CLIQUED HOLES AND OTHER GRAPHIC STRUCTURES

FOR THE NODE PACKING POLYTOPE

by

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ABSTRACT

Graph Theory is a widely studied topic. A graph is defined by two important features: nodes and edges. Nodes can represent people, cities, variables, resources, products, while the edges represent a relationship between two nodes. Using graphs to solve problems has played a major role in a diverse set of industries for many years.

Integer Programs (IPs) are mathematical models used to optimize a problem. Often this involves maximizing the utilization of resources or minimizing waste. IPs are most notably used when resources must be of integer value, or cannot be split. IPs have been utilized by many companies for resource distribution, scheduling, and conflict management.

The node packing or independent set problem is a common combinatorial optimization problem. The objective is to select the maximum nodes in a graph such that no two nodes are adjacent. Node packing has been used in a wide variety of problems, which include routing of vehicles and scheduling machines.

This thesis introduces several new graph structures, cliqued hole, odd bipartite hole, and odd k-partite hole, and their corresponding valid inequalities for the node packing polyhedron. These valid inequalities are shown to be new valid inequalities and conditions are provided for when they are facet defining, which are known to be the strongest class of valid inequalities. These new valid inequalities can be used by practitioners to help solve node packing instances and integer programs.

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Chapter 1

Introduction

Graphs have played a major role in a diverse set of industries for centuries. Leonard Euler's paper [15], *Seven Bridges of Königsberg*, is commonly understood to be the first example of utilizing a graph to solve a problem. Euler, in his pursuits, established the basic principles of graph theory. Graphs naturally simplify abstract ideas into a physical representation of nodes and edges. This enables a streamlined system for arriving at quality solutions.

When modeling a problem as a graph, the vertices of the graph can represent the problem's variables. Examples include people, machines, or resources. An edge exists between any two nodes when there is a conflict or relationship between them.

Another possibility is for the edges to represent variables and the nodes to represent locations. Commonly, this kind of graph is used for transportation problems, where the edge represents a cost of shipping between two locations, each being represented by a node.

One prime example of a graph being implemented by industry is a study performed by Proctor & Gamble. Using a combination of transportation problem modeling and node packing, P & G redesigned and improved their operations. Through graph theory and network redesign, P & G reduced the number of North American plants by 20% and saved $200 million per annum [43].

Other successful examples of applications in graph theory have shown not only their flexibility, but also their effectiveness. Graphs are used for transportation optimization in deliveries [33, 36, 65, 80, 81, 82, 86], scheduling problems [26, 47, 48, 66], resource dedication in manufacturing [14, 68], forecasting [53, 67, 74], even crime prevention [4, 73].

1.1   Node Packing Problem

The graph problem that this thesis focuses on is the the Node Packing problem. Node packing, also referred to as vertex packing or the independent set problem, is a common combinatorial optimization technique in which the objective is to select the maximum nodes in a graph such that no two are adjacent. Node packing is an extremely popular technique in optimization used in a wide variety of problems.

An interesting application of node packing involves determining where to place probes on a testing fixture for printed circuit boards. These probes are the nodes, and the edges exist wherever two nodes cannot be placed in order to prevent a short circuit. By utilizing a node packing algorithm, it was found that the probe placement was improved by

5%, and therefore improved this company's standard. Also, by utilizing node packing, software could generate a near optimal probe placement much more quickly than an experienced operator [1].

Since node packing inherently aligns with many real world problems, it has been implemented in a diverse set of areas. Whether it be routing trains through a train station in Denmark [87], scheduling machines [75], or sensor coverage [52], node packing has proven its usefulness consistently.

A feasible solution to the node packing problem is simple to achieve, but the optimal solution can be extremely difficult and computationally exhausting. This results from the fact that these problems are $\mathcal{NP}$-Complete [49]. Often, IPs are utilized in order to help solve the node packing problem.

1.2   Integer Programs

Graphs are also commonly utilized when attempting to solve Integer Programs (IPs). IPs are mathematical models that are used when resources must be of integer value, or cannot be split. This gives the user of such a tool feasibility when operating within such constraints. IPs have been utilized by many companies for resource distribution [14, 68], scheduling [3, 31, 48, 64, 70], and conflict management [27].

IPs inherent weakness is that they are $\mathcal{NP}$-complete [49], meaning that many IPs require exponential time to solve with the current computers. Consequently, much

research has focused on improving the solution time of integer programs.

The most common IP solution method is branch and bound. Branch and bound begins by solving the linear relaxation, which is basically solving the integer program without the integer constraint. Cutting planes are another method used to solve IPs. First introduced by Gomory [34], cutting planes remove parts of the linear relaxation without removing any feasible integer solutions. The valid inequalities generated by the cutting plane method are useful only if they eliminate significant areas of the linear relaxation.

The theoretically strongest cutting planes are facet defining. One method to create facet defining cutting planes is to utilize lifting. There are various types of lifting and this thesis focuses on simultaneous lifting for the node packing problem.

1.3   Motivation

Recently, a substantial amount of research has been performed on simultaneous lifting at Kansas State University [41, 44, 45, 51, 54, 71]. Simultaneous lifting is an integer programming technique used to generate strong cutting planes. One of the most commonly studied IPs has been the node packing problem. The motivating question for this thesis was whether or not these two topics could be integrated. Hence, this research tried to discover graphic structures that allow for simultaneous lifting.

1.4 Research Contributions

In attempting to find graphic structures to allow simultaneous lifting, this research discovered two new classes of graphs that define valid inequalities. These structures are called cliqued holes and odd bipartite holes. Of these structures, the odd bipartite hole and its generalization to odd k-partite holes have implications to simultaneous lifting.

The first structure, the cliqued hole, begins with an odd hole. Each of the nodes is exploded into cliques. This structure generates a valid inequality, which is called a cliqued hole inequality. Conditions for these inequalities to be facet defining are presented along with arguments for why these inequalities are not just a natural derivation of existing inequalities, but are indeed new and useful.

The odd bipartite hole consists of two odd holes, such that every vertex in one odd hole is adjacent to every vertex in the other odd hole. This characteristic displays as a complete bipartite graph between the two odd holes. The odd bipartite inequality is a valid inequality and is generated from this structure. The details for when this inequality becomes facet defining are explored. This structure is visually very stunning, and can be expanded to become a k-partite odd hole.

These structures offer new cutting planes to enable practitioners to implement in software in order to help solve the node packing problem and even general integer programs faster. It is also evident that these inequalities are stronger than many of the commonly used inequalities, which leads to the belief that these inequalities will be extremely useful.

1.5   Outline

Chapter 2 gives the background necessary to understand this thesis. First, graph theory is discussed, along with some common graph structures. Then integer programs and polyhedral theory are explained. This is followed by cutting planes, facet defining inequalities, the node packing problem, and lifting.

Chapter 3 introduces three new structures for the node packing polytope: cliqued holes, odd bipartite holes, and odd k-partite holes. Definitions and their respective valid inequalities are thoroughly discussed. Simultaneous lifting in the node packing polyhedron is also introduced.

Chapter 4 gives a conclusion of research contributions and results. This includes the major advancements of this thesis as well as areas of interest for future research.

Chapter 2

Background Information

This chapter introduces some definitions and background information necessary to understand this thesis. Various topics provide a foundation for this research, including graph theory, integer programming, polyhedral theory, node packing, cutting planes, facet defining inequalities, and lifting.

2.1 Graph Theory

A brief review of some fundamental definitions and concepts of graph theory is presented in this section. Only a minimal amount of graph theory is covered here and [2, 15, 32] provide a more detailed perspective and additional topics.

Let $G = (V, E)$ be a graph, where V is a set of nodes and E is a set of edges $e = \{u, v\} \in E$ where $u, v \in V$. A network is a graph with weights on either the nodes or edges. A directed graph provides an ordering to the edges. Thus, $E = \{(u, v) : u, v \in V\}$.

A graph is bipartite if, and only if, there exists a partition of the nodes into V_1 and V_2 such that every $\{u, v\} \in E$ has $u \in V_1$ and $v \in V_2$.

Subgraphs have been critical in graph theory research [18, 85]. A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if and only if V' is a subset of V and E' is a subset of E. A subgraph is induced if, and only if, it contains all the same edges as G over the same vertex set V'. A subgraph is spanning if $V' = V$.

A path is a set of nodes $(v_1, v_2, ..., v_p)$ such that $\{v_i, v_{i+1}\} \in E$ for all $i = 1, ..., p-1$. A cycle is a path with $v_1 = v_p$ and all other nodes are unique. A graph is acyclic if, and only if, it contains no cycles. An acyclic graph is called a forest.

A clique on p nodes, K_p, or complete graph, is a specific graph type that is of great importance to this research. A clique is a graph structure in which an edge exists between every set of two nodes. A clique is named by the number of nodes involved, therefore a K_5 is a clique with five nodes and has $\binom{5}{2} = 10$ edges.

A hole on p vertices, H_p, is a cycle $(v_1, v_2, ..., v_p)$ without any chords where a chord is an edge $\{v_i, v_j\}$ where $|i - j| \geq 2$ or an edge between v_p and any node other than v_{p-1} or v_1.

A fan on p vertices, f_p, consists of a center node and a set of nonadjacent perimeter nodes. Thus, the induced subgraph $\{v_1, ..., v_p\}$ is a fan if, and only if, the edges take the form $\{v_1, v_j\}$ for all $j = 2, ..., p$.

One useful way to solve graph problems is by modeling the problem as an integer program. Some relevant topics in integer programming are discussed next.

2.2 Integer Programs

Integer programs are often solved with the help of graph theory concepts. This nature of IPs being used to solve graphs, and vice versa, has led to a vicious cycle of each area building upon the other. Some basic definitions are sufficient and more detailed information can be found in [58].

Integer Programs contain an objective function that is either maximized or minimized. They also contain constraints in order to limit the solution space, and decision variables which must be integer. An IP takes the following form: Maximize $c^T x$, subject to $Ax \leq b, \; x \in \mathbf{Z}_+^n$, where $c \in \mathbf{R}^n, \; A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. As an example, consider the following IP.

Maximize $\;5x_1 + 3x_2,$

Subject to $3x_1 + x_2 \leq 7$

$$2x_2 \leq 5$$

$$x_1, \; x_2 \geq 0 \text{ and integer}$$

The solution to this particular IP is $x_1 = 2, \; x_2 = 1$, as it maximizes the objective function to a value of 13 while satisfying all of the constraints.

A common technique for solving IPs is to consider the linear relaxation. This is done by removing the integer constraint on the problem. Thus, the linear relaxation takes the form: Maximize $\;c^T x, \;$ subject to $Ax \leq b, \; x \in \mathbf{R}_+^n$, where $c \in \mathbf{R}^n, \; A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. As an example, the linear relaxation of the above IP is

Maximize $5x_1 + 3x_2$,

Subject to $3x_1 + x_2 \leq 7$

$$2x_2 \leq 5$$

$$x_1, \ x_2 \geq 0$$

The most common technique used to solve an integer program is called branch and bound. The algorithm begins by solving the linear relaxation with solution Z^{LR} and x^{LR}. If the solution is not integer, then two child nodes are created. Take any x_i^{LR} that isn't integer and one child node adds the inequality $x_i^{LR} \leq \lfloor x_i^{LR} \rfloor$; while the other adds the inequality $x_i^{LR} \geq \lceil x_i^{LR} \rceil$. This continues until all nodes are fathomed. A node is fathomed if it is infeasible, an integer solution or if $Z^{LR} \leq Z^{IP}$ where Z^{IP} is the best found integer solution thus far.

It is easy to see that branch and bound has an exponential running time. So researchers naturally try to restrict the space of the linear relaxation. The tool used to analyze a problem's linear relaxation is called polyhedral theory.

