ELEMENTARY CONCEPTS CONCERNING THE LEBESGUE INTEGRAL

by

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TABLE OF CONTENTS

INTRODUCTION

Before developing the Lebesgue integral, there must be a basic understanding of Lebesgue measurable sets and Lebesgue measurable functions. By considering a typical term of the Riemann sum for a real-valued function f(x) over an interval [a, b] it can be seen that this term is a product of two numbers, the value of the function f(x) at a specific point times the length of a sub-interval of the interval [a, b] which contains the point. This sub-interval is obtained by partitioning the interval [a, b], which is the domain of definition of the function f(x).

The corresponding situation with the Lebesgue integral is not as simple. A typical term of a "Lebesgue sum" for a function f(x) over an interval [a, b] is again a product of two factors, but these factors are obtained quite differently. One of the factors, say α, is a value of the function, but the value is related to a partition P for the range of the function, and not a partition of the domain. The other factor, say β, is a number that represents "length" or measure of a set E of all points x in the domain for which f(x) is between a particular pair of elements, say (α, η) of P. This measure is a generalization of length obtained by covering a set E with a countable number of open sets. The set E is not necessarily an interval. Defining the Lebesgue measure for these sets is discussed in the first part of this report.

Lebesgue measurable functions, or the functions "compatible" with Lebesgue measurable sets, are discussed in the second part of the report. Then the Lebesgue integral is defined for bounded Lebesgue measurable functions, and elementary properties are presented.

In the next part of the report the Lebesgue integral is compared with the Riemann integral, and it is shown that the set of all Riemann integrable functions is a proper subset of the set of all Lebesgue integrable functions on a closed interval. The Lebesgue integral is superior to the Riemann integral in the area of finding limits relative to integration processes. The Lebesgue integral of a derivative is shown to yield the primitive for more general conditions than the Riemann integral. The last unit illustrates a weakness of the Lebesgue integral encountered when the derivative to be integrated is not required to be bounded.

LEBESGUE MEASURABLE SETS

The discussion will be restricted to sets that are bounded subsets of the real number line R. To define the Lebesgue measure of a set, two other numbers are defined; these numbers are the outer and inner Lebesgue measure of a set. Basic to the understanding of these two numbers is the concept of length of an open interval, which will now be defined.

Definition 1. The length of an open interval (a, b) is the number b-a.

If I = (a, b), then $\ell(I)$ will denote the length of I. Hence $\ell(I)$ = b-a, whenever I = (a, b). Obviously $\ell(I)$ is a nonnegative number.

Another concept besic to the understanding of outer and inner Lebesgue meesure is the concept of e component open interval.

Definition 2. Let G be any open subset of R. If the open intervel (e, b) is contained in G and its endpoints do not belong to G,

$$(e, b) \subset G, \quad e \notin G, \quad b \notin G,$$

then this interval is seid to be e component open intervel or e component of the set G.

Example: Let $G = (0, 1) \cup (2, 3)$. Then $(0,1)$ and $(2, 3)$ ere component open intervels of the set G.

Using these two definitions, eny set $E \subset R$ thet is the union of e finite or denumereble number of disjoint component intervels cen be essigned a number equal to the sum of the lengths of the component open intervels, if such a sum exists.

Definition 3. Let E be the union of a finite or denumereble number of peirwise disjoint open intervels. Associete with E the number L(E) such that if

$$E = \bigcup_k I_k \qquad (k = 1, 2, \ldots),$$

then

$$L(E) = \sum_k \ell(I_k) \qquad (k = 1, 2, \ldots),$$

whenever this sum exists.

A reason for the preceding definition becomes apparent upon considering the following theorem.

Theorem 1. If G is an open set of real numbers then G is the union of a finite or denumerable number of disjoint open intervals, called the component open intervals of G [2, 73].[1]

Proof. Associate with every $x \in G$ an open interval I_x in the following way. Let

$$I_x = \bigcup_{\alpha} I_{\alpha}, \quad \alpha \in A,$$

for some indexing set A, such that $I_{\alpha} = (a_{\alpha}, b_{\alpha}) \subset G$ and $x \in I_{\alpha}$. Let λ be the greatest lower bound of the a_{α}, and μ be the least upper bound of the b_{α}. Then $I_x = (\lambda, \mu)$. This may be seen by assuming $y \geqslant \mu$ or $y \leqslant \lambda$. If $y \geqslant \mu$, then $y \notin I_{\alpha}$ for any $\alpha \in A$; or if $y \leqslant \lambda$, $y \notin I_{\alpha}$ for any $\alpha \in A$, hence $y \notin I_x$. Now it will be shown that if $y \in (\lambda, \mu)$, $y \in I_x$. If $y \in (\lambda, \mu)$, then either $y = x$ or $x < y < \mu$, or $\lambda < y < x$. If $y = x$, then $y \in I_x$. If $x < y < \mu$, then there is an α such that $y \in I_{\alpha}$, since μ is the least upper bound of the b_{α}'s. Also if $\lambda < y < x$ there exists an α such that $y \in I_{\alpha}$, since λ is the greatest lower bound of the a_{α}'s. Therefore $y \in I_x$. Now it will be shown that if $x \in G$ and $y \in G$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Suppose $c \in I_x \cap I_y$, then $I_x \cup I_y$ is an open interval. Since $I_x \cup I_y$ contains x, it follows that $I_x \cup I_y \subset I_x$. Also $I_x \cup I_y$ contains y, so that $I_x \cup I_y \subset I_y$. Therefore if $c \in I_x \cap I_y$, $I_y = I_x$.

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[1] Throughout the report this notation will be used: the first number indicates the number of the reference at the end of the report, and the second number indicates the page number.

Finally, any set of disjoint open intervals is finite or denumerable in number. Associate with each open interval of the set a rational number which is in the interval. Since disjoint open intervals are associated in this way with distinct rational numbers, the cardinal number of this set of open intervals does not exceed the cardinal number of the set of rational numbers, and so it is either finite or denumerable.

Since the null set $\emptyset$ is considered to be open, the number $L(G)$ associated with this set will be zero. Therefore a non-negative number $L(G)$ can be associated with every open set G; that is, $L(G) \geqslant 0$.

The definition of outer Lebesgue measure will now be given.

Definition 4. For every set S, the outer Lebesgue measure,

$$m^*(S) = \inf \{L(G) : G \supset S\},$$

where G varies over all open sets containing S [2, 154].

The following theorem can be proven for any open set G.

Theorem 2. If G is an open set, then

$$m^*(G) = L(G) \quad [2, 155].$$

Proof. Let $H \supset G$ be an open set. Then every component of G is contained in a component of H. Thus $L(H) \geqslant L(G)$. But $G \supset G$ is an open set. Hence

$$\inf \{L(H) : H \supset G\} = L(G),$$

and

$$m^*(G) = L(G).$$

Another important property of outer Lebesgue measure will be presented before defining inner Lebesgue measure.

<u>Theorem 3</u>. Let A and B be bounded subsets of R. If $A \subset B$, then

$$m^*(A) \leq m^*(B) \quad [3,\ 64].$$

Proof. Let S be a set consisting of the numbers $L(G_\alpha)$ associated with all open sets G_α containing A, where α belongs to an indexing set J. Let T be a set consisting of the numbers $L(H_\beta)$ associated with all open sets H_β containing B, where β belongs to an indexing set K. If E is an open set containing B, then E necessarily contains A, since $A \subset B$. Therefore

$$T \subset S,$$

and

$$m^*(A) = \inf(S) \leq \inf(T) = m^*(B) .$$

Now inner Lebesgue measure can be defined. Let $\Delta = [a,\ b]$ represent any bounded closed interval of R. Let $S \subset \Delta$, and $C_\Delta(S)$ represent the complement of S in the interval Δ.

<u>Definition 5</u>. For every set S the inner measure of S is the number

$$m_*(S) = (b-a) - m^*(C_\Delta(S)) \quad [4,\ 31].$$

The definition of a Lebesgue measurable set may now be given.

<u>Definition 6</u>. Let E be any bounded subset of R. The set E is Lebesgue measurable if its outer and inner measures are equal; that is,

$$m^*(E) = m_*(E) \; [4, 31].$$

The common value of these measures is called the Lebesgue measure of the set E, and is denoted m(E).

Now that the definition of Lebesgue measure has been established, it is important to consider several families of sets which are actually measurable according to this definition. In order to accomplish this goal a few elementary properties are presented. The following lemma will be useful in proving these elementary properties.

<u>Lemma 1</u>. If $I_1', I_2', \ldots, I_n'$ are a finite number of open intervals which cover $\Delta = [a, b]$, then

$$\sum_{k=1}^{n} \ell(I_k') \geqslant b - a \quad [2, 155].$$

Proof. It may be assumed without loss of generality that $I_k' \cap \Delta \neq \emptyset$, for every $k = 1, 2, \ldots, n$. Let $I_k' = (a_k, b_k)$, $k = 1, 2, \ldots, n$. It may also be assumed without loss of generality that $a \in I_1' = (a_1, b_1)$. Let $b_1 \in I_2'$, and in general

$$b_k \in I_{k+1}' = (a_{k+1}, b_{k+1}), \qquad (k = 1, 2, \ldots, n-1)$$

where $b < b_n$. Hence

$$b - a < b_n - a_1 = (b_n - b_{n-1}) + \ldots + (b_2 - b_1) + (b_1 - a_1) \leqslant \sum_{k=1}^{n} (b_k - a_k),$$

and the proof is complete.

