TRANSLATIONS IN LATTICES

by

DON F. HUNZIKER

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Department of Mathematics

KANSAS STATE UNIVERSITY
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Approved by:

S. Thomas Parker
Major Professor

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INTRODUCTION

The study of translations in lattices has been used in the study of congruences on a non-distributive lattice. There is a connection with the translational dimension of a lattice and the congruences on the lattice.

There have been no attempts to adapt the problem of computing the translational dimension of a lattice to the computer, as far as is known. This is the problem which will be considered in the following sections.

II. Definition of a Lattice

A partially ordered set S is an algebraic system in which a binary relation $x \leq y$ (read: y includes x) is defined, which satisfies the following postulates.

P_1: For all x, $x \leq x$. (reflexive property)

P_2: If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetric property)

P_3: If $x \leq y$ and $y \leq z$, then $x \leq z$. (transitive property)

A binary relation which satisfied P_1, P_2, P_3, is called an inclusion relation or an order relation. Associated with the relation $\leq$ we can conveniently introduce the relations $\geq$, $<$, $>$, defined as follows.

$x \geq y \Longleftrightarrow y \leq x$;

$x < y \Longleftrightarrow x \leq y$ and $x \neq y$;

$x > y \Longleftrightarrow y < x$.

It should be observed that the inclusion need not be defined for each pair of elements of the set. It is sufficient that it be defined for some pairs of elements. It should be emphasized that $x \not\leq y$ does not necessarily imply $x > y$ (1,1).[1] An example of a partially ordered set is the set S of all positive integers; where $x \leq y$ means that y has x as a factor.

A partially ordered set S is said to be totally ordered and is a chain if P_4: For any x and y is S, $x \leq y$ or $y \leq x$. That is, every two elements of the set are related.

[1]In this report the first number of the ordered pair will be used to indicate the reference and the second number will indicate the page, with the references numbered in the bibliography.

Let Q be a subset of the elements of a partially ordered set S. We call an element x of S an upper bound (lower bound) of Q if $y \leq x$ ($x \leq y$) for all $y \varepsilon Q$. An upper bound (lower bound) x of Q is said to be the least upper bound (greatest lower bound) of Q if every upper (lower) bound x' of Q satisfies $x \leq x'$ ($x' \leq x$). If any element y of Q is an upper (lower) bound of Q, it must be the least upper bound (greatest lower bound), since for any other upper (lower) bound x', $y \leq x'$ ($x' \leq y$) is satisfied. The least upper bound (greatest lower bound) of Q need not exist. If Q has at least one upper (lower) bound, Q is called a subset of S bounded above (bounded below); a subset which is bounded both above and below is called a bounded subset. If Q = S, we shall speak of a set bounded above or bounded below, respectively.

Definition: A lattice L is a partially ordered set S such that any two elements of S possess both a least upper bound (denoted by V) and a greatest lower bound (denoted by Λ).

Definition: A sublattice M of a lattice L is a subset such that, $x \varepsilon M$ and $y \varepsilon M$ implies $x \Lambda y \varepsilon M$ and $x V y \varepsilon M$.

Let (S, $\leq$) be a partially ordered set. Then the partially ordered set (S, $\geq$) is the dual of the partially ordered set (S, $\leq$). We have a partially ordered set in both cases, but the second has been obtained from the first by replacing $\leq$ with $\geq$. Any statement which was made about the first set can also be made about the second set by replacing $\leq$ with $\geq$ in the statement.

Since a lattice is a partially ordered set with the two operations Λ, V, by the Principle of Duality: Any statement which has been deduced from the axioms of a lattice for Λ, remains valid if Λ and V are interchanged in the statement.

Some fundamental identities for lattices:

L_1: $x \wedge y = y \wedge X$

L_2: $x V y = y V X$

Commutative law

L_3: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

L_4: $x V (y V z) = (x V y) V z$

Associative law

L_5: $x \wedge (x V y) = x$

L_6: $x V (x \wedge y) = x$

Absorption law

Proof of L_3: Write $p = y \wedge z$ and $q = x \wedge p = x \wedge (y \wedge z)$. Then $p \leq y$, $p \leq z$, $q \leq x$, $q \leq p$, and by P_3, $q \leq y$ and $q \leq z$. Thus q is a lower bound for the subset x, y, z. If r is any other lower bound of this subset, then $r \leq x$, $r \leq y$, $r \leq z$, and so r is a lower bound of y and z. But p is the greatest lower bound of y and z and consequently $r \leq p$. This shows that r is a lower bound of x and p. But q is the greatest lower bound of x and p, from which we conclude that $r \leq q$. It follows that q, or $x \wedge (y \wedge z)$, is the greatest lower bound of the subset x, y, z. In a similar manner we can show that $(x \wedge y) \wedge z$ is also the greatest lower bound of the subset x, y, z. Consequently

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z,$$

which is L_3. The other associative law is dual.

Proof of L_5: Write $p = x V y$, $q = x \wedge p = x \wedge (x V y)$, then $p \geq x$, $p \geq y$, $q \leq x$, and $q \leq p$. Since $x \leq x$, it follows that x is a lower bound of x and p, but q is the greatest lower bound of x and p, consequently $x \leq q$. Since we know $q \leq x$, it follows from antisymmetric law that $x = q = x \wedge (x V y)$. The other absorption law is dual.

