PROBABLE WORST CASE CIRCUIT DESIGN

by

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INTRODUCTION

A participant in a game of poker will have a low performance per dollar invested if he bets only on the one hand that is a sure winner. The reason, of course, is that he rarely is dealt an optimum hand. If he is more optimistic and decides to bet on hands that have a high probability of winning, the performance per dollar invested will undoubtedly increase.

Not concerned with poker, per se, this paper nevertheless is interested in posing the question: "Can this approach be applied to circuit and system design in the game of designing reliable products?"

To continue the analogy, although the poker player will never lose betting on sure hands, he cannot realize his full potential for winning. Similarly, while an electrical circuit theoretically will never be worse than the "worst case" combination of its components, its full performance capability is not being realized. The content of this report postulates a "yes" answer to the basic question concerning poker--winning ratios and circuit possibilities.

The performance of a circuit is defined by functions of the components describing the various outputs. When the components are distributed between two extremes, the output function of interest can be any value between some maximum and minimum value determined by specific combinations of component values. If the circuit designer must specify a certain value which the output will never exceed, it is obvious that the minimum value cannot

be used. However, is it necessary to use the maximum value when the probability that this value will ever be realized is approximately zero? The performance may be greatly increased if a "probable worst case" value, less than the maximum value, is used. The reliability of the circuit will be negligibly decreased as long as the probability that the output is greater than this probable worst case value is very small.

The same approach can be used with systems. An example is the transition time of cascaded circuits when the individual circuit transition times are known.[1] The possibility that all circuits would have maximum transition times is very remote.

The methods presented in this report give the circuit or logic designer a conservative estimate of the distribution of an output function, between the maximum and minimum values, caused by the distributions of the components in the circuit.

There are several methods for estimating the distribution of the output function of a circuit. Some require extensive sampling of components and use Monte Carlo methods. The composition of distributions formula

$$\sigma^2(f) = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i} \sigma(x_i) \right]^2$$

also is used but this gives results which have the greatest error in the region of interest near the worst case value.

The methods presented in this paper allow the circuit designer to choose a distribution that he knows, or believes, to be

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[1] Transition time is that time to shift from one mode of operation to another, such as from full off to full on.

conservative for each component, and therefore will give results which are highly useful and still remain generally conservative. The methods are compatible with the present circuit evaluation and are easy for the circuit designer to apply. They apply to both alternating-current and direct-current functions.

There also are valuable byproducts of the methods which will be pointed out as they arise.

THE EXACT AND APPROXIMATE MODELS OF A FUNCTION

Definition of the Problem

The problem is to obtain the disbribution of the output function between its two extreme values. If the performance of the circuit can be improved by moving one of the two extremes closer to the other, let that extreme be known as the worst case value. The other will be designated as the "best case value". Quite often both extremes satisfy this condition. However, this approach will consider only one at a time. The following definitions will be used.

f = output function of interest

$\overline{f}$ = worst case value of the function

x_i = one of the components in the function f

$\underline{f}$ = best case value of the function

$\underline{x}_i$ = value of x_i to cause f to be best case

$\overline{x}_i$ = value of x_i to cause f to be worst case

Each component will be distributed between two extreme

values. Initially, a component will be within a specified set of limits. After a length of time, known as "end of life", these limits will have changed. Since most component values can drift in either direction, the widest set of limits, i.e., the end-of-life values, will be used. These end-of-life values are normally known by the circuit designer and used to obtain $\bar{f}$. It should be noted that component values which drift in only one direction will require a different treatment and may require use of the initial extremes.

Circuit failures will be considered to be one of three types for the purpose of this report. The first is a catastrophic failure such as a component opening or shorting. The second is a catastrophic drift failure and is defined to be a circuit failure caused by a component value drifting outside of the end-of-life extremes before end of life. The third type is a drift failure caused by using the probable worst case value. The first two are failures caused by faulty components. We are interested only in failures of the third type as this is a direct measure of what must be done to obtain the improved performance.

It is assumed there are no catastrophic drift failures while obtaining the distribution of the output function. This is not the actual case but it does enable us to isolate the type of failures in which we are interested. It is noted, however, that use of the probable worst case value will slightly increase the catastrophic drift failues.

It should now be obvious that all functions are not applicable to this method of analysis as a unique combination of the

x_i's must give $\bar{f}$. This implies the function must be strictly monotonic over the range of the x_i's.[1] This may seem to be a severe restriction. However, most output functions of interest will satisfy this condition.

A function with a dependent variable can quite often be reduced to a function with all independent variables that gives conservative results. An example would be a circuit with resistors whose values change with temperature. The variable T can be eliminated by changing the two extremes on all resistors to be the maximum and minimum over the range of T. Note that this also reduces a nonmonotonic function to a strictly monotonic function.

The preceding statements can be clarified by considering a simple voltage divider (see Fig. 1).

The voltage V is

$$V = \frac{R_1}{R_1 + R_2} E$$

In terms of functional notation, $f = V$, $x_1 = R_1$, $x_2 = R_2$, and $x_3 = E$. Consider the resistors to have initial values of $R_0 \pm 1$ per cent and values of $R_0 \pm 3$ per cent at end of life. The voltage E is specified as $E_0 \pm 2$ per cent initially and $E_0 \pm 4$ per cent at end of life. The end-of-life extremes would be used for all variables with x_{i_M} the maximum extreme, and

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[1] A strictly monotonic function of one variable is one whose derivative is always greater than zero. A function of several variables is strictly monotonic if, and only if, the same defition holds for all variables when the remaining variables are considered constants. This must be true for any combination over the range of the x_i's.

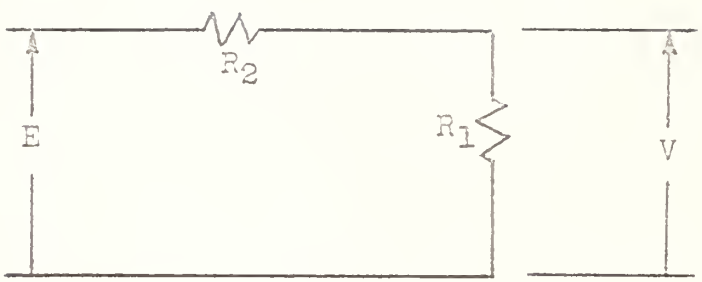

Fig. 1. Simple voltage divider.

x_{i_m} the minimum one.

$$\bar{x}_1 = x_{1_M}$$
$$\bar{x}_2 = x_{2_m}$$
$$\bar{x}_3 = x_{3_M}$$
$$\bar{f} = \frac{\bar{x}_1}{\bar{x}_1 + \bar{x}_2} \bar{x}_3$$

If the resistances vary with temperature, the maximum value of x_1 at any temperature and the minimum value of x_2 at any temperature would be used. This gives a worst case value that could never happen if both resistances change the same way with a change in temperature. It is, however, an upper bound on the worst case value and is a strictly monotonic function.

Exact Model

The exact model for a function of three variables is shown in Fig. 2. The discussion to follow is valid for a function of n variables. Due to the difficulty in visualizing an n dimensional space, however, it will be based on the three-dimensional model.

The plot of $f(x_1, x_2, x_3) = f_k$, where f_k is a constant between $\bar{f}$ and $\underline{f}$, will be a surface in three dimensions. The rectangular cube shown is the range of the three variables. Therefore any point on the surface, inside or on the boundary of the cube, is a possible combination of (x_1, x_2, x_3) over their ranges to give f_k. Each time f_k is changed, a different surface is described. This new surface does not intersect the previous one

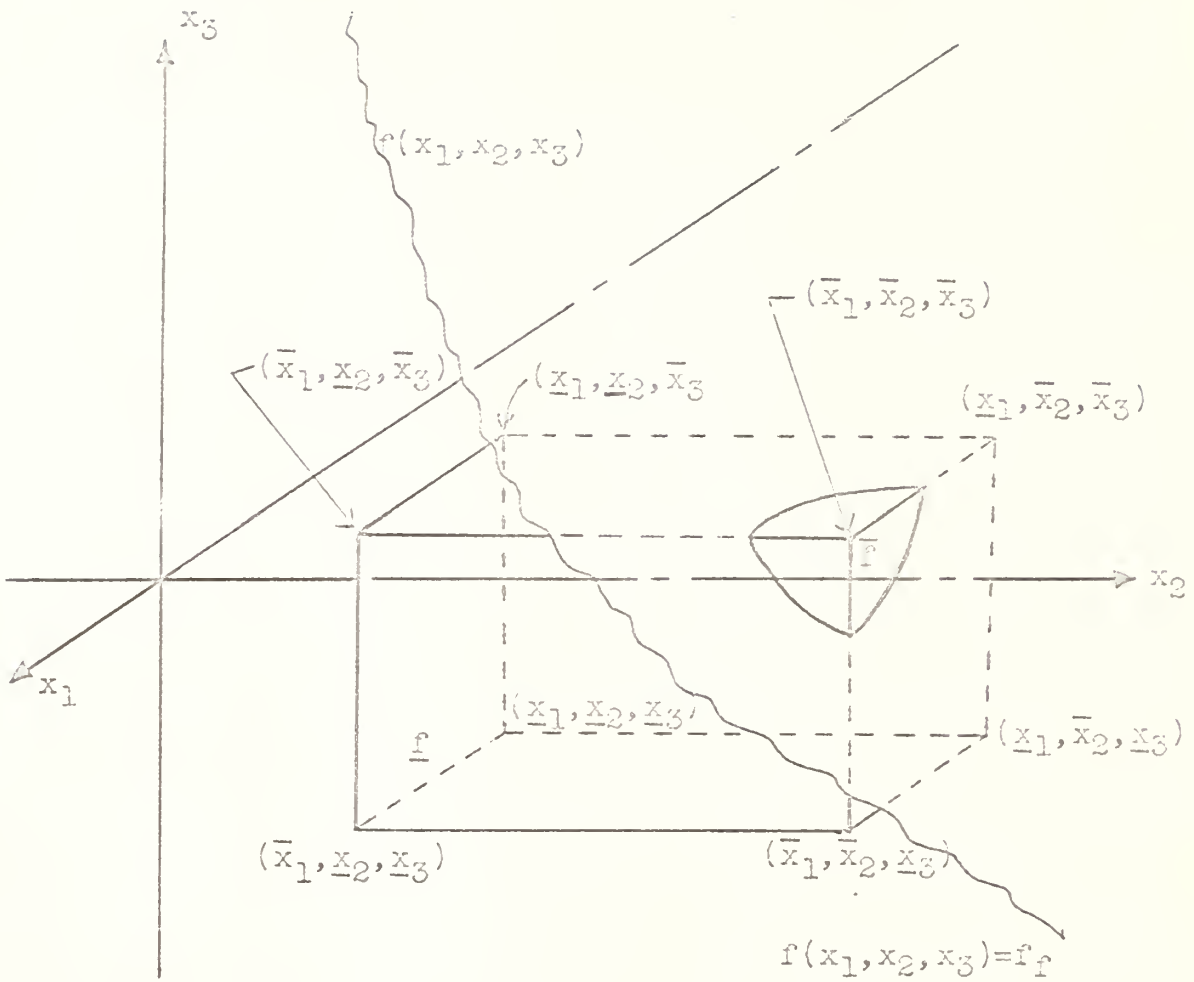

Fig. 2. Exact model for three variables.

within the cube because the function is strictly monotonic over this range.

The combination of (x_1, x_2, x_3) which will give $\bar{f}$ represents one of the 2^n corners of the cube. The function determines the corner for $\bar{f}$ and the limits on each component determine the length of the sides. In Fig. 2, $\underline{f}$ is shown closest to the origin and $\bar{f}$ the farthest away. This is an assumed combination to clarify the discussion. Any two corners directly opposite from each other could have been chosen.

It is assumed that the reader is familiar with basic probability theory. The terms and basic theory used to develop the methods presented can be found in Feller (3), or any introductory text on probability theory.

Two different distributions of the individual components will be considered in detail. The first is that x_i is either $\bar{x}_i$ or $\underline{x}_i$ with equal probability of having either value. The second is a uniform distribution between $\bar{x}_i$ and $\underline{x}_i$. The first distribution restricts the combinations of (x_1, x_2, x_3) to the 2^n corners. The probability that f equals one of these 2^n values, $P\{f = f_k\}$, is $\frac{1}{2^n}$, where n is the number of independent variables in the function. The second distribution restricts the combinations of (x_1, x_2, x_3) to the inside and boundary of the cube. The probability that f equals one of the values determined by the points in an infinitesimal volume located inside the cube is

$$\left| \frac{1}{(\bar{x}_1 - \underline{x}_1)(\bar{x}_2 - \underline{x}_2)(\bar{x}_3 - \underline{x}_3)} \right| dx_1 dx_2 dx_3$$

We are interested in $P\{f > f_k\}$, where f_k is some value between $\overline{f}$ and $\underline{f}$.

The first distribution requires counting the number of corners in the portion of the cube cut off by the surface $f(x_1, x_2, x_3) = f_k$, and dividing this number by 2^n to obtain $P\{f > f_k\}$. There is only one corner in the portion cut off in Fig. 2. Therefore $P\{f > f_k\} = \frac{1}{2^3} = \frac{1}{8}$.

In the second distribution, $P\{f > f_k\}$ is directly proportional to the volume cut off by the surface $f(x_1, x_2, x_3) = f_k$ divided by the total volume V of the cube. Notice that $P\{f \geq \overline{f}\} = \frac{0}{V} = 0$ and $P\{f > \underline{f}\} = \frac{V}{V} = 1$. This is not a proof of the above statement. It does show consistency with the obvious results at two points. The explanation is presented in Appendix II.

We now have a method for determining $P\{f > f_k\}$ which will give exact results for the two distributions. However, a few problems remain and may be stated as follows.

1. The hypersurface $f(x_1, x_2, \ldots, x_n) = f_k$ is quite difficult to visualize and plot for $n > 3$.
2. The corners to count become very numerous as n increases; for example, $2^{20} = 1{,}048{,}576$, and $2^{40} = 1{,}099{,}511{,}627{,}776$.
3. The volume of the rectangular hypercube cut off by the hypersurface $f(x_1, x_2, \ldots, x_n) = f_k$ is extremely difficult to evaluate.

Due to the problems described and the possibility that the function itself is not available in a numerical solution, the function will be approximated. The approximate function represents an approximate model, a fact that will be enlarged upon subsequently.

Approximation of the Function

The output function of interest is a function of the components $x_1, x_2, \ldots, x_n$ where there are n components that affect the function. A Taylor series will be used to approximate the function. The region of interest of the output function is near the worst case value, and therefore the expansion is about $\bar{f}$.

The expansion about $\bar{f}$ is (Brand, 1, p. 185)

$$f = \bar{f} + \sum_{i=1}^{n} \left[\left. \frac{\partial f}{\partial x_i} \right|_{\bar{f}} (x_i - \bar{x}_i) \right] + \frac{1}{2!} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \left. \frac{\partial^2 f}{\partial x_i \partial x_i} \right|_{\bar{f}} (x_i - \bar{x}_i)(x_j - \bar{x}_j) \right] + \ldots + R_n$$

where

$$\left. \frac{\partial f}{\partial x_i} \right|_{\bar{f}} \doteq \frac{\bar{f} - f_{\underline{x}_i}}{\bar{x}_i - \underline{x}_i}$$

and $f_{\underline{x}_i}$ is the value of the output function with all components $x_j = \bar{x}_j$, $j \neq i$, and $x_j = \underline{x}_j$, when $j = i$. (A smaller variation of x_i should be used when a small change in x_i causes a large change in f.) The truncation of the series and the approximation of the partial derivatives render the above series an approximation. If the function is slowly varying at $\bar{f}$, however, the method of

evaluating the partial derivatives does give a reasonable approximation due to the small magnitude of the higher order terms in this region. Since the point of expansion is $\bar{f}$, $\left.\frac{\partial f}{\partial x_i}\right|_{\bar{f}}$ will be negative when $\underline{x}_i > \bar{x}_i$, and positive when $\bar{x}_i > \underline{x}_i$.

The evaluation of the partial derivatives is most time-consuming. Fortunately there are two important byproducts that make the results more valuable. First, if $\left.\frac{\partial f}{\partial x_i}\right|_{\bar{f}}$ does not have the proper sign, the circuit designer has chosen the wrong extreme for $\bar{x}_i$. The second, and probably most important, is that the magnitude of the partial derivatives indicates the components that need closer tolerances and the ones that can have wider tolerances, resulting in a possible reduction in the cost of the components.

The first-order terms of the Taylor series will be used as the approximating function. This will give the greatest accuracy in the region close to $\bar{f}$ which is the region of interest. The approximate function is as follows.

$$f \doteq \bar{f} + \sum_{i=1}^{n} \left[\left.\frac{\partial f}{\partial x_i}\right|_{\bar{f}} (x_i - \bar{x}_i) \right] = A_0 + \sum_{i=1}^{n} A_i x_i$$

$$= A_0 + \tilde{x}_1 + \tilde{x}_2 + \ldots \tilde{x}_n$$

where

$$A_0 = \bar{f} - \sum_{i=1}^{n} A_i \bar{x}_i$$

$$A_i = \left.\frac{\partial f}{\partial x_i}\right|_{\bar{f}}$$

$$\tilde{x}_i = A_i \bar{x}_i$$

$$\underset{\sim}{x}_i = A_i \underline{x}_i$$

A cumulative density function, CDF, will be obtained using the approximate function and the final step will be to fit the CDF between $\bar{f}$ and $\underline{f}$ in a manner designed to give accurate results in a region about $\tilde{f}$. The method of fitting the CDF and a discussion of the errors involved will follow in a later section.