It is now possible to prove the following elementary property for any bounded subset of e closed interval.

<u>Theorem 4</u>. For every set $S \subset \Delta$, where $\Delta = [e, b]$,
$m^*(S) + m^*C_\Delta(S) \geq b - e$ [2, 155].

Proof. Let G end H be open sets such that $S \subset G$ and $C_\Delta(S) \subset H$. Let $I_1, I_2, \ldots$ be the component intervals of G and $J_1, J_2, \ldots$ be the component intervals of H. Since every $x \in \Delta$ is either in S or $C_\Delta(S)$, the open intervels $I_1, I_2, \ldots, J_1, J_2, \ldots$ cover Δ. But Δ is a closed bounded set, hence by the Borel Covering Theorem, e finite number of theee intervals, eey $I_{k_1}, I_{k_2}, \ldots, I_{k_m}$ end $J_{k_1}, J_{k_2}, \ldots, J_{k_n}$ cover Δ. By lemme one, the sum

$$\sum_{i=1}^{m} \ell(I_{k_i}) + \sum_{j=1}^{n} \ell(J_{k_j}) \geq b - a.$$

But

$$L(G) \geq \sum_{i=1}^{m} \ell(I_{k_i}) \text{ and } L(H) \geq \sum_{j=1}^{n} \ell(J_{k_j}),$$

hence

$$L(G) + L(H) \geq b - a.$$

It follows thet

$$m^*(S) + m^*(C_\Delta(S)) = \inf\{L(G): G \supset S\} + \inf\{L(H): H \supset C_\Delta(S)\}$$
$$= \inf\{L(G) + L(H): G \supset S,\ H \supset C_\Delta(S)\} \geq b - e.$$

The following corollery releting outer end inner meeeure is epparent.

<u>Corollery 1</u>. For every $S \subset \Delta$, where $\Delta = [e, b]$,
$m^*(S) \geq m_*(S) \geq 0$.

An elementary property of Lebesgue measure will now be proved.

Theorem 5. A set $S \subset \Delta$, $\Delta = [a, b]$, is measurable if and only if

$$m^*(S) + m^*(C_\Delta(S)) = b - a \quad [2, 156].$$

Proof. Assume the set S is measurable. Then,

$$m^*(S) = m_*(S) = (b - a) - m^*(C_\Delta(S));$$

therefore

$$m^*(S) + m^*(C_\Delta(S)) = b - a.$$

Now assume

$$m^*(S) + m^*(C_\Delta(S)) = b - a.$$

Then it follows that

$$m^*(S) = (b - a) - m^*(C_\Delta(S)) = m_*(S),$$

and S is measurable.

By combining the results of Theorem 4 and Theorem 5, the following theorems are obvious.

Theorem 6. A set $S \subset \Delta$, where $\Delta = [a, b]$, is nonmeasurable if and only if

$$m^*(S) + m^*(C_\Delta(S)) > b - a \quad [2, 156].$$

Theorem 7. Let S be any measurable subset of the interval $\Delta = [a, b]$. Then $C_\Delta(S)$ is also measurable [2, 156].

The following theorem establishes the measurability of an important family of sets.

<u>Theorem 8</u>. Let S be a subset of the interval $\Delta = [a, b]$. If $m^*(S) = 0$, then S is measurable and has measure zero $[2, 156]$.

Proof. The proof follows immediately from the corollary to Theorem 4.

The following theorem establishes the measurability of countable sets.

<u>Theorem 9</u>. Every countable set $A \subset R$ is Lebesgue measurable with $m(A) = 0$ $[4, 33]$.

Proof. Let A be the set of elements $a_1, a_2, \ldots, a_n, \ldots$. Given $\epsilon > 0$, cover the elements $a_1, a_2, \ldots$ with open intervals $I_{a_1}, I_{a_2}, \ldots, I_{a_n}, \ldots$, respectively, such that

$$\ell(I_{a_n}) < \frac{\epsilon}{2n} \qquad (n = 1, 2, \ldots).$$

Then the sum of the lengths

$$\sum_{n=1}^{\infty} \ell(I_{a_n}) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \epsilon \cdot 1 = \epsilon.$$

Since ϵ is an arbitrary positive real number, $m^*(A) = 0$.

Examples of sets which are measurable include the set of integers, the set of positive integers, the set of rational numbers, and the set of irrational numbers in the interval $(0, 1)$.

Another important family of sets is the collection of open sets. The following lemma is used to prove sets in this family are measurable.

Lemma 2. If $J_1, J_2, \ldots$ are open intervals and the open set $G = \bigcup_{n=1}^{\infty} J_n$ has components $I_1, I_2, \ldots$, then

$$\sum_{n=1}^{\infty} \ell(I_n) \leq \sum_{n=1}^{\infty} \ell(J_n) \quad [2, 157].$$

Proof. If $J_1, J_2, \ldots$ are disjoint open intervals, then they are identically the components of G and

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \ell(J_n).$$

Therefore assume that $J_1, J_2, \ldots$ are not all disjoint. Then for some J_i, J_j, $i \neq j$ there exist $x_{i,j}$ such that

$$x_{i,j} \in J_i \cap J_j, \qquad (i,j = 1, 2, \ldots).$$

Let

$$J_i' = \{x : x \in J_i \text{ and } x \notin J_j\}, \; J_{ij} = \{x : x \in J_i \cap J_j\},$$

and

$$J_j' = \{x : x \in J_j \text{ and } x \notin J_i\}.$$

The contribution of these sets to the sum of the components of G is[2]

$$\ell(J_i') + \ell(J_{ij}) + \ell(J_j'),$$

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[2] The J_i' and J_j' are half-open intervals of the form $(a, b]$ and $[a, b)$. The following definition of length is used for these half-open sets:

$$\ell(a, b] = b - a$$

and

$$\ell[a, b) = b - a.$$

whereas the contribution to the sum $\sum_{n=1}^{\infty} \ell(J_n)$ is

$$\ell(J_i') + \ell(J_{ij}) \quad + \quad \ell(J_j') + \ell(J_{ij}) \quad ,$$

since

$$J_i = J_i' \cup J_{ij} \quad \text{and} \quad J_j = J_j' \cup J_{ij} \; .$$

Therefore since all of these numbers are nonnegative, it can be seen that the contribution to $\sum_{i=1}^{\infty} \ell(J_i)$ is greater than the contribution to $\sum_{n=1}^{\infty} \ell(I_n)$. Hence

$$\sum_{n=1}^{\infty} \ell(I_n) \leq \sum_{n=1}^{\infty} \ell(J_n).$$

The following important theorem is proved.

Theorem 10. Every open set $G \subset \Delta = [a, b]$ is measurable [2, 157].

Proof. Since G is open, it can be written as the union of a finite or denumerable number of component open intervals I_k that are disjoint. Since the series

$$\sum_{k=1}^{\infty} \ell(I_k)$$

is convergent, for every $\epsilon > 0$, there is a number $n(\epsilon)$ such that

$$\sum_{k=n+1}^{\infty} \ell(I_k) < \frac{\epsilon}{2}$$

whenever $n > n(\epsilon)$. Since G is open,

$$L(G) = \sum_{k=1}^{n} \ell(I_k) + \sum_{k=n+1}^{\infty} \ell(I_k),$$

and, by substitution,

$$L(G) < \sum_{k=1}^{n} \ell(I_k) + \frac{\epsilon}{2}$$

or
$$L(G) - \frac{\epsilon}{2} < \sum_{k=1}^{n} \ell(I_k).$$

Now let $J_1, J_2, \ldots, J_m$ be the intervals in Δ complementary to $I_1, I_2, \ldots, I_n$. Also let J_k' $(k = 1, 2, \ldots, m)$, be an open interval concentric with J_k such that

$$\ell(J_k') = \ell(J_k) + \frac{\epsilon}{2m}, \qquad (k = 1, 2, \ldots, m).$$

Let $H = \bigcup_{k=1}^{m} J_k'$; then $L(H) \leqslant \sum_{k=1}^{m} \ell(J_k')$ by lemma 2. Since

$$\sum_{k=1}^{m} \ell(J_k) + \sum_{k=1}^{n} \ell(I_k) = b - a,$$

it follows that

$$\sum_{k=1}^{m} \ell(J_k') + \sum_{k=1}^{n} \ell(I_k) < (b - a) + \frac{\epsilon}{2}.$$

Thus $L(H) + L(G) < (b - a) + \epsilon$, and since $C_\Delta(G) \subset H$,

$$m^*(G) + m^*(C_\Delta(G)) < (b - a) + \epsilon.$$

Therefore, since ϵ is an arbitrary positive real number,

$$m^*(G) + m^*(C_\Delta(G)) \leqslant b - a;$$

and, by Theorem 6, G is measurable.

