

Suppose $\overline{l}(S(0)) = 0$. Then by property (4), $\overline{l}(S(m)) = 0$ for every integer m. Since

$$A = \bigcup_{m=-\infty}^{\infty} S_m.$$

by property (3), $\overline{l}(A) = 0$, so that property (5) is violated.

Suppose $\overline{l}(S(0)) = k > 0$. Let $n > 1/k$. Then by properties (3) and (4),

$$\overline{l}\left(\bigcup_{i=1}^{n} S_i\right) > 1.$$

But

$$\bigcup_{i=1}^{n} S_i \subset A$$

and by property (5),

$$\overline{l}\left(\bigcup_{i=1}^{n} S_i\right) \leq 1,$$

which is a contradiction.

Hence $\overline{I}$ (S(0)) cannot be greater than 0, nor can it be equal to 0. Also by property (2), it cannot be less than 0.

This proves that if all sets are to have a "length" then properties (2) to (5) are too severe. Therefore, if one accepts the last four properties then not every set can have a "length."

In this chapter a set function was defined first on an interval and it was then indicated that this set function could be extended to E_n. The purpose of the next chapter is to define a set function on a more general class of sets.

CHAPTER 2

MEASURE ON RINGS

The purpose of the rest of this report is to discuss "length" in a more general setting. This "length" of a set will be referred to as the measure of a set. However, before a definition of measure is given, a certain class of sets, a ring of sets, is needed.

Definition 2.1. A ring of sets is a non-empty class $\mathcal{R}$ of sets such that if $E \in \mathcal{R}$ and $F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$ and $E - F \in \mathcal{R}$.

Since $A \cap B = A - (A-B)$ for any sets A and B, a ring $\mathcal{R}$ is closed under intersections. Also, $A \Delta B = (A-B) \cup (B-A)$, the symmetric difference of two sets A and B is in a ring $\mathcal{R}$.

The class of all finite unions of semiclosed intervals of the form,

$$\left\{(x_1, x_2, \cdots, x_n) : -\infty < a_i \leq x_i < b_i < \infty,\ i = 1, 2, \cdots, n\right\},$$

in E_n is a ring. Also, the set of all finite subsets of an arbitrary set A is a ring. The class of sets $\mathcal{B}$ that was discussed in the introduction to this report is not a ring as it is not closed under finite unions.

Now a measure can be defined for a class of sets that is more general than the "length" that was discussed on rectangles in E_n.

Definition 2. 2. A measure is an extended real valued, non-negative and countably additive set function m, defined on a Ring $\mathcal{R}$ and such that $m(0) = 0$. (The empty set is denoted by 0.)

The class of all finite disjoint unions of the intervals defined in Chapter 1 form a ring. The length l defined in Chapter 1 can be extended to a measure on this ring. In the remainder of this report $\mathcal{R}$ denotes a ring and m denotes a measure on $\mathcal{R}$.

If m is a measure on $\mathcal{R}$, a set $E \in \mathcal{R}$ is said to have finite measure if $m(E) < \infty$.

If m is an extended real valued, non-negative, and additive set function defined on $\mathcal{R}$ and such that $m(E) < \infty$ for at least one $E \in \mathcal{R}$, then $m(0) = 0$. This holds since m is additive and

$$m(E) = m(E \cup 0) = m(E) + m(0).$$

A basic property of m on $\mathcal{R}$ is contained in the following theorem.

Theorem 2.1. If $m(A) \geq 0$, for all $A \in \mathcal{R}$ and $A_1 \subset A_2$, $A_1, A_2 \in \mathcal{R}$, then $m(A_1) \leq m(A_2)$.

Proof: $A_2 = A_1 \cup (A_2 - A_1)$ and therefore $m(A_2) = m(A_1 \cup (A_2 - A_1)) = m(A_1) + m(A_2 - A_1)$.

If for $E \in \mathcal{J}$, $F \in \mathcal{J}$, $E \subset F$, and v is an extended real valued set function on the class $\mathcal{J}$, then $v(E) \leq v(F)$; then v is called monotone. From Theorem 2.1 above, m defined on $\mathcal{R}$ is monotone.

Throughout the rest of this section additional properties of a measure on $\mathcal{R}$ are examined. These properties give a greater understanding of measure on $\mathcal{R}$ and are necessary in discussing "outer measure" in Chapter 3.

Definition 2.3. An extended real valued set function v on a class $\mathcal{J}$ is said to be subtractive if, for $E \in \mathcal{J}$, $F \in \mathcal{J}$, $E \subset F$, $F - E \in \mathcal{J}$, and $|v(E)| < \infty$, then $v(F - E) = v(F) - v(E)$.

Theorem 2.3. If m is a measure on $\mathcal{R}$, then m is subtractive.

Proof: If $E \in \mathcal{R}$, $F \in \mathcal{R}$, and $E \subset F$, then $F - E \in \mathcal{R}$ and

$$m(F) = m(E) + m(F - E).$$

By subtracting m(E) from both sides of the equation if $m(E) < \infty$, it is seen that m is subtractive.

Theorems 1.1 and 1.2 can now be extended for m on $\mathcal{R}$. The proofs are very similar to the proofs of Theorems 1.1 and 1.2 (3, 37).

Before more properties of m are discussed it is necessary to make some important definitions concerning the limits of sets. These properties of m that are proved deal with continuous measure. The basic ideas of continuous measure are associated with the property of countable additivity of m. This property of the countable additivity of m and also the notion of a finite measure are the main concepts in the proofs involving continuous measure.

Definition 2.4. If $\{E_n\}$ is a sequence of subsets of the space $\mathcal{U}$, the set of all those points of $\mathcal{U}$ which belong to E_n for infinitely many values of n is called the superior limit of the sequence. This superior limit is denoted by $E^* = \limsup E_n$.