Approximate Model

The three-dimensional approximate model is shown in Fig. 3. The discussion on $P\{f > f_k\}$ presented with the exact model still holds. The rectangular parallelepiped with sides of length $(\bar{x}_i - \underline{x}_i)$ is now represented by a rectangular parallelepiped with sides of length $A_i(\bar{x}_i - \underline{x}_i)$. The surfaces $f(x_1, x_2, x_3) = f_k$ are approximated by planes which represent constant values of the approximate function.

An algorithm will be developed to count the number of corners in the portion of the volume cut off by a plane $f(x_1, x_2, x_3) = f_k$ for the first distribution. An extensive development of the combination of n uniform distributions will be presented for the second distribution, which amounts to calculating the volume cut off by the plane.

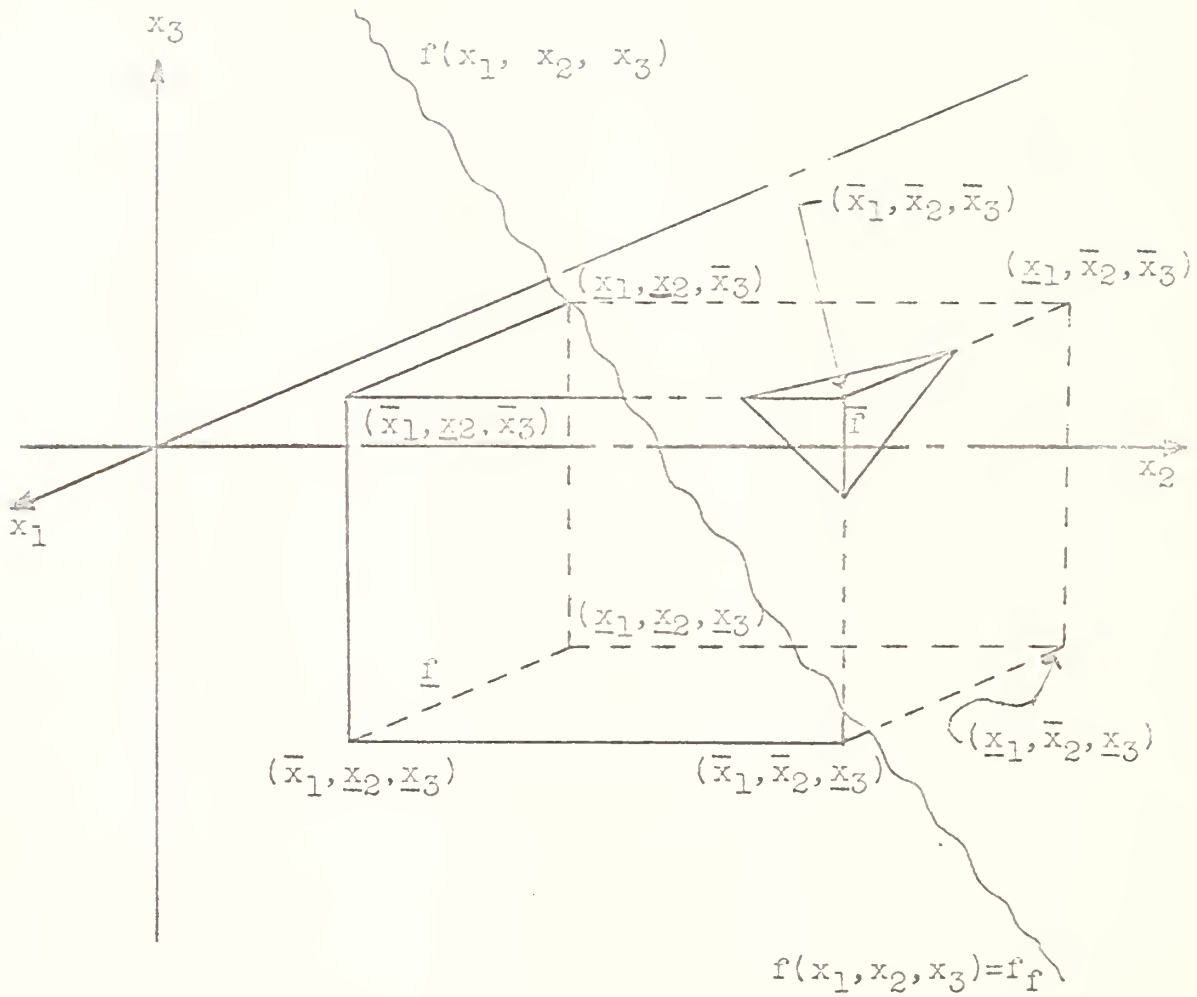

Fig. 3. Approximate model for three variables.

δ FUNCTION DISTRIBUTION OF COMPONENTS

The first distribution to be considered in detail is highly conservative. The distribution of each individual component is assumed to be one-half at $\underline{x}_i$ and one-half at $\overline{x}_i$ (DiMucci, 2, p. 2). The probability density function, PDF, for each component is as follows.

$$\text{PDF of } x_i = g(x_i) = \tfrac{1}{2}\left[\delta(x_i - \underline{x}_i) + \delta(x_i - \overline{x}_i)\right]$$

This distribution of components was considered by DiMucci (2). A PDF was obtained for a function by 2^n substitutions to cover all possible combinations of component values. The method to follow gives a good approximation to this PDF. The calculations required are reduced by a factor of nearly $\frac{2n}{2^n}$ which is a considerable reduction for large values of n. However, the method in DiMucci (2) is quite satisfactory for small values of n.

The PDF for f follows immediately since n components imply 2^n possible values for f. It is represented by the following equation with f_i being one of the 2^n constant values of f at the corners of the approximate model.

$$\text{PDF of } f(x_1, x_2, \ldots, x_n) = g(f) = \frac{1}{2^n}\sum_{i=1}^{2^n}\left[\delta(f - f_i)\right]$$

where $f_1 = \overline{f}$

and $\left|f_{i+1} - f_1\right| \geq \left|f_i - f_1\right|$

This represents 2^n possible values for $f(x_1, x_2, \ldots, x_n)$. The probability of f taking on any one of these values is $1/2^n$. The 2^n values are, of course, the constant values in the equation

$f(x_1, x_2, \ldots, x_n) = f_k$. They represent the surfaces passing through the corners of the approximate model. There are 2^n corners and consequently 2^n possible values.

The probability that $f > f_k$, where f_k is one of these 2^n values, is as follows. $\bar{f}$ will be considered the maximum value of the function.

$$P\{f > f_k\} = \frac{k - 1}{2^n} \qquad (1 \leq k \leq 2^n)$$

The approach is to obtain values of f between $\bar{f}$ and $\underline{f}$ and then determine which f_k each value represents. The inequality sign is reversed when $\bar{f}$ is the minimum value.

Taylor's theorem can be written as

$$\Delta f = df + \frac{1}{2!} d^2f + \ldots + \frac{1}{n!} d^nf + \ldots + \text{higher order terms (Brand, 1, p. 186)}$$

The approximation used is $\Delta f \doteq df$. At $\bar{f}$ the total differential is $\Delta f = df = 0$. There are n points to be obtained besides $\bar{f}$ and they will be found by taking various combinations of the partial differentials. Let

$$p_i = \left| \frac{\partial f}{\partial x_i} \right|_{\bar{f}} (\bar{x}_i - \underline{x}_i) \Bigg| = \left| \bar{x}_i - \underline{x}_i \right|$$

be the magnitude of the i^{th} partial differential (Brand, 1, p. 154). Arrange the n partial differentials in order of increasing magnitude with p_1 the smallest and p_n the largest. The j^{th} point of the n points obtained will be $f_j = \bar{f} \pm \sum_{i=1}^{j} p_i$. The minus sign is used when $\bar{f}$ is the maximum value of the function.

Observe that $\binom{n}{k}$ is the number of unique formal sums of k numbers that can be formed from n numbers where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are the binomial coefficients (Feller, 3, P. 32).

The n points obtained are exactly the values that would be found by the corresponding solution of the approximate function since $\Delta f = df$. Because $\Delta f_j = \sum_{i=1}^{j} p_i$ and is the smallest sum of j of the p_i's, all other unique Δf's obtained by taking J of the p_i's will be greater if $J \geq j$. If the number of Δf's $\geq \Delta f_j$ is known when $J < j$, the probability that $f > \bar{f} - \Delta f_j$ is easily found. The problem then is to determine how many Δf's of the 2^n possible are less than Δf_j. The 2^n possible values of Δf, when subtracted from $\bar{f}$, correspond to the 2^n possible values of $f(x_1, x_2, \ldots, x_n)$.

The first step is to arrange the n partial differentials so that $p_{i+1} > p_i$. The smallest sum of j p_i's is then calculated for $j = 1, 2, \ldots, n$. The following numbers are then found.

k_j = number of p_i's $\geq p_1 + p_2 + \cdots + p_{j+1}$ $\quad (1 \leq j)$

k_j' = number of p_i's $\geq p_2 + p_3 + \cdots + p_{j+1}$ $\quad (2 \leq j)$

k_j'' = number of p_i's $\geq p_3 + p_4 + \cdots + p_{j+1}$ $\quad (3 \leq 1)$

k_j^m = number of p_i's $\geq p_{m+1} + p_{m+2} + \cdots + p_{j+1}$ $\quad (m + 1 \leq j)$

To clarify the above, consider the following set of p_i's for n = 10.

$p_1 = 1,\ p_2 = 2,\ p_i = i \qquad (1 \leq i \leq 10)$

$k_1 = 8,\ k_2 = 5,\ k_3 = 1,\ k_j = 0 \qquad (j > 3)$

$$k_2' = 6,\ k_3' = 2,\ k_j' = 0 \qquad (j > 3)$$

$$k_3'' = 4,\ k_j'' = 0 \qquad (j > 3)$$

$$k_4''' = 2,\ k_j''' = 0 \qquad (j > 4)$$

$$k_j^m = 0 \qquad (3 < m)$$

The general formulae for the number of sums of (j - 1) and (j - h) partial differentials greater than the smallest sum of j are developed in detail in Appendix III. They will be stated here but should not be applied without fully realizing their limitations. The following definitions will be used.

$$\binom{n}{j} = 1 \qquad j = 0$$

$$\binom{n}{j} = 0 \qquad j > n$$

$$\binom{n}{j} = 0 \qquad j < 0$$

$$\binom{n}{j} = 1 \qquad j = n = 0$$

A. Number of sums of (j - 1) partial differentials equal to or greater than the smallest sum of j.

$$\binom{k_1}{1} \cdot \binom{n-k_1-2}{j-2} + \binom{k_1}{2} \cdot \binom{n-k_1-2}{j-3} + \binom{k_1}{3} \cdot \binom{n-k_1-2}{j-4} + \cdots$$

$$+ 2\binom{k_2'}{1} \cdot \binom{n-k_2'-2}{j-3} + 2\binom{k_2'}{2} \cdot \binom{n-k_2'-2}{j-4} + \cdots$$

$$+ \binom{k_3''}{1} \cdot \binom{n-k_3''-2}{j-4} + \binom{k_3''}{2} \cdot \binom{n-k_3''-2}{j-5} + \cdots$$

B. Number of sums of (j - 2) partial differentials equal to or greater than the smallest sum of j.

$$\binom{k_2}{1} \cdot \binom{n-k_2-3}{j-3} + \binom{k_2}{2} \cdot \binom{n-k_2-3}{j-4} + \cdots$$

$$+ 3\binom{k_3'}{1} \cdot \binom{n-k_3'-3}{j-4} + 3\binom{k_3'}{2} \cdot \binom{n-k_3'-3}{j-5} + \cdots$$

$$+ 3\binom{k_4''}{1} \cdot \binom{n-k_4''-3}{j-5} + 3\binom{k_4''}{2} \cdot \binom{n-k_4''-3}{j-6} + \cdots$$

$$+ \binom{k_5'''}{1} \cdot \binom{n-k_5'''}{j-6} + \binom{k_5'''}{2} \cdot \binom{n-k_5'''-3}{j-7} + \cdots$$

C. Number of sums of (j - h) partial differentials equal to or greater than the smallest sum of j.

$$\binom{h+1}{0} \cdot \binom{k_h}{1} \cdot \binom{n-k_h-h-1}{j-h-1} + \binom{h+1}{0} \cdot \binom{k_h}{2} \cdot \binom{n-k_h-h-1}{j-h-2} + \cdots$$

$$+ \binom{h+1}{1} \cdot \binom{k_{h+1}'}{1} \cdot \binom{n-k_{h+1}'-h-1}{j-h-2}$$

$$+ \binom{h+1}{1} \cdot \binom{k_{h+1}'}{2} \cdot \binom{n-k_{h+1}'-h-1}{j-h-3} + \cdots$$

$$+ \binom{h+1}{2} \cdot \binom{k_{h+2}''}{1} \cdot \binom{n-k_{h+2}''-h-1}{j-h-3}$$

$$+ \binom{h+1}{2} \cdot \binom{k_{h+2}''}{2} \cdot \binom{n-k_{h+2}''-h-1}{j-h-4} + \cdots$$

$$+ \quad . \quad . \quad .$$

$$\vdots$$

$$+ \binom{h+1}{x} \cdot \binom{k_{h+x}^{(x)}}{1} \cdot \binom{n-k_{h+x}^{(x)}-h-1}{j-h-x-1} + \cdots$$

$$\vdots$$

The last row will be for x = h + 1.

A single product term in any row is of the form

$\binom{u}{v} \cdot \binom{w}{x} \cdot \binom{y}{z}$. The first factor $\binom{u}{v}$ determines the number of rows in the formula. The second and third factors determine the number of nonzero product terms in each row. The four numbers u, v, w, and y are the same for all terms in a given row. Since x starts at one and increases, the second factor will cause all terms after the w^{th} term to be zero. This factor also can cause all terms in a row to be zero when w is zero. The effect of the third factor is not as definite as the first two. The number z decreases one with each term in the row. This can nullify the first few terms if $z > y$. It will be a nonzero number for $(y \geq z \geq 0)$, and will again be zero when z becomes negative. The reason for each factor is fully explained in Appendix III.

Notice that $v + x + z = j - h$ and $u + w + y = n$ in each term. This observation and the symmetry of the formulae suggest writing the individual formulae in matrix form. The general formula is represented by the one element in the product matrix AD, where $D = BC$. The matrices and their elements are:

$$A = \left[\binom{h+1}{0} \quad \binom{h+1}{1} \quad \binom{h+1}{2} \quad \cdots \quad \binom{h+1}{h+1} \right]$$

$$B = \begin{bmatrix} \binom{k_h}{1} & \binom{k_h}{2} & \binom{k_h}{3} & \cdots \\ \binom{k_{h+1}'}{1} & \binom{k_{h+1}'}{2} & \binom{k_{h+1}'}{3} & \cdots \\ \binom{k_{h+2}''}{1} & \binom{k_{h+2}''}{2} & \binom{k_{h+2}''}{3} & \cdots \\ \vdots & & & \end{bmatrix}$$

$$C = \begin{bmatrix} \binom{n-k_h-h-1}{j-h-1} & \binom{n-k_{h+1}'-h-1}{j-h-2} & \cdots \\ \binom{n-k_h-h-1}{j-h-2} & \binom{n-k_{h+1}'-h-1}{j-h-3} & \cdots \\ \binom{n-k_h-h-1}{j-h-3} & \binom{n-k_{h+1}-h-1}{j-h-4} & \cdots \\ \vdots & & \end{bmatrix}$$

$$D = \begin{bmatrix} \binom{k_h}{1} \cdot \binom{n-k_h-h-1}{j-h-1} + \binom{k_h}{2} \cdot \binom{n-k_h-h-1}{j-h-2} + \cdots \\ \binom{k_{h+1}'}{1} \cdot \binom{n-k_{h+1}'-h-1}{j-h-2} + \binom{k_{h+1}'}{2} \cdot \binom{n-k_{h+1}'-h-1}{j-h-3} + \cdots \\ \vdots \end{bmatrix}$$

The matrix "A" is a row matrix of (h+2) elements where $a_{1m} = \binom{h+1}{m-1}$. Matrix "D" is a column matrix of (h+2) elements formed from the main diagonal of the product matrix "BC" with $D_{\ell 1} = \left[BC\right]_{\ell\ell}$. The elements of "B" and "C" are:

$$b_{\ell m} = \binom{(\ell-1)}{k_{h+\ell-1}}{m}$$

$$c_{\ell m} = \begin{pmatrix} (m-1) \\ n-k_{h+m-1} \\ \\ j-h-\ell-m+1 \end{pmatrix} \quad -h-1$$

Both "B" and "C" are square matrices of dimension (h + 2).

The determination of the element values for A, B, and C can be programmed on a digital computer. Each element is a binomial coefficient of the form $\binom{u}{v}$. Once the numbers u and v are found, the determination of the element values is reduced to a table look-up problem. The products AD and BC are simply matrix multiplication.

The preceding procedure is carried out for as many of the n points as necessary. To clarify this statement consider the variables arranged as $\left|\bar{x}_1 - \underline{x}_1\right| \leq \left|\bar{x}_2 - \underline{x}_2\right| \leq \left|\bar{x}_3 - \underline{x}_3\right| \leq \cdots \leq \left|\bar{x}_n - \underline{x}_n\right|$. The first point obtained will be $f(\underline{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n)$. The second point will be $f(\underline{x}_1, \underline{x}_2, \bar{x}_3, \bar{x}_4, \ldots, \bar{x}_n)$. At each point this procedure will give the number of possible solutions between that value and $\bar{f}$. When $\bar{f}$ is f maximum, we are interested in $P\{f > f_j\}$.

If f_j represents one of the points obtained and the procedure gives T as the total number of solutions greater than f_j, the probability that $f > f_j$ is $T/2^n$.

