ROTATIONAL AND IRROTATIONAL FLOW IN CERTAIN GAS DYNAMICS PROBLEMS

by

CHENG-CHIEN CHOU

B. S., Cheng Kung University, Taiwan, China, 1957

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

Approved by:

Wilson Tripp
Major Professor

TABLE OF CONTENTS

INTRODUCTION

The material involved in this report is a discussion of the rotationality of the normal shock, the potential vortex, the Prandtl-Meyer expansion and the oblique shock. In the investigation of oblique shock, some expressions involving the relationship between the entropy change and the angle of rotation have been developed.

All problems investigated in this report are based on the following basic hypotheses:

(1) The compressible fluid flow field is two-dimensional.

(2) The fluid is a perfect gas with $k = 1.4$.

(3) All analyses concerning the compressible fluid motions are governed by the four basic laws which are:

(a) the law of conservation of mass

(b) Newton's second law of motion

(c) the first law of thermodynamics

(d) the second law of thermodynamics.

Some theoretical concepts and governing definitions which relate to the work of this report are described in the following pages.

Circulation

<u>Definition of Circulation.</u> (1)[1], (2), (3). The circulation Γ is defined as the line integral of the velocity vector taken

---

[1]Number in parentheses designates References at end of paper.

around a certain closed curve C enclosing a surface within a fluid region. Referring to Fig. 1, this is

$$\Gamma \equiv \oint_c V \cos \phi \, d\ell \tag{1a}$$

In vector notation it is

$$\Gamma = \oint_c \overline{V} \cdot d\overline{r} \tag{1b}$$

where $\overline{V}$ is the velocity vector and $\overline{r}$ the radius vector from a certain origin.

<u>Circulation Per Unit Area in Two Dimensions</u> (1). Consider the circulation $d\Gamma_z$ around a small square element in the x,y-plane as shown in Fig. 2. The circulation is

$$d\Gamma_z = udx + (v + \frac{\partial v}{\partial x} dx)dy - (u + \frac{\partial u}{\partial y} dy)dx - vdy$$

or simplifying,

$$d\Gamma_z = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\, dA_z \tag{2}$$

The circulation per unit area in the x,y-plane will then be given by

$$\frac{d\Gamma_z}{d\,A_z} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{3}$$

Rotational and Irrotational Fluid Flow

<u>Definition of the Fluid Rotation of a Particle at a Point</u> (1). "The fluid rotation at a point is the mean angular velocity of two

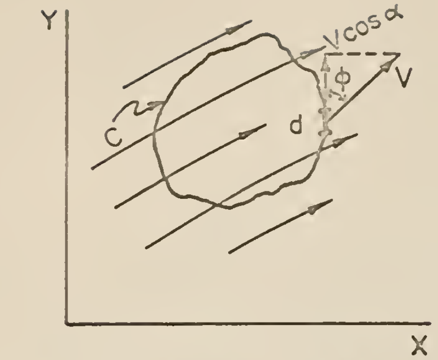

Fig. 1. Illstration of the concept of circulation.

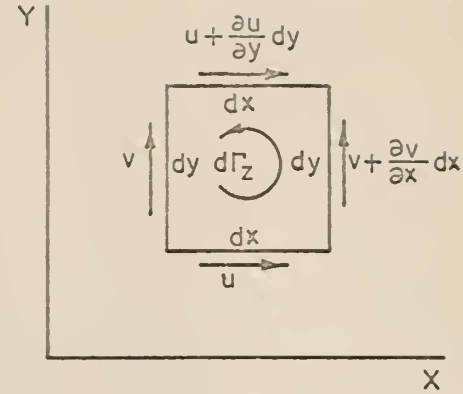

Fig. 2. Circulation for elementary closed curve.

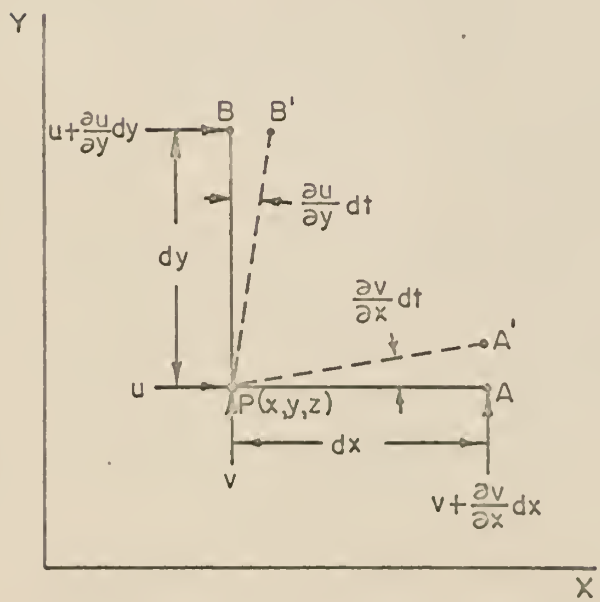

Fig. 3. Fluid rotation at a point.

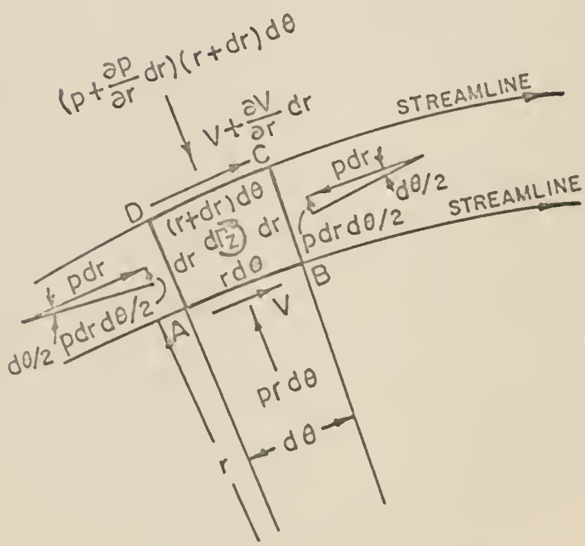

Fig. 4. Pressure force and velocities for an elementary fluid particle in a streamline coordinate system.

infinitesimal and mutually perpendicular fluid curves instantaneously passing through the point."

Rotation in Two Dimensions (3). In order to find an analytical expression for the fluid rotation at a point in a two-dimensional flow, the rotation component of a fluid particle about an axis through itself, say parallel to the z-axis, will be considered. The symbol ω_z is used to indicate this component of rotation.

Referring to Fig. 3, two line segments dx and dy passing through the point P are chosen parallel to the x- and y-axes, and they must be at right angles to each other. The particle is at P(x,y,z) and has the velocity components u and v in the x,y-plane. If the components of velocity at A and B are different from that at P, the segments PA and PB will rotate to the relative positions PA' and PB', respectively, during a certian time interval dt. Then the angular velocity of the dx segment is

$$\frac{v + \frac{\partial v}{\partial x}dx - v}{dx} = \frac{\partial v}{\partial x} \qquad \text{rad/sec}$$

and the angular velocity of the dy segment is

$$\frac{u + \frac{\partial u}{\partial y}dy - u}{dy} = -\frac{\partial u}{\partial y} \qquad \text{rad/sec}$$

using counterclockwise rotation as positive.

Hence, by definition, the rotation of a fluid particle at a point in a two-dimensional flow field is

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{4}$$

Relationship Between Circulation and Fluid Rotation. Referring to Eqs. (3) and (4), the circulation per unit area is twice the average rotation of a fluid particle, i.e.,

$$\frac{d\Gamma_z}{dA} = 2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{5}$$

The Connection Between the Rotation and the Thermodynamics Properties of the Flow (1). A system of curvilinear coordinates comprising the streamlines and the system of lines normal to the streamlines are taken for a steady and frictionless flow as shown in Fig. 4.

The rotation of the chosen element ABCD is

$$d\Gamma_z = Vrd\theta - \left(V + \frac{\partial V}{\partial r}dr\right)(r + dr)d\theta = -\left(r\frac{\partial V}{\partial r} + V\right)d\theta\, dr$$

or

$$2\omega_z = \frac{d\Gamma_z}{dA_z} = \frac{d\Gamma_z}{rd\theta\, dr} = -\frac{\partial V}{\partial r} - \frac{V}{r} \tag{6}$$

The force balance in the normal direction gives

$$\left(p + \frac{\partial p}{\partial r}dr\right)(r+dr)d\theta - prd\theta - 2\left(pdr\frac{d\theta}{z}\right) - (\rho rd\theta\, dr)\left(\frac{V^2}{r}\right)=0$$

which yields

$$\frac{\partial p}{\partial r} = \frac{\rho V^2}{r} \tag{7}$$

The combination of Eq. (6) and (7) reduces to

$$2\omega_z = -\frac{\partial V}{\partial r} - \frac{1}{\rho V}\frac{\partial p}{\partial r} \tag{8}$$

The first law of thermodynamics gives the stagnition enthalpy as

$$h_o = h + \frac{V^2}{2}$$

Its partial derivative with respect to the normal displacement is

$$\frac{\partial h_o}{\partial r} = \frac{\partial h}{\partial r} + V\frac{\partial V}{\partial r} \tag{9}$$

Since

$$Tds = dh - \frac{1}{\rho}\,dp$$

the partial derivative of enthaply h with respect to the normal displacement is

$$\frac{\partial h}{\partial r} = T\frac{\partial s}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{10}$$

Eq. (9) and (10) together yield

$$\frac{\partial p}{\partial r} = -\rho T\frac{\partial s}{\partial r} + \rho\frac{\partial h_o}{\partial r} - \rho V\frac{\partial V}{\partial r} \tag{11}$$

Substituting Eq. (11) into Eq. (8), an expression for the relationship between the fluid rotation and the thermodynamic properties of the flow is

$$\omega_z = \frac{1}{2V}\left(T\frac{\partial s}{\partial r} - \frac{\partial h_o}{\partial r}\right) \tag{12}$$

The result of Eq. (12) shows that the fluid rotation depends on the rates of change of the entropy and the stagnation enthalpy normal to the streamlines.

NOMENCLATURE

A	flow area
A^*	choking area of flow at Mach Number unity
c	speed of sound
c^*	critical speed of sound for adiabatic flow
c_p	specific heat at constant pressure
h	enthalpy per unit mass
h_o	stagnition enthalpy
k	ratio of specific heats
ℓ	length
m	mass
M	Mach Number
M^*	V/c^*
p	pressure
r	radius
$\bar{r}$	radius vector
R	gas constant
s	entropy per unit mass
t	time
T	absolute temperature
u	velocity component in x-direction
v	velocity component in y-direction
V	velocity
$\bar{V}$	velocity vector
α	Mach angle
β	angle
Γ	circulation

δ wall angle for oblique shock

θ angle

ν angle

ρ density

σ angle of oblique shock to incoming flow

ϕ angle

ω defined by Eq. (17a)

ω_z angular rotation or rate of angular rotation about z-axis

NORMAL SHOCK

Physical Equations (1)

Steady-flow Energy Equation:

$$h_x + \frac{V_x^2}{2} = h_y + \frac{V_y^2}{2} = h_o$$

Equation of Continuity:

$$\rho_x V_x = \rho_y V_y$$

Momentum Equation:

$$p_x + \rho_x V_x^2 = p_y + \rho_y V_y^2$$

where subscripts x and y denote the conditions before and after the shock, respectively (Fig. 5).

Irrotationality of Normal Shock

Consider the square control volume ABCD passing through a normal shock as shown in Fig. 6. Since there is no velocity component along the shock, it shows clearly that the circulations around both of the control volumes are zero, i.e.,

$$\Gamma_{ABCD} = - V_x \, \ell_{AB} + V_x \, \ell_{CD} = 0$$

and

$$\Gamma_{A'B'C'D'} = - V_y \, \ell_{A'B'} + V_y \, \ell_{C'D'} = 0$$

According to the definition of the rotation of a particle at a point ("The fluid rotation at a point is the mean angular velocity of two infinitesimal and mutually perpendicular fluid

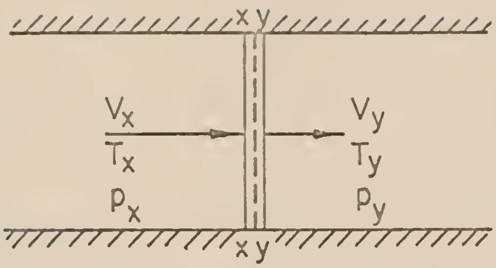

Fig. 5. Normal shock discontinuity.

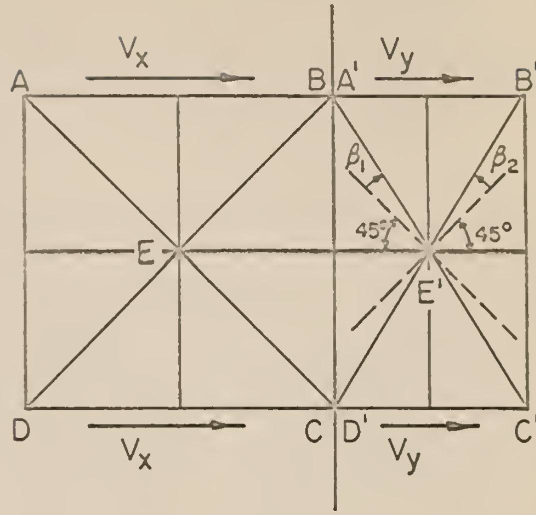

Fig. 6. Two-dimensional control volume in normal shock.