Examples of open sets include open intervals, and sets composed of a finite or denumerable number of open intervals. By Theorem 1, these are the only open sets, with the exception of the null set.

Another family of sets is now proved to be measurable.

Theorem 11. Every closed set $F \subset \Delta$, $\Delta = [e, b]$, is measur-eble [2, 158].

Proof. Since every closed set is the complement of en open set, then every cloeed set is measureble by Theorem 7.

Exemplas of closed sets include finite sets end the closed intervals. Therefora $\{1, 2, 3\}$ end $[0, 1]$ ere meesureble eets. Also eny union of e finite number of closed eets is closed, and therefore meesureble by Theorem 11.

In perticular, the closed intarval $\Delta = [e, b]$ is meesur-abla, end hes meesure b - e. This fect will now be established.

Theorem 12. If Δ is the closed interval $[e, b]$, then Δ is meesureble end $m(\Delta) = b - e$.

Proof. Since Δ is closed, Δ is maesureble by Theorem 11, end

$$m^*(\Delta) = m_*(\Delta) = (b - e) - m^*(C_\Delta(\Delta)).$$

But $C_\Delta(\Delta) = \emptyset$, end $m(\emptyset) = 0$, therefore

$$m^*(\Delta) = m_*(\Delta) = (b - e) - 0 = b - e.$$

Therefore it cen be seen thet the number b - e in the pre-ceding theoreme end definitions wes ectuelly the Lebesgua mae-sure of the intarvel.

In order to devalop the elementery properties of meesur-able functions and to esteblish the definition of the Lebesgue integrel, unions end intarsections of measureble sets must be considerad.

Theorem 13. If e bounded set E is tha union of e finite or denumereble number of meesureble sets which ere disjoint,

$$E = \bigcup_k E_k \qquad (E_k \cap E_{k'} = \emptyset,\ k \neq k'),$$

then E is meesureble end

$$m(E) = \sum_k m(E_k) \quad [3,\ 67].$$

Proof. The proof follows from the inequalities

$$\sum_k m(E_k) = \sum_k m_*(E_k) \leq m_*(E) \leq m^*(E) \leq \sum_k m^*(E_k)$$
$$= \sum_k m(E_k) \ ,$$

since outer meesure is countebly subedditive [3, 64] end the inequality for inner meesure holds [3, 65].

Theorem 14. The union of e finite number of meesureble sets is e measureble set [3, 67].

Proof. Let $E = \bigcup_{k=1}^{n} E_k$, where eech E_k is meesureble. Given $\epsilon > 0$, there exists e closed set F_k and e bounded open set G_k such thet $F_k \subset E_k \subset G_k$, end $m(G_k) - m(F_k) < \frac{\epsilon}{n}$. Set

$$F = \bigcup_{k=1}^{n} F_k,\ G = \bigcup_{k=1}^{n} G_k \ ,$$

where F end G ere closed end open sets respectively. Since $F \subset E \subset G$,

$$m(F) \leq m_*(E) \leq m^*(E) \leq m(G).$$

The set G - F is open, since it cen be represented in the form $G \cap C_G(F)$, end is therefore meesureble. Since G cen be represented es

$$G = F \cup (G - F)$$

where F and G - F are disjoint measurable sets, the preceding theorem applies and

$$m(G) = m(F) + m(G - F).$$

Therefore

$$m(G - F) = m(G) - m(F)$$

and

$$m(G_k - F_k) = m(G_k) - m(F_k).$$

Since

$$G - F \subset \bigcup_{k=1}^{n} (G_k - F_k),$$

and all these sets are open, it follows that

$$m(G - F) \leq \sum_{k=1}^{n} m(G_k - F_k),$$

or

$$m(G) - m(F) \leq \sum_{k=1}^{n} \left[m(G_k) - m(F_k) \right] < \epsilon .$$

Therefore $m^*(E) - m_*(E) < \epsilon$, and E is measurable.

The analogous theorem for intersections of measurable sets is given.

Theorem 15. The intersection of a finite number of measurable sets is a measurable set [3, 68].

Proof. Let $E = \bigcap_{k=1}^{n} E_k$, where the sets E_k are measurable sets. Let Δ be any open interval containing all the sets E_k. It can be verified that

$$C_\Delta(E) = \bigcup_{k=1}^{n} C_\Delta(E_k).$$

The sets $C_\Delta(E_k)$ ere measurable, since the sets E_k are measurable, and by Theorem 14, $C_\Delta(E)$ is measurable. Hence E is also measurable, since $C_\Delta(C_\Delta(E)) = E$.

The next two theorems establish results for unions and intersections of denumerable measurable sets.

Theorem 16. If a bounded set E is the union of a denumerable number of measurable sets, then E is measurable [3, 69].

Proof. Let $E = \bigcup_{k=1}^{\infty} E_k$. Let $A_k (k = 1, 2, \ldots)$, be sets such thet

$$A_1 = E_1,\ A_2 = E_2 - E_1,\ \ldots,\ A_k = E_k - (E_1 \cup \ldots \cup E_{k-1}),\ \ldots,$$

then

$$E = \bigcup_{k=1}^{\infty} A_k.$$

All these A_k are measurable and are disjoint, therefore E is measurable by Theorem 13.

Theorem 17. The intersection of a denumerable number of meesureble sets is measurable [3, 69].

Proof. Let $E = \bigcap_{k=1}^{\infty} E_k$, where ell the sets E_k ere meesurable. Since $E \subset E_1$, E is bounded. Let Δ be eny open intervel conteining E, end let

$$A_k = \Delta \cap E_k.$$

Then

$$E = \Delta \cap E = \Delta \cap \bigcap_{k=1}^{\infty} E_k = \bigcap_{k=1}^{\infty} (\Delta \cap E_k) = \bigcap_{k=1}^{\infty} A_k.$$

But

$$C_\Delta(E) = \bigcup_{k=1}^{\infty} C_\Delta(A_k),$$

and by epplying Theorem 7 end Theorem 16 this completes the proof.

One mey be led to believe thet all sets ere meesureble, or thet ell bounded sets are meesureble. That this is not the cese hes been proved [3, 76], [2, 165]; in fect, it cen be shown thet, "Every meesureble set of positive measure contains e nonmeesureble subset" [3, 78]. Exemples ere evaileble [1, 92], [4, 47], elthough the choice exiom is ueed to construct them [4, 50].

LEBESGUE MEASURABLE FUNCTIONS

The concept of measureble functions is elso besic to the understending of the Lebesgue integrel. In this part of the report meesurable functions ere defined, and e few elementery properties ere presented.

Definition 7. The reel-velued function $f(x)$ ie meesureble in $[e, b]$ if the sets

$$\{x : \alpha \leq f(x) < \beta\} = E[\alpha \leq f(x) < \beta]$$

ere meesureble for every peir of reel numbers α, β with $\alpha < \beta$ [4, 67].

Insteed of the set ueed above, eny one of the following sets could be used:

$$E\left[\alpha < f(x) < \beta\right],\ E\left[\alpha \leq f(x) \leq \beta\right],\ \text{or}\ E\left[\alpha < f(x) \leq \beta\right]$$

[4, 67].

The following theorem is an important consequence of this fact.

Theorem 18. If all sets of one of these four types are measurable, then the sets

$$E\left[f(x) = \alpha\right]$$

are also measurable for every real number α [4, 67].

Proof. The proof follows from the fact that

$$E\left[f(x) = \alpha\right] = \bigcap_n E\left[\alpha - \frac{1}{n} \leq f(x) < \alpha + \frac{1}{n}\right],$$

$$(n = 1, 2, \ldots).$$

The following theorem is very useful in deriving certain basic characteristics of measurable functions.

Theorem 19. In order that $f(x)$ be measurable, it is necessary and sufficient that any one of the following sets is measurable for arbitrary real numbers α and β, respectively:

$$E\left[\alpha \leq f(x)\right],\ E\left[f(x) \leq \beta\right],\ E\left[\alpha < f(x)\right],\ \text{or}\ E\left[f(x) < \beta\right]$$

[4, 68].

A few elementary properties of measurable functions can now be established.

Theorem 20. If $f(x)$ is measurable on a measurable set M, then $a - f(x)$, $a + f(x)$, $a \cdot f(x)$, and $-f(x)$ are also measurable, for any real number a [4, 68].

Proof. $-f(x)$ can be obtained from $a \cdot f(x)$ when $a = -1$; also $a - f(x) = a + (-f(x))$. Hence proofs are required only for $a + f(x)$ and $a \cdot f(x)$. The measurability of $a + f(x)$ follows from

$$E\left[\alpha \leq a + f(x)\right] = E\left[\alpha - a \leq f(x)\right],$$

which is measurable by Theorem 19. The measurability of $a \cdot f(x)$ can be established as follows: when $a = 0$, $a \cdot f(x) = 0$ is obviously measurable. For $a > 0$, it follows that

$$E\left[\alpha < a \cdot f(x)\right] = E\left[\frac{\alpha}{a} < f(x)\right],$$

which is also measurable by Theorem 19. For $a < 0$, the proof is similar.