The plot of $P\{f > f_j\}$ versus f_j is a plot of the cumulative density function obtained from the approximate function. This plot is needed out to the point where $P\{f > f_j\} = 1/2$. This point will be fitted to the mean value of the original function. The curve fitting is discussed in detail in a later section.

This method will give the circuit and logic designer a

conservative result on which to base the design of a system without depending upon a worst case performance criterion.

It still is necessary to know how changes in each component affect the performance of the circuit. Because this takes a certain amount of computer time it is believed that this information should be utilized to its fullest extent. Therefore while the above method gives conservative and valuable results, a more realistic distribution of components may be used and the results can be improved. The next section of this report utilizes less drastic distributions, and, as will be shown, the results are more gratifying.

Note that $P\left\{f < \bar{f} - p_1\right\} = 1/2^n$. This will be compared with the results obtained at the same point in the next section.

UNIFORM DISTRIBUTION OF COMPONENTS

The two extremes for a component x_i are its end-of-life minimum and maximum. At the time a circuit is built, component values are guaranteed to be between the minimum and maximum values specified by the initial tolerances. The end-of-life values are obtained from various sources. Two of these sources are (1) past history of the component, and (2) the rate of change of component value with time under specified operating conditions.

The exact distribution of a component at or near end of life is very difficult to obtain for many components. An example of this is a transistor which would be obsolete by the time this information is obtained to a high degree of accuracy.

The best approach appears to be to utilize available information to determine a distribution that meets the following three conditions.

1. It must be slightly conservative.

2. It must, for obvious reasons, give worthwhile results.

3. It must be easy to work with.

As time and the number of each component tend to infinity, the distribution will approach a normal distribution. Since both time and number are finite, a uniform distribution between $\underline{x}_i$ and $\bar{x}_i$ appears to be a good assumption with which to meet the first two conditions. The third condition is also a prime reason for using the uniform distribution. A method for handling skewed distribution is presented later.

The approximate function is rewritten as follows:

$$f(x_1, x_2, \ldots, x_n) \doteq f(X_1, X_2, \ldots, X_n)$$
$$= A_0 + X_1 + X_2 + \ldots + X_n$$

where

$$A_0 = \bar{f} - \sum_{i=1}^{n} A_i \bar{x}_i \ , \ A_i = \left.\frac{\partial f}{\partial x_i}\right|_{\bar{f}} \ , \text{ and } X_i = A_i x_i$$

The function is a sum of independent variables with the assumption that the components change independently. It should be pointed out that this is not always strictly true. The results of Appendix II are valid for this function if this assumption is valid. The PDF and CDF are represented by the following two equations, considering $\bar{f}$ as the maximum value of $f(X_1, X_2, \ldots, X_n)$ The proof of these equations is given in Appendix II.

$$\text{PDF} = g(f) = \frac{1}{\prod_{i=1}^{n} \Delta X_i} \cdot \frac{1}{(n-1)!} \sum_{j=1}^{2^n} \left[(f_j - f)^{n-1} \, U(f_j - f)(-1)^{y_i} \right]$$

where

$$U(f_j - f) = 0, \quad f > f_j$$
$$= 1, \quad f < f_j$$

f_j is one of the 2^n constant values of f at the corners of the approximate model. $f_{j+1} < f_j$ when $\bar{f}$ = f maximum = f_1.

y_i is equal to the number of $\underline{X}_i$ in the coordinates of the corner represented by f_j. If $f_j = A_0 + \underline{X}_1 + \underline{X}_2 + \underline{X}_3 + \bar{X}_4 + \bar{X}_5$, then $y_i = 3$.

$$\Delta X_1 \equiv \left| \bar{X}_i - \underline{X}_i \right|$$

The CDF can be considered two different ways. Under the above PDF, the total area between $\bar{f}$ and $\underline{f}$ is unity.[1] The CDF can be considered the area under the PDF between f_k and $\bar{f}$ as f_k goes from $\bar{f}$ to $\underline{f}$. The CDF also can be considered as the volume of the approximate model between the hyperplane $f = f_k$ and $\bar{f}$, divided by the total volume as f_k goes from $\bar{f}$ to $\underline{f}$. The first representation is valid for any distribution of components if the PDF is known. The volume representation is valid only for uniformly distributed components unless a weighting factor is placed on the differential volume which is a function of position.

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[1] This is true because $\bar{f} \geq f \geq \underline{f}$ and exist nowhere else under the stated conditions.

$$CDF = P\{f > f_k\} = \frac{1}{\prod_{i=1}^{n} \Delta X_i} \cdot \frac{1}{n!} \sum_{j=1}^{k-1} \left[(f_j - f_k)^n (-1)^{y_j} \right]$$

f_k is one of the 2^n values of f_j and $f_{j+1} < f_j$.

The entire equation represents the area under the PDF between $\bar{f}$ and f_k, and $\prod_{i=1}^{n} \Delta X_i$ is the total volume of the approximate model. Therefore the remainder of the equation is the volume cut off by the hyperplane $f = f_k$.

The special case where all ΔX_i's are equal reduces to the following when $f_N = \bar{f} - N \Delta X$. N is a number $0 \leq N \leq n$ and need not be an integer.

$$P\{f > f_N\} = \frac{1}{(\Delta X)^n} \cdot \frac{1}{n!} \left\{ \binom{n}{0} (N \Delta X)^n - \binom{n}{1} \left[(N - 1)(\Delta X) \right]^n + \binom{n}{2} \left[(N - 2)(\Delta X) \right]^n \ldots \right\}$$

$$= \frac{1}{n!} \left\{ \binom{n}{0} (N)^n - \binom{n}{1} (N - 1)^n + \binom{n}{2} (N - 2)^n - \binom{n}{3} (N - 3)^n + \ldots \right\}$$

Note that $P\{f > \bar{f} - \Delta X\} = \frac{1}{n!}$ for this case. The δ distribution would give a value of $1/2^n$. Since $\frac{1}{n!} << \frac{1}{2^n}$ for expected values of n, the results are much better and should be conservative.

The equation for the CDF is undefined if any ΔX_i is zero. It is obvious that the denominator would be zero as

$\prod_{i=1}^{n} \Delta X_i = 0$. Although not as obvious, the numerator is also zero which gives an indeterminant function. This possibility is considered in Appendix II and the results prove this equation is valid as any, or all but one, of the ΔX_i's approach zero.

Two examples are worked out in detail in the following section, which should demonstrate the application of the two methods adequately.

THEORETICAL EXAMPLES

The δ function and uniform distributions are compared in the two following examples. The values of the partial differentials obtained from an actual problem would tend to cloud the application of the methods which can be presented much more clearly if normalized magnitudes are used. The first example considers equal ΔX_i's and the second a large difference between the smallest and largest. Both examples are for n equal to ten.

Consider a circuit represented by a black box with input and output terminals where interest is concentrated on the time it takes the output to reach a certain voltage level V_0 after a step function of voltage is applied at the input. It is presumed that the output voltage versus time is obtained by a numerical solution of the defining differential equations of the circuit. Note that both distributions require evaluating ΔX_i for each component.

$$\Delta X_i \equiv \left| \overline{X}_i - \underline{X}_i \right| = p_i = \left| \left. \frac{\partial f}{\partial x_i} \right|_{\overline{f}} (\overline{x}_i - \underline{x}_i) \right|$$

where

$$\left. \frac{\partial f}{\partial x_i} \right|_{\overline{f}} \text{ is approximated by } \frac{\overline{f} - f_{\underline{X}i}}{\overline{x}_i - \underline{x}_i}$$

Therefore $\Delta X_i = \left| \overline{f} - f_{\underline{X}_i} \right|$, where $f_{\underline{X}_i}$ is the magnitude of f with all components $x_j = \overline{x}_j$, $j \neq i$, and $x_j = \underline{x}_j$ for $j = i$.

The magnitude of ΔX_i is found from the output voltage solutions shown in Fig. 4. The solid curve is a plot of output voltage versus time for all components at their extreme value which maximizes the time between application of the input voltage, and output voltage reaching a value of V_0. The dotted curve is the same solution with the i^{th} component at its opposite extreme. The magnitude of ΔX_i can be read from the graph as it is the difference in the time the two curves cross the line, output voltage = V_0. Small differences should not be a problem as the scale can be expanded and the solution read out at shorter intervals about V_0.

The same worst case curve would be used for all ΔX_i. This implies one solution for each component plus the worst case solution. The mean value and the best case solution also would be required.

Example 1.

The following results are assumed for the magnitudes of the ΔX_i's as

$$\Delta X_1 = \Delta X_2 = \ldots = \Delta X_{10} = 10 \text{ microseconds}$$

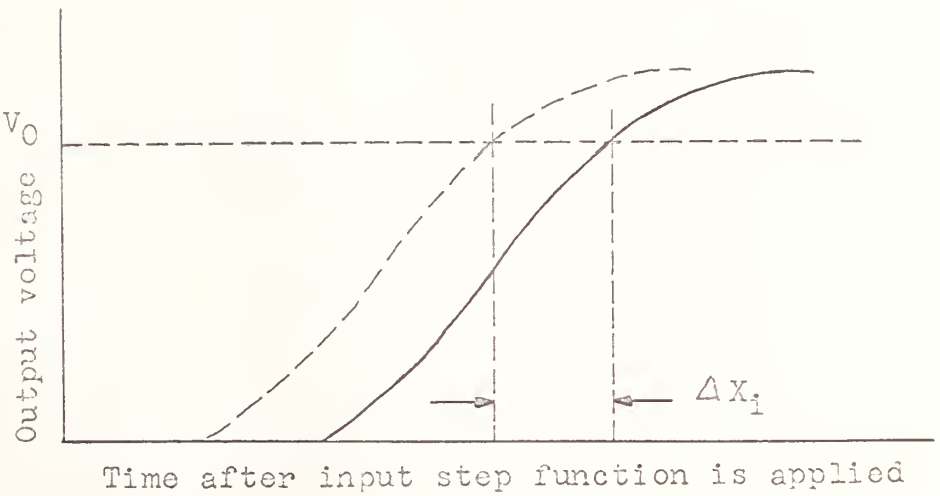


Fig. 4. Curves to obtain ΔX_i.

The same distribution will be obtained for any value of $\bar{f}$ so assume $\bar{f}$ = 100 microseconds.

δ Distribution. The first step is to calculate the K_i's. Since all p_i's are equal, the K_i's will all be zero. This means all solutions with (j - h) components at $\underline{x}_i$ will be greater than the smallest solution with j components at $\underline{x}_i$ for $0 < h \leq j$. It also implies all solutions with j components at $\underline{x}_i$ are equal for $0 \leq j \leq 10$.

There will be no solutions greater than 100 microseconds. $\binom{10}{0}$ solutions = 100 microseconds, $\binom{10}{1}$ solutions = 90 microseconds, $\binom{10}{2}$ solutions = 80 microseconds, $\binom{10}{3}$ solutions = 70 microseconds, $\binom{10}{4}$ solutions = 60 microseconds, and $\binom{10}{5}$ solutions = 50 microseconds.

Referring to Table 1, the following values are obtained.

$\binom{10}{0} = 1$, $\binom{10}{1} = 10$, $\binom{10}{2} = 45$, $\binom{10}{3} = 120$, $\binom{10}{4} = 210$,

$\binom{10}{5} = 252$.

The next step is to determine T, the number of solutions that are greater than each f_j, as $P\{f \geq f_j\} = T/2^n$ and n is equal to 10 for this example. The results are listed in Table 2 and the CDF plotted in Fig. 5.

Uniform Distribution. Because all ΔX_i's are equal, it is necessary to calculate only the value of f_N for $0 \leq N \leq n/2$. These values are: f_0 = 100 microseconds, f_1 = 90 microseconds,

Table 1. Value of $\binom{n}{x}$.

n \ x	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0	0	0
4	1	4	6	4	1	0	0	0	0	0	0
5	1	5	10	10	5	1	0	0	0	0	0
6	1	6	15	20	15	6	1	0	0	0	0
7	1	7	21	35	35	21	7	1	0	0	0
8	1	8	28	56	70	56	28	8	1	0	0
9	1	9	36	84	126	126	84	39	9	1	0
10	1	10	45	120	210	252	210	120	45	10	1

Table 2. Data for CDF plot.

f_j (μsec)	T	$P\{f > f_j\}$
100	0	0
90	1	.000977
80	11	.0107
70	56	.0547
60	176	.1719
50	386	.3770
40	638	.6230

$f_{1.5}$ = 85 microseconds, f_2 = 80 microseconds, $f_{2.5}$ = 75 microseconds, f_3 = 70 microseconds, $f_{3.5}$ = 65 microseconds, f_4 = 60 microseconds, $f_{4.5}$ = 55 microseconds, and f_5 = 50 microseconds. The CDF is found by the following calculations.

$$P\{f > f_0\} = 0$$

$$P\{f > f_1\} = \frac{1}{10!}\left\{(1)^{10}\right\} = .000000275$$

$$P\{f > f_{1.5}\} = \frac{1}{10!}\left\{(1.5)^{10} - (10)(.5)^{10}\right\} = .00000159$$

$$P\{f > f_2\} = \frac{1}{10!}\left\{(2)^{10} - (10)(1)^{10}\right\} = .000279$$

$$P\{f > f_{2.5}\} = \frac{1}{10!}\left\{(2.5)^{10} - (10)(1.5)^{10} + (45)(.5)^{10}\right\} = .00247$$

$$P\{f > f_3\} = \frac{1}{10!}\left\{(3)^{10} - (10)(2)^{10} + (45)(1)^{10}\right\} = .0128$$

$$P\{f > f_{3.5}\} = .0490$$

$$P\{f > f_4\} = .139$$

$$P\{f > f_{4.5}\} = .295$$

$$P\{f > f_5\} = .500$$

The two graphs in Figs. 5 and 6 compare $P\ f > f_j$ for the two assumed distributions. Figure 5 is for $50 \leq f_j \leq 100$ and Fig. 6 is the region between 80 and 100 microseconds expanded to give a better comparison of the two. This is the CDF that will be fitted between $\bar{f}$ and the mean value.

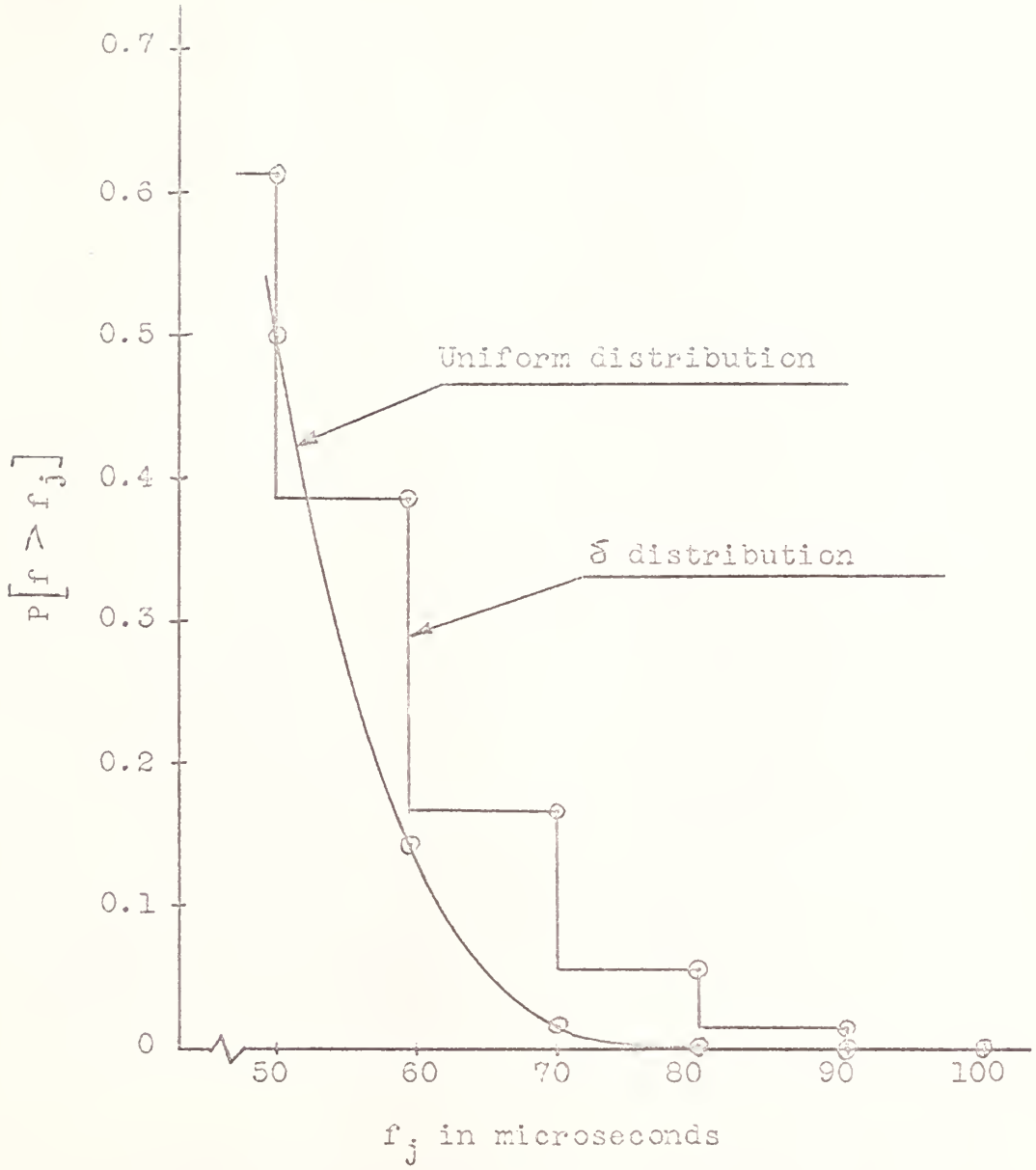


Fig. 5. Comparison of the δ and uniform distributions for equal ΔX_i's with n = 10.