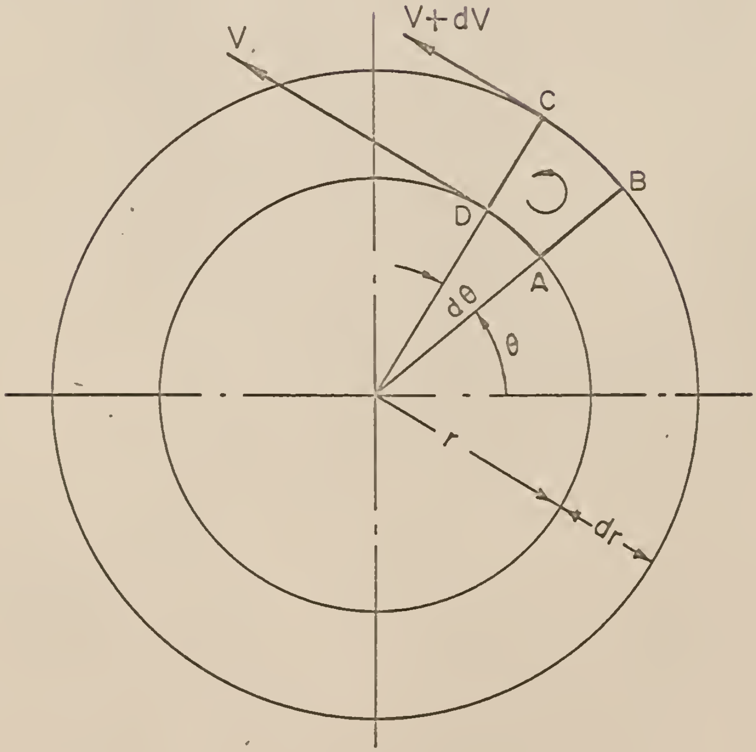

Fig. 7. Potential vortex.

curves instantaneously passing through the point."), Fig. 6 shows that the control volume ABCD, in moving through the shock to position A'B'C'D', underwent no rotation at its four corners. Also since $\beta_1 = \beta_2$, the rotation at the center of the control volume (E to E') is zero. Hence, it is concluded that the normal shock process is irrotational. However, the process is not reversible, and it is shown by this example that reversibility is not a criterion for irrotationality.

POTENTIAL VORTEX

Equation of Motion

The two-dimensional potential vortex is formed by streamlines which are concentric circles, and the tangential velocity along any streamline is inversely proportional to the radius of the streamline, i.e.,

$$Vr = C$$

where C is a constant.

Irrotationality of the Potential Vortex Motion

<u>Irrotationality Based on the Concept of Circulation around a Closed Path.</u> Consider the circulation around the element ABCD in Fig. 7. Since the line integral is zero along the sides AB and CD, the circulation around this element is

$$d\Gamma_{ABCD} = (V+dV)(r+dr)d\theta - Vrd\theta = (Vdr+rdV)d\theta$$
$$= \left[d(Vr)\right]d\theta = \left[d(C)\right]d\theta = 0$$

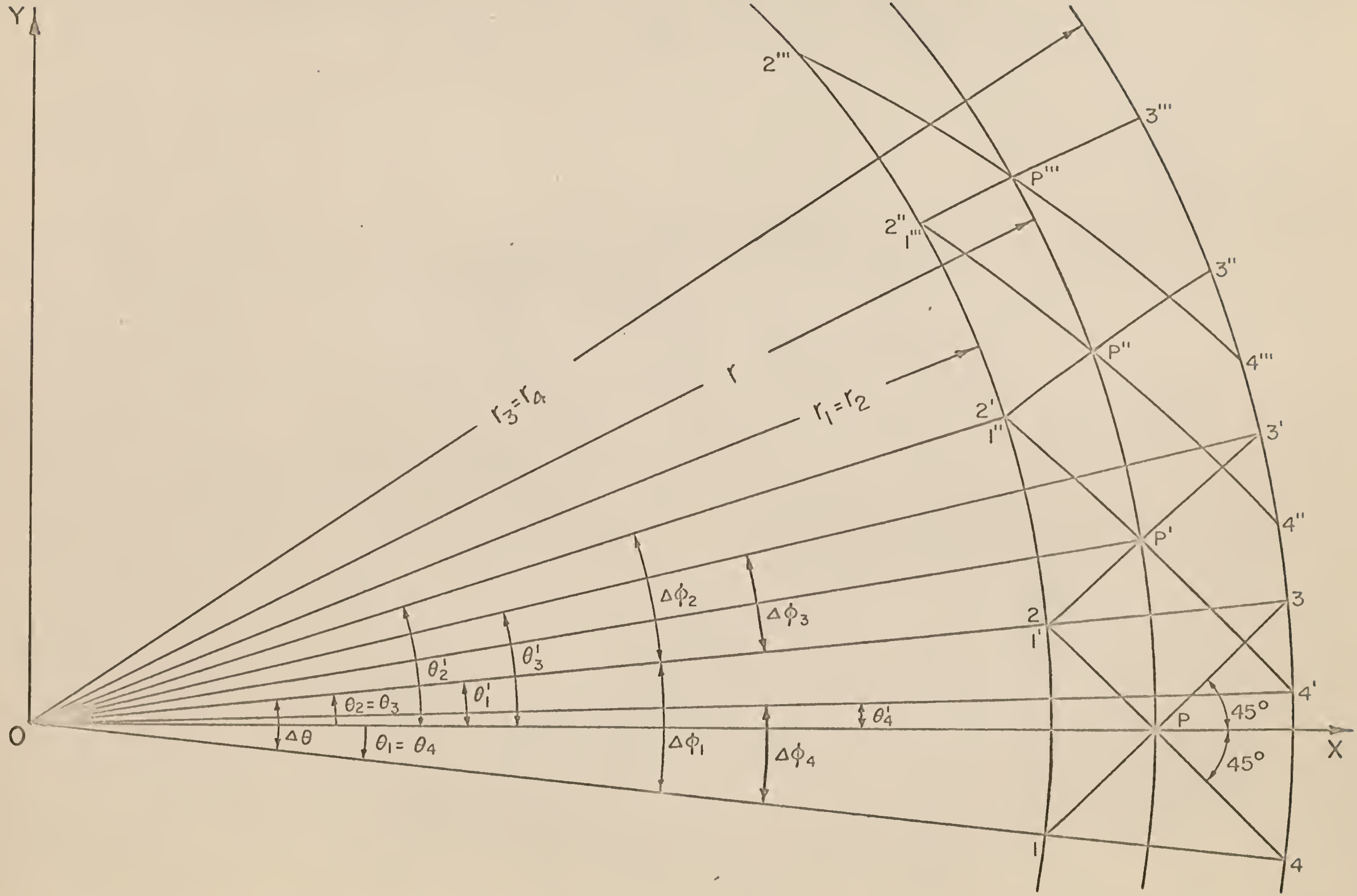

Fig.8. Example of potential vortex motion in the Cartesian coordinates system.

Thus the circulation of any element in the potential vortex motion not enclosing the origin is zero.

<u>Irrotationality Based on the Concept of the Rotation of a Particle.</u> Consider the area 1234 of Fig. 8 located within a potential vortex. The lines 1-3 and 2-4 are straight, mutually perpendicular, intersect at point P, and are inclined at 45 degrees to the x-axis. The rotationality of the point P moving with the element 1234 is investigated as follows:

Referring to Fig. 8, let the coordinates of the points 1, 2, 3 and 4 be (x_1,y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4) respectively, and the radii of the circles passing through the points 1, 2, 3 and 4 be r_1, r_2, r_3 and r_4, respectively.

Since $\quad r_1 = r_2 \quad$ and $\quad -\theta_1 = \theta_2 = \frac{\Delta\theta}{2}$,

$$x_1 = x_2 \quad \text{and} \quad -y_1 = y_2 .$$

Also

$$\tan\theta_2 = \frac{y_2}{x_2} = \frac{(r - x_2)\tan 45^o}{x_2} = \frac{r - x_2}{x_2}$$

$$x_2 = x_1 = \frac{r}{1 + \tan\theta_2} = \frac{r}{1 + \tan\frac{\Delta\theta}{2}}$$

$$y_2 = -y_1 = (r - x_2)\tan 45^o = r - x_2 = r\left(1 - \frac{1}{1 + \tan\frac{\Delta\theta}{2}}\right)$$

$$= \frac{r\tan\frac{\Delta\theta}{2}}{1 + \tan\frac{\Delta\theta}{2}} = \frac{r}{1 + \cot\frac{\Delta\theta}{2}}$$

and

$$r_2 = r_1 = \frac{y_2}{\sin\theta_2} = \frac{r\ \tan\frac{\Delta\theta}{2}}{\sin\frac{\Delta\theta}{2}\ (1 + \tan\frac{\Delta\theta}{2})}$$

$$= \frac{r}{\sin\frac{\Delta\theta}{2} + \cos\frac{\Delta\theta}{2}}$$

Let ℓ_{2p} = length from point 2 to P $= \frac{y_2}{\sin 45^\circ}$

$$= \frac{1}{\sin 45^\circ} \cdot \frac{r\ \tan\frac{\Delta\theta}{2}}{1 + \tan\frac{\Delta\theta}{2}}$$

$$r_3 - r_2 = \frac{\ell_{2P}}{\cos\angle 324} = \frac{1}{\sin 45^\circ} \cdot \frac{r\ \tan\frac{\Delta\theta}{2}}{1 + \tan\frac{\Delta\theta}{2}} \cdot \frac{1}{\cos\ (45^\circ + \frac{\Delta\theta}{2})}$$

in which

$$\sin 45^\circ \cos\ (45^\circ + \frac{\Delta\theta}{2}) = \sin 45^\circ \left[\cos 45^\circ \cos\frac{\Delta\theta}{2} - \sin 45^\circ \sin\frac{\Delta\theta}{2}\right]$$

$$= \frac{1}{2}\ (\cos\frac{\Delta\theta}{2} - \sin\frac{\Delta\theta}{2})$$

$$r_3 - r_2 = \frac{r\ \tan\frac{\Delta\theta}{2}}{1 + \tan\frac{\Delta\theta}{2}} \cdot \frac{1}{\frac{1}{2}(\cos\frac{\Delta\theta}{2} - \sin\frac{\Delta\theta}{2}}$$

$$= \frac{2r\ \tan\frac{\Delta\theta}{2}}{(1 + \tan\frac{\Delta\theta}{2})(\cos\frac{\Delta\theta}{2} - \sin\frac{\Delta\theta}{2})}$$

$$r_3 = r_2 + \frac{2r\ \tan\frac{\Delta\theta}{2}}{(1 + \tan\frac{\Delta\theta}{2})(\cos\frac{\Delta\theta}{2} - \sin\frac{\Delta\theta}{2})}$$

$$= \frac{y_2}{\sin\ \theta_2} + \frac{2r\ \tan\frac{\Delta\theta}{2}}{(1 + \tan\frac{\Delta\theta}{2})(\cos\frac{\Delta\theta}{2} - \sin\frac{\Delta\theta}{2})}$$

$$= \frac{r \tan \frac{\Delta\theta}{2}}{(1 + \tan \frac{\Delta\theta}{2}) \sin \frac{\Delta\theta}{2}} + \frac{2r \tan \frac{\Delta\theta}{2}}{(1 + \tan \frac{\Delta\theta}{2})(\cos \frac{\Delta\theta}{2} - \sin \frac{\Delta\theta}{2})}$$

$$= \frac{r \tan \frac{\Delta\theta}{2} (\cos \frac{\Delta\theta}{2} - \sin \frac{\Delta\theta}{2} + 2 \sin \frac{\Delta\theta}{2})}{(1 + \tan \frac{\Delta\theta}{2}) \left[\sin \frac{\Delta\theta}{2} (\cos \frac{\Delta\theta}{2} - \sin \frac{\Delta\theta}{2})\right]}$$

$$= \frac{r \tan \frac{\Delta\theta}{2} (\cos \frac{\Delta\theta}{2} + \sin \frac{\Delta\theta}{2})}{(1 + \tan \frac{\Delta\theta}{2})(\cos \frac{\Delta\theta}{2} - \sin \frac{\Delta\theta}{2}) \sin \frac{\Delta\theta}{2}}$$

$$= \frac{r (\cos \frac{\Delta\theta}{2} + \sin \frac{\Delta\theta}{2})}{(\cos \frac{\Delta\theta}{2} + \sin \frac{\Delta\theta}{2})(\cos \frac{\Delta\theta}{2} - \sin \frac{\Delta\theta}{2})}$$

$$= \frac{r}{\cos \frac{\Delta\theta}{2} - \sin \frac{\Delta\theta}{2}} = r_4$$

Hence,

$$x_3 = x_4 = r_3 \cos \theta_3 = r_3 \cos \frac{\Delta\theta}{2} = \frac{r}{1 - \tan \frac{\Delta\theta}{2}}$$

$$y_3 = y_4 = r_3 \sin \theta_3 = r_3 \sin \frac{\Delta\theta}{2} = \frac{r}{\cot \frac{\Delta\theta}{2} - 1}$$

and the coordinates of point P are

$$x = r$$

$$y = 0$$

When the motion proceeds in the counterclockwise direction, and when point 1 reached the original position of point 2, the corresponding new coordinates of points 1, 2, 3, 4 and P, which are 1', 2', 3', 4' and P', can be found. Let Δt be the required time interval for this displacement and $\Delta\phi_1$, $\Delta\phi_2$, $\Delta\phi_3$, $\Delta\phi_4$ and $\Delta\phi$ be the corresponding angular displacements for points 1, 2, 3, 4 and P respectively.

Since $Vr = C$

the arc displaced along any circular path during the motion

$$= r\,\Delta\phi = V\,\Delta t = \frac{C\Delta t}{r}$$

$$\therefore\ \Delta\phi = \frac{C\Delta t}{r^2}$$

For a given time interval,

$$\Delta\phi = \frac{C_1}{r^2}$$

where C_1 is a proportionality constant.

