A STUDY OF PLANT IDENTIFICATION TECHNIQUES

by

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Intelligent design of a feedback control system can be effected only if the designer is cognizant of the dynamic characteristics of the process to be controlled. In a conventional linear control system, the process can be described perfectly by a known transfer function G(s), and a feedback control system can compensate for variations due to changes in the parameters of the controlled system and to external disturbances to some extent. However, a conventional feedback control system is not capable of satisfactory performance in the presence of extreme changes of the controlled system's parameters, and when the system is subjected to large external perturbances. These changes are often in an unpredictable manner. It has been suggested that adaptive control systems might be designed in such a way as to alleviate this problem.

Usually, adaptive control systems are characterized by devices which automatically measure the dynamics of the controlled system and other devices which automatically adjust the characteristics of the controlled elements, based on a comparison of these measurements with some optimum figure of merit, so that two fundamental features found in all adaptive control systems are (1) Identification, (2) Actuation. In this report, some techniques of identification of linear systems have been studied.

Model identification is described in chapter one. In the technique, a set of simultaneous differential equations which constitute a mathematical model of the system is set up using physics laws. From this model a block diagram, or possibly a circuit diagram of an analogous electric network is derived in order to determine all the parameters of the system.

The study of dynamic transmittance identification is emphasized in chapter two. This is a much more practical approach to the identification problem, since in this restricted interpretation of the problem, attention is focused on only those characteristics which are directly of interest. In particular, concentration is focused on an evaluation of the transfer characteristics from the specific inputs to the outputs of primary interest. Therefore, in this case, the identification may be in terms of a set of coefficients of a preselected differential equation, or in terms of the values of any desired number of points on the impulse response, of in terms of the transfer function of the system. Some methods proposed by Braun [1], Mishkin [2], Kalman [3], Truxal [1], Anderson [5] and Lendaris [6] in connection with this technique are investigated in this report.

MODEL IDENTIFICATION

Generally, a physical component can be described mathematically by means of a differential equation. When a system or process is composed of various physical components, we can usually obtain a set of simultaneous differential equations which constitute a mathematical model of the system. For example [1], if a process consists of a D-C electric motor driving a load as shown in Fig. (1.1), the system may be described by the set of six differential equations.

Armature electric circuit

$$e_a = L_a\frac{di}{dt} + R_a i + Kb\frac{d\theta_m}{dt} \qquad (1.1\text{-a})$$

Energy conversion term

$$l_d = K_b i \qquad (1.1\text{-b})$$

Summation of torques at motor shaft

$$l_d = J_m\frac{d^2\theta_m}{dt^2} + B_m\frac{d\theta_m}{dt} + K_1(\theta_m - \theta_1) \qquad (1.1\text{-c})$$

Gear ratio

$$\theta_2 = \frac{1}{g}\theta_1$$

Torque transmitted through gears

$$gK_1(\theta_1 - \theta_m) = K_2(\theta_L - \theta_2) \qquad (1.1\text{-3})$$

D'Alembert's law at load

$$0 = J_L\frac{d^2\theta_L}{dt^2} + B_L\frac{d\theta_L}{dt} + K_2(\theta_L - \theta_2) \qquad (1.1\text{-f})$$

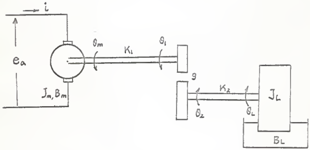

Fig. (1.1) Electromechanical process

Parameters	Variables
Motor inertia J_m	Armature voltage e_a
Motor damping B_m	Armature current i
Shaft compliances K_1, K_2	Motor angle θ_m
Load inertia J_L	Load position θ_L
Load damping B_L	Gear angles θ_1, θ_2
Armature parameters R_a, L_a	Developed torque l_d

In this set of differential equations, L_a, R_a, K_b, B_m, J_m, K_1, K_2, g, J_L and B_L are the parameters of the system. From this set of differential equations, assuming that all these parameters are known and the independent variable e_a is given, then any of the dependent variables i, i_d, θ_m, θ_1, θ_2 and θ_L can be determined.

In fact, the differential equations constitute only a model of the physical system. In the derivation of the differential equations for a system, a large number of assumptions must be made. For example Eq. (1.1) is obtained by the assumption that the motor is operating in linear region (no saturation, no static friction), that gears are ideal (no backlace, negligible inertia), and that all damping is viscous.

If the interrelation between the dependent variables in the physical model is desired, a block diagram can be constructed for this purpose. There are many block diagrams which can be constructed, depending on which parameter we are interested in. The most common block diagram, however, is one which represents the manner in which the signals flow through the system. For the example of the armature-controlled D-C motor and load, a block diagram can be derived from the differential equations in the following manner:

If p is used to represent the operator d/dt, Eq.(1.1-a) yields

$$e_a = L_a p i + R_a i + K_b p \theta_m$$

which can be written in the form

$$i = \frac{e_a - K_b p \theta_m}{L_a p + R_a} \tag{1.2}$$

A relation between the current i and the developed torque l_d is given by Eq. (1.1-b), i.e.,

$$l_d = K_b i \tag{1.3}$$

From Eq. (1.1-c), it follows that

$$l_d = J_m P^2 \theta_m + B_m P \theta_m + K_1(\theta_m - \theta_1)$$

or

$$K_1(\theta_m - \theta_1) = l_d - (J_m P^2 + B_m P)\theta_m$$

and from Eq. (1.1-f), it follows that

$$J_L P^2 \theta_L + B_L P \theta_L + K_2(\theta_L - \theta_2) = 0 \tag{1.4}$$

From Eq. (1.1-e), Eq. (1.4)yields

$$J_L P^2 \theta_L + B_L P \theta_L + g K_1(\theta_1 - \theta_m) = 0$$

or

$$\theta_L = \frac{g K_1(\theta_m - \theta_1)}{(J_L P^2 + B_L P)} \tag{1.5}$$

Looking back for Eq. (1.1-e) and Eq. (1.1-d), it follows that

$$\theta_1 - \theta_m = \frac{K_2}{g K_1}(\theta_L - \theta_2)$$

or

$$\theta_m = \theta_1 - \frac{K_2}{g K_1}(\theta_L - \theta_2)$$

$$= g\theta_2 - \frac{K_2}{g K_1}(\theta_L - \theta_2) \tag{1.6}$$

Finally, the relation between θ_2 and θ_L is derived as follows:
From Eqs. (1.5) and (1.6), it follows that

$$\theta_L = \frac{gK_1(\theta_m-\theta_1)}{J_LP^2 + B_LP} = \frac{gK_1[g\theta_2 - \frac{K_2}{gK_1}(\theta_L-\theta_2) - \theta_1]}{J_LP^2 + B_LP}$$

$$= \frac{g^2K_1\theta_2 - K_2\theta_L + K_2\theta_2 - gK_1\theta_1}{J_LP^2 + B_LP}$$

$$= \frac{g^2K_1\theta_2 - K_2\theta_L + K_2\theta_2 - gK_1g\theta_2}{J_LP^2 + B_LP}$$

$$= \frac{K_2\theta_2 - K_2\theta_L}{J_LP^2 + B_LP}$$

Thus it follows that

$$(J_LP^2 + B_LP)\theta_L = K_2\theta_2 - K_2\theta_L$$

$$\theta_L(J_LP^2 + B_LP + K_2) = K_2\theta_2$$

or

$$\theta_2 = \frac{J_LP^2 + B_LP + K_2}{K_2}\,\theta_L \qquad (1.7)$$

If a block diagram is desired to involve the succession of variables:

e_a i   ld $K_1(\theta_m-\theta_1)$ $gK_1(\theta_m-\theta_1)$ θ_L

From Eqs. (1.2), (1.3), (1.5), (1.6) and (1.7), the final block diagram of Fig. (1.2) is obtained.

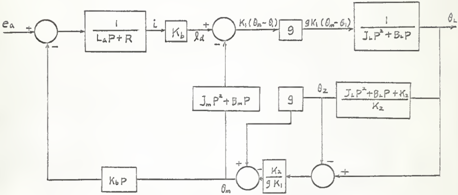

Fig. (1.2) One possible block diagram

Thus, a block diagram model is exactly analogous to the differential equation model, and by means of the diagram, the interrelationship of the various system parameters can be shown visually.

In order to identify a process, the evaluation of each parameter in the mathematical model of the system is required. Thus, for the specific example of D-C motor control of an inertial load with viscous damping mentioned above, all the parameters L_a, R_a, K_b, B_m, J_m, K_1, K_2, g, J_L and B_L must be evaluated. If all these parameters are known with reasonable accuracy and the mathematical model adequately represents the process, the performance of the process under all possible operating conditions and excitations can be determined.

In general, it is impossible to obtain such extensive data for the evaluation of all parameters. But certain parameters, such as the gear ratio, the motor inertia, the motor torque constant, and the shaft compliance in the above example are usually known with at least reasonable accuracy.

In order to evaluate the other unknown parameters, the study of the differential equations, the block diagrams, or possibly the circuit diagram of an analogous electric network is needed. To illustrate this latter procedure, consider the example mentioned above. From Eq. (1.1) and using elementary transformer theory, a circuit diagram as shown in Fig. (1.3) can be obtained. The differential equations which describe this circuit have the same form as those describing the electromechanical system shown in Fig. (1.1). However, in order to easily evaluate the parameters, the circuit diagram must be as simple as possible. Using elementary transformer theory again, the two ideal transformers in Fig. (1.3) can be removed if all impedances are reflected back into the input circuit. Then the circuit takes the form shown in Fig. (1.4).

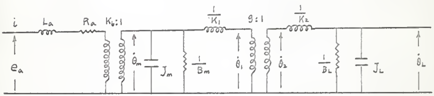

Fig. (1.3) Circuit diagram

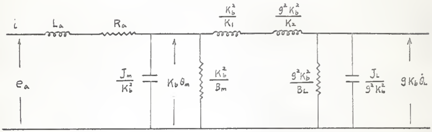

Fig. (1.4) Circuit diagram simplified from Fig. (1.3)

From the simplified diagram as shown in Fig. (1.4), it is seen that, by using the blocked rotor test, R_a and L_a can be determined by applying either a step function or sinusoidal signal at e_a. If a step function is applied, the armature current takes the form

$$i = \frac{E_a}{R_a} (1 - e^{-R_a t/L_a}) \tag{1.8}$$

where R_a can be found from the final value, that is $R_a = E_a/I$, where I is steady-state current, and L_a can be found from the time constant of the exponential rise toward the final value. Similarly, additional tests may be used to determine the values of mechanical parameters such as torque, damping, etc., of the system.

There are two difficulties in such a one-by-one determination of model parameters. First, in utilizing this method, the control engineer must be able to write the differential equations from the physical laws, and he must be able to distinguish which parameters are negligible and which are important in order to simplify the situation. This is not easy to accomplish except in relatively simple control systems. The other factor limiting the usefulness of this parameter-evaluation approach is due to the inherent nonlinearity of physical processes. In order to make the linear model useful, parameters must be measured under the linear operating conditions.

The advantages of this method are that it gives a detailed picture of the physics of the operation of the process and the effect of varying a specific parameter can be directly estimated.

Because of these difficulties and because extensive data is needed in this approach to identification problem, attention is usually paid to the evaluation of the transfer characteristics from the inputs to the outputs of primary interest. For example, in the system mentioned above, rather than attempting to evaluate each of the parameters, the over-all transfer function θ_L/E_a is sought. Such a restricted interpretation of the identification problem is referred to as "dynamic transmittance identification."

DYNAMIC TRANSMITTANCE IDENTIFICATION

There are many equivalent mathematical forms which can be used to describe a linear time-invariant system and each of these forms can be used for the solution of identification process. Among these, the following three forms will be discussed.

(1) The differential equation

(2) The impulse (or unit step) response

(3) Pulse transfer function

Fig. (2.1) shows the notation to be used in a transmission process.

Fig. (2.1) Simplified process

where r(t) and c(t) can be variables in any physical system, and they need not be in the same units. The only restriction is that the process must be linear time-invariant. Of course, some processes requiring complex adaptive control systems are time-variable. The following discussion will be confined to either the time-invariant system or the slowly varying process. By a slowly varying process it means one in which the impulse response decays to zero before the process parameters can vary significantly.