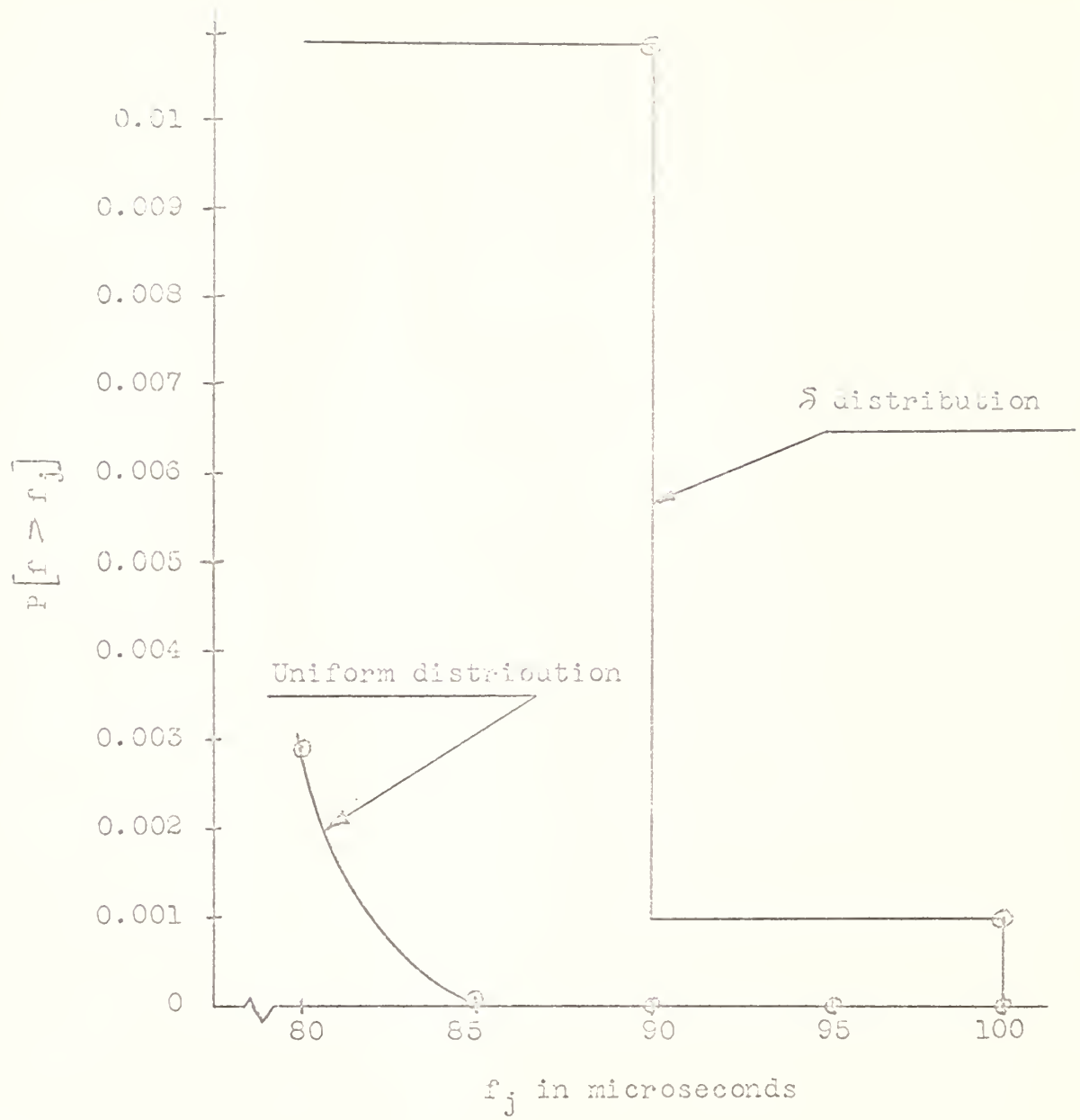


Fig. 6. Comparison of the δ and uniform distributions for equal ΔX_i's with n = 10.

Example 2.

The following results are assumed for the magnitudes of the ΔX_i's. $\Delta X_i = i$ microseconds. ($i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.$) Also assume $\bar{f} = 100$ microseconds.

δ Distribution. The first step is the calculation of the K_i's.

$\Delta X_i = p_i$

$p_1 + p_2 = 3$ microseconds; therefore $K_1 = 8$

$p_1 + p_2 + p_3 = 6$ microseconds; therefore $K_2 = 5$

$p_1 + p_2 + p_3 + p_4 = 10$ microseconds; therefore $K_3 = 1$,

$K_i = 0 \quad (i > 3)$

$p_2 + p_3 = 5$ microseconds; therefore $K_2' = 6$

$p_2 + p_3 + p_4 = 9$ microseconds; therefore $K_3' = 2$,

$K_i' = 0 \quad (i > 3)$

$p_3 + p_4 = 7$ microseconds; therefore $K_3'' = 4$,

$K_i'' = 0 \quad (i > 3)$

$p_4 + p_5 = 9$ microseconds; therefore $K_4''' = 2$,

$K_i''' = 0 \quad (i > 4)$

All other K_i's are zero.

$j = 1$. $\Delta f_1 = 1$ microsecond, $\bar{f} - \Delta f_1 = 99$ microseconds

There is one solution for $(j - 1)$. This is $\bar{f}$ and is greater than $\bar{f} - \Delta f_1$. Therefore $T = 1$ at this point.

$j = 2$. $\Delta f_2 = 3$ microseconds, $\bar{f} - \Delta f_2 = 97$ microseconds

There are $\binom{10}{1}$ solutions for f with $(j - 1)$ of the X_i's = $\underline{X}_i$ and $\binom{10}{0}$ solutions for f with $(j - 2)$ of the X_i's = X_i. The $(j - 2)$ solution is $\bar{f}$ and is greater than 97 microseconds. The

(j - 1) solutions greater than 97 microseconds are found by using the general equation with h = 1 as follows.

$$\binom{2}{0} \cdot \binom{8}{1} \cdot \binom{10-8-1-1}{2-1-1} = 1 \cdot 8 \cdot 1 = 8$$

All other terms are zero. Therefore 10 - 8 = 2 of the (j - 1) solutions are greater than 97 microseconds and T = 3 at this point.

j = 3. Δf_3 = 6 microseconds, $\bar{f} - \Delta f_3$ = 94 microseconds.

There are $\binom{10}{2}$ = 45 solutions for f with (j - 1) of the X_i's = $\underline{X}_i$, $\binom{10}{1}$ = 10 solutions for f with (j - 2) of the X_i's = $\underline{X}_i$, and $\bar{f}$ for (j - 3). The (j - 1) solutions greater than 94 microseconds are found by using the general equation with h - 1 as follows.

$$\binom{2}{0} \cdot \binom{8}{1} \cdot \binom{10-8-1-1}{3-1-1} + \binom{2}{0} \cdot \binom{8}{2} \cdot \binom{10-8-1-1}{3-1-2} = 0 + 28$$

$$+ \binom{2}{1} \cdot \binom{6}{1} \cdot \binom{10-6-1-1}{3-1-2} = 2 \cdot 6 = 12$$

All other terms are zero. Therefore 45 - 40 = 5 of the (j - 1) solutions are greater than 94 microseconds.

The (j - 2) solutions greater than 94 microseconds are found with h = 2 as follows.

$$\binom{3}{0} \cdot \binom{5}{1} \cdot \binom{10-5-2-1}{3-2-1} = 5$$

All other terms are zero. Therefore 10 - 5 = 5 of the (j - 2) solutions are greater than 94 microseconds, and T = 5 + 5 + 1

= 11 for this point.

j = 4. Δf_4 = 10 microseconds, $\bar{f} - \Delta f_4$ = 90 microseconds.

There are $\binom{10}{3}$ = 120 solutions for f with (j - 1) of the X_i's = $\underline{X}_i$, $\binom{10}{2}$ = 45 solutions for f with (j - 2) of the X_i's = $\underline{X}_i$, $\binom{10}{1}$ = 10 solutions for f with (j - 3) of the X_i's = $\underline{X}_i$, and $\bar{f}$ for (j - 4). The (j - 1) solutions greater than 90 microseconds are found with h = 1 as follows.

$$\binom{2}{0} \cdot \binom{8}{3} \cdot \binom{10-8-1-1}{4-1-3} = 56 + \binom{2}{1} \cdot \binom{6}{1} \cdot \binom{10-6-1-1}{4-1-2}$$

$$+ \binom{2}{1} \cdot \binom{6}{2} \cdot \binom{10-6-1-1}{4-1-3} = 2 \cdot 6 \cdot 2 + 2 \cdot 15 = 54$$

$$+ \binom{2}{2} \cdot \binom{4}{1} \cdot \binom{10-4-1-1}{4-1-3} = 4$$

Therefore 120 - 56 - 54 - 4 = 6 of the (j - 1) solutions are greater than 90 microseconds.

The (j - 2) solutions greater than 90 microseconds are found with h = 2 as follows.

$$\binom{3}{0} \cdot \binom{5}{1} \cdot \binom{10-5-2-1}{4-2-1} + \binom{3}{0} \cdot \binom{5}{2} \cdot \binom{10-5-2-1}{4-2-2}$$

$$= 5 \cdot 2 + 10 \cdot 1 = 20$$

$$+ \binom{3}{1} \cdot \binom{2}{1} \cdot \binom{10-2-2-1}{4-2-2} = 3 \cdot 2 = 6$$

Therefore 45 - 20 - 6 = 19 of the (j - 2) solutions are greater than 90 microseconds.

The (j - 3) solutions greater than 90 microseconds are

found with h = 3 as follows.

$$\binom{4}{0} \cdot \binom{1}{1} \cdot \binom{10-1-1-1}{4-3-1} = 1$$

Therefore 10 - 1 = 9 of the (j - 3) solutions are greater than 90 microseconds and T = 6 + 19 + 9 + 1 = 35 for this point.

In this example the general equation was applied through j = 7, illustrating the method adequately. Results for all j's are listed in Table 3.

Table 3. General equation results.

j	Δf_j	$\bar{f} - \Delta f_j$	General equation solution				Total number of Δf_i's ($i \leq j$)	T	$T/2^{10}$
			h=1	h=2	h=3	h=4			
1	1	99	0	0	0	0	1	1	.000977
2	3	97	8	0	0	0	11	3	.00195
3	6	94	40	5	0	0	56	11	.0107
4	10	90	114	26	1	0	176	35	.0352
5	15	85	204	68	5	0	386	109	.106
6	21	79	248	110	10	0	638	270	.264
7	28	72	208	111	10	0	848	519	.506

This example is a strict test for the algorithm upon which the general equation is based because of the number of solutions that fall on the same point, and the uniform difference in the p_i's. The general equation results then may be compared to the results obtained when the 2^{10} solutions are calculated with the approximate function.

Table 4. Comparison of general equation results and exact results obtained from approximate function solution.

f_j	$p\{f > f_j\}$ General equation	$p\{f > f_j\}$ Exact results
99	.000977	.000977
97	.00195	.00195
94	.0107	.00977
90	.0352	.0322
85	.106	.0967
79	.264	.246
72	.506	.500

Uniform Distribution. The first step to obtain the CDF is to determine the values for f_j and the number y_i which determines the sign of each term. Note that for y_i even, the sign is positive, and for y_i odd, the sign is negative. Also when $f_i = f_j$, if y_i and y_j are not both even or both odd, the terms cancel each other. Table 5 lists the results necessary to find the CDF. Using the information in this table the CDF is determined as follows.

$$\text{Let } C = 10! \prod_{i=1}^{10} \Delta X_i = 13,168,189,440,000$$

$$P\{f > 96\} = \frac{1}{C}\left\{(4)^{10} - (3)^{10} - (2)^{10}\right\} = .000000075$$

$$P\{f > 92\} = \frac{1}{C}\left\{(8)^{10} - (7)^{10} - (6)^{10} + (3)^{10} + (1)^{10}\right\} = .0000554$$

Table 5. Exact uniform distribution results.

f_k	Value of y_i = number of X_i's							Total: odd	Total: even	Effective total and sign	
	$0X_i$	$1X_i$	$2X_i$	$3X_i$	$4X_i$	$5X_i$	$6X_i$	$7X_i$				
100	1								0	1	1	+
99		1							1	0	1	-
98		1							1	0	1	-
97		1	1						1	1	0	
96		1	1						1	1	0	
95		1	2						1	2	1	+
94		1	2	1					2	2	0	
93		1	3	1					2	3	1	+
92		1	3	2					3	3	0	
91		1	4	3					4	4	0	
90		1	4	4	1				5	5	0	
89			5	5	1				5	6	1	+
88			4	7	2				7	6	1	-
87			4	8	3				8	7	1	-
86			3	9	5				9	8	1	-
85			3	10	6	1			11	9	2	-
84			2	10	9	1			11	11	0	
83			2	10	10	2			12	12	0	
82			1	10	13	3			13	14	1	+
81			1	9	14	5			14	15	1	+
80				8	16	7			15	16	1	+
79				7	16	9	1		16	17	1	+
78				5	18	11	1		16	19	3	+
77				4	16	14	2		18	18	0	
76				3	16	16	3		19	19	0	
75				2	14	18	5		20	19	1	-
74				1	13	19	6		20	19	1	-
73				1	10	20	9		21	19	2	-
72					9	20	10	1	21	19	2	-

$$P\{f > 88\} = \frac{1}{C}\left\{(12)^{10} - (11)^{10} - (10)^{10} + (7)^{10} + (5)^{10} + (1)^{10}\right\} = .001995$$

$$P\{f > 84\} = \frac{1}{C}\left\{(16)^{10} - (15)^{10} - (14)^{10} + (11)^{10} + (9)^{10} + (5)^{10} - (4)^{10} - (3)^{10} - (2)^{10} - 2(1)^{10}\right\} = .0200$$

$$P\{f > 80\} = .0953$$

$$P\{f > 78\} = .1704$$

$$P\{f > 76\} = .2732$$

$$P\{f > 73\} = .4658$$

$$P\{f > 72\} = .5343$$

Figures 7 and 8 compare $P\{f > f_j\}$ for the two distributions. The CDF for the δ distribution was obtained from the data in Table 5 also. Figure 7 is for $72 \leq f_j \leq 92$ and Fig. 8 for $90 \leq f_j \leq 100$ with the scale expanded to give a better comparison.

While it is obvious that the uniform distribution will give the designer a better probable worst case than the δ distribution, the reader should bear in mind that an optimistic distribution of components will only lead to grief. The importance of using an exact or conservative distribution for each component must be understood.

The two preceding examples were worked manually. Even for ten components, the calculations became lengthy. The next step will be to develop a method by which the uniform distribution can be found easily using a digital computer. It should be noted, however, that the equations do give reliable results and

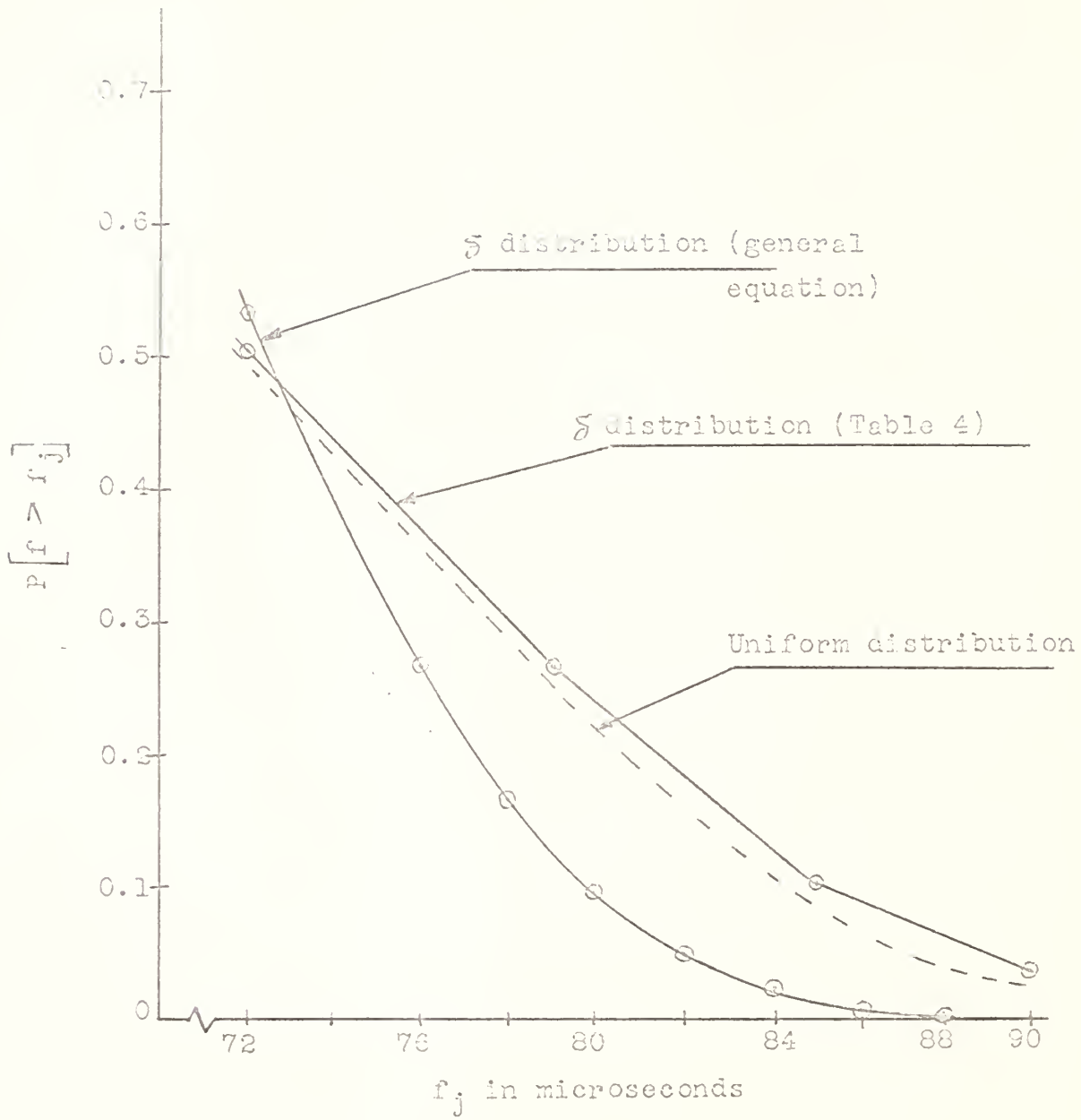


Fig. 7. Comparison of the δ and uniform distributions for unequal ΔX_i's with n = 10.