The following theorem expresses a property peculiar to Lebesgue measure.

Theorem 21. If $f(x)$ is measurable, $|f(x)|$ is also measurable [4, 68].

Proof. The proof follows from the equality

$$E\left[\,|f(x)| \geq \alpha\right] = E\left[f(x) \geq \alpha\right] \cup E\left[f(x) \leq -\alpha\right], \; \alpha \in R .$$

At times a function may be proved to be measurable by representing it as the sum of two measurable functions. To prove that the sum of two measurable functions is measurable the following theorem may be used.

Theorem 22. If f_1 and f_2 are measurable, then

$$E\left[f_1(x) > f_2(x)\right]$$

is also measurable [4, 69].

<u>Theorem</u> 23. If f_1 end f_2 ere meesureble, then $f_1 + f_2$ end $f_1 - f_2$ are elso meesureble [4, 69].

Proof. Since $f_1 - f_2 = f_1 + (-f_2)$, end $-f_2$ is meesureble by Theorem 20, the proof is required only for $f_1 + f_2$. Since

$$E\left[f_1(x) + f_2(x) > \alpha\right] = E\left[f_1(x) > \alpha - f_2(x)\right]$$

end $\alpha - f_2(x)$ is measureble by Theorem 20, it follows from Theorem 22 thet the sets

$$E\left[f_1(x) > \alpha - f_2(x)\right]$$

ere elso measureble.

The following theorem expresses another elementery property of meesureble functions.

<u>Theorem</u> 24. If $f(x)$ is meesureble, $f^2(x)$ is elso meesureble [4, 69].

Proof. Consider the following reletionship:

$$E\left[f^2(x) \geqslant \alpha\right] = E\left[f(x) \geqslant \sqrt{\alpha}\right] \cup E\left[f(x) \leqslant -\sqrt{\alpha}\right], \quad (\alpha \geqslant 0).$$

Then since $E\left[f^2(x) \geqslant \alpha\right]$ is the union of two meesureble sets, $f^2(x)$ is also meesureble.

The following theorem is en immediete consequence of the preceding theorem.

<u>Theorem</u> 25. If $f(x)$ and $g(x)$ are measureble real functions, them $f(x) \cdot g(x)$ is measurable [2, 185].

Proof. The proof follows from the equelity

$$f(x) \cdot g(x) = \frac{1}{4}\left\{\left[f(x) + g(x)\right]^2 - \left[f(x) - g(x)\right]^2\right\}.$$

The following theorem concerns functions of e very importent cless of meesurable functions.

Theorem 26. Every reel-valued function $f(x)$ continuous in $[a, b]$ is measureble on this closed intervel.

Proof. Consider the sets

$$E[f(x) \geqslant \alpha] = E_{\alpha} .$$

These sets are closed and therefore meesureble. The fact thet eech E_{α} is closed cen be shown es follows: Teke e sequence of points

$$p_{v} \in E_{\alpha}, \text{ where } p_{v} \longrightarrow p.$$

Since $p \in [e, b]$, the function $f(x)$ is continuous et p, and from $f(p_{v}) \geqslant \alpha$ it follows that

$$\lim_{v \to \infty} f(p_{v}) = f(p) \geqslant \alpha ,$$

which implies $p \in E_{\alpha}$.

The following discussion leeds to the importent conclusion thet the limit function of a sequence of meesureble functions is meesurable. This is helpful since it will be shown thet the Lebesgue integral of the limit function of a sequence of integreble functions exists, if the sequence of functions is of bounded variation.

Theorem 27. If $\{f_n(x)\}$ is e sequence of measureble functions, then $\sup [f_n(x)\colon n = 1, 2, \ldots]$ end $\inf [f_n(x)\colon n = 1, 2, \ldots]$ ere measureble if they exist [2, 185].

Proof. Let α be a real number. Then, if $f(x) = \sup [f_n(x)\colon n = 1, 2, \ldots]$, then

$$E\left[f(x) > \alpha\right] = \bigcup_{n=1}^{\infty} E\left[f_n(x) > \alpha\right]$$

is measurable, so that $\sup\left[f_n(x) : n = 1, 2, \ldots\right]$ is measurable. Similarly, $\inf\left[f_n(x) : n = 1, 2, \ldots\right]$ is measurable.

Theorem 28. If $\left\{f_n(x)\right\}$ is a sequence of measurable functions then $\limsup_{n\to\infty} f_n(x)$ and $\liminf_{n\to\infty} f_n(x)$ are measurable [2, 185].

Proof. Let

$$E\left[\limsup_{n\to\infty} f_n(x) < \alpha\right] = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E_{m,n},$$

where

$$E_{m,n} = \left[f_r(x) < \alpha - \frac{1}{n} : r = m, m + 1, \ldots\right].$$

But $E_{m,n}$ is measurable for every m,n so that $\bigcup_m \bigcup_n E_{m,n}$ is measurable and $\limsup_{n\to\infty} f_n(x)$ is measurable. Similarly, $\liminf_{n\to\infty} f_n(x)$ is measurable.

The following conclusion is established.

Corollary 1. If $\left\{f_n(x)\right\}$ is a convergent sequence of measurable functions and $f(x) = \lim_{n\to\infty} f_n(x)$, then $f(x)$ is measurable [2, 185].

DEFINITION OF THE LEBESGUE INTEGRAL

The form of the definition of the Riemann integral is not appropriate if the real function $f(x)$ is "badly" discontinuous since in any contribution to the Riemann sum the value of the function represents widely varying values of $f(x)$ over the interval. Lebesgue avoided this difficulty by applying horizontal strips instead of the vertical strips used by Riemann [4, 62]. A definition and discussion of the Lebesgue integral will now be given for which $f(x)$ is assumed to be bounded and Lebesgue measurable in $[a, b]$.

Let a partition $P = \{y_0, y_1, y_2, \ldots, y_n, y_{n+1}\}$ be given such that

$$\alpha = y_0 < y_1 < \ldots < y_n < y_{n+1} = \beta$$

where $\alpha \leq f(x) < \beta$. The notation

$$E_v = E[y_v \leq f(x) < y_{v+1}]$$

will be used to denote the set of $x \in [a, b]$ for which $y_v \leq f(x) < y_{v+1}$, where y_v, y_{v+1} are elements of P. Form the sums

$$s_P = \sum_{v=0}^{n} y_v \cdot m(E_v) \text{ and } S_P = \sum_{v=0}^{n} y_{v+1} \cdot m(E_v),$$

where $s_P \leq S_P$. Let P* be a subdivision or refinement of P, or all the points of P together with finitely many new ones. It is sufficient to consider a refinement P* of P which contains only one additional point $\bar{y}$. Let

$$\bar{y} \in (y_v, y_{v+1})$$

then

$$E_v' = E[y_v \leq f(x) < \bar{y}],\ E_v'' = E[\bar{y} \leq f(x) < y_{v+1}].$$

Hence

$$E_v = E_v' \cup E_v'',$$

where E_v' and E_v'' are disjoint. Therefore

$$m(E_v) = m(E_v') + m(E_v'');$$

and

$$y_v \cdot m(E_v) = y_v \left[m(E_v') + m(E_v'')\right] \leq y_v \cdot m(E_v') + \bar{y} \cdot m(E_v'')$$

and it follows that

$$s_P \leq s_{P*} .$$

Now consider the sum S_P, a typical term of which is $y_{v+1} \cdot m(E_v)$. Then

$$y_{v+1} \cdot m(E_v) = y_{v+1} \left[m(E_v') + m(E_v'')\right] \geq \bar{y} \cdot m(E_v') + y_{v+1} \cdot m(E_v'') ,$$

and it follows that

$$S_P \geq S_{P*}.$$

A combining of the above results yields

$$s_P \leq s_{P*} \leq S_{P*} \leq S_P.$$

The following theorem can now be proved.

<u>Theorem</u> <u>29</u>. If P' and P" are any two partitions of $[a, b]$, then

$$s_{P'} \leq S_{P''} \text{ and } s_{P''} \leq S_{P'} \quad [3, 119].$$

Proof. Form the partition $P''' = P' \cup P''$, that is, P"' is formed by using all the points of P' together with all the points of P". Thus P"' is a subdivision of P' and P" and

$$s_{P'} \leq s_{P'''} \leq S_{P'''} \leq S_{P''}$$

and

$$s_{P''} \leq s_{P'''} \leq S_{P'''} \leq S_{P'} .$$

From these inequalities it follows that

$$s_{P'} \leq S_{P''} \text{ and } s_{P''} \leq S_{P'} .$$

It is now possible to form a sequence of subdivisions $\{P_k\}$

with norm

$$d_k = \max_{(P_k)}(y_{v+1} - y_v), \qquad (k = 1, 2, \ldots),$$

such that $d_k \longrightarrow 0$, and such that

$$s_{P_1} \leq s_{P_2} \leq \ldots \leq s_{P_k} \leq \ldots \leq S_{P_k} \leq \ldots \leq S_{P_2} \leq S_{P_1} .$$

Thus s_{P_k} and S_{P_k} form bounded monotone sequences whose limits exist and

$$\lim_{k \to \infty} s_{P_k} = s \leq S = \lim_{k \to \infty} S_{P_k} .$$

Therefore

$$0 \leq S - s \leq S_{P_k} - s_{P_k} = \sum_v (y_{v+1} - y_v) \cdot m(E_v) \leq \sum_v d_k \cdot m(E_v)$$

$$= d_k \sum_v m(E_v) = d_k \cdot (b - a).$$

Since $d_k \longrightarrow 0$ as $k \longrightarrow \infty$, $d_k \cdot (b - a) \longrightarrow 0$, and $S = s$.