2.2.1 Polyhedral Theory

Many optimization methods utilize polyhedral theory. Polyhedral theory is the mathematical understanding of all feasible solutions of Linear Programs (LPs). Important definitions and background information are presented in this section.

A convex set S is a set in which every point on the line segment connecting any two

points in the set is also in the set. The set S is convex if, and only if, $\lambda s_1 + (1 - \lambda)s_2 \in$ $S \; \forall \; s_1, s_2 \in S$ and $\lambda \in [0, 1]$. A convex hull of the set S, denoted by S^{CH}, is the intersection of all convex sets containing S.

A half-space is the solution space for a single linear inequality. That is, all $x \in \mathbf{R}^n$ such that $\sum_{j=1}^{n} \alpha_j x_j \leq \beta$. A half-space is convex and a polyhedron is the intersection of a finite number of half-spaces. This leads to the conclusion that the feasible region of an LP $\{x \in \mathbf{R}^n : Ax \leq b\}$ is both convex and a polyhedron. A polytope is a polyhedron that is bounded.

Let P be the set of feasible solutions to an integer program. That is, $P = \{x \in \mathbf{Z}_+^n : Ax \leq b\}$. The set P is comprised of a countable set of points and is therefore not convex. The convex hull of P, denoted as $P^{CH} = conv(P)$, is both convex and a polyhedron. For convenience, define $N = \{1, ..., n\}$ as the set of variable indices in an IP.

To understand the importance of this thesis, it is vital to consider the linear relaxation of an integer program. The linear relaxation is the difference found between an integer program and a linear program of congruent form except for the lack of the integer constraint. Given an integer program, Maximize $c^T x$, subject to $Ax \leq b, \; x \in \mathbf{Z}_+^n$, the linear relaxation is known as IP^{LR} and is Maximize $c^T x$, subject to $Ax \leq b, \; x \in \mathbf{R}_+^n$. Define P^{LR} to be the feasible region of the linear relaxation or equivalently, $P^{LR} = \{x \in \mathbf{R}_+^n : Ax \leq b\}$. Clearly, P^{LR} is convex and a polyhedron.

The dimension of a polyhedron can be found by the number of linearly independent vectors contained within the polyhedron. Since P^{CH} is derived from a number of points,

affine independence should be used to determine its dimension.

A set of points $x_1, x_2, ..., x_d \in \mathbf{R}_+^n$ are affinely independent if, and only if, $\sum_{j=1}^{d} \lambda_j x_j = 0$ and $\sum_{j=1}^{d} \lambda_j = 0$ is solved uniquely by $\lambda_j = 0 \; \forall j = 1, 2, ..., d$. The $dim(P^{CH})$ is the maximum number of affinely independent points in P^{CH} minus one.

2.2.2 Cutting Planes and Facet Defining Inequalities

Now that a polyhedron and linear relaxation have been discussed, it is helpful to define cutting planes, valid inequalities, and facets, as these are the focus of this research. This section introduces these topics and describes their importance to integer programming research.

An inequality (α, β) takes the form $\alpha^T x \leq \beta$ and is a valid inequality of P^{CH} if, and only if, it is satisfied for all points in P. Formally, the inequality (α, β) is a valid inequality if, and only if, P lies in the half-space $\{x \in \mathbf{R}^n : \alpha^T x \leq \beta\}$. Equivalently, $\sum_{j=1}^{n} \alpha_j x_j \leq \beta$ is a valid inequality for P^{CH} if, and only if, $\sum_{j=1}^{n} \alpha_j x_j' \leq \beta$ is satisfied for every $x' \in P$.

The goal of a cutting plane is to eliminate an area of P^{LR} without eliminating any feasible integer points, points in P. Therefore, a cutting plane manages to remove non-integer solution space of P^{LR} while keeping all integer points; P that are in P^{LR}.

Every valid inequality induces a face of a polyhedron and the face consists of the points in the polyhedron that meet the inequality at equality. Let $\sum_{j=1}^{n} \alpha_j x_j \leq \beta$ be a valid inequality, then the corresponding face F of P^{CH} is $F = \{x \in P^{CH} : \sum_{j=1}^{n} \alpha_j x_j = $

$\beta\}$.

Facet defining inequalities are the most restrictive of all valid inequalities. Thus, the theoretically strongest valid inequalities are facet defining and can remove a large amount of the linear relaxation space and greatly reduce the time required to solve an integer program. Formally, let $\sum_{j=1}^{n} \alpha_j x_j \leq \beta$ be a valid inequality, then F is a facet if, and only if, the dimension of F is the dimension of P^{CH} minus one.

With P^{CH}, the only inequalities required to describe the polyhedron are the facet defining inequalities. Thus, if all facet defining inequalities are included to the linear relaxation, all basic feasible linear relaxation points are integer. Therefore, branch and bound can be reduced to solving a single linear relaxation since its solution is integer.

2.3 Node Packing

This thesis focuses on the node packing polyhedron by identifying cutting planes and facet defining inequalities. As previously mentioned, node packing is also referred to as vertex packing and the independent set problem. Numerous researchers have focused on the node packing polyhedron for a wide variety of research [17, 20, 37, 55, 56, 57, 59].

The input to the node packing problem is a graph $G = (V, E)$. The node packing problems seeks the largest set of vertices $V' \subseteq V$ such that for all $\{u, v\} \notin E$ for all $u, v \in V'$. In other words the solution to a node packing instance is the maximum number of nonadjacent nodes.

The integer programming model of the node packing polyhedron can be defined by letting $x_i = 1$ if $i \in V'$, and 0 if not, for all $i \in V$. The objective function is to maximize $\sum_{i \in V} x_i$. The constraints are $x_i + x_j \leq 1$ for all $\{i, j\} \in E$ and $x_i \in \{0, 1\}$ for all $i \in V$. Denote PNP as the set of feasible points to this problem, $PNP = \{x \in \{0, 1\}^n : x_i + x_j \leq 1$ for all $\{i, j\} \in E\}$. Now, define PNP^{CH} as the convex hull of PNP.

A conflict graph is a particular IP application associated with the node packing problem. Conflict graphs are a good example of how theory can translate to practice [6, 23, 30, 46, 61, 79]. They are often used by professionals in industry and academia to improve the solution time of IPs.

Formally, a conflict graph $G = (V, E)$ can be a defined by a set of vertices V and a set of edges E where $e = \{u, v\} \in E$. Every variable x_i in the IP is represented by a node $i \in V$. An edge $e = \{i, j\}$ exists if setting both x_i and x_j to 1 is infeasible. An example of a conflict graph is shown to better illustrate this concept. Consider the following integer program.

$$\text{Maximize } 5x_1 + 3x_2, +x_3 + 2x_4 + 4x_5$$

$$\text{Subject to } 3x_1 + 3x_2 + 5x_3 \leq 7$$

$$2x_2 + x_4 + 2x_5 \leq 3$$

$$x_3 + x_5 \leq 1$$

$$x_1, ..., x_5 \in \{0, 1\}.$$

The conflict graph for this integer program would contain several nodes, one for each

variable. The edges exist between two nodes if setting both variables to 1 is infeasible.

In this example, $\{1,3\} \in E$ since setting $x_1 = 1$ and $x_3 = 1$ violates the first constraint.

The edge $\{2,3\} \in E$ as setting $x_2 = 1$ and $x_3 = 1$ also violates the first constraint.

The edge $\{5,2\} \in E$ because setting $x_5 = 1$ and $x_2 = 1$ violates the second constraint.

Finally, the edge $\{3,5\} \in E$ because setting $x_3 = 1$ and $x_5 = 1$ violates the third

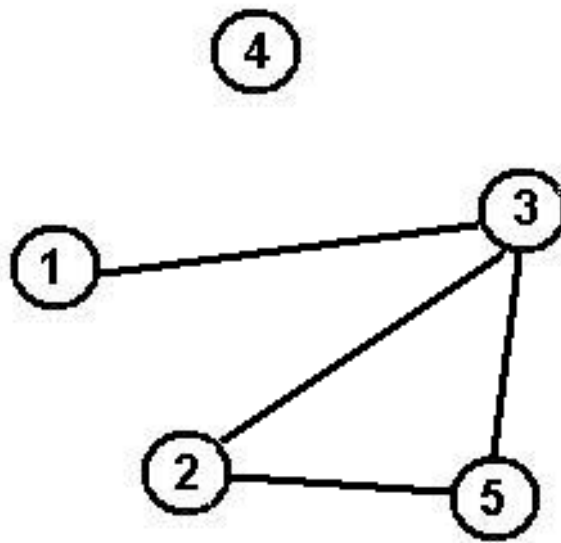constraint. The conflict graph for this example is shown in Figure 2.1.

Figure 2.1: Conflict Graph

The inequalities for this conflict graph are trivial to arrive at. If an edge exists

between two nodes x_i, x_j, then $x_i + x_j \leq 1$ is a valid inequality. The constraints for this

conflict graph are: $x_1 + x_3 \leq 1$, $x_2 + x_3 \leq 1$, $x_2 + x_5 \leq 1$, and $x_3 + x_5 \leq 1$. Observe that

these constraints are precisely the node packing constraints.

Notice that the constraints $x_2 + x_3 \leq 1$, $x_2 + x_5 \leq 1$, and $x_3 + x_5 \leq 1$ are all valid.

Now recall that a clique is a graph structure in which an edge exists between every

set of two nodes. Thus, the structure created by $\{2,3,5\}$ is a K_3. Now notice that

$x_2 + x_3 + x_5 \leq 1$ is a valid inequality to the above IP. This calss of inequalities is called

a clique inequality.

A few common subgraph structures in node packing are presented as subgraphs in

Figure 2.2. Given the graph $G = (V, E)$, denote V'' as the set of vertices in V that form

the subgraph being discussed. The basic idea is to use the graph structure to create a
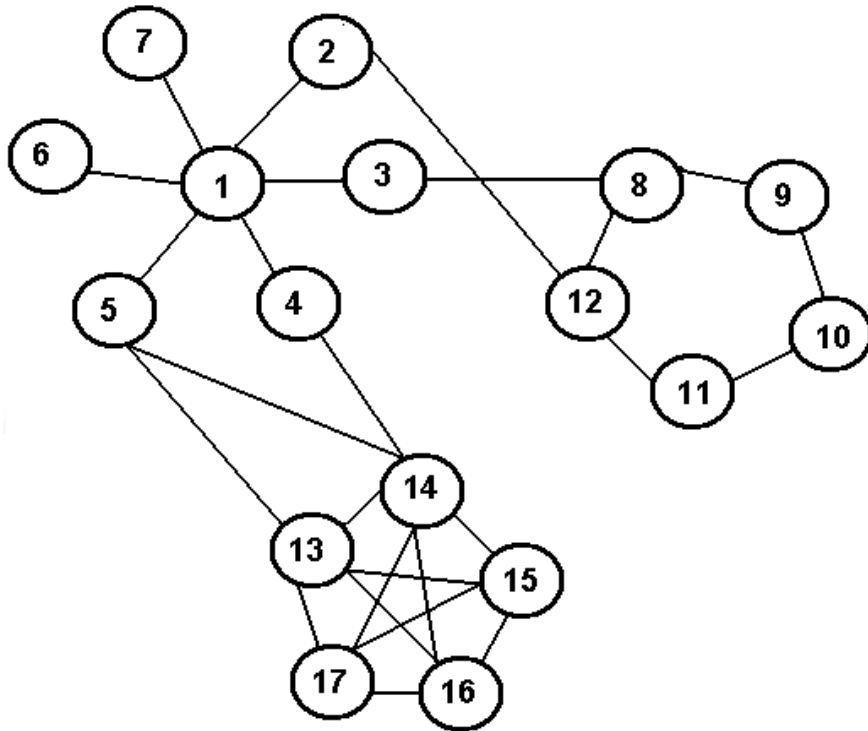
valid inequality for PNP^{CH}.

