SOME ANALYTICAL SOLUTIONS OF MAGNETOHYDRODYNAMIC ENTRANCE REGION FLOW PROBLEM

by

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CHAPTER I

Introduction

Recently attention has been directed toward the effect of a magnetic field on the flow of an electrically conducting fluid, usually referred to as magnetohydrodynamics (MHD). Magnetohydrodynamic flow has many engineering applications such as magnetohydrodynamic generators, accelerators, electromagnetic flowmeter, electromagnetic pumps, and similar devices.

Let us now consider that the electrically conducting fluid is flowing with a steady state velocity $\underline{V}$, and that a magnetic field with field density $\underline{B}_{ap}$ is applied perpendicularly to the flow (Fig. 1). Because of the interaction of the flow and the magnetic fields, an electric field $\underline{E}_{ind}$ is induced perpendicular to both $\underline{V}$ and $\underline{B}_{ap}$.

This electric field is given by the following equation:

$$\underline{E}_{ind} = \underline{V} \times \underline{B}_{ap} \ . \qquad (1)$$

For simplification assume that the electrical conductivity, σ_e, is constant in spite of the magnetic field. By Ohm's Law the current density induced in the conducting fluid, and denoted by $\underline{J}_{ind}$ is:

$$\underline{J}_{ind} = \sigma_e \underline{E}_{ind} \ . \qquad (2)$$

Simultaneously occurring with the induced current is the induced Lorentz force $\underline{F}_{ind}$ which is given by the following:

$$\underline{F}_{ind} = \underline{J}_{ind} \times \underline{B}_{ap} \ . \qquad (3)$$

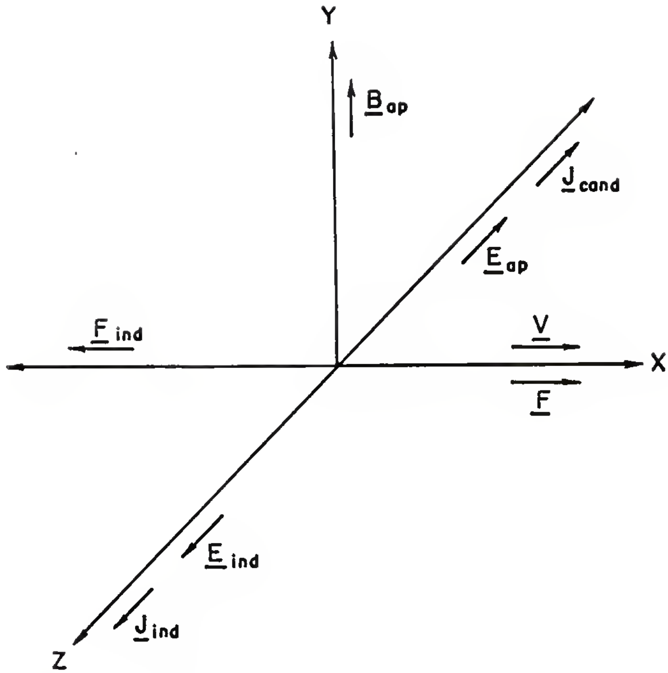

Fig. 1. Vector diagram of MHD.

This force $\underline{F}_{ind}$ occurs because the conducting fluid cuts the lines of the magnetic field. Because the vector product of equation (3) yields a vector perpendicular to both $\underline{J}_{ind}$ and $\underline{B}_{ap}$, the induced force is parallel to $\underline{V}$ but opposite in direction.

For the more general case, we further consider an electric field $\underline{E}_{ap}$ perpendicular to both $\underline{B}_{ap}$ and $\underline{V}$, but opposite in direction to $\underline{J}_{ind}$. The current density due to this applied electric field is $\underline{J}_{cond}$. The net current $\underline{J}$ through the conducting fluid is then

$$\begin{aligned} \underline{J} &= \sigma_e(\underline{E}_{ap} + \underline{V} \times \underline{B}_{ap}) \\ &= \sigma_e(\underline{E}_{ap} \times \underline{E}_{ind}) \; . \end{aligned} \tag{4}$$

The ponderomotive or Lorentz force associated with this current is then

$$\underline{F} = \underline{J} \times \underline{B}_{ap} = \sigma_e(\underline{E}_{ap} + \underline{V} \times \underline{B}_{ap}) \times \underline{B}_{ap} \; . \tag{5}$$

In equation (5), if $\underline{E}_{ap} > \underline{V} \times \underline{B}_{ap}$ the system is a magnetohydrodynamic accelerator (or pump) which may be used as a thrust-producing device. If $\underline{E}_{ap} < \underline{V} \times \underline{B}_{ap}$, it is a magnetohydrodynamic generator.

The equations of MHD flow of continuous fluid media are the ordinary electromagnetic and hydrodynamic equations, modified to take account of the interaction between the fluid motion and the magnetic field.

On the assumptions that: (1) the flow is laminar, (2) all fluid properties, ρ, C_p, K, ν, are constant, (3) the displacement current is negligible (as in most electromagnetic problems) i.e., no oscillations of very high frequency occur, (4) the permeability, μ_e, and conductivity, σ_e, are constant scalar quantities, and (5) the effect of gravitational force is negligible and the Lorentz force is the only body force on the fluid, the basic MHD equation can be written as follows (1, 2)*.

Maxwell's equations in cgs electromagnetic units:

$$\text{curl } \underline{H} = 4\pi \underline{J}, \qquad \text{div } \underline{J} = 0 \tag{6}$$

$$\text{curl } \underline{E} = -\mu_e \frac{\partial H}{\partial t}, \qquad \text{div } \underline{H} = 0. \tag{7}$$

Ohm's law for a moving fluid:

$$\underline{J} = \sigma_e(\underline{E} + \underline{V} \times \mu_e \underline{H}). \tag{8}$$

Continuity equation:

$$\text{div } \underline{V} = 0$$

The modified Navier-Stokes equation:

$$\frac{\partial \underline{V}}{\partial t} + (\underline{V} \text{ grad})\underline{V} = -\frac{1}{\rho} \text{grad } p + \nu\nabla^2\underline{V} + \frac{1}{\rho}(\underline{J} + \mu_e\underline{H}) \tag{9}$$

For steady-two-dimensional flow with the usual Prandtl boundary-layer assumptions that:

(1) $\frac{\partial^2 u}{\partial x^2}$ is very small compared with $\frac{\partial^2 u}{\partial y^2}$ so it can be neglected;

* Numbers in parentheses refer to references at the end of the chapter.

(2) the transverse velocity v is small in comparison with u; and

(3) consequently the pressure gradient $\frac{\partial p}{\partial x}$ is a function of x alone,

equation (9) is resolved into the following three equations in the x, y, and z-directions, respectively:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{\mu_e}{\rho}(J_y H_z - J_z H_y) , \tag{10}$$

$$0 = -\frac{\partial p}{\partial y} + \mu_e (J_z H_x - J_x H_z) , \tag{11}$$

$$0 = \mu_e (J_x H_y - J_y H_x) . \tag{12}$$

Next, Ohm's law may be written as:

$$J_x = \sigma_e (E_x + \mu_e v H_z) , \tag{13}$$

$$J_y = \sigma_e (E_y - \mu_e u H_z) , \tag{14}$$

$$J_z = \sigma_e (E_z + \mu_e u H_y - \mu_e v H_x) . \tag{15}$$

If the magnetic field is steady, $\frac{\partial H}{\partial t} = 0$, and from equation (7), curl $\underline{E} = 0$, that is

$$\frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} , \tag{16}$$

$$\frac{\partial E_z}{\partial x} = \frac{\partial E_x}{\partial z} , \tag{17}$$

$$\frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y} . \tag{18}$$

Equation (6) becomes:

$$4\pi J_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} , \qquad (19)$$

$$4\pi J_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} , \qquad (20)$$

$$4\pi J_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} . \qquad (21)$$

For the simplification of the foregoing equations further assumptions are made as follows: (1) Variations in the z-direction are assumed to be zero, (2) the electric field term, E_y, measured across the insulated duct walls is zero, but small local values may exist in the midstream region; however, these will be considered negligible and E_y is taken as zero at all points across the duct. This implies J_y is also zero.

Thus taking $E_y = 0$, $J_y = 0$, and $\frac{\partial}{\partial z} = 0$, in Ohm's law and Maxwell's equations, it can be shown that

$$E_z = \text{constant} = E_0 ,$$

$$B_y = \text{constant magnetic field applied} = B_0 ,$$

$$H_z = E_x = J_x = 0 .$$

Equations (10), (11), and (12) reduce to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma_e B_0}{\rho} (E_0 + uB_0 - vB_x) , \qquad (22)$$

$$\frac{\partial p}{\partial y} = \sigma_e B_x (E_0 + uB_0 - vB_x) , \qquad (23)$$

$$0 = 0 . \qquad (24)$$

Equation (21) may be written as,

$$4\pi\sigma_e(E_0 + uB_0 - vB_x) = -\frac{\partial H_x}{\partial y} . \tag{25}$$

Unlike the case of the fully developed velocity profile, the foregoing equations are coupled by the appearance of v, the transverse-velocity component. Since the applied external magnetic field has no x- or z-component, and $B_z = 0$, it follows that the component of the magnetic induction in the x-direction, B_x, is induced only by J_z. It is assumed that B_x is negligible in comparison with the applied field B_0. These assumptions reduce the number of the equations to two:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{26}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma_e B_0}{\rho}(E_0 + uB_0) \tag{27}$$

which become the basic governing equations for magnetohydrodynamic channel flow.

In this report we consider a laminar flow of a conducting fluid with constant properties entering a semi-infinite, nonconducting flat duct with a normal transverse applied magnetic field. The initial velocity profile is assumed to be uniform or a nonmagnetically fully developed parabolic profile, and at a large distance downstream from the entrance, the velocity profile is the fully developed Hartmann profile. There is also an external variable resistance connecting the two perfectly conducting end plates (electrodes) which are displaced to infinity (Fig. 2).

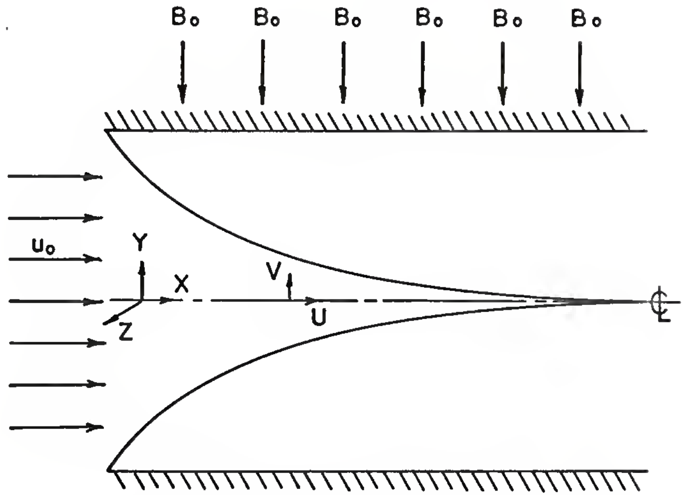

Fig. 2 Geometry of parallel plate channel at entrance region with transverse magnetic field and uniform velocity.

In steady, laminar, non-magnetohydrodynamic flow in the entrance of the channel, boundary layers form at both walls due to viscous friction. Near the entrance, these boundary layers form much like those for flow over a flat plate. However, in channel flow the restriction of the flow near the walls is compensated by an increase in the rate of flow in the center of the channel. The boundary layers then increase in size as the flow progresses down the channel, until they merge into each other to asymptotically form the Poiseuille velocity profile.

In the case of flow of a conducting fluid under the effect of a transverse magnetic field, <u>i.e.</u>, the magnetohydrodynamic flow, the boundary layers will form on both channel walls in a similar manner, but the force due to the magnetic field adds an effect to the flow. As in the non-magnetic case, the thickness of the boundary layers will gradually increase as the velocity profile asymptotically approaches the Hartmann fully developed profile.

In many magnetohydrodynamic applications, <u>e.g.</u>, MHD generators, liquid metal pumps, plasma engines, etc., the flow of fluid is seldom fully developed and is of the boundary layer type with variable longitudinal pressure gradients. For this reason a study of the velocity fields, boundary layer development, and friction factors in the entrance region of MHD channel is of practical importance and has been a subject of investigation in recent years.

In this entrance region we wish to obtain expressions for:

(1) the pressure drop between any two sections, (2) the velocity distribution at any section, and (3) the value of x at which the fully developed flow is attained.

The two simultaneous equations, equations (26) and (27) with appropriate boundary conditions, are sufficient to solve for the unknowns, u and v. The pressure ceases to be an unknown function since it can now be evaluated from the potential flow in the central core by the aid of the Bernoulli equation.

Although equations (26) and (27) are much simpler than the original equations representing governing equations, their exact analytical solutions have not been found so far. However, a number of different approximate solutions has been published in channel entrance flow problem. Hwang and Fan have published a bibliography of hydrodynamic entrance region flow, a thorough review and classification of the literature on the subject which has appeared in the last hundred years (3).

The approximate solutions may be classified in four general categories -- the momentum integral method, linearization method, matching method, and finite difference method.

In the momentum integral method the flow is divided into a boundary layer part near the wall and a potential flow part in the central core. A parabolic velocity profile (or any other similar velocity profile) is assumed in the boundary layer and is joined with the center core velocity profile which is assumed to be a straight line. A momentum integral equation is derived based on the momentum conservation principle. This approach was devised and applied by Schiller (4) for flow in a circular tube and is similar to the Karman-Pohlhausen momentum integral method applied to a flat plate. Schiller's method was applied to the magnetohydrodynamic flow by Maciulaitis and Loeffler (5), by Moffatt (6) in 1964, and by Tan (7) in 1965.

In the second category of solutions, the inertia terms of boundary layer equations are linearized. This category of solutions is capable of providing continuous solutions for the velocity of distribution and pressure drop in the entrance region of nonmagnetic flow. This class of solutions for circular tubes is mainly due to the work of Langhaar (8, 9) in 1940 and Targ (10) in 1951. This category of solutions was applied to magnetohydrodynamic flow in the entrance region by Hsueh (11) in 1963 and by Snyder (12) in 1965.

The third group of solutions is constructed by matching the boundary layer solutions which are valid near the entrance with the perturbations of the fully developed solutions which are valid for downstream. This class of solution was originally given by Schlichting (13) in 1934 for a flat duct. This matching method was applied to magnetohydrodynamic flow at the entrance region by Barsness (14) in 1960 and by Roidt and Cess (15) in 1962.

The fourth approach involves reduction of the continuity and momentum equations to finite difference equations which are solved numerically on an electronic digital computer. This method was used by Bodoia and Osterle (16) in 1961. Hwang and

Fan (17) in 1963, Shohat *et al*. (18) in 1962, and Hwang, Li and Fan (19) in 1966 applied this method to MHD channel flow.

The purpose of this report is to study laminar magnetohydrodynamic flow in the entrance region of a flat duct by presenting in detail the solutions according to the momentum integral method (5), Targ's linearization method (12), and the matching method (15).

NOMENCLATURE

a	Channel half-height
B	Magnetic induction
e	Electric field magnitude factor, $E_0 = - eu_0B_0$
E	Electric field intensity
H	Magnetic field intensity
j	Electric current density
M	Hartmann number, $M = Ba(\frac{\sigma}{\rho\nu})^{\frac{1}{2}}$
p	Fluid pressure
u	x-component of velocity
v	y-component of velocity
x, y, z	Space coordinate
ν	Kinematic viscosity
ρ	Fluid density
μ	Dynamic viscosity
μ_e	Magnetic permeability
σ_e	Electric conductivity

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CHAPTER II

Matching Method (1)

In this analysis we consider the laminar flow of an electrically conductive fluid with constant properties entering a semi-infinite, nonconducting flat duct with a normal transverse applied magnetic field. The fluid is assumed to enter the duct with a uniform velocity profile. There is an external variable resistance connecting the two perfectly conducting end plates (electrodes) which are displaced to infinity.

The process of analyzing the velocity field in the entrance region is patterned after a method developed by Schlichting (2) for non-MHD flow in which the flow is divided into two sections: flow near the inlet or upstream section, and a downstream section which approaches the fully developed flow. In the upstream section, the flow is analogous to boundary layer flow over a flat plate with a pressure gradient; hence, the solution is modeled after the Blasius series solution. In the downstream section, the velocity distribution is assumed to be the sum of the fully developed profile and a deviation of the profile from its asymptotic parabolic distribution. The integration is then performed in the upstream direction in order to find this deviation velocity. After obtaining both solutions in the form of series expansions, they are joined at the point where both solutions are valid. In this way an approximate description of the flow field in the entire magnetohydrodynamic entrance region is obtained.

For the magnetohydrodynamic flow the governing equations are

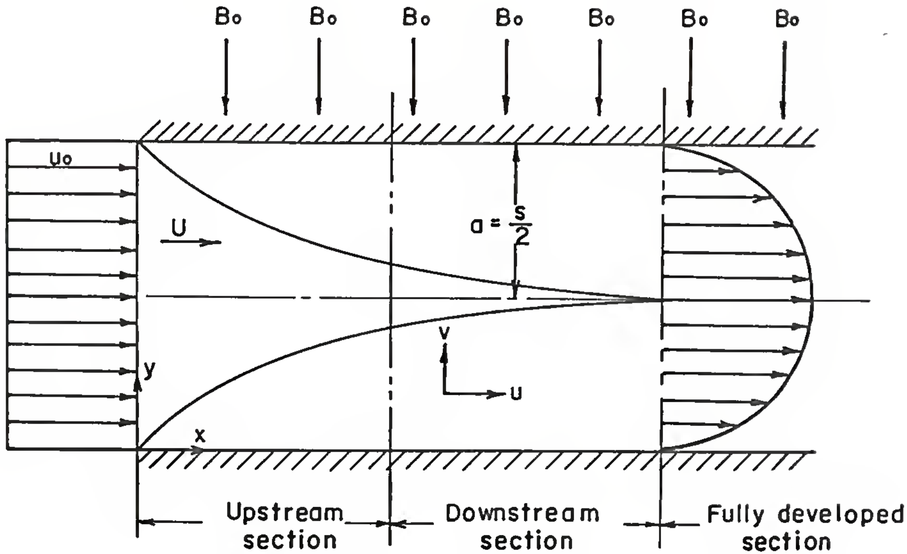

Fig. 1 Configuration of an MHD duct-entrance region with parallel plate for matching method.

(see Chapter I):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{1}$$

and

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0}{\rho} (E_0 + uB_0) . \tag{2}$$

Upstream Solution

The upstream section is divided into the subregions of inviscid potential flow and viscous boundary layer flow. In the potential flow where u=U, v=0, and U is only a function of x, equation (2) becomes

$$U \frac{dU}{dx} = -\frac{1}{\rho} \frac{dp}{dx} - \frac{\sigma B_0^2}{\rho} U - \frac{\sigma B_0 E_0}{\rho} . \tag{3}$$

Solution of equation (3) for $\frac{dp}{dx}$ and substitution into equation (2) yields

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \frac{\sigma B_0^2}{\rho} (U-u) + \nu \frac{\partial^2 u}{\partial y^2} . \tag{4}$$

Let the transformation variable ξ be

$$\xi = \sqrt{\frac{\nu x}{s^2 u_0}} .$$

and assume the potential flow velocity, U, to be in the following form:

$$U(x) = u_0(K_0 + K_1\xi + K_2\xi^2 + K_3\xi^3 + \ldots) \tag{5}$$

in which K_0 is equal to one, since $U = u_0$ at $\xi = 0$, and K_1, K_2, ... are unknown constants to be determined.