The Lebesgue integral can now be defined.

<u>Definition</u> <u>8</u>. The common value $S = s$ is called the Lebesgue integral of $f(x)$ in $[a, b]$, denoted

$$\int_a^b f(x)\, dx,$$

and is equivalent to

$$\int_a^b f(x)\, dx = \lim_{d_k \to 0} \sum_v y_v \cdot m(E_v)$$

$$= \lim_{d_k \to 0} \sum_v y_{v+1} \cdot m(E_v) ,$$

and also

$$= \lim_{d_k \to 0} \sum_v \lambda_v \cdot m(E_v) ,$$

where λ satisfies the inequality $y_v \leq \lambda_v \leq y_{v+1}$ [4, 64].

A function f(x) for which s = S in [a, b] is said to be Lebesgue integrable or summable in [a, b].

It will now be proved that the Lebesgue integral, as defined above, is independant of the sequence of subdivisions used, and any sequence of partitions with norms $d_k \longrightarrow 0$ may be employed.

Consider any two sequences of partitions $\{P_k\}$, $\{P'_k\}$ with norms d_k and $d'_k \longrightarrow 0$, respectively. The corresponding sums are S_{P_k}, s_{P_k} and $S_{P'_k}$, $s_{P'_k}$. Form a third partition P''_k by combining the points of P_k and P'_k. Thus P''_k is a subdivision of P_k and of P'_k; moreover, P''_{k+1} is a subdivision of P''_k. Let $s_{P''_k}$ and $S_{P''_k}$ be the sums corresponding to P''_k and $d''_k \longrightarrow 0$ be the norm of P''_k. Also set

$$s'' = \lim_{d''_k \to 0} s_{P''_k} \text{ and } S'' = \lim_{d''_k \to 0} S_{P''_k} .$$

Then s" = S", and

$$s_{P_k} \leq s_{P''_k} \leq s'' = S'' \leq S_{P''_k} \leq S_{P_k}$$

$$s_{P'_k} \leq s_{P''_k} \leq s'' = S'' \leq S_{P''_k} \leq S_{P'_k} .$$

Since $S_{P_k} - s_{P_k} \leq d_k(b - a)$ and $S_{P'_k} - s_{P'_k} \leq d'_k \cdot (b - a)$, it follows for every $\epsilon > 0$ there exists a k_0 such that

$$S_{P_k} - s_{P_k} < \epsilon \text{ and } S_{P'_k} - s_{P'_k} < \epsilon$$

whenever $k \geq k_0$. It then follows that

$$S_{P_k} - S'' < \epsilon , \qquad S'' - s_{P_k} < \epsilon ,$$

$$S_{P'_k} - S'' < \epsilon , \qquad S'' - s_{P'_k} < \epsilon , \text{ for } k \geq k_0 .$$

Therefore

$$\lim_{d_k\to 0} s_{P_k} = \lim_{d_k\to 0} S_{P_k} = \lim_{d'_k\to 0} s_{P'_k} = \lim_{d'_k\to 0} S_{P'_k} = S'' = s'' .$$

Thus two completely arbitrary sequences of partitions $\{P_k\}$ and $\{P'_k\}$ have the same limit, which implies the integral is independent of the sequence used.

In the definition of the Lebesgue integral the interval $[a, b]$ can be replaced by a measurable set M. Then the E_v's are defined as

$$E_v = \{x \in M : y_v \leq f(x) < y_{v+1}\} ,$$

and m(M) replaces b - a. The notation for the integral is

$$\int_M f(x)\, dx.$$

With a few additional assumptions the Lebesgue integral can be generalized to include unbounded measurable functions [4, 66]. The y-axis can be subdivided by means of a partition P such that

$$\cdots < y_{-v} < \cdots < y_{-2} < y_{-1} < y_0 < y_1 < \cdots < y_v < y_{v+1} < \cdots$$

with $y_v \longrightarrow \infty$ as $v \longrightarrow \infty$ and $y_v \longrightarrow -\infty$ as $v \longrightarrow -\infty$. It must be assumed that the set of differences $(y_{v+1} - y_v)$ is bounded, and call the least upper bound of this set the norm d of P. Now form a sequence of such partitions $\{P_k\}$ with $d_k \longrightarrow 0$. A final assumption must be made, that the infinite sums s_{P_k} and S_{P_k} converge. Under these additional assumptions the previous discussion can be modified, and the value S = s is again the Lebesgue integral. It is helpful to know that, since $\{s_{P_k}\}$ and $\{S_{P_k}\}$ are monotone increasing and decreasing sequences, if for any particular value of k, say k_0, the sums S_{P_k} and s_{P_k} are finite, then the corresponding sums are finite for all $k \geqslant k_0$.

ELEMENTARY PROPERTIES OF THE LEBESGUE INTEGRAL

To expand the concept of the Lebesgue integral, a few elementary properties are presented. Most of the properties established in this section are for a real function $f(x)$ which is assumed measurable and bounded on a measurable set M. The exception is the last theorem where $|f(x)|$ is assumed measurable and bounded.

The following theorem is obtained as a direct result of the limitations placed on $f(x)$ when defining the Lebesgue integral in the preceding part of this report.

Theorem 30. Every function $f(x)$ which is bounded and measurable in $[a, b]$ is summable in $[a, b]$ [4, 64].

The following elementary property is proved.

Theorem 31. If $f(x)$ is measurable and bounded on M, then $f(x)$ is summable on each measurable subset M_1 of M [4, 74].

Proof. Using the definition of a partition previously stated, let P be a partition such that

$$\alpha = y_0 < y_1 < \cdots < y_n < y_{n+1} = \beta ,$$

where $\alpha \leqslant f(x) < \beta$. It can be seen that

$$\{x \in M_1 : y_v \leqslant f(x) < y_{v+1}\} = \{x \in M : y_v \leqslant f(x) < y_{v+1}\} \cap M_1$$
$$= E_v \cap M_1 .$$

Since $E_v \cap M_1 \subset E_v$,

$$m(E_v \cap M_1) \leq m(E_v) .$$

Therefore the Lebesgue sums involving $m(E_v \cap M_1)$ converge, since the Lebesgue sums in terms of M converge.

The following theorem is sometimes celled the first law of the mean.

Theorem 32. If $f(x)$ is meesurable end bounded on M ($\alpha \leq f(x) < \beta$ for ell $x \in M$), then

$$\alpha \cdot m(M) \leq \int_M f(x)\, dx \leq \beta \cdot m(M) \quad [3, 121].$$

Proof. Let $\{P_k\}$ be a sequence of pertitions with norms $d_k \longrightarrow 0$. It hes been shown thet

$$s_{P_1} \leq s_{P_2} \leq \ldots \leq s_{P_k} \leq \ldots \leq \int_M f(x)\, dx \leq \ldots \leq S_{P_k} \leq \ldots$$
$$\leq S_{P_2} \leq S_{P_1} .$$

Let P_1 be the undivided intervel $[\alpha, \beta]$. Then $s_{P_1} = \alpha \cdot m(M)$ and $S_{P_1} = \beta \cdot m(M)$, and this esteblishes the theorem.

The following corollaries are both useful and descriptive of the Lebesgue integral.

Corollery 1. If $f(x) \geqslant 0$ on M, $\int_M f(x)\, dx \geqslant 0$.

Proof. This follows from the theorem by letting $\alpha = 0$.

Corollary 2. If $m(M) = 0$, then $\int_M f(x)\, dx = 0$.

Corollary 3. If $f(x) = C$, e constent on M, then

$$\int_M C\, dx = C \cdot m(M).$$

Proof. This can be seen by letting the interval $[\alpha, \beta] = [C, C+\epsilon]$, where $\epsilon > 0$. In particular, if $C = 1$, then

$$\int_M C\, dx = \int_M 1 \cdot dx = m(M).$$

The next theorem asserts the additivity of the Lebesgue integral.