Given $\Delta\phi_1 = \Delta\theta$ and $r_1 = r_2$, then $\Delta\phi_2 = \Delta\phi_1 = \Delta\theta$

Also $\dfrac{\Delta\phi_3}{\Delta\phi_1} = \left(\dfrac{r_1}{r_3}\right)^2$ and $r_3 = r_4$

$$\therefore\ \Delta\phi_3 = \Delta\phi_4 = \left(\frac{r_1}{r_3}\right)^2 \Delta\phi_1 = \left(\frac{\dfrac{r}{\sin\frac{\Delta\theta}{2} + \cos\frac{\Delta\theta}{2}}}{\dfrac{r}{\cos\frac{\Delta\theta}{2} - \sin\frac{\Delta\theta}{2}}}\right)^2 \Delta\theta$$

$$= \left(\frac{\cos\frac{\Delta\theta}{2} - \sin\frac{\Delta\theta}{2}}{\sin\frac{\Delta\theta}{2} + \cos\frac{\Delta\theta}{2}}\right)^2 \Delta\theta = \left(\frac{1 - \sin\Delta\theta}{1 + \sin\Delta\theta}\right)\Delta\theta$$

For the same reason,

$$\Delta\phi = \left(\frac{r_1}{r}\right)^2 \Delta\phi_1 = \left(\frac{\dfrac{r}{\sin\frac{\Delta\theta}{2} + \cos\frac{\Delta\theta}{2}}}{r}\right)^2 \Delta\theta = \frac{\Delta\theta}{1 + \sin\Delta\theta}$$

The new position angles are

$$\theta_1' = \Delta\phi_1 + \theta_1 = \Delta\theta - \frac{\Delta\theta}{2} = \frac{\Delta\theta}{2}$$

$$\theta_2' = \Delta\phi_2 + \theta_2 = \Delta\theta + \frac{\Delta\theta}{2} = \frac{3\Delta\theta}{2}$$

$$\theta_3' = \Delta\phi_3 + \theta_3 = \frac{(1-\sin\Delta\theta)\Delta\theta}{1+\sin\Delta\theta} + \frac{\Delta\theta}{2} = \frac{(3-\sin\Delta\theta)\Delta\theta}{2(1+\sin\Delta\theta)}$$

$$\theta_4' = \Delta\phi_4 + \theta_4 = \frac{(1-\sin\Delta\theta)\Delta\theta}{1+\sin\Delta\theta} - \frac{\Delta\theta}{2} = \frac{(1-3\sin\Delta\theta)\Delta\theta}{2(1+\sin\Delta\theta)}$$

$$\theta' = \Delta\phi + \theta = \frac{\Delta\theta}{1+\sin\Delta\theta}$$

Hence the new coordinates of points 1, 2, 3, 4 and P after time Δt are

$$\begin{cases} x_1' = r_1 \cos \theta_1' \\ y_1' = r_1 \sin \theta_1' \end{cases}$$

$$\begin{cases} x_2' = r_2 \cos \theta_2' \\ y_2' = r_2 \sin \theta_2' \end{cases}$$

$$\begin{cases} x_3' = r_3 \cos \theta_3' \\ y_3' = r_3 \sin \theta_3' \end{cases}$$

$$\begin{cases} x_4' = r_4 \cos \theta_4' \\ y_4' = r_4 \sin \theta_4' \end{cases}$$

and $$\begin{cases} x' = r \cos \theta' \\ y' = r \sin \theta' \end{cases}$$

They are all in terms of r and $\Delta\theta$.

In general the position angles for the n-th position after a time interval of $n(\Delta t)$ are

$$\theta_1^{(n)} = n\,\Delta\phi_1 + \theta_1$$

$$\theta_2^{(n)} = n\,\Delta\phi_2 + \theta_2$$

$$\theta_3^{(n)} = n\,\Delta\phi_3 + \theta_3$$

$$\theta_4^{(n)} = n\,\Delta\phi_4 + \theta_4$$

$$\theta^{(n)} = n\,\Delta\phi + \theta$$

The new coordinates of points 1, 2, 3, 4 and P will then be

$$\begin{cases} x_1^{(n)} = r_1 \cos \theta_1^{(n)} \\ y_1^{(n)} = r_1 \sin \theta_1^{(n)} \end{cases}$$

$$\begin{cases} x_2^{(n)} = r_2 \cos \theta_2^{(n)} \\ y_2^{(n)} = r_2 \sin \theta_2^{(n)} \end{cases}$$

$$\begin{cases} x_3^{(n)} = r_3 \cos \theta_3^{(n)} \\ y_3^{(n)} = r_3 \sin \theta_3^{(n)} \end{cases}$$

$$\begin{cases} x_4^{(n)} = r_4 \cos \theta_4^{(n)} \\ y_4^{(n)} = r_4 \sin \theta_4^{(n)} \end{cases}$$

and $\begin{cases} x^{(n)} = r \cos \theta^{(n)} \\ y^{(n)} = r \sin \theta^{(n)} \end{cases}$

At any instant the coordinates for the points which originally lay along the lines 1-3 and 2-4 (lines which were originally perpendicular and intersected at P) can be obtained by use of the procedure given above. Therefore, it will be possible either to plot or to set up equations for the curved lines 1'-P'-3' and 2'-P'-4', 1"-P"-3" and 2"-P"-4", etc. for different intervals of time. Then, by employing the definition of fluid rotation, the rotationality at the point P can be studied by examining the slopes of the tangents of the curvilinear lines passing through P, which were originally perpendicular.

A numerical example, to show that the potential vortex motion is irrotational, is given below, in which the motion is described in the time interval Δt.

Let $r = 1$ and $\Delta\theta = 12^\circ$, then

$$\theta_1 = \theta_4 = -6^\circ$$

$$\theta_2 = \theta_3 = 6^\circ$$

$$r_2 = r_1 = \frac{r}{\sin\frac{\Delta\theta}{2} + \cos\frac{\Delta\theta}{2}} = \frac{r}{\sin 6^\circ + \cos 6^\circ}$$

$$= \frac{1}{0.10453 + 0.99452} = \frac{1}{1.09905} = 0.909876$$

$$r_3 = r_4 = \frac{r}{\cos\frac{\Delta\theta}{2} - \sin\frac{\Delta\theta}{2}} = \frac{1}{\cos 6^\circ - \sin 6^\circ}$$

$$= \frac{1}{0.99452 - 0.10453} = \frac{1}{0.88999} = 1.123608$$

and

$$\theta_1' = \frac{\Delta\theta}{2} = 6^\circ$$

$$\theta_2' = \frac{3\,\Delta\theta}{2} = 18^\circ$$

$$\theta_3' = \frac{(3-\sin\Delta\theta)\,\Delta\theta}{2(1+\sin\Delta\theta)} = \frac{3-\sin 12^\circ}{1+\sin 12^\circ} \times 6^\circ = \frac{3-0.20791}{1+0.20791} \times 6^\circ$$

$$= \frac{2.79209}{1.20791} \times 6^\circ = 13.869024 = 13^\circ\ 52.14144'$$

$$\theta_4' = \frac{(1 - 3 \sin\Delta\theta)\Delta\theta}{2(1 + \sin\Delta\theta)} = \frac{1 - 0.62373}{1 + 0.20791} \times 6^\circ = \frac{0.37627}{1.20791} \times 6^\circ$$

$$= 1.869029 = 1^\circ\ 52.14174'$$

$$\theta' = \frac{12^\circ}{1 + \sin\Delta\theta} = \frac{12^\circ}{1.20791} = 9.934515 = 9^\circ\ 56.07090'$$

Hence,

$$x_1' = r_1 \cos\theta_1' = (0.909876) \cos 6^\circ = (0.909876)(0.99452)$$
$$= 0.904890$$
$$y_1' = r_1 \sin\theta_1' = (0.909876) \sin 6^\circ = (0.909876)(0.10453)$$
$$= 0.0951184$$

$$x_2' = r_2 \cos\theta_2' = (0.909876) \cos 18^\circ = (0.909876)(0.95106)$$
$$= 0.865347$$
$$y_2' = r_2 \sin\theta_2' = (0.909876) \sin 18^\circ = (0.909876)(0.30902)$$
$$= 0.281170$$

$$x_3' = r_3 \cos\theta_3' = (1.123608)(0.970850) = 1.090855$$
$$y_3' = r_3 \sin\theta_3' = (1.123608)(0.239701) = 0.269330$$

$$x_4' = r_4 \cos\theta_4' = (1.123608)(0.999468) = 1.123010$$
$$y_4' = r_4 \sin\theta_4' = (1.123608)(0.032611) = 0.036642$$

$$x' = r \cos\theta' = \cos\theta' = 0.984956$$
$$y' = r \sin\theta' = \sin\theta' = 0.172520$$

Let the equation of the curve passing through the points 1', P' and 3' be

$$y = ax^2 + bx + c$$

Then, the three constants in the equation can be determined by substituting the coordinates of the points 1', P' and 3' into it. There is obtained

$$0.0951184 = a(0.904890)^2 + b(0.904890) + c$$
$$= a(0.818826) + b(0.904890) + c$$
$$0.172520 = a(0.984956)^2 + b(0.984956) + c$$
$$= a(0.970138) + b(0.984956) + c$$
$$0.269330 = a(1.090855)^2 + b(1.090855) + c$$
$$= a(1.189960) + b(1.090855) + c$$

The solution of these three simultaneous equations for the constants a and b is

$$\begin{cases} a = -0.284851 \\ b = 1.505052 \end{cases}$$

Hence the slope of the tangent of the line 1'-P'-3' passing through P' is

$$\tan \phi_{t1} = \frac{dy}{dx} = 2ax + b = 2(-0.284851)(0.984956) + 1.505052$$
$$= 0.943921$$
$$\phi_{t1} = 43^\circ\ 20.856'$$
$$\beta_1 \text{ (the clockwise angle of rotation at P)} = 2^\circ\ 39.144'$$

Let the equation of the curve passing through the points 2', P' and 4' be

$$y = a'x^2 + b'x + c'$$

which results in

$$0.281170 = a'(0.865347)^2 + b'(0.865347) + c'$$
$$= a'(0.748825) + b'(0.865347) + c'$$
$$0.172520 = a'(0.984956)^2 + b'(0.984956) + c'$$
$$= a'(0.970138) + b'(0.984956) + c'$$

$$0.036642 = a'(1.123010)^2 + b'(1.123010) + c'$$

$$= a'(1.261151) + b'(1.123010) + c'$$

The solution of these three simultaneous equations for the constants a' and b' is

$$\begin{cases} a' = -0.294526 \\ b' = -0.363414 \end{cases}$$

Hence the slope of the tangent of the line 2'-P'-4' passing through P' is

$$\tan \phi_{t2} = \frac{dy}{dx} = 2a'x + b = 2(-0.294526)(0.98456) - 0.363414$$

$$= -0.943404$$

$$\phi_{t2} = 47^\circ\ 40.083'$$

β_2 (the counterclockwise angle of rotation at P')

$$= 2^\circ\ 40.083'$$

The results show that β_1 nearly equals β_2, and that, with fairly good accuracy, the fluid rotation at the point P is shown to be zero.

PRANDTL-MEYER EXPANSION

Characteristic Equations (1)

Flow with pressure waves of one family is known as simple-wave flow, corner-type flow or Prandtl-Meyer flow. Figure 9 shows a case of Prandtl-Meyer flow in which a left-running Mach wave turns the flow through the negative angle $d\theta$, with corresponding infinitesimal changes in all stream properties. Consider $d\theta$ to be an infinitesimal increment. It follows from the geometry of the figure that

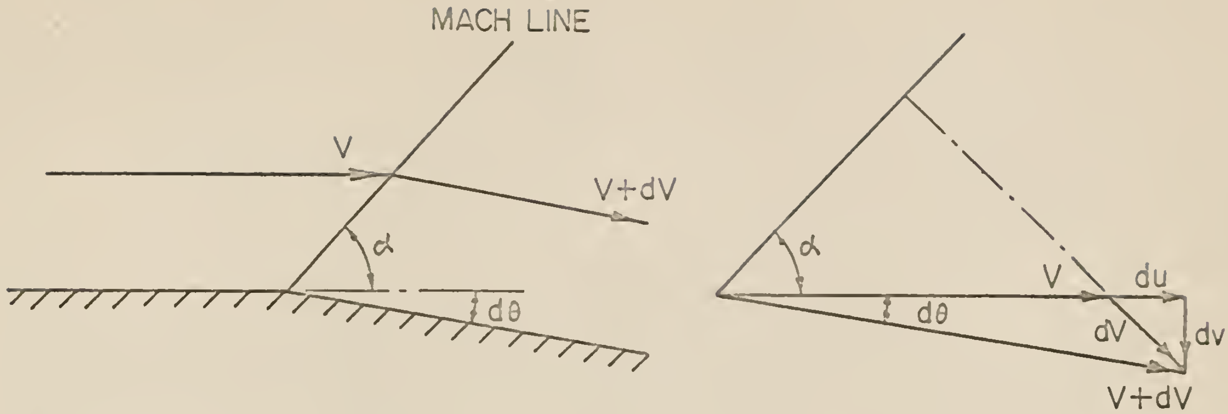

Fig. 9. Infinitesimal Mach wave.