Figure 2.2: Example Graph

A clique, as defined earlier, is a set of nodes $i \in V''$ that are all adjacent to each other.

In Figure 2.2, the nodes $\{13, 14, 15, 16, 17\}$ form a K_5. Since every node is adjacent to

every node, at most one node can be chosen in a node packing. Thus, a clique has a valid

inequality of the form $\sum_{i \in K_n} x_i \leq 1$. In this case, $x_{13} + x_{14} + x_{15} + x_{16} + x_{17} \leq 1$ is the valid inequality. If the clique is maximal, then this inequality becomes facet defining.

In Figure 2.2, $\{1, 2, 3, 4, 5, 6, 7\}$ forms a fan, f_7. The valid inequality for a fan is $\sum_{i=2}^{n} x_i + (n-1)x_1 \leq (n-1)$. For this example, the valid fan inequality is $6x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 6$. This inequality is valid since if the center node, x_1, is in the node packing, then none of the peripheral nodes, $x_2, ..., x_7$, can be in the node packing. Conversely, if all the peripheral nodes are in the node packing, then the center node cannot.

A hole is a cycle with no chords, and is denoted as H_n, where n is the number of nodes in the hole. In Figure 2.2, the nodes $\{8, 9, 10, 11, 12\}$ form an H_5. Since n is an odd number, this is an odd hole. Clearly, an odd hole has a valid inequality of the form $\sum_{i=1}^{n} x_i \leq \lfloor \frac{n}{2} \rfloor$ and can be facet defining. For this example, the valid inequality is $\sum_{i=1}^{5} x_i \leq 2$.

For node packing, perfect graphs are considered to be of great importance. A perfect graph can be defined in terms of its fractional node-packing polytope. That is, given a graph $G = (V, E)$, its fractional node-packing polytope is given by $P = \{x \in \mathbf{R}_+^n : Kx \leq 1\}$, where the clique matrix K of graph G is the $(0, 1)$ incidence matrix whose rows correspond to all the cliques in G and columns correspond to the nodes of G. The graph G is perfect if, and only if, its fractional node-packing polytope is integral [58].

The motivation of this research focuses on finding valid inequalities that are strong and of high dimension for PNP^{CH}. One method to obtain such an inequality is called

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lifting.

2.4 Lifting

First introduced by Gomory [35], lifting is a common method used to increase the dimension of a cut. Lifting takes a valid inequality and, by altering some of the coefficients and possibly the right hand side, strengthens the inequality. Lifting is also used to determine cutting planes with potential to be facet-defining inequalities. In fact, lifting typically increases the dimension of the face of an inequality. Other researchers have made many advancements in lifting [7, 11, 12, 13, 19, 21, 22, 24, 25, 29, 38, 39, 41, 50, 60, 63, 69, 84].

Three categories of lifting exist; exact versus approximate, up versus down, and sequential versus simultaneous. Given the three categories and the two choices, there are a total of 8 (2^3) different ways to lift an inequality.

The restricted space is vital when considering a lifting technique. Define the restricted space of P^{conv} on the set of $D \subseteq N$ as $P^{conv}_{D,K} = conv\{x \in P : x_j = k_j$ for all $j \in D\}$ where $k_j \in \mathbf{Z}$ and $K = (k_1, k_2, ..., k_{|D|})$. Instead of observing the entire polyhedron, only a subset of variables is considered. This implies that $x_j = k_j$ for all $j \in D$. In other words, the variables with indices in D have fixed values.

The basic procedure to lift is to begin with a lifting set $D \subset N$, a set K, and a valid inequality $\sum_{i \in N \setminus D} \alpha_i x_i + \alpha \sum_{i \in D} \alpha_i x_i \leq \beta$ over $P^{CH}_{D,K}$. The general form for a lifted inequality is $\sum_{i \in N \setminus D} \alpha_i x_i + \alpha \sum_{i \in D} \alpha'_i x_i \leq \beta'$, which is valid over P^{CH}.

Exact lifting requires calculating the coefficients with complete accuracy. Thus, exact lifting should increase the dimension of the inequality, as there must exist a point not in the restricted space that meets the exact lifted inequality at equality [11, 28, 71]. Since exact lifting typically requires solving an integer program, a common practice is to reduce the accuracy of the lifting coefficient in hopes of obtaining the coefficient in a more timely manner. This method is called approximate lifting [25].

Sequential lifting changes the coefficients for one variable at a time, so $|D| = 1$. Simultaneous lifting alters the coefficients of a group of variables at the same time, therefore $|D| \geq 2$. Sequential lifting is by far the most commonly used [9, 10, 42, 62, 76, 77]. Albeit, a substantial amount of research has recently been performed on more efficient methods of simultaneous lifting [8, 40, 41, 72, 78].

Uplifting assumes that there is a valid inequality of $P_{D,K}^{conv}$ where $K = (0, 0, ..., 0)$. Uplifting leaves the right hand side of the valid inequality consistent and seeks to increase the coefficients associated with variables in D. Whereas, down lifting assumes a valid inequality of $P_{D,K}^{conv}$ where $K = (u_{j_1}, u_{j_e}, ..., u_{j_{|D|}})$ where u_j is the upper bound for variable j. Down lifting, on the other hand, often decreases the values of both the right hand side of the valid inequality and the coefficients for the variables in D. There is also a middle lifting, which is roughly a combination of both up and down lifting [77].

2.4.1 Sequential Lifting

The most commonly used lifting method is sequential uplifting [9, 10, 42, 62, 76, 77]. Sequential uplifting a binary variable begins by formulating an IP. In this case, the valid inequality is the objective function of the IP, and the constraints are those given. The variable to be lifted is set to 1, thus another constraint is created to represent this. Next, the IP is solved and the objective value, Z^*, is computed. To determine the lifting coefficient α, it follows that $\alpha = \beta - Z^*$. Every time a new variable is lifted, a constraint is substituted to set that variable to 1, and the objective function is updated and α is recalculated. The process repeats for each variable that is to be lifted. It is clear that the order of lifting is important, as different orders result in different coefficients.

An example of sequential lifting is demonstrated next. Consider the structure shown in Figure 2.3 and consider PNP^{CH}. To begin lifting, note that x_1, x_2, x_3, x_4, x_5 form an odd hole. This example begins with the odd hole inequality $\sum_{i=1}^{5} x_i \leq 2 = \lfloor \frac{m}{2} \rfloor$ and lifts from there.
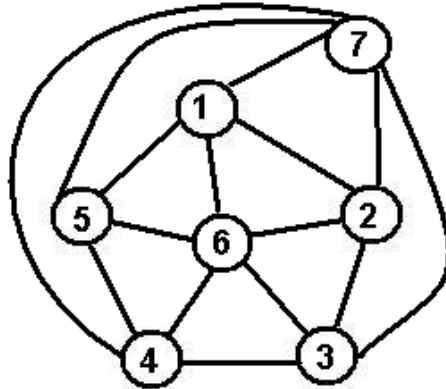

Figure 2.3: Lifting Example

To sequentially lift x_6 into this inequality, solve the following integer program.

Maximize $x_1 + x_2, +x_3 + x_4 + x_5$

Subject to $x_i + x_j \leq 1$ for all $\{i, j\} \in E$

$x_6 = 1$

$x_1, ..., x_6 \in \{0, 1\}$.

The solution to this IP is 0. Thus, the coefficient given to x_6 is $\alpha_6 = \beta - Z^* = 2 - 0 =$ 2. Thus the new valid inequality is $x_1 + x_2, +x_3 + x_4 + x_5 + 2x_6 \leq 2$. This is commonly referred to as a wheel inequality and it takes the general form $\lfloor \frac{m}{2} \rfloor x_{n+1} + \sum_{i=1}^{n} x_i \leq \lfloor \frac{m}{2} \rfloor$ where x_{n+1} is the central node and the peripheral nodes x_i form a hole of size n.
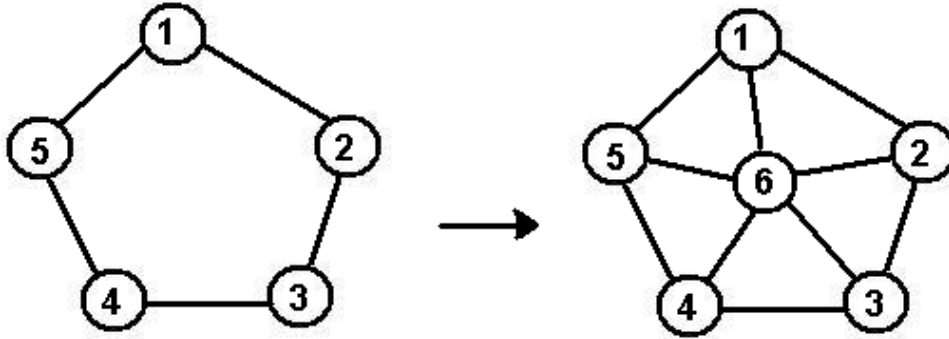

Figure 2.4: An Odd Hole Lifted to a Wheel

To lift in x_7, solve the following updated IP.

Maximize $x_1 + x_2, +x_3 + x_4 + x_5 + 2x_6$

Subject to $x_i + x_j \leq 1$ for all $\{i, j\} \in E$

$x_7 = 1$

$x_1, ..., x_7 \in \{0, 1\}$.

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The solution to this IP is 2. Thus, the coefficient given to x_7 is $\alpha_7 = \beta - Z^* = 2 - 2 = 0$. Thus the new valid inequality is $x_1 + x_2, +x_3 + x_4 + x_5 + 2x_6 + 0x_7 \leq 2$. Clearly, either 2 nonadjacent nodes from the original odd hole, or both x_6 and x_7 can be selected into our solution and still satisfy feasibility.

Note that had x_7 been lifted in first, the weights on x_6 and x_7 would be reversed. This demonstrates the importance of order when performing a lift, as the resulting inequality can be affected.
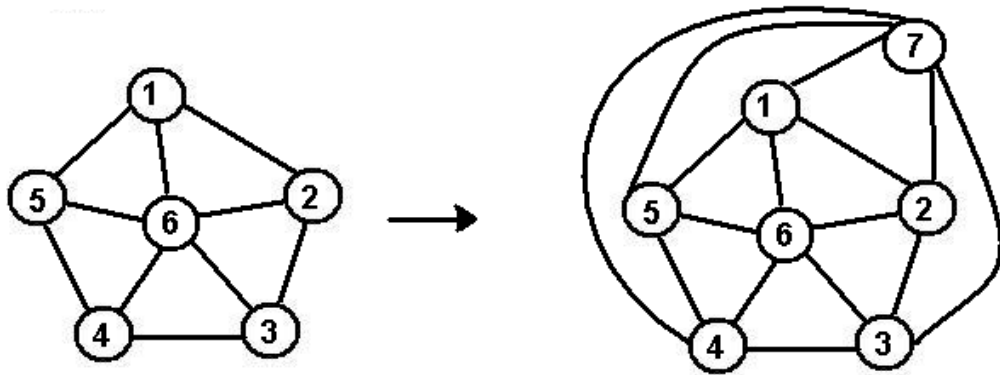
Figure 2.5: Final Lift

2.4.2 Simultaneous Lifting

Simultaneous lifting is another approach to generate cutting planes. This method originated in 1978 by Zemel [83]. Zemel's method only lifted integer programs with binary variables, and still involved solving exponentially many IPs. Although this method is accurate, it cannot be applied in practical instances, so it is only useful in theory.