Theorem 33. If $f(x)$ is measurable and bounded on M and M is the union of countably many disjoint and measurable sets

$$M = \bigcup_{k=1}^{\infty} M_k, \qquad (M_k \cap M_{k'} = \emptyset,\ k \neq k'),$$

then

$$\int_M f(x)\, dx = \sum_{k=1}^{\infty} \int_{M_k} f(x)\, dx \quad [3,\ 121].$$

Proof. Consider first the simple case in which there are only two disjoint sets:

$$M = M_1 \cup M_2\ , \qquad (M_1 \cap M_2 = \emptyset).$$

Since $f(x)$ is bounded, $\alpha \leq f(x) < \beta$ on the set M. Let P be a partition of the interval $[\alpha, \beta]$ and define the sets

$$E_v = E[y_v \leq f(x) < y_{v+1}] \text{ on } M,$$
$$E_v' = E[y_v \leq f(x) < y_{v+1}] \text{ on } M_1,$$

and
$$E_v'' = E[y_v \leq f(x) < y_{v+1}] \text{ on } M_2.$$

Obviously

$$E_v = E_v' \cup E_v'' \qquad (E_v' \cap E_v'' = \emptyset),$$

and therefore

$$\sum_{v=0}^{n} y_v \cdot m(E_v) = \sum_{v=0}^{n} y_v \cdot m(E_v') + \sum_{v=0}^{n} y_v \cdot m(E_v'').$$

Let $\{P_k\}$ be a sequence of pertitions with norms d_k. Then as $d_k \longrightarrow 0$,

$$\int_M f(x)\,dx = \int_{M_1} f(x)\,dx + \int_{M_2} f(x)\,dx.$$

Therefore the theorem holds for the case of two disjoint sets. Applying the technique of methemeticel induction, the theorem cen be generelized to the case of en arbitrery finite number "n". The denumereble cese is ell that is left to consider. For this cese

$$M = \bigcup_{k=1}^{\infty} M_k.$$

By e property of meesureble sets,

$$m(M) = \sum_{k=1}^{\infty} m(M_k),$$

but since this series converges,

$$\sum_{k=n+1}^{\infty} m(M_k) \longrightarrow 0 \text{ es } n \longrightarrow \infty .$$

Denote

$$\bigcup_{k=n+1}^{\infty} M_k = R_n.$$

Since the theorem is alreedy proved for a finite number of component terms, it is possible to write the following equelity:

$$\int_M f(x)\,dx = \sum_{k=1}^{n} \int_{M_k} f(x)\,dx + \int_{R_n} f(x)\,dx.$$

Then, by Theorem 32,

$$\alpha \cdot m(R_n) \leq \int_{R_n} f(x)\,dx \leq \beta \cdot m(R_n),$$

end the meesure, $m(R_n)$, of the set R_n epproeches zero es

$n \longrightarrow \infty$. It follows that

$$\int_{R_n} f(x)\ dx \longrightarrow 0,$$

as $n \longrightarrow \infty$, which yields the conclusion.

The following useful property is proved for real functions $f(x)$ and $g(x)$.

<u>Theorem 34</u>. If $f(x)$ and $g(x)$ are measurable and bounded on M, then $f(x) + g(x)$ is summable and

$$\int_M (f(x) + g(x))\ dx = \int_M f(x)\ dx + \int_M g(x)\ dx \quad [2,\ 217].$$

Proof. Let $\alpha \leq f(x) < \beta$, and $\delta \leq g(x) < \tau$. Let P and Q be partitions of $[\alpha, \beta]$ and $[\delta, \tau]$, respectively, such that

$$\alpha = y_0 < y_1 < \ldots < y_n < y_{n+1} = \beta,$$

and
$$\delta = \bar{y}_0 < \bar{y}_1 < \ldots < \bar{y}_N < \bar{y}_{N+1} = \tau.$$

Also set

$$E_v = E\left[y_v \leq f(x) < y_{v+1}\right],$$
$$\bar{E}_i = E\left[\bar{y}_i \leq g(x) < \bar{y}_{i+1}\right] \qquad (v = 0, 1, 2, \ldots, n;\ i = 0, 1, 2, \ldots, N).$$

Define

$$T_{i,v} = E_v \cap \bar{E}_i \qquad (v = 0, 1, 2, \ldots, n;\ i = 0, 1, 2, \ldots, N).$$

Obviously the set

$$M = \bigcup_{i,v} T_{i,v}$$

and the sets $T_{i,v}$ are disjoint, hence

$$\int_M (f(x) + g(x))\,dx = \sum_{i,v} \int_{T_{i,v}} (f(x) + g(x))\,dx.$$

On the set $T_{i,v}$

$$y_v + \bar{y}_i \leqslant f(x) + g(x) < \bar{y}_{i+1} + y_{v+1},$$

end the first lew of the meen implies

$$(y_v + \bar{y}_i) \cdot m(T_{i,v}) \leqslant \int_{T_{i,v}} (f(x) + g(x))\,dx$$

$$\leqslant (y_{v+1} + \bar{y}_{i+1}) \cdot m(T_{i,v}).$$

A combinetion of these inequelities yields

$$\sum_{i,v} (y_v + \bar{y}_i) \cdot m(T_{i,v}) \leqslant \int_M (f(x) + g(x))\,dx$$

$$\leqslant \sum_{i,v} (y_{v+1} + y_{i+1}) \cdot m(T_{i,v}).$$

Consider the sum

$$\sum_{i,v} y_v \cdot m(T_{i,v}),$$

which can be written in the form

$$\sum_{v=0}^{n-1} y_v \left(\sum_{i=0}^{N-1} m(T_{i,v})\right),$$

where

$$\sum_{i=0}^{N-1} m(T_{i,v}) = m\left[\bigcup_{i=0}^{N-1} T_{i,v}\right] = m\left[\bigcup_{i=0}^{N-1} \bar{E}_i \cap E_v\right] = m\left[E_v \cap \bigcup_{i=0}^{N-1} \bar{E}_i\right]$$

$$= m(E_v \cap M) = m(E_v);$$

so thet the original sum mey also be written es

$$\sum_{v=0}^{n-1} y_v \cdot m(E_v).$$

Hence the originel sum is the Lebesgue sum s_p of the function $f(x)$. Denote this sum s_f. The other sums in the inequality cen be denoted end evelueted enalogously, so thet the inequelity cen be written

$$s_f + s_g \leq \int_M (f(x) + g(x))\ dx \leq S_f + S_g.$$

By increesing the number of points of the pertitions P end Q end by teking the limit in the inequelities ebove, the theorem is proved.

It is now possible to prove the following elementary property.

Theorem 35. If f(x) is meesureble end bounded on M end C is e constent, then

$$\int_M C \cdot f(x)\ dx = C \int_M f(x)\ dx \quad [3,\ 125].$$

Proof. If C = 0, the theorem is obvious. Consider the cese $C > 0$. Since f(x) is bounded, $\alpha \leq f(x) < \beta$. Let P be e pertition of the segment $[\alpha,\ \beta]$ end let

$$E_v = E[y_v \leq f(x) < y_{v+1}].$$

It follows thet

$$\int_M C \cdot f(x)\ dx = \sum_{n=0}^{n-1} \int_{M_k} C \cdot f(x)\ dx.$$

On the sets E_v the inequelities

$$C \cdot y_v \leq C \cdot f(x) < C \cdot y_{v+1} ,$$

hold. Thus by the first law of the mean,

$$C \cdot y_v \cdot m(E\ ) \leq \int_{M_k} C \cdot f(x)\ dx \leq C \cdot y_{v+1} \cdot m(E_v).$$

Combining these inequalities yields

$$C \cdot s \leq \int_M C \cdot f(x)\, dx \leq C \cdot S,$$

where s and S are the Lebesgue sums for f(x). The theorem is obtained from this last inequality by taking S - s arbitrarily small. Finally, consider $C < 0$. Here

$$0 = \int_M \left[C \cdot f(x) + (-C) \cdot f(x)\right] dx = \int_M C \cdot f(x)\, dx$$
$$+ (-C) \int_M f(x)\, dx,$$

and the proof is completed.

Another useful property of the Lebesgue integral is the fact that equivalent functions have equal integrals. Two functions are said to be equivalent, denoted $f(x) \sim g(x)$, if $f(x) = g(x)$ on M except for a set of measure zero. The property will now be stated as a theorem.

Theorem 36. If f(x) is measurable and bounded on M and $f(x) \sim g(x)$ on M, then g(x) is summable on M and

$$\int_M f(x)\, dx = \int_M g(x)\, dx \quad [4, 75].$$

Proof. By definition $f(x) = g(x)$ on M - Z, where Z is a set of measure zero. Then

$$\int_M f(x)\, dx = \int_{M-Z} f(x)\, dx + \int_Z f(x)\, dx.$$

Since

$$\int_Z f(x)\, dx = \int_Z g(x) = 0 \text{ and } \int_{M-Z} f(x)\, dx = \int_{M-Z} g(x)\, dx,$$

it follows that

$$\int_M f(x)\ dx = \int_{M-Z} g(x)\ dx + \int_Z g(x)\ dx = \int_M g(x)\ dx.$$

An application of this theorem will now be given. Consider the problem of finding the Lebesgue integral of

$$f(x) = \begin{cases} 1 \text{ for irrational } x \\ 0 \text{ for rational } x, \end{cases} \text{ in the interval } M = [0, 1].$$

Let $g(x) = 1$ in $[0, 1]$. Then $f(x) \sim g(x)$ in $[0, 1]$. By Corollary 3 of Theorem 32

$$\int_M g(x)\ dx = \int_M 1\ dx = 1 \cdot (1 - 0) = 1.$$

Hence by the preceding theorem $f(x)$ is also summable and

$$\int_M g(x)\ dx = \int_M f(x)\ dx = 1.$$

The following theorem is fundamental to the Lebesgue integral.