Let the other transformation variable η be

$$\eta = \frac{y}{2}\sqrt{\frac{u_0}{\nu x}}$$

and assume that the stream function, $\psi(x,y)$, can be expanded into a series in ξ in a form similar to that for U(x). Thus,

$$\psi = u_0 s\left\{\xi f_1(\eta) + \xi^2 f_2(\eta) + \xi^3 f_3(\eta) + \xi^4 f_4(\eta) + \ldots\right\} \quad (6)$$

where s is the width, and the f_n are functions of η only. To satisfy continuity equation, equation (1), u and v are given by

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \quad (7)$$

From equation (5) we have

$$U\frac{dU}{dx} = u_0(1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + \ldots)\left\{u_0(K_1 + 2\xi K_2 + 3\xi^2 K_3 + \ldots)\right\}\frac{d\xi}{dx}$$

where

$$\frac{d\xi}{dx} = \frac{1}{2}\sqrt{\frac{\nu}{s^2 u_0 x}} = \frac{1}{2}\sqrt{\frac{\nu x}{s^2 u_0}}/x = \frac{1}{2}\frac{\xi}{x}.$$

Therefore, we have

$$U\frac{dU}{dx} = \frac{u_0^2\xi}{2x}(1 + K_1\xi + K_2\xi^2 + \ldots)(K_1 + 2\xi K_2 + 3\xi^2 K_3 + \ldots)$$

$$= \frac{u_0^2\xi}{2x}\left\{K_1 + (K_1^2 + 2K_2)\xi + 3(K_1K_2 + K_3)\xi^2 + \ldots.\right\}. \quad (8)$$

From equations (6) and (7) we have

$$u(x,y) = \frac{\partial\psi}{\partial y} = u_0 s \left\{ \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \ldots \right\} \frac{\partial\eta}{\partial y} .$$

where

$$\frac{\partial\eta}{\partial y} = \frac{1}{2}\sqrt{\frac{u_0}{\nu x}} = \frac{1}{2}\sqrt{\frac{s^2 u_0}{\nu x}}/s = \frac{1}{2s\xi} .$$

Therefore, we have

$$u(x,y) = \frac{u_0}{2}\left\{ f_1'(\eta) + \xi f_2'(\eta) + \xi^2 f_3'(\eta) + \ldots \right\} . \tag{9}$$

Similarly, from equations (6) and (7) we obtain

$$\begin{aligned} v(x,y) &= -\frac{\partial\psi}{\partial x} \\ &= -u_0 s \left\{ f_1(\eta) + 2\xi f_2(\eta) + 3\xi^2 f_3(\eta) + \ldots \right\} \frac{d\xi}{dx} \\ &\quad -u_0 s \left\{ \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \xi^3 f_3'(\eta) + \ldots \right\} \frac{d\eta}{dx} \end{aligned}$$

where

$$\frac{d\xi}{dx} = \frac{1}{2}\frac{\xi}{x} ,$$

$$\frac{\partial\eta}{\partial x} = \frac{y}{2}\sqrt{\frac{u_0}{\nu}}\left(-\frac{1}{2}x^{-3/2}\right) = -\frac{\eta}{2x} .$$

Therefore, the transverse velocity v(x,y) becomes

$$\begin{aligned} v(x,y) = &-\frac{u_0 \xi s}{2x}\left\{ f_1(\eta) + 2\, f_2(\eta) + 3\xi^2 f_3(\eta) + \ldots \right\} \\ &+ \frac{u_0 s \xi \eta}{2x}\left\{ f_1'(\eta) + \xi f_2'(\eta) + \xi^2 f_3'(\eta) + \ldots \right\} . \end{aligned} \tag{10}$$

From equation (9) we obtain

$$\frac{\partial u}{\partial x} = \frac{u_0}{2}\left\{f_2'(\eta) + 2\xi f_3(\eta) + 3\xi^2 f_4'(\eta) + \ldots\right\}\frac{d\xi}{dx}$$

$$+ \frac{u_0}{2}\left\{f_1''(\eta) + \xi f_2''(\eta) + \xi^2 f_3''(\eta) + \ldots\right\}\frac{\partial\eta}{\partial x}$$

$$= \frac{u_0\xi}{4x}\left\{f_2'(\eta) + 2\xi f_3(\eta) + 3\xi^2 f_4'(\eta) + \ldots\right\}$$

$$- \frac{u_0}{4x}\left\{f_1''(\eta) + \xi f_2''(\eta) + \xi^2 f_3''(\eta) + \ldots\right\}. \tag{11}$$

$$\frac{\partial u}{\partial y} = \frac{u_0}{2}\left\{f_1''(\eta) + \xi f_2''(\eta) + \xi^2 f_3''(\eta) + \ldots\right\}\frac{\partial\eta}{\partial y}$$

$$= \frac{u_0}{4s\xi}\left\{f_1''(\eta) + \xi f_2''(\eta) + \xi^2 f_3''(\eta) + \ldots\right\}. \tag{12}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_0}{4s\xi}\left\{f_1'''(\eta) + \xi f_2'''(\eta) + \xi^2 f_3'''(\eta) + \ldots\right\}\frac{\partial\eta}{\partial y}$$

$$= \frac{u_0}{8s^2\xi^2}\left\{f_1'''(\eta) + \xi f_2'''(\eta) + \xi^2 f_3'''(\eta) + \ldots\right\}. \tag{13}$$

Substituting equations (5) through (13) into equation (4), we obtain

$$\frac{\xi u_0^2}{8x}\left\{f_1'(\eta) + f_2'(\eta) + \xi^2 f_3'(\eta) + \ldots\right\}\left\{f_2'(\eta) + 2\xi f_3'(\eta) + 3\xi^2 f_4'(\eta) + \ldots\right\}$$

$$- \frac{\eta u_0^2}{8x}\left\{f_1'(\eta) + \xi f_2'(\eta) + \xi^2 f_3'(\eta) + \ldots\right\}\left\{f_1''(\eta) + \xi f_2''(\eta) + \xi^2 f_3''(\eta) + \ldots\right\}$$

$$- \frac{u_0^2}{8x}\left\{f_1''(\eta) + f_2''(\eta) + \xi^2 f_3''(\eta) + \ldots\right\}\left\{f_1(\eta) + 2\xi f_2(\eta) + 3\xi^2 f_3(\eta) + \ldots\right\}$$

$$+ \frac{\eta u_0^2}{8x}\left\{f_1''(\eta)+\xi f_2''(\eta)+\xi^2 f_3''(\eta)+\ldots\right\}\left\{f_1'(\eta)+\xi f_2'(\eta)+\xi^2 f_3'(\eta)+\ldots\right\}$$

$$= \frac{\xi u_0^2}{2x}\left\{K_1 + (K_1^2 + 2K_2)\xi + 3(K_1K_2 + K_3)\xi^2 + \ldots\right\}$$

$$+ \frac{\sigma_e B_0^2 u_0}{\rho}\left\{(1+K_1\xi+K_2\xi^2+K_3\xi^3+\ldots) - \frac{1}{2}[f_1'(\eta)+\xi f_2'(\eta) +\xi^2 f_3'(\eta)+\ldots]\right\}$$

$$+ \nu\frac{u_0}{8s^2\xi^2}\left\{f_1'''(\eta)+\xi f_2'''(\eta)+\xi^2 f_3'''(\eta)+\ldots\right\}$$

where

$$\frac{\sigma_e B_0^2 u_0}{\rho} = \left(\frac{\sigma_e B_0^2 a^2}{\rho\nu}\right)\left(\frac{\nu}{a^2}\right) u_0$$

$$= M^2 \left(\frac{4}{s^2}\right)\left(\frac{s^2\xi^2 u_0}{x}\right) u_0$$

$$= M^2 \left(\frac{4\xi^2 u_0^2}{x}\right)$$

and

$$\frac{\nu u_0}{8s^2\xi^2} = \frac{\nu u_0}{8s^2 \frac{\nu x}{s^2 u_0}} = \frac{u_0^2}{8x}\,.$$

Substituting these into the above equations and simplifying, we obtain

$$\xi\left\{f_1'(\eta)+\xi f_2'(\eta)+\xi^2 f_3'(\eta)+\dots\right\}\left\{f_2'(\eta)+2\xi f_3'(\eta)+3\xi^2 f_4'(\eta)+\dots\right\}$$

$$-\left\{f_1''(\eta)+\xi f_2''(\eta)+\xi^2 f_3''(\eta)+\dots\right\}\left\{f_1(\eta)+2\xi f_2(\eta)+3\xi^2 f_3(\eta)+\dots\right\}$$

$$= 4\xi\left\{K_1 + (K_1^2 + 2K_2)\xi + 3(K_1K_2 + K_3)\xi^2 + \dots\right\}$$

$$+32M^2\xi^2\left\{(1+K_1\xi+K_2\xi^2+K_3\xi^3+\dots) - \tfrac{1}{2}(f_1'(\eta)+\xi f_2'(\eta)+\xi^2 f_3'(\eta)+\dots)\right\}$$

$$+\left\{f_1'''(\eta)+\xi f_2'''(\eta)+\xi^2 f_3'''(\eta)+\dots\right\}. \qquad (14)$$

Equating like powers of ξ gives

for ξ^0,

$$f_1'''(\eta) + f_1''(\eta)f(\eta) = 0 \qquad (15)$$

for ξ^1,

$$f_1'(\eta)f_2'(\eta) - 2f_1''(\eta)f_2(\eta) - f_2''(\eta)f_1(\eta) - f_2'''(\eta) = 4K_1 \qquad (16)$$

for ξ^2,

$$f_2'^2(\eta)+2f_1'(\eta)f_3'(\eta) - 3f_3(\eta)f_1''(\eta) - 2f_2(\eta)f_2''(\eta) - f_1(\eta)f_3''$$

$$= 8K_2 + 4K_1^2 + f_3'''(\eta) + 16M^2(2 - f_1'(\eta)) \qquad (17)$$

for ξ^3,

$$3f_1'(\eta)f_4'(\eta) + 3f_2'(\eta)f_3'(\eta) - 4f_1''(\eta)f_4(\eta) - 3f_3(\eta)f_2''(\eta)$$

$$- 2f_2(\eta)f_3''(\eta) - f_1(\eta)f_4''(\eta)$$

$$= 12K_3 + 12K_1K_2 + f_4'''(\eta) + 16M^2(2K - f_2'(\eta)) \qquad (18)$$

where $M = (\sigma_e/\rho\nu)^{\frac{1}{2}}\, B_0\, \frac{S}{2}$ is the Hartmann number. The boundary conditions for equations (15) through (18) are determined from $u=0$ and $v=0$ at the wall ($y=0$), and become

$$f_1(0) = f_2(0) = \dots = f_n(0) = 0$$

$$f_1'(0) = f_2'(0) = \dots = f_n'(0) = 0$$

At the edge of the boundary layer, $\eta \to \infty$ and $u = U$, so that equating equations (5) and (9) gives

$$u_0(K_0 + K_1\xi + K_2\xi^2 + K_3\xi^3 + \dots)$$

$$= \frac{u_0}{2} \; f_1'(\eta) + \xi f_2'(\eta) + \xi^2 f_3'(\eta) + \dots$$

$$f_1'(\eta) + \xi f_2'(\eta) + \xi^2 f_3'(\eta) + \xi^3 f_4'(\eta) + \dots$$

$$= 2 + 2K_1\xi + 2K_2\xi^2 + 2K_3\xi^3 + \dots \tag{19}$$

Collecting coefficients of like powers of ξ yields

$$f_1'(\infty) = 2$$

$$f_2'(\infty) = 2K_1$$

$$\vdots$$

$$f_n'(\infty) = 2K_{n-1} .$$

For large η, the foregoing relations may be integrated to yield the approximate relations

$$
\begin{aligned}
f_1 &\sim 2\eta + A_1 \\
f_2 &\sim 2K_1\eta + A_2 \\
&\vdots \\
f_n &\sim 2K_{n-1}\eta + A_n \; .
\end{aligned}
\tag{20}
$$

Further, from continuity, one obtains

$$u_0 a = \int_0^a u\,dy = \left[\psi(x,y)\right]_{y=a} \; .$$

At the center line of the duct, η is large and equations (20) are valid. In addition, at y=a one finds that

$$
\begin{aligned}
\eta &= \frac{\frac{s}{2}}{2} \sqrt{\frac{u_0}{\nu x}} \\
&= \frac{1}{4}\sqrt{\frac{u_0 s^2}{\nu x}} = \frac{1}{4\xi} \; .
\end{aligned}
$$

Using these relations in the foregoing equation together with equations (6) and (20) yields

$$\psi(x,y) = 2u_0 a \left\{ \xi f_1(\eta) + \xi^2 f_2(\eta) + \xi^3 f_3(\eta) + \ldots \right\}$$

or

$$1 = 2\left\{ \xi\left(\frac{1}{2\xi} + A_1\right) + \xi^2\left(\frac{K_1}{2\xi} + A_2\right) + \xi^3\left(\frac{K_2}{2\xi} + A_3\right) + \ldots \right\}$$

or

$$1 = 1 + 2\xi A_1 + K_1\xi + 2A_2\xi^2 + \xi^2 K_2 + 2A_3\xi^3 + \ldots \; ,$$

and thus,

$$K_1 = -2A_1 , \qquad K_2 = -2A_2 , \ldots , K_n = -2A_n .$$

Substituting these into equations (20), we obtain

$$f_n = 2K_{n-1}\eta - \frac{K_n}{2}$$

for large values of η. The K_n-values needed in each equation can be determined from the solution of the preceding equation of the set (15) to (18), and the entire upstream flow field is thus described.

Downstream Solution

After Schlichting, the velocity in the downstream region is taken as

$$u = u_f(y') + u'(x,y') , \tag{22}$$

in which $u_f(y')$ and $u'(x,y')$ are the fully developed velocity and deviation velocity, respectively.

In the region of fully developed flow, the momentum equation may be written

$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma_e B_0^2 u}{\rho} - \frac{\sigma_e B_0 E_0}{\rho} \tag{23}$$

and the solution by Hartmann (3) is

$$u_f = \frac{dp/dx + \sigma_e B_0 E_0}{\sigma_e B_0^2}\left(1 - \frac{\cosh(My'/a)}{\cosh M}\right) \tag{24}$$

where $M = \left(\frac{\sigma_e}{\rho\nu}\right)^{\frac{1}{2}} B_0 a$, the Hartmann number.

Integrating equation (24) from zero to a with respect to y' and remembering the assumptions that dp/dx, σ_e, B_0, and E_0 are constant, we obtain

$$\int_0^a u_f dy' = \frac{dp/dx + \sigma B_0 E_0}{\sigma_e B_0^2} \int_0^a \left[1 - \frac{\cosh(\frac{My'}{a})}{\cosh M}\right] dy'$$

$$= \frac{dp/dx + \sigma B_0 E_0}{\sigma_e B_0^2} \left[y' - \frac{a}{M\cosh M} \sinh(\frac{My'}{a})\right]_0^a$$

$$= \frac{a(\frac{dp}{dx} + \sigma B_0 E_0)}{\sigma_e B_0^2} \left[\frac{M - \tanh M}{M}\right]. \qquad (25)$$

The continuity integral

$$\int_0^a u_f dy' = u_0 a$$

can be introduced into the left hand side of equation (25) to reduce it to the form

$$u_0 a = \frac{a(\frac{dp}{dx} + \sigma_e B_0 E_0)}{\sigma_e B_0^2} \left[\frac{M - \tanh M}{M}\right]$$

from which we can obtain

$$\frac{\frac{dp}{dx} + \sigma_e B_0 E_0}{\sigma_e B_0^2} = \frac{u_0 M}{M - \tanh M} .$$

Substituting the last expression back into equation (24) we finally obtain

$$u_f(y') = \frac{u_0 M}{M - \tanh M}\left[1 - \frac{\cosh(M \frac{y'}{a})}{\cosh M}\right]. \tag{26}$$

When M is equal to zero, this can be written as

$$\lim_{M \to 0} u_f(y) = \lim_{M \to 0} \frac{u_0 M}{M - \tanh M}\left[1 - \frac{\cosh(M \frac{y'}{a})}{\cosh M}\right]$$

$$= \lim_{M \to 0} \frac{u_0 M[\cosh M - \cosh(M \frac{y'}{a})]}{[M - \tanh M][\cosh M]}.$$

This can be evaluated according to L'Hospital's rule as follows

$$\lim_{M \to 0} \frac{u_0 M[\cosh M - \cosh(M \frac{y'}{a})]}{[M - \tanh M]\cosh M}$$

$$= \lim_{M \to 0} \frac{u_0[\cosh M - \cosh(M \frac{y'}{a})] + u_0 M[\sinh M - (\frac{y'}{a})\sinh(M \frac{y'}{a})]}{(1-\mathrm{sech}^2 M)\cosh M - (M - \tanh M)\sinh M}$$

$$= \lim_{M \to 0} \frac{u_0[\sinh M - (\frac{y'}{a})^2 \sinh(M \frac{y'}{a})] + u_0[\sinh M - (\frac{y'}{a})\sinh(M \frac{y'}{a})] +}{2\mathrm{sech}M\mathrm{sech}M\tanh M\cosh M + (1-\mathrm{sech}^2 M)\sinh M + (1-\mathrm{sech}^2 M)}$$

$$\frac{u_0 M[\cosh M - (\frac{y'}{a})^2 \cosh(M \frac{y'}{a})]}{\sinh M + (M - \tanh M)\cosh M}$$

$$= \lim_{M \to 0} \frac{u_0[\cosh M - (\frac{y'}{a})^2 \cosh(M \frac{y'}{a})] + u_0[\cosh M - (\frac{y'}{a})^2 \cosh(M \frac{y'}{a})] +}{2\mathrm{sech}^3 M - 2\tanh M\mathrm{sech}M\tanh M + 2\mathrm{sech}^2 M\tanh M\sinh M + (1-\mathrm{sech}^2 M)}$$

$$\frac{u_0[\cosh M - (\frac{y'}{a})^2 \cosh(M \frac{y'}{a})] + u_0 M[\sinh M - (\frac{y'}{a})^3 \sinh(M \frac{y'}{a})]}{\cosh M + 2\mathrm{sech}^2 M^2 \tanh M\sinh M + (1-\mathrm{sech}^2 M)\cosh M + (1-\mathrm{sech}^2 M)}$$

$$\overline{\cosh M + (M - \tanh M)\sinh M}$$

$$= \frac{u_0\left[1-(\frac{y'}{a})^2\right] + u_0\left[1-(\frac{y'}{a})^2\right] + u_0\left[1-(\frac{y'}{a})^2\right]}{2}$$

$$= \frac{3}{2}\, u_0\left[1-(\frac{y'}{a})^2\right] .$$

That is

$$u_f(y) = \frac{3}{2}\, u_0\left[1-(\frac{y'}{a})^2\right] \qquad (27)$$

which is the equation for the velocity profile for non-magnetic, fully-developed channel flow.