In the 1990s, research was spurred and has continued in simultaneous lifting. One such research developed sequence independent lifting [8, 40, 72]. Sequence independent

lifting ignores the order in which the variables are lifted and does not require solving any integer programs. Utilizing a super-additive function for a cover inequality and setting a lower bound for every coefficient allows for all variables being lifted simultaneously from one simple expression. While sequence independent lifting is a faster technique to create cutting planes, it is only approximate, and the valid inequalities formed are not guaranteed to be facet defining. These lifting coefficients can therefore be strengthened.

Exact simultaneous lifting tries to work efficiently and with precision. In simultaneous lifting, a set of variables is added to the inequality. This method has the bonus of lifting multiple variables at the same time, while reducing the number of optimization problems that need to be solved. These inequalities are often stronger cuts. Once the lifted inequality is formed, it can be utilized as a cutting plane to reduce the solution space of the IP.

As mentioned before, simultaneous lifting has been a focus of much research at Kansas State University. Dr. Easton and Hooker worked on the background concepts regarding simultaneous lifting research [44]. They presented a linear time algorithm to simultaneously lift a set of variables into a cover inequality for a binary knapsack problem.

Later, Sharma built upon this Easton and Hooker by performing additional theoretical research and computational studies [71]. Sharma's technique presents the advantage of assisting in selecting which sets of variables to lift. The algorithm generates numerous inequalities and runs in quadratic time. Sharma showed impressive computational

results.

Gutierrez [41] developed a lifting technique to exactly lift sets of bounded integer variables simultaneously by solving a single IP. Gutierrez's algorithm sets α high, such as $\alpha = M$. An integer program is solved where the objective is the left hand side of the proposed simultaneously lifted inequality, with the α value. If $Z \leq \beta$, then the algorithm terminates and reports α as the lifting coefficient. If $Z > \beta$, then the x^* from the IP is used to solve for a new α and the process repeats.

Now reconsider Figure 2.3. Implementation of simultaneous lifting can result in a different inequality than sequential lifting. Recall that with sequential lifting, order has an effect on the resulting inequality. To lift in x_6 and x_7 simultaneously, let $\alpha = M$ and solve the following IP.

Maximize $x_1 + x_2, +x_3 + x_4 + x_5 + M(x_6 + x_7)$

Subject to $x_i + x_j \leq 1$ for all $\{i, j\} \in E$

$x_1, ..., x_7 \in \{0, 1\}$.

The solution to this IP is $Z = 2M > 2$ from the solution $x_1 = ... = x_5 = 0$ and $x_6 = x_7 = 1$. Now this x point is used to solve $x_1 + x_2, +x_3 + x_4 + x_5 + \alpha(x_6 + x_7) = 2$, which implies $0 + \alpha(2) = 2$. Thus, $\alpha = 1$. Now the following IP is solved.

Maximize $x_1 + x_2, +x_3 + x_4 + x_5 + 1(x_6 + x_7)$

Subject to $x_i + x_j \leq 1$ for all $\{i, j\} \in E$

$x_1, ..., x_7 \in \{0, 1\}$.

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The solution is now $Z = 2$ and so the simultaneous lifting terminates and reports the simultaneously lifted inequality $x_1 + x_2, +x_3 + x_4 + x_5 + x_6 + x_7 \leq 2$. This is clearly a different inequality than obtained by sequential lifting. One problem is that the face of this inequality only has dimension 5, which is not facet defining. To create structures that enable this process to be facet defining is the motivation for this research.

Facet Defining Structures for the Node Packing Polytope

This chapter introduces the cliqued hole, odd bipartite hole, and the odd k-partite hole. These structures and their valid inequalities are the basis of advancements in this research. Implementation of simultaneous lifting in the node packing problem is also discovered. These structures all present valid inequalities that can be facet defining. The first structure discussed is the cliqued hole.

3.1   Cliqued Hole

The basic idea of a cliqued hole is to generate a valid inequality by combining both a clique and an odd hole. Briefly, each node in an odd hole can be expanded into a clique structure in the obvious way. Formally, let a graph with q nodes be a cliqued hole of

size m if, and only if, the nodes can be partitioned into m sets, $P_1, P_2, ..., P_m$, such that

$E = \cup_{i=1}^{m}\{\{u, v\} : u, v \in P_i\}\bigcup\cup_{i=1}^{m}\{\{u, v\} : u \in P_i, \ v \in P_{(i \ mod \ m)+1}\}$. Denote such a

cliqued hole by $CH_{m,P}$ where $P = (p_1, p_2, ..., p_m)$ and $\sum_{i=1}^{m} p_i = q$, $p_i \geq 1$ and $p_i \in \mathbf{Z}$ for

all $i = 1, ..., m$. For convenience denote the vertices in P_i as v_{i_j} for $j = 1, ..., |P_i|$ for all

$i = 1, ..., m$.

In order to better understand the cliqued hole structure and its definitions several

diagrams are shown regarding how to create a $CH_{5,P}$ where $P = (2, 2, 2, 2, 2)$. Consider

an odd hole of size five as shown in the first stage in Figure 3.1, so $m = 5$. The

next step shows that each node is exploded into a clique of size 2. This means that

$p_1 = 2, p_2 = 2, p_3 = 2, p_4 = 2$ and $p_5 = 2$, so the size of the clique for each node in the

hole is 2. The last stage in the diagram shows that since an edge existed between each

hole in the original structure, there must exist an edge between each node in each clique

and every node in every adjacent clique. Also, $q = 2 + 2 + 2 + 2 + 2 = 10$, as there now

exist a total of 10 nodes and 2*2+2*2+2*2+2*2+2*2=20 edges.

Figure 3.1 shows that the basic centering structure is a hole. Each node on the hole

is a node that is then exploded into cliques, in this case uniform size of 2. Each node is

then cliqued with the adjacent nodes. This cliquing of neighbors is what provides the

valid inequality, which is to say that the overall structure follows the constraints set up

by the underlying hole structure.
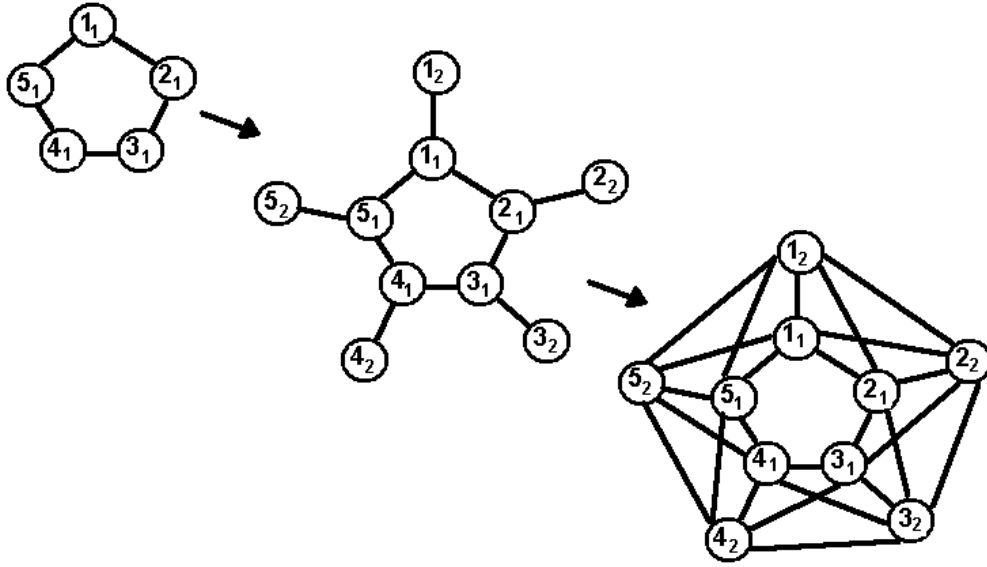
Figure 3.1: Cliqued Hole Transition

Figure 3.2 presents a more complex cliqued hole, a $CH_{5,P}$ where $P = (3, 2, 3, 4, 1)$. This means that $p_1 = 3, p_2 = 2, p_3 = 3, p_4 = 4$, and $p_5 = 1$ as well as $q = 13$. This graph has 3*2+2*3+3*4+4*1+1*3=31 edges. The size of m and P are of no importance until we consider the inequality of interest. Once a facet-defining cliqued hole inequality is desired, it will be important only to note that m must be an odd hole in order to generate sufficient points to prove it is facet defining.

Now that the cliqued hole has been defined, it is relevant to show that there exists a valid inequality called the cliqued hole inequality. Formally,

Theorem 3.1.1. *Given a graph $G = (V, E)$ with an induced cliqued hole of the form $CH_{m,P}$ with $P = (p_1, ..., p_{|p|})$, then $\sum_{i \in CH_{m,P}} x_i \leq \lfloor \frac{m}{2} \rfloor$ is a valid inequality for the node packing polyhedron.*

Proof: Assume that $G = (V, E)$ has an induced cliqued hole of the form $CH_{m,P}$ in any

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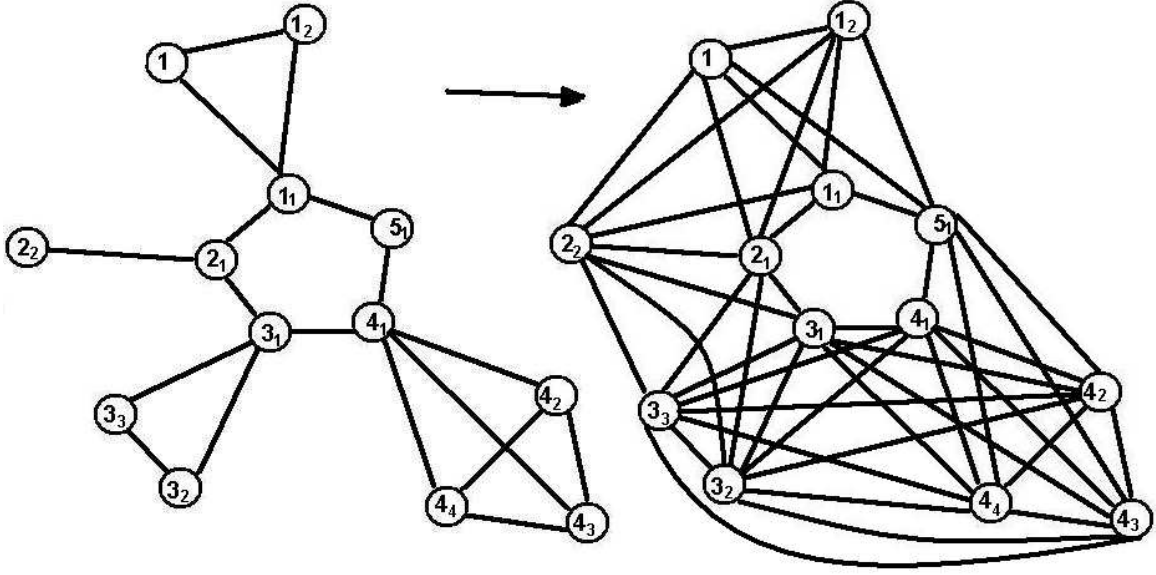
Figure 3.2: Non-uniform Cliqued Hole Transition

node packing. There can be at most one node selected from any P_i structure due to P_i forming a clique for $i = 1, ..., m$. Since every node in P_i is adjacent to every node in $P_{(i \bmod m)+1}$ and also adjacent to each node in $P_{(i-2) \bmod m+1}$ for $i = 1, ..., m$. Thus, there can exist at most $\lfloor \frac{m}{2} \rfloor$ nodes selected in this substructure in a node packing and the result follows.