We obtain another important formula by replacing y in L_5 by $x \wedge y$.

$x = x \wedge (x V (x \wedge y))$ but $x V (x \wedge y) = x$, hence $x = x \wedge x$ and dually we can obtain $x = x \vee x$. Hence we have the idempotent laws,

L_7: $x = x \wedge x$

L_8: $x = x \vee x$.

If $y \leq x$, then y is a lower bound for x and y. It is also the greatest lower bound for x and y, since any other lower bound must be included by y.

A finite lattice is one with a finite number of elements. Any finite lattice, being a partially ordered set, can be represented by a Hasse diagram. For example a five element lattice L defined by $I = a \vee b = a \vee c$, $b \leq c$, and $0 = a \wedge b = a \wedge c$ is represented by

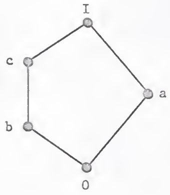

Fig. 1

Another example of a lattice is the set N of positive integers. Let $a \wedge b$ and $a \vee b$ $(a, b \varepsilon N)$ denote the greatest common divisor and the least common multiple, respectively, of the numbers a and b. With respect to these two operations, N is a lattice.

Theorem 2.1: In a lattice L

(1) $a \leq b \Longleftrightarrow a \wedge b = a$ $(a, b \varepsilon L)$. (7,38)

Proof: $a \leq b \Rightarrow a$ is a lower bound for $(a, b) \Rightarrow a = a \wedge b$.

Conversly, $a = a \wedge b \Rightarrow a \leq b$.

If a lattice L has an element 0 (I) such that every element x of L satisfies the inequality $0 \leq x (x \leq I)$, then 0 (I) is called the least or null (greatest or universal) element of L. These elements will also be called the bound elements of L. In a lattice the terms "minimal element" and "least element" (similarly "maximal element" and "greatest element") mean the same thing.

Definition: A distributive lattice is a lattice for which L_9 and L_{10} hold, where

L_9: For any triplet of elements a, b, c of the lattice,

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

L_{10}: For any triplet of elements a, b, c of the lattice,

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Theorem 2.2. A lattice L for which either L_9 or L_{10} is satisfied is distributive (7, 79).

Proof: If, say, L_9 holds for a lattice L, then for any triplet a, b, c of L, $(a \vee b) \wedge (a \vee c) = ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) = a \vee ((a \wedge b) \wedge c)$. But $(a \vee b) \wedge c = c \wedge (a \vee b)$ and, since L_9 holds $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$. Hence $a \vee ((a \vee b) \wedge c) = a \vee ((a \wedge c) \vee (b \wedge c)) = (a \vee (a \wedge c)) \vee (b \wedge c) = a \vee (b \wedge c)$ and, hence, L_{10} also holds. L_9 follows from L_{10} by the dual of the above argument.

Theorem 2.3. A lattice L is distributive, if, and only if, it has no sublattice S isomorphic with either one of the lattices shown in Figures 1 and 2.

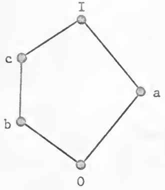

Fig. 1

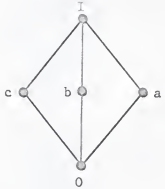

Fig. 2

Proof: Assume we have a sublattice S which is isomorphic to Figure 1; there exist elements a, b, $c \in S$ as shown. Hence $b \leq c$ and $b \vee (a \wedge c) \neq (b \vee a) \wedge (b \vee c)$ since $b \vee (a \wedge c) = b \vee 0 = b$ and $(b \vee a) \wedge (b \vee c) = I \wedge c = c$, which implies L is not distributive.

Assume we have a sublattice S which is isomorphic to Figure 2; then there exists elements a, b, c of L such that

$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$ since $a \vee (b \wedge c) = a \vee 0 = a$ and

$(a \vee b) \wedge (a \vee c) = I \wedge I = I$, which implies L is not distributive.

The converse can be shown to be true. (7, 91)

Definition: A modular lattice is a lattice L in which the following identity holds:

L_{11}: For any triplet of elements a, b, c of a lattice satisfying $a \leq c$ the identity $a \vee (b \wedge c) = (a \vee b) \wedge c$ holds.

It is clear that any distributive lattice is modular. (6, 13)

Theorem 2.4. A lattice is modular if, and only if, it contains no sub-lattice isomorphic to the pentagonal lattice of Figure 1. (6, 13)

Proof: The lattice of Figure 1 is non-modular since $b<c$ and $b \vee (a \wedge c) = b \vee 0 = b < c = (b \vee a) \wedge c = I \wedge c$, hence the modular identity fails to hold.

The converse can be shown to be true. (6, 13)

Theorem 2.5. If $a \wedge b = a \wedge c$ and, $a \vee b = a \vee c$ and $b \leq c$ implies $b = c$ for any choice of elements a, b, c, then the lattice is modular.

Proof: If it were nonmodular, it would contain a pentagonal sublattice such as Figure 1 in which $b \neq c$ although $a \wedge b = a \wedge c$ and, $a \vee b = a \vee c$ and $b \leq c$.