For the deviation velocity $u'(x,y')$, Schlichting assumed a series of the form

$$u'(x,y') = u_0 C_1 e^{-4\lambda_1 \xi^2}\, \varphi_1'(y'/a) + u_0 C_2 e^{-4\lambda_2 \xi^2}\, \varphi_2'(y'/a) + \ldots \qquad (28)$$

where the constants C and λ, the function $\varphi(y'/a)$, and the region over which (28) holds are to be determined. In the downstream section the deviation velocity is never very large and hence the higher order terms in (28) may be neglected as an approximation. Thus,

$$u'(x,y') = u_0 C_1 e^{-4\lambda_1 \xi^2}\, \varphi_1'(y'/a) . \qquad (29)$$

Substituting equation (22) into (2), we have

$$(u_f+u')\left(\frac{\partial u'}{\partial x}\right) + v\left(\frac{du_f}{dy'} + \frac{\partial u'}{\partial y'}\right)$$

$$= -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{d^2u_f}{dy'^2} + \frac{\partial^2 u'}{\partial y'^2}\right)$$

$$- \frac{\sigma_e B_0^2 u_f}{\rho} - \frac{\sigma B_0^2 u'}{\rho} - \frac{\sigma_e B_0 E_0}{\rho} \qquad (30)$$

Differentiating equation (30) with respect to y', the pressure term vanishes because it is a function of x alone.

$$\frac{du_f}{dy'}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y'}\frac{\partial u'}{\partial x} + (u_f+u')\frac{\partial^2 u'}{\partial x \partial y'}$$

$$+ \frac{\partial v}{\partial y'}\left(\frac{du_f}{dy'} + \frac{\partial u'}{\partial y'}\right) + v\left(\frac{d^2u_f}{dy'^2} + \frac{\partial^2 u'}{\partial y'^2}\right)$$

$$= \nu\left(\frac{d^3u_f}{dy'^3} + \frac{\partial^3 u'}{\partial y'^3}\right) - \frac{\sigma_e B_0^2}{\rho}\left(\frac{du_f}{dy'} + \frac{\partial u'}{\partial y'}\right) - \frac{\sigma_e B_0}{\rho}\frac{\partial E_0}{\partial y'} \qquad (31)$$

where $\frac{\partial E_0}{\partial y'} = 0$ since E_0 is assumed to be a function of z-direction only.

From equation (22), we have

$$\frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x}\,. \qquad (32)$$

And according to the continuity equation, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y'} = 0$, equation (32) becomes

$$\frac{\partial v}{\partial y'} = -\frac{\partial u'}{\partial x}\,. \qquad (33)$$

Substituting equation (33) into equation (31) and simplifying, we obtain

$$(u_f+u')\frac{\partial^2 u'}{\partial x\,\partial y'} + v\left(\frac{d^2u_f}{dy'^2}+\frac{\partial^2 u'}{\partial y'^2}\right)$$

$$= \nu\left(\frac{d^3u_f}{dy'^3}+\frac{\partial^3 u'}{\partial y'^3}\right) - \frac{\sigma_e B_0^2}{\rho}\left(\frac{du_f}{dy'}+\frac{\partial u'}{\partial y'}\right) \tag{34}$$

From equation (29), we have

$$\frac{\partial u'}{\partial x} = u_0 C_1 e^{-4\lambda_1\xi^2}\,\varphi_1'\left(\frac{y'}{a}\right)(-8\lambda_1\xi)\,\frac{d\xi}{dx}$$

$$= u_0 C_1 e^{-4\lambda_1\xi^2}\,\varphi_1'\left(\frac{y'}{a}\right)(-8\lambda_1\xi)\left(\frac{1}{2}\,\frac{\xi}{x}\right)$$

$$= \frac{-4\lambda_1\xi^2}{x}\,u_0 C_1 e^{-4\lambda_1\xi^2}\,\varphi_1'\left(\frac{y'}{a}\right) . \tag{35}$$

$$\frac{\partial u'}{\partial y'} = \frac{u_0 C_1}{a}\,e^{-4\lambda_1\xi^2}\,\varphi_1''\left(\frac{y'}{a}\right) \tag{36}$$

$$\frac{\partial^2 u'}{\partial x\,\partial y'} = -\frac{4u_0\lambda_1 C_1\xi^2}{ax}\,e^{-4\lambda_1\xi^2}\,\varphi_1''\left(\frac{y'}{a}\right) \tag{37}$$

$$\frac{\partial^2 u'}{\partial y'^2} = \frac{u_0 C_1}{a^2}\,e^{-4\lambda_1\xi^2}\,\varphi_1'''\left(\frac{y'}{a}\right) \tag{38}$$

$$\frac{\partial^3 u'}{\partial y'^3} = \frac{u_0 C_1}{a^3}\,e^{-4\lambda_1\xi^2}\,\varphi_1^{iv}\left(\frac{y'}{a}\right) . \tag{39}$$

From equation (26), we have

$$\frac{du_f}{dy'} = -\frac{u_0 M^2}{a(M-\tanh M)}\left(\frac{\sinh\left(\frac{My'}{a}\right)}{\cosh M}\right) \tag{40}$$

$$\frac{d^2 u_f}{dy'^2} = - \frac{u_0 M^3}{a^2(M-\tanh M)} \left(\frac{\cosh(\frac{My'}{a})}{\cosh M}\right) \tag{41}$$

$$\frac{d^3 u_f}{dy'^3} = - \frac{u_0 M^4}{a^3(M-\tanh M)} \left(\frac{\sinh(\frac{My'}{a})}{\cosh M}\right) . \tag{42}$$

From equation (33), we get

$$\begin{aligned} V &= - \int \frac{\partial u'}{\partial x}\, dy' \\ &= - \int \frac{-4u_0\lambda_1 C_1 \xi^2}{x}\, e^{-4\lambda_1 \xi^2}\, \varphi_1'(\frac{y'}{a})\, dy' \\ &= \frac{4u_0\lambda_1 C_1 \xi^2}{x}\, e^{-4\lambda_1 \xi^2} \int \varphi_1'(\frac{y'}{a})\, dy' \\ &= \frac{4au_0\lambda_1 C_1 \xi^2}{x}\, e^{-4\lambda_1 \xi^2}\, \varphi_1(\frac{y'}{a}) . \end{aligned} \tag{43}$$

And from the transformation variable

$$\xi = \sqrt{\frac{\nu x}{s^2 u_0}} ,$$

we get

$$\nu = \frac{\xi^2 s^2 u_0}{x} . \tag{44}$$

Substituting equations (28) and (31), and equations (38) through (45) into equation (36) and neglecting the higher order terms, we obtain,

$$\left[\frac{u_0 M}{M - \tanh M}\left(1 - \frac{\cosh(\frac{My'}{a})}{\cosh M}\right) + u_0 C_1 e^{-4\lambda_1 \xi^2} \varphi_1'(\frac{y'}{a})\right]$$

$$x\left[-\frac{4u_0\lambda_1 C_1 \xi^2}{ax} e^{-4\lambda_1\xi^2} \varphi_1''(\frac{y'}{a})\right]$$

$$+\frac{4au_0\lambda_1 C_1\xi^2}{x} e^{-4\lambda_1\xi^2} \varphi_1(\frac{y'}{a})\left[-\frac{u_0 M^3}{a^2(M - \tanh M)}\left(\frac{\cosh(\frac{My'}{a})}{\cosh M}\right)\right.$$

$$\left.+\frac{u_0 C_1}{a^2} e^{-4\lambda_1\xi^2} \varphi_1'''(\frac{y'}{a})\right]$$

$$= \frac{4\xi^2 a^2 u_0}{x}\left[-\frac{u_0 M^4}{a^2(M - \tanh M)}\left(\frac{\sinh(\frac{My'}{a})}{\cosh M}\right) + \frac{u_0 C_1}{a^3} e^{-4\lambda_1\xi^2} \varphi_1^{iv}(\frac{y'}{a})\right]$$

$$-\frac{\sigma_e B_0^2}{\rho}\left[-\frac{u_0 M^4}{a(M - \tanh M)}\left(\frac{\sinh(\frac{My'}{a})}{\cosh M}\right) + \frac{u_0 C_1}{a} e^{-4\lambda_1\xi^2} \varphi_1''(\frac{y'}{a})\right]$$

in which, in the last term,

$$\frac{\sigma B_0^2}{\rho} = \left(\frac{\sigma_e}{\rho\nu}\right) B_0^2 a^2 \frac{\nu}{a^2}$$

$$= \frac{M^2}{a^2}\cdot\nu$$

$$= \frac{M^2}{a^2}\cdot\frac{4\xi^2 a^2 u_0}{x}$$

$$= \frac{4u_0\xi^2 M^2}{x}\quad.$$

Thus, the previous equation can be rewritten as follows

$$
\begin{aligned}
&\frac{4u_0^2\lambda_1 C_1\xi^2}{ax}\, e^{-4\lambda_1\xi^2}\,\Psi_1''(\tfrac{y'}{a})\left\{\frac{M}{M-\tanh M}\left[1-\frac{\cosh(\frac{My'}{a})}{\cosh M}\right]+C_1 e^{-\lambda_1\xi^2}\Psi'(\tfrac{y'}{a})\right\} \\
&+\frac{4u_0\lambda_1 C_1\xi^2}{ax}\, e^{-4\lambda_1\xi^2}\,\Psi_1(\tfrac{y'}{a})\left\{\frac{M^3}{M-\tanh M}\cdot\frac{\cosh(\frac{My'}{a})}{\cosh M}-C_1 e^{-\lambda_1\xi^2}\Psi_1'''(\tfrac{y'}{a})\right\} \\
&=\frac{4u_0^2\xi^2}{ax}\left\{\frac{M^4}{M-\tanh M}\left[\frac{\sinh(\frac{My'}{a})}{\cosh M}\right]-C_1 e^{-4\lambda_1\xi^2}\Psi_1^{iv}(\tfrac{y'}{a})\right\} \\
&+\frac{4u_0^2\xi^2 M^2}{ax}\left\{-\frac{M^2}{M-\tanh M}\left[\frac{\sinh(\frac{My'}{a})}{\cosh M}\right]+C_1 e^{-4\lambda_1\xi^2}\Psi_1''(\tfrac{y'}{a})\right\}. \qquad (45)
\end{aligned}
$$

Dividing both sides of the above equation by $4u_0^2\xi^2/ax$ and simplifying yield

$$
\begin{aligned}
&\lambda_1 C_1 e^{-4\lambda_1\xi^2}\frac{M}{M-\tanh M}\left[1-\frac{\cosh(\frac{My'}{a})}{\cosh M}\right]\Psi_1''(\tfrac{y'}{a})+\lambda_1 C_1^2 e^{-8\lambda_1\xi^2}\Psi_1''(\tfrac{y'}{a})\cdot\Psi'(\tfrac{y'}{a}) \\
&+\lambda_1 C_1 e^{-4\lambda_1\xi^2}\frac{M^3}{M-\tanh M}\left[\frac{\cosh(\frac{My'}{a})}{\cosh M}\right]\Psi_1(\tfrac{y'}{a})-\lambda_1 C_1^2 e^{-8\lambda_1\xi^2}\Psi_1(\tfrac{y'}{a})\cdot\Psi_1'''(\tfrac{y'}{a}) \\
&=-C_1 e^{-4\lambda_1\xi^2}\Psi_1^{iv}(\tfrac{y'}{a})+M^2 C_1 e^{-4\lambda_1\xi^2}\Psi_1''(\tfrac{y'}{a}) \quad . \qquad (46)
\end{aligned}
$$

In the above equation, second order terms of Ψ_1 can be neglected. This is the same approximation made by Schlichting (2) in the solution of the non-magnetic problem. Also, dividing both sides of the above equation by $\lambda_1 C_1 e^{-4\lambda_1\xi^2}$ we finally obtain

$$\Psi_1^{iv} + \left\{ \frac{M\lambda_1}{M - \tanh M} \left[1 - \frac{\cosh(\frac{My'}{a})}{\cosh M} \right] - M^2 \right\} \Phi_1''$$

$$+ \frac{M^3\lambda_1}{M - \tanh M} \left[\frac{\cosh(\frac{My'}{a})}{\cosh M} \right] \Phi_1 = 0 \qquad (47)$$

with λ_1 as its eigenvalue. The boundary conditions

$$v = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0$$

$$v = 0 \quad \text{and} \quad u = 0 \quad \text{at} \quad y = a$$

are equivalent to

$$\Phi_1 = \Psi_1'' = 0 \qquad \text{at} \qquad y = 0 \qquad (48)$$

$$\Psi_1 = \Psi_1' = 0 \qquad \text{at} \qquad y = \pm a\ . \qquad (49)$$

The other boundary condition is chosen as

$$\Psi_1' = 1 \qquad \text{at} \qquad y = 0\ . \qquad (50)$$

This is possible because C_1 is still free. Having these boundary conditions, equation (47) can be solved by a power series method.

<u>Upstream Pressure Distributions</u>

Near the entrance of the duct, the boundary layers are very thin. Most of the flow is composed of the accelerating potential flow in the central core of the duct. Therefore, the pressure distribution can be obtained from equation (3),

$$-\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx} \quad \frac{\sigma_e B_0^2}{\rho} U + \frac{\sigma B_0 E_0}{\rho} . \tag{51}$$

In the above equation, according to the definitions

$$M^2 = \frac{\sigma_e B_0^2 a^2}{\rho \nu} ,$$

$$\nu = \frac{4\xi^2 a^2 u_0}{x} ,$$

and

$$E_0 = -e u_0 B_0 .$$

one can obtain

$$\frac{\sigma B_0^2}{\rho} = \frac{\sigma B_0^2 a^2}{\rho \nu} \cdot \frac{\nu}{a^2}$$

$$= M^2 \frac{\nu}{a^2}$$

$$= \frac{4M^2 \xi^2 u_0}{x}$$

and

$$\frac{\sigma_e B_0 E_0}{\rho} = -\frac{\sigma B_0^2 e u_0}{\rho}$$

$$= -\frac{\sigma B_0^2 a^2}{\rho \nu} \cdot \frac{u_0}{a^2} \cdot e$$

$$= -M^2 \frac{4 \ {}^2 u_0}{x} \cdot e .$$

Also by definition we have

$$\frac{d\xi}{dx} = \frac{1}{2}\frac{\xi}{x}$$

or

$$dx = \frac{2x}{\xi}\,d\xi \,.$$

Hence, according to equation (5), equation (51) can be rewritten as:

$$-\frac{1}{\rho}\,dp = UdU + 8u_0^2M^2\xi\,(1+K_1\xi^2 + K_2\xi^3 + K_3\xi^4 + \ldots)d\xi$$
$$- 84u_0^2M^2\xi d\xi \,.$$

Integrating the last equation from $x = x_0$ at which $P = P_0$ and $\xi = 0$ to any section in the upstream, we obtain

$$-\frac{1}{\rho}\int_{P_0}^{P} dp = \frac{1}{2}\int_0^{\xi} d(U^2) + 8u_0^2M^2\int_0^{\xi}(1+K_1\xi^2 + K_2\xi^3$$
$$+ K_3\xi^4 + \ldots)d\xi - 8eu_0^2M^2\int_0^{\xi}\xi\,d\xi$$

or

$$\frac{P_0 - P}{\rho} = \frac{1}{2}\,u_0^2(K_0 + K_1\xi + K_2\xi^2 + K_3\xi^3 + \ldots)^2$$
$$+ 8u_0^2M^2\left(\frac{\xi^2}{2} + K_1\frac{\xi^3}{3} + K_2\frac{\xi^4}{4} + K_3\frac{\xi^5}{5} + \ldots\right)$$
$$- 8u_0^2eM^2\left(\frac{\xi^2}{2}\right) \,.$$

Finally, the pressure drop may be expressed as:

$$\frac{P_0 - P}{\frac{1}{2}\rho u_0^2} = 2K_1\xi + (2K_2 + K_1^2 + 8M^2\,[1-e])\xi^2$$

$$+ (2K_3 + 2K_1K_2 + \frac{16}{3}M^2K_1)\xi^3$$

$$+ (2K_4 + 2K_1K_3 + K_2^2 + 4M^2K_2)\xi^4 + \ldots \qquad (52)$$

Downstream Pressure Distribution

At the end of the upstream section, the potential-flow region is quite small and the acceleration of the remaining core flow in the downstream section is assumed to be negligible. This permits the use of the momentum equation for the region of fully developed flow

$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{{}^2u}{y^2} - \frac{\sigma_e B_0^2 u}{\rho} - \frac{\sigma B_0 E_0}{\rho}$$

in the downstream section. By taking a force summation over a differential volume of fluid according to the above equation we can obtain

$$-\frac{1}{\rho}\frac{dp}{dx} = -\frac{\nu}{a}\left(\frac{\partial u}{\partial y'}\right)_{wall} + \frac{\sigma_e B_0^2 u_0}{\rho} + \frac{\sigma_e B_0 E_0}{\rho}$$

or

$$-\frac{dp}{dx} = -\frac{\rho\nu}{a}\left(\frac{\partial u}{\partial y'}\right)_{wall} + \sigma_e B_0^2 u_0 - \sigma_e B_0^2 e u_0$$

or

$$-\frac{dp}{dx} = -\frac{\rho\nu}{a}\left(\frac{\partial u}{\partial y'}\right)_{wall} + \sigma_e B_0^2 u_0(1-e) \quad . \tag{53}$$

This is due to the fact that the pressure varies with x only and $\frac{dp}{dx}$ can be evaluated at the walls, $y' = \pm a$. According to the solution for the downstream velocity profiles equation (53) can also be written as follows

$$-dp = -\frac{\rho\nu}{a}\left(\frac{\partial u_f}{\partial y'} + \frac{\partial u'}{\partial y'}\right)_{wall} dx + \frac{\sigma_e B_0^2 a^2}{\rho} \cdot \frac{\rho\nu}{a^2} \cdot u_0(1-e)dx$$

$$= -\frac{\rho\nu}{a}\left\{-\frac{u_0 M^2}{a(M - \tanh M)}\left[\frac{\sinh(\frac{My'}{a})}{\cosh M}\right]\right.$$

$$\left. + \frac{u_0 C_1}{a} e^{-4\lambda_1 \xi^2} \Psi_1''\left(\frac{y'}{a}\right)_{wall}\right\} dx$$

$$+ \frac{4M^2\xi^2 u_0^2}{x}(1-e)dx$$

$$= -\frac{\rho\nu u_0}{a^2}\left(-\frac{M^2\frac{\sinh M}{\cosh M}}{M - \tanh M} + C_1 e^{-4\lambda_1\xi^2} \Psi_1''(1)\right)\frac{2x}{\xi}\, d\xi$$

$$+ \frac{4M^2\xi^2 u_0^2 e}{x} \cdot \frac{2x}{\xi}(1-e)d\xi$$

$$= -\frac{\rho\nu u_0}{a^2} \cdot \frac{2x4a^2u_0\xi^2}{\xi\nu}\left(-\frac{M^2 \tanh M}{M-\tanh M} + C_1 e^{-4\lambda_1\xi^2}\Psi_1''(1)\right) d\xi$$

$$+ 8M^2 u_0^2 \rho (1-e)\xi \, d\xi$$

or

$$- dp = 8\rho u_0^2 \frac{M^2 \tanh M}{M - \tanh M} \xi d\xi + 8C_1 \rho u_0^2 \Psi''(1) e^{-4\lambda_1 \xi^2} \xi \, d\xi$$

$$+ 8M^2 \rho u_0^2 (1-e) \xi \, d\xi \tag{54}$$

Letting the value of ξ at the jointing point be denoted by ξ_1, and integrating equation (54) from $\xi = \xi_1$ at which $P = P_1$, to $\xi = \xi$ at which $P = P$, one can obtain

$$\int_{P_1}^{P} -dp = 8\rho u_0^2 \frac{M^2 \tanh M}{M - \tanh M} \int_{\xi_1}^{\xi} \xi \, d\xi + 8C_1 \rho u_0^2 \Psi_1''(1) \int_{\xi_1}^{\xi} e^{-4\lambda_1 \xi^2} \xi \, d\xi$$

$$+ 8M^2 \rho u_0^2 (1-e) \int_{\xi_1}^{\xi} \xi \, d\xi$$

or

$$\frac{P_1 - P}{\frac{1}{2} \rho u_0^2} = 8 \left\{ \frac{M^2 \tanh M}{M - \tanh M} + (1-e) M^2 \right\} (\xi^2 - \xi_1^2)$$

$$+ \frac{2C_1 \Psi_1''(1)}{\lambda_1} \left(e^{-4\lambda_1 \xi^2} - e^{-4\lambda_1 \xi_1^2}\right) \tag{55}$$

as the pressure distribution in the downstream section.