$\square$

Returning to the $CH_{5,P}$ where $P = (3, 4, 3, 2, 1)$, the valid inequality would be $x_{1_1} + x_{1_2} + x_{1_3} + x_{2_1} + x_{2_2} + x_{3_1} + x_{3_2} + x_{3_3} + x_{4_1} + x_{4_2} + x_{4_3} + x_{4_4} + x_{5_1} \leq \lfloor \frac{5}{2} \rfloor = 2$. This means that at most two nodes can be selected into the solution space. Similarly, the $CH_{5,P}$ when $P = (2, 2, 2, 2, 2)$ valid inequality would be $x_{1_1} + x_{1_2} + x_{2_1} + x_{2_2} + x_{3_1} + x_{3_2} + x_{4_1} + x_{4_2} + x_{5_1} + x_{5_2} \leq 2$. Thus, the size of the clique doesn't change the right hand side, only the number of variables in the left hand side. Therefore, all variables in the clique can be

simultaneously lifted with a coefficient of 1.

The question remains as to whether or not this structure is a new structure or just a collection of existing structures. Finding one linear relaxation point that satisfies all inequalities from known structures, but violates this cliqued hole inequality is sufficient to show that these inequalities are a new class of inequalities for PNP^{CH}. This linear relaxation point is formed by using the $CH_{5,P}$ with $P = (3, 2, 3, 4, 1)$.

When looking at a cliqued hole, several known structures are easily recognizable. First, there are numerous cliques. The induced subgraph of each P_i is a clique. There are also cliques between any two adjacent P_is. Recall a K_n clique inequality takes the form $\sum_{i \in K_n} x_i \leq 1$. Thus, adding all of the clique inequalities in a cliqued hole would result in adding the following constraints.

$$x_{1_1} + x_{1_2} + x_{1_3} \leq 1.$$

$$x_{2_1} + x_{2_2} \leq 1.$$

$$x_{3_1} + x_{3_2} + x_{3_3} \leq 1.$$

$$x_{4_1} + x_{4_2} + x_{4_3} + x_{4_4} \leq 1.$$

$$x_{1_1} + x_{1_2} + x_{1_3} + x_{2_1} + x_{2_2} \leq 1.$$

$$x_{2_1} + x_{2_2} + x_{3_1} + x_{3_2} + x_{3_3} \leq 1.$$

$$x_{3_1} + x_{3_2} + x_{3_3} + x_{4_1} + x_{4_2} + x_{4_3} + x_{4_4} \leq 1.$$

$$x_{4_1} + x_{4_2} + x_{4_3} + x_{4_4} + x_{5_1} \leq 1.$$

$$x_{5_1} + x_{1_1} + x_{1_2} + x_{1_3} \leq 1.$$

Lastly, there are a number of odd holes of size five. In fact this cliqued hole has $3 * 2 * 3 * 4 * 1 = 72$ induced odd holes. The number of odd holes can be computed by multiplying the p_is. Recall the odd hole H_n inequality is $\sum_{i \in H_n} x_i \leq \lfloor \frac{n}{2} \rfloor$. Thus, 72 more constraints can be added. The first four and the final constraint are listed below.

$$x_{1_1} + x_{2_1} + x_{3_1} + x_{4_1} + x_{5_1} \leq 2$$

$$x_{1_2} + x_{2_1} + x_{3_1} + x_{4_1} + x_{5_1} \leq 2$$

$$x_{1_3} + x_{2_1} + x_{3_1} + x_{4_1} + x_{5_1} \leq 2$$

$$x_{1_1} + x_{2_2} + x_{3_1} + x_{4_1} + x_{5_1} \leq 2$$

$$.$$
$$.$$
$$.$$

$$x_{1_3} + x_{2_2} + x_{3_3} + x_{4_4} + x_{5_1} \leq 2$$

Comparing the benefit of the cliqued hole inequality to the standard linear relaxation of the node packing problem is trivial. Assigning each variable to $\frac{1}{2}$ results in a valid relaxation point. Clearly, this point violates this cliqued hole inequality as $6.5 > 2$. Thus, the cliqued holed inequality is vastly superior to the standard formulation.

Now if all 9 clique inequalities and all 72 odd hole inequalities are added to the standard linear relaxation of the node packing problem, then the following point is in the linear relaxation. For each variable x_{i_j} assign a value of $\frac{1}{2} * \frac{1}{p_i}$ for each $j \in P_i$ and all $i \in \{1, ..., m\}$. Thus, the variables are assigned as follows $x_{1_1} = x_{1_2} = x_{1_3} = \frac{1}{3} * \frac{1}{2} = \frac{1}{6}$, $x_{2_1} = x_{2_2} = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$, $x_{3_1} = x_{3_2} = x_{3_3} = \frac{1}{3} * \frac{1}{2} = \frac{1}{6}$, $x_{4_1} = x_{4_2} = x_{4_3} = x_{4_4} = \frac{1}{4} * \frac{1}{2} = \frac{1}{8}$,

$x_{5_1} = 1 * \frac{1}{2} = \frac{1}{2}$. This point clearly satisfies each of the 81 inequalities and all edge constraints for the node packing problem. However, this point violates the cliqued hole inequality as $2.5 > 2$. Since none of these known cuts eliminate this point, the cliqued hole inequality is a previously undiscovered class of valid inequalities.
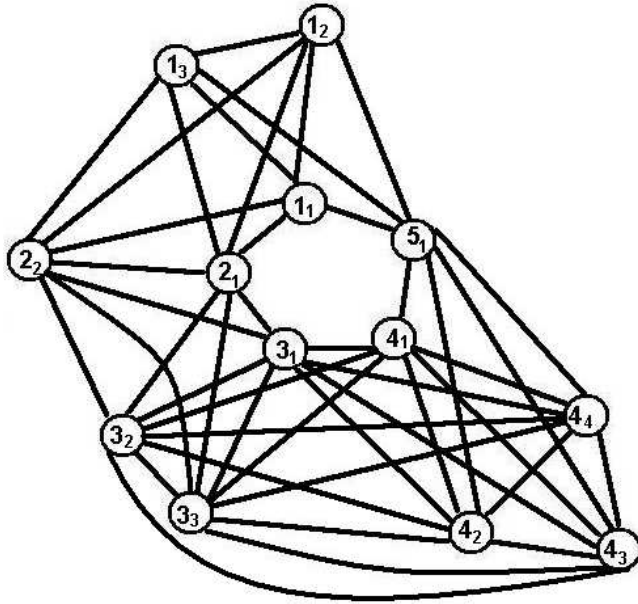
Figure 3.3: $CH_{(}5, P)$ with $P = (3, 2, 3, 4, 1)$

Since the cliqued hole inequality is a new class of inequalities, it is important to determine if it is a proper face for the node packing polyhedron, and determine under what conditions it is facet defining. To achieve this let e_i be defined as the ith identity point. Clearly, if $x = e_i$, then the corresponding node packing solution is vertex i as the lone vertex in a node packing problem. The following theorem provides a lower bound on the dimension of a cliqued hole inequality.

Theorem 3.1.2. *Given a graph $G = (V, E)$ with an induced cliqued hole of the form $CH_{m,P}$, then $\sum_{i \in CH_{m,P}} x_i \leq \lfloor \frac{m}{2} \rfloor$ defines a face of dimension at least $q - 1$ for the node*

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packing polyhedron.

Proof: Given a node packing problem, let $CH_{m,P}$ be an induced subgraph. From Theorem 3.1.2 $\sum_{i \in C_{m,P}} x_i \leq \lfloor \frac{m}{2} \rfloor$ is a valid inequality. It suffices to find $|CH_{m,P}|$ affinely independent points that meet this inequality at equality.

Let the first set of m points be the standard odd hole points for each of the first vertices in each P_i. Thus, the points are given by $e_{i_1} + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor - 1} e_{[(i+2j-1)mod\ m+1]_1}$ for each $i = 1, ..., m$. These points are well known to be affinely independent.

For each vertex $v_{i_j} \in V(CH_{m,P})$ where $j \geq 2$ and any $i \in \{1, ..., m\}$, include the point $e_{i_j} + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor - 1} e_{[(i+2j-1)mod\ m+1]_1}$. Clearly, these points meet the cliqued hole inequality at equality and are affinely independent. Thus the dimension of any cliqued hole inequality is at least $q - 1$ in PNP^{CH}.

$\square$

To illustrate Theorem 3.1.2, consider again the $CH_{5,P}$ where $P = (3, 2, 3, 4, 1)$. The affinely independent points for Figure 3.3 are as follows:

$$
\begin{array}{c|ccccc|cccccccc}
x_{1_1} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{2_1} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
x_{3_1} & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{4_1} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
x_{5_1} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
x_{1_2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{1_3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{2_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
x_{3_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
x_{3_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
x_{4_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
x_{4_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
x_{4_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
$$

It is also of great interest to determine if the cliqued hole inequality not only defines a high dimension face, but also can define a facet of PNP^{CH}. Thus, what conditions on nodes not in the cliqued hole must exist for this to be facet defining. Formally,

Theorem 3.1.3. *Given a graph $G = (V, E)$ with an induced cliqued hole of the form $CH_{m,P}$, then $\sum_{i \in V(CH_{m,P})} x_i \leq \lfloor \frac{m}{2} \rfloor$ defines a facet of the node packing polyhedron if for any node v_k not contained within the $CH_{m,P}$, v_k is non-adjacent to at least $\lfloor \frac{m}{2} \rfloor$ nodes in $CH_{m,P}$ that are not adjacent to each other.*

Proof: Assume that $CH_{m,P}$ is an induced subgraph of $G = (V, E)$ such that for any node $v_k \in V \setminus V(CH_{m,P})$, v_k is non-adjacent to at least $\lfloor \frac{m}{2} \rfloor$ nodes in $V(CH_{m,P})$ that are not adjacent to each other. Denote these $\lfloor \frac{m}{2} \rfloor$ nodes as V''_k.

The cliqued hole inequality $\sum_{i \in V(CH_{m,P})} x_i \leq \lfloor \frac{m}{2} \rfloor$ from Theorem 3.1.1 defines a face of dimension at least $q - 1$. To these q points include the point $e_k + \sum_{l \in V''_k} e_l$ for each $v_k \in V \setminus V(CH_{m,P})$. Each of these points is feasible, meets the inequality at equality, and are clearly affinely independent.
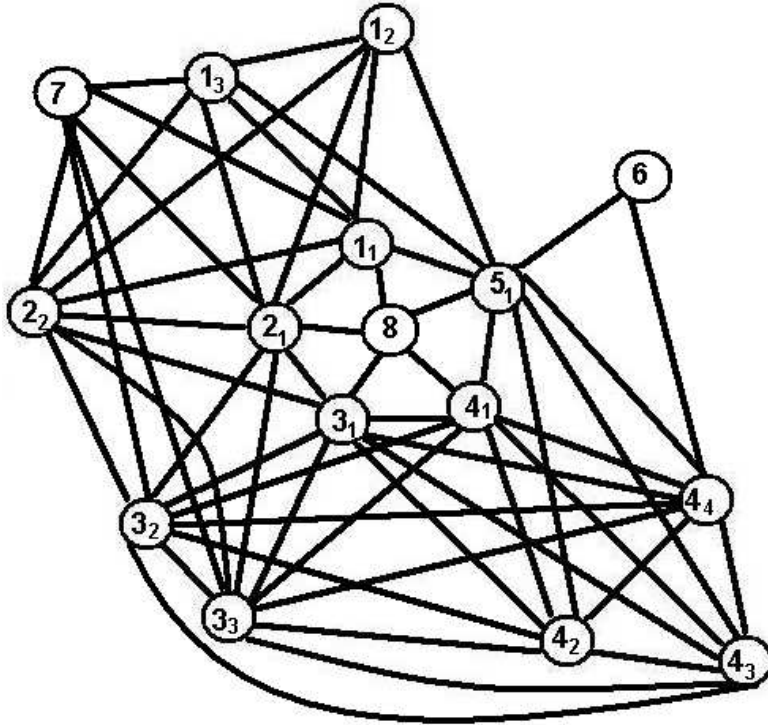
□

Figure 3.4: Facet Defining $CH_{5,P}$

To illustrate the conditions described in Theorem 3.1.3, consider Figure 3.4. Note that 3 new nodes have been added to the graph from Figure 3.3. The affinely independent points are as follows:

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x_{1_1}	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
x_{2_1}	0	1	0	0	1	0	0	0	0	0	1	1	1	1	0	0
x_{3_1}	1	0	1	0	0	1	1	0	0	0	0	0	0	0	1	0
x_{4_1}	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	0
x_{5_1}	0	0	1	0	1	0	0	0	1	1	0	0	0	0	0	0
x_{1_2}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
x_{1_3}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
x_{2_2}	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
x_{3_2}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
x_{3_3}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
x_{4_2}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
x_{4_3}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
x_{4_4}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
x_6	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
x_7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
x_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Note that for $x_6 = 1$, it is relevantly trivial to find a point that meets the cliqued hole inequality at equality. This involves picking up e_{2_1} and e_{4_1}, as they are two non adjacent nodes in the hole. The point for $x_7 = 1$ is slightly more demanding, as more edges exist. Note that the point includes e_{3_1} and e_{1_2}, which shows that a point for the hole and a peripheral node can be used to meet the facet defining conditions.