III. Translations in Lattices

In any lattice L we shall denote the elementary translations defined by $a \varepsilon L$ by ρ_a: $x \to x \wedge a$ and $\sigma_a : x \to x \vee a$. A translation is then the composition of finitely many such mappings. We shall write all translations as operators on the right.

Since $\rho_a \rho_b = \rho_{a\wedge b}$, $\sigma_a \sigma_b = \sigma_{a\vee b}$ identically, any finite product of elementary translations can be reduced to one of the following forms:

(A) $\rho_{c_1} \sigma_{c_2} \rho_{c_3} \sigma_{c_4} - - - - \rho_{c_{n-1}} \sigma_{c_n}$, $1 \leq n$, n even;

(B) $\rho_{c_1} \sigma_{c_2} \rho_{c_3} \sigma_{c_4} - - - - \sigma_{c_{n-1}} \rho_{c_n}$, $1 \leq n$, n odd;

(C) $\sigma_{c_1} \rho_{c_2} \sigma_{c_3} \rho_{c_4} - - - - \sigma_{c_{n-1}} \rho_{c_n}$, $1 \leq n$, n even;

(D) $\sigma_{c_1} \rho_{c_2} \sigma_{c_3} \rho_{c_4} - - - - \rho_{c_{n-1}} \sigma_{c_n}$, $1 \leq n$, n odd.

The reduction is unique, although the translation itself can be written in possibly more than one of the forms above.

For any positive integer n, we denote by $R_n(L)$ the set of all translations which can be expressed under the form (A) or (B) (with the same n); $S_n(L)$ is the set of all translations which can be expressed under the form (C) or (D) (with the same n). We set $T_n(L) = R_n(L) \cup S_n(L)$, so that the set $T(L)$ of all translations of L is the union of all $T_n(L)$. We shall write these sets R_n, S_n, T_n, and T as if there is only one lattice under consideration.

The order of a translation τ is the smallest integer n such that $\tau \varepsilon T_n$. It is also the smallest integer p such that τ can be written as a product of p elementary translations. For example, for any $a \varepsilon L$, the constant mapping $K_a : x \to a$ can be written as $K_a = \rho_a \sigma_a = \sigma_a \rho_a$ (which is the absorption law) and is a translation of order two unless a is a maximum or minimum of element of L.

The translations of L are order-preserving, since: if $x \leq y$, then $x\rho_a = x \wedge a = x \wedge y \wedge a \leq y \wedge a = y\rho_a$ (since $x = x \wedge y$ by Theorem 2.1. Then $x\sigma_a \leq y\sigma_a$ by the dual. The composition of order preserving maps is also order preserving, hence all translations are order preserving.

The translations are also bounded except possibly for the elementary translations. If, for instance, $\tau = \tau'\rho_c\sigma_d$, where $\tau' \varepsilon T$, or it may be the identity translation (which implies the order of $\tau \geq 2$), then $x\tau = x\tau'\rho_c\sigma_d = (x\tau' \wedge c) V d$ and $d \leq x\tau \leq cVd$ for all $x \varepsilon L$ since

(1) $(x\tau' \wedge c) \leq c$ because if $c \leq x\tau'$ implies $x\tau' \wedge c = c$ and if $x\tau' \leq c$ implies $x\tau' \wedge c = x\tau' \leq c$. Hence $(x\tau' \wedge c) Vd \leq cVd$ or $x\tau \leq cVd$.

(2) $d \leq (x\tau' \wedge c) Vd$ since $d \leq$ least upper bound of $(x\tau' \wedge c, d)$.

Combining the two results we have $d \leq x\tau \leq cVd$, for all $\tau \varepsilon T$ except possibly when τ is an elementary translation. The elementary translations of R_1 are bounded above ($x\rho_a = x \wedge a \leq x V a$, $x V a \varepsilon L$ for all $x \varepsilon L$), but are bounded below if, and only if, L has a least element. The elementary translations of S_1 are bounded below ($x\sigma_a = x V a \geq x \wedge a$, $x \wedge a \varepsilon L$ for all $x \varepsilon L$), but are bounded above if, and only if, L has a greatest element.

Theorem 3.1. For any $n \geq 2$, $R_n \subseteq R_{n+1}$ and $S_n \subseteq S_{n+1}$, so that $T_n \subseteq T_{n+1}$. The corresponding inclusions for n = 1 hold if, and only if, L is bounded. In this case, $T_n \subseteq R_{n+1} \cap S_{n+1}$ for all n.

Proof: Take $\tau \varepsilon R_n$, and assume that $n \geq 2$ or that L is bounded. From above, if $n \geq 2$ or if the translation is bounded, then we can find a, $b \varepsilon L$ such that $a \leq x\tau \leq b$ for all $x \varepsilon L$. Now $\tau = \tau\rho_b = \tau\sigma_a$ for all $x \varepsilon L$ and either $\tau\rho_b$ or $\tau\sigma_a$ is in

R_{n+1}. This implies $R_n \subseteq R_{n+1}$. In the case when L is bounded, with minimum element 0, we also have $\tau = \sigma_o \tau \varepsilon S_{n+1}$, which implies $R_n \subseteq S_{n+1}$. Dually, $S_n \subseteq R_{n+1}$, hence $T_n \subseteq R_{n+1} \cap S_{n+1}$.