For the complete solution of the pressure distribution, only C_1 and ξ_1 remain to be evaluated. These values are determined in the process of matching the upstream and the downstream solutions.

Matching of the Upstream and the Downstream Solutions

The upstream and the downstream solutions of velocity and pressure distribution are to be matched at a point where both solutions are valid. Because there are two undetermined quantities, C_1 and ξ_1, in the equations to be joined, two conditions are needed to solve for them. One condition is based on the fact that the centerline velocities represented by both solutions are equal at the joining point. The other condition is that the slopes of the pressure distribution, $\frac{dp}{dx}$, for the two solutions match at the joining point. These two conditions are used to develop two simultaneous equations from which C_1 and ξ_1 can be determined. It is a reasonable assumption that the upstream solution is more accurate than the other; hence, this solution is extended as far downstream as possible before joining. K_4 is the highest order coefficient solved for in the potential flow velocity expansion and it is assumed that when $K_4\xi^4$ is equal to 5 per cent of U/U_0 in equation (5) the ξ^5 term can be safely neglected. Thus

$$U = u_0(1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + K_4\xi^4) \ . \tag{56}$$

The downstream centerline velocity, from equations (28) and (31) is

$$\begin{aligned} U &= u_f(y') + u'(x, y') \\ &= u_f(0) + u_0 C e^{-4\lambda_1 \xi^2} \Phi_1'(0) \ . \end{aligned}$$

According to equation (12) and the boundary condition represented by equation (50), the above equation can be written as

$$U = \frac{u_0 M}{M - \tanh M}\left(1 - \frac{1}{\cosh M}\right) + u_0 C e^{-4\lambda_1 \xi^2} . \quad (57)$$

Equating the centerline velocities represented by equations (56) and (57) gives

$$1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + K_4\xi^4$$

$$= \frac{M}{M - \tanh M}\left(1 - \frac{1}{\cosh M}\right) + C_1 e^{-4\lambda_1 \xi^2} \quad (58)$$

The upstream pressure slope equation is

$$\frac{dp}{dx} = -\rho U \frac{dU}{dx} - \sigma_e B_0^2 U - \sigma_e B_0 E_0 .$$

Substituting the centerline velocity into the upstream section and using the definitions

$$M^2 = \frac{\sigma_e B_0^2 a^2}{\rho \nu}$$

and

$$E_0 = -e u_0 B_0 ,$$

we can obtain

$$\frac{dp}{dx} = -\rho u_0^2 (1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + K_4\xi^4)(K_1 + 2K_2\xi$$

$$+ 3K_3\xi^2 + 4K_4\xi^3)\frac{d\xi}{dx} + M^2 \frac{\rho\nu u_0}{a^2}(1 + K_1\xi$$

$$+ K_2\xi^2 + K_3\xi^3 + K_4\xi^4) + M^2 \frac{\rho\nu u_0 e}{a^2} .$$

Since $\xi^2 = \frac{\nu x}{4a^2 u_0}$ and $\frac{d\xi}{dx} = \frac{\xi}{2x}$, the upstream pressure slope equation can be reduced to

$$\frac{dp}{dx} = \frac{\rho u_0 \nu}{a^2}\left\{- \frac{1}{8\xi}(1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + K_4\xi^4)(K_1 + 2K_2\xi + 3K_3\xi^2 + 4K_4\xi^3) - M^2(1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + K_4\xi^4) + M^2 e\right\}. \tag{59}$$

On the other hand, from equation (53), the downstream pressure drop is

$$\begin{aligned}\frac{dp}{dx} &= \frac{\rho\nu}{a}\left(\frac{\partial u}{\partial y'}\right)_{wall} - \sigma_e B_0^2 u_0(1-e) \\ &= \frac{\rho\nu}{a}\left(\frac{\partial u_f}{\partial y'} + \frac{\partial u'}{\partial y'}\right)_{wall} - \frac{\sigma_e B_0^2 u^2}{\rho\nu} \cdot \frac{\rho\nu u_0}{a^2}(1-e) \\ &= \frac{\rho\nu u_0}{a^2}\left\{- \frac{M^2 \tanh M}{M - \tanh M} + C_1 e^{-4\lambda_1\xi^2}\Psi_1''(1) - M^2(1-e)\right\}\end{aligned} \tag{60}$$

Equating equations (59) and (60), we have

$$\frac{1}{8\xi}(1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + K_4\xi^4)(K_1 + 2K_2\xi + 3K_3\xi^2 + 4K_4\xi^3) - M^2(1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + K_4\xi^4) + \frac{M^2 \tanh M}{M - \tanh M} + M^2 - C_1 e^{-4\lambda_1\xi^2}\Psi_1''(0) = 0 \tag{61}$$

Eliminating $C_1 e^{-4\lambda_1\xi^2}$ from equations (58) and (61) gives a seventh order equation in ξ

$$(1 + K_1\xi + K_2\xi^2 + K_3\xi^3 + K_4\xi^4)(K_1 + 2K_2\xi + 3K_3\xi^2 + 4K_4\xi^3)$$
$$- 8M^2(\xi + K_1\xi^2 + K_2\xi^3 + K_3\xi^4 + K_4\xi^5) + \frac{8M^2 \tanh M}{M - \tanh M}\xi$$
$$+ 8M^2\xi + \frac{8M}{M - \tanh M}\left(1 - \frac{1}{\cosh M}\right)\Phi_1''(1)\xi - 8(1 + K_1\xi + K_2\xi^2$$
$$+ K_3\xi^3 + K_4\xi^4)\Phi_1''(1)\xi = 0$$

or

$$K_1 + \left\{ 2K_2 + K_1^2 + \frac{8M^2 \tanh M}{M - \tanh M}\xi + \frac{8M}{M - \tanh M}\left(1 - \frac{1}{\cosh M}\right)\Phi_1''(1)\right.$$
$$\left. - 8\Phi_1''(1)\ \xi + 3K_3 + 3K_2K_1 + K_1 - 8M^2K_1 - 8K_2\Phi_1''(1)\right\}\xi^2$$
$$+ \left\{4K_1 + 4K_1K_3 + 2K_2^2 - 8M^2K_2 - 8K_2\Phi_1''(1)\right\}\xi^3$$
$$+ \left\{5K_1K_4 + 5K_2K_3 - 8MK_4 - 8K_3\Phi_1''(1)\right\}\xi^4$$
$$+ \left\{6K_2K_4 + 3K_3^2 - 8M^2K_4 - 8K_4\Phi_1''(1)\right\}\xi^5 + \left\{7K_3K_4\right\}\xi^6$$
$$+ \left\{4K_4^2\right\}\xi^7 = 0 \qquad (62)$$

Solving this equation for the various values of M gives ξ at the joining points. Substituting the values of ξ into the equation for the upstream centerline velocity gives the values of C_1 for each magnetic field strength calculated.

NOMENCLATURE

a	Channel half-height, $a = \frac{s}{2}$
B	Magnetic field intensity
e	Electric field magnitude factor, $E_0 = - euB_0$
E	Electric field strength
f_n	Blasius coefficient, function of η
j	Electric current density
M	Hartmann number, $M = B_0\ a(\frac{\sigma}{\rho\nu})^{\frac{1}{2}}$
p	Fluid pressure
s	Channel height, $s = 2a$
u	x-component of velocity
u_0	Fluid velocity at inlet
u_f	Fully developed velocity
U	Potential flow velocity
v	y-component of velocity
w	z-component of velocity
y	Direction component measured from channel wall
y'	Direction component measured from channel center line
η	Transformation variable, $\eta = \frac{y}{2}\sqrt{\frac{u_0}{\nu x}}$
λ	Eigenvalue
ν	Kinematic viscosity
ρ	Fluid density
σ	Electrical conductivity
ψ	Blasius stream function
Ψ	Deviation velocity function
τ_w	Wall shear stress

ξ Transformation variable, $\xi = \left(\frac{\nu x}{u_0 s^2}\right)^{\frac{1}{2}}$

Subscripts

0 = constant magnitude

1 = joining point

2 = entrance length

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CHAPTER III

Schiller's Method

In the analysis by momentum integration (1), we also consider a laminar flow of a conducting fluid with constant properties entering a semi-infinite, nonconducting flat duct with a normal transverse applied magnetic field. In this case, the velocity profile is approximated by a curve comprised of two parabolas and a straight line (see Fig. 1); the vertices of the parabolas lie on the border of the boundary layer. The parabolic velocity profile in the boundary layer is assumed to be

$$\frac{u}{U} = 1 - \left[\left(\frac{y}{\delta}\right) - 1 \right]^2 , \tag{1}$$

where U is the velocity of the freestream and is a function of x, the coordinate along the direction of the flow; δ is the thickness of the boundary layer.

In this study two different boundary conditions are considered. One is the uniform velocity distribution across the entry plane, and the other, the velocity at the entry, is the fully developed non-MHD one which has a parabolic profile. Both develop in the entrance region of the channel to the Hartmann velocity profile which is the fully developed velocity of MHD flow.

First of all, the momentum integral equation for flow in the duct is derived. Consider the flow region within the boundary layer between section ab and cd, which are dx apart (see Fig. 2). Let the x-axis be in the same direction as the flow, and let the

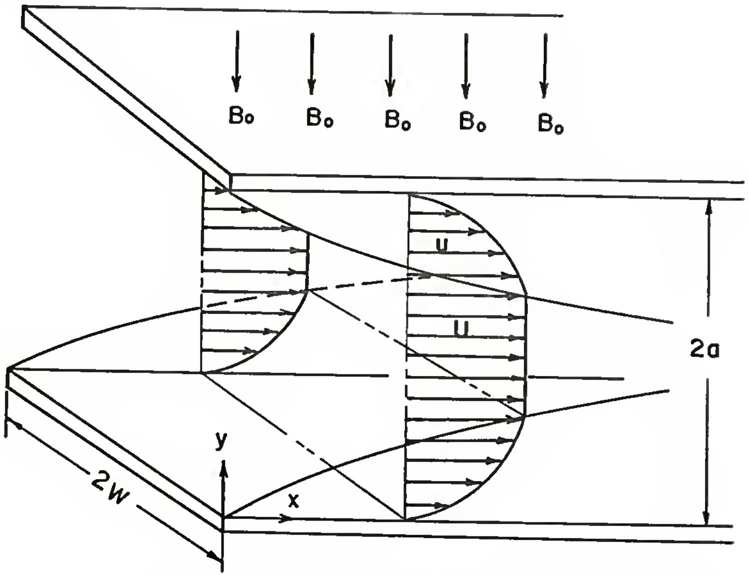

Fig. I Velocity profile in the entrance region of an MHD duct.

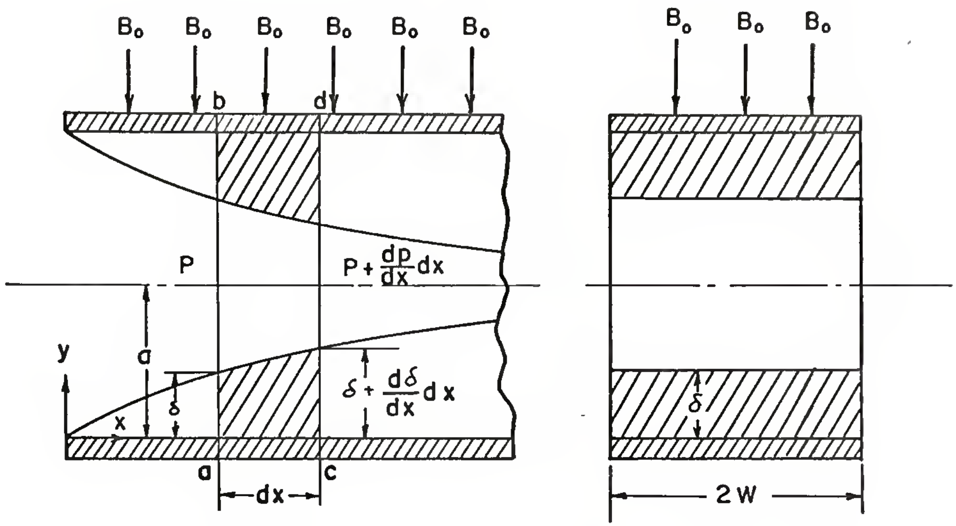

Fig. 2 Region for deriving momentum integral equation.

y-axis be measured from the wall. Then the mass flow rate into the section ab within the boundary layer is

$$M_{in} = \int_0^\delta 2u \times 2w \times \rho \, dy$$

$$= 4w \rho \int_0^\delta u dy .$$

The mass flow rate out of the region is

$$M_{out} = 4w \rho \int_0^\delta u dy + 4w \rho \frac{d}{dx} [\int_0^\delta u dy] dx .$$

The difference between M_{in} and M_{out} is

$$M_{in} - M_{out} = -4w \rho \frac{d}{dx} [\int_0^\delta u dy] dx$$

which is the mass flow rate into the region from the center core.

Therefore, the momentum flux into the region due to the bulk flow (or convective flow) is

$$-4w \rho U \frac{d}{dx} [\int_0^\delta u dy] dx + 4w \rho \int_0^\delta u^2 dy .$$

The momentum flux out of the region due to the bulk flow is

$$4w \rho \int_0^\delta u^2 dy + 4w \rho \frac{d}{dx} [\int_0^\delta u^2 dy] dx .$$

Thus the net momentum flux out of the region is

$$4w \rho \frac{d}{dx} [\int_0^\delta u^2 dy] dx + 4w \rho U \frac{d}{dx} [\int_0^\delta u dy] dx . \qquad (2)$$

For steady-flow, the net momentum flux out of the region is equal to the net force acting on the region in the same direction. The forces exerted on the region are the pressure force, the shear force of the wall, and the Lorentz force. Hence, when Hall effects and induced magnetic fields are assumed to be small enough to be negligible, we get

$$\Sigma F_x = -2\tau_w(2w)dx + 4w\left(\int_0^\delta jBdy\right)dx$$
$$+ P(2\delta)2w + \left(P + \varepsilon \frac{dp}{dx}dx\right) \cdot 2w \cdot 2d\delta$$
$$- \left(P + \frac{dp}{dx}dx\right)(2\delta + 2d\delta)2w \qquad (3)$$

where $0 < \varepsilon < 1$. In equation (3), the first term of the right-hand side represents the shear force of the wall. The second term is the Lorentz force in which j is the current density; B is the imposed magnetic field intensity. The third term is the force acting on the boundary layer at section ab, the fourth term is the force acting on the boundary layer from the free-stream, and the last term is the force acting on the boundary layer at section cd.

Rearranging and neglecting the infinitesimal terms of higher orders yields

$$\Sigma F_x = -4w\tau_w\,dx + 4w\left(\int_0^\delta jBdy\right)dx$$
$$-4\delta w\frac{dp}{dx}dx\,. \qquad (4)$$

According to the momentum theorem, equation (2) is set equal to the right-hand side of equation (4) and we can obtain the balance of forces and momentum fluxes:

$$4w\rho \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]dx + 4w\rho U \frac{d}{dx}\left[\int_0^\delta u dy\right]dx$$

$$= -4w\tau_w\, dx + 4w\left(\int_0^\delta jB dy\right)dx - 4\delta w \frac{dp}{dx}\, dx$$

or

$$\tau_w + \delta \frac{dp}{dx} - \int_0^\delta jB dy = -\rho U \frac{d}{dx}\int_0^\delta u dy - \rho \frac{d}{dx}\int_0^\delta u^2 dy \,. \qquad (5)$$

The freestream momentum equation is

$$\frac{dp}{dx} = -\rho U \frac{dU}{dx} + j_\infty B \qquad (6)$$

and Ohm's law is

$$j = \sigma(E - uB) \,. \qquad (7)$$

These provide the means for expressing the pressure gradient and the Lorentz force in terms of velocities and the magnetic field. That is, a rearrangement of equation (5) and the substitution of equations (6) and (7) into equation (5) give

$$\tau_w = -\rho U \frac{d}{dx}\int_0^\delta u dy - \rho \frac{d}{dx}\int_0^\delta u^2 dy + \int_0^\delta jB dy - \delta \frac{dp}{dx}$$

$$= \rho U \frac{d}{dx}\int_0^\delta (U-u)dy - \rho U \frac{d}{dx}\int_0^\delta U dy$$

$$+ \rho \frac{d}{dx}\int_0^\delta (uU-u^2)dy - \rho \frac{d}{dx}\int_0^\delta uU dy$$

$$+ \int_0^\delta \sigma(E-uB)B dy$$

$$- \delta \left[- \rho U \frac{dU}{dx} + \sigma(E-UB)B \right]$$

$$= \rho U \frac{d}{dx} \int_0^\delta (U-u) dy - \delta \rho U \frac{dU}{dx}$$

$$+ \rho \frac{d}{dx} \int_0^\delta (uU-u^2) dy - \rho \frac{d}{dx} U \int_0^\delta u dy$$

$$+ \sigma EB \int_0^\delta dy - \sigma B^2 \delta U \int_0^1 \frac{u}{U} d(\frac{y}{\delta})$$

$$+ \delta U \frac{dU}{dx} - \delta EB\sigma + \delta\sigma UB^2$$

$$= \rho U \frac{d}{dx} \int_0^\delta (U-u) dy + \rho \frac{d}{dx} \int_0^\delta (uU-u^2) dy$$

$$+ \delta\sigma UB^2 - \delta\sigma UB^2 \int_0^1 \frac{u}{U} d(\frac{y}{\delta}) \ .$$

Introducing the displacement thickness δ^* and the momentum thickness θ defined by

$$\delta^* U = \int_0^\delta (U-u) dy$$

and

$$\theta U^2 = \int_0^\delta u(U-u) dy \ ,$$

respectively, reduces the equation into the form

$$\tau_w = \delta\sigma B^2 U \left[1 - \int_0^1 \frac{u}{U} d(\frac{y}{\delta}) \right] + \delta^* \rho U \frac{dU}{dx} + \rho \frac{d}{dx} (U^2 \theta) \qquad (8)$$

In order to solve this equation for the boundary layer thickness, we use the assumed velocity profile expressed in equation (1). That is, it is assumed that the velocity profile consists of the parabolic variation

$$\frac{u}{U} = 1 - \left[\left(\frac{y}{\delta}\right) - 1 \right]^2$$

across the boundary layer and a uniform core velocity.