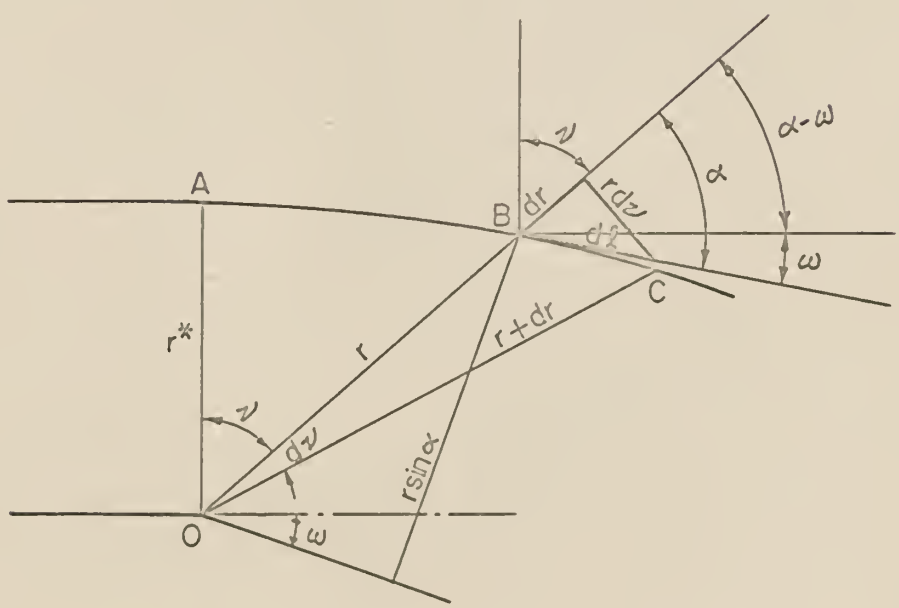

Fig. 10. Construction of streamlines for Prandtl-Meyer flow.

$$dv = - Vd\theta$$

$$du = dV$$

$$\frac{du}{dv} = \tan \alpha = \frac{1}{\sqrt{M^2 - 1}}$$

where α is the Mach angle and M is the approaching Mach number. Elimination of du and dv from this set of equations yields

$$\frac{1}{V} \frac{dV}{d\theta} = - \frac{1}{\sqrt{M^2 - 1}} \tag{13}$$

From the definition of Mach Number

$$V^2 = C^2M^2 = kRTM^2$$

$$2VdV = kRT\, d(M^2) + kRM^2\, dT$$

and

$$2\frac{dV}{V} = \frac{kRT}{V^2} d(M^2) + \frac{kRM^2}{V^2} dT = \frac{d(M^2)}{M^2} + \frac{dT}{T} \tag{14}$$

As

$$T_o = T(1 + \frac{k - 1}{2} M^2)$$

$$\frac{dT}{T} = \frac{\frac{k-1}{2} d\,(M^2)}{1 + \frac{k-1}{2} M^2} \tag{15}$$

The combination of Eqs. (13), (14) and (15) yields

$$d\theta = - \frac{\sqrt{M^2-1}\ d(M^2)}{2M^2\,(1 + \frac{k-1}{2} M^2)} \tag{16}$$

The integration of the above equation by standard methods gives

$$\theta = - \sqrt{\frac{k+1}{k-1}} \tan^{-1} \sqrt{\frac{k-1}{k+1} (M^2-1)} + \tan^{-1} \sqrt{M^2-1} + \text{constant} \tag{17}$$

It is often more convenient to work with the dimensionless velocity $M^* = \frac{V}{c^*}$ (where c^* is the acoustic velocity at Mach Number unity) rather than the Mach Number M. Using the adiabatic relation that

$$M^2 = \frac{\frac{2}{k+1} M^{*2}}{1 - \frac{k-1}{k+1} M^{*2}}$$

Eq. (17) may be put into the form

$$\theta = - \omega (M^*) + \text{constant} \tag{17a}$$

where

$$\omega (M^*) = \sqrt{\frac{k+1}{k-1}} \tan^{-1} \sqrt{\frac{M^{*2} - 1}{\frac{k+1}{k-1} - M^{*2}}} - \tan^{-1} \sqrt{\frac{M^{*2} - 1}{1 - \frac{k-1}{k+1} M^{*2}}}$$

It follows that, crossing the same family of waves with a known initial condition,

$$\theta_2 - \theta_1 = - (\omega_2 - \omega_1)$$

The streamline shapes can be examined by letting r denote the distance, measured along the Mach line, between a point on the wall and a point on a certain streamline (Fig. 10). Then the distance r^* is proportional to the minimum cross sectional area for isentropic flow, and the distance $r(\sin\alpha)$ is proportional to the cross-sectional area for flow at any other section, i.e.

$$\frac{r \sin\alpha}{r^*} = \frac{A}{A^*}$$

Since

$$\frac{A}{A^*} = \frac{1}{M} \left[\left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k+1}{2(k-1)}}$$

and

$$\sin\alpha = \frac{1}{M}$$

$$\frac{r}{r^*} = \left[\left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k+1}{2(k-1)}} \tag{18}$$

If the initial flow is not at Mach Number unity, the distances along the Mach lines are related by

$$\frac{r_2}{r_1} = \frac{M_2 \; (A/A^*)_2}{M_1 \; (A/A^*)_1} \tag{19}$$

Rotationality of Prandtl-Meyer Expansion

The Prandtl-Meyer expansion is examined from two viewpoints, one of which describes this expansion as being irrotational while the other describes the flow as being rotational.

<u>Geometry of a Differential Element in the Flow Field</u>. Referring to Fig. 10, the following expressions for the differential element in the flow field are developed.

$$\sin \alpha = \frac{1}{M}$$

$$\cos \alpha = \frac{\sqrt{M^2 - 1}}{M} \tag{20}$$

and

$$d\alpha = d\left(\sin^{-1} \frac{1}{M}\right) = \frac{-dM}{M\sqrt{M^2-1}} = \frac{-d(M^2)}{2M^2\sqrt{M^2-1}} \tag{21}$$

From Eq. (14) and

$$\frac{T_o}{T} = \left(1 + \frac{k-1}{2} M^2\right)$$

$$\frac{dV}{V} = \frac{1}{2}\left[\frac{d(M^2)}{M^2} + \frac{d(\frac{T_o}{T})}{\frac{T_o}{T}}\right] = \frac{1}{2}\left[\frac{d(M^2)}{M^2} + \frac{\frac{k-1}{2}\, d(M^2)}{1 + \frac{k-1}{2} M^2}\right]$$

$$= \frac{d(M^2)}{2M^2\,(1 + \frac{k-1}{2} M^2)} \tag{22}$$

As
$$\nu = \frac{\pi}{2} - (\alpha - \omega)$$

$$d\nu = d\omega - d\alpha \tag{23}$$

From Eq. 16

$$d\omega = -d\theta = \frac{\sqrt{M^2 - 1}\;\; d(M^2)}{2M^2\,(1 + \frac{k-1}{2} M^2)} \tag{24}$$

From Eqs. (21), (23) and (24)

$$d\nu = \frac{(k+1)\; d\,(M^2)}{4\sqrt{M^2 - 1}\;(1 + \frac{k-1}{2} M^2)} \tag{25}$$

The logarithmic differentiation of Eq. (18) gives

$$\frac{dr}{r} = d\ln\left(\frac{r}{r^*}\right) = d\;\ln\left[\left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k+1}{2(k-1)}}$$

$$= \frac{\frac{k+1}{2(k-1)} \cdot \frac{k-1}{2}\, d(M^2)}{1 + \frac{k-1}{2} M^2} = \frac{(k+1)\; d\,(M^2)}{4(1 + \frac{k-1}{2} M^2)}$$

From the geometry of Fig. 10, the length $d\ell$ on the curved path may be expressed by

$$d\ell = \frac{r\,d\nu}{\sin\alpha}$$

After suitable substitution and simplification of the corresponding terms, there results

$$d\ell = \frac{r^* M\,\left(\frac{2}{k+1}\right)^{\frac{3-k}{2(k-1)}}\left(1 + \frac{k-1}{2} M^2\right)^{\frac{3-k}{2(k-1)}}\; d(M^2)}{2\sqrt{M^2 - 1}} \tag{26}$$

Irrotationality Based on the Concept of the Circulation Around a Closed Path. Since the velocity is constant in the direction of flow before and after the Prandtl-Meyer corner flow, the flow in these two regions must be irrotational. The problem now is to examine the circulation around a certain closed path within the Prandtl-Meyer fan. As shown in Fig. 10, the closed path OCB is chosen to be the test element. In accordance with the definition of circulation

$$d\Gamma_{OCB} = (V+dV)\cos(\alpha+d\alpha)(r+dr) - V\cos\alpha(r) - \frac{V+(V+dV)}{2}\,d\ell$$

$$= (V+dV)(\cos\alpha\cos d\alpha - \sin\alpha\sin d\alpha)(r+dr) - rV\cos\alpha - Vd\ell - \frac{dVd\ell}{2}$$

$$= rV\cos\alpha\cos d\alpha + rdV\cos\alpha\cos d\alpha + Vdr\cos\alpha\cos d\alpha + drdV\cos\alpha\cos d\alpha - rV\sin\alpha\sin d\alpha - rdV\sin\alpha\sin d\alpha - Vdr\sin\alpha\sin d\alpha - drdV\sin\alpha\sin d\alpha - rV\cos\alpha - Vd\ell - \frac{dVd\ell}{2}$$

When $d\alpha$ is very small, $\cos d\alpha$ approaches unity and $\sin d\alpha$ approaches $d\alpha$. In the mean time, all the terms higher than second order may be neglected. It will be found that

$$d\Gamma_{OCB} = rdV\cos\alpha + Vdr\cos\alpha - rV\sin\alpha\,d\alpha - Vd\ell$$

$$\frac{d\Gamma_{OCB}}{rV} = \frac{dV}{V}\cos\alpha + \frac{dr}{r}\cos\alpha - \sin\alpha\,d\alpha - \frac{d\ell}{r}$$

$$= \frac{dV}{V}\cos\alpha + \frac{dr}{r}\cos\alpha - \sin\alpha\,d\alpha - \frac{\frac{rd\nu}{\sin\alpha}}{r}$$

$$= \frac{dV}{V}\cos\alpha + \frac{dr}{r}\cos\alpha - \sin\alpha\,d\alpha - \frac{d\nu}{\sin\alpha}$$

$$= \frac{d(M^2)}{2M^2(1+\frac{k-1}{2}M^2)}\frac{\sqrt{M^2-1}}{M} + \frac{(k+1)d(M^2)}{4(1+\frac{k-1}{2}M^2)}\frac{\sqrt{M^2-1}}{M}$$

$$-\frac{1}{M}\frac{-d(M^2)}{2M^2\sqrt{M^2-1}} - M\frac{(k+1)d(M^2)}{4\sqrt{M^2-1}\,(1+\frac{k-1}{2}M^2)}$$

$$= \frac{2(M^2-1)d(M^2)+M^2(M^2-1)(k+1)d(M^2)+2(1+\frac{k-1}{2}M^2)d(M^2)-M^4(k+1)d(M^2)}{4M^3\sqrt{M^2-1}\,(1+\frac{k-1}{2}M^2)}$$

$$= \frac{[2M^2-2+M^4(k+1)-kM^2-M^2+2+kM^2-M^2-M^4(k+1)]\ d(M^2)}{4M^3\sqrt{M^2-1}\ (1+\frac{k-1}{2}M^2)}$$

$$= 0$$

The result shows that the circulation around a closed path in the Prandtl-Meyer fan is also zero.

Rotationality Based on the Concept of the Rotation of a Fluid Particle that Passed Through the Prandtl-Meyer Fan. Consider the two-dimensional, square, control volume aebcfd of Fig. 11 which passes through the Prandtl-Meyer fan to position a'e'b'c'f'd'. Let V be the velocity at any time along a certain streamline in the flow passage. Then $V = \frac{d\ell}{dt}$ and also

$$V = Mc = M\sqrt{kRT} = M\sqrt{\frac{kRT_o}{\frac{T_o}{T}}} = M\sqrt{\frac{kRT_o}{1+\frac{k-1}{2}M^2}}$$

The combination of the above two relations gives

$$dt = \frac{d\ell}{M\sqrt{\frac{kRT_o}{1+\frac{k-1}{2}M^2}}} = \frac{\sqrt{1+\frac{k-1}{2}M^2}\ d\ell}{M\sqrt{kRT_o}}$$

