MATRIX ANALYSIS OF CONTINUOUS PARABOLIC ARCHES

by 149

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SYNOPSIS

The purpose of this report is to present a method of analyzing continuous arches with varying cross section on slender piers by obtaining influence lines of the redundants. The analysis is based on the displacement method.

First, the single fixed arch theory is presented. Then, the continuous arch is analyzed by applying the displacement method. Finally, a numerical example consisting of two unsymmetrical parabolic continuous arches with a central slender pier is given to illustrate the use of the method.

Influence lines for the redundants are drawn. The influence coefficients are checked by using the energy method. The comparison shows that the results agree closely with each other.

A detailed flow diagram is given to demonstrate the solution of the continuous arch problem by a digital computer. The flow diagram is based on the following assumptions: (1) it applies to any number of arch spans, with interior arch joints on piers; (2) the equation of the centroidal axis of each arch can be expressed as $y = \alpha x^2 + \beta x + \gamma$ with origin at either end; (3) the ratio of the moment of inertia at any section of the arch to the moment of inertia at the crown is $\sec\theta$, where θ is the angle between the tangent to the arch and the horizontal axis; and (4) the ratio of the moment of inertia of the pier to that of the crown is constant.

INTRODUCTION

Matrix analysis is a relatively new approach to structural analysis. The main advantage of analyzing a structural system by the matrix method is that the analyses can be performed by a computer conveniently. The matrix method is particularly easy to handle if a structure must be analyzed for the effects of several loading patterns, such as that used in determining influence lines.

Fig. 1 Typical continuous arch on slender piers.

"An arch is a girder (beam or truss) usually curved in form, that develops reactions with inwardly directed horizontal components under the action of vertical loads alone."[1] A typical continuous arch on slender piers is shown in Fig. 1. The effect of slender piers is, in general, to decrease the horizontal thrusts, to increase the crown moments and to throw stress onto adjacent arch spans and onto other piers. The slender piers are a necessary provision for a large span.

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1. John I. Parcel and Robert B. B. Moorman, Analysis of Statically Indeterminate Structures, John Wiley and Sons, Inc., New York, 1962, p. 457.

The matrix method presented in this report is the displacement method. Basically, it makes use of the single fixed arch theory, namely the method of analyzing a single fixed arch. The analysis requires a knowledge of the equations of the centroidal axis of each single arch and of the relative moments of inertia.

This method also involves application of the Müller-Breslau Principle and numerical integration. Although it is quite tedious to use for manual computations, the basic idea of each method is simple. It should be introduced whenever a computer is available.

SINGLE FIXED ARCH THEORY

An unsymmetrical fixed arch is shown in Fig. 2(a). It is statically indeterminate to the third degree. Figure 2(c) indicates the statically determinate base structure obtained by making a free end at point O.

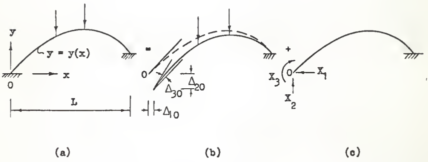

Fig. 2 Fixed arch analysis by superposition.

In the following discussion, stresses are assumed to be below the elastic limit, that is, Hooke's law applies.

Superposition can be used as long as Hooke's law applies. From Fig. 2, the following three equations exist:

$$X_1 \delta_{11} + X_2 \delta_{12} + X_3 \delta_{13} = \Delta_{10} ,$$

$$X_1 \delta_{21} + X_2 \delta_{22} + X_3 \delta_{23} = \Delta_{20} ,$$

$$X_1 \delta_{31} + X_2 \delta_{32} + X_3 \delta_{33} = \Delta_{30} .$$

In matrix notation,

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{bmatrix} \quad \ldots\ldots\ldots (1)$$

where

X_1 = horizontal force at origin point 0,

X_2 = vertical force at point 0,

X_3 = moment at point 0,

δ_{ij} = displacement in base structure at point 0 in X_i direction, due to X_j = unit acting only,

Δ_{i0} = displacement in base structure at point 0 in X_i direction, due to all external loads acting.