(1) Evaluation of the Coefficients of the Differential Equation

The mathematical equations describing the dynamic behavior of a system can usually be reduced to a differential equation involving the input and the output. This equation relates the output plus its derivatives to the input plus its derivatives, and formulates the basic relationship between the input and output, i.e., the relationship between the cause and the effect. For example, assuming that the system is third order, the equation can be written as

$$\frac{d^3C}{dt^3} + a_2\frac{d^2C}{dt^2} + a_1\frac{dC}{dt} + a_oC = b_2\frac{d^2r}{dt^2} + b_1\frac{dr}{dt} + b_or \qquad (2.1)$$

In order to identify the process described by Eq. (2.1), the coefficients of a's and b's must be evaluated. Generally speaking, the evaluation of the a's is far more important and difficult than evaluating the b's, since the right side of the differential equation usually is known from the process, and once the a's are evaluated, the b's may be evaluated in relatively simple manner, because the a's determine the form of time variations while the b's determine the relative amplitude of the system response, so that the concentration is on the evaluation of the a's.

In Eq. (2.1), the a's are the coefficients in the characteristic polynomial of the process. For example, the zeros of the polynomial

$$s^3 + a_2s^2 + a_1s + a_o$$

are the natural frequencies of the above process. If these zeros are denoted as z_1, z_2, z_3, the transient response of the process consists of the terms of the forms

$$e^{z_1 t} \qquad e^{z_2 t} \qquad e^{z_3 t}$$

Thus, evaluation of the natural frequencies present in the transient response may be used to determine the characteristic polynomial. Take the second order system for example, in this case

$$\frac{C(s)}{R(s)} = \frac{W_n^{\,2}}{s^2 + 2\xi w_n s + w_n^{\,2}} \tag{2.2}$$

or

$$\frac{d^2c}{dt^2} + 2\xi w_n \frac{dc}{dt} + w_n^{\,2} c = w_n^{\,2} r \quad . \tag{2.3}$$

The impulse response corresponding to Eq. (2.3) and Eq. (2.2) is shown in Fig. (2.2).

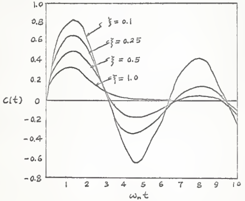

Fig. (2.2) Impulse response

From this impulse response, it is possible to determine the differential equation.

If the order and the nature of the right-hand side of the differential equation are known, an experimental method as shown in Fig. (2.2) can be used to determine the coefficients a_i for the equation

$$\frac{d^3C}{dt^3} + a_2\frac{d^2C}{dt^2} + a_1\frac{dC}{dt} + a_oC = b_or \qquad (2.4)$$

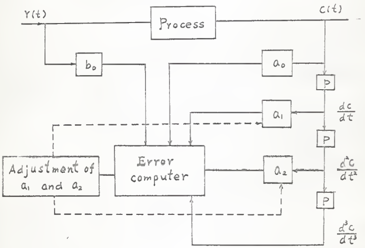

Fig. (2.3) Experimental evaluation of Eq. (2.4)

From Eq. (2.4), it is seen that, if the system is stable, a_o can be evaluated by the steady-state response with a step function input. For example, if the step input amplitude is R, the steady-state output C is b_oR/a_o, and a_o can be obtained.

In Fig. (2.3) the signals $\frac{dC}{dt}$, $\frac{d^2C}{dt^2}$, and $\frac{d^3C}{dt^3}$ can be generated by successive differentiations of the response, and by the use of trial and error method, a_1 and a_2 can be determined in such a way as to minimize the deviation from the differential equation (2.4). For example, in Eq. (2.4), if the input r is a step function, a_1 and a_2 can be approximately evaluated by minimizing the simi-infinite integral of the square of the error defined as

$$e = \frac{d^3C}{dt^3} + a_2\frac{d^2C}{dt^2} + a_1\frac{dC}{dt} + a_oC - b_or$$

In such an approach, for example, if a_1 and a_2 of Eq. (2.4) must be determined, then a two-dimensional space must be searched for the purpose of adjusting the values of a_1 and a_2 so as to realize the minimum of

$$\int_o^\infty e^2dt$$

Thus, this method involves an extensive calculation, particularly for the higher-order system, and is usually confined to low-order differential equations which are known a priori.

(2) Evaluation of the Parameters of the Impulse (or Unit Step) Response

When the process is a linear time-invariant two port transducer, the desired information often sought for the identification process is the impulse response. The unit step response can also be used to identify the dynamics of a process.

(a) Determination of the impulse (or unit step) response by the evaluation of the convolution integral

For a linear time-invariant system, the convolution integral

$$c(t) = \int_{-\infty}^{t} Y(t) g(t-\tau) dt \qquad (2.5)$$

must be solved to obtain $g(t)$, the process' impulse response, if $r(t)$, the excitation, and $c(t)$, the response, are known. Three techniques proposed by Braun, Mishkin and Kalman by the use of convolution integral to the identification problem are discussed.

Braun's Method

Braun [1] has described an identification technique in which stored energy does not affect the measurement. In Eq. (2.5), $g(t)$ must be evaluated from known values of $c(t)$ and $r(t)$. If $c(t)$, $r(t)$ and $g(t)$ all can be expanded in Maclaurin series, then Eq. (2.5) may be solved for the coefficients of the Maclaurin series expansion of $g(t)$ in terms of the coefficients of the series for $c(t)$ and the coefficients of the series for $r(t)$. Because the function $r(t)$ and $c(t)$ and their derivatives can be evaluated, the Maclaurin series expansion of $r(t)$ and $c(t)$ can be determined.

In practice, in the series expansions of $c(t)$ and $g(t)$, the determination of a finite (and usually small) number of terms is desired. In that case the applied forcing function will not be exactly the value needed to force the system output $c(t)$ to be equal to that which would result if an infinite number of terms of the series of $r(t)$ were used. Let $c_d(t)$ denote the desired system output. Therefore, a correction $\Delta r(t)$ is added to the system forcing function at $t=0$. The magnitude of $\Delta r(t)$ is chosen in such a way as to make $c(t)=c_d(t)$, where $c(t)$ is the output when a finite terms of the series of $r(t)$ are used.

This being the case, the system forcing function may be written as

$$\gamma(t) = \gamma_1(t) + \Delta r(t) \tag{2.6}$$

where $r_1(t)$ is the forcing function before correction and $\Delta r(t)$ the correction applied at $t=0$. In other words, $r(t)=r_1(t)$ for $t<0$. Substitution of Eq. (2.6) into Eq. (2.5), $c(t)$ yields

$$c(t) = \int_{-\infty}^{t} \gamma_1(\tau)g(t-\tau)d\tau + \int_{0}^{t} \Delta\gamma(\tau)g(t-\tau)d \tag{2.7}$$

$$= c_1(t) + c_2(t)$$

where

$$c_1(t) = \int_{-\infty}^{t} \gamma_1(\tau)g(t-\tau)d\tau \tag{2.8}$$

and

$$c_2(t) = \int_{0}^{t} \Delta\gamma(\tau)g(t-\tau)d\tau \tag{2.9}$$

Since $\Delta r(t)=0$ for $t<0$, the solution of $g(t)$ from Eq. (2.9) does not require a large-capacity memory, such that attention will be concentrated on Eq. (2.9).

The Maclaurin series expansion of $c_2(t)$ for $t>0$ can be determined as

$$c_2(t) = c_{20} + c_{21}t + c_{22}\frac{t^2}{2!} + \ldots = \sum_{n=0}^{\infty} c_{2n}\frac{t^n}{n!} \qquad (2.10)$$

where

$$c_{2r} = \left.\frac{d^r c_2(t)}{dt^r}\right|_{t=0_+} = c_2^{(r)}(0_+)$$

From Eq. (2.7), it follows that

$$c_2(t) = c(t) - c_1(t) \qquad (2.11)$$

At the instant $t=0_+$, Eq. (2.11) becomes

$$c_2(0_+) = c_{20} = c(0_+) - c_1(0_+) \qquad (2.12)$$

The forcing function $r_1(t)$ is assumed continuous at $t=0$, therefore $c_1(t)$ is continuous at $t=0$, thus

$$c_1(0_-) = c_1(0_+) \qquad (2.13)$$

But

$$c_1(t) = c(t) \quad \text{for } t < 0$$

therefore

$$c_1(0_-) = c(0_-) \qquad (2.14)$$

On substitution of Eqs. (2.13) and (2.14) into Eq. (2.12), $c_2(0_+)$ becomes

$$c_2(0_+) = c(0_+) - c(0_-) \tag{2.15}$$

By similar reasoning,

$$c_2^{(r)}(0_+) = c_{2r} = c^{(v)}(0_+) - c^{(r)}(0_-) \quad \text{for all } r \geq 0 \tag{2.16}$$

Eq. (2.16) can be used to determine the coefficients of the Maclaurin series expansion of $c_2(t)$ be making measurements of the derivatives of $c(t)$ just before and just after the instant $t=0$.

Since $\Delta r(t)$ is known, its Maclaurin series expansion can be determined as

$$\Delta r(t) = \Delta R_{-1}\,\delta(t) + \Delta R_0 + \Delta R_1 t + \Delta R_2 \frac{t^2}{2!} + \ldots \quad \text{for } t>0 \tag{2.17}$$

where $\delta(t)$ is unit impulse applied at $t=0$. The addition of the impulse is to correct the d-c level of the output at the beginning of the control interval.

Assuming that $g(t)$ is expandable in Maclaurin series expansion, it can be shown as

$$g(t) = G_0 + G_1 t + G_2 \frac{t^2}{2!} + \ldots = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \tag{2.18}$$

After substitution of Eqs. (2.17) and (2.18) into Eq. (2.9), and integration of Eq. (2.9) term by term, $c_2(t)$ becomes

$$c_2(t) = \int_0^t \quad [\Delta R_{-1}\delta(\tau) + \Delta R_0 + \Delta R_1 \quad + \Delta R_2 \frac{\;^2}{2!} + \ldots]$$

$$\cdot \ [G_0 + G_1(t-\tau) + G_2\frac{(t-\tau)^2}{2!} + \ldots]d\tau$$

$$= \int_0^t \quad \Delta R_{-1}G_0\delta(\tau) + [\Delta R_0 G_0 + \Delta R_{-1}G_1\delta(\tau)(t-\tau)]$$

$$+ \ [\Delta R_1 G_0 \tau + \Delta R_0 G_1(t-\tau) + \Delta R_{-1}G_2\delta(\tau)\frac{(t-\tau)^2}{2!}] + \ldots d\tau$$

$$= \Delta R_{-1}G_0 + (\Delta R_{-1}G_1 + \Delta R_0 G_0)t$$

$$+ \ (\Delta R_{-1}G_2 + \Delta R_0 G_1 + \Delta M_1 G_0)\frac{t^2}{2!} + \ldots \tag{2.19}$$

where the fact $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$ is used.

Eq. (2.10) and Eq. (2.19) both express $c_2(t)$ in power-series expansions. From the uniqueness properties of the coefficients of power series, and since they are assumed to be convergent and have the same sum $c_2(t)$, the coefficients of like powers of t in both series must be equal

$$\begin{aligned} c_{20} &= \Delta R_{-1}G_0 \\ c_{21} &= \Delta R_{-1}G_1 + \Delta R_0 G_0 \\ c_{22} &= \Delta R_{-1}G_2 + \Delta R_0 G_1 + \Delta R_1 G_0 \end{aligned} \tag{2.20}$$

Solving Eq. (2.20) for G_0, $G_1, \ldots,$ it is found that

$$G_0 = \frac{C_{20}}{\Delta R_{-1}}$$

$$G_1 = \frac{C_{21} - \Delta R_0 G_0}{\Delta R_{-1}}$$

$$G_2 = \frac{C_{22} - \Delta R_0 G_1 - \Delta R_1 G_0}{\Delta R_{-1}}$$

. . .