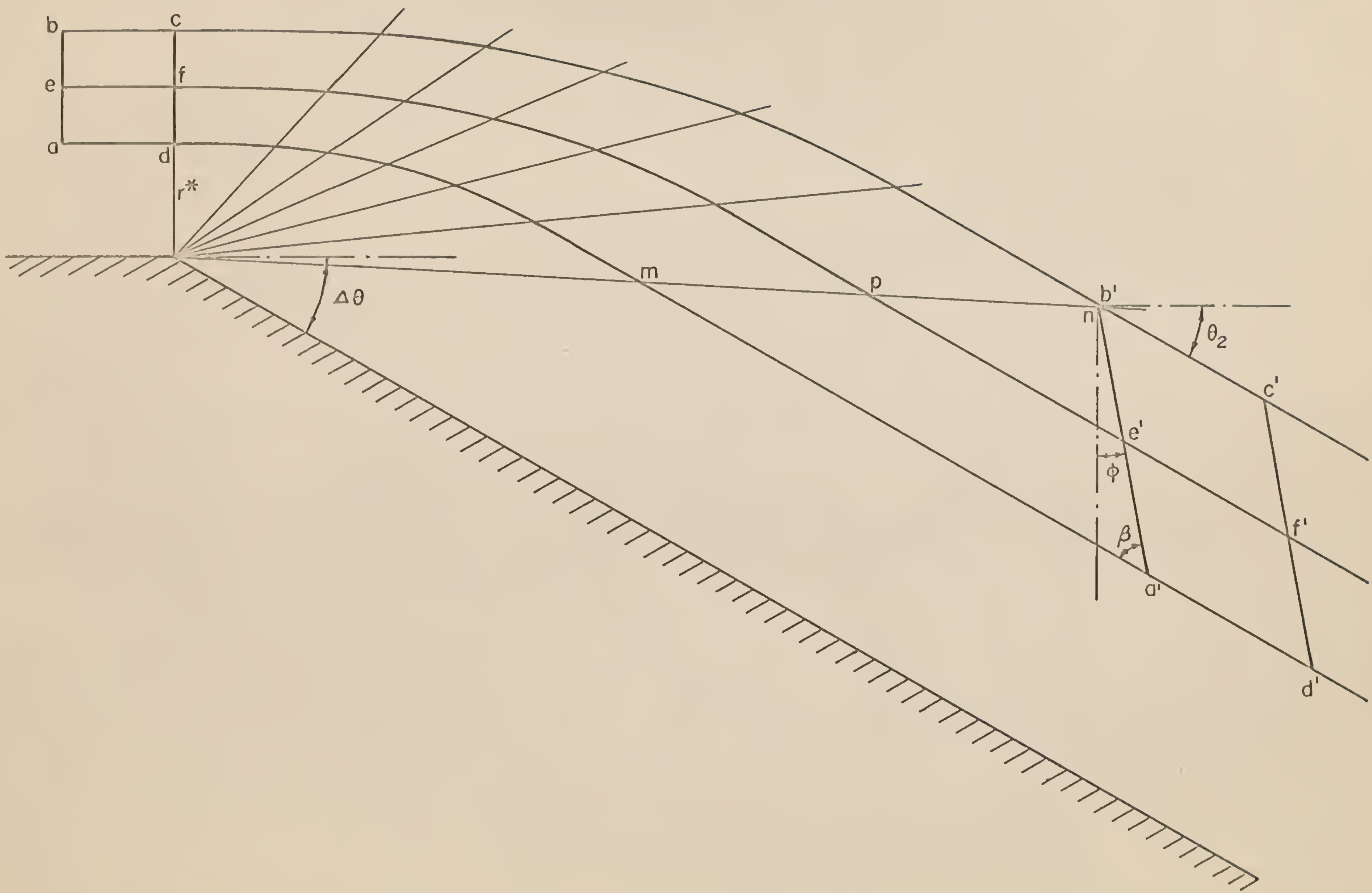

Fig. 11. Rotationality of Prandtl-Meyer flow.

Where $d\ell$ can be replaced by the expression from Eq. (26), hence

$$dt = \frac{r^*\left(\frac{2}{k+1}\right)^{\frac{3-k}{2(k-1)}} \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}} d(M^2)}{2\sqrt{kRT_o}\ \sqrt{M^2 - 1}}$$

$$= C_1 \times \frac{\left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}} d\,(M^2)}{\sqrt{M^2 - 1}} \qquad (27)$$

where

$$C_1 = \frac{\left(\frac{2}{k+1}\right)^{\frac{3-k}{2(k-1)}}}{2\sqrt{kR}} \cdot \frac{r^*}{T_o}$$

For $k = 1.4$ and $R = 1716$ $\text{ft-lb}_f/\text{slug}\ ^oF$,

$$C_1 = 7.0847 \times 10^{-3} \frac{r^*}{\sqrt{T_o}} \qquad (28)$$

The integration of Eq. (27) yields (4)

$$t = C_1\left\{\sqrt{M^2-1}\ \sqrt{1+0.2\ M^2}\left[\tfrac{1}{3}(1+0.2\ M^2)^2 + \tfrac{1}{2}(1+0.2\ M^2) + 0.9\right] + \frac{1.08}{0.2}\ln\left[\sqrt{0.2(M^2-1)} + \sqrt{1+0.2\ M^2}\right]\right\} + \text{constant} \qquad (29)$$

which is the time required for a particle to travel from the upstream side of the Prandtl-Meyer fan, having an approach Mach Number of unity, to any other position in the fan.

As the general solution of the rotation of a fluid particle, that has passed through the Prandtl-Meyer fan, is complicated, a numerical example is presented below to show how Eq. (29) can be used to determine the rotation of a fluid particle as it passes through the Prandtl-Meyer fan.

Given $r^* = 1$ ft, $k = 1.4$, $T_o = 1{,}000^{\circ}R$

M_1 = Initial Mach Number = 1.0

M_2 = Final Mach Number = 2.1339

Since

$$\frac{r}{r^*} = \left[\left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k+1}{2(k-1)}}$$

$$= \left[\left(\frac{2}{2.4}\right)\left(1 + 0.2 M^2\right)\right]^{\frac{2.4}{2(0.4)}}$$

$$= \frac{(1 + 0.2 M^2)^3}{1.728}$$

and $\nu = \frac{\pi}{2} - (\alpha - \omega)$, the flow patterns are plotted on Fig. 11 by employing the data in Table 1.

Table 1. Calculation data for the streamline configuration in Prandtl-Meyer fan.

M	r/r^*	ω	α	ν
1.0000	1.00000	0	90	0
1.2565	1.31821	5	52.7383	42.2617
1.4393	1.63718	10	44.1770	55.8230
1.6047	2.01234	15	38.5474	66.4526
1.7750	2.50677	20	34.2904	75.7096
1.9503	3.15888	25	30.8469	84.1531
2.1339	4.03682	30	27.9451	92.9451

Substituting the given conditions into Eqs. (28) and (29) and performing appropriate calculations, the results are

$$C_1 = 2.240377$$

and t_{dm} (the time for point d to traverse the Prandtl-Meyer fan)

$$= 2.179589 \times 10^{-3} \text{ seconds}$$

It follows that

t_{fp} (the time for point f to traverse the Prandtl-Meyer fan)

$$= 1.5t_{dm} = 3.269383 \times 10^{-3} \text{ seconds}$$

t_{cn} (the time for point c to traverse the Prandtl-Meyer fan)

$$= 2t_{dm} = 4.359178 \times 10^{-3} \text{ seconds}$$

Since

$$V = M\sqrt{\frac{kRT_o}{1 + \frac{k-1}{2} M^2}} \quad ,$$

the corresponding velocity for $M_1 = 1.0$ and $M_2 = 2.1339$ are $V_1 = 1414.92$ and $V_2 = 2392.81$ feet per second. Hence

$$t_{bc} = t_{ef} = t_{ad} = \frac{1}{1414.92} = 0.706753 \times 10^{-3} \text{ seconds}$$

and $t_{bb'}$ (the time for point b to traverse the Prandtl-Meyer fan)

$$= t_{cn} + t_{bc} = 5.065913 \times 10^{-3} \text{ seconds}$$

Let $\ell_{nc'}$, $\ell_{pf'}$, $\ell_{md'}$, $\ell_{pe'}$ and $\ell_{ma'}$ be the distances traveled beyond the Prandtl-Meyer fan by the points c, f, d, e and a, respectively, in time $t_{bb'}$. They are

$$\ell_{nc'} = \frac{V_2}{V_1} = \frac{2392.81}{1414.92} = 1.69114 \text{ ft.}$$

$$\ell_{pf'} = (t_{bb'} - t_{fp})V_2 = (5.065931 - 3.269383) \times 10^{-3}(2392.81)$$
$$= 4.29880 \text{ ft.}$$

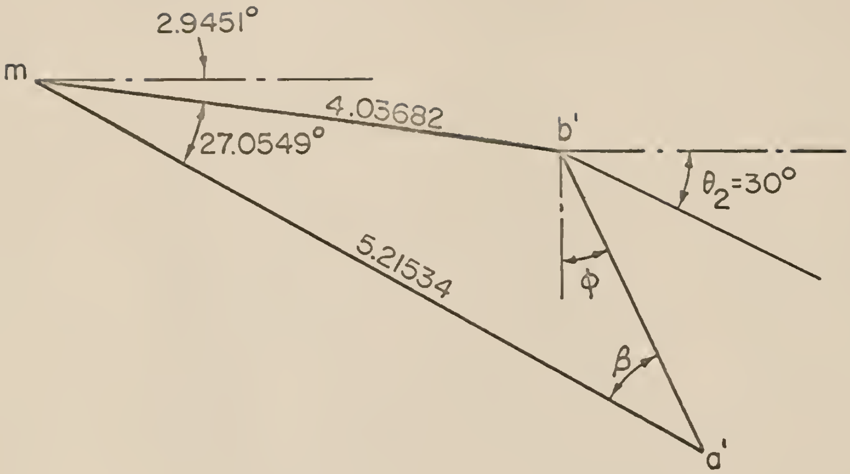

Fig. 11a. Angle of rotation for Prandtl-Meyer flow.

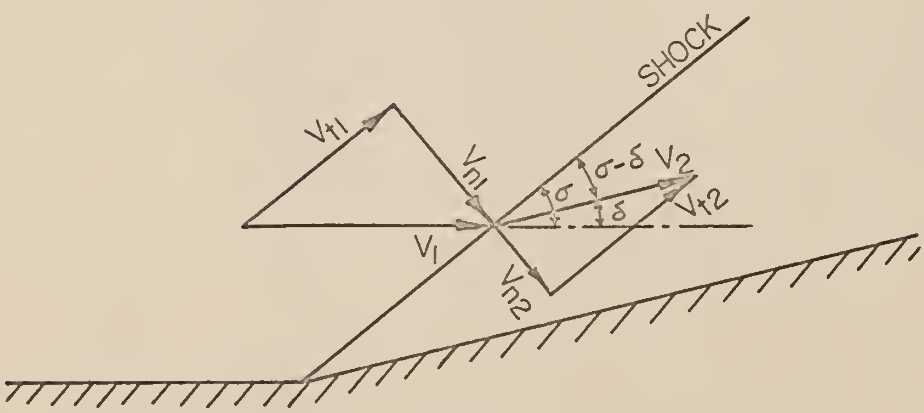

Fig. 12. Oblique shock.

$$\ell_{md'} = (t_{bb'} - t_{dm})V_2 = 6.90647 \text{ ft.}$$

$$\ell_{pc'} = (t_{bb'} - t_{fp} - t_{ef})V_2 = 2.60766 \text{ ft.}$$

and

$$\ell_{ma'} = (t_{bb'} - t_{dm} - t_{ad})V_2 = 5.21533 \text{ ft.}$$

Referring to Fig. 11, the rotation of the points a, b, c, d, e and f after traversing the Prandtl-Meyer fan is

$$\omega_z = \frac{1}{2}(\phi - \theta_2)$$

Consider the triangle a'mb' as shown in Fig. 11a.

$$\tan \beta = \frac{(4.03682) \sin 27.0549^\circ}{5.21534 - (4.03682) \cos 27.0549^\circ} = 1.133215$$

$$= 48^\circ\ 34.45'$$

and

$$\phi = \frac{\pi}{2} - \theta_2 - \beta = 11^\circ\ 25.55'$$

Finally, the angle of rotation for all points is

$\omega_z = -\ 9^\circ\ 17.225'$ (in the clockwise direction)

OBLIQUE SHOCK

Governing Equations (Fig. 12) (1)

Equation of Continuity.

$$\rho_1 V_{n1} = \rho_2 V_{n2} \tag{30}$$

Momentum Equations.

$(\rho_1 V_{n1})V_{t1} = (\rho_2 V_{n2})V_{t2}$ in tangential direction

$$\therefore\ V_{t1} = V_{t2} \tag{31}$$

$p_1 - p_2 = \rho_2 V_{n2}^2 - \rho_1 V_{n1}^2$ in normal direction (32)

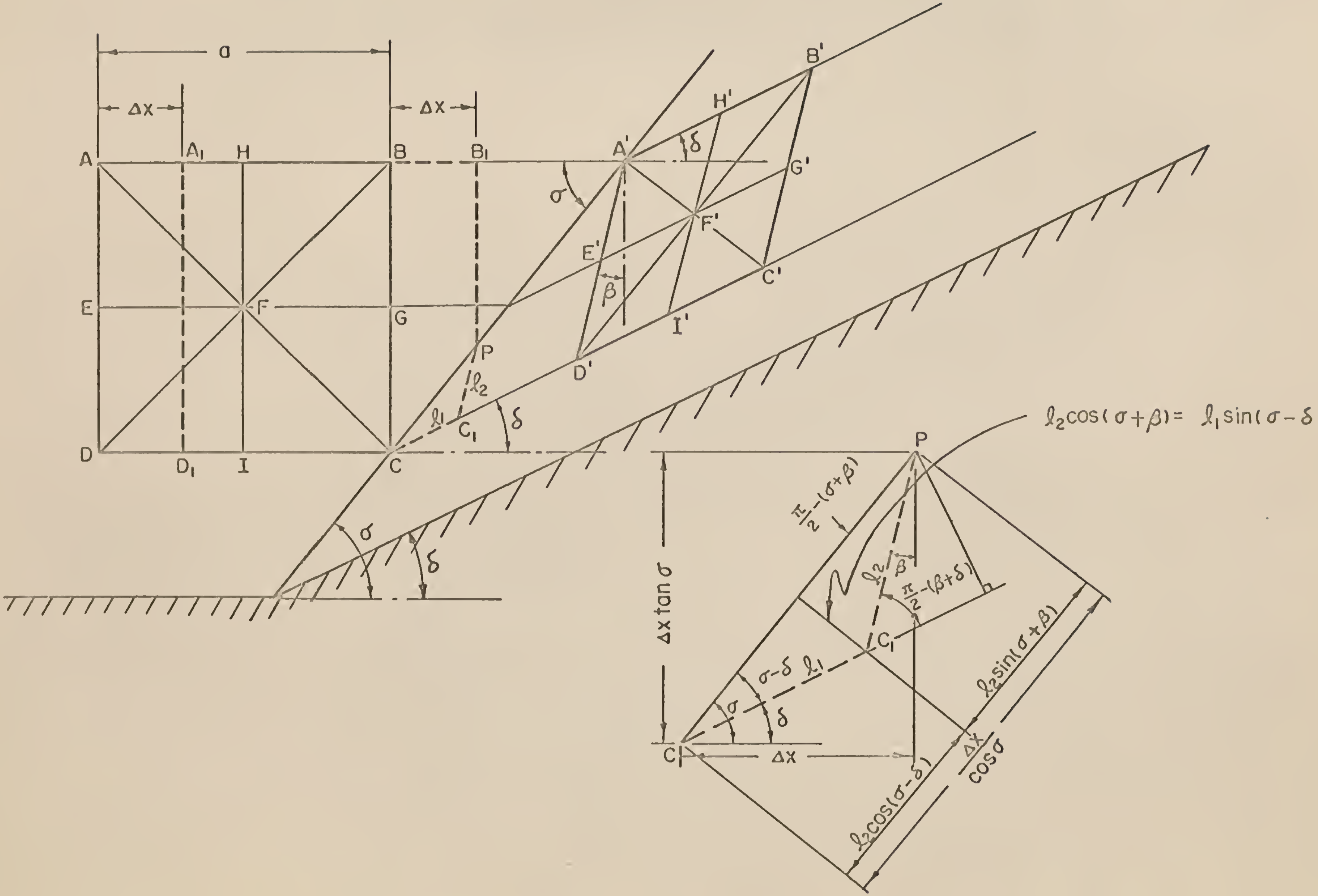

Fig. 13. A two-dimensional square control volume passing through the oblique shock.