The δ terms can be determined by using the dummy unit load method,

$$\delta_{11} = \int_{Area} \frac{m_1 m_1}{EI}\, ds = \int_A \frac{y^2\, ds}{EI} = \int_0^L \frac{y^2\, dx}{EI \cos\theta},$$

$$\delta_{22} = \int_{Area} \frac{m_2 m_2}{EI}\, ds = \int_A \frac{x^2\, ds}{EI} = \int_0^L \frac{x^2\, dx}{EI \cos\theta},$$

$$\delta_{33} = \int_{Area} \frac{1}{EI}\, ds = \int_0^L \frac{dx}{EI \cos\theta},$$

$$\delta_{12} = \delta_{21} = \int_{Area} \frac{m_1 m_2}{EI}\, ds = \int_A \frac{xy\, ds}{EI} = \int_0^L \frac{xy\, dx}{EI \cos\theta},$$

$$\delta_{13} = \delta_{31} = \int_{Area} \frac{m_1}{EI}\, ds = \int_A \frac{y\, ds}{EI} = \int_0^L \frac{y\, dx}{EI \cos\theta},$$

$$\delta_{23} = \delta_{32} = \int_{Area} \frac{m_2}{EI}\, ds = \int_A \frac{x\, ds}{EI} = \int_0^L \frac{x\, dx}{EI \cos\theta}.$$

Defining

$$\int_0^L \frac{y^2\,dx}{EI\cos\theta} = I_x\,,\quad \int_0^L \frac{x^2\,dx}{EI\cos\theta} = I_y\,,\quad \int_0^L \frac{dx}{EI\cos\theta} = A\,,$$

$$\int_0^L \frac{xy\,dx}{EI\cos\theta} = I_{xy},\quad \int_0^L \frac{y\,dx}{EI\cos\theta} = A\bar{y}\,,\quad \int_0^L \frac{x\,dx}{EI\cos\theta} = A\bar{x}\,.$$

Eq. (1) becomes,

$$\begin{pmatrix} I_x & I_{xy} & A\bar{y} \\ I_{xy} & I_y & A\bar{x} \\ A\bar{y} & A\bar{x} & A \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots (2)$$

X_1, X_2, and X_3, which are obtained by setting $\Delta_{10}=1$, $\Delta_{20}=0$, and $\Delta_{30}=0$, will cause a unit displacement in the X_1 direction only when X_1, X_2, and X_3 are applied simultaneously at 0. In other words, this is the way to determine the reactions at the origin point 0 due to a unit horizontal displacement at point 0.

The same argument applies to the other two cases, that is, $\Delta_{10}=0$, $\Delta_{20}=1$, $\Delta_{30}=0$ and $\Delta_{10}=0$, $\Delta_{20}=0$, $\Delta_{30}=1$.

Substituting for Δ_{10}, Δ_{20}, and Δ_{30} in Eq.(2),

$$\begin{pmatrix} I_x & I_{xy} & A\bar{y} \\ I_{xy} & I_y & A\bar{x} \\ A\bar{y} & A\bar{x} & A \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then,

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{bmatrix} I_x & I_{xy} & A\bar{y} \\ I_{xy} & I_y & A\bar{x} \\ A\bar{y} & A\bar{x} & A \end{bmatrix}^{-1} = \begin{pmatrix} H_1 & H_2 & H_3 \\ V_1 & V_2 & V_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \quad \ldots\ldots\ldots(3)$$

where

H_1, V_1, and M_1 will cause a unit displacement at 0 in the X_1 direction,
H_2, V_2, and M_2 will cause a unit displacement at 0 in the X_2 direction,
H_3, V_3, and M_3 will cause a unit displacement at 0 in the X_3 direction.

To obtain the influence lines, Müller-Breslau's Principle plays an important role. "The ordinates of the influence line for any stress element (such as axial force, shear, moment, or reaction) of any structure are proportional to those of the deflection curve which is obtained by removing the restraint corresponding to that element from the structure and introducing in its place a corresponding deformation into the primary structure which remains."[2] Furthermore, "In the case of an indeterminate structure, this principle is limited to structures the material of which is elastic and follows Hooke's law."[2] In other words, Müller-Breslau's Principle states that "an influence line may be drawn by producing artificially a unit displacement corresponding to the 'stress' for which the influence line is desired. The term 'stress' includes reaction, thrust, moment or shear, as the case may be."[3]

2. Charles H. Norris and John B. Wilbur, Elementary Structural Analysis, McGraw-Hill Book Co., New York, 1960, p.493.

3. P. C. A. Concrete Information ST 41-2, "Concrete Building Frames Analyzed by Moment Distribution," p.2.

Thus, the influence line for the horizontal force X_1 is the deflection curve caused by H_1, V_1, and M_1; the influence line for the vertical force X_2 is the deflection curve caused by H_2, V_2, and M_2; and the influence line for the moment X_3 is the deflection curve caused by H_3, V_3, and M_3.

To find the deflection curves, the conjugate beam method is used. The deflection curve of the real beam is obtained by double integration of the area of the elastic weight on the conjugate beam. That is

$$\text{defl'n} = \int_A \int \frac{M}{EI}\, ds \ ,$$

where

$$M = M_1 + V_1\, x + H_1\, y \ ,$$

or $$M = M_2 + V_2\, x + H_2\, y \ ,$$

or $$M = M_3 + V_3\, x + H_3\, y \ .$$

This can be performed by a digital computer. The Gaussian 5-point Integration Formula[4] is used for the first integration and Simpson's Rule is used for the second integration.