If conversely $T_1 \subseteq T_2$, then every elementary translation is bounded, since every $\tau \varepsilon T_2$ is bounded and every $\tau' \varepsilon T_1$ is equal to some $\tau \varepsilon T_2$, this implies τ' is bounded. Hence L is bounded.

Theorem 3.2. If L is distributive, then $R_2 = S_2$, so that $T = T_1 \cup T_2$.

Proof: The identity $(x \wedge a) \vee b = (x \vee b) \wedge (a \vee b)$ shows $R_2 \subseteq S_2$, since $x\rho_a\sigma_b = x\sigma_b\rho_{a\vee b}$. Dually we show $S_2 \subseteq R_2$, since we have the identity $(x \vee a) \wedge b = (x \wedge b) \vee (a \wedge b)$ which implies $x\sigma_a\rho_b = x\rho_b\sigma_{a\wedge b}$. From this follows inductively that $T_n \subseteq T_2$ for all $n \geq 2$. Indeed it holds for $n = 2$, and if $T_p \subseteq T_2$, where $p \geq 2$, then $R_{p+1} = R_1 S_p \subseteq R_1 R_2 \subseteq R_2$. Since $R_{p+1} = R_1 S_p$ and $S_p \subseteq R_2$, this implies $R_1 S_p \subseteq R_1 R_2$, but $R_1 R_2 = \rho_a\rho_b\sigma_s = \rho_{a\wedge b}\sigma_c \subseteq R_2$, dually $S_{p+1} \subseteq T_2$. Therefore $T = T_1 \cup T_2$.

IV. Translational Dimension and Its Properties

If a lattice L is such that $T(L) = T_1(L) \cup T_n(L)$ for some n, then the smallest such integer is, by definition, the translational dimension of L, tr. dim L. If no such n exists, we define tr. dim L = ∞. If the tr. dim of L is finite, it is also the smallest integer n such that any translation of L can be written as a product of at most n elementary translations.

For example a distributive lattice has tr. dim 1 or 2, by Theorem 3.2. Any finite lattice has a finite translational dimension.

Theorem 4.1. Tr. dim L $\leq$ n if, and only if, $T_{n+1}(L) \subseteq T_n(L)$.

Proof: Take a lattice L and suppose first that tr. dim L = m $\leq$ n. Then $T = T_1 \cup T_m \subseteq T_1 \cup T_n$ by Theorem 3.1 and $T_{n+1} \subseteq T$ from the definition of translational dimension. If n = 1, then $T_{n+1} \subseteq T_n$ since $T_{n+1} \subseteq T = T_1 \cup T_n$. If L is bounded, then $T_{n+1} \subseteq T = T_1 \cup T_m = T_m$ since L is bounded, hence $T_{n+1} \subseteq T_n$. Otherwise the translations of T_1 which are bounded are in T_2, hence in T_n the translations of T which are not bounded are not in any T_n except T_1, so that $T_{n+1} \subseteq T_n$.

Conversely if $T_{n+1} \subseteq T_n$, then $T_{n+p} \subseteq T_n$ for all p, by induction; indeed it holds for p = 1, and if $T_{n+p} \subseteq T_n$, then $T_{n+p+1} \subseteq T_{n+p}\, T_1 \subseteq T_n\, T_1 \subseteq T_{n+1} \subseteq T_n$. Therefore, $T = T_1 \cup T_n$, and tr. dim 1 $\leq$ n.

Theorem 4.2. (a) If $R_n(L) = S_n(L)$, then tr. dim L $\leq$ n; (b) if tr. dim L $\leq$ n and if L is bounded, then $R_{n+1}(L) = S_{n+1}(L)$.

Proof: (a) Assume that $R_n = S_n$. Then $R_{n+1} = R_1\, S_n = R_1\, R_n \subseteq R_n$ and dually $S_{n+1} = S_1\, R_n = S_1\, S_n \subseteq S_n$. Therefore, $T_{n+1} \subseteq T_n$ and tr. dim L $\leq$ n by Theorem 4.1.

Proof: (b) If tr. dim L $\leq$ n and L is bounded, then by Theorem 4.1 $T_{n+1} \subseteq T_n$ and by Theorem 3.1 $R_{n+1} \subseteq T_n \subseteq S_{n+1} \subseteq T_n \subseteq R_{n+1}$ which implies $R_{n+1} = S_{n+1}$.

Theorem 4.3. Let f be a lattice homorphism of the lattice L onto the lattice L'. Then the translational dimension of L' is less than or equal to the translational dimension of L.

Proof: Let tr. dim L = n. If n = ∞, there is nothing to prove. If $n \neq \infty$, let $\tau' \varepsilon T_{n+1}(L')$. Then τ' is the product of n+1 elementary translations defined, say by a_1', ---, $a_{n+1}' \varepsilon L'$. For each i we can find $a_i \varepsilon L$ such that $f(a_i) = a_i'$, since f is surjective; then the translation $\tau \varepsilon T_{n+1}(L)$ similarly defined by a_1, ---, a_{n+1} is such that $f(x\tau) = (f(x))\tau'$ for all $x \varepsilon L$. But $T_{n+1}(L) \subseteq T_n(L)$ by Theorem 4.1, so that τ is the product of n elementary translations defined, say by b_1, ---, b_n. Since f is onto, τ' may now be written as the product of n elementary translations defined by $f(b_1)$, ---, $f(b_n)$, therefore, $\tau' \varepsilon T_n(L')$. This shows that $T_{n+1}(L') \subseteq T_n(L)$ and by Theorem 4.1 tr. dim L' $\leq$ n = tr. dim L.