Energy Equation.

$$c_p(T_1 - T_2) = (V_2^2 - V_1^2)/2 \tag{33}$$

Ratio of Downstream to Upstream Density.

$$\frac{\rho_2}{\rho_1} = \frac{\tan \sigma}{\tan(\sigma - \delta)} \tag{34}$$

Ratio of Downstream to Upstream Pressure.

$$\frac{p_2}{p_1} = \left(\frac{k+1}{k-1}\ \frac{\rho_2}{\rho_1} - 1\right) \Big/ \left(\frac{k+1}{k-1} - \frac{\rho_2}{\rho_1}\right) \tag{35}$$

Ratio of Downstream to Upstream Velocity.

$$\frac{V_2}{V_1} = \frac{\cos \sigma}{\cos(\sigma - \delta)} \tag{36}$$

Entropy Change Across Oblique Shock.

$$\frac{\Delta S}{R} = \frac{1}{k-1}\left(\ln \frac{p_2}{p_1} - k \ln \frac{\rho_2}{\rho_1}\right) \tag{37}$$

Rotationality of Flow Passing Through an Oblique Shock

Irrotationality Based on the Concept of Circulation Around a Closed Path. Since the velocity is constant in the direction of flow before and after the shock, the flow therefore must be irrotational. Now, consider a case when part of the control volume ABCD (Fig. 13) has passed through the shock. Referring to Fig. 13, A_1, B_1, C_1 and D_1 are the corresponding new position of the points A, B, C and D when the flow proceeds a certain distance Δx. First, consider circulation around the closed path BB_1PC.

$$\Gamma_{BB_1PC} = (V_1 \cos\sigma)\left(\frac{\Delta x}{\cos\sigma}\right) - V_1\ \Delta x = 0$$

Secondly, consider circulation around the closed path CPC_1.

$$\Gamma_{CPC_1} = V_2\, \ell_1 + V_2 \cos\left[\frac{\pi}{2} - (\beta+\delta)\right]\ell_2 - V_2 \cos(\sigma-\delta)\frac{\Delta x}{\cos\sigma}$$

$$= V_2\,\ell_1 + V_2\,\ell_2 \sin(\beta+\delta)$$
$$- V_2 \cos(\sigma-\delta)\left[\ell_1 \cos(\sigma-\delta) + \ell_2 \sin(\sigma+\beta)\right]$$

$$= V_2\,\ell_1 + V_2\,\ell_2 \sin(\beta+\delta) - V_2\,\ell_1 \cos^2(\sigma-\delta)$$
$$- V_2\,\ell_2 \cos(\sigma-\delta) \sin(\sigma+\beta)$$
$$= V_2\,\ell_1 + V_2\,\ell_2 \sin(\beta+\delta) - V_2\,\ell_1\left[1 - \sin^2(\sigma-\delta)\right]$$
$$- V_2\,\ell_2 \cos(\sigma-\delta) \sin(\sigma+\beta)$$
$$= V_2\,\ell_2 \sin(\beta+\delta) + V_2\,\ell_1 \sin^2(\sigma-\delta)$$
$$- V_2\,\ell_2 \cos(\sigma-\delta) \sin(\sigma+\beta)$$
$$= V_2\,\ell_2 \sin(\beta+\delta) + V_2 \sin(\sigma-\delta)\left[\ell_2 \cos(\sigma+\beta)\right]$$
$$- V_2\,\ell_2 \cos(\sigma-\delta) \sin(\sigma+\beta)$$
$$= V_2\,\ell_2 \sin(\beta+\delta) - V_2\,\ell_2\left[\sin(\sigma+\beta)\cos(\sigma-\delta)\right.$$
$$\left.- \cos(\sigma+\beta)\sin(\sigma-\delta)\right]$$
$$= V_2\,\ell_2 \sin(\beta+\delta) - V_2\,\ell_2 \sin(\beta+\delta) = 0$$

Finally, the circulation around $A_1B_1C_1D_1$ is

$$\Gamma_{A_1B_1C_1D_1} = V_1(a - \Delta x) + V_2\,\ell_1 + V_2 \sin(\beta+\delta)\,\ell_2 - V_1 a$$
$$= - V_1\ \Delta x + V_2\,\ell_1 + V_2\,\ell_2 \sin(\beta+\delta)$$
$$= - V_1\ \Delta x + V_2\,\ell_1 + V_2\,\ell_2 \sin\left[(\sigma+\beta) - (\sigma-\delta)\right]$$
$$= - V_1\ \Delta x + V_2\,\ell_1 + V_2\,\ell_2\left[\sin(\sigma+\beta)\cos(\sigma-\delta)\right.$$
$$\left.- \cos(\sigma+\beta)\sin(\sigma-\delta)\right]$$
$$= - V_1 \Delta x + V_2\,\ell_1 + V_2\,\ell_2 \sin(\sigma+\beta)\cos(\sigma-\delta)$$
$$- V_2\left[\ell_2 \cos(\sigma+\beta)\right]\sin(\sigma-\delta)$$

$$= - V_1 \Delta x + V_2 \ell_1 \cos^2 (\sigma - \delta)$$
$$+ V_2 \ell_2 \sin (\sigma + \beta) \cos (\sigma - \delta)$$
$$= - V_1 \Delta x + V_2 \left[\ell_1 \cos^2 (\sigma - \delta)\right.$$
$$\left. + \ell_2 \sin (\sigma + \beta) \cos (\sigma - \delta)\right]$$
$$= - V_1 \Delta x + \frac{V_1 \cos \sigma}{\cos (\sigma - \delta)} \left[\ell_1 \cos^2 (\sigma - \delta)\right.$$
$$\left. + \ell_2 \sin (\sigma + \beta) \cos (\sigma - \delta)\right]$$
$$= - V_1 \Delta x + V_1 \cos \sigma \left[\ell_1 \cos (\sigma - \delta) + \ell_2 \sin (\sigma + \beta)\right]$$
$$= - V_1 \Delta x + V_1 \cos \sigma \cdot \frac{\Delta x}{\cos \sigma} = 0$$

Thus the circulation around each subarea of total area BB_1PC_1C is zero, and the circulation around the total area is likewise zero.

Rotationality Based on the Concept of the Rotation for a Fluid Particle. Referring to Fig. 13, the rotationality of a square control volume ABCD in the two-dimensional flow passage passing through an oblique shock will be investigated. In accordance with the definition of fluid rotation at a point, the only case for which the rotation at the points A, B, C etc. after the shock equals to zero is that $\beta = \delta$ as shown in the figure.

Let V_1 = the velocity before the shock

V_2 = the velocity after the shock

Since $t_{BA'} = t_{CD'}$ $\qquad \frac{\ell_{BA'}}{V_1} = \frac{\ell_{CD'}}{V_2}$

i.e. $\quad \ell_{CD'} = \ell_{BA'} \frac{V_2}{V_1}$

It follows that

$$\cot\beta = \frac{\ell_{CB} - \ell_{CD'}\sin\delta}{\ell_{BA'} - \ell_{CD'}\cos\delta} = \frac{\ell_{BA'}\tan\sigma - \ell_{BA'}\dfrac{V_2}{V_1}\sin\delta}{\ell_{BA'} - \ell_{BA'}\dfrac{V_2}{V_1}\cos\delta}$$

$$\frac{\tan\sigma - \dfrac{V_2}{V_1}\sin\delta}{1 - \dfrac{V_2}{V_1}\cos\delta}$$

The elimination of V_1/V_2 from Eq. (36) yields

$$\cot\beta = \frac{\tan\sigma - \dfrac{\cos\sigma}{\cos(\sigma-\delta)}\sin\delta}{1 - \dfrac{\cos\sigma}{\cos(\sigma-\delta)}\cos\delta}$$

$$= \frac{\tan\sigma\cos(\sigma-\delta) - \cos\sigma\sin\delta}{\cos(\sigma-\delta) - \cos\sigma\cos\delta}$$

$$= \frac{\tan\sigma(\cos\sigma\cos\delta + \sin\sigma\sin\delta) - \cos\sigma\sin\delta}{\cos\sigma\cos\delta + \sin\sigma\sin\delta - \cos\sigma\cos\delta}$$

$$= \frac{\sin\sigma\cos\delta + \dfrac{\sin\delta}{\cos\sigma}(\sin^2\sigma - \cos^2\sigma)}{\sin\sigma\sin\delta}$$

$$= \cot\delta + \frac{2\sin^2\sigma - 1}{\sin\sigma\cos\sigma} = \cot\delta - 2\cot(2\sigma) \qquad (38)$$

Equation (38) shows that there are two cases for which all fluid particles in the control volume ABCD undergo no rotation in passing through the shock wave:

(1) $\cot 2\sigma = 0$, which means $\sigma = 45^\circ$, and

(2) the trivial case of $\delta = 0^\circ$.

For cases in which $\beta \neq \delta$, the oblique shock will cause a rotation such that

$$\omega_z = \text{angle of rotation after the oblique shock} = \tfrac{1}{2}(\delta - \beta) \qquad (39)$$

where ω_z can be positive or negative. Positive ω_z gives a counterclockwise rotation while negative ω_z gives a clockwise rotation.

Working Charts.

(1) Angle of rotation ω_z for known shock angle σ and wall angle δ :

By employing Eqs. (38) and (39), the ordinate ω_z was plotted against the abscissa σ , for various values of δ , on Fig. 14. A sample calculation is given below:

Assume $\sigma = 15^\circ$ and $\delta = 5^\circ$. From Eq. (38),

$$\cot \beta = \cot 5^\circ - 2 \cot 30^\circ = 11.430 - 2(1.7321) = 7.9658$$

$$\therefore \quad \beta = 7.1573^\circ$$

Hence, the angle of rotation as obtained from Eq. (39) is

$$\omega_z = \frac{5 - 7.1573}{2} = -1.07856^\circ \text{ (in the clockwise direction)}$$

(2) Entropy change $\frac{\Delta s}{R}$ across shocks with different shock angle σ and wall angle δ : By employing data for $\frac{p_2}{p_1}$ and $\frac{\rho_2}{\rho_1}$ for oblique shocks as given in Keenan and Kaye's "Gas Tables " (5) (and employing Eqs. (34) and (35) for $\delta > 30^\circ$, as Keenan and Kaye did not tabulate data beyond $\delta = 30^\circ$), values of $\frac{\Delta s}{R}$ were calculated from Eq. (37) and plotted on Fig. 15. A sample calculation is given below:

Assume $\sigma = 15^\circ$ and $\delta = 5^\circ$. From Tables 56, 58 and 59, Keenan and Kaye's Gas Tables, the following data were found to be