A detailed flow diagram for single fixed arch theory is given in the first part of the flow diagram of the displacement method in Appendix A, to illustrate the computer approach.

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4. S. D. Conte, Elementary Numerical Analysis, McGraw-Hill Book Company, New York, 1965, p.138.

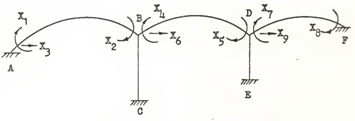


Fig. 3 The choice of redundants.

Figure 3 shows a series of arches on slender piers. The arches may or may not be identical and symmetrical. It is assumed that the equation of the centroidal axis of each arch is given and that the relative moments of inertia are known. It is also assumed that the material obeys Hooke's law.

The structure shown in Fig. 3 is statically indeterminate to the ninth degree. In other words, nine redundants must be removed to obtain a statically determinate structure, or base structure. Among numerous base structures, the one shown in Fig. 3 is suggested. The reason for this selection is that with this base structure identical redundant patterns can be assigned to each single arch.

The redundants for the continuous arches can be analyzed by the displacement method once the stiffness values are known. These values can be conveniently determined by applying single fixed arch theory.

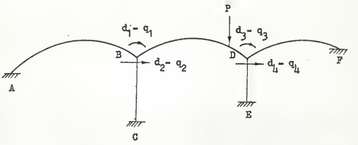


Fig. 4 Degrees of freedom.

Figure 4 shows the four degrees of freedom of the structure shown in Fig. 3. The degrees of freedom at the joints are indicated by arrows d_1, d_2, d_3, and d_4. The positive sense of rotations, displacements, moments, and forces is as indicated by these arrows. The q terms are the unbalanced joint moments or unbalanced horizontal forces due to the external load P.

The displacement method of analysis may be divided into the following operations:

(a) Analysis of each single arch assuming that they are fixed ended and determination of the end moments and thrusts due to $d_i = 1$, $i=1, 2, 3, 4$.

This can be done by applying the single fixed arch theory discoussed in the previous section. Furthermore, influence lines for the reactions of each single arch can also be obtained.

(b) Determination of the moments and horizontal forces on the pier due to $d_i = 1$, $i=1, 2, 3, 4$.

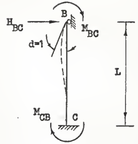


Fig. 5 Pier subjected to a unit rotation.

Figure 5 shows a pier subjected to a unit rotation, with no translation and with the far end fixed.

Then, by the slope-deflection method,

$$M_{BC} = \frac{2EI}{L}\,(2\theta_B + \theta_C) = \frac{2EI}{L}\,(2) = \frac{4EI}{L}\,,$$

$$M_{CB} = \frac{2EI}{L}\,(\theta_B + 2\theta_C) = \frac{2EI}{L}\,,$$

$$H_{BC} = \frac{-1}{L}\,(M_{BC} + M_{CB}) = -\frac{6EI}{L^2}\,.$$

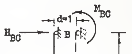


Fig. 6 Pier subjected to a unit translation.

Figure 6 shows a pier subjected to a unit translation, with no rotation and the far end fixed.

Also, by the slope-deflection method,

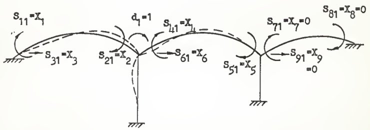
$$M_{BC} = \frac{2EI}{L}\left(-3\,\frac{1}{L}\right) = -\frac{6EI}{L^2},$$

$$M_{CB} = \frac{2EI}{L}\left(-3\,\frac{1}{L}\right) = -\frac{6EI}{L^2},$$

$$H_{BC} = -\frac{1}{L}\left(M_{BC} + M_{CB}\right) = \frac{12EI}{L^3}.$$

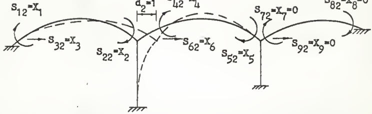
(c) Determination of [S], the stiffness matrix for a single arch and pier; and of [K], the stiffness matrix of the whole structure.

The values of the S terms are found in step (a). For instance, S_{ij} is the value of end moment or thrust for a single arch in X_i direction due to $d_j = 1$. Figure 7 shows the determination of S_{ij} due to $d_j = 1$, $j = 1, 2, 3, 4$.

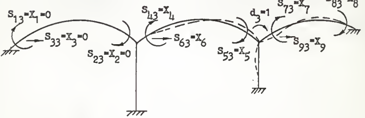


(a) $d_1 = 1$.