Theorem 4.4. For any lattice L, tr. dim L = 1 if, and only if, L has one or two elements.

Proof: If L has more than two elements, then there is a constant translation $K_a = \rho_a\sigma_a = \sigma_a\rho_a$ which is not elementary; hence tr. dim $\geq$ 2.

Conversely, it is readily verified that, if L has one or two elements (either 0 and I, or just one element), every translation is elementary.

Theorem 4.5. A lattice L is distributive if, and only if, tr. dim L $\leq$ 2.

Proof: If L is distributive, then tr. dim L $\leq$ 2 by Theorem 3.2.

Conversely, assume that L is not distributive. Then L contains a five element non-distributive sublattice which is isomorphic to one of the lattices shown in Figure 1 or Figure 2.

If L contains a sublattice {0, a, b, c, I} isomorphic to Figure 1, where $a \wedge b = a \wedge c = 0$, $a \vee b = a \vee c = I$, $b < c$, then the translation $x \to ((x \wedge a) \vee b) \wedge c$ has order 3. Indeed, assume that $((x \wedge a) \vee b) \wedge c = (x \wedge d) \vee e$ for all $x \in L$. Letting $x = b \wedge d \wedge e$, we obtain

$$(((b \wedge d \wedge e) \wedge a) \vee b) \wedge c = ((b \wedge d \wedge e) \wedge d) \vee e$$

$$((b \wedge d \wedge e \wedge a) \vee b) \wedge c = (b \wedge d \wedge e \wedge d) \vee e$$

$$(((d \wedge e \wedge a) \wedge b) \vee b) \wedge c = ((b \wedge d \wedge d) \wedge e) \vee e$$

$$b = (b \wedge c) = e.$$

Letting $x = a$, we obtain, $c = (a \vee b) \wedge c = (a \wedge d) \vee e = (a \wedge d) \vee b$. Therefore $a \wedge d \leq c$. Since $a \wedge d \leq a$, also $a \wedge d \leq a \wedge c = 0 < b$ and $c = (a \wedge d) \vee b = b$, hence $c = b$, a contradiction.

Assume that $((x \wedge a) \vee b) \wedge c = (x \vee d) \wedge e$ for all $x \in L$. Letting $x = a \vee e$, we obtain $c = (((a \vee e) \wedge a) \vee b) \wedge c = (a \vee b) \wedge c = ((a \vee e) \vee d) \wedge e = e$ or $c = (a \vee b) \wedge c = e$. Letting $x = c$ thus yields $b = ((c \wedge a) \vee b) \wedge c = (c \vee d) \wedge e = (c \vee d) \wedge c = c$, another contradiction.

Similarly, if L contains a sublattice {0, a, b, c, I} isomorphic to Figure 2, where $a \wedge b = a \wedge c = b \wedge c = 0$ and $a \vee b = a \vee c = b \vee c = I$, then the translation $x \to ((x \wedge a) \vee b) \wedge c$ has order 3. Indeed, assume that $((x \wedge a) \vee b) \wedge c = (x \wedge d) \vee e$ for all $x \in L$. Letting $x = b \wedge e$, we obtain

$$(((b \wedge e) \wedge a) \vee b) \wedge c = ((b \wedge e) \wedge d) \vee e$$

$$(((a \wedge e) \wedge b) \vee b) \wedge c = ((b \wedge d) \wedge e) \vee e$$

$$0 = b \wedge c = e.$$

Letting $x = a$ now yields

$$c = ((a \wedge a) \vee b) \wedge c = (a \wedge d) \vee e = (a \wedge d) \vee 0 \leq a$$

which is impossible, hence a contradiction.

Similarly, assume that $((x \wedge a) \vee b) \wedge c = (x \vee d) \wedge e$ for all $x \in L$. Letting $x = a \vee e$, we obtain

$$c = (((a \vee e) \wedge a) \vee b) \wedge c = (a \vee b) \wedge c = ((a \vee e) \vee d) \wedge e = e.$$

Letting $x = c$, we obtain

$$0 = ((c \wedge a) \vee b) \wedge c = (c \vee d) \vee e = (c \vee d) \wedge c = c$$

which is a contradiction.

In either case, L has a translation of order 3, which completes the proof.

Theorem 4.6. If tr. dim L $\leq$ 3 and if L is modular, then L is distributive.

Proof: This theorem was proven by P. A. Grillet in a paper published by him (3, 13).

Theorem 4.7. If L is modular, then tr. dim L $\neq$ 3.

Proof: If L is modular with tr. dim L = 3, then L is distributive by Theorem 4.6, hence tr. dim $\leq$ 2, a contradiction.

Theorem 4.8. A modular lattice L generated by 3 elements has translational dimension of 2 or 4.

Proof: Since L has at least 3 elements, tr. dim L $\geq$ 2 by Theorem 4.3. Also L is the homorphic image of the free modular lattice L' on 3 generators whose tr. dim is 4 (see Example 7.5); therefore, tr. dim L $\leq$ 4 since if f is a lattice homorphism of L' onto L, then tr. dim L $\leq$ tr. dim L' = 4. Finally, tr. dim L $\neq$ 3 by Theorem 4.7.