First of all, we can substitute $\mu\left(\frac{\partial u}{\partial y}\right)_{wall}$ for the shear, which is evaluated from the assumed velocity profile. Thus

$$\begin{aligned} \tau_w &= \mu\left(\frac{\partial u}{\partial y}\right)_{wall} \\ &= \mu U \frac{d}{dy}\left[1 - \left(\frac{y}{\delta} - 1\right)^2 \right]_{y=0} \\ &= -\frac{2\mu U}{\delta}\left[\frac{y}{\delta} - 1 \right]_{y=0} \\ &= \frac{2}{\delta}\mu U \, . \end{aligned} \tag{9}$$

The terms on the right-hand side of equation (8) can also be evaluated as follows:

$$\begin{aligned} \int_0^1 \frac{u}{U}\, d\left(\frac{y}{\delta}\right) &= \int_0^1 \left[1 - \left\{ \left(\frac{y}{\delta}\right) - 1 \right\}^2 \right] d\left(\frac{y}{\delta}\right) \\ &= \int_0^1 \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \right] d\left(\frac{y}{\delta}\right) \\ &= \left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3 \right]_0^1 \\ &= \frac{2}{3} \end{aligned} \tag{10}$$

$$\theta = \int_0^\delta \frac{u}{U}\,(1 - \frac{u}{U})\,dy$$

$$= \int_0^\delta \left[2(\frac{y}{\delta}) - (\frac{y}{\delta})^2\right]\left[1 - 2(\frac{y}{\delta}) + (\frac{y}{\delta})^2\right]dy$$

$$= \int_0^\delta \left[2(\frac{y}{\delta}) - 5(\frac{y}{\delta})^2 + 4(\frac{y}{\delta})^3 - (\frac{y}{\delta})^4\right]dy$$

$$= \left[\frac{y^2}{\delta} - \frac{5}{3}\,\frac{y^3}{\delta^2} + \frac{y^4}{\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \delta\left[1 - \frac{5}{3} + 1 - \frac{1}{5}\right]$$

$$= \frac{2\delta}{15} = \frac{2a}{15}\,(\frac{\delta}{a}) \qquad (11)$$

$$\frac{d\theta}{dx} = \frac{2a}{15}\,\frac{d}{dx}\,(\frac{\delta}{a}) \qquad (12)$$

The core velocity is allowed to vary in the x-direction in order to satisfy the conservation of mass equation applied across the entire channel cross section. If $\bar{u}$ is the average velocity, the principle of conservation of mass gives

$$4aw\bar{u} = 4w\int_0^\delta u\,dy + 4wU(a-\delta)$$

$$= 4wU\int_0^\delta \left[2(\frac{y}{\delta}) - (\frac{y}{\delta})^2\right]dy + 4wU(a-\delta)$$

$$= 4wU\left[\frac{y^3}{\delta} - \frac{1}{3}\,\frac{y^3}{\delta^2}\right]_0^\delta + 4wU(a-\delta)$$

$$= 4wU\delta(\frac{2}{3}) + 4wU(a-\delta)$$

$$= 4wUa(1 - \frac{1}{3}\,\frac{\delta}{2})$$

or

$$\frac{\bar{u}}{U} = 1 - \frac{1}{3}\frac{\delta}{a} . \tag{13}$$

Thus,

$$\frac{dU}{dx} = \bar{u}\frac{d}{dx}\frac{1}{(1 - \frac{1}{3}\frac{\delta}{a})}$$

$$= \frac{\bar{u}}{3(1 - \frac{1}{3}\frac{\delta}{a})^2}\frac{d}{dx}(\frac{\delta}{a}) \tag{14}$$

$$\rho \frac{d}{dx}(U^2\theta) = \rho U^2 \frac{d\theta}{dx} + 2\rho U\theta \frac{dU}{dx} .$$

From equations (11) through (14) this can be valuated as

$$\rho \frac{d}{dx}(U^2\theta) = \rho U^2 \cdot \frac{2a}{15}\frac{d}{dx}(\frac{\delta}{a})$$

$$+ 2\rho U \cdot \frac{2a}{15}(\frac{\delta}{a})\frac{\bar{u}}{3(1 - \frac{1}{3}\frac{\delta}{a})^2}\frac{d}{dx}(\frac{\delta}{a})$$

$$= \frac{2a\rho U^2}{15}\frac{d}{dx}(\frac{\delta}{a}) + \frac{4a\bar{u}\rho U(\frac{\delta}{a})}{45(1 - \frac{1}{3}\frac{\delta}{a})^2}\frac{d}{dx}(\frac{\delta}{a}) . \tag{15}$$

$$\delta^* = \int_0^\delta \left[1 - \frac{u}{U}\right]dy$$

$$= \int_0^\delta \left[1 - 2(\frac{y}{\delta}) + (\frac{y}{\delta})^2\right]dy$$

$$= \left[y - \frac{y^2}{\delta} + \frac{1}{3}\frac{y^3}{\delta^2}\right]_0^\delta$$

$$= \left[\delta - \delta + \frac{\delta}{3}\right] = \frac{\delta}{3} . \tag{16}$$

Substituting equations (9), (10), (12), (14), (15), and (16) into equation (8) we obtain

$$\frac{2}{\delta}\mu U = \delta B^2 U\sigma\left[1 - \frac{2}{3}\right] + \frac{\delta}{3}\rho U \frac{\bar{u}}{3(1 - \frac{1}{3}\frac{\delta}{a})^2}\frac{d}{dx}\left(\frac{\delta}{a}\right)$$

$$+ \frac{2a\rho U^2}{15}\frac{d}{dx}\left(\frac{\delta}{a}\right) + \frac{4a\bar{u}\rho U(\frac{\delta}{a})}{45(1 - \frac{1}{3}\frac{\delta}{a})^2}\frac{d}{dx}\left(\frac{\delta}{a}\right) .$$

By multiplying both sides with $\frac{\delta}{2\mu U}$ and noting that $(\delta/\mu)^{\frac{1}{2}}Ba = M$, $4a\bar{u}\rho/\mu = R_e$, and $\bar{u}/(1 - \frac{1}{3}\frac{\delta}{a}) = U$, the above equation becomes

$$dx = \frac{\sigma B^2 U\delta^2}{6\mu U}dx + \frac{\delta^2\rho U\bar{u}}{18\mu U(1 - \frac{1}{3}\frac{\delta}{a})^2}d\left(\frac{\delta}{a}\right) + \frac{a\ U^2\delta}{15\mu U}d\left(\frac{\delta}{a}\right)$$

$$+ \frac{4a\bar{u}\rho U(\frac{\delta}{a})^2}{90(1 - \frac{1}{3}\frac{\delta}{a})^2\mu U}d\left(\frac{\delta}{a}\right)$$

$$= \frac{\sigma B^2 a^2}{\mu}\cdot\frac{(\frac{\delta}{a})^2}{6}dx + \frac{4\rho a\bar{u}}{\mu}\cdot\frac{(\frac{\delta}{a})^2\ a}{72(1 - \frac{1}{3}\frac{\delta}{a})^2}d\left(\frac{\delta}{a}\right) + \frac{4a\rho\bar{u}}{\mu}$$

$$\cdot\frac{a(\frac{\delta}{a})}{60(1 - \frac{1}{3}\frac{\delta}{a})}d\left(\frac{\delta}{a}\right) + \frac{4a\rho\bar{u}}{\mu}\cdot\frac{a(\frac{\delta}{a})^2}{90(1 - \frac{1}{3}\frac{\delta}{a})^2}d\left(\frac{\delta}{a}\right)$$

$$= \left(\frac{\delta}{a}\right)^2\frac{M^2}{6}dx + \frac{R_e a(\frac{\delta}{a})^2}{72(1 - \frac{1}{3}\frac{\delta}{a})^2}d\left(\frac{\delta}{a}\right) + \frac{R_e a(\frac{\delta}{a})}{60(1 - \frac{1}{3}\frac{\delta}{a})}d\left(\frac{\delta}{a}\right)$$

$$+ \frac{R_e a(\frac{\delta}{a})^2}{90(1 - \frac{1}{3}\frac{\delta}{a})^2}d\left(\frac{\delta}{a}\right)$$

$$= \left(\frac{\delta}{a}\right)^2\frac{M^2}{6}dx + \left[\frac{R_e a\ 675(\frac{\delta}{a})^2 + 810(\frac{\delta}{a})(1 - \frac{1}{3}\frac{\delta}{a}) + 540(\frac{\delta}{a})^2}{48600(1 - \frac{1}{3}\frac{\delta}{a})^2}\right]d\left(\frac{\delta}{a}\right)$$

$$= \left(\frac{\delta}{a}\right)^2 \frac{M^2}{6}\, dx + R_e a \left[\frac{7(\frac{\delta}{a})^2 + 6(\frac{\delta}{a})}{360(1 - \frac{1}{3}\frac{\delta}{a})^2} \right] d\left(\frac{\delta}{a}\right)$$

or

$$\frac{dx}{aR_e} = \frac{7(\frac{\delta}{a})^2 + 6(\frac{\delta}{a})}{360(1 - \frac{M^2}{6}\frac{\delta^2}{a^2})(1 - \frac{1}{3}\frac{\delta}{a})^2} \,. \tag{17}$$

Equation (17) is to be integrated from x=0 to x with the boundary conditions for the uniform entry velocity, when x=0, δ=0 and when x=x, δ=δ. Thus

$$\frac{1}{aR_e}\int_0^x dx = \int_0^{\frac{\delta}{a}} \frac{7(\frac{\delta}{a})^2 + 6(\frac{\delta}{a})}{360\left[1 - (\frac{M^2}{6})(\frac{\delta^2}{a})\right]\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2} d\left(\frac{\delta}{a}\right) \,. \tag{18}$$

In order to carry out the integration on the right-hand side of equation (18), an integration by partial fractions is performed.

Let

$$\frac{7(\frac{\delta}{a})^2 + 6(\frac{\delta}{a})}{360\left[1 - \frac{M^2}{6}(\frac{\delta}{a})^2\right]\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2}$$

$$= \frac{1}{360}\left\{ \frac{A(\frac{\delta}{a}) + B}{\left[1 - \frac{M^2}{6}(\frac{\delta}{a})^2\right]} + \frac{C}{\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]} + \frac{D}{\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2} \right\}$$

$$= \frac{1}{360}\left\{ \frac{\left[A(\frac{\delta}{a})+B\right]\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2 + C\left[1 - \frac{M^2}{6}(\frac{\delta}{a})^2\right]\left[1 - \frac{1}{3}(\frac{\delta}{a})\right] + D\left[1 - \frac{M^2}{6}(\frac{\delta}{a})^2\right]}{\left[1 - \frac{M^2}{6}(\frac{\delta}{a})^2\right]\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2} \right\}.$$

Then we have

$$A(\tfrac{\delta}{a}) - \tfrac{2}{3} A(\tfrac{\delta}{a})^2 + \tfrac{1}{9} A(\tfrac{\delta}{a})^3 + B - \tfrac{2}{3} B(\tfrac{\delta}{a}) + \tfrac{1}{9} B(\tfrac{\delta}{a})^2 + C$$

$$- \tfrac{1}{3} C(\tfrac{\delta}{a}) - \tfrac{M^2}{6} C(\tfrac{\delta}{a})^2 + \tfrac{M^2}{6} \cdot \tfrac{1}{3} C(\tfrac{\delta}{a})^3 + D - \tfrac{M^2}{6} D(\tfrac{\delta}{a})^2$$

$$= 7(\tfrac{\delta}{a})^2 + 6(\tfrac{\delta}{a}) .$$

Equating coefficients of like powers of $(\frac{\delta}{a})$ in the above equation we obtain

$$\begin{cases} B + C + D = 0 , \\ A - \frac{2}{3} B - \frac{1}{3} C = 6 , \\ - \frac{2}{3} A + \frac{1}{9} B - \frac{M^2}{6} C - \frac{M^2}{6} D = 7 , \\ \frac{1}{9} A + \frac{M^2}{18} C = 0 . \end{cases}$$

Solving these simultaneous equations, we obtain

$$A = \frac{3(\frac{M^2}{6}) \left[144 + 162(\frac{M^2}{6})\right]}{\left[1 - 9(\frac{M^2}{6})\right]^2} ,$$

$$B = \frac{63 + 891(\frac{M^2}{6})}{\left[1 - 9(\frac{M^2}{6})\right]^2} ,$$

$$C = - \frac{144 + 162(\frac{M^2}{6})}{\left[1 - 9(\frac{M^2}{6})\right]^2} .$$

$$D = \frac{81\left[1 - 9(\frac{M^2}{6})\right]}{\left[1 - 9(\frac{M^2}{6})\right]^2} .$$

Hence the integration of the right-hand side of equation (18) becomes (2)

$$\int_0^{\frac{\delta}{a}} \frac{7(\frac{\delta}{a})^2 + 6(\frac{\delta}{a})}{360\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2} d(\tfrac{\delta}{a})$$

$$= \int_0^{\frac{\delta}{a}} \frac{3(\frac{M^2}{6})\left[144 + 162(\frac{M^2}{6})\right]}{360\left[1 - 9(\frac{M^2}{6})\right]^2} \cdot \frac{(\frac{\delta}{a})}{\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]} d(\tfrac{\delta}{a})$$

$$+ \int_0^{\frac{\delta}{a}} \frac{63 + 891(\frac{M^2}{6})}{360\left[1 - 9(\frac{M^2}{6})\right]^2} \cdot \frac{1}{\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]} d(\tfrac{\delta}{a})$$

$$+ \int_0^{\frac{\delta}{a}} \frac{-144 - 162(\frac{M^2}{6})}{360\left[1 - 9(\frac{M^2}{6})\right]^2} \cdot \frac{1}{\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]} d(\tfrac{\delta}{a})$$

$$+ \int_0^{\frac{\delta}{a}} \frac{81\left[1 - 9(\frac{M^2}{6})\right]}{360\left[1 - 9(\frac{M^2}{6})\right]^2} \cdot \frac{1}{\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2} d(\tfrac{\delta}{a})$$

$$= \left[\frac{-144 - 27M^2}{240(1 - \frac{3}{2}M^2)^2} \ln\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]\right]_0^{\frac{\delta}{a}}$$

$$+ \left[\frac{63 + 891(\frac{M^2}{6})}{120(1 - \frac{3}{2} M^2)^2 M^2} \ln \left[\frac{\sqrt{6} + M(\frac{\delta}{a})}{\sqrt{6} - M(\frac{\delta}{a})} \right] \right]_0^{\frac{\delta}{a}}$$

$$+ \left[\frac{144 + 27M^2}{120(1 - \frac{3}{2} M^2)^2} \ln \left[1 - \frac{1}{3}(\frac{\delta}{a})\right] \right]_0^{\frac{\delta}{a}}$$

$$+ \left[\frac{27(1 - \frac{3}{2} M^2)}{40(1 - \frac{3}{2} M^2)^2} \cdot \frac{1}{[1 - \frac{1}{3}(\frac{\delta}{a})]} \right]_0^{\frac{\delta}{a}}$$

$$= - \frac{(144 + 27M^2)}{240(1 - \frac{3}{2} M^2)^2} \ln \left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]$$

$$+ \frac{63 + 891(\frac{M^2}{6})}{120(1 - \frac{3}{2} M^2)^2 M^2} \ln \frac{\sqrt{6} + M(\frac{\delta}{a})}{\sqrt{6} - M(\frac{\delta}{a})}$$

$$+ \frac{144 + 27M^2}{120(1 - \frac{3}{2} M^2)^2} \ln \left[1 - \frac{1}{3}(\frac{\delta}{a})\right]$$

$$+ \frac{27(1 - \frac{3}{2} M^2)}{40(1 - \frac{3}{2} M^2)^2} \left[\frac{\frac{1}{3}(\frac{\delta}{a})}{1 - \frac{1}{3}(\frac{\delta}{a})} \right].$$

Finally equation (18) becomes

$$\frac{x}{aR_e} = - \frac{(144 + 27M^2)}{240(1 - \frac{3}{2} M^2)^2} \ln \left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]$$

$$+ \frac{63 + 891(\frac{M^2}{6})}{120(1 - \frac{3}{2} M^2)^2 M^2} \ln \left[\frac{\sqrt{6} + M(\frac{\delta}{a})}{\sqrt{6} - M(\frac{\delta}{a})} \right]$$

$$+ \frac{144 + 27M^2}{120(1 - \frac{3}{2} M^2)^2} \ln \left[1 - \frac{1}{3}(\frac{\delta}{a})\right]$$

$$+ \frac{27(1 - \frac{3}{2} M^2)}{40(1 - \frac{3}{2} M^2)^2} \left[\frac{\frac{1}{3}(\frac{\delta}{a})}{[1 - \frac{1}{3}(\frac{\delta}{a})]}\right]$$

or

$$\frac{x}{aR_e} = \frac{3}{20(2M^2 - \frac{2}{3} - \frac{3}{2} M^4)} \Big\{- (\frac{16}{3} + M^2) \ \ln \ (1 - \frac{\delta}{3a})$$

$$- \frac{(1 - \frac{3}{2} M^2)\frac{\delta}{a}}{1 - (\frac{\delta}{3a})} - (11M^2 + \frac{14}{3}) \frac{1}{2\sqrt{6} M} \ln \frac{\sqrt{6} + M(\frac{\delta}{a})}{\sqrt{6} - M(\frac{\delta}{a})}$$

$$+ (\frac{8}{3} + \frac{M^2}{2}) \ \ln \left[1 - (\frac{M^2}{6}) \frac{\delta^2}{a^2}\right]\Big\}. \tag{19}$$

Thus the velocity distribution in the entrance region for the case of uniform entry velocity profile is completely defined by equations (1) and (19).