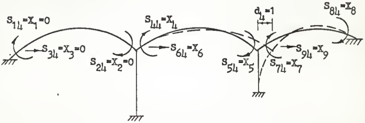
Fig. 7 Determination of [S].



(b) $d_2 = 1$.



(c) $d_3 = 1$.



(d) $d_4 = 1$.

Fig. 7 Determination of [S]. (Continued)

The stiffness coefficients of [K] at the joints are then obtained by combination of the S_{ij} values and the stiffnesses of the piers obtained in step (b). In other words, K_{ij} is the value of moment or force in the q_i direction due to $d_j = 1$. For example,

$$K_{11} = S_{21} + S_{41} + \left(\frac{4EI}{L}\right)_{pier} ,$$

$$K_{21} = S_{31} + S_{61} + \left(\frac{-6EI}{L^2}\right)_{pier} .$$

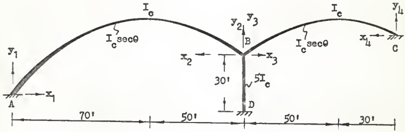
(d) Determination of the resultant reactions, $\{FT\}$.

Since [K] is obtained in step (c), the unknown rotations and displacements can be found from $\{D\} = [K]^{-1}\{q\}$. With the rotations and displacements known, correction moments and forces at the ends of the members may be obtained, making use of the single arch stiffness [S]. The reactions, due to the correction of $\{D\}$, are given by $\{FD\} = [S]\{D\}$. Then the final moments and forces are obtained by $\{FT\} = \{FF\} + \{FD\}$, Where $\{FF\}$ is the fixed end moment or force due to the external load P.

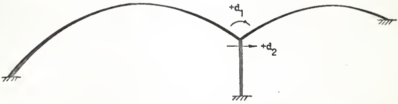
Influence coefficients of the redundants are determined by varying the position of P, where P is a unit load. A numerical example will be given to demonstrate the method of analysis.

A complete flow diagram for determining the influence coefficient of the redundants by the displacement method is given in Appendix A. This flow diagram can be used to analyze any number of arches with interior arch joints on piers. The flow diagram has been formulated based on the following assumptions: (1) the equation of the centroidal axis of every single arch can be expressed in the form $y = \alpha x^2 + \beta x + \gamma$, with origin at each end point; (2) the ratio of I, the moment of inertia at any section of a single arch, to I_c, the moment of inertia at the crown of the arch, is equal to $\sec\theta$, where θ is the angle between the horizontal axis and the tangent to the arch at the corresponding section; (3) the ratio of I_p, the moment of inertia of the pier, to I_c is constant.

NUMERICAL EXAMPLE



(a) Dimensions.



(b) Degrees of freedom.

Fig. 8 Numerical example.

Given: A two-span continuous parabolic arch on a slender pier is given in Fig. 8.

For arch AB: $y_1 = -0.008x_1^2 + 1.12x_1$ origin at pt. A

$y_2 = -0.008x_2^2 + 0.8x_2$ origin at pt. B

For arch BC: $y_3 = -0.008x_3^2 + 0.8x_2$ origin at pt. B

$y_4 = -0.008x_4^2 + 0.48x_4$ origin at pt. C

Required: Influence lines for M_{AB}, M_{BA}, H_{AB}, M_{BC}, M_{CB}, and H_{BC}.

Solution:

Step 1. Determination of the end moment and thrust for each arch due to $d_1 = 1$, and $d_2 = 1$.

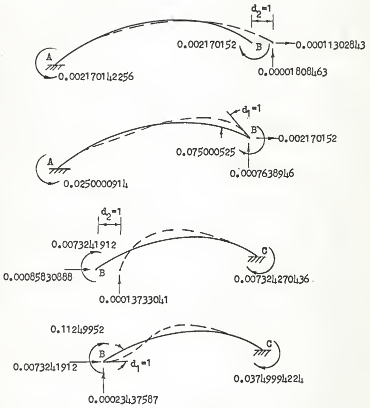


Fig. 9 End moments and forces of each arch due to $d_1 = 1$, $d_2 = 1$. (in terms of EI_c)

The moment and forces of each arch at support B due to $d_1 = 1$ or $d_2 = 1$ are determined by applying Eq. (3) from the single fixed arch theory. The moment at the opposite end, such as support A or support C, is then calculated from the equations of statics. The results are shown in Fig. 9.

Step 2. Determination of the end moment and horizontal force for the pier due to $d_1 = 1$, $d_2 = 1$. The results are illustrated in Fig. 10.