V. Adapting the Problem to the Computer

From what has been presented so far, it appears that one might benefit from looking at the translational dimension and the cardinal number of T.

Some facts have been shown about the translational dimension of the lattice L under homomorphic images and it is also known: (i) a lattice L is distributive if, and only if, the translational dimension L ≤ 2; (ii) if the translational dimension L ≤ 3 and if L is modular, then L is distributive; (iii) a modular lattice L generated by three elements has a translational dimension of two or four. The translational dimension was introduced principally to help characterize finite simple lattices.

The development of a computer method by which the computation of the translational dimension of a lattice can be carried out with the minimum of time and work is the objective. The computation of the translational dimension is not a difficult problem, but by straight-forward, hand methods it requires a great amount of work and there is considerable chance for errors.

The problem of computing the translational dimension of Figure 1 is a fairly simple, but long problem. To compute the translations of R_1 alone requires twenty-five separate calculations, for example:

$\wedge$	0	a	b	c	I
GLB(x, 0)	0	0	0	0	0
GLB(x, a)	0	a	0	0	a
GLB(x, b)	0	0	b	b	b
GLB(x, c)	0	0	b	c	c
GLB(x, I)	0	a	b	c	I

This gives all the translations associated with R_1. The translations for S_1 must be computed in a similar manner. After the translations for R_1 and S_1 have been computed, the different translations are grouped in a set T. The translations in R_2 and S_2 are computed next. In a five element lattice, such as the one that is being considered, it requires 5^3 separate computations for both R_2 and S_2, respectively, since $\tau \in R_2$ is written as $X\tau = X\rho_a\sigma_b = (X \wedge a) \vee b$, and $\tau' \in S_2$ is written as $X\tau' = X\sigma_a\rho_b = (x \vee a) \wedge b$, where X varies over all of the elements of the lattice, while a and b remain fixed. After X has taken on the value of each of the elements of the lattice, b is held fixed and a takes on another value. Then X is allowed to vary over all the elements of the lattice. The process is continued until a has taken on the value of each element of the lattice, and then one allows b to vary and the process is continued in the same manner until b has taken on the value of each element of the lattice.

Either after all the translations of R_2 and S_2 have been computed, or as they are being computed, each new translation must be added to the set T. This is quite a task in itself, as one can see, since, if we have an N element lattice, the number of possible new translations which are contained in, say, R_3 is N^3 or 125 in the case being considered. The number of computations required to compute the translations of R_3 is N^4 or, in the case being considered, is 625.

The computer will lend itself readily to this type of problem. The computer can compute the translations and sort out all the different translations much more quickly than the same job could be done by hand. The computer does not make errors, which is another important factor.

Since one is now interested only in finite lattices, we know that our computations are bounded. This is known because by Theorem 4.1, the translational dimension $L \leq n$ if, and only if, $T_{n+1}(L) \subseteq T_n(L)$. This says that $T_1(L)$, $T_2(L)$, ---, $T_{n+1}(L)$ can be computed and all of the new translations from $T_1(L)$,

$T_2(L)$, ---, $T_n(L)$ can be added to the set T. If $T_{n+1}(L)$ is the first set of translations which contributes no new translations, the translational dimension of L is equal to n.

VI. The Program

The program which is of interest here is one which, for a finite lattice, will allow the computation of the translational dimension, will give a list of the different translations, and the number of new translations that are picked up at R_n and S_n for each value of n respectively, plus the total number of different translations. In the process of computing the translational dimension, the solutions to the other problems are answered, so one can concentrate on the problem of computing the translational dimension of the lattice.

One of the first steps which should be taken when attempting to program a problem on the computer is to define the problem, which is as stated above. The program which has been developed to handle this problem, will handle any n-element lattice which has a translational dimension of seven or less, since only lattices of a small translational dimension are of interest.

The first item which should be considered is how to define the relations of GLB and LUB for each pair of elements of the lattice L. The elements of a lattice can be represented by numbers, instead of non-numeric characters; this allows the elements to be handled more readily by the computer. The GLB and LUB can be defined for each ordered pair $(x,y) \varepsilon LxL$. These ordered pairs can be read into the computer, so the computer will have a basis for its calculations. After these ordered pairs have been read into the computer, one can set up the equations to compute each translation and, if the translation is new, add it to the set T, which is the set of all the different translations computed.