Definition 2.5. The set of all those points of the space $\mathcal{U}$ which belong to E_n for all but a finite number of values of n is called the inferior limit of the sequence $\{E_n\}$. This inferior limit is denoted by $E_* = \liminf E_n$.

If $E^* = E_*$ then the sequence is said to have a limit which is denoted by $\lim_{n\to\infty} E_n$. In this case $E_* = E^* = \lim_{n\to\infty} E_n$.

For an example of these limits, let $A_n = \left[0, \frac{n}{n+1}\right]$, $n = 1, 2, \cdots$, then $\bigcap_{k=n}^{\infty} A_k = \left[0, \frac{n}{n+1}\right]$ and $\bigcup_{k=n}^{\infty} A_k = [0, 1)$. Thus $\liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = [0, 1) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup E_n$. If however $A_n = [0, 1]$ for odd values of n and $A_n = [-1, 0]$ for even values of n then $\liminf E_n \neq \limsup E_n$ (2, 8).

Now the theorems which show the relation of this limit concept to m are proved.

<u>Definition 2.6.</u> An extended real valued set function v defined on a class $\mathcal{D}$ is continuous from below at a set E in $\mathcal{D}$ if, for every increasing sequence $\{E_n\}$ of sets in $\mathcal{D}$ for which $\lim_{n\to\infty} E_n = E$, then $\lim_{n\to\infty} v(E_n) = v(E)$. Similarly v is continuous from above at E in $\mathcal{D}$ if, for every decreasing sequence $\{E_n\}$ of sets in $\mathcal{D}$ for which $|v(E_m)| < \infty$ for at least one m and for which $\lim_{n\to\infty} E_n = E$, then $\lim_{n\to\infty} v(E_n) = v(E)$.

<u>Theorem 2.4.</u> If m is a measure on $\mathcal{R}$ and if $\{E_n\}$ is an increasing sequence of sets in $\mathcal{R}$ for which $\lim_{n\to\infty} E_n \in \mathcal{R}$, then $m(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m(E_n)$.

<u>Proof:</u> Let $E_0 = 0$, then

$$m(\lim_{n\to\infty} E_n) = m(\bigcup_{i=1}^{\infty} E_i) = m(\bigcup_{i=1}^{\infty} (E_i - E_{i-1})) = \sum_{i=1}^{\infty} m(E_i - E_{i-1})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} m(E_i - E_{i-1}) = \lim_{n\to\infty} m(\bigcup_{i=1}^{n} E_i - E_{i-1})) = \lim_{n\to\infty} m(E_n).$$

Theorem 2.4 states that if m is a measure then it is continuous from below. Now a similar theorem asserts that m is also continuous from above.

Theorem 2.5. If m is a measure on $\mathcal{R}$, and if $\{E_n\}$ is a decreasing sequence of sets in $\mathcal{R}$ of which at least one has finite measure and for which $\lim_{n\to\infty} E_n \in \mathcal{R}$, then $m(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m(E_n)$.

Proof: Assume $m(E_k) < \infty$, then $m(E_n) \leq m(E_k) < \infty$ for $n \geq k$. Therefore

$$m(\lim_{n\to\infty} E_n) < \infty.$$

Because $\{E_k - E_n\}$ is an increasing sequence and m is monotone and subtractive, then

$$m(E_k) - m(\lim_{n\to\infty} E_n) = m(E_k - \lim_{n\to\infty} E_n) = m(\lim_{n\to\infty} (E_k - E_n))$$

$$= \lim_{n\to\infty} m(E_k - E_n) = \lim_{n\to\infty} (m(E_k) - m(E_n)) = m(E_k) - \lim_{n\to\infty} m(E_n).$$

Since $m(E_k) < \infty$, then $m(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m(E_n)$.

Let S be the set of all rational numbers x for which $0 \leq x \leq 1$, and let $\mathcal{P}$ be the class of all semiclosed intervals of the form $\{x : x \in S,\ a \leq x < b\}$ where $0 \leq a \leq b \leq 1$, and a and b are rational. Define w on $\mathcal{P}$ by $w(\{x: a \leq x < b\}) = b - a$. This set function w is finitely additive and continuous from above and below but it is not countably additive (3, 40).

The following theorem is a converse of Theorems 2.4 and 2.5.

<u>Theorem 2.6.</u> Let m be a finite, non-negative, and additive set function on a ring $\mathcal{R}$. If m is either continuous from below at every E in $\mathcal{R}$, or continuous from above at 0, then m is a measure on $\mathcal{R}$.

<u>Proof</u>: Since m is additive on a ring $\mathcal{R}$ then m is finitely additive. Let $\{E_n\}$ be a disjoint sequence of sets in $\mathcal{R}$, whose union E is also in $\mathcal{R}$. Write

$$F_n = \bigcup_{i=1}^{n} E_i \text{ and } G_n = E - F_n.$$

If m is continuous from below, then, since $\{F_n\}$ is an increasing sequence of sets in $\mathcal{R}$ with $\lim_{n\to\infty} F_n = E$, then

$$m(E) = \lim_{n\to\infty} m(F_n) = \lim_{n\to\infty} \sum_{i=1}^{n'} m(E_i) = \sum_{i=1}^{\infty} m(E_i).$$

If m is continuous from above at 0, then since $\{G_n\}$ is a decreasing sequence of sets in $\mathcal{R}$ with $\lim_{n\to\infty} G_n = 0$, and since m is finite, then

$$m(E) = \left(\sum_{i=1}^{n} m(E_i)\right) + m(G_n) = \lim_{n\to\infty} \sum_{i=1}^{n} m(E_i) + \lim_{n\to\infty} m(G_n)$$

$$= \sum_{i=1}^{\infty} m(E_i).$$

Let X be the set of all positive integers and let $\mathcal{U}$ be the class of all finite subsets of X and their complements. For E in $\mathcal{U}$ write v(E) = 0 or v(E) = ∞ according as E is finite or infinite. The set function v is continuous from above at 0 but it is not countably additive. Therefore Theorem 2.6 is not true if infinite values are admitted (3, 40). With this example, Chapter 2 is concluded. In the next chapter a set function over an extended class of sets is discussed.