When the entry velocity profile at the channel entry plane is the one which is nonmagnetically fully developed, the same equation (17) is integrated. Only in this case, the boundary conditions are that when $x=0$, $\delta=a$ and that when $x=x$, $\delta=\delta$. Thus

$$\frac{1}{aR_e} \int_0^x dx = \int_{\frac{a}{a}=1}^{\frac{\delta}{a}} \frac{7(\frac{\delta}{a})^2 + 6(\frac{\delta}{a})}{360\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]\left[1 - \frac{1}{3}(\frac{\delta}{a})^2\right]} d(\frac{\delta}{a}). \tag{20}$$

The integration of the right-hand side of equation (20) can be performed in the same manner as in the case of uniform entry velocity profile.

$$\int_{1}^{\frac{\delta}{a}} \frac{7(\frac{\delta}{a})^2 + 6(\frac{\delta}{a})}{360\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2}\, d(\tfrac{\delta}{a})$$

$$= \int_{1}^{\frac{\delta}{a}} \frac{3(\frac{M^2}{6})\left[144 + 162(\frac{M^2}{6})\right]}{360\left[1 - 9(\frac{M^2}{6})\right]^2} \cdot \frac{(\frac{\delta}{a})}{\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]}\, d(\tfrac{\delta}{a})$$

$$+ \int_{1}^{\frac{\delta}{a}} \frac{63 + 891(\frac{M^2}{6})}{360\left[1 - 9(\frac{M^2}{6})\right]^2} \cdot \frac{1}{\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]}\, d(\tfrac{\delta}{a})$$

$$+ \int_{1}^{\frac{\delta}{a}} \frac{-144 - 162(\frac{M^2}{6})}{360\left[1 - 9(\frac{M^2}{6})\right]^2} \cdot \frac{1}{\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]}\, d(\tfrac{\delta}{a})$$

$$+ \int_{1}^{\frac{\delta}{a}} \frac{81\left[1 - 9(\frac{M^2}{6})\right]}{360\left[1 - 9(\frac{M^2}{6})\right]^2} \cdot \frac{1}{\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]^2}\, d(\tfrac{\delta}{a})$$

$$= \left[\frac{-144 - 27M^2}{240(1 - \frac{3}{2}M^2)^2}\, \ln\left[1 - (\frac{M^2}{6})(\frac{\delta}{a})^2\right]\right]_1^{\frac{\delta}{a}}$$

$$+ \left[\frac{63 + 891(\frac{M^2}{6})}{120(1 - \frac{3}{2}M^2)^2\, M^2}\, \ln\left[\frac{\sqrt{6} + M(\frac{\delta}{a})}{\sqrt{6} - M(\frac{\delta}{a})}\right]\right]_1^{\frac{\delta}{a}}$$

$$+ \left[\frac{144 + 27M^2}{120(1 - \frac{3}{2}M^2)^2}\, \ln\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]\right]_1^{\frac{\delta}{a}}$$

$$+ \left[\frac{27(1 - \frac{3}{2}M^2)}{40(1 - \frac{3}{2}M^2)^2} \cdot \frac{1}{\left[1 - \frac{1}{3}(\frac{\delta}{a})\right]}\right]_1^{\frac{\delta}{a}}$$

$$= - \frac{144 + 27M^2}{240(1 - \frac{3}{2} M^2)^2} \left\{ \ell n \left[1 - \frac{M^2}{6}(\frac{\delta}{a})^2\right] - \ell n \left[1 - \frac{M^2}{6}\right] \right\}$$

$$+ \frac{63 + 891(\frac{M^2}{6})}{120(1 - \frac{3}{2} M^2)^2 M^2} \left\{ \ell n \frac{\sqrt{6} + M(\frac{\delta}{a})}{\sqrt{6} - M(\frac{\delta}{a})} - n \frac{\sqrt{6} + M}{\sqrt{6} - M} \right\}$$

$$+ \frac{144 + 27M^2}{120(1 - \frac{3}{2} M^2)^2} \left\{ \ell n \left[1 - \frac{1}{3}(\frac{\delta}{a})\right] - \ell n \left[1 - \frac{1}{3}\right] \right\}$$

$$+ \frac{27(1 - \frac{3}{2} M^2)}{40(1 - \frac{3}{2} M^2)^2} \left[\frac{1}{[1 - \frac{1}{3}(\frac{\delta}{a})]} - \frac{2}{3} \right]$$

$$= - \frac{144 + 27M^2}{240(1 - \frac{3}{2} M^2)^2} \left[\ell n \frac{1 - \frac{M^2}{6} (\frac{\delta}{a})^2}{1 - \frac{M^2}{6}} \right]$$

$$+ \frac{63 + 891(\frac{M^2}{6})}{120(1 - \frac{3}{2} M^2)^2 M^2} \ell n \frac{[\sqrt{6} + M(\frac{\delta}{a})](\sqrt{6} - M)}{[\sqrt{6} - M(\frac{\delta}{a})](\sqrt{6} + M)}$$

$$+ \frac{144 + 27M^2}{120(1 - \frac{3}{2} M^2)^2} \ell n \frac{1 - (\frac{\delta}{3a})}{\frac{2}{3}}$$

$$+ \frac{27(1 - \frac{3}{2} M^2)}{40(1 - \frac{3}{2} M^2)^2} \cdot \frac{\frac{1}{2}[(\frac{\delta}{a}) - 1]}{[1 - \frac{1}{3}(\frac{\delta}{a})]} .$$

Finally equation (20) becomes

$$\frac{x}{aR_e} = - \frac{144 - 27M^2}{240(1 - \frac{3}{2} M^2)^2} \left[\ell n \frac{1 - \frac{M^2}{6} (\frac{\delta}{a})^2}{1 - \frac{M^2}{6}} \right]$$

$$+ \frac{63 + 891(\frac{M^2}{6})}{120(1 - \frac{3}{2} M^2)^2 M^2} \ell n \frac{[\sqrt{6} + M(\frac{\delta}{a})] (\sqrt{6} - M)}{[\sqrt{6} - M(\frac{\delta}{a})] (\sqrt{6} + M)}$$

$$+ \frac{144 + 27M^2}{120(1 - \frac{3}{2} M^2)^2} \ell n \frac{1 - (\frac{\delta}{3a})}{\frac{2}{3}}$$

$$+ \frac{27(1 - \frac{3}{2} M^2)}{40(1 - \frac{3}{2} M^2)^2} \cdot \frac{\frac{1}{2} [(\frac{\delta}{a}) - 1]}{[1 - \frac{1}{3}(\frac{\delta}{a})]}$$

or

$$\frac{x}{aR_e} = \frac{3}{20(2M^2 - \frac{2}{3} - \frac{3}{2} M^4)} \Big\{ - (\frac{16}{3} + M^2) \ell n \frac{1 - \frac{\delta}{3a}}{\frac{2}{3}}$$

$$- \frac{3[(\frac{\delta}{a}) - 1] (1 - \frac{3}{2} M^2)}{2[1 - (\frac{\delta}{3a})]} - \frac{11M^2 + \frac{14}{3}}{2\sqrt{6} M} \times$$

$$\ell n \frac{[\sqrt{6} - (\frac{\delta}{a}) M] (\sqrt{6} - M)}{[\sqrt{6} + (\frac{\delta}{a}) M] (\sqrt{6} + M)} + (\frac{8}{3} + \frac{M^2}{2}) \times$$

$$\ell n \frac{1 - (\frac{M^2}{6}) (\frac{\delta^2}{a^2})}{1 - (\frac{M^2}{6})} \Big\} . \qquad (21)$$

Thus the velocity distribution for the situation where the velocity profile at the channel entry plane is nonmagnetically fully developed is completely defined by equations (1) and (21).

In these cases, it should be noted that the nondimensional length parameter x/aR_e, familiar from Schlichting's nonmagnetic solution, is a function of δ/a and the Hartmann number M only.

The Hartmann number can be viewed as a measure of the magnetic interaction.

The Length of the Entrance Region

Since the boundary layer grows from zero until it attains the asymptotic value and joins at the channel center, when the flow is fully developed $\delta = a$ or $\frac{\delta}{a} = 1$. Therefore, for the case of uniform entry velocity profile, from equation (19) we obtain

$$\begin{aligned} f(M, \tfrac{\delta}{a})\Big|_{\frac{\delta}{a}=1} &= \frac{3}{20(2M^2 - \frac{2}{3} - \frac{3}{2} M^4)} \Big\{ - (\tfrac{16}{3} + M^2) \ n \tfrac{2}{3} \\ &\quad - \tfrac{3}{2}(1 - \tfrac{3}{2} M^2) - (11M^2 + \tfrac{14}{3}) \frac{1}{2(6)^{\frac{1}{2}} M} \times \\ &\quad n \frac{(6)^{\frac{1}{2}} + M}{(6)^{\frac{1}{2}} - M} + (\tfrac{8}{3} + \tfrac{M^2}{2}) \ n \ (1 - \tfrac{M^2}{6}) \Big\} \end{aligned}$$

Therefore, the entrance length x_e for the case of uniform entry velocity profile is

$$\begin{aligned} \frac{x_e}{aR_e} &= f(M, \tfrac{\delta}{a})\Big|_{\frac{\delta}{a}=1} \\ &= \frac{3}{20(2M^2 - \frac{2}{3} - \frac{3}{2} M^4)} \Big\{ - (\tfrac{16}{3} + M^2) \ \ell n \tfrac{2}{3} \\ &\quad - \tfrac{3}{2}(1 - \tfrac{3}{2} M^2) - (11M^2 + \tfrac{14}{3}) \frac{1}{2(6)^{\frac{1}{2}} M} \ \ell n \frac{(6)^{\frac{1}{2}} + M}{(6)^{\frac{1}{2}} - M} \\ &\quad + (\tfrac{8}{3} + \tfrac{M^2}{2}) \ \ell n \ (1 - \tfrac{M^2}{6}) \Big\} \end{aligned} \tag{22}$$

According to Schiller, the entrance length is defined as the distance required for the friction factor to come within 10% of the final, fully developed value. From equation (1) we know that the velocity gradient at the wall will be

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = U\left[\frac{2}{\delta} - 2\frac{y}{\delta}\right]_{y=0} = \frac{2U}{\delta} .$$

In equation (21), the minimum value of $(\frac{\delta}{a})$ is decided by the fact that the argument of the logarithmic function should not be zero or a negative value. For instance, the minimum value of $(\frac{\delta}{a})$ for the Hartmann number, M, of 4 is

$$\left(\frac{\delta}{a}\right)_{min} = \frac{\sqrt{6}}{4}$$

or

$$\delta = \frac{\sqrt{6}}{4} a . \tag{23}$$

If the velocity profile corresponding to $(\frac{\delta}{a})_{min}$ is assumed to be the fully developed one, the friction factor at the fully developed condition for M=4 is

$$\left(\frac{\partial u}{\partial y}\right)_{y=0 \text{ at F.D.}} = \left.\frac{2U}{\delta}\right|_{F.D.} = \frac{2U}{(\sqrt{6}/4)a} . \tag{24}$$

Substituting equations (23) and (13) into equation (24) yields

$$\left(\frac{\partial u}{\partial y}\right)_{y=0 \text{ at F.D.}} = \frac{\dfrac{2\bar{u}}{1 - \sqrt{6}/12}}{(\sqrt{6}/4)a} = \frac{\bar{u}}{a}\,\frac{\dfrac{24}{12 - \sqrt{6}}}{\dfrac{\sqrt{6}}{4}}$$

$$= \frac{\bar{u}}{a}\left(\frac{96}{12\sqrt{6} - 6}\right) = \frac{\bar{u}}{a}\left(\frac{16}{2\sqrt{6} - 1}\right) . \tag{25}$$

The friction factor at any other section is, by virtue of equation (13)

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{2U}{\delta'} = \frac{2\bar{u}}{\delta(1 - \frac{1}{3}\frac{\delta}{a})} \, . \tag{26}$$

When the Hartmann number, M, equals 4, the value of $\frac{\delta}{a}$ at the location of 10% deviation from the fully developed profile is obtained, (by use of equations (25) and (26)),

$$\frac{\bar{u}}{a}\left(\frac{16}{2\sqrt{6} - 1}\right) \times 0.9 = \frac{2\bar{u}}{\delta(1 - \frac{1}{3}\frac{\delta}{a})} \, ,$$

or

$$\left(\frac{\delta}{a}\right)^2 - 3\left(\frac{\delta}{a}\right) = - \frac{20}{3} \cdot \frac{3.896}{16} = - 1.625 \, .$$

Hence

$$\begin{aligned} \frac{\delta}{a} &= \frac{3 \pm \sqrt{9 - 6.5}}{2} \\ &= 1.5 - 0.79 \\ &= 0.71 \, . \end{aligned} \tag{27}$$

Finally substituting equation (27) into equation (21), we obtain

$$\frac{x_e}{aR_e} = 0.0185884$$

for the case of the nonmagnetically fully developed entry velocity profile at M=4.

The Pressure Drop

From the Bernoulli equation, the difference in pressure between any two locations in the entrance region is

$$P_1 - P_2 = \frac{\rho}{2} U_2^2 - \frac{\rho}{2} U_1^2 \tag{24}$$

where P_1 and P_2 are the pressures at x_1 and x_2, respectively, and U_1 and U_2 are the velocities at the central core at x_1 and x_2, respectively.

From equation (13), we have

$$U_1 = \frac{\bar{u}}{(1 - \frac{1}{3}\frac{\delta_1}{a})}, \qquad U_2 = \frac{\bar{u}}{(1 - \frac{1}{3}\frac{\delta_2}{a})}.$$

Hence the equation (24) becomes

$$P_1 - P_2 = \frac{\rho}{2} \bar{u}^2 \left[\frac{1}{(1 - \frac{1}{3}\frac{\delta_2}{a})^2} - \frac{1}{(1 - \frac{1}{3}\frac{\delta_1}{a})^2} \right]. \tag{25}$$

The pressure drop at the entrance region and at the fully developed region can be obtained as follows.

For $x \leq f(M,1)$, the pressure difference between the inlet edge and any location in the entrance region is obtained from equation (25), as

$$P_0 - P = \frac{\rho}{2} \bar{u}^2 \left[\frac{1}{(1 - \frac{1}{3}\frac{\delta}{a})^2} - \frac{1}{(1 - \frac{1}{3}\frac{\delta_0}{a})^2} \right].$$

Since $\frac{\delta_0}{a} = 0$

$$P_0 - P = \frac{}{2}\,\bar{u}^2 \left[\frac{1}{(1 - \frac{1}{3}\frac{\delta}{a})^2} - 1 \right]$$

If $x = x_e = f(M,1)$, $\frac{\delta}{a} = 1$. Therefore, the total pressure drop in the entrance region becomes

$$P_0 - P = \frac{5}{8}\,\bar{u}^2 \,. \tag{26}$$

NOMENCLATURE

a	Channel half-height
B	Magnetic induction
E	Electric field strength
j	Electric current density
M	Hartmann number, $M = B\, a(\frac{\sigma}{\mu})^{\frac{1}{2}}$
P	Fluid pressure
R_e	Reynolds number, $R_e = 4aU\rho/\mu$
u	x-component of velocity
W	Half-width of channel
x, y, z	Space coordinate
δ	Boundary-layer thickness
δ^*	Displacement thickness
ρ	Fluid density
σ	Fluid electric conductivity
θ	Momentum thickness
μ	Fluid viscosity
τ_w	Wall shear stress

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CHAPTER IV

Targ's Method

The principal difficulty in analyzing the velocity distribution in the entrance region is due to the nonlinear inertia terms in the equation of motion. Because of this nonlinearity of the problem, all solutions obtained to date have been approximate solutions.

A new method of analysis of entrance region flows, evolved from ideas proposed by Targ (Slezkin (1)), has recently been described by Sparrow, Lin, and Lundgren (2). The method involves a linearization of the inertia terms by introducing a stretched coordinate in the direction of flow. A boundary-layer model is not necessary in the analysis, and velocity solutions are obtained which are continuous in the transverse and longitudinal directions from the channel entrance to the fully developed region. This new technique is applied by Snyder (3) to the investigation of MHD flow in the entrance region of a parallel-plate channel.

The geometry of the MHD parallel-plate channel is the same as shown in Fig. 1 of Chapter I. A constant magnetic field B_0 is applied in the y direction, and the channel is assumed to be infinite in the x direction. The initial velocity profile is also assumed to be uniform and at a great distance downstream from the entrance the velocity profile is the fully developed Hartmann profile.

As can be seen from Chapter I, the governing equations of

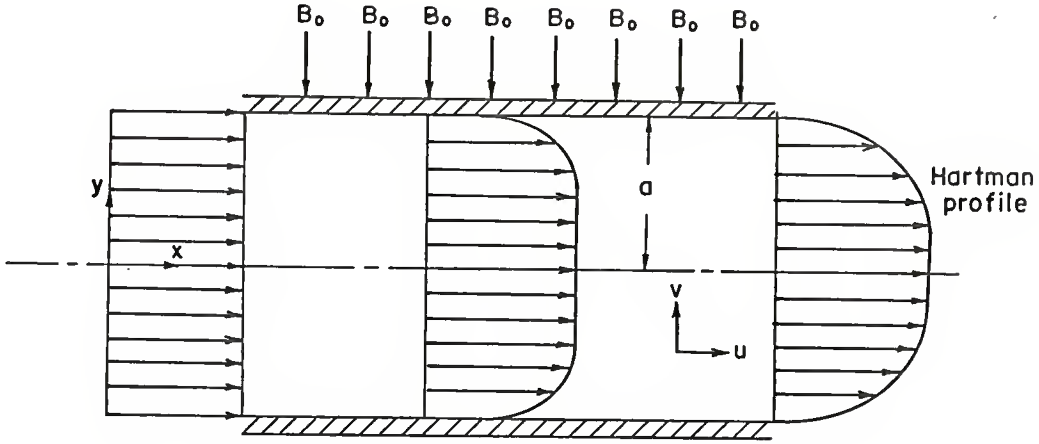

Fig. 1 Configuration of an MHD duct-entrance region with a parallel plate.

motion for laminar magnetohydrodynamic flow are

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x}\left[\int_A u dA\right] = 0 \, , \tag{1}$$

momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2} + \frac{\sigma_e B_0}{\rho}(E_0 - uB_0) . \tag{2}$$

The major assumptions implicit in equations (1) and (2) are the same as those described in the previous chapters.

According to the new technique, equation (2) is linearized as follows:

$$\epsilon(x)\,\bar{U}\frac{\partial u}{\partial x} = \Lambda(x) + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}\left(\frac{E_0}{B_0} - u\right) \tag{3}$$

where $\bar{U}$ is the mean velocity defined as

$$\bar{U} = \frac{1}{a}\int_0^a u dy \tag{4}$$

and $\epsilon(x)$ is a yet undetermined function of x which weights the mean velocity $\bar{U}$, while $\Lambda(x)$ is a second undetermined function which includes the pressure gradient as well as the residual of inertia terms after the linearization.

Integrating equation (3) over the upper half cross-sectional area of the channel, A, gives

$$\epsilon(x)\overline{U}\,\frac{\partial}{\partial x}\int_A u\,dA \;=\; \int_A \Lambda(x)\,dA + \nu\int_A \frac{\partial^2 u}{\partial y^2}\,dA + \frac{\sigma_e B_0^2}{\rho}\int_A \left(\frac{E_0}{B_0} - u\right)dA$$

where, in accordance with the mass conservation equation (1)

$$\frac{\partial}{\partial x}\int u\,dA \;=\; 0\ .$$

Hence,

$$0 \;=\; \Lambda(x)A + \nu\int_A \frac{\partial^2 u}{\partial y^2}\,dA + \frac{\sigma_e B_0^2}{\rho}\int_A \left(\frac{E_0}{B_0} - u\right)dA. \qquad (5)$$

By applying the divergence theorem to the integral involving $\partial^2 u/\partial y^2$ in the above equation we obtain

$$\nu\int_A \frac{\partial^2 u}{\partial y^2}\,dA \;=\; \nu\oint_c \frac{\partial u}{\partial n}\,d\ell\ .$$

Substituting this into equation (5) and solving it for (x), we obtain

$$\Lambda(x) \;=\; -\frac{\nu}{A}\oint_c \frac{\partial u}{\partial n}\,d\ell + \frac{\sigma_e B_0^2\,\overline{U}}{\rho}\left(1 - \frac{E_0}{\overline{U}B_0}\right) \qquad (6)$$

in which the contour c is the circumstance of the channel half area, $\partial u/\partial n$ is the normal derivative of the velocity evaluated along c, and d is an element of the circumference. For the parallel-plate channel, the only nonvanishing contribution to $\partial u/\partial n$ is $(\partial u/\partial y)_{y=a}$.