Finally, $x_8 = 1$ is a very interesting case. Since x_8 is adjacent to every node in the original odd hole, it forms a wheel. It may be assumed that the wheel inequality mentioned earlier would lift in, but in fact this violates the cliqued hole inequality, and x_8 would indeed lift in with a 0. Thus, it is feasible to select any two non-adjacent peripheral nodes, in this case e_{2_2} and e_{4_3} to create the point needed for facet defining.

It may seem that such a structure as shown in Figure 3.3 would be a rare occurance in most graphs. However, there are more likely forms of the cliqued hole, such as shown in Figure 3.5. This structure is a much simpler example of a cliqued hole and would be

expected to occur much more frequently in a graph.

An interesting concept is that odd holes have been some of the least useful cutting planes [5]. Notice that the cliqued odd holes would greatly strengthen an odd hole inequality and may assist in fixing this problem. In this example, the hole inequality is $x_{1_1} + x_{2_1} + x_{3_1} + x_{4_1} + x_{5_1} \leq 2$ and the cliqued hole inequality is $x_{1_1} + x_{1_2} + x_{2_1} + x_{3_1} + x_{3_2} + x_{4_1} + x_{5_1} \leq 2$. Again, this adds in two more variables without increasing the right hand side value, thus it is a much stronger inequality.
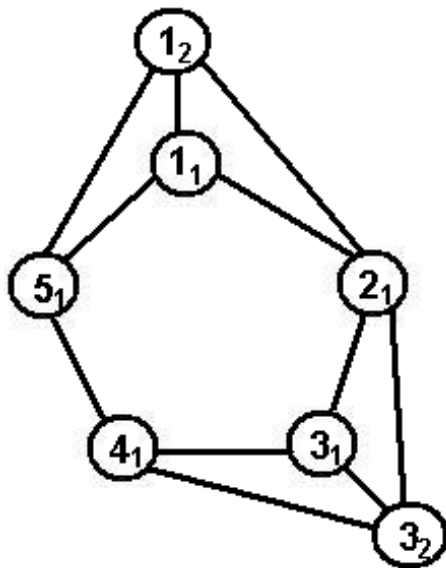

Figure 3.5: A Simple Cliqued Hole

It is recommended that the cliqued hole inequalities should be implemented in future software to help solve the node packing problem. Even more exciting is that these inequalities can be implemented in conflict graphs as a general class of cutting planes for binary integer programs.

3.2 Odd Bipartite Hole

The odd bipartite hole is another new graphic structure that can generate previously undiscovered valid inequalities for PNP^{CH}. The general concept of the structure is that there exist two odd holes, H_p and H_q such that every vertex in H_p is adjacent to every vertex in H_q. The odd bipartite hole is notable in that it allows for simultaneous lifting in the node packing problem.

Formally, a graph $G = (V, E)$ is an odd bipartite p, q hole ($OBP_{p,q}$) if, and only if, the following conditions hold. The nodes can be partitioned into H_p and H_q such that H_p and H_q form induced odd holes. Additionally, E contains $\{v_i, v_j\}$ for every $v_i \in H_p$ and $v_j \in H_q$. Figure 3.6 shows an $OBP_{5,7}$.
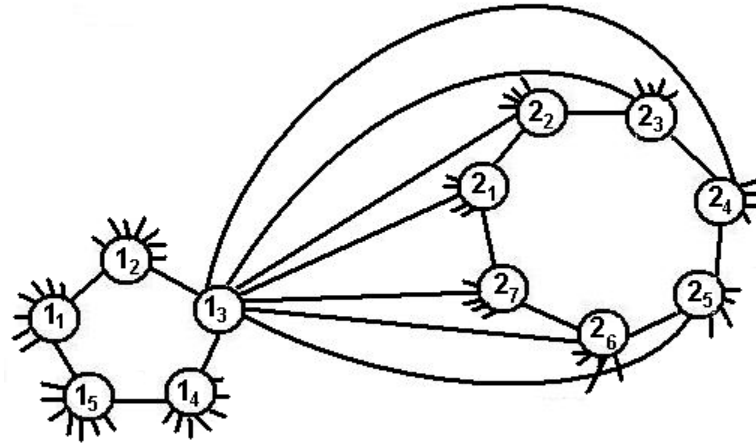

Figure 3.6: Odd Bipartite Hole

An OBP is basically a complete bipartite graph, but instead of just being adjacent to the nodes in the other partition, each node is also adjacent to two neighbors in the

same partition, and each partition contains an odd number of nodes. This generates a very interesting inequality and the physical depiction can be quite stunning, see Figure 3.7. This representation better illustrates the bipartite nature of the structure. How the structure came to fruition stemmed directly from looking at an example of a bipartite graph and inserting the additional edges to form the two odd holes H_p and H_q.
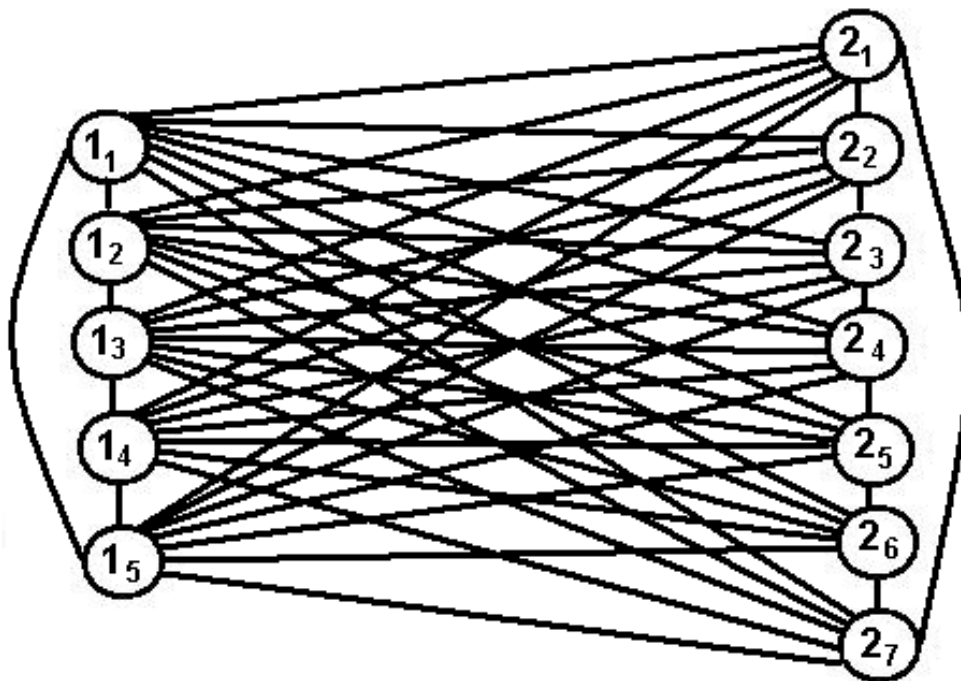
Figure 3.7: Odd Bipartite Hole Alternate

To generate a valid inequality, observe that since every vertex in H_p is adjacent to every vertex in H_q, a node packing can only have vertices in one of the holes. Thus, the valid inequality is a combination of the two odd hole inequalities.

Theorem 3.2.1. *Given a graph $G = (V, E)$ with an induced odd bipartite hole of the form $OBP_{p,q}$, then $\sum_{i \in V(H_p)} x_i + \sum_{i \in V(H_q)} \frac{p-1}{q-1} x_i \leq \lfloor \frac{p}{2} \rfloor$ is a valid inequality for PNP^{CH}.*

Proof: Assume that $G = (V, E)$ has an odd bipartite hole of the form $OBP_{p,q}$, with the

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odd holes denoted as H_q and H_p. Since every node in H_q has an edge to every node in H_p and vice versa, every node packing can contain vertices from only one of the odd holes.

If only vertices in H_p are considered, then the $\sum_{i \in V(H_p)} x_i + \sum_{i \in V(H_q)} \frac{p-1}{q-1} x_i \leq \lfloor \frac{p}{2} \rfloor$ reduces to $\sum_{i \in V(H_p)} x_i \leq \lfloor \frac{p}{2} \rfloor$, which is just the odd hole inequality, and is valid. Whereas, if only vertices in H_q are considered, then this inequality reduces to $\sum_{i \in V(H_q)} \frac{p-1}{q-1} x_i \leq \lfloor \frac{p}{2} \rfloor$. Due to H_q being an odd hole, at most $\frac{q-1}{2}$ vertices can be selected in any node packing. Thus, $\sum_{i \in V(H_q)} \frac{p-1}{q-1} x_i \leq \frac{p-1}{q-1} \frac{q-1}{2} = \frac{p-1}{2} = \lfloor \frac{p}{2} \rfloor$ and the result follows.

$\square$

Now consider Figure 3.7. In this example, note that $OBP_{7,5}$ and $OBP_{5,7}$ are both valid representations of the structure. For simplicity, denote it as $OBP_{5,7}$. The inequality is $\sum_{i \in V(H_5)} x_i + \sum_{i \in V(H_7)} \frac{5-1}{7-1} x_i \leq \lfloor \frac{5}{2} \rfloor$, or equivalently $x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + \frac{2}{3}(x_{2_1} + x_{2_2} + x_{2_3} + x_{2_4} + x_{2_5} + x_{2_6} + x_{2_7}) \leq 2$.

Comparing the benefit of the odd bipartite hole inequality to the standard linear relaxation of the node packing problem is trivial. Again, a linear relaxation point that meets the existing points known valid inequalities, but violates this OBH inequality is sufficient to show that these inequalities were previously undiscovered.