Theorem 37. If $f(x)$ is measurable and bounded on M, then $|f(x)|$ is summable on M and

$$\left|\int_M f(x)\ dx\right| \leq \int_M |f(x)|\ dx \quad [4, 76].$$

Proof. Set $M^+ = M[f(x) \geq 0]$ and $M^- = M[f(x) < 0]$. Then by Theorem 33,

$$\int_M f(x)\ dx = \int_{M^+} f(x)\ dx + \int_{M^-} f(x)\ dx,$$

and therefore

$$\int_M f(x)\ dx = \int_{M^+} |f(x)|\ dx - \int_{M^-} |f(x)|\ dx,$$

since $f(x) = -|f(x)|$ when $f(x)$ is negetive. Since the integrels on the right side of the statement above exist, then the sum of the integrels exists end by Theorem 33 egein

$$\int_{M^+} |f(x)|\, dx + \int_{M^-} |f(x)|\, dx = \int_M |f(x)|\, dx.$$

This states that $|f(x)|$ is summeble on M. Note that

$$\int_{M^+} |f(x)|\, dx \geqslant 0 \text{ and } \int_{M^-} |f(x)|\, dx \geqslant 0.$$

Then

$$\left|\int_M f(x)\, dx\right| = \left|\int_{M^+} |f(x)|\, dx - \int_{M^-} |f(x)|\, dx\right|$$

$$\leq \left|\int_{M^+} |f(x)|\, dx + \int_{M^-} |f(x)|\, dx\right|$$

$$= \int_{M^+} |f(x)|\, dx + \int_{M^-} |f(x)|\, dx = \int_M |f(x)|\, dx,$$

end the theorem is esteblished.

The converse of the preceding theorem is elso proved.

Theorem 38. If $f(x)$ is meesureble on M and $|f(x)|$ is meesureble end bounded, then $f(x)$ is also summeble on M [4, 77].

Proof. If $f(x)$ is measureble, then the sets M^+ and M^- ere meesureble. Since $|f(x)|$ is summeble,

$$\int_M |f(x)|\, dx = \int_{M^+} |f(x)|\, dx + \int_{M^-} |f(x)|\, dx.$$

However, if these two integrels on the right exist,

$$\int_{M^+} |f(x)|\, dx - \int_{M^-} |f(x)|\, dx$$

exists end equels $\int_M f(x)\ dx$.

COMPARISON OF THE RIEMANN AND LEBESGUE INTEGRALS

For the purpose of compering the Riemenn and Lebesgue integrels, the definition of the upper end lower Riemenn integrele, end the definition of the Riemenn integrel will be essumed to be known to the reader. The Riemenn integrels will be denoted by the prefix "R".

The definition of the upper end lower Lebesgue integral is es follows.

<u>Definition 9</u>. The upper end lower Lebesgue integrels of the function $f(x)$ defined on e meesureble set M ere

$$\overline{\int}_M f(x)\ dx = \inf \{S_P\}$$

end

$$\underline{\int}_M f(x)\ dx = \sup \{s_P\},$$

respectively [2, 205].

The following reletionship between the Riemenn and Lebesgue integrele will now be given.

<u>Theorem 39</u>. If M is a closed interval, then for every bounded function $f(x)$ the following inequelities hold:

$$R\ \overline{\int}_M f(x)\ dx \geqslant \overline{\int}_M f(x)\ dx \geqslant \underline{\int}_M f(x)\ dx \geqslant R\underline{\int}_M f(x)\ dx \quad [2,\ 206].$$

As e result of this theorem it cen be seen thet if the

Riemann integral exists, the upper and lower Lebesgue integrals are equal to each other and to the Riemann integral. Hence the Lebesgue integral exists whenever the Riemann integral exists, and has the same value. The converse of this preceding statement is not true, as may be seen by considering again the previous example, known as the Dirichlet function. Let

$$f(x) = 0 \quad \text{for } x \text{ irrational}$$
$$f(x) = 1 \quad \text{for } x \text{ rational} \quad \text{in } [0, 1].$$

Since $f(x)$ is a constant function of the set R^* of rationals and $m(R^*) = 0$, the Lebesgue integral $\int_M f(x)\, dx = 0$, where $M = [0, 1]$.

For the upper and lower Riemann integrals of $f(x)$,

$$R \overline{\int_0^1} f(x)\, dx = 1 \text{ and } R \underline{\int_0^1} f(x)\, dx = 0,$$

so that the Riemann integral of $f(x)$ does not exist.

Therefore the existence of the Lebesgue integral does not imply the existence of the Riemann integral. Thus the Lebesgue integral is more general than the Riemann integral, at least for bounded functions.

The Lebesgue integral is superior to the Riemann integral in the area of finding limits relative to integration processes. Let $\{f_n(x)\}$ be a sequence of summable functions on M which converge to $f(x)$. Does

$$\int_M f(x)\, dx = \lim_{n\to\infty} \int_M f_n(x)\, dx?$$

To see that the preceding equality does not hold necessarily, consider the following example: Let $M = [0, 1]$ and

$$f_n(x) = \begin{cases} 0 \text{ outside } (0, \frac{1}{n}) \\ n \text{ for } x = \frac{1}{2n} \\ \text{linear in } \left[0, \frac{1}{2n}\right] \text{ and } \left[\frac{1}{2n}, \frac{1}{n}\right] \end{cases}$$

$$(n = 1, 2, \ldots).$$

Then $f(x) = \lim_{n \to \infty} f_n(x) = 0$ since $f_n(x) = 0$ for $x \leq 0$, and, for each $x > 0$, n can be taken so large that $\frac{1}{n} < x$, and hence $f_n(x) = 0$. Thus $\int_M f(x)\,dx = 0$, but $\int_M f_n(x)\,dx = \frac{1}{2} \cdot \frac{1}{n} \cdot n = \frac{1}{2}$. Therefore it can be seen that without additional conditions the limit and integration processes cannot be interchanged.

A general condition under which the limit and integration processes may be interchanged for Lebesgue integration is known as the uniform boundedness of a sequence.

Definition 10. A sequence $\{f_n(x)\}$ is called uniformly bounded on M if $|f_n(x)| \leq C$, $n = 1, 2, \ldots$, where C is a constant independent of n and of $x \in M$ [2, 103].

The bounded convergence theorem for the Lebesgue integral may now be given.

Theorem 40. If the sequence of summable functions $\{f_n(x)\}$ converges to $f(x)$ and is uniformly bounded on M, then $f(x)$ is also summable on M and

$$\int_M f(x)\,dx = \lim_{n\to\infty} \int_M f_n(x)\,dx \quad [4,\ 82].$$

Proof. The function $f(x)$, as the limit of a convergent sequence of meesureble functions, is e meesureble function. All functions involved ere bounded end meesureble, hence they ere summeble. Since the sequence $\{f_n(x)\}$ is uniformly bounded on M, there is e $C > 0$ such thet for every n end every $x \in M$, $|f_n(x)| \leq C$. Also, for every $x \in M$, $|f(x)| \leq C$. Let $\epsilon > 0$ be given. By the Theorem of Egoroff [2, 187], there is e meesureble set $T \subset M$ such thet

$$m(M - T) < \frac{\epsilon}{4C},$$

end $\{f_n(x)\}$ converges uniformly on T to $f(x)$ [2, 223]. There is a number N such thet for every $n > N$ end every $x \in T$,

$$|f(x) - f_n(x)| < \frac{\epsilon}{2 \cdot m(T)} .$$

Hence for every $n > N$,

$$\left|\int_M f(x)\,dx - \int_M f_n(x)\,dx\right| = \left|\int_T f(x)\,dx + \int_{M-T} f(x)\,dx - \int_T f_n(x)\,dx - \int_{M-T} f_n(x)\,dx\right| \leq \left|\int_T (f(x)\,dx - f_n(x))\,dx\right| + \left|\int_{M-T} (f(x) - f_n(x))\,dx\right| < \frac{\epsilon}{2 \cdot m(T)} \cdot m(T) + \frac{\epsilon}{4C} \cdot 2C = \epsilon .$$

Hence for every $n > N$,

$$\left|\int_M f(x)\,dx - \int_M f_n(x)\,dx\right| < \epsilon ,$$

and the theorem is proved.

This theorem is not true for Riemann integrals, for in general the limit function $f(x)$ is not Riemann integrable under these conditions, as may be seen by the following example.