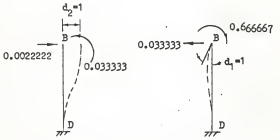


Fig. 10 End moments and horizontal forces due to $d_1 = 1$, $d_2 = 1$. (in terms of EI_c)

Horizontal force due to $d_1 = 1$, is

$$H = -\frac{6\ EI}{L^2} = -\frac{6\ E(5I_c)}{30^2} = -0.033333\ EI_c.$$

Moment due to $d_1 = 1$, is

$$M = \frac{4\ EI}{L} = \frac{4\ E(5I_c)}{30} = 0.666667\ EI_o.$$

Horizontal force due to $d_2 = 1$, is

$$H = \frac{12\ EI}{L^3} = \frac{12\ E(5I_c)}{30^3} = 0.0022222\ EI_o.$$

Moment due to $d_2 = 1$, is

$$M = -\frac{6\ EI}{L^2} = -\frac{6\ E(5I_c)}{30^2} = -0.033333\ EI_c.$$

The values determined in Step 1 and Step 2 are listed in Table I.

Table I Stiffness of single arch and pier.

		$d_1 = 1$	$d_2 = 1$
ARCH AB	M_{AB}	-0.0250000914	-0.002170142256
	M_{BA}	+0.075000525	+0.00217052
	H_{BA}	+0.00217052	+0.0001130 2843
ARCH BC	M_{CB}	-0.0374999422 4	-0.007324270436
	M_{BC}	+0.11249952	+0.00732419 12
	H_{BC}	+0.00732419 12	+0.00085830888
PIER BD	M_{BD}	+0.666667	-0.033333
	H_{BD}	-0.033333	+0.0022222

Step 3. Determination of the stiffness for single arches and pier. They are determined in Step 1 and Step 2, namely,

$$[\ S\ ] = \begin{bmatrix} -0.0250000914 & -0.002170142256 \\ +0.075000525 & +0.00217052 \\ +0.00217052 & +0.0001130 2843 \\ +0.11249952 & +0.0073241912 \\ -0.0374999422 4 & -0.007324270436 \\ +0.0073241912 & +0.00085830888 \end{bmatrix}$$

Step 4. Determination of [K], the stiffness matrix of the whole structure. Hence $[K]^{-1}$ follows.

$$K_{11} = 0.075000525 + 0.11249952 + 0.666666667 = 0.854166712 \; EI_c \, ,$$

$$K_{21} = 0.002170152 + 0.0073241912 - 0.03333333 = -0.0238389901 \; EI_c \, ,$$

$$K_{12} = 0.002170152 + 0.0073241912 - 0.03333333 = -0.0238389901 \; EI_c \, ,$$

$$K_{22} = 0.00011302843 + 0.00085830888 + 0.002222222 = 0.00319355953 \; EI_c \, .$$

Or,

$$[K] = EI_c \begin{bmatrix} 0.854166712 & -0.0238389901 \\ -0.0238389901 & 0.00319355953 \end{bmatrix}$$

Then,

$$[K]^{-1} = \frac{1}{EI_c} \begin{bmatrix} 1.4788182 & 11.038946 \\ 11.038946 & 395.53273 \end{bmatrix}$$

Step 5. Set up [q] and [FF].

By applying single fixed arch theory, the influence lines for M_{AB}, M_{BA}, H_{BA} in single arch AB and influence lines for M_{BC}, M_{CB}, H_{BC} in single arch BC are obtained in Appendix C.

[q], the matrix of unbalanced joint moments and unbalanced horizontal forces, is set up from the influence coefficients of M_{BA}, M_{BC}, H_{BA}, H_{BC}; while [FF], the matrix of moments or thrust due to fixed ends, is set up from the influence coefficients of M_{AB}, M_{BA}, H_{BA}, M_{BC}, M_{CB}, H_{BC}. [q] and [FF] are given in Table II and Table III respectively.

Table II Matrix q_{ij}

j \ i	1	2
1	+0.	+0.
2	+0.663706	+0.060493907
3	+2.1570424	+0.2086422
4	+3.840006	+0.40000031
5	+5.214817	+0.5975311
6	+5.925919	+0.771605
7	+5.759987	+0.900000
8	+4.6459	+0.9679008
9	+2.65475	+0.9678999
10	-0.00009	+0.8999978
11	-2.96309	+0.7716017
12	-5.73645	+0.5975266
13	-7.68019	+0.399996
14	-8.01205	+0.208637
15	-5.8076	+0.060488
16	+0.	+0.
17	+4.85994	-0.189866
18	+5.11994	-0.600026
19	+2.93993	-1.033621
20	-0.00005	-1.3500243
21	-2.500032	-1.464864
22	-3.840012	-1.3500151
23	-3.780003	-1.0336038
24	-2.56	-0.6000052
25	-0.8999988	-0.18984518
26	+0.	+0.