CHAPTER 3

OUTER MEASURE

In Chapter 2, some of the more important properties of m on $\mathcal{R}$ were proved. It was seen that many of these properties were related to the concept of countable additivity. In this section m will be extended to a larger class of sets. In order to do this, the measure on these sets will be defined by relaxing the concept of countable additivity. Before this measure is discussed, it is necessary to define several other concepts. One of these is the extension of a ring by using countable unions instead of finite unions. The measure m discussed in the previous sections was defined on a class of sets called a ring. In this section other special classes of sets are considered.

Definition 3.1. A non-empty class $\mathcal{J}$ of sets is hereditary if, whenever $G \in \mathcal{J}$ and $F \subset G$, then $F \in \mathcal{J}$.

An example of a hereditary class is the class of all subsets of some subset G of a space $\mathcal{U}$.

Definition 3.2. An extended real valued set function v on any class of sets is countably subadditive if for every sequence $\{E_i\}$ of sets in $\mathcal{J}$ whose union is also in $\mathcal{J}$, then $v(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} v(E_i)$.

In comparing Definition 3.2 with Definition 1.2, it is seen that countable subadditivity is more general than additivity.

Definition 3.3. A σ-ring is a non-empty class $\mathcal{J}$ of sets such that if $E \in \mathcal{J}$ and $G \in \mathcal{J}$, then $E - G \in \mathcal{J}$, and if $E_i \in \mathcal{J}$, $i = 1, 2, \cdots$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{J}$.

The hereditary classes display a very important property. A hereditary class is a σ-ring if and only if it is closed under the formation of countable unions.

If $\mathcal{J}$ is any class of sets, the hereditary σ-ring generated by $\mathcal{J}$ is denoted by $\mathcal{H}(\mathcal{J})$. This notation is used throughout the remainder of this report. This is the smallest hereditary σ-ring containing $\mathcal{J}$ and is of importance in defining "outer" measure.

Consider any hereditary σ-ring $\mathcal{H}$. A special set function, which is an extension of measure, is now defined on $\mathcal{H}$.

Definition 3.5. An outer measure is an extended real valued, non-negative, monotone, and countably subadditive set function, defined on a hereditary σ-ring $\mathcal{H}$, and such that the outer measure of the null set is zero.

In Theorem 3.1, a relationship between a measure and an outer measure is stated. An outer measure m_0 can be defined as a lower bound on sums of m and may be thought of as induced by a measure m.

Theorem 3.1. If m is a measure on $\mathcal{R}$ and if, for every set E in $\mathcal{H}(\mathcal{R})$,

$$m_0(E) = \inf\left\{\sum_{i=1}^{\infty} m(E_n) : E_n \in \mathcal{R},\ n = 1, 2, \cdots,\ E \subset \bigcup_{n=1}^{\infty} E_n\right\},$$

then m_0 is an extension of m to an outer measure on $\mathcal{H}(\mathcal{R})$.

Proof: If $E \in \mathcal{R}$, then $E \subset E \cup 0 \cup 0 \cup \cdots$ and $m_0(E) \leq m(E) + m(0) + m(0) + \cdots = m(E)$. If $E \in \mathcal{R}$, $E_n \in \mathcal{R}$, $n = 1, 2, \cdots$, and $E \subset \bigcup_{n=1}^{\infty} E_n$, then $m(E) \leq \sum_{n=1}^{\infty} m(E_n)$, so that $m(E) \leq m_0(E)$. This last statement is a result of the generalization of Theorem 1.2. Thus $m_0(E) = m(E)$ for all $E \in \mathcal{R}$ or m_0 is an extension of m. Because m_0 is an extension of m, it follows that $m_0(0) = 0$.

Now it is shown that m_0 is monotone. If $E \in \mathcal{H}(\mathcal{R})$, $F \in \mathcal{H}(\mathcal{R})$, $E \subset F$, and $\{E_n\}$ is a sequence of sets in $\mathcal{R}$ which covers F, then $\{E_n\}$ also covers E. Thus $m_0(E) \leq m_0(F)$. To complete the proof, it is necessary to prove that m_0 is countably subadditive. Let E and E_i be sets in $\mathcal{H}(\mathcal{R})$ such that $E \subset \bigcup_{i=1}^{\infty} E_i$. Let ϵ be an arbitrary positive number, and choose, for each $i = 1, 2, \cdots$ a sequence $\{E_{ij}\}$ of sets in $\mathcal{R}$ such that $E_i \subset \bigcup_{j=1}^{\infty} E_{ij}$ and $\sum_{j=1}^{\infty} m(E_{ij}) \leq m_0(E_i) + \frac{\epsilon}{2^i}$. Now, since the sets E_{ij} form a countable class of sets in $\mathcal{R}$ which covers E,

$$m_0(E) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} m(E_{ij}) \leq \sum_{i=1}^{\infty} m_0(E_i) + \epsilon .$$

Since ϵ is arbitrary,

$$m_0(E) \leq \sum_{i=1}^{\infty} m_0(E_i).$$

The following examples illustrate properties of m_0. In the first two examples exactly one condition in the definition of m_0 is violated, but the third example is an outer measure.