In general,

$$G_r = \frac{C_{2r} - (\Delta R_0 G_{r-1} + \Delta R_1 G_{r-2} + \ldots + \Delta R_{r-2} G_1 + \Delta R_{r-1} G_0)}{\Delta R_{-1}} \quad (2.21)$$

for all $r \geq 1$ where $\Delta R_{-1} \neq 0$

If $R_{-1} = 0$, ΔR_{-1} may be made zero in Eq. (2.20). In this case, the solution of Eq. (2.20) for G_0, G_1, ..., yields

$$G_0 = \frac{C_{21}}{\Delta R_0}$$

$$G_1 = \frac{C_{22} - \Delta R_1 G_0}{\Delta R_0}$$

$$G_2 = \frac{C_{23} - (\Delta R_1 G_1 + \Delta R_2 G_0)}{\Delta R_0}$$

. . .

In general,

$$G_r = \frac{C_{2,r+1} - (\Delta R_1 G_{r-1} + \Delta R_2 G_{r-2} + \ldots + \Delta R_{r-1} G_1 + \Delta R_r G_0)}{\Delta R_0} \qquad (2.22)$$

for all $r \geq 1$, where $\Delta R_0 \neq 0$

From Eq. (2.21) and Eq. (2.22), the coefficients of the series expansion of the impulse response g(t) are evaluated.

The advantage for this technique is that the existance of stored energy in the system does not affect the measurement and that no extraneous signal is required. But the difficulty usually encountered in the measurement of high-order derivatives and the requirement for an impulse function in $\Delta r(t)$ are the disadvantages of the method.

Mishkin and Haddad's Method

Mishkin and Haddad [2] described an identification technique that uses a computer. Such an approach is represented by Fig. (2.4). In this technique, the computer monitors the signals c(t) and a(t) and approximately solves the convolution integral for the system's unit step response every T seconds.

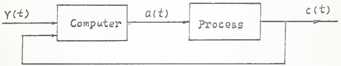

Fig. (2.4) Computer controlled system

If the unit step response of the process is considered, the convolution integral reads

$$C(t) = \int_{-\infty}^{t} a'(\tau)u(t-\tau)\,d\tau \tag{2.23}$$

where the prime denotes differentiation and $u(t)$ is the unit step response. Eq. (2.23) may be rewritten in the form

$$C(t) = \int_{-\infty}^{0-} a'(\tau)u(t-\tau)\,d\tau + \int_{0-}^{t} a'(\tau)u(t-\tau)\,d\tau \tag{2.24}$$

$$= c_1(t) + c_2(t)$$

where

$$c_1(t) = \int_{-\infty}^{0-} a'(\tau)u(t-\tau)\,d\tau \tag{2.25}$$

$$c_2(t) = \int_{0-}^{t} a'(\tau)u(t-\tau)\,d\tau \tag{2.26}$$

The integral $c_1(t)$ represents the process response due to stored energy. It is this term which makes the measurement problem particularly troublesome. The solution can be done in the following manner:

The Maclaurin series of $u(t-\tau)$ has the form

$$u(t-\tau) = u(-\tau) + \frac{t}{2!}u'(-\tau) + \frac{t^2}{2!}u''(-\tau) + \ldots \tag{2.27}$$

Substitution of Eq. (2.27) into Eq. (2.25) yields

$$c_1(t) = \int_{-\infty}^{0-} a'(\tau)u(-\tau)d\tau + \frac{t}{1!}\int_{-\infty}^{0-} a'(\tau)u'(-\tau)d \qquad (2.28)$$

$$+ \frac{t^2}{2!}\int_{-\infty}^{0-} a'(\tau)u''(-\tau)d\tau + \ldots$$

Expanding $c_1(t)$ in a Taylor series about $t=0_-$, it is found that

$$c_1(t) = c_1(0) + c_1'(0)t + \frac{c_1''(0)}{2!}t^2 + \ldots \qquad (2.29)$$

where

$$c_1(0) = \int_{-\infty}^{0-} a'(\tau)u(-\tau)d\tau = c(t)_{\,t=0_-} = c(0_-)$$

$$c_1'(0) = \int_{-\infty}^{0-} a'(\tau)u'(-\tau)d\tau = \left.\frac{dc(t)}{dt}\right|_{t=0_-} = c'(0_-)$$

$$c_1''(0) = \int_{-\infty}^{0-} a'(\tau)u''(-\tau)d\tau = \left.\frac{d^2c(t)}{dt^2}\right|_{t=0_-} = c''(0_-)$$

Hence

$$c_1(t) = c(0_-) + tc'(0_-) + \frac{c''(0_-)}{2!}t^2 + \ldots \qquad (2.30)$$

$$= \sum_{i=0}^{\infty} \frac{c^{(i)}(0_-)}{i!}t^i$$

Eq. (2.30) involves the sum of an infinite Taylor series. In practice, we want a finite number of terms. This can be done by choosing some interval $0<t<T$, such that the series converges rapidly. n is used to represent this finite number of terms. Thus, Eq. (2.24) becomes

$$c(t) = \sum_{i=0}^{n} \frac{t^i c^{(i)}(0_-)}{i!} + c_2(t) \qquad (2.31)$$

In order to simplify the evaluation of $c_2(t)$, and to measure the unit step response at every T seconds, a "staircase" function is used for actuating signal a(t).

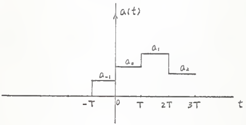

Fig. (2.5) Staircase form for a(t)

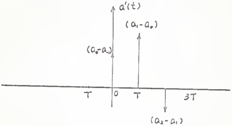

Fig. (2.6) Derivative of staircase function

Let a(t) be the staircase function shown in Fig. (2.5). Then a'(t) must be a string of impulses occuring at t=0,T, 2T,.... etc. as shown in Fig. (2.6). And in the interval $0_{-} \le t \le T_{-}$, $c_2(t)$ becomes

$$c_2(t) = \int_{0_{-}}^{t} (a_0 - a_{-1})\delta(\tau)u(t-\tau)d\tau$$

$$= \Delta a_0 u(t)_1 \qquad 0 < t < T \tag{2.32}$$

where

$$\Delta a_0 = a_0 - a_{-1}$$

and $u(t)_1$ is the process' unit step response in the interval $0<t<T$.

The evaluation of the unit step response at the instant $t=T_{-}$, from Eq. (2.30) and Eq. (2.32), Eq. (2.31) yields

$$c(T_{-}) = \sum_{i=0}^{n} \frac{T^i c^{(i)}(0_{-})}{i!} + \Delta a_0 u(T)_1 \tag{2.33}$$

The last term is obtained since the unit step response is continuous at t=T.

Solving Eq. (2.33) for $u(t)_1$ yields

$$u(T)_1 = \frac{c(T_{-}) - \sum_{i=0}^{n} \frac{T^i c^{(i)}(0_{-})}{i!}}{\Delta a_0} \tag{2.34}$$

Here, $u(T)_1$ is the unit step response at t=T when a unit step applied T seconds earlier.

The response at t=2T with a unit step input applied at t=T can be derived in the similar manner. In this case, Eq. (2.23) may be rewritten as

$$c(t) = \int_{-\infty}^{T_-} a'(\tau)u(t-\tau)d\tau + \int_{T_-}^{t} a'(\tau)u(t-\tau)d\tau \tag{2.34}$$

$$= c_3(t) + c_4(t)$$

where

$$c_3(t) = \int_{-\infty}^{T_-} a'(\tau)u(t-\tau)d$$

$$c_4(t) = \int_{T_-}^{t} a'(\tau)u(t-\tau)d\tau$$

The Taylor series expansion of $u(t-\tau)$ at t=T yields

$$u(t-\tau) = u(T-\tau) + u'(T-\tau)(t-T) + \frac{u''(T-\tau)}{2!}(t-T)^2 + \ldots \tag{2.35}$$

Now the expansion of $c_3(t)$ in a Taylor series about $t=T_-$, yields

$$c_3(t) = c_3(T) + c_3'(T)(t-T) + \frac{c_3''(T)}{2!}(t-T)^2 + \ldots \tag{2.36}$$

where

$$c_3(T) = \int_{-\infty}^{T_-} a'(\tau)u(T-\tau)d\tau = c(t)\Big|_{t=T_-} = c(T_-)$$

$$c_3'(T) = \int_{-\infty}^{T_-} a'(\tau)u(T-\tau)d\tau = c'(t)\Big|_{t=T_-} = c'(T_-)$$

$$c_3''(T) = \int_{-\infty}^{T_-} a'(\tau)u(T-\tau)d\tau = c''(t)\Big|_{t=T_-} = c''(T_-)$$

Therefore

$$c_3(t) = c(T_-) + c'(T_-)(t-T) + \frac{c''(T)}{2!}(t-T)^2 + \ldots$$

$$= \sum_{i=0}^{n} \frac{c^{(i)}(T_-)}{i!}(t-T)^i \tag{2.37}$$

Considering the interval $T_-<t<2T_-$, $c_4(t)$ can be expressed as

$$c_4(t) = \int_{T_-}^{t} (a_1-a_0)\delta(\tau-T)u(t-\tau)d\tau = \Delta a_1 u(t-T)$$

$$= \Delta a_1 u(t)_2 \qquad T < t < 2T \tag{2.38}$$

where $\Delta a_1 = a - a_0$ and the subscript 2 denotes that the measurement is taken at the second interval. Let $t=2T_-$. From Eq. (2.37) and Eq. (2.38), Eq. (2.34) becomes

$$c(2T_{-}) = \sum_{i=0}^{n} \frac{c^{(i)}(T_{-})}{i!}(2T_{-} - T)^{i} + \Delta a_1 u(t)_2$$

$$= \sum_{i=0}^{n} \frac{c^{(i)}(T_{-})T^{i}}{i!} + \Delta a_1 u(T)_2 \tag{2.39}$$

Solving $u(T)_2$ from Eq. (2.39), it is found that

$$u(T)_2 = \frac{c(2T_{-}) - \sum_{i=0}^{n} \frac{T^{i} c^{(i)}(T_{-})}{i!}}{\Delta a_1} \tag{2.40}$$

Similarly, in general

$$u(T)_k = \frac{c(kT_{-}) - \sum_{i=0}^{n} \frac{T^{i} c^{(i)}[(K-1)T]}{i!}}{\Delta a_{k-1}} \tag{2.41}$$

Thus, the unit step response at t=kT can be evaluated from Eq. (2.41). This can be solved using relatively simple computing equipment.

Kalman's Method

Kalman [3] described a technique employing a pulse transfer function. For the block diagram of Fig. (2.7), the relationship between a(t) and c(t) is described by the convolution integral

$$c(t) = \int_{-\infty}^{t} a'(\tau)u(t-\tau)d\tau \tag{2.42}$$

where u(t) is the unit step response; u(t)=0 when t<0.

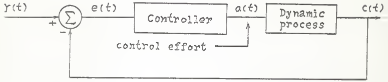

Fig. (2.7) Block diagram of simplest control system

Solution of the identification problem here requires solution of the integral equation for u(t) which usually is a difficult task. Kalman handled such problems by using digital techniques in which sampled values of c(t) and a(t) are used, such that the solution of an integral equation can be replaced by the solution of a set of algebraic equations.

In this technique, the sampling instants are denoted by $t=kT$, $k=0, 1, 2, \ldots$, where T is called the sampling period. Then the sampled values of a(t) and c(t) are

$a(0), a(T), a(2T), \ldots a(kT), \ldots$

$c(0), c(T), c(2T), \ldots c(kT), \ldots$

$k=0,1,2,\ldots$

For the sake of simplification, one sets $a_k = a(kT)$ and $c_k = c(kT)$.