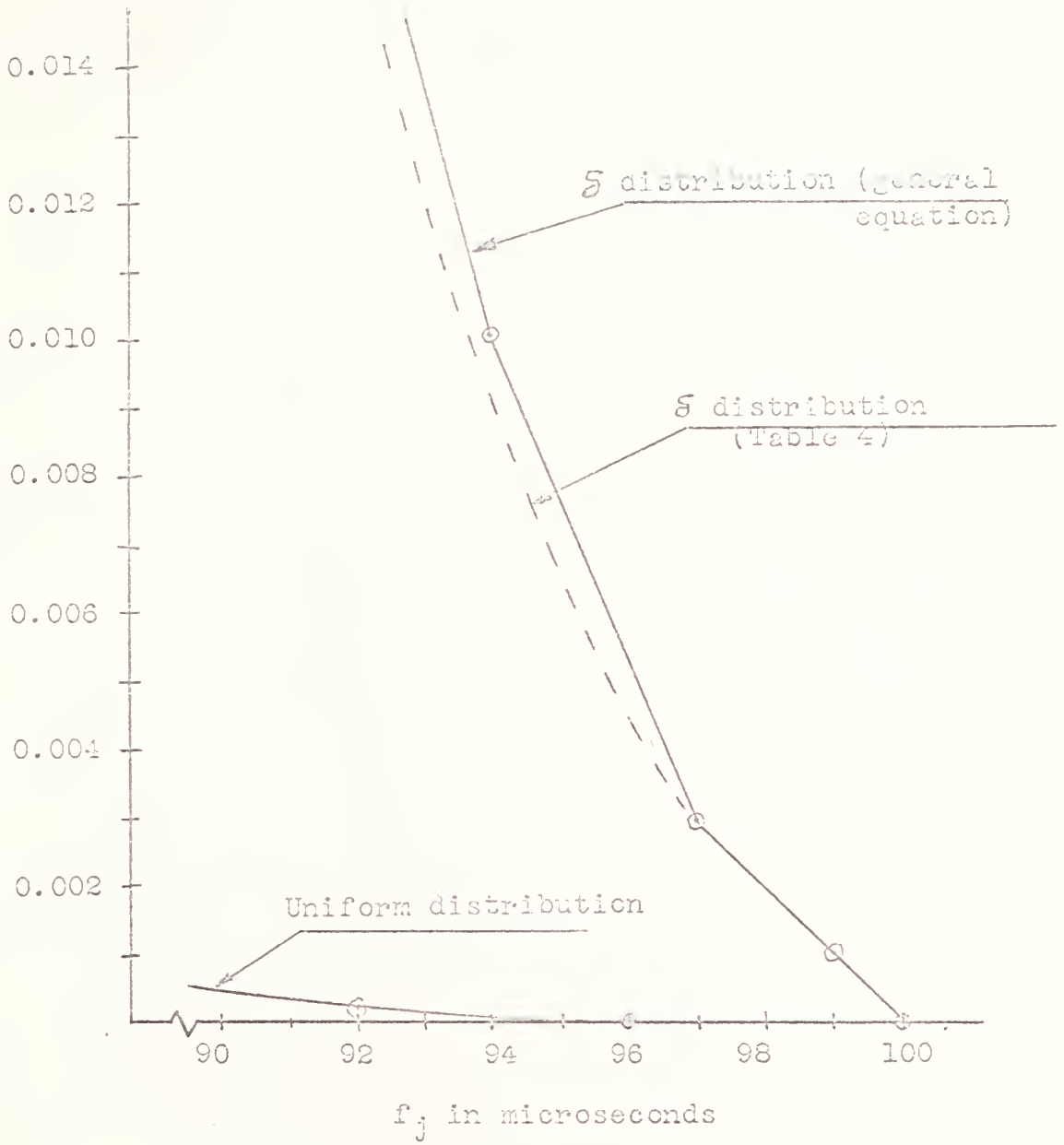


Fig. 8. Comparison of the δ and uniform distributions for unequal ΔX_i's with n = 10.

calculations can be carried out by hand when a computer is not available, and the number of variables in the function is not excessive.

COMPUTER METHODS

Uniform Distributions

Given a function $f = X_1 + X_2 + \ldots + X_n$ where each X_i is uniformly distributed between $\underline{X}_i$ and $\overline{X}_i$, it is quite easy to obtain the PDF of f with a digital computer.

Consider that the following PDF for $f' = X_1 + X_2 + \ldots + X_{(k-1)}$ has been obtained by the method presented below. The PDF for $f = X_1 + X_2 + \ldots + X_k$ can be obtained by the following procedure.

The PDF of f is the area under the point-by-point product of the two PDF's shown in Fig. 9 as f is varied from $\overline{f}$ to $\underline{f}$. The value of the PDF of f' is known for each point g_i' and the area under each curve is equal to one.

First multiply the value of each point g_i' of the PDF of f' by $h = \left| \frac{1}{\overline{X}_k - \underline{X}_k} \right|$ to obtain a new value at each point $g_i'' = g_i' \cdot h$.

The points at which the value of the PDF of f' is known are $\triangle F$ apart. Consider $\frac{|\overline{X}_k - \underline{X}_k|}{\triangle F} = m$ and let m be rounded to the closest integer. $\triangle F$ should be small enough to give the

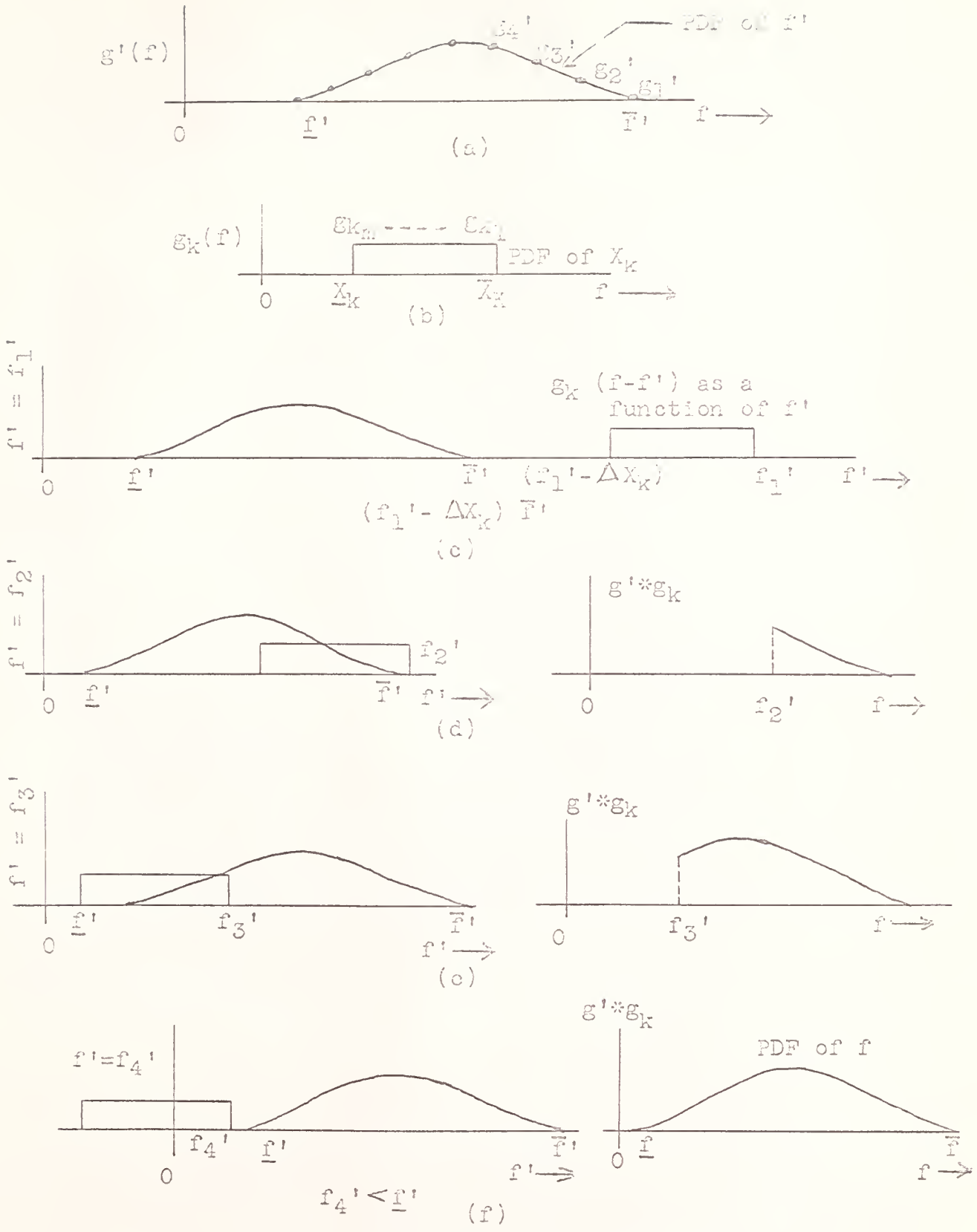


Fig. 9. Convolution of PDF of X_k upon PDF of f'.

desired accuracy. The combination should be started with the two smallest ΔX_i's and then f' always combined with the smallest ΔX_i remaining. Each time m becomes large enough to cause excessive calculations, ΔF will be multiplied by an integer, $I \cdot \Delta F$, and only the points g_1', $g_{(I+1)}'$, $g_{(2I+1)}'$, ..., $g_{(il+1)}'$, ..., are needed until $I \cdot \Delta F$ is once again increased.

The first point obtained for the PDF of f will be for $\bar{f}$ and that value will be $g_1 = 0$. As long as $i \leq m$, the values for g_i are as follows.

$$g_2 = \frac{g_2''}{2} \cdot \Delta F$$

$$g_3 = g_2 + \frac{g_3'' + g_2''}{2} \cdot \Delta F$$

$$g_i = g_{(i-1)} + \frac{g_i'' + g_{(i-1)}''}{2} \cdot \Delta F$$

The area under the product curve out to the point $g_{(i-1)}''$ is $g_{(i-1)}$ and $\frac{g_i'' + g_{(i-1)}''}{2} \cdot \Delta F$ is the trapezoidal area under the new product curve obtained as f is decreased by the value ΔF to shift the PDF of X_k ΔF units to the left. The magnitude of this new trapezoidal area should be saved each time as it will be used later.

When $i > m$ the trapezoidal area $\frac{g_{(i-m)}'' + g_{(i-m-1)}''}{2} \cdot \Delta F$ must be subtracted from the preceding equation as follows.

$$g_i = g_{(i-1)} + \frac{g_i'' + g_{(i-1)}''}{2} \cdot \Delta F$$

$$- \frac{g_{(i-m)}'' + g_{(i-m-1)}''}{2} \cdot \Delta F \qquad (i > m)$$

Note that this holds for all i when $g_{(i-m)}'' = 0$ for $m \geq i$. Each trapezoidal area subtracted has been calculated and saved.

This method, when carried out for all X_i, will give a PDF for f between $\bar{f}$ and $\underline{f}$. The area under the PDF curve between $\bar{f}$ and f_j is equal to $P\{f > f_j\}$.

Uniform and δ Distributions

It may be desirable to use the δ distribution for one or more components if the uniform distribution is not considered conservative for that component.

If g' is the PDF for $f' = X_1 + X_2 + \ldots + X_{(k-1)}$ and the k^{th} variable has a PDF $g_k = \frac{\delta(X_k - \underline{X}_k) + \delta(X_k - \bar{X}_k)}{2}$, then the PDF of k variables will be as follows.

$$g = \int_{-\infty}^{+\infty} g'(f') g_k(f - f') df'$$

$$= \frac{1}{2}\left\{\int_{-\infty}^{\infty} g'(f')\,\delta(f - f' - \underline{X}_k)df' + \int_{-\infty}^{\infty} g'(f')\,\delta(f - f' - \bar{X}_k)df'\right\}$$

(See Appendix I.)

But

$$\int_{-\infty}^{\infty} g'(f')\,\delta(c - f')df' = g'(c) \int_{c-\epsilon}^{c+\epsilon} \delta(c - f')df'$$

where ϵ is a very small number and

$\int_{c-\epsilon}^{c+\epsilon} \delta(c - f')df' = 1$ (Goldman, 4, p. 101), $c = f - \underline{X}_k$ or $f - \overline{X}_k$

Therefore $g = \frac{1}{2}\left\{g'(f - \underline{X}_k) + g'(f - \overline{X}_k)\right\}$

$\underline{f} = \underline{f}' + \underline{X}_k$, $\overline{f} = \overline{f}' + \underline{X}_k$

The first term is g'(f') translated to the right a distance equal to $\underline{X}_k$ and the second term is g'(f') translated to the right $\overline{X}_k$. The f' and f axes are considered to be common. Both terms are divided by two. Figure 10 is a pictorial representation of how g is obtained from g' when the value of $\underline{X}_k$ and $\overline{X}_k$ are known.

The combination with the computer is as follows:

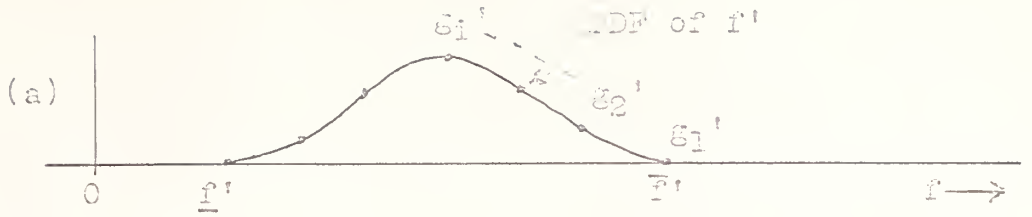
$\left|\overline{X}_{zk} - \underline{X}_k\right| = m\Delta F$

$g_i'' = g_i'/2$

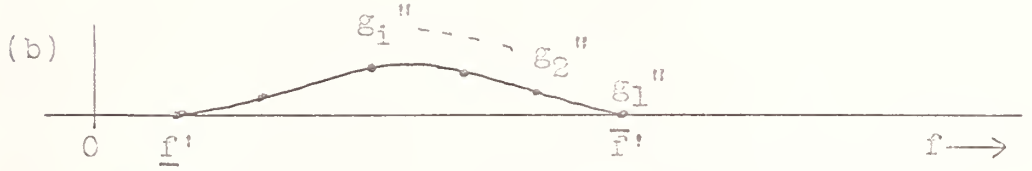
$g_i = g_i'' + g_{(i-m)}''$ where $g_x'' = 0$ when $\left[\frac{\overline{f}' - \underline{f}'}{\Delta F} + 1\right] < X \leq 0$

General Distribution

Any general distribution can be used for a component. However, more computations are required to form the product curve by a point-by-point multiplication of the two curves as $g(X_k)$ is shifted through g'. The procedure is outlined as follows.

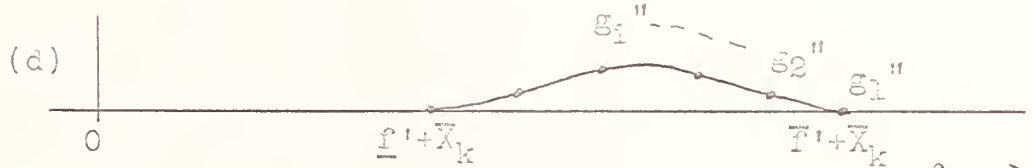


Divide value of g' by two at each point in (a) to obtain (b).



Translate (b) to the right by the value of $\underline{X}_k$ to obtain (c).

Translate (b) to the right by the value of $\overline{X}_k$ to obtain (d).



Add (c) and (d) to obtain (e).

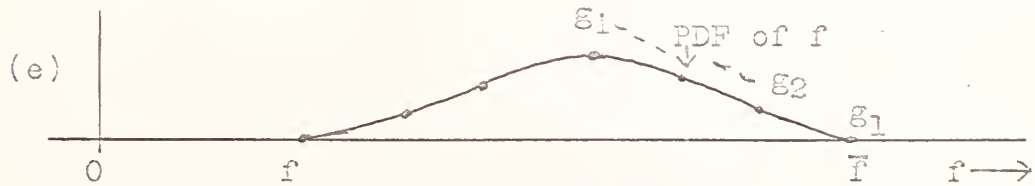


Fig. 10. Step-by-step formation of PDF of f.