Let $A = \{x, y\}$ be a set consisting of exactly two distinct points x and y, $\mathcal{U}$ be the class of all subsets of A, and v_0 be defined by the relations $v_0(0) = 0$, $v_0(\{x\}) = v_0(\{y\}) = 10$ and $v_0(A) = 1$. This v_0 is not an outer measure since it is not monotone.

Let A be arbitrary, $\mathcal{U}$ is the class of all subsets of A. For every $E \varepsilon \mathcal{U}$, let $v_0(E) = 1$. In this example v_0 satisfies all the conditions of the definition of outer measure except $v_0(0) = 1 \neq 0$.

Let A be arbitrary, $\mathcal{U}$ is the class of all countable subsets of A, $m_0(E)$ is the number of points in E. ($m_0(E) = \infty$ if E is infinite.) Then m_0 is an outer measure.

CHAPTER 4

PROPERTIES OF MEASURABLE SETS

In this chapter an important concept of a certain class of sets in relation to m_0 is defined and discussed. As before, let m_0 be an outer measure on a hereditary σ-ring $\mathcal{H}$. Definition 4.1 is due to Carathéodory.

<u>Definition 4.1.</u> A set E in $\mathcal{H}$ is m_0-measurable if, for every set A in $\mathcal{H}$,

$$m_0(A) = m_0(A \cap E) + m_0(A \cap E'),$$

where E' is the complement of E.

It is very difficult to gain an understanding of this concept of m_0-measurability. This concept is, however, a valuable tool in deriving theorems in this chapter, and the extension theorem of the following chapter. As is given in Definition 3.5, m_0 is not necessarily a countably additive set function. To try to satisfy the requirement of additivity the sets which split every set additivity are singled out for study. This is a very loose description of the concept of m_0-measurability.

Now some properties of the sets which are m_0-measurable are proved.

Theorem 4.1. If m_0 is an outer measure on $\mathcal{H}$ and if $\bar{S}$ is the class of all m_0-measurable sets, then $\bar{S}$ is a ring.

Proof: If E and F are in $\bar{S}$ and $A \in \mathcal{H}$, then

(a) $$m_0(A) = m_0(A \cap E) + m_0(A \cap E')$$

(b) $$m_0(A \cap E) = m_0(A \cap E \cap F) + m_0(A \cap E \cap F')$$ and

(c) $$m_0(A \cap E') = m_0(A \cap E' \cap F) + m_0(A \cap E' \cap F')$$

from the definition of m_0-measurability. By substituting (b) and (c) into (a)

(d) $$m_0(A) = m_0(A \cap E \cap F) + m_0(A \cap E \cap F') + m_0(A \cap E' \cap F) + m_0(A \cap E' \cap F')$$

is obtained.

If A is replaced by $A \cap (E \cup F)$ in (d) then

(e) $$m_0(A \cap (E \cup F)) = m_0(A \cap E \cap F) + m_0(A \cap E \cap F') + m_0(A \cap E' \cap F),$$

since the first three terms on the right hand side remain unchanged and

$$(A \cap (E \cup F)) \cap E' \cap F' = 0.$$

Now by substituting (e) into (d) and using $E' \cap F' = (E \cup F)'$, then

(f) $$m_0(A) = m_0(A \cap (E \cup F)) + m_0(A \cap (E \cup F)')$$

is obtained. This proves that $E \cup F \varepsilon \bar{S}$.

The next step in this proof is to show that $E - F \varepsilon \bar{S}$. To show this, replace A in (d) by $A \cap (E - F)' = A \cap (E' \cup F)$ to obtain

(g) $m_0(A \cap (E - F)') = m_0(A \cap E \cap F) + m_0(A \cap E' \cap F) + m_0(A \cap E' \cap F')$.

Since $E \cap F' = E - F$,

$$m_0(A) = m_0(A \cap (E - F)) + m_0(A \cap (E - F)')$$

from the substitution of (g) into (d). Hence $E - F \varepsilon \bar{S}$. Now substitute $E = 0$ into (a) to complete the proof (3, 45).

In the next theorem, Theorem 4.1 is generalized by replacing finite unions by countable unions. First a lemma is stated which is used in the proof.

<u>Lemma 4.1.</u> If m_0 is an outer measure on a hereditary σ-ring $\mathcal{H}$ and if a set E in $\mathcal{H}$ is such that, for every A in $\mathcal{H}$, $m_0(A) \geq m_0(A \cap E) + m_0(A \cap E')$ then E is m_0-measurable (3, 45).

<u>Theorem 4.2.</u> If m_0 is an outer measure on $\mathcal{H}$ and if $\bar{S}$ is the class of all m_0-measurable sets, then $\bar{S}$ is a σ-ring.

Proof: Define $E = \bigcup_{n=1}^{\infty} E_n$ such that $E_i \cap E_j \neq 0$, $i \neq j$. Replace E and F in

(a) $$m_0(A\cap(E\cup F)) = m_0(A\cap E\cap F) + m_0(A\cap E\cap F') + m_0(A\cap E'\cap F)$$

by E_1 and E_2 which are disjoint sets. Equation (a) reduces to

(b) $$m_0(A\cap(E_1\cup E_2)) = m_0(A\cap E_1) + m_0(A\cap E_2).$$

By mathematical induction, it follows that

(c) $$m_0(A\cap\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m_0(A\cap E_i)$$

for every positive integer n.