A stretched axial coordinate x* is defined by the relation

$$dx \;=\; \epsilon(x)\,dx^*\ . \qquad (7)$$

Combining equations (3), (6), and (7) we obtain the linearized momentum equation:

$$\overline{U}\,\frac{\partial x}{\partial x^*}\,\frac{\partial u}{\partial x} = -\frac{\nu}{A}\oint_c \frac{\partial u}{\partial n}\,d\ell + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma_e B_0^2\,\overline{U}}{\rho}\left(1 - \frac{E_0}{\overline{U}B_0}\right) + \frac{\sigma_e B_0^2}{\rho}\left(\frac{E_0}{B_0} - u\right)$$

or

$$\overline{U}\,\frac{\partial u}{\partial x^*} = \nu\left[\frac{\partial^2 u}{\partial y^2} - \left(\frac{\nu}{a}\,\frac{\partial u}{\partial y}\right)_{y=a}\right] + \frac{\sigma_e B_0^2\,\overline{U}}{\rho}\left(1 - \frac{u}{\overline{U}}\right). \tag{8}$$

It is also convenient to introduce dimensionless variables

$$\alpha = \frac{u}{\overline{U}}, \qquad \eta = \frac{y}{a}, \qquad \beta = \left(\frac{1}{Re}\right)\left(\frac{x^*}{a}\right)$$

$$Re = \frac{\rho\overline{U}a}{\mu}, \qquad \text{and} \qquad M = B_0\,a\left(\frac{\sigma_e}{\rho\nu}\right)^{\frac{1}{2}}. \tag{9}$$

Hence,

$$\frac{\partial u}{\partial x^*} = \frac{\partial \frac{u}{\overline{U}}}{\partial \frac{x^*}{Re\;a}}\left(\frac{\overline{U}}{Re\;a}\right) = \frac{\overline{U}}{Re\;a}\left(\frac{\partial \alpha}{\partial \beta}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial \frac{u}{\overline{U}}}{\partial \frac{y}{a}}\left(\frac{\overline{U}}{a}\right) = \frac{\overline{U}}{a}\,\frac{\partial \alpha}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\overline{U}}{a}\,\frac{\partial}{\partial y}\,\frac{\partial \alpha}{\partial \eta} = \frac{\overline{U}}{a}\,\frac{\partial}{\partial \frac{y}{a}}\,\frac{\partial \alpha}{\partial \eta} = \frac{\overline{U}}{a^2}\,\frac{\partial^2 \alpha}{\partial \eta^2}.$$

Substitution of these gives the dimensionless form of equation (8) as follows:

$$\frac{\overline{U}^2}{Re\ a}\left(\frac{\partial\alpha}{\partial\beta}\right) = \frac{\nu\overline{U}}{a^2}\left[\frac{\partial^2\alpha}{\partial\eta^2} - \left(\frac{\partial\alpha}{\partial\eta}\right)_{\eta=1}\right] + \frac{\sigma_e B_0^2\ \overline{U}}{\rho}\left(1 - \frac{u}{\overline{U}}\right)$$

or

$$\frac{\partial\alpha}{\partial\beta} = \frac{\partial^2\alpha}{\partial\eta^2} - \left(\frac{\partial\alpha}{\partial\eta}\right)_{\eta=1} + M^2(1-\alpha)\ . \qquad (10)$$

The boundary conditions are

$$\begin{aligned} \eta &= 1 \quad : \quad \alpha = 0\ , \\ \eta &= 0 \quad : \quad \frac{\partial\alpha}{\partial\eta} = 0\ , \\ \beta &= 0 \quad : \quad \alpha = 1\ . \end{aligned} \qquad (11)$$

That is, equation (10) is to be solved subject to the no-slip boundary condition that u=0 on the duct wall. In addition, the velocity profile at the inlet will be assumed to be uniform across the section, i.e., $u = \overline{U}$.

Although the relationship between the actual axial coordinate x and the stretched coordinate x* is not yet determined, it will be convenient to set this question aside until later and proceed with the solution of equation (10).

Equation (10) is a linear equation with constant coefficients and is thus amenable to solution by means of the Laplace transform. Defining the transform of α as

$$\bar{\alpha} = \int_0^\infty \alpha e^{-s\beta}\, d\beta$$

then we get

$$L(\alpha') = s\bar{\alpha} - \alpha(0)$$

$$= s\bar{\alpha} - 1$$

in which

$$\alpha(\beta=0) = 1 ;$$

$$L(\alpha) = \bar{\alpha} ; \qquad \text{and}$$

$$L(M^2) = \frac{1}{s} M^2 .$$

Hence, equation (10) is then transformed into the form

$$s\bar{\alpha} - 1 = \frac{\partial^2\bar{\alpha}}{\partial\eta^2} - \left(\frac{\partial\bar{\alpha}}{\partial\eta}\right)_{\eta=1} + M^2\left(\frac{1}{s} - \bar{\alpha}\right)$$

or

$$\frac{d^2\bar{\alpha}}{d\eta^2} - (M^2 + s)\bar{\alpha} = \left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} - \frac{(M^2 + s)}{s} . \tag{12}$$

To solve this linear differential equation of second order, we first find by observation the complementary function

$$\bar{\alpha}_c = c_1 e^{(M^2+s)^{\frac{1}{2}}\eta} + c_2 e^{-(M^2+s)^{\frac{1}{2}}} .$$

Then the general solution is found to be

$$\bar{\alpha} = c_1 e^{(M^2+s)^{\frac{1}{2}}\eta} + c_2 e^{-(M^2+s)^{\frac{1}{2}}\eta}$$

$$- \frac{1}{(M^2+s)}\left[\left(\frac{\partial\bar{\alpha}}{\partial\eta}\right)_{\eta=1} - \frac{(M^2+s)}{s}\right] . \tag{13}$$

The boundary conditions (11) can be transformed as follows:

$$\eta = 1 \quad : \quad \bar{\alpha} = \left(\int_0^\infty \alpha e^{-s\beta}\, d\beta\right)_{\eta=1} = 0 \qquad (14)$$

$$\eta = 0 \quad : \quad \frac{\partial \bar{\alpha}}{\partial \eta} = \frac{\partial}{\partial \eta}\int_0^\infty \alpha e^{-s\beta}\, d\beta$$

$$= \int_0^\infty \frac{\partial \alpha}{\partial \eta} e^{-s\beta}\, d\beta$$

$$= 0 \qquad (15)$$

With these transformed boundary conditions (14) and (15), the constants in the solution (13) are to be found.

From (14), we have

$$c_1 e^{(M^2+s)^{\frac{1}{2}}} + c_2 e^{-(M^2+s)^{\frac{1}{2}}} - \frac{1}{(M^2+s)}\left[\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} - \frac{(M^2+s)}{s}\right] = 0 \; .$$

From (15) we have

$$c_1 (M^2+s)^{\frac{1}{2}} - c_2 (M^2+s)^{\frac{1}{2}} = 0$$

or

$$c_1 = c_2 \; .$$

Solving these two simultaneous equations for c_1 and c_2, we obtain

$$c_1 \left[e^{(M^2+s)^{\frac{1}{2}}} + e^{-(M^2+s)^{\frac{1}{2}}}\right] = \frac{\left[\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} - \frac{(M^2+s)}{s}\right]}{(M^2+s)}$$

or

$$c_1 = c_2 = \frac{\left[\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} - \frac{(M^2+s)}{s}\right]}{(M^2+s)\left[e^{(M^2+s)^{\frac{1}{2}}} + e^{-(M^2+s)^{\frac{1}{2}}}\right]} \; .$$

Substitution of these constants into equation (13) gives

$$\bar{\alpha} = \frac{\left[\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} - \frac{(M^2+s)}{s}\right]\left\{\left[e^{(M^2+s)^{\frac{1}{2}}\eta} + e^{-(M^2+s)^{\frac{1}{2}}\eta}\right] - \left[e^{(M^2+s)^{\frac{1}{2}}} + e^{-(M^2+s)^{\frac{1}{2}}}\right]\right\}}{(M^2+s)\left[e^{(M^2+s)^{\frac{1}{2}}} + e^{-(M^2+s)^{\frac{1}{2}}}\right]}$$

$$= \frac{\left[\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} - \frac{(M^2+s)}{s}\right]\left[\cosh(M^2+s)^{\frac{1}{2}}\eta - \cosh(M^2+s)^{\frac{1}{2}}\right]}{(M^2+s)\left[\cosh(M^2+s)^{\frac{1}{2}}\right]}$$

From this we get

$$\frac{d\bar{\alpha}}{d\eta} = \frac{\left[\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} - \frac{(M^2+s)}{s}\right](M^2+s)^{\frac{1}{2}}\left[\sinh(M^2+s)^{\frac{1}{2}}\eta\right]}{(M^2+s)\left[\cosh(M^2+s)^{\frac{1}{2}}\right]}$$

or

$$\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} = \frac{\left[\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} - \frac{(M^2+s)}{s}\right](M^2+s)^{\frac{1}{2}}\left[\sinh(M^2+s)^{\frac{1}{2}}\eta\right]}{(M^2+s)\left[\cosh(M^2+s)^{\frac{1}{2}}\right]}$$

Solving for $\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1}$, we obtain

$$\left(\frac{d\bar{\alpha}}{d\eta}\right)_{\eta=1} = \frac{(M^2+s)\sinh(M^2+s)^{\frac{1}{2}}}{s\left[\sinh(M^2+s)^{\frac{1}{2}} - (M^2+s)^{\frac{1}{2}}\cosh(M^2+s)^{\frac{1}{2}}\right]} .$$

As a result the general solution of equation (12) becomes

$$\bar{\alpha} = \frac{\left[\frac{(M^2+s)\sinh(M^2+s)^{\frac{1}{2}}}{s\left[\sinh(M^2+s)^{\frac{1}{2}} - (M^2+s)^{\frac{1}{2}}\cosh(M^2+s)^{\frac{1}{2}}\right]} - \frac{(M^2+s)}{s}\right]\left[\cosh(M^2+s)^{\frac{1}{2}}\eta - \cosh(M^2+s)^{\frac{1}{2}}\right]}{(M^2+s)\cosh(M^2+s)^{\frac{1}{2}}}$$

$$= \frac{\left[(M^2+s)\right]\left[\cosh(M^2-s)^{\frac{1}{2}}\eta - \cosh(M^2+s)^{\frac{1}{2}}\right]}{s\left[(M^2+s)^{\frac{1}{2}}\sinh(M^2+s)^{\frac{1}{2}} - (M^2+s)\cosh(M^2+s)^{\frac{1}{2}}\right]} . \qquad (16)$$

Carrying out the inverse transform (4), gives α in the following two equivalent forms:

$$\alpha = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{s\beta}\, \frac{(M^2+s)\,[\cosh(M^2+s)^{\frac{1}{2}}\eta-\cosh(M^2+s)^{\frac{1}{2}}]\,ds}{s\,[(M^2+s)^{\frac{1}{2}}\sinh(M^2+s)^{\frac{1}{2}}-(M^2+s)\cosh(M^2+s)^{\frac{1}{2}}]} \tag{17}$$

or

$$\alpha = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \frac{2\lambda^2 e^{(\lambda^2-M^2)\beta}\,(\cosh\lambda\eta-\cosh\lambda)\,d\lambda}{(\lambda^2-M^2)(\sinh\lambda - \lambda\cosh\lambda)} \tag{18}$$

in which

$$\lambda = (M^2+s)^{\frac{1}{2}}, \quad s = \lambda^2-M^2, \quad \text{and} \quad ds = 2\lambda d\lambda .$$

The integrals of equations (17) and (18) can be evaluated by residue theorem (5) which states that if f(s) is a single-valued function that is analytic inside a simple closed path c and on c, except for a finite number of singular points $a_1, a_2, \ldots, a_m$ interior to the region bounded by c, we have

$$\int_c f(s)ds = 2\pi i \sum_{j=1,\, s=a_j}^{m} \text{Res } f(s)$$

where Res f(s) denotes the residue of f(s) at a_n and where the integration is in the counterclockwise sense around c.

From equation (17) it is clear that a simple pole occurs at s=0 with a residue

$$\operatorname*{Res}_{s=0} f(s) = \operatorname*{Res}_{s=0} \frac{P(s)}{q(s)}$$

$$= \frac{P(0)}{q'(0)}$$

$$= \left\{ \frac{e^{s\beta}(M^2+s)^{\frac{1}{2}} \left[\cosh(M^2+s)^{\frac{1}{2}}\eta\right.}{\sinh(M^2+s)^{\frac{1}{2}} - (M^2+s)^{\frac{1}{2}}\cosh(M^2+s)^{\frac{1}{2}} + \frac{s}{2}\ (M^2+s)^{\frac{1}{2}}} \right.$$

$$\left. \frac{\left. - \cosh(M^2+s)^{\frac{1}{2}}\right]}{\cosh(M^2+s)^{\frac{1}{2}} - (M^2+s)^{-\frac{1}{2}}\cosh(M^2+s)^{\frac{1}{2}} - \sinh(M^2+s)^{\frac{1}{2}}]} \right\}_{s=0}$$

$$= \frac{M(\cosh M\eta - \cosh M)}{(\sinh M - M\cosh M)} \,. \tag{19}$$

From equation (18), if λ is pure imaginary, say $\lambda = i\gamma$ where γ is real, it is clear that poles will occur where

$$\sinh \lambda - \lambda \cosh \lambda = 0$$

or

$$\frac{\sinh(i\gamma)}{\cosh(i\gamma)} = i\gamma$$

<u>i.e.</u>,

$$\gamma = \tan \gamma \,. \tag{20}$$

The roots of equation (20) give an infinite set of eigenvalues at which poles occur in the integrand of equation (18). If γ_n is the nth root of equation (20), the residue corresponding to this root is

$$R_n = \frac{2e^{-(\gamma_n^2+M^2)\beta}\ (\cos \gamma_n\eta - \cos \gamma_n)}{(\gamma_n^2+M^2)\ \cos \gamma_n} \,. \tag{21}$$

The solution for α obtained as the sum of all residues of the integrand in the inversion integral becomes

$$\alpha = \frac{M(\cosh M \quad - \cosh M)}{(\sinh M - M\cosh M)}$$

$$+ 2 \sum_{n=1}^{\infty} \frac{e^{-(\gamma_n^2+M^2)\beta}(\cos\gamma_n\eta - \cos\gamma_n)}{(\gamma_n^2+M^2)\cos\gamma_n} . \qquad (22)$$

The first term of equation (22) is recognized as the Hartmann profile and represents the fully developed solution, and the series corresponds to the difference between the actual velocity profile and the fully developed profile. The limiting form of equation (22) for $M \longrightarrow 0$ can be written as

$$\lim_{M\to 0} \alpha = \lim_{M\to 0} \frac{M(\cosh M\eta - \cosh M)}{(\sinh M - M\cosh M)}$$

$$+ \lim_{M\to 0} 2M \frac{e^{-(\gamma_n^2+M^2)\beta}(\cos\gamma_n\eta - \cos\gamma_n)}{(\gamma_n^2+M^2)\cos\gamma_n}$$

in which, the first term in the right side can be evaluated according to L'Hospital's rule. Thus

$$\lim_{M\to 0} \frac{M(\cosh M\eta - \cosh M)}{(\sinh M - M\cosh M)}$$

$$= \lim_{M\to 0} \frac{(\cosh M\eta - \cosh M) + M(\eta\sinh M\eta - \sinh M)}{\cosh M - M\sinh M - \cosh M}$$

$$= \lim_{M\to 0} \frac{(\eta\sinh M\eta - \sinh M) + \eta\sinh M - \sinh M + M(\eta^2\cosh M\eta - \cosh M)}{-M\cosh M - \sinh M}$$

$$= \lim_{M \to 0} \frac{\zeta^2 \cosh M\zeta - \cosh M + \zeta^2 \cosh M\zeta - \cosh M + \zeta^2 \cosh M\zeta}{-M\sinh M - \cosh M - \cosh M}$$

$$\underline{-\cosh M - M(\zeta^3 \sinh M\zeta - \sinh M)}$$

$$= \frac{\zeta^2 - 1 + \zeta^2 - 1 + \zeta^2 - 1}{-2} = \frac{3}{2}(1-\zeta^2)$$

Hence,

$$\lim_{M \to 0} \alpha = \frac{3}{2}(1-\zeta^2) + 2 \sum_{n=1}^{\infty} \frac{e^{-\gamma_n^2 \beta} (\cos \gamma_n \zeta - \cos \gamma_n)}{\gamma_n^2 \cos \gamma_n}$$

which is identical to the velocity profile for the case of zero magnetic field dealt with in (2).

Stretched Axial Coordinate

(Evaluation of ϵ)

The solution given by equation (22) is in terms of the stretched coordinate β and the relation between β and the physical coordinate $(\frac{1}{Re})(x/a)$ requires a knowledge of the relationship between ϵ and β. This relationship will be found by imposing an additional physical constraint, namely, that the local pressure gradient $\partial p/\partial x$ calculated from momentum considerations be the same as that calculated from mechanical energy considerations. For an exact velocity solution, a unique value of the pressure gradient would be derived no matter whether one employs momentum or mechanical-energy considerations. However, since entrance-region analyses are necessarily approximate, the local pressure gradients calculated on these different bases need not necessarily be the same.