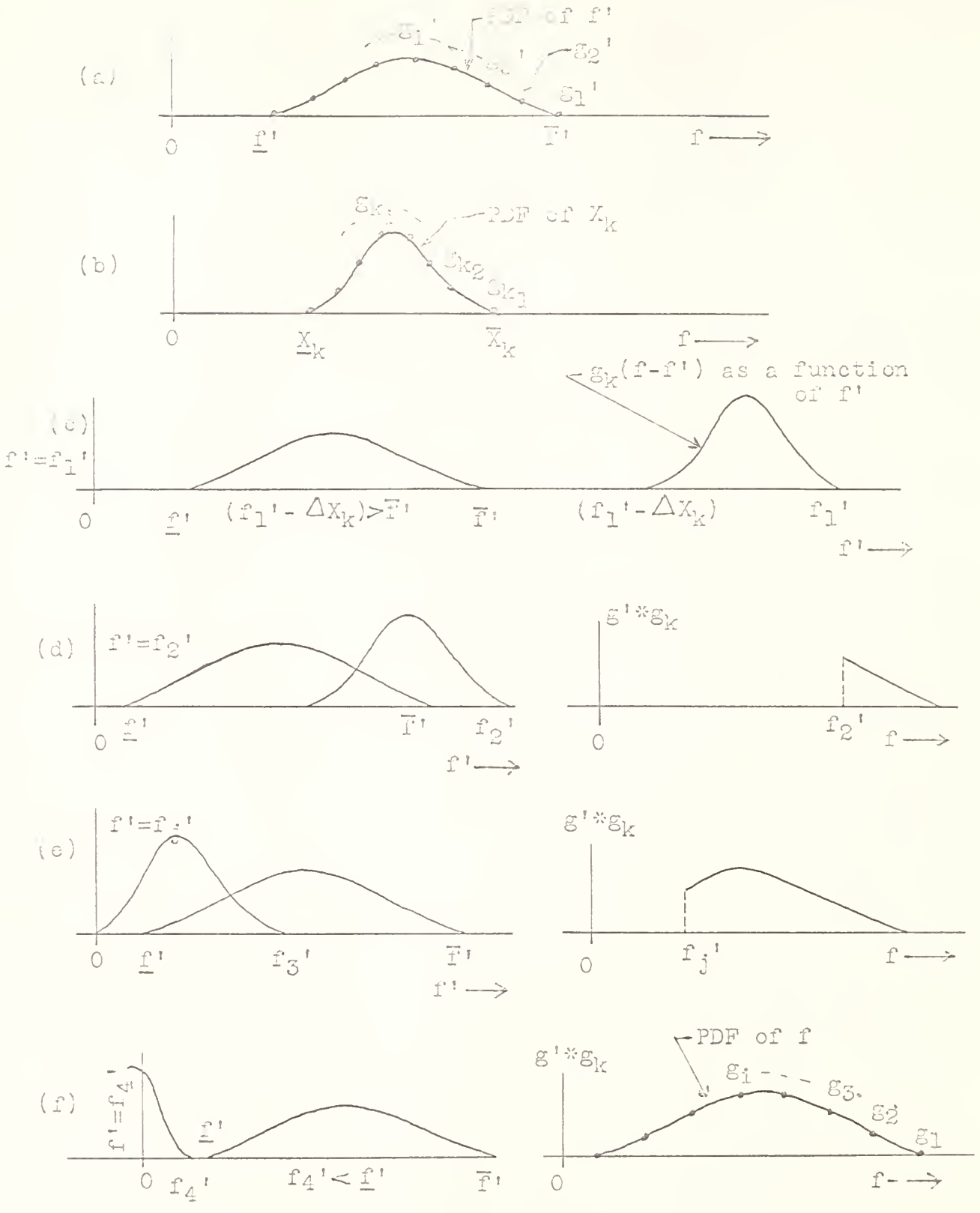


Fig. 11. Convolution of PDF of generally distributed X_k upon PDF of f'.

$$g_1 = 0$$

$$g_2 = \frac{g_2' \cdot g(X_k)_1 + g_1' \cdot g(X_k)_2}{2} \cdot \Delta F$$

$$g_3 = \frac{g_3' \cdot g(X_k)_1 + 2g_2' \cdot g(X_k)_2 + g_1' \cdot g(X_k)_3}{2} \cdot \Delta F$$

$$g_i = \frac{\begin{array}{l} g_i' \cdot g(X_k)_1 + 2\, g(i-1)' \cdot g(X_k)_2 + \cdots \\ + 2\, g_2' \cdot g(X_k)_{(i-1)} + g_1' - g(X_k)_i \end{array}}{2} \cdot \Delta F$$

When $g' = g(X_k) = 0$ at $\underline{f}'$, $\overline{f}'$, $\underline{X}_k$, and $\overline{X}_k$, the first and last term will always be zero and the equation can be simplified as follows.

$$g_i = \Big[g(i-1)' \cdot g(X_k)_2 + g(i-2)' \cdot g(X_k)_3 + \cdots + g_3' \cdot g(X_k)_{(i-2)} + g_2' \cdot g(X_k)_{(i-1)} \Big] \cdot \Delta F$$

$$g_i' = 0 \qquad 1 \geq i \geq \left[\frac{\overline{f}' - \underline{f}'}{\Delta F} + 1 \right]$$

$$g(X_k)_i = 0 \qquad 1 \geq i \geq \left[\frac{|\overline{X}_k - \underline{X}_k|}{\Delta F} + 1 \right]$$

This equation can be solved on a digital computer and is correct as written. The equation must be modified slightly if either g' or $g(X_k)$ is not zero at its extremes.

EVALUATION OF RESULTS

Error Considerations

The exact values of errors due to assumptions and

approximations cannot be determined. If they could, there would be no need to assume or approximate and the exact results could be obtained. It is therefore necessary to point out the causes of errors.

An error can be either conservative or optimistic, depending on the circuit function approximated. Since the methods are applied to a function in general, the emphasis will be on determining what causes a conservative or optimistic error.

The Resultant Cumulative Density Function

The final result of the methods presented in a CDF between $\overline{f}$ and $\underline{f}$. When the higher order terms, neglected in the approximating function, are negligible over the range of interest, the CDF is obtained to a high degree of accuracy. This, of course, assumes that exact component distributions were used. If the higher order terms are not negligible, this CDF cannot be an exact representation of the true CDF. When this is the case, as it should be for most functions, the CDF obtained is corrected to better represent the true CDF. The implication is not that the CDF will be exact after the correction, but that it will be more accurate.

Obviously the first problem is to determine when the higher order terms are not negligible. This is indicated by $\overline{f}$ and $\underline{f}$ of the approximate solution being different than the values obtained from the original function. The value $\overline{f} = f(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n)$ with $\overline{X}_i = \frac{1}{2}\left[\overline{X}_i + \underline{X}_i\right]$.

The correction shifts $\bar{\underline{f}}$ and $\underline{f}$ on the CDF obtained from the approximate function to the value of $\bar{\underline{f}}$ and $\underline{f}$ obtained from the original function. The remaining points on the approximate function CDF between $\bar{f}$ and $\bar{\underline{f}}$ are then shifted a corresponding distance with the point at $\bar{f}$ remaining fixed. Note that the approximate $\bar{\underline{f}}$ will be the point where $P\{f > f_j\} = 1/2$.

A graphic representation of the shifting will help clarify the above statements. Let the approximate function be f* and the original function be f. Plot the points ($\underline{f}$*, $\underline{f}$), ($\bar{\underline{f}}$*, $\bar{\underline{f}}$), and ($\bar{f}$*, $\bar{f}$). Draw a curve with a constant change in slope passing through these three points. The value of f_j corresponding to f_j* can be found as shown in Fig. 12. Since the rate of change of slope is constant, the curve is an arc of the circle passing through the three points.

Refer to Fig. 2 for the following discussion. The family of surfaces $f(x_1, x_2, x_3) = f_k$ is now approximated by a family of planes. The surface passing through the point ($\bar{\underline{x}}_1$, $\bar{\underline{x}}_2$, $\bar{\underline{x}}_3$) is represented by a plane passing through that point because $P\{f > \bar{\underline{f}}\}$ is now one-half. A plane passing through this point must cut the volume into two equal parts.

If the main diagonal between ($\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$) and ($\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$) is drawn, the surfaces $f(x_1, x_2, x_3) = \bar{f} - \dfrac{n(\bar{f} - \underline{f})}{N}$, (n = 1, 2, ..., N) are now approximated by planes. There is a direct relationship between the constant rate of change of slope in Fig. 12 and the rate of change of lengths on this diagonal cut off by these planes. If the slope is constant, the lengths are

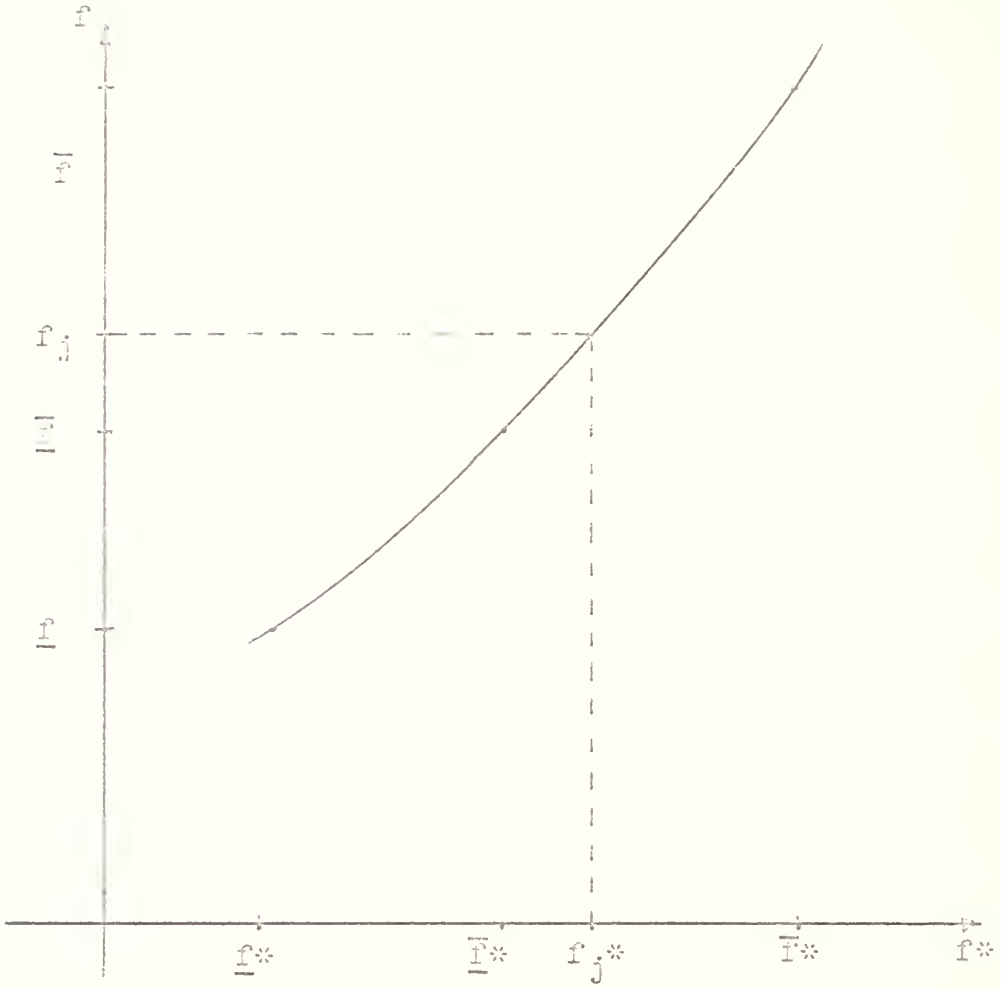


Fig. 12. Fitting the CDF.

all the same. If the slope is increasing as in Fig. 12, the shortest length is next to the point ($\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$) and the longest at the point ($\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$) because there are more planes between $\bar{f}$ and $\underline{\bar{f}}$ than $\underline{\bar{f}}$ and $\underline{f}$. A decreasing slope is opposite.

As stated previously, we now have a better approximation. The question now becomes, "When will the approximation be within of the exact value?", where is an arbitrarily small position quantity. Consider a function with surfaces $f(x_1, x_2, x_3) = \bar{f} - \frac{n(\bar{f} - \underline{f})}{N}$, (n = 1, 2, ..., N), passing through the exact model. If these surfaces are good approximations to parallel planes within the model and the distance between surfaces along the main diagonal is either constant or increasing (decreasing) at a constant rate, then the CDF should be nearly exact after fitting. When the surfaces do not behave this way throughout the entire model, the region of high accuracy is reduced. We are interested in the region near $\bar{f}$ and hopefully the surfaces will be well-behaved in this region.

The two causes of error are then nonplanor surfaces and nonuniformly changing distances between the surfaces. Consider the nonplanor surfaces first. If the volume of the exact model cut off by the surface is less than the volume cut off by the plane passing through the same point on the main diagonal, the approximate CDF is conservative. If the volume cut off is greater, the CDF is optimistic.

Now consider the surfaces as planes but failing in the uniform increase in distance condition. The approximate CDF

will be exact at $\bar{f}$, $\underline{\bar{f}}$, and $\underline{f}$ when the true surfaces are planes. If the true surface is farther from $\bar{f}$ than the approximate surface between $\bar{f}$ and $\underline{\bar{f}}$, then the CDF is optimistic over this region. If the true surfaces are closer, the CDF is conservative.

The method of fitting and evaluating the fit of the CDF has been discussed. It is left to the designer to inspect the function at the time of application for analysis of specific results.

Errors Due to Assumptions and Approximations

The first and most definite error is due to the distribution of components assumed. Two values of a component which will be the two extremes after a fixed time are utilized. These are the end-of-life values used. Since no distribution is given, the most practical distribution to assume and use is a uniform distribution between the two extremes. If a distribution of a component is known to be concentrated nearer one extreme, the method for general distributions should be used, providing that extreme causes $\bar{f}$. The uniform distribution will be conservative for all others.

The next error to be considered is the approximation of the various partial derivatives. The function in question must be strictly monotonic between $\bar{f}$ and $\underline{f}$ before a worst case value can be considered. This means the sign of a partial derivative does not change over the region of interest. The mean-value theorem states the function will have a partial derivative of this magnitude at some point between $\bar{f}$ and the value of f obtained

by changing the one component (Brand, 1, p. 113). This error can be minimized by decreasing the change in the component. It should be pointed out that this partial derivative is generally a monotonic function and the error is conservative when the partial derivative has its maximum magnitude at $\bar{f}$. The conservative error is in reference to the distribution obtained with the true values of the partial derivatives at $\bar{f}$. The small partial derivatives obtained by the approximation shift the distribution of f towards $\bar{f}$.

SUMMARY

The methods presented all require the designer to choose a distribution for each component. If the actual distribution is known it should be used. The problem still remains as to how to utilize the results.

The final result of any of the methods is a plot of probability of failure versus output values for a circuit or system. The designer then has to decide what probability of failure he wants to accept to gain a better worst case value. It is felt that the magnitude of this probability of failure should be left to the designer. Many parameters, such as catastrophic failure rate, enter into the decision. Therefore it seems best to make this decision at the time of application of the method when these parameters will be better defined.

It should be noted that n identical circuits with probability of failure P, cascaded in such a way that the output

functions add independently, will have a combined probability of failure $<<<P$. An example is five circuits with uniformly distributed outputs, a probable worst case transition time of 100 microseconds, and $P\{T > 100\} = 1/10$. If these five circuits are cascaded, the probable worst case transition time is 500 microseconds. The probability of failure $P\{T > 500\}$ would be .00026. Suppose $\overline{T}$ is 110 microseconds for each circuit. Fifty microseconds are gained from the true worst case cascaded transition time with a probability of failure of 1/3850 before end of life. This gain should be realized!

All methods are much easier to apply with the aid of a digital computer than by hand when the circuit contains many components. The "hand" procedure and equations have been presented for interested readers who either do not have access to a computer or who wish to obtain a probable worst case of a circuit with a moderate number of components.

The methods can be applied to any output function that can be approximated by a Taylor series. For this reason the methods apply equally as well to direct-current and alternating-current functions although the partial derivatives probably are easier to evaluate for the steady-state than the transient function.

APPENDICES

APPENDIX I

COMBINATION OF GENERAL DISTRIBUTIONS

The material in this Appendix is presented for the reader with little or no background in basic probability theory. The development is not intended to be completely general with the specific application of the end result as the motive.

Functions of Two Independent Variables

Let $f(x_1, x_2)$ be a strictly monotonic function of the two independent variables X_1 and X_2 with F_k a constant value of $f(X_1, X_2)$. We are interested in the probability that $f = F_k$, $P\{f = F_k\}$. Now, if $F_k = f(X_{1_k}, X_{2_k})$, then $P\{f = F_k\} = P\{X_1 = X_{1_k}\} \cdot P\{X_2 = X_{2_k}\}$ when there is only one unique combination of X_1 and X_2 to give F_k. This statement is based on the assumption that if two events, A and B, are statistically independent, the probability of both events A and B occurring simultaneously is $P\{AB\} = P\{A\} \cdot P\{B\}$. This is consistent since one must have X_{1_k} and X_{2_k} to have F_k.

When there is more than one unique combination of X_1 and X_2 which gives F_k, the preceding must be changed. The following is based on the probability theorem $P\{A_1 \cdot B_1$ or $A_2 \cdot B_2$ or . . . or $A_i \cdot B_i$ . . .$\} = P\{A_1\} \cdot P\{B_1\} + P\{A_2\} \cdot P\{B_2\} + \ldots + P\{A_i\} \cdot P\{B_i\} + \ldots$. Therefore

$$P\{f = F_k\} = \sum \left[P\{X_1 = X_{1_k}\} \cdot P\{X_2 = X_{2_k}\} \right]$$

where the summation indicates X_1 and X_2 ranging over all combinations of values so that $f(X_{1_k}, X_{2_k}) = F_k$.