Let $F_n = \bigcup_{i=1}^{n} E_i$, $(n = 1, 2, \cdot\cdot\cdot)$, and then by Lemma 4.1

(d) $$m_0(A) = m_0(A\cap F_n) + m_0(A\cap F'_n) \geq \sum_{i=1}^{n} m_0(A\cap E_i) + m_0(A\cap E').$$

Since this is true for all n,

(e) $$m_0(A) \geq \sum_{i=1}^{\infty} m_0(A\cap E_i) + m_0(A\cap E') \geq m_0(A\cap E) + m_0(A\cap E').$$

Since every countable union of sets in a ring may be written as a disjoint countable union of sets in a ring, (e) holds for every countable union of sets in a ring. Therefore $\mathfrak{S}$ is a σ-ring.

Now it is possible to show that the sets in $\mathfrak{S}$ satisfy the condition of countable additivity.

Theorem 4.3. If $A \in \mathcal{H}$ and if $\{E_n\}$ is a disjoint sequence of sets in $\overline{S}$ with $\bigcup_{n=1}^{\infty} E_n = E$, then

$$m_0(A \cap E) = \sum_{n=1}^{\infty} m_0(A \cap E_n).$$

Proof: Equation (e) in Theorem 4.2 states that E is m_0-measurable. Therefore

$$\text{(a)} \qquad \sum_{i=1}^{\infty} m_0(A \cap E_i) + m_0(A \cap E')$$

$$= m_0(A \cap E) + m_0(A \cap E').$$

If A is replaced by $A \cap E$ in (a), then

$$\text{(b)} \qquad \sum_{i=1}^{\infty} m_0(A \cap E_i) = m_0(A \cap E).$$

Note that $m_0(A \cap E')$ may be infinite and therefore cannot be subtracted from both sides of (a).

Before the last theorem in this chapter is stated, it is necessary to make the following definition.

Definition 4.2. A measure m is called complete if for $E \in \mathcal{R}$, $F \subset E$, and $m(E) = 0$, then $F \in \mathcal{R}$.

In Theorem 4.4 it is shown that all sets of measure zero belong to $\overline{S}$.

Theorem 4.4. If m_0 is an outer measure on $\mathcal{H}$ and if $\overline{\mathcal{S}}$ is the class of all m_0-measurable sets, then every set of outer measure zero belongs to $\overline{\mathcal{S}}$ and the set function $\overline{m}$, defined for E in $\overline{\mathcal{S}}$ by $\overline{m}(E) = m_0(E)$ is a complete measure on $\overline{\mathcal{S}}$. This measure $\overline{m}$ is said to be induced by the outer measure m_0.

Proof: Let $E \in \mathcal{H}$ and $m_0(E) = 0$, then for every $A \in \mathcal{H}$,

$$m_0(A) = m_0(E) + m_0(A) \geq m_0(A \cap E) + m_0(A \cap E').$$

Therefore $E \in \overline{\mathcal{S}}$ from Lemma 4.1.

If A is replaced by E in

$$\sum_{i=1}^{\infty} m_0(A \cap E_i) + m_0(A \cap E') = m_0(A \cap E) + m_0(A \cap E'),$$

then $\overline{m} = m_0$ is countably additive. If $E \in \overline{\mathcal{S}}$, $F \subset E$, and $\overline{m}(E) = m_0(E) = 0$, then $m_0(F) = 0$, so that $F \in \overline{\mathcal{S}}$, which proves that $\overline{m}$ is complete.

CHAPTER 5

INDUCED MEASURES

In this chapter some of the properties of induced measures are discussed. In these results a relationship between m_0 and $\overline{m}$ is established. Throughout this chapter $\mathcal{S}(\mathcal{E})$ shall denote the σ-ring generated by any class $\mathcal{E}$ of sets. For example $\mathcal{S}(\mathcal{R})$ is the smallest σ-ring containing $\mathcal{R}$, where $\mathcal{R}$ is a ring. Also m denotes a measure on a Ring $\mathcal{R}$, m_0 denotes an outer measure on a Hereditary σ-ring $\mathcal{H}$, and $\bar{m}$ denotes the induced measure on the class of all m_0-measurable sets, $\bar{\mathcal{S}}$.

<u>Theorem 5.1.</u> Every set in $\mathcal{S}(\mathcal{R})$ is m_0-measurable.

<u>Proof</u>: Let $E \varepsilon \mathcal{R}$, $A \varepsilon \mathcal{H}(\mathcal{R})$ and $\epsilon > 0$, then by the definition of m_0, there exists a sequence $\{E_n\}$ of sets in $\mathcal{R}$ such that $A \subset \bigcup_{n=1}^{\infty} E_n$ and

$$m_0(A) + \epsilon \geq \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} (m(E_n \cap E) + m(E_n \cap E')$$

$$= \sum_{n=1}^{\infty} m(E_n \cap E) + \sum_{n=1}^{\infty} m(E_n \cap E') \geq m_0(A \cap E) + m_0(A \cap E').$$

Since this is true for all $\epsilon > 0$, then E is m_0-measurable. Now since $\bar{\mathcal{S}}$ is a σ-ring, and since $\mathcal{R} \subset \bar{\mathcal{S}}$, then $\mathcal{S}(\mathcal{R}) \subset \bar{\mathcal{S}}$.

The following theorem states that the outer measure m_0 on $\mathcal{H}(\mathcal{R})$ can be induced by an induced measure $\bar{m}$ on $\bar{\mathcal{S}}$ or on $\mathcal{S}(\mathcal{R})$.