Table III Matrix FF_{ij}

j \ i	1	2	3	4	5	6
1	+0.	+0.	+0.	+0.	+0.	+0.
2	-5.80789	-0.663706	-0.0604939	+0.	+0.	+0.
3	-8.01236	-2.1570424	-0.2086422	+0.	+0.	+0.
4	-7.68051	-3.840006	-0.40000031	+0.	+0.	+0.
5	-5.73677	-5.214817	-0.5975311	+0.	+0.	+0.
6	-2.96343	-5.925919	-0.771605	+0.	+0.	+0.
7	-0.0004	-5.759987	-0.9000	+0.	+0.	+0.
8	+2.65448	-4.6459	-0.9679008	+0.	+0.	+0.
9	+4.64567	-2.65475	-0.9678999	+0.	+0.	+0.
10	+5.759802	+0.00009	-0.8999978	+0.	+0.	+0.
11	+5.925783	+2.96309	-0.7716017	+0.	+0.	+0.
12	+5.214722	+5.73645	-0.5975266	+0.	+0.	+0.
13	+3.83995	+7.68019	-0.399996	+0.	+0.	+0.
14	+2.1570159	+8.01205	-0.208637	+0.	+0.	+0.
15	+0.6636984	+5.8076	-0.060488	+0.	+0.	+0.
16	+0.	+0.	+0.	+0.	+0.	+0.
17	+0.	+0.	+0.	-4.85994	-0.900001	+0.189866
18	+0.	+0.	+0.	-5.11994	-2.559999	+0.600026
19	+0.	+0.	+0.	-2.93993	-3.779982	+1.033621
20	+0.	+0.	+0.	+0.00005	-3.839956	+1.3500243
21	+0.	+0.	+0.	+2.500032	-2.499904	+1.464864
22	+0.	+0.	+0.	+3.840012	+0.00015	+1.3500151
23	+0.	+0.	+0.	+3.780003	+2.94021	+1.0336038
24	+0.	+0.	+0.	+2.56	+5.12026	+0.6000052
25	+0.	+0.	+0.	+0.8999988	+4.86028	+0.18984518
26	+0.	+0.	+0.	+0.	+0.	+0.

Step 6. Determination of [FT], the matrix of influence coefficients for M_{AB}, M_{BA}, H_{BA}, M_{BC}, M_{CB}, and H_{BC}.

[FT] is obtained in the following manner.

$$[\,FT\,] = [\,FF\,] + [\,FD\,]$$

$$= [\,FF\,] + [\,S\,]\,[\,D\,]$$

$$= [\,FF\,] + [\,S\,]\,[\,K\,]^{-1}\,[\,q\,]$$

The results of [FT] are drawn in Fig. 11 through Fig. 16.

These results are checked in Appendix B using the energy method. A comparison of the influence coefficients obtained from the two methods shows excellent agreement (see Figs 11-16).

Energy Method	Displacement Method
+0.0	+0.0
+0.27015	+0.27018
+0.83647	+0.83658
+1.40257	+1.40276
+1.76510	+1.76533
+1.81372	+1.81397
+1.53117	+1.53139
+0.99320	+0.99335
+0.36862	+0.36869
-0.08074	-0.08073
+0.0	+0.0
+0.94874	+0.94892
+2.40805	+2.40850
+3.85338	+3.85415
+4.88537	+4.88643
+5.22975	+5.23106
+4.73745	+4.73891
+3.38450	+3.38600
+1.27209	+1.27350
-1.37346	-1.37224
-4.20065	-4.19974
-6.73289	-6.73229
-8.36842	-8.36820
-8.38033	-8.38045
-5.91658	-5.91695
+0.0	+0.0

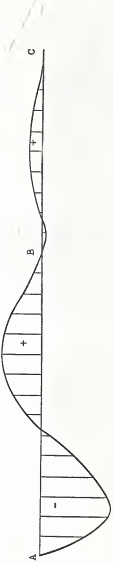


Fig. 11 Influence line for M_{AB}.