The following definitions will be used:

g_1 = PDF of X_1

g_2 = PDF of X_2

g = PDF of $f(X_1, X_2)$

Consider Δf, ΔX_1, and ΔX_2 to be small increments about F_k, X_{1_k}, and X_{2_k}.

$$P\{f = F_k\} \doteq g\,\Delta f \doteq \sum \left[g_{1_k}\,\Delta X_1\; g_{2_k}\,\Delta X_2 \right]$$

$$g \doteq \sum \left[g_{1_k}\, g_{2_k}\, \frac{\Delta X_2}{\Delta f}\, \Delta X_1 \right] \qquad (1)$$

This expression for g must be interpreted as follows. The summation is over all possible combinations of X_1 and X_2 that give F_k. g_{1_k} and g_{2_k} are the values of g_1 and g_2 at the various values of X_{1_k} and X_{2_k} that give F_k.

The PDF's g_1 and g_2 are respective functions of X_1 and X_2 and define the distribution of these two variables for $-\infty < X_1, X_2 < +\infty$. Suppose we solve $f(X_1, X_2)$ for X_2 and $X_2 = \phi(f, X_1)$. This can be done as $f(X_1, X_2)$ is strictly monotonic by hypothesis, and therefore $F(X_1, X_2) = f(X_1, X_2) - F_k$ satisfies the Existence theorem (Brand, 1, p. 165). If X_2 is replaced by ϕ in Eq. (1), then any value for X_1 assigns a unique value to g_1 and g_2 for a given F_k. The summation can be replaced by an integration with limits as shown and in the limit as ΔX_1, ΔX_2, and Δf tend to zero, the approximation becomes an equality.

$$g = \int_{-\infty}^{\infty} g_1(X_1)\, g_2\left[\phi(f, X_1)\right] \left|\frac{\partial \phi}{\partial f}\right| dX_1 \qquad (2)$$

Restrictions must be placed on $\frac{\partial \phi}{\partial f}$. A PDF is always positive; therefore $\left|\frac{\partial \phi}{\partial f}\right|$ must be used. $\frac{\partial \phi}{\partial f}$ must be defined over the range of X_1 and X_2. Since f is strictly monotonic, $\frac{\partial \phi}{\partial f} = \frac{1}{\partial f / \partial \phi}$ (Brand, 1, p. 105). Also $\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial X_2}$ does not change sign and is not zero over the range of X_1 and X_2.

Consider briefly the interpretation of Eq. (2). During integration, f is treated as a constant and the final result will be a function of f. For any value F_k, choosing a value of X_1 defines the corresponding value of X_2 to give F_k. The integration process selects all values of X_1 and X_2. The two functions g_1 and g_2 are zero except where X_1 and X_2 can exist. This expression, when interpreted exactly as written, gives the results we seek. It is of extreme importance that g_i be zero over the range where X_i cannot exist.

The application to the following three elementary functions is shown.

1. $f = X_1 + X_2 \qquad \phi = f - X_1 \qquad \left|\frac{\partial \phi}{\partial f}\right| = 1$

$$g = \int_{-\infty}^{\infty} g_1(X_1)\, g_2(f - X_1)\, dX_1$$

2. $f = X_1 X_2 \qquad \phi = \frac{f}{X_1} \qquad \left|\frac{\partial \phi}{\partial f}\right| = \left|\frac{1}{X_1}\right|$

$$g = \int_{-\infty}^{\infty} g_1(X_1)\, g_2\left(\frac{f}{X_1}\right) \frac{dX_1}{|X_1|} \qquad (X_1 \neq 0)$$

3. $f = \dfrac{X_1}{X_2} \qquad \phi = \dfrac{X_1}{f} \qquad \left|\dfrac{\partial \phi}{\partial f}\right| = \dfrac{|X_1|}{f^2}$

$$g = \int_{-\infty}^{\infty} g_1(X_1)\, g_2\left(\frac{X_1}{f}\right) \frac{|X_1|}{f^2}\, dX_1 \qquad (f \neq 0)$$

APPENDIX II

COMBINATION OF UNIFORM DISTRIBUTIONS

A function of n independent variables is considered in this Appendix. The function is of the form $f = X_1 + X_2 + \ldots + X_n$. Each variable is uniformly distributed between its two extremes $\overline{X}_i$ and $\underline{X}_i$. Notice that $\Delta X_i = \left|\Delta X_i\right|$ when $\overline{f} = f$ maximum as will be considered.

Integral Development of CDF

The following definitions will be used throughout this Appendix:

$f_i = X_1 + X_2 + \ldots + X_i$

g_i = PDF of X_i

$g(f_i)$ = PDF of f_i

$\Delta X_i = \overline{X}_i - \underline{X}_i$

Either of the following two equations is valid for g_i.

$$g_i = \frac{1}{\Delta X_i}\left[U(X_i - \underline{X}_i) - U(X_i - \overline{X}_i)\right]$$

$$= \frac{1}{\Delta X_i}\left[U(\overline{X}_i - X_i) - U(\underline{X}_i - X_i)\right]$$

$$g(f_2) = \int_{-\infty}^{\infty} g_1(X_1)\, g_2(f_2 - X_1)\, dX_1, \quad X_2 = \emptyset = f_2 - X_1, \quad \frac{\partial \emptyset}{\partial f^2} = 1$$

$$= \frac{1}{\prod_{i=1}^{2} \Delta X_i}\left[\int_{f_2-\overline{X}_2}^{\overline{X}_1} dX_1 \Bigg|_{f_2 < \overline{X}_1+\overline{X}_2} - \int_{f_2-\overline{X}_2}^{\underline{X}_1} dX_1 \Bigg|_{f_2 < \underline{X}_1+\overline{X}_2} - \int_{f_2-\underline{X}_2}^{\overline{X}_1} dX_1 \Bigg|_{f_2 < \overline{X}_1+\underline{X}_2} + \int_{f_2-\underline{X}_2}^{\underline{X}_1} dX_1 \Bigg|_{f_2 < \underline{X}_1+\underline{X}_2}\right]$$

$$= \frac{1}{\prod_{i=1}^{2} \Delta X_i}\left[(\overline{X}_1+\overline{X}_2-f_2)\ U(\overline{X}_1+\overline{X}_2-f_2) - (\underline{X}_1+\overline{X}_2-f_2)\ U(\underline{X}_1+\overline{X}_2-f_2)\right.$$

$$\left. - (\overline{X}_1+\underline{X}_2-f_2)\ U(\overline{X}_1+\underline{X}_2-f_2) + (\underline{X}_1+\underline{X}_2-f_2)\ U(\underline{X}_1+\underline{X}_2-f_2)\right]$$

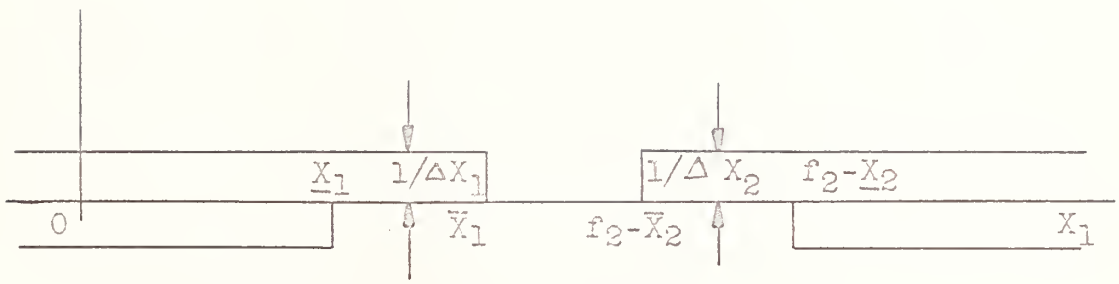
and $U(a - f_2) = 0, \quad f_2 > a$

$$= 1, \quad f_2 < a$$

This represents the area under the product curve $g_1 \cdot g_2$ for different values of f_2 when g_2 is turned around on a common axis with g_1 and then shifted through g_1. The plot of $g(f_2)$ is shown in Fig. 14.

$$f_3 = X_1 + X_2 + X_3 = f_2 + X_3 \qquad X_3 = \emptyset = f_3 - f_2 \qquad \frac{\partial \emptyset}{\partial f_3} = 1$$

$$g(f_3) = \int_{-\infty}^{\infty} g(f_2)\, g_3(f_3 - f_2)\, df_2$$



$$f_2 > \bar{X}_1 + \bar{X}_2$$

Fig. 13. Convolution of g_2 upon g_1.

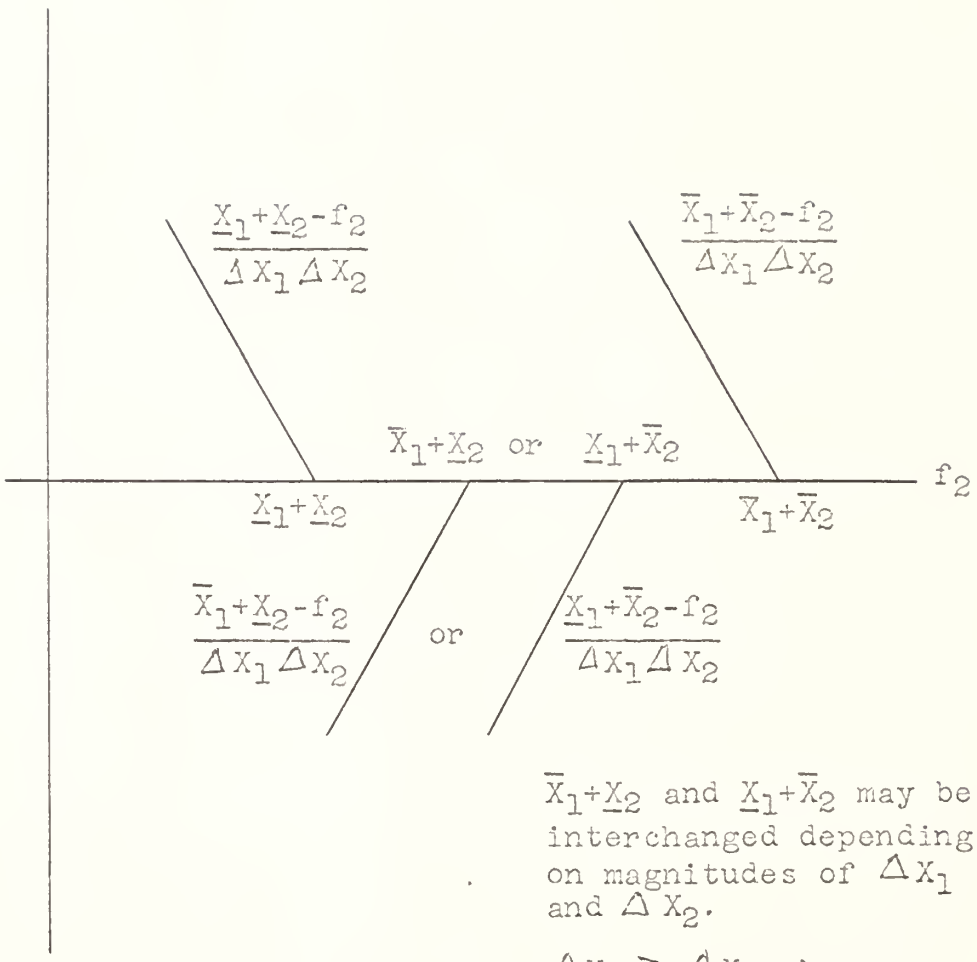


Fig. 14. Plot of terms of $g(f_2)$.

$$= \frac{1}{\prod_{i=1}^{3} \Delta X_i} \left[\int_{f_3-\bar{X}_3}^{\bar{X}_1+\bar{X}_2} (\bar{X}_1+\bar{X}_2-f_2)\,df_2 \Big|_{f_3<(\bar{X}_1+\bar{X}_2+\bar{X}_3)} - \int_{f_3-\bar{X}_3}^{\underline{X}_1+\bar{X}_2} (\underline{X}_1+\bar{X}_2-f_2)\,df_2 \Big|_{f_3<(\underline{X}_1+\bar{X}_2+\bar{X}_3)} \right.$$

$$- \int_{f_3-\bar{X}_3}^{\bar{X}_1+\underline{X}_2} (\bar{X}_1+\underline{X}_2-f_2)\,df_2 \Big|_{f_3<(\bar{X}_1+\underline{X}_2+\bar{X}_3)} - \int_{f_3-\underline{X}_3}^{\bar{X}_1+\bar{X}_2} (\bar{X}_1+\bar{X}_2-f_2)\,df_2 \Big|_{f_3<(\bar{X}_1+\bar{X}_2+\underline{X}_3)}$$

$$+ \int_{f_3-\underline{X}_3}^{\underline{X}_1+\bar{X}_2} (\underline{X}_1+\bar{X}_2-f_2)\,df_2 \Big|_{f_3<(\underline{X}_1+\bar{X}_2+\underline{X}_3)} + \int_{f_3-\bar{X}_3}^{\underline{X}_1+\underline{X}_2} (\underline{X}_1+\underline{X}_2-f_2)\,df_2 \Big|_{f_3<(\underline{X}_1+\underline{X}_2+\bar{X}_3)}$$

$$\left. + \int_{f_3-\underline{X}_3}^{\bar{X}_1+\underline{X}_2} (\bar{X}_1+\underline{X}_2-f_2)\,df_2 \Big|_{f_3<(\bar{X}_1+\underline{X}_2+\underline{X}_3)} - \int_{f_3-\underline{X}_3}^{\underline{X}_1+\underline{X}_2} (\underline{X}_1+\underline{X}_2-f_2)\,df_2 \Big|_{f_3<(\underline{X}_1+\underline{X}_2+\underline{X}_3)} \right]$$

$$g(f_3) = \frac{1}{2\prod_{i=1}^{3} \Delta X_i} \left[(\bar{X}_1+\bar{X}_2+\bar{X}_3-f_3)^2\, U(\bar{X}_1+\bar{X}_2+\bar{X}_3-f_3) \right.$$

$$- (\underline{X}_1+\bar{X}_2+\bar{X}_3-f_3)^2\, U(\underline{X}_1+\bar{X}_2+\bar{X}_3-f_3) - (\bar{X}_1+\underline{X}_2+\bar{X}_3-f_3)^2$$

$$U(\bar{X}_1+\underline{X}_2+\bar{X}_3-f_3) + (\underline{X}_1+\underline{X}_2+\bar{X}_3-f_3)^2\, U(\underline{X}_1+\underline{X}_2+\bar{X}_3-f_3)$$

$$- (\bar{X}_1+\bar{X}_2+\underline{X}_3-f_3)^2\, U(\bar{X}_1+\bar{X}_2+\underline{X}_3-f_3) + (\underline{X}_1+\bar{X}_2+\underline{X}_3-f_3)^2$$

$$U(\underline{X}_1+\bar{X}_2+\underline{X}_3-f_3) + (\bar{X}_1+\underline{X}_2+\underline{X}_3-f_3)^2\, U(\bar{X}_1+\underline{X}_2+\underline{X}_3-f_3)$$

$$\left. - (\underline{X}_1+\underline{X}_2+\underline{X}_3-f_3)^2\, U(\underline{X}_1+\underline{X}_2+\underline{X}_3-f_3) \right]$$

The PDF has the general form for f_n as follows.

$$g(f_n) = \frac{1}{(n-1)!} \frac{1}{\prod_{i=1}^{n} \Delta X_i} \sum_{j=1}^{2^n} \left[(f_j - f_n)^{(n-1)} \, U(f_j - f_n)(-1)^{y_j} \right]$$

$$U(f_j - f_n) = 0, \quad f_n > f_j$$
$$= 1, \quad f_n < f_j$$

$f_n = X_1 + X_2 + \ldots + X_n$ and is a variable

f_j is a constant equal to one of the 2^n possible combinations of the X_i's. $f_1 = \bar{f}$ and $f_{(j+1)} < f_j$.

y_i = number of $\underline{X}_i$'s in f_j

This is correct for n = 1, 2, or 3. If it is true for n variables, then a general term of $g(f_{n+1})$ will be as follows.

$$\frac{(-1)^{y_j}(-1)^{y_k}}{(n-1)! \prod_{i=1}^{n+1} \Delta X_i} \int_{f_{n+1}-k}^{f_j} (f_j - f_n)^{(n-1)} \, df_n \Big|_{f_n < f_j + k}$$

$$= \frac{(-1)^{(y_i + y_k)}}{n! \prod_{i=1}^{n+1} \Delta X_i} \left[(f_j + k - f_{n+1})^n \, U(f_j + k - f_{n+1}) \right]$$

This is exactly the same form because k is either $\bar{X}_{n+1}$ or $\underline{X}_{n+1}$. The exponent of (-1) is unchanged if k is $\bar{X}_{n+1}$ as $y_k = 0$ and is increased by one if k is $\underline{X}_{n+1}$. The interpretation of y_i therefore remains unchanged. The constant $(f_j + k)$ is one of the 2^{n+1} possible combinations of the extremes of the X_i's. Since the formula is true for n = 1 and the truth of the formula for n variables implies its truth for n + 1 variables, the formula is true for all positive integers (Brand, 1, p. 5).