An expression for the pressure gradient will be derived first from momentum considerations. The momentum equation will be integrated across the channel half section. The integration of the inertia term is facilitated by combining the inertia term and the continuity equation:

$$\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$$

$$= u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

where

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .$$

Hence,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} . \tag{23}$$

Substituting this into the momentum equation (2) and integrating the same equation over the upper half of the channel gives

$$\int_0^a \frac{\partial uu}{\partial x} dy + \int_0^a \frac{\partial uv}{\partial y} dy$$

$$= -\frac{1}{\rho} \int_0^a \frac{\partial p}{\partial x} dy + \nu \int_0^a \frac{\partial^2 u}{\partial y^2} dy + \int_0^a \frac{\sigma_e B_0}{\rho} (E_0 - uB_0) dy$$

$$\frac{\partial}{\partial x} \int_0^a u^2 dy + (uv)\Big|_{y=a} - (uv)\Big|_{y=0}$$

$$= -\frac{a}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{u}{y}\right)\Big|_{y=a} - \nu \left(\frac{\partial u}{\partial y}\right)\Big|_{y=0} + \frac{a\sigma_e B_0}{\rho} (E_0 - \overline{U} B_0)$$

in which $(uv)\Big|_a$ and $(uv)\Big|_0$ are zero, because u and v are all zero at the wall and v is zero at y=0; and $\left(\frac{\partial u}{\partial y}\right)\Big|_0$ is also zero because u is symmetric at the center line. It is also to be noted that $\int_0^a ud = \overline{U}a$. Hence,

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{a} \frac{\partial}{\partial x} \int_0^a u^2 dy - \left(\frac{\nu}{a} \frac{\partial u}{\partial y}\right)_{y=a} - \frac{\sigma_e B_0^2 \overline{U}}{\rho} \left(\frac{E_0}{\overline{U} B_0} - 1\right). \tag{24}$$

Next, a mechanical-energy equation may be constructed by multiplying through the momentum equation (2) by the velocity u. The mechanical-energy equation may then be integrated over the upper half of the channel.

$$\int_0^a u^2 \frac{\partial u}{\partial x}\, dy + \int_0^a uv \frac{\partial u}{\partial y}\, dy$$

$$= -\frac{1}{\rho}\int_0^a u \frac{\partial p}{\partial x}\, dy + \nu\int_0^a u \frac{\partial^2 u}{\partial y^2}\, dy + \int_0^a \frac{\sigma_e B_0}{\rho}(E_0 - uB_0)u\,dy$$

in which

$$\int_0^a u^2 \frac{\partial u}{\partial x}\, dy = \int_0^a \frac{\partial u^3}{\partial x}\, dy - \int_0^a u \frac{\partial u^2}{\partial x}\, dy$$

$$= \frac{\partial}{\partial x}\int_0^a u^3 dy - \int_0^a u \frac{\partial u^2}{\partial x}\, dy$$

$$\int_0^a vu \frac{\partial u}{\partial y}\, dy = \int_0^a \frac{\partial u^2 v}{\partial y}\, dy - \int_0^a u \frac{\partial uv}{\partial y}\, dy$$

$$= u^2 v\Big|_0^a - \int_0^a u \frac{\partial uv}{\partial y}\, dy$$

$$= -\int_0^a u \frac{\partial uv}{\partial y}\, dy\,.$$

Hence,

$$\int_0^a u^2 \frac{\partial u}{\partial x}\, dy + \int_0^a uv \frac{\partial u}{\partial y}\, dy$$

$$= \frac{\partial}{\partial x}\int_0^a u^3 dy - \int_0^a u \frac{\partial u^2}{\partial x}\, dy - \int_0^a u \frac{\partial uv}{\partial y}\, dy$$

$$= \frac{\partial}{\partial x}\int_0^a u^3 dy - \int_0^a u\left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\right) dy$$

$$= \frac{\partial}{\partial x}\int_0^a u^3 dy - \int_0^a u\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) dy$$

$$= \frac{\partial}{\partial x} \int_0^a \frac{u^3}{2}\, dy \,.$$

$$-\frac{1}{\rho} \int_0^a u \frac{\partial p}{\partial x}\, dy = -\frac{1}{\rho} \frac{\partial p}{\partial x} \int_0^a u dy$$

$$= -\frac{1}{\rho} \bar{U}\, a \frac{\partial p}{\partial x} \,.$$

$$\nu \int_0^a u \frac{\partial^2 u}{\partial y^2}\, dy = \nu \int_0^a \frac{\partial}{\partial y} \left(u \frac{\partial u}{\partial y}\right) dy - \nu \int_0^a \left(\frac{\partial u}{\partial y}\right)^2 dy$$

$$= \nu u \frac{u}{y}\Big|_0^a - \nu \int_0^a \left(\frac{\partial u}{\partial y}\right)^2 dy$$

$$= -\nu \int_0^a \left(\frac{\partial u}{\partial y}\right)^2 dy$$

because $(u)_{y=a} = 0$ and $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$.

$$\int_0^a \frac{\sigma_e B_0}{\rho} (E_0 - uB_0) u dy = \frac{\sigma_e B_0 E_0}{\rho} \int_0^a u dy - \frac{\sigma B_0^2}{\rho} \int_0^a u^2 dy$$

$$= \frac{\sigma B_0 E_0}{\rho} (\bar{U} a) - \frac{\sigma B_0^2}{\rho} \int_0^a u^2 dy$$

$$= \frac{\sigma B_0^2\, \bar{U}}{\rho} \left(\frac{E_0 a}{B_0} - \frac{1}{\bar{U}} \int_0^a u^2 dy\right) \,.$$

Substituting this back we may obtain the pressure gradient from the mechanical energy equation as

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{\bar{U} a} \frac{\partial}{\partial x} \int_0^a \frac{u^3}{2}\, dy + \frac{\nu}{\bar{U} a} \int_0^a \left(\frac{\partial u}{\partial y}\right)^2 dy$$

$$- \frac{\sigma B_0^2 \overline{U}}{\rho} \left(\frac{E_0}{\overline{U} B_0} - \frac{1}{a \overline{U}^2} \int_0^a u^2 dy \right) . \tag{25}$$

The pressure gradients given by equations (24) and (25) may now be equated to give

$$\frac{1}{a} \frac{\partial}{\partial x} \int_0^a u^2 dy - \left(\frac{\nu}{a} \frac{\partial u}{\partial y} \right)_{y=a} - \frac{\sigma_e B_0^2 \overline{U}}{\rho} \left(\frac{E_0}{\overline{U} B_0} - 1 \right)$$

$$= \frac{1}{\overline{U} a} \frac{\partial}{\partial x} \int_0^a \frac{u^3}{2} dy + \frac{\nu}{\overline{U} a} \int_0^a \left(\frac{\partial u}{\partial y} \right)^2 dy$$

$$- \frac{\sigma B_0^2 \overline{U}}{\rho} \left(\frac{E_0}{\overline{U} B_0} - \frac{1}{a \overline{U}^2} \int_0^a u^2 dy \right) .$$

Replacing (d/dx) by $(\frac{1}{\epsilon})(d/dx^*)$ and rearranging, we get

$$\epsilon = \frac{\frac{1}{a} \frac{\partial}{\partial x^*} \left(\int_0^a u^2 dy - \frac{1}{\overline{U}} \int_0^a \frac{u^3}{2} dy \right)}{\left(\frac{\nu}{a} \frac{u}{y} \right)_{y=a} + \frac{\nu}{\overline{U} a} \int_0^a \left(\frac{\partial u}{\partial y} \right)^2 dy - \frac{\sigma_e B_0^2 \overline{U}}{\rho} + \frac{\sigma_e B_0^2 \overline{U}}{\rho a} \int_0^a \frac{u^2}{\overline{U}^2} dy} \tag{26}$$

in which, from equation (9), we have

$$\frac{\partial}{\partial x^*} = \frac{\partial}{\partial \frac{x^*}{R_e a}} \cdot \frac{1}{R_e a} = \frac{1}{R_e a} \frac{\partial}{\partial \beta} ,$$

$$u^2 = \alpha^2 \overline{U}^2 ,$$

$$u^3 = \alpha^3 \overline{U}^3 ,$$

$$dy = a d\eta ,$$

$$\frac{\partial u}{\partial y} = \frac{\partial \frac{u}{\overline{U}}}{\partial \frac{y}{a}} \frac{\overline{U}}{a} = \frac{\overline{U}}{a} \frac{\partial \alpha}{\partial \eta},$$

and

$$\frac{\sigma_e B_0^2 \overline{U}}{\rho} = \frac{\sigma_e B_0^2 a^2}{\mu} \cdot \frac{\overline{U}\nu}{a^2} = M^2 \frac{\overline{U}\nu}{a^2}.$$

Substituting these into equation (26) we obtain the expression for ϵ in terms of dimensionless variables.

$$\epsilon(M,\beta) = \frac{\frac{1}{R_e a^2} \frac{\partial}{\partial x^*} \left(\int_0^1 \alpha^2 \overline{U}^2 \, a d\eta - \frac{1}{\overline{U}} \int_0^1 \frac{\alpha^3 \overline{U}^3}{2} \, a d\eta \right)}{\left(\frac{\nu}{a} \cdot \frac{\overline{U}}{a} \frac{\partial \alpha}{\partial \eta} \right)_{\eta=1} + \frac{\nu}{\overline{U}a} \cdot \frac{\overline{U}^2}{a^2} \cdot a \int_0^1 \left(\frac{\partial \alpha}{\partial \eta}\right)^2 d\eta + \frac{\overline{U}\nu}{a^2} M^2 \int_0^1 (\alpha^2 - 1) d\eta}$$

$$= \frac{\frac{\partial}{\partial \beta} \int_0^1 \left(\alpha^2 - \frac{\alpha^3}{2}\right) d\eta}{\left(\frac{\partial \alpha}{\partial \eta}\right)_{\eta=1} + \int_0^1 \left(\frac{\partial \alpha}{\partial \eta}\right)^2 d\eta + M^2 \int_0^1 (\alpha^2 - 1) d\eta}. \qquad (27)$$

The right side of equation (27) is a known function of β from equation (22), and thus the variation of ϵ with β is specified. It is clear that the $(\epsilon - \beta)$ relationship will also depend on the Hartmann number M. The physical coordinate x is related to the dimensionless stretched coordinate β by the following integration:

From equation (7),

$$dx = \epsilon \cdot dx^*$$

where

$$dx^* = R_e a \, d\beta \ .$$

Hence,

$$\int_0^x dx = R_e a \int_0^\beta \epsilon \, d\beta$$

or

$$x = R_e a \int_0^\beta \epsilon \, d\beta \ . \qquad (28)$$

Equation (28) can be evaluated numerically.

Equations (22), (27), and (28) thus give the solution for the velocity profile in the entrance region. From this basic solution, the calculation of other physical quantities of interest is possible.

Pressure Drop

The pressure distribution may be obtained by integrating either the momentum equation (24) or the mechanical-energy equation (25). The momentum equation has a somewhat simpler form and will be employed in the following development. The pressure drop between the duct inlet (pressure P_0) and any axial location (pressure P) may be determined by integrating equation (24) from x=0 to x=x.

$$- \int_0^x \frac{dP}{\rho} = \int_0^x \left\{ \frac{1}{a} \frac{\partial}{\partial x} \int_0^a u^2 dy \right\} dx - \frac{\nu}{a} \int_0^x \left(\frac{\partial u_f}{\partial y}\right)_{y=a} dx$$
$$- \frac{\nu}{a} \int_0^x \left(\frac{\partial (u-u_f)}{\partial y}\right)_{y=a} dx + \int_0^x \frac{\sigma_e B_0^2 \bar{U}}{\rho} \left(\frac{E_0}{\bar{U} B_0} - 1\right) dx$$

or

$$\frac{P_0 - P}{\rho} = \int_0^x \left\{ \frac{1}{a} \frac{\partial}{\partial x} \int_0^a u^2 dy \right\} dx - \frac{\nu}{a} \int_0^x \left(\frac{\partial u_f}{\partial y}\right)_{y=a} dx$$
$$- \frac{\nu}{a} \int_0^x \left(\frac{\partial (u-u_f)}{\partial y}\right)_{y=a} dx - \frac{\sigma_e B_0^2 \bar{U} x}{\rho} \left(\frac{E_0}{\bar{U} B_0} - 1\right) \qquad (29)$$

where the identity

$$u = u_f + (u - u_f)$$

has been used in evaluating $(\partial u/\partial y)_{y=a}$. Employing the dimensionless variables expressed by equations (9) we have

$$\int_0^x \frac{1}{a}\frac{\partial}{\partial x}\int_0^a u^2 dy\, dx = \frac{1}{a}\int_0^1 a\overline{U}^2 \alpha^2 \, d\eta$$

$$= \overline{U}^2 \int_0^1 \alpha^2 \, d\eta \, ,$$

$$\frac{\nu}{a}\int_0^x \left(\frac{\partial u_f}{\partial y}\right)_{y=a} dx = \frac{\nu}{a}\left(\frac{\partial u_f}{\partial y}\right)_{y=a}\int_0^x dx$$

$$= \frac{\nu}{a}\left(\frac{\partial \frac{u_f}{\overline{U}}}{\partial \frac{y}{a}}\right)_{y=a} \frac{\overline{U}}{a} \cdot x$$

$$= \frac{\overline{U}\nu x}{a^2}\left(\frac{\partial \alpha_f}{\partial \eta}\right)_{\eta=1}$$

$$= \frac{\overline{U}^2}{R_e}\left(\frac{x}{a}\right)\left(\frac{\partial \alpha_f}{\partial \eta}\right)_{\eta=1} \, ,$$

$$\frac{\nu}{a}\int_0^x \left(\frac{\partial (u-u_f)}{\partial y}\right)_{y=a} dx$$

$$= \frac{\nu}{a}\int_0^\beta \left(\frac{\partial \left(\frac{u}{\overline{U}} - \frac{u_f}{\overline{U}}\right)\overline{U}}{\partial \left(\frac{y}{a}\right) a}\right)_{y=a} \epsilon R_e \, a d\beta$$

$$= \frac{\nu}{a} R_e \overline{U}\int_0^\beta \epsilon \left(\frac{\partial (\alpha-\alpha_f)}{\partial \eta}\right)_{\eta=1} d\beta$$

$$= \overline{U}^2 \int_0^\beta \epsilon \left(\frac{\partial (\alpha-\alpha_f)}{\partial \eta}\right)_{\eta=1} d\beta \, ,$$

and

$$\frac{\sigma B_0^2\ \overline{U}x}{\rho}\left(\frac{E_0}{\overline{U}B_0} - 1\right) = M^2\,\frac{\overline{U}x\nu}{a^2}\left(\frac{E_0}{\overline{U}B_0} - 1\right)$$

$$= M^2\,\frac{\overline{U}^2}{R_e}\left(\frac{x}{a}\right)\left(\frac{E_0}{\overline{U}B_0} - 1\right) .$$

Substituting these into equation (29), we can obtain the pressure distribution in dimensionless form:

$$\frac{P_0 - P}{\frac{1}{2}\,\rho\,\overline{U}^2} = \frac{2}{R_e}\left(\frac{x}{a}\right)\left\{ M^2\left(1 - \frac{E_0}{\overline{U}B_0}\right) - \left(\frac{\partial\alpha_f}{\partial\eta}\right)_{\eta=1}\right\} +$$

$$+\ 2\left\{\int_0^1 \alpha^2 d\eta - \int_0^\beta \epsilon\left(\frac{(\alpha-\alpha_f)}{\partial\eta}\right)_{\eta=1} d\beta\right\} \qquad (30)$$

where the first bracket term represents the pressure drop that would result from a fully developed flow and the second bracket term represents the correction due to the entrance flow.

Since, from equation (22),

$$\alpha_f = \frac{M(\cosh M\eta - \cosh M)}{(\sinh M - M\cosh M)}$$

$\left(\frac{\partial\alpha_f}{\partial\eta}\right)_{\eta=1}$ can be evaluated as

$$\left(\frac{\partial\alpha_f}{\partial\eta}\right)_{\eta=1} = M^2\,\frac{\sinh M}{M\cosh M - \sinh M} .$$

And by defining

$$\phi = \frac{E_0}{\overline{U}B_0}$$

and

$$K(M,\frac{1}{R_e}\,\frac{x}{a}) \;=\; 2\left\{\int_0^1 \alpha^2 d\eta - 1 - \int_0^\beta \epsilon\,(\frac{\partial(\alpha-\alpha_f)}{\partial\eta})_{\eta=1}\right\}d\beta \quad .$$

Equation (30) can be written as

$$\frac{P_0 - P}{\frac{1}{2}\,\rho\,\overline{U}^2} = \frac{2M^2}{R_e}\,(\frac{x}{a})\left\{\,(1-\phi) + \frac{\sinh M}{M\cosh M - \sinh M}\right\}$$

$$+ \; K(M,\,\frac{1}{R_e}\,\frac{x}{a}) \;. \tag{31}$$

The coefficient of $(\frac{1}{R_e})(\frac{x}{a})$ in equation (31) is seen to be a function of M and ϕ and may be defined as the fully developed friction factor. Thus we may write

$$\frac{P_0 - P}{\frac{1}{2}\,\rho\,\overline{U}^2} \;=\; f\,\frac{1}{R_e}\,(\frac{x}{a}) + K(M,\,\frac{1}{R_e}\,\frac{x}{a}) \tag{29}$$

where

$$f \;=\; 2M^2\left\{(1-\phi) + \frac{\sinh M}{M\cosh M - \sinh M}\right\}.$$

From this it can be observed that it is the fully developed friction factor, f, not K, that is influenced by the electric field. The term K may be considered as a correction term to be added to the fully developed pressure drop to account for the entrance effects.

NOMENCLATURE

A	One-half the channel cross-sectional area
a	Channel half-height
B	Magnetic field intensity
E	Electric field strength
K	Pressure correction term
M	Hartmann number, $M = B_0 \, a(\frac{\sigma}{\rho \nu})^{\frac{1}{2}}$
P	Fluid pressure
R_e	Reynold's number, $R_e = \rho \bar{U} a/\mu$
s	Laplace transform variable
u	x-component of velocity
v	y-component of velocity
$\bar{U}$	Mean velocity
x, y, z	Space coordinates
x^*	Stretched axial coordinate
β	$\beta = x^*/R_e a$
γ_n	Eigenvalues of $\tan \gamma_n = \gamma_n$
$\epsilon(x)$	Scalar factor
η	Dimensionless transverse coordinate, $\eta = \frac{y}{a}$
α	Dimensionless x-component of velocity, $\alpha = \frac{u}{\bar{U}}$
α_0	α evaluated at $y=0$
$\bar{\alpha}$	Laplace transform of α
λ	$\lambda = (M^2+s)^{\frac{1}{2}}$
$\Lambda(x)$	Defined by equation (3)
μ	Viscosity
ν	Kinematic viscosity

ρ	Density
σ_e	Electrical conductivity
ϕ	$\phi = E_0/\bar{U}B_0$

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SOME ANALYTICAL SOLUTIONS OF MAGNETOHYDRODYNAMIC ENTRANCE REGION FLOW PROBLEM

by

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Recently attention has been directed toward the effect of a magnetic field on the flow of an electrically conductive fluid, usually referred to as magnetohydrodynamics (MHD). MHD is important because it finds many applications in engineering, such as an MHD generator, the pumping of conducting fluids, plasma confinement for the fusion reaction, propulsion and flight control for rocket and hypersonic aerodynamic vehicles.

In many MHD applications, the flow of the extremely high temperature fluids is seldom fully developed. For this reason, a study of the velocity fields, boundary layer development, and friction factors in the entrance region of MHD channel is of practical importance and has been a subject of investigation in recent years. Although the exact analytical solution is not available, many approximate solutions have been presented. The approximation methods of solution can be classified into four different categories: the momentum integral method, the linearization method, the matching method, and the finite difference method.

The purpose of this report is to study laminar MHD flow in the entrance region of a flat duct by presenting in detail the solutions according to the momentum integral method, the linearization method, and the matching method.

A brief introduction of basic governing equations of MHD flow -- Maxwell's equations, the continuity equation, and the modified Navier-Stokes equation -- is given in Chapter 1.

In Chapter 2, Schlichting's matching method applied by Roidt and Cess to MHD entrance region flow is presented in detail. The flow field is divided into two sections and an appropriate analysis utilized in each. In the section near the inlet a boundary-layer formulation of the equation is used and a solution developed in a series stream function with Blasius' function as coefficients. When this solution becomes unwieldy, an exponential velocity deviation from the fully developed flow is assumed and this is joined to the boundary-layer solution to complete the description of the flow.

Chapter 3 contains a detailed account of Schiller's integral method applied by Maciulaitis and Loeffler to the problem. A momentum integral equation is derived first. Then by applying the assumed velocity profile, which involves the boundary layer thickness, to the momentum integral equation, a relation between the boundary layer thickness is obtained. The solution is completed after carrying out the integration. Two cases, the developing of velocity from a uniform profile at the entry to Hartmann velocity and the developing of velocity from a parabolic one at the entry to the Hartmann one, are considered.

In Chapter 4, the linearization method evolved from ideas proposed by Targ and applied by Snyder is presented in detail. The nonlinear partial differential equation of motion is linearized by the assumption that the convective terms are a function only of the direction of flow. A stretched axial coordinate is introduced in the linearization of the convective terms is the key point of the method.