Energy Method	Displacement Method
+0.0	+0.0
-0.44148	-0.44152
-1.35694	-1.35705
-2.25258	-2.25276
-2.79418	-2.79442
-2.80712	-2.80737
-2.27632	-2.27654
-1.34632	-1.34648
-0.32121	-0.32130
+0.33531	+0.33528
+0.0	+0.0
+5.12633	+5.12634
+7.28354	+7.28330
+7.31942	+7.31889
+5.97126	+5.97039
+3.86576	+3.86461
+1.51910	+1.51774
-0.66308	-0.66455
-2.38566	-2.38715
-3.46409	-3.46548
-3.82435	-3.82555
-3.50295	-3.50389
-2.64695	-2.64759
-1.51395	-1.51429
-0.47209	-0.47218
+0.0	+0.0

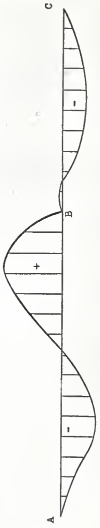


Fig. 12 Influence line for M_{BA}. +

Energy Method	Displacement Method
+0.0	+0.0
-0.01705	-0.01705
-0.05260	-0.05261
-0.08781	-0.08782
-0.10980	-0.10981
-0.11171	-0.11172
-0.09269	-0.09270
-0.05786	-0.05787
-0.01838	-0.01838
+0.00863	+0.00862
+0.0	+0.0
-0.08222	-0.08222
-0.23000	-0.23002
-0.40673	-0.40676
-0.58201	-0.58207
-0.73176	-0.73183
-0.83812	-0.83820
-0.88952	-0.88961
-0.88065	-0.88074
-0.81245	-0.81253
-0.69214	-0.69221
-0.53321	-0.53326
-0.35538	-0.35542
-0.18468	-0.18470
-0.05338	-0.05338
+0.0	+0.0

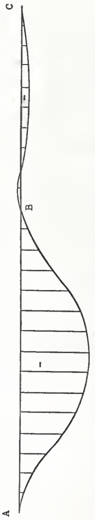


Fig. 13 Influence line for H_{BA} $(=-H_{AB})$. $\bullet\!\longrightarrow +$

Energy Method	Displacement Method
+0.0	+0.0
-0.10825	-0.10823
-0.55625	-0.55620
-1.43247	-1.43240
-2.69689	-2.69680
-4.18093	-4.18085
-5.58755	-5.58749
-6.49113	-6.49111
-6.33757	-6.33761
-4.44424	-4.44430
+0.0	+0.0
-1.18541	-1.18539
-1.11738	-1.11721
-0.24351	-0.24316
+1.05438	+1.05491
+2.46031	+2.46100
+3.72411	+3.72492
+4.66140	+4.66228
+5.15365	+5.15453
+5.14810	+5.14892
+4.65783	+4.65854
+3.76173	+3.76228
+2.60448	+2.60485
+1.39659	+1.39679
+0.41440	+0.41445
+0.0	+0.0

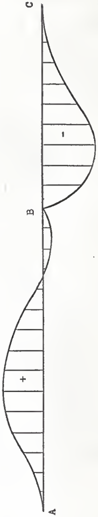


Fig. 14 Influence line for M_{BC} .

Energy Method	Displacement Method
+0.0	+0.0
+5.61124	+5.61152
+7.45553	+7.45579
+6.87745	+6.87766
+4.99324	+4.99340
+2.69084	+2.69096
+0.62983	+0.62991
-0.75855	-0.75845
-1.27132	-1.27123
-0.93390	-0.93381
+0.0	+0.0
+0.59136	+0.59135
+0.40143	+0.40132
-0.27726	-0.27750
-1.19609	-1.19646
-2.15036	-2.15084
-2.97926	-2.97983
-3.56591	-3.56653
-3.83732	-3.83794
-3.76440	-3.76499
-3.36199	-3.36249
-2.68882	-2.68922
-1.84754	-1.84780
-0.98468	-0.98483
-0.29072	-0.29076
+0.0	+0.0

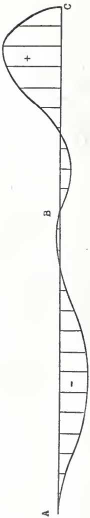


Fig. 15 Influence line for M_{CB}. +

Energy Method	Displacement Method
+0.0	+0.0
+0.09176	+0.09177
+0.29581	+0.29582
+0.52237	+0.52238
+0.70456	+0.70457
+0.79834	+0.79836
+0.78253	+0.78255
+0.65883	+0.65885
+0.45176	+0.45178
+0.20873	+0.20874
+0.0	+0.0
-0.09251	-0.09250
-0.07501	-0.07499
+0.01215	+0.01218
+0.13463	+0.13468
+0.26410	+0.26417
+0.37823	+0.37830
+0.46067	+0.46076
+0.50110	+0.50119
+0.49519	+0.49527
+0.44460	+0.44467
+0.35701	+0.35706
+0.24608	+0.24611
+0.13148	+0.13150
+0.03890	+0.03891
+0.0	+0.0

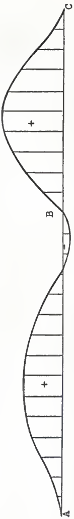


Fig. 16 Influence line for H_{BC} $(=-H_{CB})$. $\cdot\longrightarrow +$

CONCLUSIONS

The following conclusions can be made from the work completed in this report.