The probability that $f > f_k$, $P\{f > f_k\}$, is the area under $g(f)$ between f_k and $\bar{f}$. The subscript n is now dropped from f_n. This is the cumulative density function, CDF, of $f = X_1 + X_2 + \ldots + X_n$.

$$P\{f > f_k\} = \int_{f_k}^{\bar{f}} g(f)df = \frac{1}{n} \cdot \frac{1}{(n-1)!} \frac{1}{\prod_{i=1}^{n} \Delta X_i}$$

$$\sum_{j=1}^{k-1} \left[(f_j - f_k)^n (-1)^{y_j} \right]$$

This is valid when f_k is one of the 2^n values for f_j. When f_k is not one of these values, the summation is from $j = 1$ to the number N so that $f_j > f_k$ and $f_{j+1} < f_k$, $(j=N)$. If this is not true for any j, $1 \leq j \leq 2^n$, then $\underline{f} > f_k > \bar{f}$.

Laplace Transform Development of CDF

The following is an alternate approach to obtain the preceding results using the Laplace transformation.

$$g(f_2) = \int_{-\infty}^{\infty} g_1(X_1) g_2(f_2 - X_1) dX_1$$

Using $g_i = \frac{1}{\Delta X_i} \left[U(X_i - \underline{X}_i) - U(X_i - \bar{X}_i) \right]$ the limits of $\binom{+}{-}\infty$ can be reduced to the following since $g_i = 0$ for $\underline{X}_i > X_i > \bar{X}_i$. As long as $\underline{X}_i \geq 0$.

$$\mathcal{L}\left[g(f_2)\right] = \mathcal{L}\left[\int_0^{f_2} g_1(X_i)g_2(f_2-X_1)dX_1\right] = \mathcal{L}\left[g_1 * g_2\right]$$

$$= \mathcal{L}\left[g_1\right] \cdot \mathcal{L}\left[g_2\right]$$

$$\mathcal{L}\left[g_i\right] = \frac{1}{\Delta X_i}\left[\frac{e^{-\underline{X}_i s} - e^{-\overline{X}_i s}}{s}\right]$$

Therefore

$$\mathcal{L}\left[g(f_2)\right] = \frac{1}{\Delta X_1}\ \frac{1}{\Delta X_2}\ \frac{1}{s^2}\left[e^{-(\underline{X}_1+\underline{X}_2)s} - e^{-(\underline{X}_1+\overline{X}_2)s}\right.$$

$$\left. - e^{-(\overline{X}_1+\underline{X}_2)s} + e^{-(\overline{X}_1+\overline{X}_2)s}\right]$$

As long as $\underline{f}_2 \geq 0$,

$$\mathcal{L}\left[g(f_3)\right] = \mathcal{L}\left[\int_0^{f_3} g(f_2)g_3(f_3-f_2)df_2\right] = \mathcal{L}\left[g(f_2)\right] \cdot \mathcal{L}\left[g_3\right]$$

$$= \frac{1}{\prod_{i=1}^{3} \Delta X_i}\ \frac{1}{s^3}\left[e^{-(\underline{X}_1+\underline{X}_2+\underline{X}_3)s} - e^{-(\underline{X}_1+\overline{X}_2+\underline{X}_3)s}\right.$$

$$- e^{-(\overline{X}_1+\underline{X}_2+\underline{X}_3)s} + e^{-(\overline{X}_1+\overline{X}_2+\underline{X}_3)s} - e^{-(\underline{X}_1+\underline{X}_2+\overline{X}_3)s}$$

$$\left. + e^{-(\underline{X}_1+\overline{X}_2+\overline{X}_3)s} + e^{-(\overline{X}_1+\underline{X}_2+\overline{X}_3)s} - e^{-(\overline{X}_1+\overline{X}_2+\overline{X}_3)s}\right]$$

Each time a new distribution is combined, the following changes will occur in the equation for $\mathcal{L}\left[g(f_{k-1})\right]$.

$$\prod_{i=1}^{k-1} \Delta X_i \longrightarrow \prod_{i=1}^{k} \Delta X_i$$

$$\frac{1}{s^{k-1}} \longrightarrow \frac{1}{s^k}$$

$$\pm e^{-(a)s} \rightarrow \binom{+}{-} e^{-(a+\underline{X}_k)s} \text{ or } \binom{-}{+} e^{-(a+\overline{X}_k)s}$$

The number of terms of the form $\binom{+}{-} e^{-as}$ goes from 2^{k-1} to 2^k. The $\binom{+}{-}$ is determined by $(-1)^{y_j}$ where y_j is the number of $\overline{X}_i$'s making up the constant a_j. Therefore

$$g(f_n) = \mathcal{L}^{-1}\left[g(s)\right] = \frac{1}{\prod_{i=1}^{n} \Delta X_i} \frac{1}{(n-1)!} \sum_{j=1}^{2^n} \left[(f_n - a_j)^{(n-1)} U(f_n - a_j)(-1)^{y_j} \right]$$

This is a symmetric function about $\frac{\underline{f}_n + \overline{f}_n}{2}$ between $\underline{f}_n$ and $\overline{f}_n$. Also $g(f_n)$ is zero for $\underline{f}_n > f_n > \overline{f}_n$. Therefore the same function with the same area will be given if written as follows with y_j the number of $\underline{X}_i$'s making up a_j.

$$g(f_n) = \frac{1}{\prod_{i=1}^{n} \Delta X_i} \frac{1}{(n-1)!} \sum_{j=1}^{2^n} \left[(a_j - f_n)^{(n-1)} U(a_j - f_n)(-1)^{y_j} \right]$$

This is exactly the same PDF obtained in the previous section. Since $P\{f > f_k\}$ is desired, this form requires integration of considerably fewer terms in the region close to $\overline{f}$ because $\overline{f}$ is considered f maximum. If $\overline{f}$ had been f minimum, the first form would be more desirable.

Justification for Very Small ΔX_i's

Consider the extreme case where $X_1 >>>> X_i$ for $2 \leq i \leq n$. Let all ΔX_i's except ΔX_1 be equal to ΔX_0. In the limit as ΔX_0 goes to zero, ΔX_1 will be equal to Δf and f should have

the same distribution as X_1 since all other X_i's are constants. If the general equation for $P\{f > f_k\}$ gives the same distribution for f as X_1, it can be stated that the equation is valid for all values of ΔX_i.

$$P\{f > f_k\} = \frac{1}{n!} \frac{1}{\Delta X_1 (\Delta X_0)^{(n-1)}} \left[(\bar{f}-f_k)^n - \binom{n-1}{1} (\bar{f}-\Delta X_0-f_k)^n \right.$$

$$\left. + \binom{n-1}{2} (\bar{f}-2\Delta X_0-f_k)^n \ldots \right]$$

The limit as ΔX_0 goes to zero is of the form $\frac{0}{0}$ and is indeterminate. Application of L'Hospital's rule (n - 1) times and taking the limit as ΔX_0 goes to zero, gives the following.

$$P\{f > f_k\} = \frac{1}{n!\Delta X_1 (n-1)!} - \left[\binom{n-1}{1} (n!)(\bar{f}-f_k)(-1)^{(n-1)} \right.$$

$$\left. + \binom{n-1}{2} (n!)(\bar{f}-f_k)(-2)^{n-1} \ldots + (n!)(\bar{f}-f_k)\binom{n-1}{n-1}(n-1)^{n-1} \right]$$

$$= \left[\frac{\bar{f}-f_k}{\Delta X_1} \right] \left[\frac{1}{(n-1)!} \right] \left[\binom{n-1}{n-1} (n-1)^{(n-1)} - \binom{n-1}{n-2} (n-2)^{(n-1)} \right.$$

$$\left. + \binom{n-1}{n-3} (n-3)^{(n-1)} \ldots \right]$$

The order in the last bracket has been reversed. This expression is equal to (n - 1)! and the proof is complete (Feller, 3, p. 63).

Volume Model Representation

The equation for $P\{f \geq f_k\}$ is rewritten below.

$$P\{f \geq f_k\} = \left\{ \frac{1}{\prod_{i=1}^{n} \Delta X_i} \right\} \left\{ \frac{1}{n!} \sum_{j=1}^{k-1} \left[(f_j - f_k)^n (-1)^{y_j} \right] \right\}$$

The denominator of the first bracket on the right is the volume of an n dimensional rectangular hypercube with sides of length ΔX_i. Consider this hypercube located in an n dimensional coordinate system with $(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n)$ the coordinates of the closest corner to the origin and $(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n)$ the furthest corner from the origin. A hyperplane passing through this cube perpendicularly to the diagonal between the two corners will cut off a portion of the volume between the plane and $\overline{f}$. The volume of the portion cut off is represented by the second bracket on the right when the equation for the hyperplane is $X_1 + X_2 + X_3 + \ldots + X_n = f_k$. This representation is easily seen in three dimensions but is difficult to visualize in more than three dimensions.

APPENDIX III

ALGORITHM FOR ORDERING SUMS

Given a set of n numbers, $(p_1, p_2, \ldots, p_n)$, there are 2^n possible numbers that can be formed by taking all possible unique formal sums of 0, 1, 2, 3, ..., n numbers of the set. When n is

three, the eight possible unique formal sums are as follows: 0, p_1, p_2, p_3, $p_1 + p_2$, $p_2 + p_3$, $p_1 + p_3$, $p_1 + p_2 + p_3$. The problem is to order these sums starting with the smallest and ending with the largest. This is quite simple for small values of n. However, when n = 40, 2^n = 1,099,511,627,776, and the ordering is time-consuming. The algorithm here presented will give nearly exact results of the number of sums less than a certain sum for n points between the smallest and largest of the 2^n sums.

The first step is to arrange the n numbers so that $p_{(i+1)} \geq p_i$. The smallest sum of k numbers is then calculated for k = 1, 2, 3, ..., n. The following numbers are then found.

K_j = number of p_i's $\geq p_1 + p_2 + \cdots + p_{(j+1)}$ $\quad (1 \leq j)$

K_j' = number of p_i's $\geq p_2 + p_3 + \cdots + p_{(j+1)}$ $\quad (2 \leq j)$

K_j'' = number of p_i's $\geq p_3 + p_4 + \cdots + p_{(j+1)}$ $\quad (3 \leq j)$

$\vdots$

K_j^m = number of p_i's $\geq p_{(m+1)} + p_{(m+2)} + \cdots + p_{(j+1)}$ $\quad (m+1 \leq j)$

Once the above numbers have been found, the following positive statements can be made.

1. The number of sums of (j-1) numbers $\geq$ the smallest sum of j numbers.

   a. Any sum containing at least one p_i in the K_1 set that is not in the smallest sum of j p_i's and neither p_1 or p_2. If $p_k \geq p_1 + p_2$, then $p_3 + p_4 + p_k \geq p_1 + p_2 + p_3 + p_4$.

   b. Any sum containing at least one p_i in the K_2' set

that is not in the smallest sum of j p_i's and not p_3. Each sum will contain either p_1 or p_2. If $p_k \geq p_2 + p_3$, then $p_1 + p_4 + p_k \geq p_1 + p_2 + p_3 + p_4$. Also if $p_k \geq p_2 + p_4$, then $p_1 + p_3 + p_k \geq p_1 + p_2 + p_3 + p_4$. This is quite likely; therefore p_3 is not excluded. This will help to balance out the sums with no p_i in K_2' that may also satisfy the above conditions. Note that K_2' includes many p_i's in K_x''s where $X \geq 3$ and this implies $p_k \geq p_2 + p_3 + p_4$. Also the (j - 1) sums with more than one p_i in the K_2' set need not exclude p_3.

c. Any sum containing at least one p_i in the K_3'' set that is not in the smallest sum of j numbers and not p_3 or p_4. Each sum will contain both p_1 and p_2. If $p_k \geq p_3 + p_4$, then $p_1 + p_2 + p_k \geq p_1 + p_2 + p_3 + p_4$. Also $p_1 + p_2 + p_5 + p_k \geq p_1 + p_2 + p_3 + p_4 + p_5$, and if $p_k \geq p_4 + p_5$, then $p_1 + p_2 + p_3 + p_k \geq p_1 + p_2 + p_3 + p_4 + p_5$. This will be true for most p_i in K_3''. For the same reasons as above, p_3 and p_4 are not excluded.

a, b, and c are formulated as follows.

$$1 \cdot \binom{K_1}{1}\binom{n-K_1-2}{j-2} + 1 \cdot \binom{K_1}{2}\binom{n-K_1-2}{j-3} + 1 \cdot \binom{K_1}{3}\binom{n-K_1-2}{j-4} + \cdots$$

$$+ 2\binom{K_2'}{1}\binom{n-K_2'-2}{j-3} + 2 \cdot \binom{K_2'}{2}\binom{n-K_2'-2}{j-4} + 2 \cdot \binom{K_2'}{3}\binom{n-K_2-2}{j-5} + \cdots$$

$$+ 1 \cdot \binom{K_3''}{1}\binom{n-K_3''-2}{j-4} + 1 \cdot \binom{K_3''}{2}\binom{n-K_3''-2}{j-5} + 1 \cdot \binom{K_3''}{3}\binom{n-K_3''-2}{j-6} + \cdots$$

Note that a product term of any row divides the set of numbers into three subsets. The middle term $\binom{K_n}{1}$ of the first row is the basic building block. In the number of sums of (j-h) numbers greater than the smallest sum of j numbers, the first K_i that can be used is K_h. This is because a sum of (j-h) numbers must have one number greater than the (h+1) smallest numbers of the smallest sum of j numbers. When j = 6 and h = 3, if $p_k \geq p_1 + p_2 + p_3 + p_4$, then $p_5 + p_6 + p_k \geq p_1 + p_2 + p_3 + p_4 + p_5 + p_6$.

The first row represents taking the subset of the K_h numbers one at a time, two at a time, out to all at a time. The subset of the smallest (h + 1) numbers is isolated from the sums in the first row as $\binom{h+1}{0} = 1$. The right factor represents the remainder of the $(n-K_h-h-1)$ numbers, taken j - (h + number of K_h numbers in that term) at a time to complete a sum of (j-h) numbers.

The second row allows the inclusion of one of the (h+1) subset numbers in each term, $\binom{h+1}{1}$. This necessitates having at least one $K_{(h+1)}$' number in each (j - h) sum. To be completely positive the remaining numbers should be chosen from the $(n - K_{(h+1)}' - h - 1 - 1)$ numbers remaining but as mentioned previously, this number is not excluded.

A product term in any row is of the form $\binom{u}{v}\binom{w}{x}\binom{y}{z}$. The number w is equal to K_h in the first row for the sums of (j - h) numbers greater than the smallest sum of j numbers. The number u + w + y is always equal to n for any term. The number v + x + z is always equal to (j - h). The equation for

the number of sums of (j - h) numbers greater than the smallest sum of j numbers is as follows. (Notice how the equation is built upon the number h.)

2. Number of sums of (j - h) $\geq$ smallest sum of j.

$$\binom{h+1}{0}\binom{K_h}{1}\binom{n-K_h-h-1}{j-h-1} + \binom{h+1}{0}\binom{K_h}{2}\binom{n-K_h-h-1}{j-h-2} + \cdots$$

$$+ \binom{h+1}{1}\binom{K_{h+1}'}{1}\binom{n-K_{h+1}'-h-1}{j-h-2} + \binom{h+1}{1}\binom{K_{h+1}'}{2}\binom{n-K_{h+1}'-h-1}{j-h-3} + \cdots$$

$$+ \cdots$$

$$\vdots$$

$$+ \binom{h+1}{X}\binom{K_{h+X}^{(X)}}{1}\binom{n-K_{h+X}^{(X)}-h-1}{j-h-X-1} + \cdots$$

$$\vdots$$

The last row is for X = h + 1. To have a completely positive statement the $\binom{n - K_{h+X}^{(X)} - h - 1}{j - h - X - 1}$ factor should be $\binom{n - K_{h+X}^{(X)} - h - 1 - X}{j - h - X - 1}$.

REFERENCES

1. Brand, L.
Advanced calculus. New York: John Wiley & Sons, Inc., 1958.

2. DiMucci, V. A.
Statistical circuit design. Computer Department, Internal Report EPA FZB.00, General Electric Co., Phoenix, Arizona, October 11, 1961.

3. Feller, W.
An introduction to probability theory and its application, Vol. I. New York: John Wiley & Sons, Inc., 1957.

4. Goldman, S.
Transformation calculus and electrical transients. Englewood, New Jersey: Prentice-Hall, 1949.

PROBABLE WORST CASE CIRCUIT DESIGN

by

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A digital computer is a complex combination of circuits. The performance of each circuit must be well defined and a given set of rules for each circuit must be followed in order for the entire system to function properly. If the set of rules is strict enough, the system will be extremely reliable. This is the case when "worst case" circuit analysis is applied to the individual circuits.

If the tolerances of the components are taken into account and a probable worst case value for a circuit output function considered, the strictness of the set of rules can be reduced with very little effect on the reliability. A strictly monotonic output function will have a worst case value which is found by choosing a unique combination of one or the other extreme values of the circuit components entering into that output function. The values of each component in a specific type of circuit are distributed in some manner between two extremes. The probability is very small that a circuit will ever be built that has this worst case value for an output function.

When the output function is expanded in a Taylor series about the worst case value, a linear combination of variables is obtained if only the constant and first order terms are retained. The distribution of this type of function is readily found for two kinds of component value distributions. The first distribution considers the component value to be either its maximum or minimum value. The second is a uniform distribution between the two extremes.

Once the distribution is found for the approximate function for either component distribution, a systematic curve fitting can be applied using values obtained from the exact output function. Using an n dimensional model of the function, the distribution obtained after the curve fitting process can be evaluated as to the region of near exactness.

As long as conservative component value distributions are used, the output function distribution can be safely used. A probable worst case value for the output function can then be found once the acceptable probability of failure is decided upon.

This method will enable development of a set of rules for the circuit design which will increase the performance capability of the circuit and have a negligible effect on the reliability of the entire system.

Date Due