If the sequences of numbers a_k is used to represent the continuous function a(t), some method of interpolation is required. In what follows, a(t) is assumed to be the output of a "sample-and-hold" circuit, i.e., a(t) is given by

$$a(t) = a_k \qquad kT \leq t < (k+1)T$$

In a manner similar to that we used to handle the "staircase" function in the previous section, Eq. (2.42) can be written as

$$c(t) = \sum_{=-\infty}^{T<t} u(t-\ell T)(a_\ell - a_{\ell-1}) \tag{2.45}$$

Considering only sampled values of c(t) and u(t), and noting that u(kT)=0 for all k<0, Eq. (2.45) yields

$$c_k = \sum_{\ell=-\infty}^{\ell=k} u(kT-\ell T)(a_\ell - a_{\ell-1})$$

$$= \sum_{\ell=-\infty}^{\ell=k} u\,[(k-\ell)T\,](a_\ell - a_{\ell-1})$$

$$= \sum_{\ell=-\infty}^{\ell=k} u_{k-\ell}\,a_\ell - \sum_{\ell=-\infty}^{\ell=k} u_{k-\ell}\,a_{\ell-1}$$

$$= \sum_{\ell=-\infty}^{\ell=k} u_{k-\ell}\,a_\ell - \sum_{\ell=-\infty}^{\ell=k-1} u_{k-\ell-1}\,a_\ell$$

$$= \sum_{\ell=-\infty}^{\ell=k} u_{k-\ell}\,a_\ell - \sum_{=-\infty}^{=k} u_{k-\ell-1}\,a_\ell$$

$$= \sum_{\ell=-\infty}^{\ell=k} (u_{k-\ell} - u_{k-\ell-1})\,a_\ell$$

$$= \sum_{\ell=-\infty}^{\ell=k} g_{k-\ell}\,a_\ell \tag{2.46}$$

Because the unit step response can be used to identify the process, the process dynamics may now be characterized by the infinite set of numbers g_0, g_1, ... g_k ... which can be determined by solving an infinite set of simultaneous linear algebraic equations given by Eq. (2.46). In practice, it is not possible to solve an infinite set of equations, but in the case of a stable process $g_k \to 0$ as $k \to \infty$, so that only a finite set of linear algebraic equations has to be solved to obtain g_k, although the number of equations required is perhaps still much too large to be useful.

A different way to represent a dynamic process utilizes the pulse transfer function which relates the system output and input at the sampling instants. It is known that if any fixed linear system whose input a(t) and output c(t) are related by a linear differential equation, then its input a_k and output c_k at discrete instants t=kT can be described by the difference equation

$$c_k + b_1 c_{k-1} + \dots + b_n c_{k-n} = d_0 a_k + d_1 a_{k-1} + \dots + d_q a_{k-q} \tag{2.47}$$

In general, n=q. Then, Eq. (2.47) can be rewritten as

$$c_k = d_0 a_k + d_1 a_{k-1} + \dots + d_n a_{k-n} - b_1 c_{k-1} - \dots - b_n c_{k-n} \tag{2.48}$$

By comparing Eq. (2.48) and Eq. (2.46), it is seen that when the system is known to be governed by a difference equation, much fewer d_i and b_i than g_k are needed to represent the system.

Using the notation $z^i c_k = c_{k+i}$, (where i is any integer), and refering back to Eqs. (2.48) and (2.46), it is possible to make much more efficient use of the available data by employing the pulse transfer function G(z), defined by

$$G(Z) = \frac{C(Z)}{A(Z)} = \frac{d_1 Z^{-1} + \ldots + d_n Z^{-n}}{1 + b_1 Z^{-1} + \ldots + b_n Z^{-n}}\,, \tag{2.49}$$

$$= g_1 Z^{-1} + g_2 Z^{-2} + \ldots + g_k Z^{-k} + \ldots$$

Here $d_0 = g_0 = 0$, since physical systems cannot respond instantaneously.

The coefficients in Eq. (2.48) are the same as the coefficients in Eq. (2.46). From Eq. (2.49), it is possible to characterize the process' dynamics by the d_i and b_i, i.e., by 2n numbers rather than by an infinity of numbers. For this reason, Kalman used the d_i and b_i to identify the process.

The accuracy of the process characterization depends upon the value of n in Eq. (2.49). n should be chosen sufficiently large so that the d_i and b_i represent the process with some desired accuracy. This is a matter of approximation and n can be regarded as the design parameter to be selected by the designer. For the purpose of illustration, it will be assumed that n=2.

Now the determination of the coefficients of Eq. (2.48), i.e., the d_i and b_i is desired. This can be done by using a substantial amount of measured data in order to minimize the effect of measurement errors. Now suppose a particular guess for the d_i and b_i at the Nth sampling instant is made. These assumed values of the coefficients will be denoted by $d_i(N)$ and $b_i(N)$. It is possible to compute all the past values of c_k using this set of coefficients in Eq. (2.48). This value of c_k will be called $c_k^*(N)$. It is given by

$$c_k^*(N) = d_1(N)a_{k-1} + d_2(N)a_{k-2} + \dots + d_n(N)a_{k-n}$$
$$-b_1(N)c_{k-1} - b_2(N)c_{k-2} - \dots - b_n(N)c_{k-n}$$
$$k = 0,1,2,\dots,N. \qquad (2.50)$$

The mean-square error is a useful measure of the accuracy of the set of coefficients chosen. The mean-square error at the sampling instants is defined as

$$\frac{1}{N}\sum_{k=0}^{N}\varepsilon_k{}^2(N) = \frac{1}{N}\sum_{k=0}^{N}[c_k - c_k{}^*(N)]^2 \qquad (2.51)$$

where $\varepsilon_k{}^2(N)$ denotes the square error between measured past values c_k and values computed from Eq. (2.50). The $d_i(N)$ and $b_i(N)$ will be chosen to minimize Eq. (2.51).

Since the process dynamics may change with time, the older data should not be given the same importance as more recent data. In order to meet this requirement, the mean-square error given by Eq. (2.51) must be modified to include a weighting function $w(t)$ which is a continuous, monotonically decreasing function of time such that

$$w(0) = 1$$

$$0 < w(t) < 1 \qquad 0 < t < \infty \tag{2.52}$$

$$w(\infty) = 0$$

$$\int_0^\infty w(t)\,dt < \infty$$

If the value of w(t) at the kth sampling instant is denoted by w_k, the final criterion of determining the coefficients may be stated that choosing $d_i(N)$, $b_i(N)$ in such a way that the expression

$$E(N) = \sum_{k=0}^{k=N} \epsilon_k^{\,2}(N) w_{N-k} \tag{2.53}$$

is a minimum.

The required computation would be considerably simplified if the coefficients of Eq. (2.48) are not recomputed at every sampling interval, but rather at every qth interval, where q is an integer. From Eqs. (2.48) and (2.50), assuming n=2, it follows that

$$\epsilon_{qj}^{\,2}(N) = [\,c_{qj} - c^*_{qj}(N)]^2 = c_{qj}^2 - 2c_{qj}c^*_{qj}(N) + c^{*2}_{qj}(N)$$

$$= c_{qj}^2 - 2c_{qj}[d_1(N)a_{qj-1} + d_2(N)a_{qj-2} - b_1(N)c_{qj-1} - b_2(N)$$

$$c_{qj-2}] + [\,d_1(N)a_{qj-1} + d_2(N)a_{qj-2} - b_1(N)c_{qj-2} -$$

$$b_2(N)c_{qj-2}]^2$$

$$= C_{qj}^2 + b_1^2(N)C_{qj-1}^2 + b_2^2(N)C_{qj-2}^2 + 2b_1(N)C_{qj}C_{qj-1}$$

$$+ 2b_2(N)C_{qj}C_{qj-2} + 2b_1(N)b_2(N)C_{qj-1}C_{qj-2}$$

$$- 2d_1(N)C_{qj}a_{qj-1} - 2d_2(N)C_{qj}a_{qj-2}$$

$$-2b_1(N)d_1(N)C_{qj-1}a_{qj-1} - 2b_1(N)d_2(N)C_{qj-1}a_{qj-2}$$

$$-2b_2(N)d_1(N)C_{qj-2}a_{qj-1} - 2b_2(N)d_2(N)C_{qj-2}a_{qj-2}$$

$$+ d_1^2(N)a_{qj-1}^2 + d_2^2(N)a_{qj-2}^2 + 2d_1(N)d_2(N)a_{qj-1}a_{qj-2} \tag{2.54}$$

The measured values of c's and a's occurred in Eq. (2.54) always in terms of the type

$$C_{qj-r} \qquad C_{qj-r}a_{qj-s} \qquad a_{qj-r}a_{qj-s} \tag{2.55}$$

where r,s=0,1,2. If we let q=n+1=3, a set of pseudo-correlation functions may be defined as

$$\phi_{N-r}^{cc}(r-s) = \sum_{j=1}^{j=N/3} C_{3j-r}C_{3j-s}W_{N-3j}$$

$$\phi_{N-r}^{ca}(r-s) = \sum_{j=1}^{j=N/3} C_{3j-r}a_{3j-s}W_{N-3j} \tag{2.56}$$

$$\phi_{N-r}^{aa}(r-s) = \sum_{j=1}^{j=N/3} a_{3j-r}a_{3j-s}W_{N-3j}$$

From Eqs. (2.53), (2.54) and (2.56), the function E(N) may be written as

$$E(N) = \phi_N^{cc}(0) + b_1^2(N)\phi_{N-1}^{cc}(0) + b_2^2(N)\phi_{N-2}^{cc}(0)$$

$$+ 2b_1(N)\phi_N^{cc}(-1) + 2b_2(N)\phi_N^{cc}(-2)$$

$$+ 2b_1(N)b_2(N)\phi_{N-1}^{cc}(-1) - 2d_1(N)\phi_N^{ca}(-1)$$

$$- 2d_2(N)\phi_N^{ca}(-2) - 2b_1(N)d_1(N)\phi_{N-1}^{ca}(0)$$

$$- 2b_1(N)d_2(N)\phi_{N-1}^{ca}(-1) - 2b_2(N)d_1(N)\phi_{N-2}^{ca}(1)$$

$$- 2b_2(N)d_2(N)\phi_{N-2}^{ca}(0) + d_1^2(N)\phi_{N-1}^{aa}(0) + d_2^2(N)\phi_{N-2}^{aa}(0)$$

$$+ 2d_1(N)d_2(N)\phi_{N-1}^{aa}(-1) \qquad (2.57)$$

At this stage, the pseudo-correlation is to be computed. This can be done by choosing the weighting function as

$$w_{sj} = \alpha^j \qquad (0 < \alpha < 1) \qquad (2.58)$$

Then

$$\phi_{sj-r}^{cc}(r-s) = \sum_{i=1}^{i=j} C_{3i-r}C_{3i-s}W_{3j-3i}$$

$$\phi_{3(j-1)-r}^{cc}(r-s) = \sum_{i=1}^{i=j} C_{3i-r}C_{3i-s}W_{3j-3-3i}$$

and

$$\phi^{cc}_{3j-r}(r-s) - \alpha\phi^{cc}_{3(j-1)-r}(r-s)$$

$$= c_{3-r}c_{3-s}W_{3j-3} + c_{3\cdot 2-r}c_{3\cdot 2-s}W_{3j-3\cdot 2t}\ldots$$

$$+ c_{3j-r}c_{3j-s}W_0 - \alpha[c_{3-r}c_{3-s}W_{3(j-2)}$$

$$+ c_{3\cdot 2-r}c_{3\cdot 2-s}W_{3(j-3)} + \ldots + c_{3j-3-r}c_{3j-3-s}W_0]$$

$$= c_{3-r}c_{3-s}[W_{3(j-1)} - W_{3(j-2)}] + c_{3\cdot 2-r}c_{3\cdot 2-s}[W_{3(j-2)}$$

$$- W_{3(j-3)}] + \ldots + c_{3j-3-r}c_{3j-3-s}[W_3 - \alpha W_0] + (c_{3j-r}c_{3j-s})W_0$$

$$= c_{3j-r}c_{3j-s}W_0$$

$$= c_{3j-r}c_{3j-s}$$

In the similar manner, it can be shown that each pseudo-correlation function satisfies a first-order difference equation of the type

$$\phi^{ca}_{3j-r}(r-s) - \alpha\phi^{ca}_{3(j-1)-r}(r-s) = c_{3j-r}a_{3j-s} \tag{2.59}$$

Eq. (2.59) can be used to compute the pseudo-correlation functions and it requires only a knowledge of the corresponding function three samples earlier plus the values of c_{N-2}, c_{N-1}, c_N, a_{N-2} and a_{N-1}.