Observe that the maximum clique in the graph is of size 4, e.g. $\{1_1, 1_2, 2_1, 2_2\}$. There are 35 such clique inequalities. (Note, any K_3 inequalities are dominated by a K_4 inequality.) There are no odd hole inequalities other than the two odd original holes since any odd hole that contained vertices from both odd holes would have chords. There

are 12 wheel inequalities generated by including one vertex from one of the holes with all of the vertices from the other hole. These wheel inequalities clearly dominate the odd hole inequalities. Thus, the known cutting planes are:

$$x_{1_1} + x_{1_2} + x_{2_1} + x_{2_2} \leq 1$$

$$x_{1_1} + x_{1_2} + x_{2_2} + x_{2_3} \leq 1$$

$$.$$
$$.$$
$$.$$

$$x_{1_5} + x_{1_1} + x_{2_7} + x_{2_1} \leq 1$$

$$x_{1_1} + x_{1_2} + ...x_{1_5} + 2x_{2_1} \leq 2$$

$$.$$
$$.$$
$$.$$

$$x_{1_1} + x_{1_2} + ...x_{1_5} + 2x_{2_7} \leq 2$$

$$x_{2_1} + x_{2_2} + ...x_{2_7} + 3x_{1_1} \leq 3$$

$$.$$
$$.$$
$$.$$

$$x_{2_1} + x_{2_2} + ...x_{2_7} + 3x_{1_5} \leq 3$$

For each variable let $x_i = \frac{1}{4}$ for each $i \in V(H_5) \cup V(H_7)$. Clearly, this point satifies all the K_4 inequalities and the wheel inequalities. Evaluating this point in the OBP inequality results in $\frac{29}{12} > 2$. Since none of these known cuts eliminate this point, the

odd bipartite hole inequality is a new and unique valid inequality.

The obvious question now is how strong of a face is defined by the odd bipartite hole inequality.

Theorem 3.2.2. *Given a graph $G = (V, E)$ with an odd bipartite hole of the form $OBP_{p,q}$, then $\sum_{i \in V(H_p)} x_i + \sum_{i \in V(H_q)} \frac{p-1}{q-1} x_i \leq \lfloor \frac{p}{2} \rfloor$ defines a face of dimension at least $q + p - 1$ for the node packing polyhedron.*

Proof: Given a node packing problem, let $OBP_{p,q}$ be an induced subgraph. From Theorem 3.2.1 $\sum_{i \in V(H_p)} x_i + \sum_{i \in V(H_q)} \frac{p-1}{q-1} x_i \leq \lfloor \frac{p}{2} \rfloor$ is a valid inequality. It suffices to find $|V(OBP_{q,p})| = q + p$ affinely independent points that meet this inequality at equality.

Let the first set of p points be the standard odd hole points for H_p. These points are given by $\sum_{j=0}^{\lfloor \frac{p}{2} \rfloor - 1} e_{1_{(i+2j-1)}\ mod\ p+1}$ for each $i = 1, ..., p$. Next, include the standard odd hole points for H_q. These points are given by $\sum_{j=0}^{\lfloor \frac{q}{2} \rfloor - 1} e_{2_{(i+2j-1)}\ mod\ q+1}$ for each $i = 1, ..., q$.

Both sets of points are well known to be affinely independent. Clearly, these points meet the OBP inequality at equality and are affinely independent. Thus the dimension of any OBP inequality is at least $q + p - 1$ in PNP^{CH}.

□

To emphasize the implications of Theorem 3.2.2, consider the affinely independent points for Figure 3.6.

$$\begin{array}{c|ccccc|ccccccc}
x_{1_1} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{1_2} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{1_3} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{1_4} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{1_5} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
x_{2_1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
x_{2_2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
x_{2_3} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
x_{2_4} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
x_{2_5} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
x_{2_6} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
x_{2_7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{array}$$

Now that we have shown the odd bipartite hole inequality is not only valid, but defines a face, it is of great interest to determine whether it is also facet defining. The following theorem provides such conditions.

Theorem 3.2.3. *Given a graph $G = (V, E)$ with an induced odd bipartite hole of the form $OBP_{p,q}$, then $\sum_{i \in V(H_p)} x_i + \sum_{i \in V(H_q)} \frac{q-1}{p-1} x_i \leq \lfloor \frac{p}{2} \rfloor$ defines a facet of the node packing polyhedron if for each $v_k \in V \setminus V(OBP_{p,q})$, there exists a $i \in \{1, ..., p\}$ such that v_k is not adjacent to the nodes v_{1_i}, $v_{1_{(i+1) \mod p+1}}$, $v_{1_{(i+3) \mod p+1}}, \ldots$ $v_{1_{(i+\frac{p-3}{2}) \mod p+1}}$ or there exists an $j \in \{1, ..., q\}$ such that v_k is not adjacent to the nodes v_{2_j}, $v_{2_{(j+1) \mod q+1}}$, $v_{2_{(j+3) \mod q+1}}, \ldots v_{2_{(j+\frac{q-3}{2}) \mod q+1}}$.*

Proof: Assume that $OBP_{p,q}$ is an induced subgraph of $G = (V, E)$ and that each $v_k \in V \setminus V(OBP_{p,q})$ satisfies the above condition. To find $|V|$ affinely independent points that meet $\sum_{i \in V(H_p)} x_i + \sum_{i \in V(H_q)} \frac{q-1}{p-1} x_i \leq \lfloor \frac{p}{2} \rfloor$ at equality, begin by using the $p + q$ points used in Theorem 3.2.2. To these points add in either $e_k + e_{1_i} + e_{1_{(i+1) \mod p+1}} + e_{1_{(i+3) \mod p+1}} + ... + e_{1_{(i+\frac{p-3}{2}) \mod p+1}}$ or the point $e_k + e_{2_i} + e_{2_{(i+1) \mod q+1}} + e_{2_{(i+3) \mod q+1}} + ... + e_{2_{(i+\frac{q-3}{2}) \mod q+1}}$ depending upon the above condition. These points clearly are feasible,

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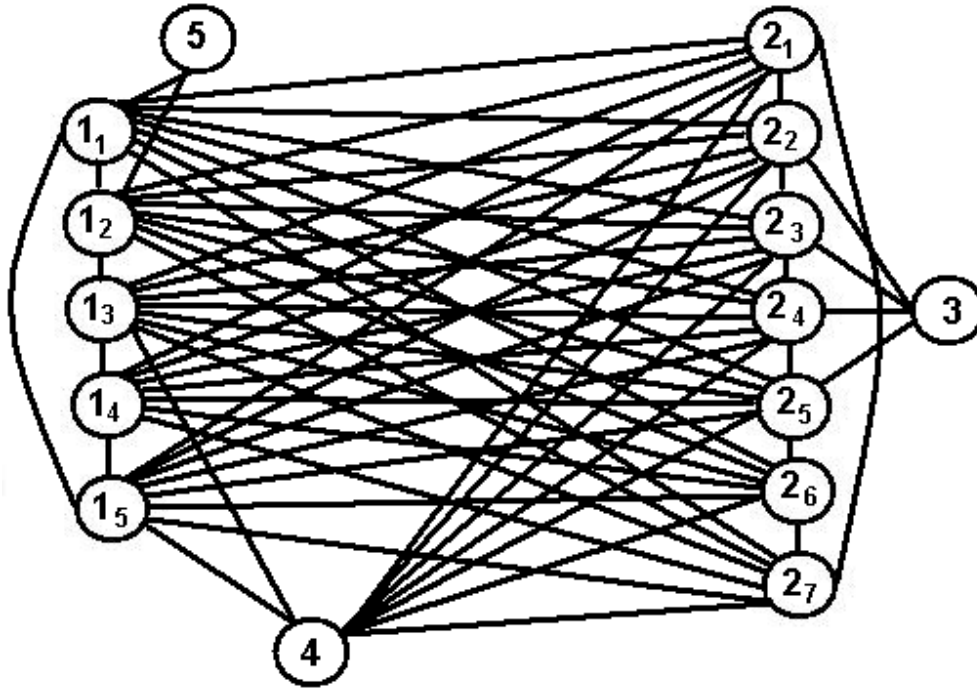meet the inequality at equality and are affinely independent.

□

Figure 3.8: Facet Defining

To better illustrate the results of Theorem 3.2.3, consider Figure 3.8. Notice that the

graph structure in Figure 3.6 is an induced subgraph. The affinely independent points

for Figure 3.8 are as follows.

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	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0
x_{1_1}	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0
x_{1_2}	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0
x_{1_3}	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0
x_{1_4}	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0
x_{1_5}	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0
x_{2_1}	0	0	0	0	0	1	0	0	1	0	1	0	0	0	1
x_{2_2}	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0
x_{2_3}	0	0	0	0	0	1	0	1	0	0	1	0	0	0	1
x_{2_4}	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0
x_{2_5}	0	0	0	0	0	1	0	1	0	1	0	0	0	0	1
x_{2_6}	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0
x_{2_7}	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0
x_3	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
x_4	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
x_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

For this example, notice that x_3 is not adjacent to any nodes from H_5, thus the point includes e_{1_1} and e_{1_3}, which meets the odd bipartite hole inequality at equality. Since x_4 is adjacent to every node in H_7 and two nonadjacent nodes in H_5, the obvious solution involves two remaining non-adjacent nodes, e_{1_2} and e_{1_4}. The simplest point is for x_5, as it is only adjacent to two nodes in H_5, thus any 3 non-adjacent nodes in H_7.

These odd bipartite holes achieve the motivational goal of this research in that they are inequalities that can only be achieved by simultaneous lifting. To show this, consider the odd hole inequality generated by H_5, $x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} \leq 2$. There are 7 other variables that could be lifted sequentially into this inequality. Thus, there are $7! = 5{,}040$ possible inequalities that could be generated.

Without loss of generality, we could lift x_{2_1} first. This creates a wheel and as in section 2.3.1 it would lift in with a 2. Thus, the new sequentially lifted inequality is $x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_{2_1} \leq 2$. The variables x_{2_3}, x_{2_4}, x_{2_5} and x_{2_6} would all lift in with a 0 coefficient. This can be seen since any of these nodes and node 2_1 can be in

a feasible node packing and so Z from the lifting IP would be 2 and $2 - 2 = 0$.

If x_{2_2} is lifted before x_{2_7}, then it is lifted with a 2 coefficient, because nodes $2_1, 1_1, 1_2, ..., 1_5$ cannot be in a node packing with node 2_2. Since 2_2 and 2_7 can be in a node packing, x_{2_7} lifts in with a 0. Obviously switching the order of lifting x_{2_2} and x_{2_7} would flip the 2 and 0 coefficients.

Thus, there are only two sequentially lifted inequalities if x_{2_1} is lifted first. Of the $6! = 720$ possible remaining sequences for lifting, there will only be 2 distinct inequalities. They are $x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_{2_1} + 2x_{2_7} \leq 2$ and $x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_{2_1} + 2x_{2_2} \leq 2$. Following a similar line of logic, of the $7! = 5040$ sequentially lifted inequalities, there are only 7 unique inequalities. They are

$$x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_{2_1} + 2x_{2_2} \leq 2,$$

$$x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_{2_2} + 2x_{2_3} \leq 2,$$

$$x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_{2_3} + 2x_{2_4} \leq 2,$$

$$x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_2 x_{2_4} + 2x_{2_5} \leq 2,$$

$$x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_2 x_{2_5} + 2x_{2_6} \leq 2,$$

$$x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_2 x_{2_6} + 2x_{2_7} \leq 2 \text{ and}$$

$$x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + 2x_2 x_{2_7} + 2x_{2_1} \leq 2.$$

Averaging these sequentially lifted inequalities results in $x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + \frac{4}{7}(x_{2_1} + x_{2_2} + ... + x_{2_7}) \leq 2$. The odd bipartite hole inequality is $x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + \frac{2}{3}(x_{2_1} + x_{2_2} + ... + x_{2_7}) \leq 2$, which strictly dominates this average sequentially lifted

inequality. Furthermore, applying simultaneously lifting to the variables $\{x_{2_1}, ..., x_{2_7}\}$ would also result in a coefficient of $\frac{2}{3}$, which is the same inequality. Consequently, this thesis has discovered the first subgraphic structures that would allow unique inequalities for simultaneous lifting in PNP^{CH}.