The program which follows will produce all the results that were set up as objectives. Some results of the program are listed in Section VII.

```
      IMPLICIT INTEGER (A-Z)
      DIMENSION T(625,32), GLB(32,32), LUB(32,32)
      DIMENSION NS(32,32)
    1 FORMAT(32I2)
    2 FORMAT('OTRANSLATIONS',32I3)
    3 FORMAT('-N=',10X,I10)
    4 FORMAT('1AT END OF R1,N=',I10)
    5 FORMAT('-AT END OF S1,N=',I10)
    6 FORMAT('-AT END OF R2,N=',I10)
    7 FORMAT('-AT END OF S2,N=',I10)
    8 FORMAT('-AT END OF R3,N=',I10)
    9 FORMAT('-AT END OF S3,N=',I10)
   10 FORMAT('-AT END OF R4,N=',I10)
   11 FORMAT('-AT END OF S4,N=',I10)
   12 FORMAT('-AT END OF R5,N=',I10)
   13 FORMAT('-AT END OF S5,N=',I10)
   14 FORMAT('-AT END OF R6,N=',I10)
   15 FORMAT('-AT END OF S6,N=',I10)
   16 FORMAT('-AT END OF R7,N=',I10)
   17 FORMAT('-AT END OF S7,N=',I10)
   18 FORMAT('-AT END OF R8,N=',I10)
   19 FORMAT('-AT END OF S8,N=',I10)
   20 FORMAT('-AT END OF R9,N=',I10)
   21 FORMAT('-AT END OF S9,N=',I10)
   25 FORMAT(I5)
      READ(1,25)(R)
      READ(1,1)((GLB)A,B),B=1,R),A=1,R),((LUB(A,B),B=1,R,)A=1,R)
      WRITE(3,2)((GLB(A,B),B=1,R),A=1,R),((LUB(A,B),B=1,R),A=1,R)
      DO 30 A=1,R
      DO 30 B=1,R
   30 NS(A,B)=GLB(A,B)
      GO TO 500
   31 DO 32 A=1,R
      DO 32 B=1,R
   32 NS(A,B)=LUB(A,B)
      GO TO 510
   40 DO 49 RR1=1,R1
      DO 49 J=1,R
      DO 45 B=1,R
      MM=T(RR1,B)
   45 NS(1,B)=LUB(MM,J)
      KK=1
      GO TO 530
   49 CONTINUE
      R2=N
      WRITE(3,6)(R2)
      Q=R1+1
   50 DO 59 SS1=Q,S1
      DO 59 J=1,R
      DO 55 B=1,R
      MM=T(SS1,B)
```

```
55 NS(1,B)=GLB(MM,J)
   KK=2
   GO TO 530
59 CONTINUE
   S2=N
   WRITE(3,7)(S2)
   IF(S2-S1) 999,999,60
60 Q=S1+1
61 DO 69 RR2=Q,R2
   DO 69 J=1,R
   DO 65 B=1,R
   MM=T(RR2,B)
65NS(1,B)=GLB(MM,J)
   KK=3
   GO TO 530
69 CONTINUE
   R3=N
   WRITE(3,8)(R3)
   Q=R2+1
70 DO 79 SS2=Q,S2
   DO 79 J=1,R
   DO 75 B=1,R
   MM=T(SS2,B)
75 NS(1,B)=LUB(MM,J)
   KK=4
   GO TO 530
79 CONTINUE
   S3=N
   WRITE(3,9)(S3)
   IF(S3-S2) 999,999,80
80 Q=S2+1
81 DO 89 RR3=Q,R3
   DO 89 J=1,R
   DO 85 B=1,R
   MM=T(RR3,B)
85 NS(1,B)=LUB(MM,J)
   KK=5
   GO TO 530
89 CONTINUE
   R4=N
   WRITE(3,10)(R4)
   Q=R3+1
90 DO 99 SS3=Q,S3
   DO 99 J=1,R
   DO 95 B=1,R
   MM=T(SS3,B)
95 NS(1,B)=GLB(MM,J)
   KK=6
   GO TO 530
99 CONTINUE
   S4=N
   WRITE(3,11)(S4)
```

```
      IF(S4-S3) 999,999,100
  100 Q=S3+1
  101 DO 109 RR4=Q,R4
      DO 109 J=1,R
      DO 105 B=1,R
      MM=T(RR4,B)
  105 NS(1,B)=GLB(MM,J)
      KK=7
      GO TO 530
  109 CONTINUE
      R5=N
      WRITE(3,12)(R5)
      Q=R4+1
  110 DO 119 SS4=Q,S4
      DO 119 J=1,R
      DO 115 B=1,R
      MM=T(SS4,B)
  115 NS(1,B)=LUB(MM,J)
      KK=8
      GO TO 530
  119 CONTINUE
      S5=N
      WRITE(3,13)(S5)
      IF(S5-S4) 999,999,120
  120 Q=S4 +1
  121 DO 129 RR5=Q,R5
      DO 129 J-1,R
      DO 125 B=1,R
      MM=T(RR5,B)
  125 NS(1,B=LUB(MM,J)
      KK=9
      GO TO 530
  129 CONTINUE
      R6=N
      WRITE(3,14)(R6)
      Q=R5+1
  130 DO 139 SS5=Q,S5
      DO 139 J=1,R
      DO 135 B=1,R
      MM=T(SS5,B)
  135 NS(1,B)=GLB(MM,J)
      KK=10
      GO TO 530
  139 CONTINUE
      S6=N
      WRITE(3,15)(S6)
      IF(S6=S4) 999,999,140
  140 Q=S5+1
  141 DO 149 RR6=Q,R6
      DO 149 J=1,R
      DO 145 B=1,R
      MM=T(RR6,B)
```

```
145 NS(1,B)=GLB(MM,J)
    KK=11
    GO TO 530
149 CONTINUE
    R7=N
    WRITE(3,16)(R7)
    Q=R6+1
150 DO 159 SS6=Q,S6
    DO 159 J=1,R
    DO 155 B=1,R
    MM=T(SS6,B)
155 NS(1,B)=LUB(MM,J)
    KK=12
    GO TO 530
159 CONTINUE
    S7=N
    WRITE(3,17)(S7)
    IF(S7-S6) 999,999,160
160 Q=S6+1
161 DO 169 RR7=Q,R7
    DO 169 J=1,R
    DO 165 B=1,R
    MM=T(RR7,B)
165 NS(1,B)=LUB(MM,J)
    KK=13
    GO TO 530
169 CONTINUE
    R8=N
    WRITE(3,18)(R8)
    Q=R7+1
170 DO 179 SS7=Q,S7
    DO 179 J=1,R
    DO 175 B=1,R
    MM=T(SS7,B)
175 NS(1,B)=GLB(MM,J)
    KK=14
    GO TO 530
179 CONTINUE
    S8=N
    WRITE(3,17)(S8)
    GO TO 999
500 N=0
    DO 502 A=1,R
    N=N+1
    DO 502 B=1,R
502 T(N,B)=NS(A,B)
    R1=N
    WRITE(3,4)(R1)
    GO TO 31
510 A=1
512 W=0
513 W=W+1
```

```
      GO TO 516
  515 A=A+1
      W=1
  516 B=0
  517 B=B+1
      IF(W-N) 519,519,523
  519 IF(B-R) 520,520,515
  520 IF(A-R) 521,521,526
  521 IF(T(W,B)-NS(A,B)) 513,517,513
  523 N=N+1
      DO 525 B=1,R
  525 T(N,B)=NS(A,B)
      GO TO 512
  526 S1=N
      WRITE(3,5)(S1)
      GO TO 40
  530 W=0
  531 W=W+1
      IF(WON) 532,532,537
  532 IF(T(W,32)-NS(1,32)) 531,533,531
  533 B=0
  534 B=B+1
  535 IF(B-R) 536,536,539
  536 IF(T(W,B)-NS(1,B)) 531,534,531
  537 N=N+1
      DO 538 B=1,R
  538 T(N,B)=NS(1,B)
  539 GO TO (49,59,69,79,89,99,109,110,129,139,149,159,169,179,KK
  999 WRITE(3,2)((T(W,B),B=1,R),W=1,N)
      WRITE(3,3)(N)
      STOP
      END
```