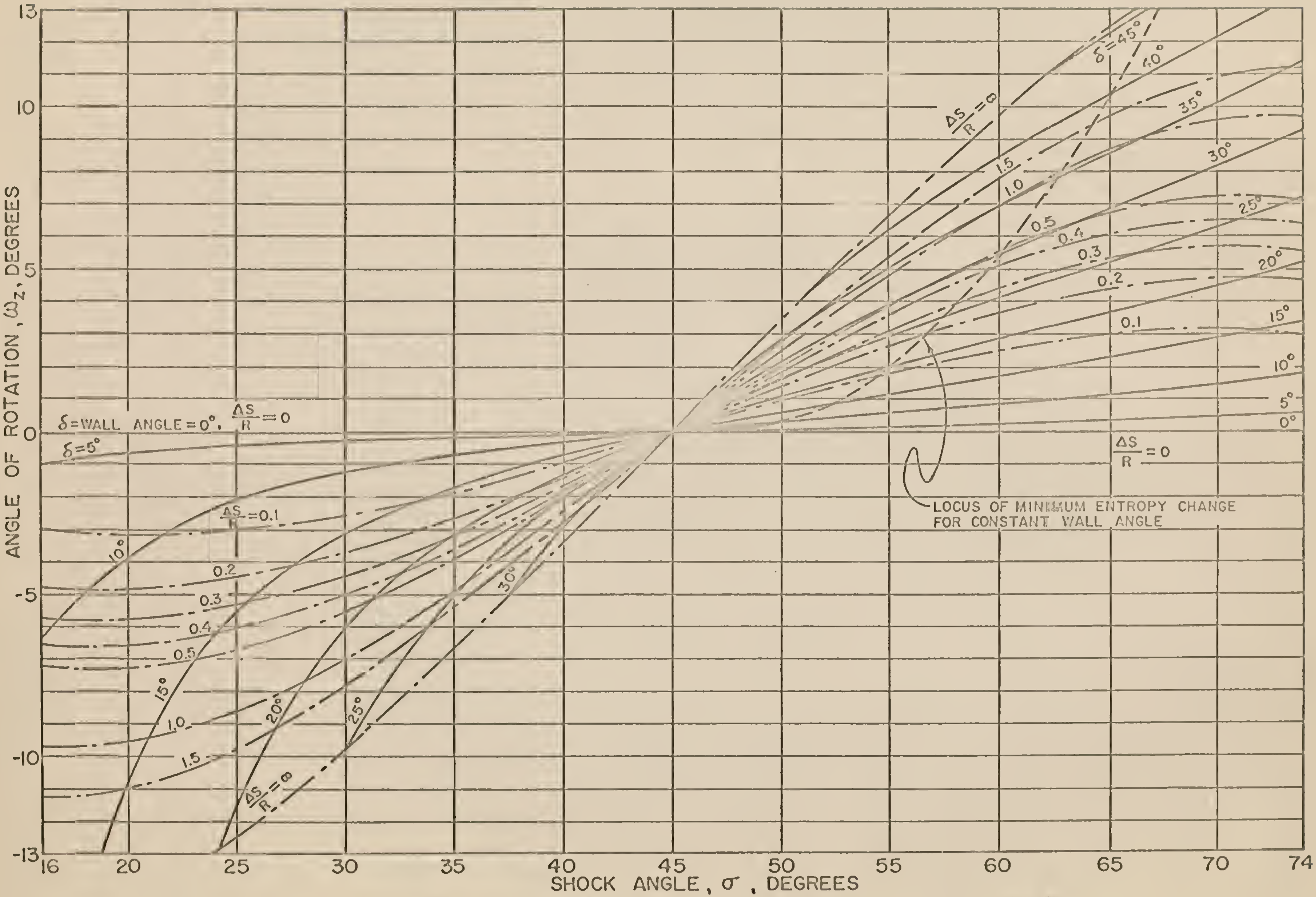

Fig. 14. Rotationality and entropy change of a fluid particle passing through oblique shocks.

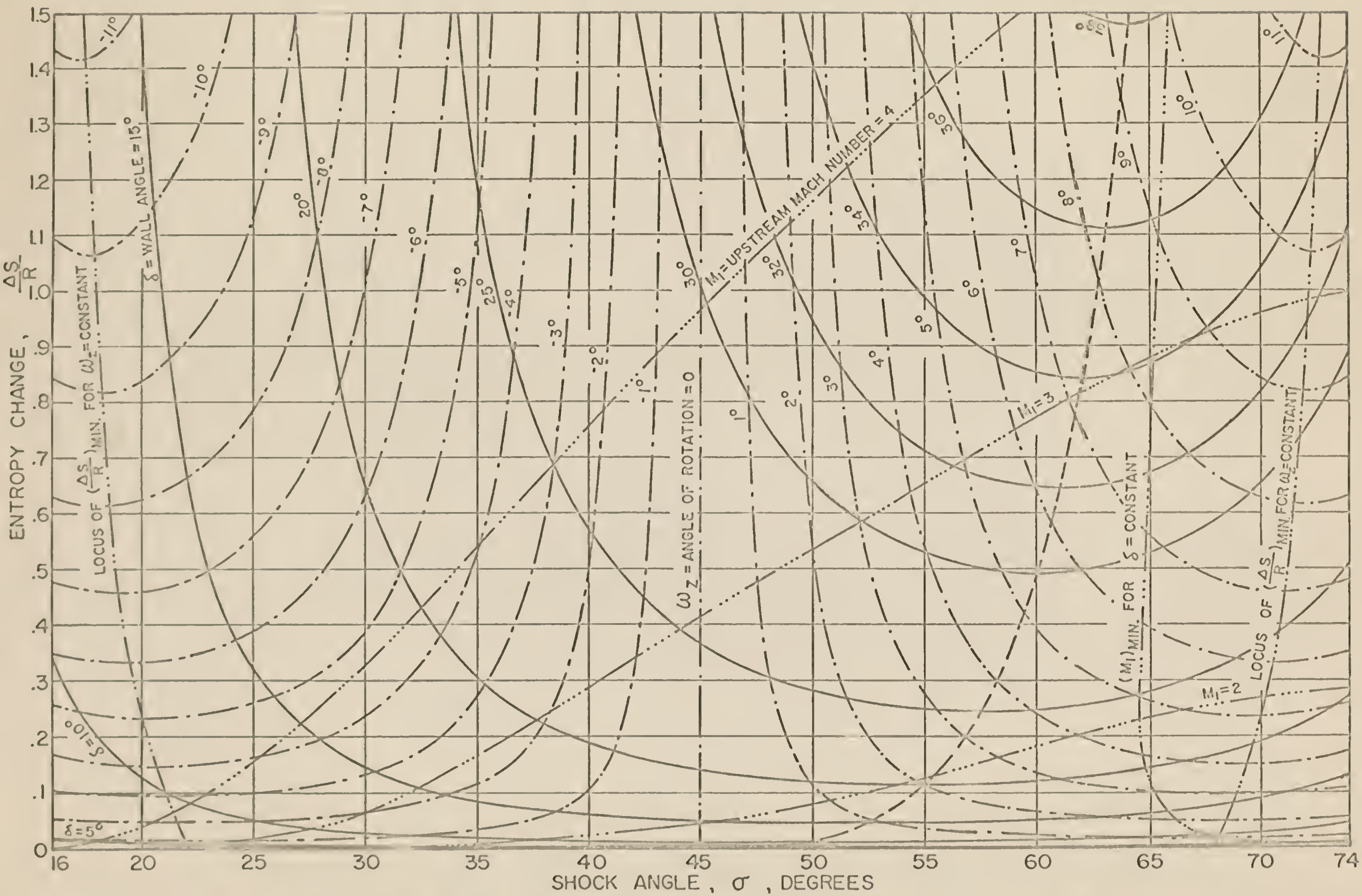

Fig. 15. Entropy change and rotationality of a fluid particle passing through oblique shocks.

M_1 = upstream Mach Number = 5.032

$p_2/p_1 = 1.812$ and $\rho_2/\rho_1 = 1.502$

Then, the numerical value of $\frac{\Delta s}{R}$ was obtained by substituting the above data into Eq. (37), i.e.

$$\frac{\Delta s}{R} = \frac{1}{1.4 - 1}\left[\ln (1.812) - (1.4)\ln (1.520)\right] = 0.02063$$

Since each constant δ curve in the figure shows that a certain minimum entropy exists at a certain value of σ, it will be convenient to set the partial derivative of $\frac{\Delta s}{R}$ with respect to σ equal to zero and plot the locus of minimum entropy at constant δ. Eq. (37) gives

$$\frac{\Delta s}{R} = \frac{1}{k-1}\left(\ln \frac{p_2}{p_1} - k \ln \frac{2}{1}\right)$$

$$\therefore \quad \frac{\partial\left(\frac{\Delta s}{R}\right)}{\partial \sigma} = \frac{1}{k-1}\left[\frac{p_1}{p_2}\,\frac{\partial\left(\frac{p_2}{p_1}\right)}{\partial \sigma} - k\,\frac{\rho_1}{\rho_2}\,\frac{\partial\left(\frac{\rho_2}{\rho_1}\right)}{\partial \sigma}\right]$$

$\frac{\partial\left(\frac{p_2}{p_1}\right)}{\partial \sigma}$ can be obtained from Eq. (35)

$$\frac{\partial\left(\frac{p_2}{p_1}\right)}{\partial \sigma} = \frac{\left(\frac{k+1}{k-1} - \frac{\rho_2}{\rho_1}\right)\left(\frac{k+1}{k-1}\right)\frac{\partial\left(\frac{\rho_2}{\rho_1}\right)}{\partial \sigma} + \left(\frac{k+1}{k-1}\,\frac{\rho_2}{\rho_1} - 1\right)\frac{\partial\left(\frac{\rho_2}{\rho_1}\right)}{\partial \sigma}}{\left(\frac{k+1}{k-1} - \frac{\rho_2}{\rho_1}\right)^2}$$

The combination of the above two equations shows that when

$$\frac{\partial\left(\frac{\rho_2}{\rho_1}\right)}{\partial \sigma} = 0 \quad , \qquad \frac{\partial\left(\frac{\Delta s}{R}\right)}{\partial \sigma} = 0$$

Consequently, the result shows that when $\dfrac{\partial(\frac{\rho_2}{\rho_1})}{\partial\sigma} = 0$, $\dfrac{\Delta s}{R}$ will be a minimum. From Eq. (34)

$$\frac{\rho_2}{\rho_1} = \frac{\tan\sigma}{\tan(\sigma-\delta)}$$

$$\frac{\partial(\frac{\rho_2}{\rho_1})}{\partial\sigma} = \frac{\tan(\sigma-\delta)\sec^2\sigma - \tan\sigma\sec^2(\sigma-\delta)}{\tan^2(\sigma-\delta)} = 0$$

The above expression yields $\dfrac{\sin(\sigma-\delta)}{\cos\sigma} = \dfrac{\sin\sigma}{\cos(\sigma-\delta)}$

The only solution of this equation has to be

$$\frac{\sin(\sigma-\delta)}{\cos\sigma} = \frac{\sin\sigma}{\cos(\sigma-\delta)} = 1$$

The final result is $\quad 2\sigma - \delta = \dfrac{\pi}{2}$ (40)

The locus of $(\frac{\Delta s}{R})_{min}$ for constant δ were plotted on Fig. 15 by using Eq. (40).

For k = 1.4, Eq. (35) yields

$$\frac{p_2}{p_1} = \frac{6(\frac{\rho_2}{\rho_1}) - 1}{6 - \frac{\rho_2}{\rho_1}} \tag{41}$$

When ρ_2/ρ_1 approaches 6, p_2/p_1 approaches infinity. This will cause the entropy $\frac{\Delta s}{R}$ to approach infinity, also. Thus when $\rho_2/\rho_1 = 6$, Eqs. (34) and (35) result in $\sigma = 67.80^\circ$, which is the asymptote of the locus of $(\frac{\Delta s}{R})_{min}$.

(3) Constant ω_z lines on $(\frac{\Delta s}{R})-\sigma$ coordinates: From Eqs. (38) and (39), there is obtained

$$\cot (2\sigma) = \tfrac{1}{2}\left[\cot\delta - \cot(\delta - 2\omega_z)\right] \tag{42}$$

The substitution of Eqs. (34) and (41) into Eq. (37) yields

$$\frac{\Delta s}{R} = 2.5\ln\left[\frac{6\tan\sigma - \tan(\sigma-\delta)}{6\tan(\sigma-\delta) - \tan\sigma}\right] - 3.5\ln\left[\frac{\tan\sigma}{\tan(\sigma-\delta)}\right] \tag{43}$$

The entropy change $\frac{\Delta s}{R}$ can be defined then by the known angle of rotation ω_z and δ. Therefore, the constant ω_z line can be plotted on $(\frac{\Delta s}{R})-\sigma$ coordinates as shown in Fig. 15. A sample calculation is shown below:

Given $\omega_z = -1^\circ$ and $\delta = 20^\circ$

$$\cot 2\sigma = \tfrac{1}{2}\left[\cot 20^\circ - \cot 22^\circ\right] = \tfrac{1}{2}(2.7475 - 2.4751)$$

$$= \tfrac{1}{2}(0.2724) = 0.1362$$

$$\therefore \quad \sigma = 41.12^\circ$$

Hence $$\frac{\Delta s}{R} = 2.5\ln\left[\frac{6\tan(41.12^\circ) - \tan(21.12^\circ)}{6\tan(21.12^\circ) - \tan(41.12^\circ)}\right]$$

$$-3.5\ln\left[\frac{\tan(41.12^\circ)}{\tan(21.12^\circ)}\right] = 0.1747$$

Since $$\cot\delta - \cot(\delta - 2\omega_z) = \frac{\sin(\delta - 2\omega_z)\cos\delta - \cos(\delta-2\omega_z)\sin\delta}{\sin\delta\,\sin(\delta - 2\omega_z)}$$

$$= \frac{\sin\left[(\delta - 2\omega_z) - \delta\right]}{\sin\delta\,\sin(\delta - 2\omega_z)} = \frac{-\sin 2\omega_z}{\sin\delta\,\sin(\delta - 2\omega_z)}$$

The above equation combined with Eq. (42) yields

$$\cot 2\sigma = \frac{-\sin 2\,\omega_z}{2 \sin\delta \sin(\delta - 2\omega_z)}$$

which, when rearranged is

$$\tan 2\sigma \sin 2\,\omega_z = -2 \sin\delta \sin(\delta - 2\omega_z)$$

$$= -2 \sin \tfrac{1}{2}\left[2(\delta - \omega_z) + 2\,\omega_z\right] \sin \tfrac{1}{2}\left[2(\delta - \omega_z) - 2\,\omega_z\right]$$

$$= \cos 2\,(\delta - \omega_z) - \cos 2\,\omega_z$$

$$\therefore \qquad \delta = \omega_z + \tfrac{1}{2} \cos^{-1}(\cos 2\,\omega_z + \tan 2\sigma \sin 2\,\omega_z)$$

The partial derivative of δ with respect to σ is

$$\left(\frac{\partial\delta}{\partial\sigma}\right)_{\omega_z} = -\frac{\sin 2\omega_z \sec^2(2\sigma)}{\sqrt{1 - (\cos 2\,\omega_z + \tan 2\sigma \sin 2\,\omega_z)^2}} \qquad (44)$$

where

$$1 - (\cos 2\,\omega_z + \tan 2\sigma \sin 2\,\omega_z)^2$$

$$= 1 - \cos^2(2\omega_z) - 2\cos 2\omega_z \tan 2\sigma \sin 2\omega_z - \tan^2(2\sigma)\sin^2(2\omega_z)$$

$$= \sin^2(2\omega_z)\left[1 - \tan^2(2\sigma)\right] - 2\cos 2\omega_z \sin 2\omega_z \tan 2\sigma$$

$$\left(\frac{\partial\delta}{\partial\sigma}\right)_{\omega_z} = \pm \frac{\sec 2\sigma}{\sqrt{\dfrac{\sin^2(2\omega_z)\,\dfrac{\cos^2(2\sigma) - \sin^2(2\sigma)}{\cos^2(2\sigma)}}{\sin^2(2\omega_z)\,\dfrac{1}{\cos^2(2\sigma)}} - \dfrac{2\cos 2\omega_z \sin 2\omega_z\,\dfrac{\sin 2\sigma}{\cos 2\sigma}}{\sin^2(2\omega_z)\,\dfrac{1}{\cos^2(2\sigma)}}}}$$

$$= \pm \frac{1}{\cos 2\sigma \sqrt{\cos 4\sigma - \cot 2\omega_z \sin 4\sigma}}$$

where the positive sign will be chosen for negative values of ω_z, while the negative sign will be chosen for positive ω_z.