Now the values of $d_1(N)$, $d_2(N)$, $b_1(N)$, and $b_2(N)$ are chosen to make $E(N)$ in Eq. (2.57) a minimum. These values are determined from the conditions

$$\frac{\alpha E(N)}{\alpha d_i} = 0 \qquad \frac{\alpha E(N)}{\alpha b_i} = 0 \quad \text{for } i = 1,2,\ldots n. \tag{2.60}$$

Application of Eq. (2.60) to Eq. (2.57) leads to four linear equations in $d_1(N)$, $d_2(N)$, $b_1(N)$ and $b_2(N)$ as follows

$$d_1(N)\phi^{aa}_{N-1}(0) + d_2(N)\phi^{aa}_{N-1}(-1) - b_1(N)\phi^{ca}_{N-1}(0) - b_2(N)\phi^{ca}_{N-2}(1) =$$

$$\phi^{ca}_{N}(-1), \; d_1(N)\phi^{aa}_{N-1}(-1) - d_2(N)\phi^{aa}_{N-2}(0) - b_1(N)\phi^{ca}_{N-2}(0) = \phi^{ca}_{N}(-2).$$

$$(2.61)$$

$$- d_1(N)\phi^{ca}_{N-1}(0) - d_2(N)\phi^{ca}_{N-1}(-1) + b_1(N)\phi^{cc}_{N-1}(0) + b_2(N)\phi^{cc}_{N-1}(-1)$$

$$= -\phi^{cc}_{N}(-1), \; - d_1(N)\phi^{ca}_{N-2}(1) - d_2(N)\phi^{ca}_{N-2}(0) + b_1(N)\phi^{cc}_{N-1}(-1)$$

$$+ b_2(N)\phi^{cc}_{N-2}(0) = -\phi^{cc}_{N}(-2)$$

The straightforward solution of Eq. (2.61) for the desired coefficients requires considerable computation. Kalman, using the Gauss-Seidel iteration procedure, found the coefficients as follows:

$$d_1(N) = \frac{-d_2(N)\phi^{aa}_{N-1}(-1)+b_1(N-3)\phi^{ca}_{N-1}(0)+b_2(N-3)\phi^{ca}_{N-2}(1)+\phi^{ca}_{N}(-1)}{\phi^{aa}_{N-2}(0)}$$

$$d_2(N) = \frac{-d_1(N)\phi^{aa}_{N-1}(-1)+b_1(N-3)\phi^{ca}_{N-1}(-1)+b_2(N-3)\phi^{ca}_{N-2}(0)+\phi^{ca}_{N}(-2)}{\phi^{aa}_{N-2}(0)}$$

$$b_1(N) = \frac{d_1(N)\phi^{ca}_{N-1}(0)+d_2(N)\phi^{ca}_{N-1}(-1)-b_2(N-3)\phi^{cc}_{N-1}(-1)-\phi^{cc}_{N}(-1)}{\phi^{cc}_{N-1}(0)}$$

$$b_2(N) = \frac{d_1(N)\phi^{ca}_{N-1}(1)+d_2(N)\phi^{ca}_{N-2}(0)-b_1(N)\phi^{cc}_{N-1}(-1)-\phi^{cc}_{N}(-2)}{\phi^{cc}_{N-2}(0)} \qquad (2.62)$$

Eq. (2.62) constitutes Kalman's solution of the identification problem. A relatively small special-purpose digital computer is used to realize this technique.

(b) Determination of the impulse response using cross-correlation.

Truxal [4] described a method to measure the impulse response of a linear process by the use of cross-correlation. In this technique, a white noise is used as the process input.

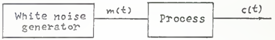

Fig. (2.8) Measurement of g(t) by white noise input

Fig. (2.8) shows the block diagram for this procedure. The output of the linear process is given as

$$c(t) = \int_{-\infty}^{\infty} g(x)m(t-x)\,dx \tag{2.63}$$

The cross-correlation function between m(t) and c(t) is given by

$$\phi_{mc}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} m(t-\tau)c(t)\,dt \tag{2.64}$$

Substituting Eq. (2.63) into Eq. (2.64), $\phi_{mc}(\tau)$ becomes

$$\phi_{mc}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} m(t-\tau)\,dt \int_{-\infty}^{\infty} g(x)m(t-x)\,dx \tag{2.65}$$

By interchanging the order of integration, $\phi_{mc}(\tau)$ becomes

$$\phi_{mc}(\tau) = \int_{-\infty}^{\infty} g(x)\,dx \left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} m(t-\tau)m(t-x)\,dt\right] \tag{2.66}$$

The term within the square brackets is the autocorrelation function of the input m(t) with argument $(\tau-x)$, that is

$$\phi_{mm}(\tau-x) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} m(t-\tau)m(t-x)\,dt$$

Thus, $\phi_{mc}(\tau)$ may be written as

$$\phi_{mc}(\tau) = \int_{-\infty}^{\infty} g(x)\,\phi_{mm}(\tau-x)\,dx \tag{2.67}$$

In this process, a white noise is injected as the input. It will be shown that the autocorrelation of white noise is an impulse.

Some random function $x(t)$ is considered, and let a new function $x_T(t)$ be defined by

$$x_T(t) = \begin{cases} x(t) & |t| \leq T \\ 0 & |t| > T \end{cases} \tag{2.68}$$

In the limit as T becomes infinite, the function $x_T(t)$ becomes equal to $x(t)$; however, $x_T(t)$ possesses the advantage of having only a finite total energy for finite values of T.

The ensemble average of $x_T(t)x_T(t+\tau)$ is defined as

$$<x_T(t)x_T(t-\tau)> = \frac{1}{2T}\int_{-\infty}^{\infty} x_T(t-\tau)\,dt \tag{2.69}$$

If $$x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}_T(w)\,\epsilon^{jwt}dw \tag{2.70}$$

Substituting Eq. (2.70) into Eq. (2.69), one obtains

$$< x_T(t)x_T(t+\ )> = \frac{1}{2T}\int_{-\infty}^{\infty}\left\{\left[\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}_T(w)e^{jwt}dw\right]\cdot x_T(t+\tau)\right\}\,dt \tag{2.71}$$

Interchanging the order of integration, and, in addition, the substitution $t'=t+\tau$ is made, Eq. (2.71) yields

$$<x_T(t)x_T(t+\tau)> = \frac{1}{2T}\int_{-\infty}^{\infty}\left\{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}_T(w)e^{jw(t'-\tau)}dw\right]\cdot x_T(t')\right\}dt'$$

$$= \frac{1}{2T} \int_{-\infty}^{\infty} \frac{dw}{2\pi} \overline{X}_T(w) \varepsilon^{-jwt} \int_{-\infty}^{\infty} x_T(t') \cdot e^{jwt'} dt'$$

$$= \frac{1}{2T} \int_{-\infty}^{\infty} \overline{X}_T(w) \overline{X}_T^*(w) \varepsilon^{-jwt} \frac{dw}{2\pi}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\overline{X}_T(w)|^2}{2T} e^{-jwt} dw \tag{2.72}$$

Now if T is allowed to approach infinity, the Eq. (2.72) defines the autocorrelation function of x(t), i.e.,

$$\phi_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\overline{X}_T(w)|^2}{2T} \varepsilon^{-jw\tau} dw \tag{2.73}$$

From Eq. (2.73), the power density spectrum $\Phi_{xx}(w)$ is

$$\Phi_{xx}(w) = \lim_{T \to \infty} \frac{|\overline{X}_T(w)|^2}{2T} \tag{2.74}$$

Now the white noise is defined as a random signal with a flat frequency spectrum, i.e., $\Phi_{xx}(w) = k$ for all w. Then from Eq. (2.74) and Eq. (2.73), for white noise, it follows that

$$\phi_{xx}(\tau) = \frac{K}{2\pi} \int_{-\infty}^{\infty} e^{-jwt} dw = K\delta(\tau) \tag{2.75}$$

where

$$\delta(x-x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j(x-x_0)u} du$$

and $\delta(t)$ is a unit impulse at t=0.

It is shown that the autocorrelation function of a white noise is an impulse. With this in mind, looking back at Eq. (2.67), if m(t) is white noise whose power density spectrum is constant at a value K, $\phi_{mm}(\tau)$ yields

$$\phi_{mm}(\tau) = K\delta(\tau)$$

and $\phi_{mc}(\tau)$ becomes

$$\phi_{mc}(\tau) = K\delta(\tau) \tag{2.76}$$

Thus, from Eq. (2.76), it is seen that the cross-correlation function between the process input and output, when the input is white noise, is proportional to the value of g(t) at the time $t=\tau$. It is possible to obtain the value of g(t) at any desired instant by just varying the length of the delay τ. A simple mechanization to realize this procedure is shown in Fig. (2.9).

Although the white noise is a physically unrealizable phenomenon, any signal whose power density spectrum is constant over a frequency range considerably greater than the system bandwidth may be considered white noise. Therefore, the technique mentioned above is useful in the experimental identification of a process in the laboratory.

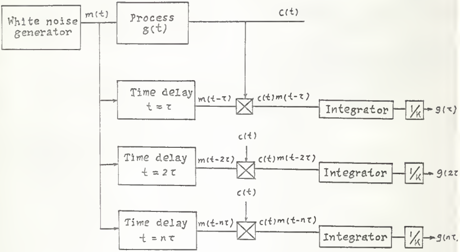

Fig. (2.9) Mechanization of identification with cross-correlation

If an adaptive system is considered, the test signal may be mixed with the normal operating input. Such a situation is shown in Fig. (2.10).

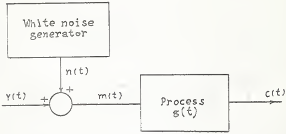

Fig. (2.10) Identification by cross-correlation

In Fig. (2.10), the signal r(t) is the normal operating input, and n(t) is white noise. Then, the process input is

$$m(t) = r(t) + n(t) \tag{2.77}$$

From Eq. (2.63), the output is

$$c(t) = \int_{-\infty}^{\infty} g(x)[r(t-x) + n(t-x)]\,dx$$

$$= \int_{-\infty}^{\infty} g(x)r(t-x)\,dx + \int_{-\infty}^{\infty} g(x)n(t-x)\,dx$$

$$= c_r(t) + c_n(t) \tag{2.78}$$

where

$$c_r(t) = \int_{-\infty}^{\infty} g(x)r(t-x)\,dx \tag{2.79}$$

$$c_n(t) = \int_{-\infty}^{\infty} g(x)n(t-x)\,dx \tag{2.80}$$

The cross-correlation between n(t) and c(t) is

$$\phi_{nc}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} n(t-\tau)c(t)\,dt \tag{2.81}$$

After substitution of Eq. (2.78) into Eq. (2.79), one obtains

$$\phi_{nc}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} n(t-\tau)\cdot c_r(t)\,dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} n(t-\tau)\,c_n(t)\,dt \tag{2.82}$$

Substitution of Eqs. (2.79) and (2.80) into Eq. (2.82), $\phi_{nc}(\tau)$ yields

$$\phi_{nc}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} n(t-\tau)g(x)\,\gamma(t-x)\,dx$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} n(t-\tau)g(x)n(t-x)\,dx$$

$$= \int_{-\infty}^{\infty} g(x)\,\phi_{nr}(\tau-x)\,dx + \int_{-\infty}^{\infty} g(x)\phi_{nn}(\tau-x)\,dx \tag{2.83}$$

Since the inputs r(t) and n(t) are uncorrelated, $\phi_{nr}(\tau) = 0$. In addition, n(t) is white noise, so Eq. (2.83) becomes

$$\phi_{nc}(\tau) = \int_{-\infty}^{\infty} g(x)\phi_{nn}(\tau - x)\,dx = Kg(\tau) \tag{2.84}$$

Again, a simple realization of Eq. (2.84) for any desired time is shown in Fig. (2.11).

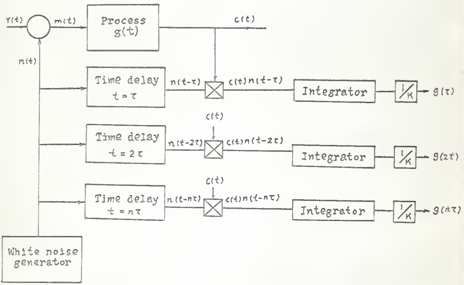

Fig. (2.11) Identification in the present of r(t) using cross-correlation

In this technique, the measurement of g(t) is independent of r(t), so that stored energy is not considered, since it is due to r(t). But the compensating disadvantage is that an extraneous signal in this process is required.