VII. Some Specific Problems

Some results which have been produced by the computer program will be presented at this time.

Example 7.1. The non-modular five-element lattice of Figure 1 has Card T = 18 and a translational dimension of L = 3.

Example 7.2. The modular, non-distributive five-element lattice of Figure 2 has Cart T = 42 and a translational dimension of L = 4.

Example 7.3. The modular, non-distributive lattice of Figure 3, which is generated by three elements, has a Card T = 62 and a translational dimension of L = 4.

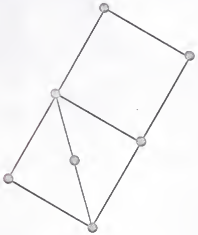

Fig. 3

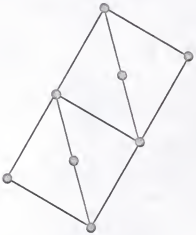

Fig. 4

Example 7.4. The non-modular lattice of Figure 5 has a Card T = 83 and a translational dimension of L = 7.

Example 7.5. The free modular lattice with three generators of Figure 6 has a Card T = 445 and a translational dimension of L = 4.

Example 7.6. The modular, non-distributive lattice with four generators of Figure 4 has a Card T = 136 and a translational dimension of L = 6.

PLATE I

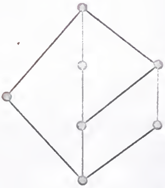

Fig. 5

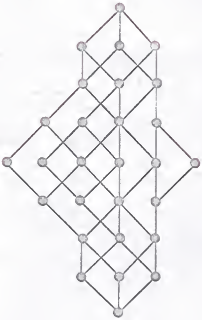

Fig. 6

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TRANSLATIONS IN LATTICES

by

DON F. HUNZIKER

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AN ABSTRACT OF A MASTER'S REPORT

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KANSAS STATE UNIVERSITY
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1968

Some introductory statements are made about partially ordered sets and the ordering relations which they possess. From the definition of a partially ordered set, the idea of a lattice is introduced.

The concepts of distributive and modular lattices are discussed. These are the two types of lattices which are of interest in the main portion of the paper. Some of the properties of the two operations in the distributive and modular lattices are described in the same section.

The paper then deals with the idea of translations in a lattice L. The translations of a lattice L are studied to find the smallest n such that any translation of L is the product of at most n elementary translations. The number n is called the translational dimension of L.

Some theorems are presented regarding the connection of the translational dimension of a lattice and whether the lattice is distributive or modular.

The main portion of the paper is dedicated to the adaptation of the problem of finding the translational dimension of a lattice to the computer. The program which has been developed will calculate the translational dimension of lattices with a small number of elements, say twenty-five, and a translation dimension of seven or less. A very definite time factor is involved if the lattice has more elements than twenty-five.

Included in the paper are several lattices which have been processed by the program. The translational dimension and the number of translations which are computed from each different lattice are noted. These results agree with those results computed by hand methods for the same lattices by Dr. Grillet, except that a few translations were picked up which had been omitted in the hand computations.