<u>Theorem 5.2.</u> If $E \varepsilon \mathcal{H}(\mathcal{R})$, then

$$m_0(E) = \inf \left\{ \bar{m}(F) : E \subset F \varepsilon \bar{\mathcal{S}} \right\}$$

$$= \inf \left\{ \bar{m}(F) : E \subset F \varepsilon \mathcal{S}(\mathcal{R}) \right\}.$$

<u>Proof</u>: For $F \varepsilon \mathcal{R}$, $m(F) = \bar{m}(F)$ then

$$m_0(E) = \inf\left\{\sum_{n=1}^{\infty} m(E_n) : E \subset \bigcup_{n=1}^{\infty} E_n,\ E_n \in \mathcal{R},\ n = 1, 2, \cdots\right\}$$

$$\geq \inf\left\{\sum_{n=1}^{\infty} \bar{m}(E_n) : E \subset \bigcup_{n=1}^{\infty} E_n,\ E_n \in S(\mathcal{R}),\ n = 1, 2, \cdots\right\}.$$

If $\bigcup_{n=1}^{\infty} E_n \supset E$, $E_n \in S(\mathcal{R})$, then $F = \bigcup_{n=1}^{\infty} E_n \supset E$, $F \in S(\mathcal{R})$, $\bar{m}(F) \leq \sum_{n=1}^{\infty} \bar{m}(E_n)$.

Therefore $\inf\left\{\bar{m}(F) : E \subset F,\ F \in S(\mathcal{R})\right\}$

$$\leq \inf\left\{\sum_{n=1}^{\infty} \bar{m}(E_n) : E \subset \bigcup_{n=1}^{\infty} E_n,\ E_n \in S(\mathcal{R}),\ n = 1, 2, \cdots\right\}.$$

Therefore $m_0(E) \geq \inf\left\{\bar{m}(F) : E \subset F,\ F \in S(\mathcal{R})\right\}$

$$\geq \inf\left\{\bar{m}(F) : E \subset F,\ F \in \bar{S}\right\}.$$

If $E \subset F$, $F \in \bar{S}$, then $m_0(E) \leq m_0(F) = \bar{m}(F)$. Hence $\inf\left\{\bar{m}(F) : E \subset F,\ F \in \bar{S}\right\} \geq m_0(E)$.

The main results in this chapter deal with the relationship between induced measures. If the induced measure is formed from the outer measure m_0 and then the outer measure $\bar{m}_0$ induced by $\bar{m}$ is formed, what is the relationship between m_0 and $\bar{m}_0$? In general, these two measures are not the same. Before this question is examined, however, it is necessary to define a measurable cover and a σ-finite measure.

<u>Definition 5.1.</u> If $E \in \mathcal{H}(\mathcal{R})$ and $F \in S(\mathcal{R})$, then F is a measurable

cover of E if $E \subset F$, and if for every set G in $S(\mathcal{R})$ for which $G \subset F - E$, then $\overline{m}(G) = 0$.

The term measurable cover is a good description because a measurable cover of a set E is, in a vague sense, a minimal set in $S(\mathcal{R})$ which covers E.

Definition 5.2. The measure m of $E \in \mathcal{R}$ is σ-finite if there exists a sequence $\{E_n\}$ of sets in $\mathcal{R}$ such that

$$E \subset \bigcup_{n=1}^{\infty} E_n \text{ and } m(E_n) < \infty, \; n = 1, 2, \cdots .$$

If the measure of every $E \in \mathcal{R}$ is σ-finite, m is called σ-finite on $\mathcal{R}$. If the space $X \in \mathcal{R}$, $\mathcal{R}$ is a ring of sets in X, and m(X) is σ-finite, then m is called totally σ-finite.

The following theorem illustrates another relationship between m and m_0.

Theorem 5.3. If a measure m on $\mathcal{R}$ is σ-finite then m_0 on $\mathcal{H}(\mathcal{R})$ is also σ-finite.

Proof: Let E be any set in $\mathcal{H}(\mathcal{R})$. Then there exists a sequence $\{E_i\}$ of sets in $\mathcal{R}$ such that $E \subset \bigcup_{i=1}^{\infty} E_i$ (3, 41). Since m is σ-finite, there exists for each $i = 1, 2, \cdots$, a sequence $\{E_{ij}\}$ of sets in $\mathcal{R}$ such that $E_i \subset \bigcup_{j=1}^{\infty} E_{ij}$ and $m(E_{ij}) < \infty$. Therefore $E \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_{ij}$ and $m_0(E_{ij}) =$

$m(E_{ij}) < \infty$.

<u>Theorem 5.4.</u> If a set E in $\mathcal{H}(\mathcal{R})$ is of σ-finite outer measure, then there exists a set F in $S(\mathcal{R})$ such that $m_0(E) = \bar{m}(F)$ and such that F is a measurable cover of E.

<u>Proof:</u> First consider the case where $m_0(E) < \infty$. It follows from Theorem 5.2 that for $n = 1, 2, \cdots$, there exists a set F_n in $S(\mathcal{R})$ such that

$$E \subset F_n \text{ and } \bar{m}(F_n) \leq m_0(E) + 1/n.$$

If $F = \bigcap_{n=1}^{\infty} F_n$, then $E \subset F \in S(\mathcal{R})$ and $m_0(E) \leq \bar{m}(F) \leq \bar{m}(F_n) \leq m_0(E) + 1/n$. Since n is arbitrary, $m_0(E) = \bar{m}(F)$. If $G \in S(\mathcal{R})$ and $G \subset F - E$, then $E \subset F - G$ and therefore $\bar{m}(F) = m_0(E) \leq \bar{m}(F - G) = \bar{m}(F) - \bar{m}(G) \leq \bar{m}(F)$. Since $m(F)$ is finite, F is a measurable cover of E.