Another method which uses the cross-correlator to measure the unit impulse in the identification process will be discussed. This technique was described by Anderson and Buland and Cooper [5]. The block diagram of the method is shown in Fig. (2.12). The excitation $x_i(t)$ is assumed to be a sample of an ergodic random process.

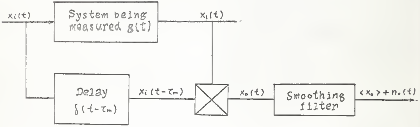

Fig. (2.12) A basic cross-correlator

In Fig. (2.12), the system output $x_1(t)$ can be obtained from the convolution integral as

$$x_1(t) = \int_0^\infty x_i(t-\lambda)g(\lambda)\,d\lambda \tag{2.85}$$

Then, the function $x_1(t)$ is multiplied by the delayed input to produce

$$x_0(t) = \int_0^\infty x_i(t-\tau_m)x_i(t-\lambda)g(\lambda)\,d\lambda \tag{2.86}$$

The ensemble average value of $x_0(t)$ is

$$\langle x_0 \rangle = \int_0^\infty E[x_i(t-\tau_m)x_i(t-\lambda)]g(\lambda)d\lambda \tag{2.87}$$

$$= \int_0^\infty \phi_{ii}(\tau_m-\lambda)g(\lambda)d\lambda$$

where $\phi_{ii}(\tau)$ is the autocorrelation function of $x_i(t)$.

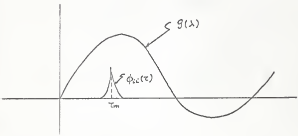

Fig. (2.13) Showing the convolution required to obtain x_0

A graphical representation of the convolution is shown in Fig. (2.13). When the input $x_i(t)$ is wide band, the autocorrelation of it would be sufficiently narrow, then

$$\langle x_0 \rangle = Kg(\tau_m) \tag{2.88}$$

where K is the area under the $\phi_{ii}(\tau)$ function.

Eq. (2.88) states that the value $\langle x_0 \rangle$ is nearly proportional to the impulse response at $t=\tau_m$. Therefore the measurement of $\langle x_0 \rangle$ can be used to measure the impulse response of the system.

If the complete identification which needs to measure the impulse response at any instant is required, a number of correlation channels with different values of delay are used in parallel.

There are disadvantages in this correlator, because it requires an ideal multiplier and an ideal delay filter. These are both difficult to achieve. Another disadvantage is that the long smoothing time is required to reduce the random components in the output.

In order to alleviate such difficulties, a discrete-interval binary noise is used as the test signal because of the simplicity with which the functions of generation, multiplication and delay can be accomplished. Such an excitation function has only two possible values (say +X and -X) as shown in Fig. (2.14).

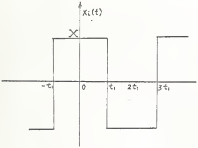

Fig. (2.14) Discrete-interval binary noise

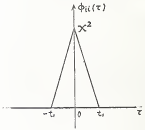

Fig. (2.15) Autocorrelation of $x_1(t)$

For Fig. (2.14), if the minimum interval width is t_1 seconds, the autocorrelation function for this type of binary noise is given by [4]

$$\phi_{ii}(\tau) = X^2[1 - \frac{|\tau|}{t_1}] \quad ; \quad -t_1 < \tau < t_1 \tag{2.89}$$

$$= 0 \quad ; \quad |\tau| > t_1$$

This autocorrelation function is shown in Fig. (2.15). It is clear that it can be made as narrow as desired by choosing t_1 sufficiently small.

By using the binary noise as process input as mentioned above, the mechanization of the cross-correlator can be simplified, but the reduction of smoothing time must be done in another way, i.e., a representative sample Nt_1 seconds long from the discrete-interval binary noise to form a periodic noise signal is chosen. Such periodic excitation has an autocorrelation function $\phi_{ii}(\tau)$ which is periodic and produces a multiplier output $x_0(t)$ over one period that is the same as the average over all time, and the smoothing time need be no longer than Nt_1. The autocorrelation function of periodic discrete-interval binary noise is shown in Fig. (2.16).

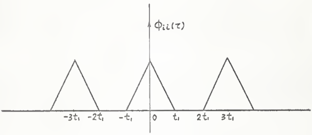

Fig. (2.16) Autocorrelation of periodic discrete-interval binary noise

(3) Evaluation of the Transfer Function from the State Equations

Lendaris [6] described an identification technique which used the state equations.

Suppose that a linear system is described by the following differential equation,

$$\frac{d^n c}{dt^n} - a_{n-1}\frac{d^{n-1}c}{dt^{n-1}} - a_{n-2}\frac{d^{n-2}c}{dt^{n-2}} - \cdots - a_1\frac{dc}{dt} - a_0 = b_m\frac{d^m r}{dt^m}$$

$$+ b_{m-1}\frac{d^{m-1}r}{dt^{m-1}} + \cdots + b_1\frac{dr}{dt} + b_0 \qquad (2.90)$$

For a linear system, a transfer function as well can be used to describe the system. The transfer function of Eq. (2.90) is

$$G(s) = \frac{c(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n - a_{n-1}s^{n-1} - \cdots - a_1 s - a_0} \qquad (2.91)$$

In general, $G(s)$ is a rational function, and thus can be expressed as a ratio of polynomials. Let $G(s)=N(s)/P(s)$, where

$$N(s) = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0$$

$$P(s) = s^n - a_{n-1} s^{n-1} - \ldots - a_1 s - a_0$$

In order to identify the system, the coefficients of Eq. (2.91) must be determined by means of the data which are obtained from the measurements of the system response when some signal injected as the input. Usually, the measurement of data is performed in time domain, so that rather than using s-plane characterization described by Eq. (2.91), a set of state equations can be used to characterize the system in the time domain.

In general, the system may be represented by the state equations:

$$\dot{\underline{x}} = \underline{A}\underline{x}(t) + \underline{B}\underline{Y}(t) \qquad (2.92)$$

$$\underline{y}(t) = \underline{c}\underline{x}(t) + \underline{D}\underline{Y}(t)$$

where

$\underline{A}$ is the essential matrix of the system, as the structure of this matrix decides the nature of the state transition matrix.

$\underline{B}$ is a coupling matrix; the structure of this matrix determines how the input is coupled to the various state variables.

$\underline{r}(t)$ is the input vector.

$\underline{y}(t)$ is the output vector.

$\underline{C}$ is also a coupling matrix, coupling the state variables to the output.

$\underline{D}$ is again a coupling matrix, as it directly couples the input vector to the output vector.

With this concept in mind, the system described by Eq. (2.90) will be considered. The state vector can be chosen as

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} = \begin{bmatrix} c(t) \\ \dot{c}(t) \\ \cdot \\ \cdot \\ \cdot \\ c^{(n-1)}(t) \end{bmatrix} \tag{2.93}$$

and define an input vector

$$\underline{\gamma}(t) = \begin{bmatrix} \gamma(t) \\ \dot{\gamma}(t) \\ \cdot \\ \cdot \\ \cdot \\ \gamma^{m}(t) \end{bmatrix} \tag{2.94}$$

Then the system can be characterized by

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}\ (t) + \underline{B}\ \underline{\gamma}\ (t)$$

where

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & & & & 0 \\ 0 & 0 & 0 & & & & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & 1 \\ a_0 & a_1 & a_2 & . & . & . & a_{n-1} \end{bmatrix}$$

and

$$\underline{B} = \begin{bmatrix} 0 & . & . & . & . & 0 \\ 0 & & & & & 0 \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ b_0 & b_1 & . & . & . & b_m \end{bmatrix}$$

As the unit step function is used as an input signal in this technique, how the input discontinuity affects the system output will be considered.

Let the input $r(t)=u(t)$, the unit function. Then

$$\underline{v}(t) = \begin{bmatrix} u(t) \\ \delta(t) \\ \dot{\delta}(t) \\ . \\ . \\ . \\ \delta^{(m-1)}(t) \end{bmatrix}$$

where $\delta(t)$ is the Dirac delta function, and from Eq. (2.90), one has

$$c^{(n)}(t) = \sum_{K=0}^{n-1} a_k c^{(k)}(t) + \sum_{K=0}^{m} b_k u^{(k)}(t) \tag{2.96}$$

Now if all $u^{(k)}(t)$ terms in Eq. (2.96) are separated and express what remains as a power series in t, then make a succession of integrations, one obtains

$$c^{(n)}(t) = \sum_{k=0}^{m} d_k u^{(k)}(t) + \sum_{k=1}^{\infty} g_n t^k$$

$$c^{(n-1)}(t) = \sum_{k=0}^{m-1} d_{k+1} u^{(k)}(t) + d_0 t + \sum_{k=1}^{\infty} \frac{g_k}{(k+1)} t^{k+1}$$

$$c^{(n-2)}(t) = \sum_{k=0}^{m-2} d_{k+2} u^{(k)}(t) + d_1 t + d_0 \frac{t^2}{2!} + \sum_{k=1}^{\infty} \frac{g_k}{(k+1)(k+2)} t^{k+2}$$

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$$c^{(n-m+1)}(t) = \sum_{k=0}^{1} d_{k+m-1} u^{(k)}(t) + d_{m-2} t + d_{m-3} \frac{t^2}{2!} + \dots$$

$$+ d_0 \frac{t^{m-1}}{(m-1)!} + \sum_{k=1}^{\infty} \frac{g_k}{(k+1)(k+2)\dots(k+m-1)} t^{k+m-1}$$

$$c^{(n-m)}(t) = d_m u^{(0)}(t) + d_{m-1}t + d_{m-2}\frac{t^2}{2!} + \ldots + d_0 \frac{t^m}{m!}$$

$$+ \sum_{k=1}^{\infty} \frac{g_k}{(k+1)(k+2)\ldots(k+m)} t^{k+m}$$

$$c^{(n-m-1)}(t) = d_m t + d_{m-1}\frac{t^2}{2!} + d_{m-2}\frac{t^3}{3!} + \ldots + d_0 \frac{t^{m+1}}{(m+1)!}$$

$$+ \sum_{k=1}^{\infty} \frac{g_k}{(k+1)(k+2)\ldots(k+m+1)} t^{k+m+1}$$

(From here down, each term of all derivatives has a factor of t or its powers) (2.97)

Substituting Eq. (2.97) into Eq. (2.96) and considering only those terms contributing to the discontinuity at t=0, one obtains

$$\sum_{k=0}^{m} d_k u^{(k)}(0) = \sum_{k=0}^{n-1} a_k c^{(k)}(0) + \sum_{k=0}^{m} b_k u^{(k)}(0)$$

Thus,

$$d_0 u^{(0)} + d_1 u^{(1)}(0) + d_2 u^{(2)}(0) + \ldots + d_m u^{(m)}(0) = a_0 c^{(0)}(0)$$

$$+ a_1 c^{(1)}(0) + a_2 c^{(2)}(0) + \ldots + a_{n-1} c^{(n-1)}(0) + b_0 u^{(0)}(0)$$

$$+ b_1 u^{(1)}(0) + b_2 u^{(2)}(0) + \ldots + b_m u^{(m)}(0)$$

$$= a_{n-m}c^{(n-m)}(0) + a_{n-m+1}c^{(n-m+1)}(0) + \ldots + a_{n-1}c^{(n-1)}(0)$$

$$+ b_0u^{(0)}(0) + b_1u^{(1)}(0) + b_2u^{(2)}(0) + \ldots + b_mu^{(m)}(0)$$

$$= a_{n-m}d_mu^{(0)}(0) + a_{n-m+1}[d_{m-1}u^{(0)}(0) + d_mu^{(1)}(0)] + \ldots$$

$$+ a_{n-1}[d_1u^{(0)}(0) + d_2u^{(1)}(0) + \ldots + d_mu^{(m-1)}(0)]$$

$$+ b_0u^{(0)}(0) + b_1u^{(1)}(0) + b_2u^{(2)}(0) + \ldots + b_mu^{(m)}(0)$$

$$= (a_{n-m}d_m + a_{n-m+1}d_{m-1} + \ldots + a_{n-1}d_1 + b_0)u^{(0)}(0)$$

$$+ (a_{n-m+1}d_m + \ldots + a_{n-1}d_2 + b_1)u^{(1)}(0) + \ldots + b_mu^{(m)}(0) \tag{2.98}$$

Comparing both sides of Eq. (2.98), it is found that

$$d_0 - a_{n-m}d_m - a_{n-m+1}d_{m-1} - \ldots - a_{n-1}d_1 = b_0$$

$$d_1 - a_{n-m+1}d_m - \ldots - a_{n-1}d_2 = b_1 \tag{2.99}$$

$$\cdot \quad \cdot \quad \cdot$$

$$d_{m-1} - a_{n-1}d_m = b_{m-1}$$

$$d_m = b_m$$

If the a's and b's are known, the d's can be obtained from Eq. (2.99).