Assume the rational numbers in $[0, 1]$ to be ordered in a sequence $r_1, r_2, \ldots, r_m, \ldots$, and set

$$f_n(x) = \begin{cases} 0 \text{ for } x = r_1, r_2, \ldots, r_m \\ 1 \text{ otherwise} \end{cases} \quad \text{in } [0, 1].$$

Thus the $f_n(x)$ are Riemann integrable. However,

$$\lim_{n \to \infty} f_n(x) = f(x) = 0 \text{ for rational } x$$
$$= 1 \text{ otherwise} \quad \text{in } [0, 1],$$

and $f(x)$ is not Riemann integrable [2, 210].

A further generalization of Theorem 40 is possible for the Lebesgue integral. This theorem is known as the "dominated convergence theorem".

<u>Theorem 41</u>. If the sequence of summable functions $\{f_n(x)\}$ converges to $f(x)$ and if

$$|f_n(x)| \leq F(x) \qquad (n = 1, 2, \ldots)$$

on M, where $F(x)$ is summable on M, then $f(x)$ is also summable on M and

$$\int_M f(x)\, dx = \lim_{n \to \infty} \int_M f_n(x)\, dx \quad [4, 83].$$

Another area in which the Lebesgue integral is superior to the Riemann integral is in the relation between integration and differentiation. Consider a function $f(x)$ which is continuous in $[a, b]$ and define

$$F(x) = \int_a^x f(t)\ dt \text{ with } x \in [a,\ b].$$

$F(x)$ is a primitive or antiderivative of $f(x)$, for either the Lebesgue or Riemann integrals, since the following theorem is true in both cases.

Theorem 42. If $f(x)$ is continuous at $x_0 \in (a,\ b)$, then $F'(x_0)$ exists and equals $f(x_0)$ [4, 86].

If the function $f(x)$ is required to be a bounded derivative, then the Riemann integral does not necessarily yield the primitive, while the following theorem can be proved for the Lebesgue integral.

Theorem 43. Every bounded derivative in $[a,\ b]$ is summable and the Lebesgue integral yields the primitive (antiderivative) up to an additive constant. That is, if $F'(x)$ is bounded in $[a,\ b]$, then for every $x \in [a,\ b]$

$$\int_a^x F'(t)\ dt = F(x) - F(a) \quad [4,\ 87].$$

Proof. Since $F'(x)$ is measurable and bounded in $[a,\ b]$, it is summable in $[a,\ b]$. There is a theorem of Dini which states that if $F'(x)$ is bounded in $[a,\ b]$, then

$$\frac{F(x + h) - F(x)}{h}, \ (h > 0),$$

has the same bounds there as $F'(x)$ [4, 87]. Thus using a null sequence $\{h_v\}$, it follows by Theorem 40 that

$$\int_a^x F'(t)\,dt = \int_a^x \lim_{h_v \to 0} \frac{F(t + h_v) - F(t)}{h_v}\,dt$$

$$= \lim_{h_v \to 0} \int_a^x \frac{F(t + h_v) - F(t)}{h_v}\,dt$$

$$= \lim_{h_v \to 0} \left[\frac{1}{h_v} \left(\int_a^x F(t + h_v)\,dt - \int_a^x F(t)\,dt \right) \right].$$

Set $t + h_v = \tau$ in the first integral of the last expression. Then

$$\int_a^x F'(t)\,dt = \lim_{h_v \to 0} \left[\frac{1}{h_v} \left(\int_{a+h_v}^{x+h_v} F(\tau)\,d\tau - \int_a^x F(t)\,dt \right) \right].$$

Since $F(x)$ is continuous in $[a, b]$, then its primitive $\Phi(x)$ exists there, that is, $\Phi'(x) = F(x)$, and hence

$$\int_a^x F'(t)\,dt = \lim_{h_v \to 0} \left[\frac{\Phi(x + h_v - \Phi(x)}{h_v} - \frac{\Phi(a + h_v) - \Phi(a)}{h_v} \right]$$

$$= \Phi'(x) - \Phi'(a) = F(x) - F(a).$$

To show that the preceding theorem does not hold true for Riemann integration, the following example is given.

Let $s(x)$ be the so-called signum function defined as follows:

$$s(x) = 1 \text{ if } x > 0$$

$$s(x) = -1 \text{ if } x < 0$$

$$s(x) = 0 \text{ if } x = 0.$$

Let $M = [-1, 1]$, then $s(x)$ is bounded in M, but the primitive does not exist [1, 42].

WEAKNESSES OF THE LEBESGUE INTEGRAL

A weakness in the Lebesgue integral for a bounded function $f(x)$ occurs as a result of Theorem 37, which states that the integral of $|f(x)|$ also exists whenever $f(x)$ is summable. However, from elementary calculus there are improper integrals for which this property does not hold. For example,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}, \text{ but } \int_0^\infty \frac{|\sin x|}{x}\,dx$$

does not exist.

For bounded derivatives the Lebesgue integral is satisfactory, as was stated in Theorem 43; however, unbounded derivatives $F'(x)$ are not necessarily summable. The following is an example of an unbounded derivative which is not summable [4, 89]. Let

$$F(x) = x^2 \sin \frac{1}{x^2} \text{ for } x \neq 0$$

$$= 0 \text{ for } x = 0.$$

Then

$$F'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} \text{ for } x \neq 0$$

$$= 0 \text{ for } x = 0,$$

since

$$F'(0) = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h^2}}{h} = \lim_{h \to 0} h \sin \frac{1}{h^2} = 0 .$$

Now consider the integration of $F'(x)$ between 0 and

$a = \sqrt{2/\pi}$. This first term of $F'(x)$ is continuous in $[0, a]$; however,

$$\int_0^a \frac{2}{x} \cos \frac{1}{x^2}\, dx$$

does not exist. To show this, assume the integral did exist, then by Theorem 37

$$\int_0^a \frac{2}{x} \left|\cos \frac{1}{x^2}\right| dx \tag{1}$$

also exist. It can be proven that this integral is continuous for every $x \in (0, a)$, [4, 86]; hence

$$\int_0^a \frac{2}{x} \left|\cos \frac{1}{x^2}\right| dx = \lim_{\epsilon \to 0^+} \int_\epsilon^a \frac{2}{x} \left|\cos \frac{1}{x^2}\right| dx . \tag{2}$$

The zeros of the integrand in (1) are at $x = \sqrt{\dfrac{2}{(2n + 1)\pi}}$

$(n = 0, 1, \ldots)$, thus the right member of (2) may be written as

$$\sum_{n=0}^{\infty} \int_{\sqrt{2/(2n+3)\pi}}^{\sqrt{2/(2n+1)\pi}} \frac{2}{x} \left|\cos \frac{1}{x^2}\right| dx . \tag{3}$$

Making the change of variables $\dfrac{1}{x^2} = z$ in (3), yields

$$\sum_{n=0}^{\infty} \int_{(2n+1)\pi/2}^{(2n+3)\pi/2} \frac{\cos z}{z}\, dz > \sum_{n=0}^{\infty} \int_{(4n+3)\pi/4}^{(4n+5)\pi/4} \frac{|\cos z|}{z}\, dz.$$

This last sum is greater than

$$\sum_{n=0}^{\infty} \frac{1}{2}\sqrt{2} \cdot \frac{1}{(4n+5)\pi/4} \cdot \frac{\pi}{2} = \sqrt{2} \sum_{n=0}^{\infty} \frac{1}{4n+5} > \frac{\sqrt{2}}{5} \sum_{n=0}^{\infty} \frac{1}{n+1} .$$

This last series diverges, hence (1) is infinite. Since (1) does not exist,

$$\int_0^a \frac{2}{x} \cos \frac{1}{x^2}\, dx$$

cannot exist, by the contrapositive of Theorem 37.

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ELEMENTARY CONCEPTS CONCERNING THE
LEBESGUE INTEGRAL

by

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The first pert of the report is e discussion of Lebesgue meesureble sets, restricted to the reel number line. A definition of Lebesgue meesure is given in terms of outer and inner Lebesgue meesure. After e few elementary properties of Lebesgue meesure ere esteblished, certein femilies of sets which are meesurable eccording to the definition ere considered. For exemple, open end closed sets ere meesureble sets.

The next pert of the report is e discussion of Lebesgue measureble functions, the functions "competible" with Lebesgue meesurable sets. A few elementery properties of Lebesgue meesureble functions ere presented.

In the third pert of the report the Lebesgue integrel is defined. It is shown thet the Lebesgue integral as defined is independent of the sequence of pertitions used.

The fourth pert of the report is devoted to en elementery discussion of the Lebesgue integral. A few of the properties of the Lebesgue integrel ere presented, end the Lebesgue integrel is compered with the Riemann integral. It is shown thet whenever the Riemenn integrel exists on e closed intervel, the Lebesgue integrel exists. The converse is shown not to be true by presenting en exemple. The Lebesgue integrel is elso shown to be superior to the Riemenn integral in the area of finding limits reletive to integration processes. The Lebesgue and Riemann integrals ere elso compared relative to the reletion between integretion and differentietion. It is shown that the Lebesgue integral of e derivative yields the primitive in e closed interval for more general conditions than the Riemann integral. The last unit illustrates a weakness of the Lebesgue integral encountered when a derivative to be integrated is not required to be bounded.