If $m_0(E) = \infty$, let $E \subset \bigcup_{n=1}^{\infty} E_n$ where $m_0'(E_n) < \infty$, $n = 1, 2, \cdots$. Let $A_1 = E_1$ and $A_n = E_n - \bigcup_{k=1}^{n-1} E_k$, $n \geq 2$. Note that $A_i \cap A_j = 0$, $i \neq j$, $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\} \subset \mathcal{H}(\mathcal{R})$. Since $A_n \subset E_n$, $m_0(A_n) \leq m_0(E_n) < \infty$. Let $B_n = E \cap A_n$, $n = 1, 2, \cdots$. Then $E = \bigcup_{n=1}^{\infty} B_n$, $B_n \in \mathcal{H}(\mathcal{R})$ and $m_0(B_n) < \infty$.

For each B_n there exists $F_n \in S(\mathcal{R})$ such that $F_n \supset B_n$, $\overline{m}(F_n) = m_0(B_n)$. F_n is a measurable cover of B_n. Let $F = \bigcup_{n=1}^{\infty} F_n$, then $m_0(E) = \overline{m}(F)$. Let $G \in S(\mathcal{R})$ such that $G \subset F - E = \bigcup_{n=1}^{\infty} F_n - E$. Let $G_n = G \cap F_n$. Then $G_n \subset F_n - E \subset F_n - B_n$. Since F_n is a measurable cover of B_n, $\overline{m}(G) = \overline{m}(\bigcup_{n=1}^{\infty} G_n) \leq \sum_{n=1}^{\infty} \overline{m}(G_n) = 0$. Therefore F is a measurable cover of E.

<u>Lemma 5.1</u>. If both F_1 and F_2 are measurable covers of $E \in \mathcal{H}(\mathcal{R})$, then $\overline{m}(F_1 \Delta F_2) = 0$, where $F_1 \Delta F_2$ denotes the symmetric difference of F_1 and F_2.

<u>Proof</u>: Since $E \subset F_1 \cap F_2 \subset F_1$ then $F_1 - (F_1 \cap F_2) \subset F_1 - E$. Also since F_1 is a measurable cover of E then $\overline{m}(F_1 - (F_1 \cap F_2)) = 0$. By the same relationship, $\overline{m}(F_2 - (F_1 \cap F_2)) = 0$. Therefore $\overline{m}(F_1 \Delta F_2) = 0$.

<u>Theorem 5.5</u>. If $E \in \mathcal{H}(\mathcal{R})$ and F is a measurable cover of E, then $m_0(E) = \overline{m}(F)$.

<u>Proof</u>: If $m_0(E) = \infty$, then $m_0(E) = m_0(F)$. If $m_0(E) < \infty$, then from Theorem 5.4 there exists a measurable cover F_0 of E with $\overline{m}(F_0) = m_0(E)$. But Lemma 5.1 implies that every two measurable covers

have the same measure. Therefore it can be concluded that $m_0(E) = \bar{m}(F)$.

The following theorem states that any σ-finite measure m on $\mathcal{R}$ can always be extended to a unique measure $\bar{m}$, called the extension of m on $\mathcal{S}(\mathcal{R})$. This theorem is the most important result obtained using the extension procedures of this chapter. Note that in the previous chapter it was proved that m can be extended to $\bar{\mathcal{S}}$ which is in general larger than $\mathcal{S}(\mathcal{R})$ (3, 55).

Theorem 5.6. If m is a σ-finite measure on $\mathcal{R}$, then there is a unique measure $\bar{m}$ on $\mathcal{S}(\mathcal{R})$ such that, for E in $\mathcal{R}$, $\bar{m}(E) = m(E)$. The measure $\bar{m}$ is σ-finite (3, 54).

Proof: Theorem 4.4 states there is a measure on the sets in $\bar{\mathcal{S}}$, and Theorem 5.1 states that every set in $\mathcal{S}(\mathcal{R})$ is m_0-measurable. Hence, measure $\bar{m}$ exists.

To prove that $\bar{m}$ is unique, assume m_1 and m_2 are two measures on $\mathcal{S}(\mathcal{R})$ such that $m_1(E) = m_2(E)$ whenever $E \varepsilon \mathcal{R}$, and let $\mathcal{M}$ be the class of all sets E in $\mathcal{S}(\mathcal{R})$ for which $m_1(E) = m_2(E)$. If one of the two measures is finite, and if $\{E_n\}$ is a monotone sequence of sets in $\mathcal{M}$, then $\lim_{n\to\infty} E_n \varepsilon \mathcal{M}$, since

$$m_i(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m_i(E_n), \ (i = 1, 2,)$$

Since this means that $\mathcal{M}$ is a monotone class, and since $\mathcal{M}$ contains $\mathcal{R}$, it also contains $S(\mathcal{R})$.

For the general case, let A be any fixed set in $\mathcal{R}$, of finite measure with respect to one of the two measures m_1 and m_2. Since $\mathcal{R} \cap A$[1] is a ring and $S(\mathcal{R}) \cap A$[2] is the σ-ring $\mathcal{R} \cap A$ generates, it follows that the ideas in the first part of this proof apply to $\mathcal{R} \cap A$ and $S(\mathcal{R} \cap A)$. Therefore, if $E \in S(\mathcal{R}) \cap A$ then $m_1(E) = m_2(E)$. Since every $E \in S(\mathcal{R})$ can be covered by a countable disjoint union of sets with finite measure, then every $E \in S(\mathcal{R})$ has the same property; that is, $m_1(E) = m_2(E)$.