Now suppose a unit step function is applied at t=0, with $\underline{x}(0_-)=0$. Then from Eq. (2.97) and the fact that higher-order derivatives of u(t) are zero for time $t \geq 0_+$, $x(0_+)$ yields

$$\underline{x}(0_+) = \begin{bmatrix} c(0_+) \\ \dot{c}(0_+) \\ \cdot \\ \cdot \\ \cdot \\ c^{(n-m-1)}(0_+) \\ c^{(n-m)}(0_+) \\ \cdot \\ \cdot \\ \cdot \\ c^{(n-1)}(0_+) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ d_m \\ d_{m-1} \\ \cdot \\ \cdot \\ d_1 \end{bmatrix} = \underline{d} \tag{2.100}$$

$\underline{x}(0_+)=\underline{d}$ will be called as unit discontinuity vector.

It has been shown that the state vector for $t=0_+$ is given by Eq. (2.100). The state vector for $t>0_+$ will be investigated.

Let a step function $r(t)=r_o \cdot u(t)$ be applied at t=0. Since $\dot{r}(t)=\ddot{r}(t)=\ldots=r^{(m)}(t)=0$ for $t>0_+$, one obtains

$$\underline{v}(t) = \begin{bmatrix} \gamma_o \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{B}\ \underline{v}(t) = \gamma_o \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_0 \end{bmatrix}$$

Let

$$\underline{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ b_0 \end{bmatrix}$$

Then for $t > 0_+$, the state equation can be written as

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \gamma_0 \underline{b} \tag{2.101}$$

The solution of Eq. (2.101) is given as

$$\underline{x}(t) = e^{\underline{A}t}\underline{x}(0_+) + \gamma_0 \int_0^t e^{\underline{A}(t-\tau)}.\underline{b}d\tau \tag{2.102}$$

Looking back at Eq. (2.93), it is clear that the output $c(t)$ can be determined by Eq. (2.102) at any time for $t > 0_+$.

In order to identify the system, some measurements of the output must be made. To simplify the situation, these measurements may as well be taken at $t=kT$, where $k=0,1,2,\ldots$ and T is the sampling period. Therefore, for the step input function $r=r_0 u(t)$, from Eq. (2.102), one has, for $t=T_-$,

$$\underline{x}(T_-) = e^{\underline{A}T}\underline{x}(0_+) + \gamma_0 \int_0^T e^{\underline{A}(T-\tau)}.\underline{b}d\tau \tag{2.103}$$

Let $\underline{\phi} = e^{\underline{A}T}$ and $\underline{g} = \int_0^T e^{\underline{A}(T-\tau)}\underline{b}d\tau$ and from Eq. (2.100), in this case, $\underline{x}(0_+)=r_0\underline{d}$ with $\underline{x}(0_-)=0$, Eq. (2.103) can be rewritten as

$$\underline{x}(T_-) = r_0\underline{\phi}\,\underline{d} + \gamma_0\underline{g} \tag{2.104}$$

Let

$$\underline{x}_k = \underline{x}(kT_+) \tag{2.105}$$

and with the idea in mind that

$$\underline{x}_k = \underline{x}(kT_-) + \text{discontinuity term at time } kT.$$

It follows that

$$\underline{x}_0 = \underline{x}(0_-) + a_0\underline{d}$$

$$\underline{x}_1 = \underline{x}(T_-) = \underline{\phi}\,\underline{x}_0 + \gamma_0\underline{g}$$

$$\underline{x}_2 = \underline{x}(2T_-) = \underline{\phi}\,\underline{x}_1 + \gamma_0\underline{g}$$

and, in general

$$\underline{x}_{k+1} = \underline{\phi}\,\underline{x}_k + \gamma_0\underline{g} = \underline{\phi}^{k+1}\underline{x}_0 + \gamma_0(\underline{\phi}^k + \underline{\phi}^{k-1} + \ldots + \underline{I})\underline{g} \tag{2.106}$$

$$k = 0,\ 1,\ 2,\ \ldots$$

where a_0 is due to the non-zero initial condition at t=0.

The Cayley-Hamilton theorem states that every matrix satisfies its own characteristic equation, For the system which is concerned, it implies that $P(\underline{A})=0$. Thus,

$$\underline{A}^n = a_{n-1}\underline{A}^{n-1} + a_{n-2}\underline{A}^{n-2} + \ldots + a_1\underline{A} + a_0\underline{I} \tag{2.107}$$

Suppose that matrix $\underline{A}$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$, then the transition matrix is

$$\underline{\Phi} = e^{\underline{A}T} = \underline{M}e^{\underline{\Lambda}T}\underline{M}^{-1}$$

where $\underline{M}$ is modal matrix and $\underline{\Lambda}$ is diagonal matrix.

The eigenvalues of $\underline{\Phi} = e^{\underline{A}T}$ can be derived as follows

$$|Z\underline{I} - e^{\underline{A}T}| = |Z\underline{M}\,\underline{I}\,\underline{M}^{-1} - \underline{M}e^{\underline{\Lambda}T}\underline{M}^{-1}|$$

$$= |\underline{M}(Z\underline{I} - e^{\underline{\Lambda}T})\underline{M}^{-1}|$$

$$= |\underline{M}|\cdot|Z\underline{I} - e^{\underline{\Lambda}T}|\cdot|\underline{M}^{-1}|$$

Since $|\underline{M}| \neq 0$, one obtains

$$|Z\underline{I} - e^{\underline{\Lambda}T}| = \begin{vmatrix} Z-e^{\lambda_1 T} & 0 & 0 & \ldots & 0 \\ 0 & Z-e^{\lambda_2 T} & & 0 & 0 \\ \vdots & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & Z-e^{\lambda_n T} \end{vmatrix} = 0$$

Thus, the eigenvalues of $\underline{\Phi}$ are

$$Z_1 = e^{\lambda_1 T} \quad , \quad Z_2 = e^{\lambda_2 T} \quad , \quad \ldots \quad Z_n = e^{\lambda_n T} \qquad (2.108)$$

The characteristic equation for $\underline{\Phi}$ is

$$(Z-Z_1)(Z-Z_2)\ldots(Z-Z_n) = Z^n - \xi_{n-1}Z^{n-1} - \ldots - \xi_1 Z - \xi_0 \qquad (2.109)$$

Again, the Cayley-Hamilton theorem implies

$$\underline{\phi}^n = \xi_{n-1}\underline{\phi}^{n-1} + \ldots + \xi_1\underline{\phi} + \xi_0\underline{I} \quad (2.109)$$

The essential of this technique lies in the fact that if some experimental procedures can be used to determine $\{\xi_i\}$ in Eq. (2.109), then, in turn, $\{\lambda_i\}$ can be determined from Eq. (2.108), so that P(s) is identified. These procedures can be accomplished in the following manner.

Because the first component of $\underline{x}(t)$ is the system response which may be readily measured at any instant, an "Observation" vector $\underline{0}$ is introduced as follows

$$\underline{0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (2.110)$$

Then, the magnitude of the output at any instant t=kT can be expressed as the inner product of $\underline{x}_k$ and $\underline{0}$.

Again, a system subjected to a step input $r(t) = r_0 \cdot u(t)$, with $x(0\_) \neq 0$, will be considered. Recalling Eq. (2.106), it follows that

$$\underline{x}_k = \underline{\phi}^k\underline{x}_0 + \gamma_0(\underline{\phi}^{k-1} + \underline{\phi}^{k-2} + \ldots + \underline{I})\underline{g}$$

Let

$$\underline{w}_k = \underline{x}_{k+1} - \underline{x}_k = (\underline{\Phi}^{K+1} - \underline{\Phi}^k)\underline{x}_0 + \gamma_0\underline{\Phi}^k\underline{g} \tag{2.111}$$

The observed magnitude of the response at time t = kT can be written as

$$\psi_k = \langle \underline{x}_k, \underline{0} \rangle \tag{2.112}$$

Let

$$\begin{aligned} \theta_k = \psi_{k+1} - \psi_k &= \langle \underline{x}_{k+1}, \underline{0} \rangle - \langle \underline{x}_k, \underline{0} \rangle \\ &= \langle \underline{x}_{k+1} - \underline{x}_k, \underline{0} \rangle \\ &= \langle \underline{w}_k, \underline{0} \rangle \\ &= \langle (\underline{\Phi}^{k+1} - \underline{\Phi}^k)\underline{x}_0 + \gamma_0\underline{\Phi}^k\underline{g}, \underline{0} \rangle \end{aligned} \tag{2.113}$$

From Eq. (2.109), it follows that

$$\underline{\Phi}^{n+\ell} = \xi_{n-1}\underline{\Phi}^{n+\ell-1} + \ldots + \xi_1\underline{\Phi}^{\ell+1} + \xi_0\underline{\Phi} \tag{2.114}$$

Then, from Eq. (2.113), one obtains

$$\begin{aligned} \theta_n &= \langle \underline{\Phi}^{n+1}\underline{x}_0 - \underline{\Phi}^n\underline{x}_0 + \gamma_0\underline{\Phi}^n\underline{g}, \underline{0} \rangle \\ &= \langle \underline{\Phi}^{n+1}\underline{x}_0, \underline{0} \rangle - \langle \underline{\Phi}^n\underline{x}_0 \underline{0} \rangle + \langle \gamma_0\underline{\Phi}^n\underline{g}, \underline{0} \rangle \\ &= \xi_0 \langle \underline{\Phi}\,\underline{x}_0, \underline{0} \rangle + \xi_1 \langle \underline{\Phi}^2\underline{x}_0, \underline{0} \rangle + \ldots + \xi_{n-1} \langle \underline{\Phi}^n\underline{x}_0, \underline{0} \rangle \\ &\quad - \xi_0 \langle \underline{I}\,\underline{x}_0, \underline{0} \rangle - \xi_1 \langle \underline{\Phi}\,\underline{x}_0, \underline{0} \rangle - \ldots - \xi_{n-1} \langle \underline{\Phi}^{n-1}\underline{x}_0, \underline{0} \rangle \\ &\quad + \xi_0\gamma_0 \langle \underline{I}\,\underline{g}, \underline{0} \rangle + \xi_1\gamma_0 \langle \underline{\Phi}\,\underline{g}, \underline{0} \rangle + \ldots + \xi_{n-1}\gamma_0 \langle \underline{\Phi}^{n-1}\underline{g}, \underline{0} \rangle \end{aligned}$$

$$= \xi_0 <(\underline{\Phi}-\underline{I})\underline{x}_0 + \gamma_0 \underline{I}\,\underline{g}, \underline{0}> + \xi_1 <(\underline{\Phi}^2-\underline{\Phi})\underline{x}_0 + \lambda_0 \underline{\Phi}\,\underline{g}, \underline{0}> + \dots$$

$$+ \xi_{n-1} <(\underline{\Phi}^n - \underline{\Phi}^{n-1})\underline{x}_0 + \gamma_0 \underline{\Phi}^{n-1}\underline{g}, \underline{0}> \tag{2.115}$$

$$= \xi_0 \theta_0 + \xi_1 \theta_1 + \dots + \xi_{n-1}\theta_{n-1}$$

Similarly,

$$\begin{aligned} \theta_{n+1} &= \xi_0 \theta_1 + \xi_1 \theta_2 + \dots + \xi_{n-1}\theta_n \\ &\vdots \\ \theta_{2n-1} &= \xi_0 \theta_{n-1} + \xi_1 \theta_n + \dots + \xi_{n-1}\theta_{2n-2} \end{aligned} \tag{2.116}$$

From Eqs. (2.115) and (2.116), the $\{\xi_i\}$ can be expressed in matrix form as

$$\begin{bmatrix} \xi_0 \\ \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_{n-1} \end{bmatrix} = \begin{bmatrix} \theta_0 & \theta_1 & \cdot & \cdot & \cdot & \theta_{n-1} \\ \theta_1 & \theta_2 & \cdot & \cdot & \cdot & \theta_n \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \theta_{n-1} & \theta_n & \cdot & \cdot & \cdot & \theta_{2n-2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \theta_n \\ \theta_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \theta_{2n-1} \end{bmatrix} \tag{2.117}$$

Since $\{\theta_i\}$ can be measured from the system output, the $\{\xi_i\}$ can be determined from Eq. (2.117). The solution is unique if and only if the simultaneous equations are linearly independent.