Then, let $\dfrac{\partial(\frac{\Delta s}{R})}{\partial\sigma} = 0$, and the locus of minimum $(\frac{\Delta s}{R})$ for constant angle of rotation ω_z can be found. Employing Eq. (43), the expression for

$\dfrac{\partial(\frac{P_2}{P_1})}{\partial\sigma}$ for ω_z constant is

$$\frac{\partial(\frac{P_2}{P_1})}{\partial\sigma} = \frac{\tan(\sigma-\delta)\sec^2\sigma - \tan\sigma\sec^2(\sigma-\delta)(1 - \frac{\partial\delta}{\partial\sigma})}{\tan^2(\sigma-\delta)}$$

Since it has been shown before that the condition for

$$\frac{\partial(\frac{\Delta s}{R})}{\partial\sigma} = 0 \qquad \text{is} \qquad \frac{\partial(\frac{P_2}{P_1})}{\partial\sigma} = 0,$$

the above expression becomes

$$\tan(\sigma-\delta)\sec^2\sigma - \tan\sigma\sec^2(\sigma-\delta)(1 - \frac{\partial\delta}{\partial\sigma}) = 0$$

From which

$$\frac{\sin(\sigma-\delta)}{\cos(\sigma-\delta)\cos^2\sigma} = \frac{\sin\sigma}{\cos\sigma\cos^2(\sigma-\delta)}(1 - \frac{\partial\delta}{\partial\sigma})$$

and

$$\frac{\sin 2(\sigma-\delta)}{\sin 2\sigma} = 1 - \left(\frac{\partial\delta}{\partial\sigma}\right)_{\omega_z}$$

where $\left(\frac{\partial\delta}{\partial\sigma}\right)_{\omega_z}$ is given by Eq. (44).

The loci so formed are plotted on Fig. 15.

(4) Constant $\frac{\Delta s}{R}$ lines on $\omega_z - \sigma$ coordinates: The lines of constant entropy change in Fig. 14 are plotted by using the

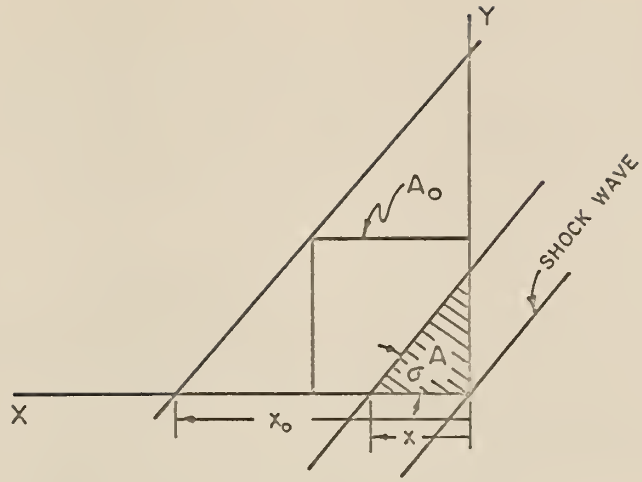

Fig. 16. Fraction of area passing through a shock when the flow proceeds.

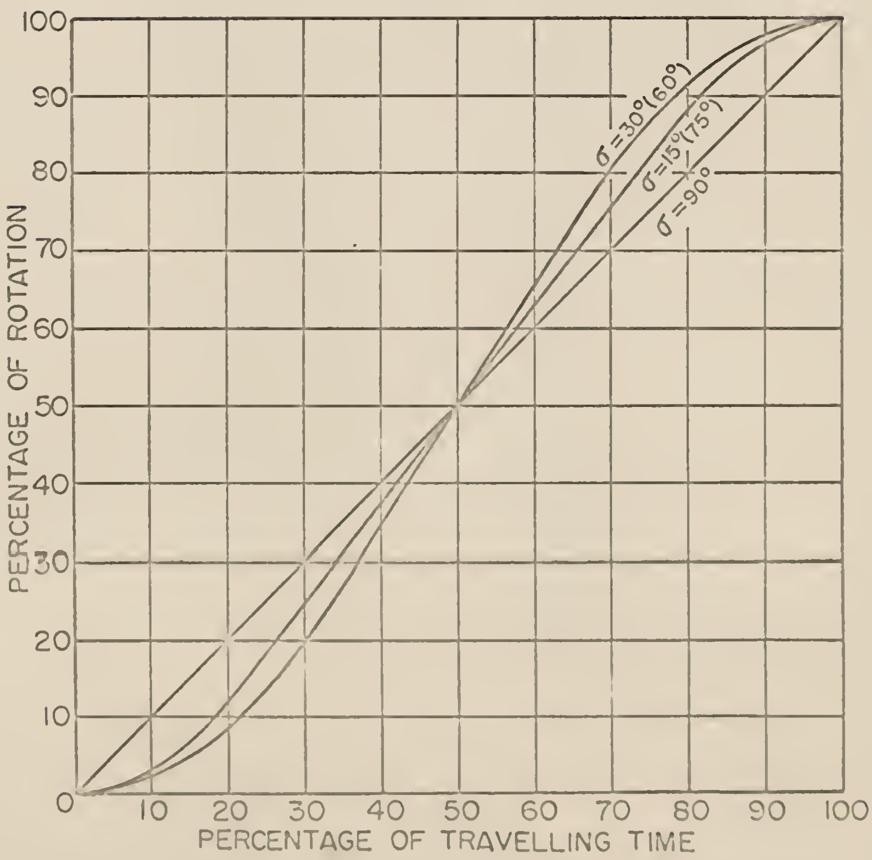

Fig. 17. Percentage of rotation vs. percentage of travelling time.

corresponding data from Fig. 15, except that the line of

$\frac{\Delta s}{R} = \infty$ is found from the equation

$$\frac{\rho_2}{\rho_1} = \frac{\tan \sigma}{\tan(\sigma - \delta)} = 6$$

(5) Constant upstream Mach Number (M_1) lines and the locus of minimum M_1 for constant δ : These lines are plotted on Fig. 15 from data obtained from Chart 2-1, Dailey and Wood's "Computation Curves for Compressible Fluid Problems" (6). The locus of minimum M_1 for constant δ distinguishes the strong and weak shocks. The region to the right of this line is the domain of strong shocks, while that to the left is the domain of weak shocks. This locus of minimum M_1 for constant δ is the locus of the common tangents of the constant M_1 and constant δ lines (see Chart 2-1, Reference 6).

(6) Percentage of rotation vs. percentage of time for a square control volume passing through oblique shocks: Since the flow pattern taken is two-dimensional, the percentage of rotation for a control volume, which is the shape of square before reaching the chock wave, is proportional to the fraction of mass (or area) which passes through the shock. Since the upstream velocity V_1 is constant, the traveling time is proportional to the distance traveled in the direction of flow. Let

A_o = the area of the square control volume

A = the area passed through the shock at time t

x = the distance traveled by the control volume in time t

x_o = the distance traveled by the control volume when it entirely passes the shock

t_o = the total time required for the control volume to pass the shock

Referring to Fig. 16, the following expressions are found from the geometry of the figure:

(a) $$\frac{A}{A_o} = (1 + \csc 2\sigma)\left(\frac{x}{x_o}\right)^2$$

when $\sigma > \frac{\pi}{4}$ and $0 \leqq \frac{x}{x_o} \leqq \frac{1}{1 + \tan\sigma}$

or when $\sigma < \frac{\pi}{4}$ and $0 \leqq \frac{x}{x_o} \leqq \frac{}{1 + \cot\sigma}$

(b) $$\frac{A}{A_o} = (1 + \cot\sigma)\left(\frac{x}{x_o}\right) - \frac{\cot\sigma}{2}$$

when $\sigma > \frac{\pi}{4}$ and $\frac{1}{1 + \tan\sigma} \leqq \frac{x}{x_o} \leqq \frac{1}{1 + \cot\sigma}$

$$\frac{A}{A_o} = (1 + \tan\sigma)\left(\frac{x}{x_o}\right) - \frac{\tan\sigma}{2}$$

when $\sigma < \frac{\pi}{4}$ and $\frac{1}{1 + \cot\sigma} \leqq \frac{x}{x_o} \leqq \frac{1}{1 + \tan\sigma}$

(c) $$\frac{A}{A_o} = 1 - (1 + \csc 2\sigma)\left(1 - \frac{x}{x_o}\right)^2$$

when $\sigma > \frac{\pi}{4}$ and $\frac{1}{1 + \cot\sigma} \leqq \frac{x}{x_o} \leqq 1$

or when $\sigma < \frac{\pi}{4}$ and $\frac{1}{1 + \tan} \leqq \frac{x}{x_o} \leqq 1$

The percentage rotation of the control volume = $\frac{A}{A_o}$ x 100 and the percentage of traveling time = $\frac{x}{x_o}$ x 100. Hence, the curves in Fig. 17, for five different shock angles, are plotted by employing the above analytical expressions (Table 2).

Table 2. Calculation data for percentage of rotation vs. percentage of traveling time.

σ	Percentage of Traveling Time	Percentage of Rotation
	10.0	3.00
	20.0	12.00
	21.2	13.35
$15^o(75^o)$	78.9	86.65
	80.0	88.00
	90.0	97.00
	10.0	2.14
	20.0	8.60
	30.0	19.30
	36.6	29.05
$30^o(60^o)$	63.4	70.95
	70.0	80.70
	80.0	91.40
	90.0	97.86

CONCLUSION

The definition of circulation and fluid rotation in two-dimensional flow state that if the circulation around a closed path in a certain region is zero the rotation at any point over that region should also be zero. This is shown to be true in the two cases of the normal shock and the potential vortex. But, the work done in this report has shown, on a numerical basis, that the circulations everywhere in Prandtl-Meyer flow and in oblique shock flow are zero while the fluid particles passing through those regions in a certain time interval undergo angular rotations. The reason for the contradiction has not been found. The author recommends that the problem needs further investigation.

ACKNOWLEDGMENT

The author wishes to extend his hearty thanks to Dr. Wilson Tripp for his advice, direction and correction during the work of this report.

REFERENCES

1. Ascher H. Shapiro, "The Dynamics and Thermodynamics of Compressible Fluid Flow", Vol. I, The Ronald Press Company, New York, N.Y., 1953.

2. H. W. Liepmann and A. Roshko, "Elements of Gasdynamics", John Wiley & Sons, Inc., New York, N.Y., 1957.

3. Victor L. Streeter, "Fluid Dynamics", First edition, McGraw-Hill Book Company, New York, N.Y., 1948.

4. Charles D. Hodgman, "Mathematical Tables from Handbook of Chemistry and Physics", Eleventh edition, Chemical Rubber Publishing Company, Cleveland, Ohio, 1960.

5. Joseph H. Keenan and Joseph Kaye, "Gas Tables", John Wiley & Sons, Inc., New York, N.Y., 1948.

6. C. L. Dailey and F. C. Wood, "Computation Curves for Compressible Fluid Problems", John Wiley & Sons, Inc., New York, N.Y., 1949.

ROTATIONAL AND IRROTATIONAL FLOW
IN CERTAIN GAS DYNAMICS PROBLEMS

by

CHENG-CHIEN CHOU

B. S., Cheng Kung University, Taiwan, China, 1957

AN ABSTRACT OF
A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

The purpose of the work in this report is to investigate the rotationality of the flow for the normal shock, the potential vortex, the Prandtl-Meyer flow and the oblique shock. The fluid considered in all cases is a perfect gas. Some expressions to show the relationship between the entropy change and the fluid rotation have been developed for the case of flow passing through oblique shocks.

The rotationalities for these four problems are examined from two viewpoints, one of which is based on the concept of the circulation around a closed path in the flow while the other is based on the concept of the fluid rotation of a particle at a point in the flow. The definitions of circulation and fluid rotation in two-dimensional flow state that if the circulation around a closed path in a certain region is zero the rotation at any point over that region should also be zero. This has been shown to be true in the cases of the normal shock and the potential vortex. However, the work done in this report has also shown, on a numerical basis, that when the circulations everywhere in Prandtl-Meyer flow and in flow passing through oblique shock are zero the fluid particles traveling in those regions in a certain time interval really involve angular rotations.

The reason for the contradiction has not been found.

Date Due