Now an example is given to illustrate the restriction of σ-finiteness on $\mathcal{R}$ in Theorem 5.6 (3, 57). Let $\mathcal{R}$ be a ring of subsets of a countable set A, with the property that every non-empty set in $\mathcal{R}$ is infinite and such that $S(\mathcal{R})$ is the class of all subsets of A. If, for every subset E of A, $m_1(E)$ is the number of points in E and $m_2(E) = 2m_1(E)$, then m_2 and m_1 agree on $\mathcal{R}$ but not on $S(\mathcal{R})$. Therefore, the uniqueness assertion of Theorem 5.6 is not true without the restriction of σ-finiteness on $\mathcal{R}$, even for measures which are totally σ-finite on $S(\mathcal{R})$.

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[1] $\mathcal{R} \cap A$ denotes the class of all sets of the form $E \cap A$ with E in $\mathcal{R}$.

[2] If $\mathcal{R}$ is a ring and if A is any subset of the space, then $S(\mathcal{R} \cap A) = S(\mathcal{R} \cap A)$.

CHAPTER 6

CONCLUSION

The purpose of this chapter is to apply the general theory discussed in the main part of the report to the measure discussed in Chapter 1. This measure is a classical case in measure theory, Lebesgue measure. Throughout this chapter the space is the real line, $\mathcal{P}$ is the class of all bounded, semi-closed intervals of the form $[a, b)$ and v is the set function on $\mathcal{P}$ defined by $v([a, b)) = b - a$. Note that v is the length of the line segment bounded by the points a and b as in Chapter 1.

Now let $\mathcal{T}$ be the class of all finite disjoint unions of sets of $\mathcal{P}$. Since the union and the difference of any two sets of $\mathcal{T}$ is a set of the same form, $\mathcal{T}$ is a ring.

If $A = I_1 \cup I_2 \cup \cdots \cup I_n$ where I_k $(k = 1, 2, \cdots, n)$ are sets from $\mathcal{T}$ then $v(A) = v(I_1) + v(I_2) + \cdots + v(I_n)$.

The set function v is countably additive on $\mathcal{T}$; if a = b then $v(0) = 0$. Since all the conditions for a measure are satisfied, the set function v is called a measure in the remainder of this chapter.

Now consider the class $\mathcal{S}$ which is the hereditary σ-ring generated by $\mathcal{T}$. This class $\mathcal{S}$ is the class of all subsets of the line. Define an outer measure v_0 on $\mathcal{S}$ as in Definition 3.5. A set E in $\mathcal{S}$ is v_0-measurable if for every set F in $\mathcal{S}$,

$$v_0(F) = v_0(F \cap E) + v_0(F \cap E').$$

The class of all sets which are v_0-measurable is denoted by $\bar{\mathcal{S}}$. The set function $\bar{v}$ defined for E in $\bar{\mathcal{S}}$ by $\bar{v}(E) = v_0(E)$ is now a complete measure.

$\bar{\mathcal{S}}$ could also be obtained the following way. Note that this method was not completely covered in this report. Define $\mathcal{S}(\mathcal{J})$ to be the smallest σ-ring containing $\mathcal{J}$. The sets of $\mathcal{S}(\mathcal{J})$ are commonly called the Borel sets of the line. By Theorem 5.6 there is a unique measure $\bar{v}$ on $\mathcal{S}(\mathcal{J})$ which is an extension of v on $\mathcal{J}$. Now enlarge $\mathcal{S}(\mathcal{J})$ by including all the sets of measure zero. This is the class of sets $\bar{\mathcal{S}}$. The sets of $\bar{\mathcal{S}}$ are the Lebesgue measurable sets of the line. The measure v is called Lebesgue measure (3, 56).

Historically, the concept of measure was approached in the manner of this last chapter. However, the fundamental concepts of measure are perhaps clearer in a more general setting and are not complicated by a "mass of comparatively trivial detail" (4, 190).

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SOME ELEMENTARY CONCEPTS IN MEASURE THEORY

by

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The purpose of this report is to gain an understanding of some of the elementary concepts of measure theory.

In the introduction, the length of a set in the collection of semi-closed intervals on the real line is considered. From the study of this set function, natural questions arise which lead to the definition of measure.

A measure m is defined on a ring of sets $\mathcal{R}$ and some properties of m on $\mathcal{R}$ are proved. Then the problem of extending a measure is considered.

A special set function, outer measure, is defined on a hereditary σ-ring, $\mathcal{H}(\mathcal{R})$. This outer measure is an extension of a measure on $\mathcal{R}$ to a set function on $\mathcal{H}(\mathcal{R})$. Since an outer measure is not necessarily additive, the sets which split every set additively are singled out and called the class of measurable sets $\overline{\mathcal{S}}$. This is Caratheodory's method. The outer measure restricted to $\overline{\mathcal{S}}$ is a measure. This measure, called the induced measure, is proven to be a complete measure.

Next, the class of sets $\mathcal{S}(\mathcal{R})$ which is the σ-ring generated by $\mathcal{R}$, is considered. It is proven that every set in $\mathcal{S}(\mathcal{R})$ is measurable. The measure on $\mathcal{S}(\mathcal{R})$ is, in general, not the same as the measure on $\overline{\mathcal{S}}$. The most important result is that a σ-finite measure on $\mathcal{R}$ can always be extended to a unique measure on $\mathcal{S}(\mathcal{R})$.

In the conclusion, the general theory discussed in the main part of this report is applied to the length defined on the collection of semi-closed intervals. These intervals are extended to a ring $\mathcal{R}$ of sets.

The hereditary σ-ring $\mathcal{H}(\mathcal{R})$ generated by this ring is the class of all subsets of the line.. If the outer measure is formed on $\mathcal{H}(\mathcal{R})$ then the measure on $\bar{S}$ is complete and is called Lebesgue measure. In this case the sets of $\bar{S}$ are called the Lebesgue measurable sets and the sets of $S(\mathcal{R})$ are called the Borel sets.