Once the $\{\xi_i\}$ has been determined, the $\{\lambda_i\}$ can be determined as we described before, and the identification of P(s) is accomplished.

After P(s) is identified, N(s) will be identified. This can be done in following manner:

Looking back at Eq. (2.99), because P(s) can be identified first, so that $\{a_i\}$ is known. To identify N(s), i.e., to determine $\{b_i\}$, $\{d_i\}$, the unit discontinuity vector, must be identified.

It has been shown that [7], [8],

$$\underline{\phi}^r = e^{r\underline{A}T} = \sum_{k=1}^{s} \sum_{\ell=0}^{m_k-1} (rT)^{\ell} Z_k{}^r \underline{Z}_{k\ell} \tag{2.118}$$

$$\underline{g} = \sum_{k=1}^{s} \sum_{\ell=0}^{m_k-1} a_{k\ell} \underline{Z}_{k\ell} \underline{b} \tag{2.119}$$

$$f(\underline{A}) = \sum_{k=1}^{s} \sum_{\ell=0}^{m_k-1} f^{(\ell)}(\lambda_k) \underline{Z}_k \tag{2.120}$$

where s is the number of the distinct eigenvalues of $\underline{A}$, and m_k is the multiplicity of the eigenvalue λ_k .

And

$$a_{k0} = \frac{(Z_k-1)}{\lambda_k} \qquad \text{where} \quad Z_k = e^{\lambda_k T} \tag{2.121}$$

$$a_{k\ell} = \Big\{ (-1)^{\ell} \ell! \frac{(Z_k-1)}{\lambda_k^{\ell+1}} + Z_k\Big(\frac{T^{\ell}}{\lambda_k} - \ell \frac{T^{\ell-1}}{\lambda_k^2} + \ell(\ell-1)\frac{T^{\ell-2}}{\lambda_k^3} + \dots$$

$$(-1)^{\ell-1}\ell! \frac{T}{\lambda_k^{\ell}}\Big) \Big\} \tag{2.122}$$

for $\ell = 1, 2, 3, \dots m_k$

$\underline{Z}_{k\ell} = \frac{(\underline{A}-\lambda_k \underline{I})}{\ell!} \underline{E}_k$ (where $\underline{E}_k$ is a projection, and in this case,

$\underline{E}_k = Q_k(\underline{A}) = \frac{m(\lambda)n_k(\lambda)}{(\lambda-\lambda_k)m_k}$, in which $m(\lambda)$ is the minimal polynomial of the matrix $\underline{A}$) (2.123)

From Eqs. (2.118), (2.119) and (2.122), and since T, λ_k and z_k are known, the a'_k s can be computed. Therefore, in order to evaluate $\underline{\phi}^{\gamma}$ and $\underline{g}$, the values of $\underline{Z}_k$ are needed. Referring to Eq. (2.120), and knowing that there are n unknowns $\underline{Z}_k$, the following n functions are chosen.

$$
\begin{aligned}
& f_1(\lambda) = (\lambda-\lambda_1), \qquad f_2(\lambda) = (\lambda-\lambda_1)^2 \\
& \qquad \ldots, f_{m_1}(\lambda) = (\lambda-\lambda_1)^{m1} \\
& f_{m_1+1}(\lambda) = (\lambda-\lambda_2), \qquad f_{m_1+2}(\lambda) = (\lambda-\lambda_2)^2 \\
& \qquad \ldots, f_{m_1+m2}(\lambda) = (\lambda-\lambda_2)^{m2} \\
& \quad \vdots \\
& f_{n-m_s+1}(\lambda) = (\lambda-\lambda_s)_1 \qquad f_{n-m_s+2}(\lambda) = (\lambda-\lambda_s)^2 \\
& \qquad \ldots, f_n(\lambda) = (\lambda-\lambda_s)^{m_s}
\end{aligned}
\tag{2.124}
$$

Substitution of Eq. (2.124) into Eq. (2.120), one obtains

$$
\begin{aligned}
& (\underline{A}-\lambda_1\underline{I}) = \underline{Z}_{11} + (\lambda_2-\lambda_1)\underline{Z}_{20} + \underline{Z}_{21} + \ldots + (\lambda_s-\lambda_1)\underline{Z}_{s0} + \underline{Z}_{s1} \\
& (A-\lambda_1\underline{I})^2 = Z\underline{Z}_{12} + (\lambda-\lambda_1)^2\underline{Z}_{20} + 2(\lambda-\lambda_1)\underline{Z}_{21} + 2\underline{Z}_{22} + \ldots + 2\underline{Z}_{s2} \\
& \quad \vdots \\
& (\underline{A}-\lambda_s\underline{I})^{ms} = (\lambda_s-\lambda_1)^{ms}\underline{Z}_{10} + m_s(\lambda_s-\lambda_1)^{ms-1}\underline{Z}_{11} + \ldots + (m_s!)\underline{Z}_{sm_s}
\end{aligned}
\tag{2.125}
$$

These n simultaneous equations can be used to solve for the n unknowns $\underline{Z}_{k\ell}$, and in turn, $\underline{\phi}^{\gamma}$ and $\underline{g}$ can be determined.

Now $\underline{d}$ will be identified. Again, consider the system response with the input $r(t)=r_0 \cdot u(t)$ and $\underline{x}(0\_)\neq 0$.

Let

$$\underline{\Phi}_k = (\underline{\phi}^k + \underline{I}^{k-1} + \ldots + \underline{I}) \tag{2.126}$$

Then, from Eqs. (2.106) and (2.126), it follows that

$$\underline{x}_k = \underline{\phi}^k [\underline{x}(0\_) + a_0 d] + \gamma_0(\underline{\phi}^{k-1} + \ldots + \underline{I})\underline{g}$$

$$= \underline{\phi}^k \underline{x}(0\_) + a_0 \underline{\phi}^k \underline{d} + \gamma_0 \underline{\Phi}_{k-1} \underline{g} \qquad k = 1, 2, \ldots \tag{2.127}$$

Now let

$$\underline{\phi}^k = \phi_{ij}{}^{(k)} \qquad i, j = 1, 2, \ldots n$$

$$\underline{\Phi}_k = \phi_{(k)ij} \qquad i, j = 1, 2, \ldots n$$

$$\underline{g} = \begin{bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{bmatrix} \quad \text{and } \underline{x}(0\_) = \begin{bmatrix} x_1(0\_) \\ x_2(0\_) \\ \cdot \\ \cdot \\ \cdot \\ x_n(0\_) \end{bmatrix}$$

Looking back to Eqs. (2.100), (2.112) and (2.127), one has

$$\psi_1 = \langle \underline{\Phi}^{(1)} \underline{x}(0\_) + a_0 \underline{\Phi}^{(1)} \underline{d} + \gamma_0 \underline{g},\ \underline{0} \rangle$$

$$= \phi_{11}^{(1)} x_1(0\_) + \phi_{1]}^{(1)} x_2(0\_) + \ldots \phi_{1n}^{(1)} x_n(0\_) + a_0 \phi_{1,n-m+1}^{(1)}$$

$$d_m + a_0 \phi_{1,n-m+2}^{(1)} d_{m-1} + \ldots + a_0 \phi_{1n}^{(1)} d_1 + \gamma_0 g$$

$$\psi_2 = \langle \underline{\Phi}^2 \underline{x}(0\_) + a_0 \underline{\Phi}^2 \underline{d} + \gamma_0 \underline{\Phi}_1 \underline{g},\ \underline{0} \rangle$$

$$= \phi_{11}^{(2)} x_1(0\_) + \ldots + \phi_{1n}^{(2)} x_n(0\_) + a_0 \phi_{1,n-m+1}^{(2)} d_m + a_0$$

$$\phi_{1,n-m+2}^{(2)} d_{m-1} + \ldots + a_0 \phi_{1n}^{(2)} d_1 + \gamma_0 \phi_{(1)11} + \gamma_0 \phi_{(1)12} g_2$$

$$+ \ldots + \gamma_0 \phi_{(1)1n} g_n$$

$$\vdots$$

$$\psi_{n+m} = \phi_{11}^{(n+m)} x_1(0\_) + \ldots + \phi_{1n}^{(n+m)} x_n(0\_) + a_0 \phi_{1,n-m+1}^{(n+m)} d_m + a_0 \phi_{1,n-m+2}^{(n+m)}$$

$$d_{m-1} + \ldots + a_0 \phi_{1n}^{(n+m)} d_1 + \gamma_0 \phi_{(n+m-1)12} g_1 + \gamma_0 \phi_{(n+m-1)12} g_2 + \ldots + \gamma_0 \phi_{(n+m-1)1n} g_n$$

Since $g_1, g_2, \ldots g_n$, all the $\phi_{ij}^{(\gamma)}$ and $\phi_{(\gamma)ij}$ can be computed, and $\psi_1, \psi_2, \ldots \psi_{n+m}$ can be measured from the system response, the values of

$d_m, d_{m-1}, \dots d_1$

and

$x_1(0\_), x_2(0\_), \dots x_n(0\_)$

can be determined by solving (n+m) simultaneous equations (2.128).

Thus, d_m, $d_{m+}..d_1$ have been determined. From the simultaneous equations (2.99), the b_i's can be determined. This completes the identification of N(s).

In summary, by taking some measurements of the system response, P(s) can be identified first and then N(S) is identified. The step function is used as system input in this technique.

CONCLUSION

The requirement of process identification is of central importance in the design of adaptive control systems. Several techniques which have been described in the literature for obtaining a process identification have been presented in this report.

The parameter-evaluation approach to the identification problem is somewhat theoretical and often not a practical one, since it is difficult to accurately evaluate all the parameters in the system, particularly when the system is in operation. The inherent nonlinearity of physical processes also limits the usefulness of this technique. But the advantage of the technique is perhaps that it possesses a detailed picture of the physics of the operation of the process, and it is useful when one desires to estimate directly the effects of varying a specific parameter.

The dynamic transmittance identification approach which focuses its attention on determining a specified dynamic input-output relation is achieved in terms of a set of coefficients of a preselected differential equation or in terms of a time-domain representation of the dynamic response to some specified test signal such as white noise and discrete-interval binary noise. The response sought usually is the impulse or unit step response.

The problem of automatically determining the technique of identification which is most suitable for the task at hand is still one of the most interesting aspects of the design of adaptive control system.

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A STUDY OF PLANT IDENTIFICATION TECHNIQUES

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The main purpose of this report is to study the identification problem as applied to control systems.

Model identification is presented in the first chapter. In this technique, a set of differential equations is established to constitute a mathematical model of the system. From this model, all parameters of the process may be evaluated. The block diagram and circuit diagram are derived for this purpose.

With the complexity that is inherent in most control systems, it is more practical to attempt to evaluate the transfer characteristics from the specific inputs and outputs of primary interest. For this purpose, the impulse or unit step response is the desired information. The solution of the convolution integral and the application of the basic cross-correlator are presented.

Finally, since information concerning the system is usually in the time domain, a method that utilizing the state equations to identify the transfer function G(s) of the system is presented.