3.2.1 Odd k-partite Hole

Since the OBP consists of two odd holes that have a complete bipartite graph between them, a natural continuation is to add more odd holes to the structure. Denote an odd k-partite hole with k odd holes $H_1, ..., H_k$ as $OkPp_1, ..., p_k$ where p_i is the $|H_i|$. A depiction of a $OkP_{5,7,9}$ is shown in Figure 3.9. This shows 3 odd holes and would commonly be referred to as a tripartite odd hole.

As expected, the results from the previous section extend trivially. Again, only vertices from one odd hole can be in the node packing. Thus,

Corollary 3.2.4. *Given a graph $G = (V, E)$ with an induced odd k-partite hole of the form $OkP_{p_1, ..., p_k}$, then $\sum_{i \in V(H_{p_1})} x_i + \sum_{i \in V(H_{p_2})} \frac{p_1 - 1}{p_2 - 1} x_i + ... + \sum_{i \in V(H_{p_k})} \frac{p_1 - 1}{p_k - 1} x_i \leq \lfloor \frac{p_1}{2} \rfloor$ is a valid inequality for the node packing polyhedron.*

$\square$

Clearly, the face defining results from the odd bipartite hole extend trivially. There exists many affinely independent points that meet the odd k-partite hole inequality at equality. Thus,
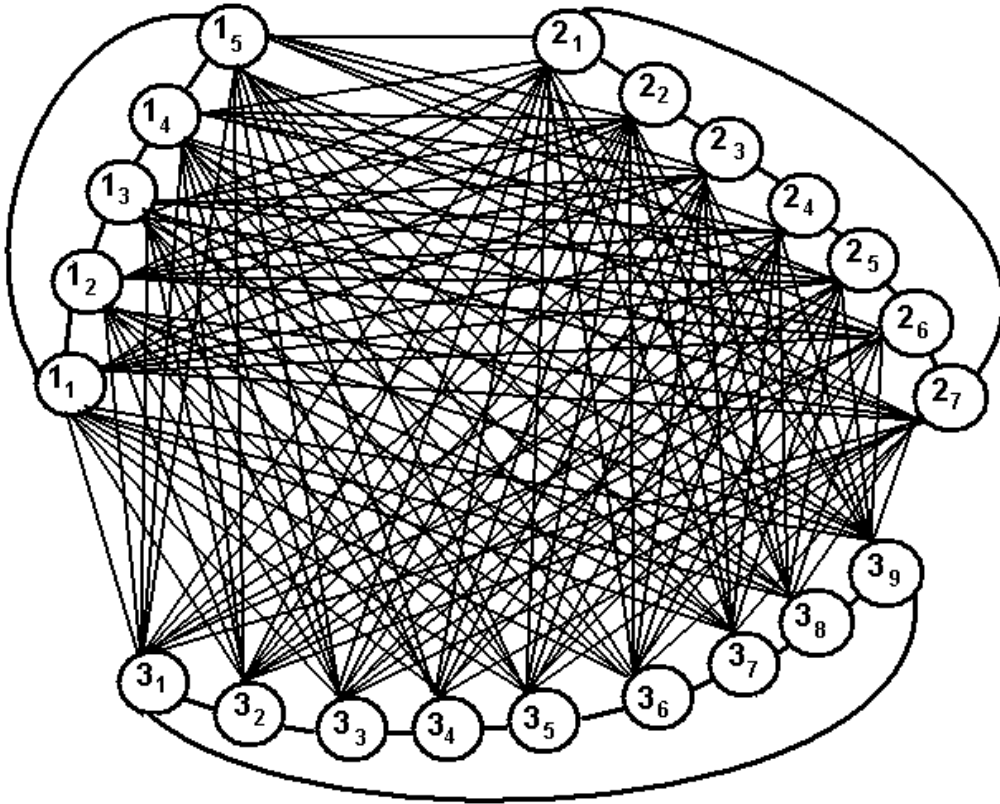
Figure 3.9: Odd Tripartite Hole

Corollary 3.2.5. *Given a graph $G = (V, E)$ with an odd k-partite hole of the form $OkP_{p_1,\ldots,p_k}$, then $\sum_{i \in V(H_{p_1})} x_i + \sum_{i \in V(H_{p_2})} \frac{p_1-1}{p_2-1} x_i + \ldots + \sum_{i \in V(H_{p_k})} \frac{p_1-1}{p_k-1} x_i \leq \lfloor \frac{p_1}{2} \rfloor$ defines a face of dimension at least $p_1 + p_2+, \ldots, +p_k - 1$ for the node packing polyhedron.*

□

Given an odd k-partite hole, it is trivial to find an affinely independent point for any peripheral point to show it is facet defining, as an extension of the previous section. Therefore,

Corollary 3.2.6. *Given a graph $G = (V, E)$ with an induced odd k-partite hole of the form $OkP_{p_1,\ldots,p_k}$, then $\sum_{i \in V(H_{p_1})} x_i + \sum_{i \in V(H_{p_2})} \frac{p_1-1}{p_2-1} x_i + \ldots + \sum_{i \in V(H_{p_k})} \frac{p_1-1}{p_k-1} x_i \leq \lfloor \frac{p_1}{2} \rfloor$ defines a facet of the node packing polyhedron, if for each $v_r \in V \setminus V(OkP_{p_1,\ldots,p_k})$, there exists a p_i for $i \in \{1, \ldots, k\}$ such that v_r is not adjacent to the nodes v_{i_s}, $v_{i_{(s+1) \ mod p_i + 1}}$, $v_{i_{(s+3) \ mod p_i + 1}}, \ldots, v_{i_{(s+\frac{p_i-3}{2}) \ mod p_i + 1}}$ for some $s \in \{1, \ldots, p_i\}$.*

□

In Figure 3.9, the valid facet defining inequality is $\sum_{i \in V(H_{p_1})} x_i + \sum_{i \in V(H_{p_2})} \frac{5-1}{7-1} x_i + \sum_{i \in V(H_{p_3})} \frac{5-1}{9-1} x_i \leq 2$. This simplifies down to $x_{1_1} + x_{1_2} + x_{1_3} + x_{1_4} + x_{1_5} + \frac{2}{3}(x_{2_1} + x_{2_2} + x_{2_3} + x_{2_4} + x_{2_5} + x_{2_6} + x_{2_7}) + \frac{1}{2}(x_{3_1} + x_{3_2} + x_{3_3} + x_{3_4} + x_{3_5} + x_{3_6} + x_{3_7} + x_{3_8} + x_{3_9}) \leq 2$. Observe that this is an example of simultaneous lifting of multiple sets. Kubik [51] provided a psuedopolynomial time technique to perform simultaneously lifting over multiple sets, which is clearly the case for this odd k-partitite hole.

Chapter 4

Conclusion and Future Research

The goal of this thesis was to develop, present, and investigate several new structures for the node packing problem. From this stemmed each structure's facet defining inequality, as well as interesting conditions in which these structures may exist. Additionally, it was found that fractional values could be used for some of the structures, which indicates the possibility for simultaneous fractional lifting for the node packing polytope.

A cliqued hole $CH_{m,P}$ with $P = (p_1, p_2, ..., p_m)$ where $\sum_{i=1}^{m} p_i = q$, $p_i \geq 1$ and $p_i \in \mathbf{Z}$ for all $i = 1, ..., m$ has the valid inequality $\sum_{i \in V(CH_{m,P})} x_i \leq \lfloor \frac{m}{2} \rfloor$. This inequality is of great interest, as it combines two structures with known inequalities into one unique, much stronger inequality. Conditions are also provided for this inequality to be facet defining.

The odd bipartite hole $OBP_{p,q}$ consists of two odd holes, H_p and H_q such that every vertex in H_p is adjacent to every vertex in H_q. The valid inequality generated for this structure is $\sum_{i \in V(H_q)} x_{q_i} + \sum_{i \in V(H_p)} \frac{p-1}{q-1} x_{p_i} \leq 2$. This structure is visually very stunning,

and conditions for these inequalities to be facet defining are provided. Furthermore, this structure can be expanded upon to become a odd k-partite hole.

4.1 Future Research

Several areas are of great interest for further research. Computational studies should be performed to determine if the inequalities formed through this thesis provide any computational advantages. If they do, then the amount of savings should also be determined.

Much of this research began in the interest of determining whether fractional simultaneous lifting was feasible and viable for PNP^{CH}. Section 4.3 provides a partial answer, but do other fractional simultaneous lifting structures exist?

Indeed, each structure can be modified in order to supply new and intriguing structures for academic investigation. Some that come to mind include removing a single or multiple edges in the OBP, and exploding the edges on the CH to a structure other than cliques, such as wheels or odd holes.

To help spur research in this area, we propose a modification of the odd bipartite hole. The basic idea is to not have all of the edges between the two holes. Consider the graph depicted in Figure 4.1. Notice how each vertex of the center hole is adjacent to six consecutive nodes in the outer hole in a specific manner.

This structure enables two valid inequalities. They are $2\sum_{i\in V(H_5)} x_i + \sum_{i\in V(H_{15})} x_i \leq$

7 and $3\sum_{i \in V(H_5)} x_i + \sum_{i \in V(H_{15})} x_i \leq 8$. Neither of these inequalities are facet defining, but they still induce fairly large faces. So what edges should be added or removed to generate facet defining inequalities. Furthermore, this structure begins the idea that one structure could potentially produce multiple inequalities, which would be a version of synchronized simultaneously lifting, which is described in Bolton's master's thesis [16]. Much research in this area could be beneficial and lead to new structures.
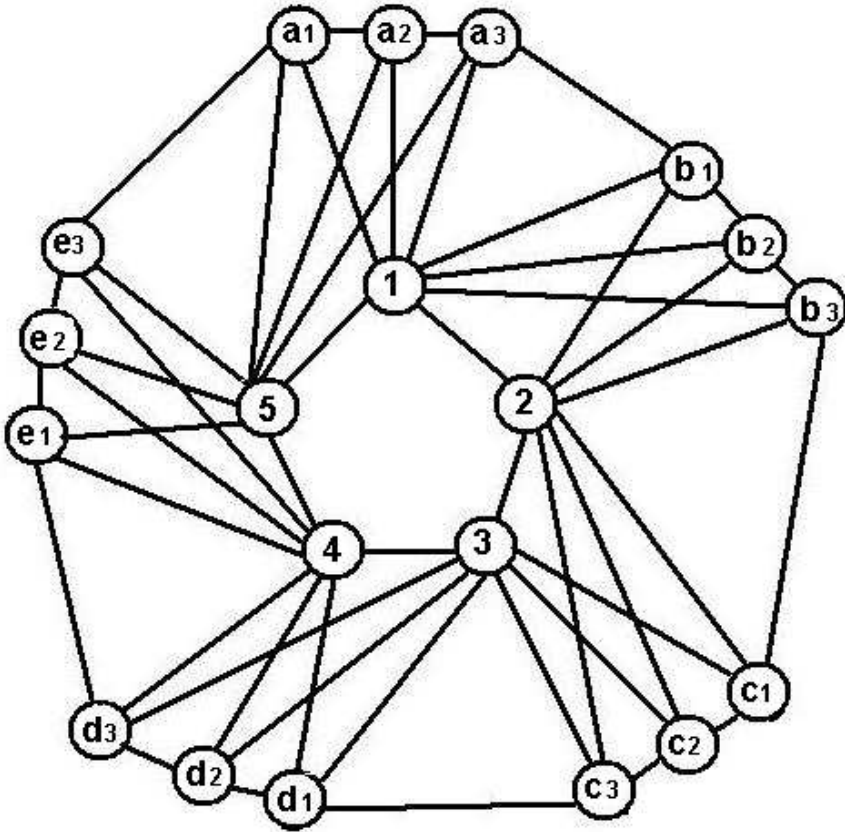
Figure 4.1: Moddified Odd Bipartite Hole

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