(1) Since the influence lines for the redundants are obtained, the moment, shear and thrust at any section due to any loading condition on the structure can be calculated.

(2) The matrix analysis presented in this report can be preformed conveniently by a digit computer rather than by hand computations.

(3) The method, presented in detail in the flow diagram, may be applied to any continuous arch, symmetrical or unsymmetrical.

(4) The flow diagram, given in appendix A, is based on the assumptions that the arches are parabolic; that the ratio of I, the moment of inertia at any section of the arch, to I_c, the moment of inertia at the crown, is secθ. However, with minor modifications the method is applicable to any other shape of arches and applicable to the oase where the ratio of I to I_c is a function of x.

(5) In this study, the influence coefficients of the redundants, obtained by the displacement method and by the energy method, have been shown to agree very closely. It is evident that any structure may be analyzed by one method, and then independently checked by the other.

ACKNOWLEDGMENT

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NOTATION

E modulus of elasticity of the material.

I moment of inertia at any section of arch.

I_c moment of inertia at crown.

I_p moment of inertia of pier.

θ the angle between the tangent at any section of the arch and horizontal axis.

A total area of an arch divided by EI.

$\bar{x}$ the dietance from centroid to y-axis.

$\bar{y}$ the distance from centroid to x-axis.

I_x moment of inertia of the area A with respect to the x-axis.

I_y moment of inertia of the area A with respect to the y-axis.

I_{xy} product of inertia of the area A with respect to x- and y-axes.

X_1 horizontal force at origin point O.

X_2 vertical force at origin point O.

X_3 moment at origin point O.

δ_{mn} displacement in baee structure at O in X_m direction, due to X_n= unit acting only.

Δ_{i0} displacement in baee etructure at O in X_i direction, due to all external loads acting.

m_1 moment anywhere in the structure due to X_1= unit.

m_2 moment anywhere in the etructure due to X_2= unit.

d_j degree of freedom.

α, β, γ arbitrary constants.

q_i unbalanced joint moment or horizontal force in d_i direction.

$X_1, X_2, \cdots, X_i$ redundants.

S_{ij} the moment or force in X_i direction due to $d_j = 1$.

K_{ij} the moment or force required at joint in q_i direction to cause $d_j = 1$.

P external load.

D displacement of joint.

FF moments or forces due to fixed ends.

FD moments or forces due to displacement D.

FT total moments or forces, sum of FF and FD.

[ ] matrix notation.

{ } column matrix.

[S] stiffness matrix of single arch.

[K] stiffness matrix of whole structure.

APPENDIX A

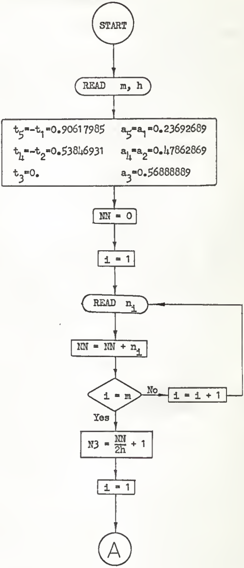
FLOW DIAGRAM FOR THE DISPLACEMENT METHOD

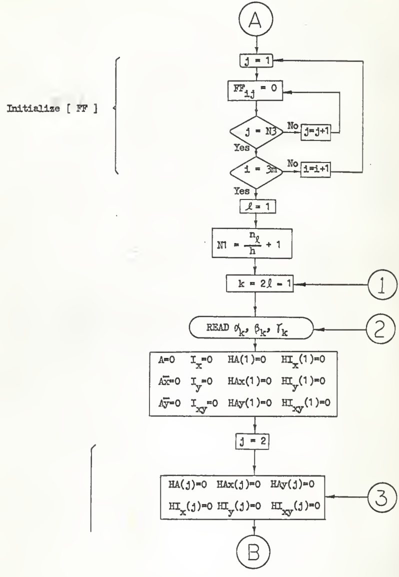
I. Single Arch Analysis

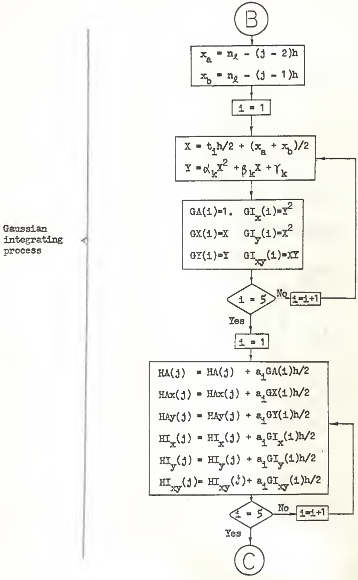
Number of arches & interval length

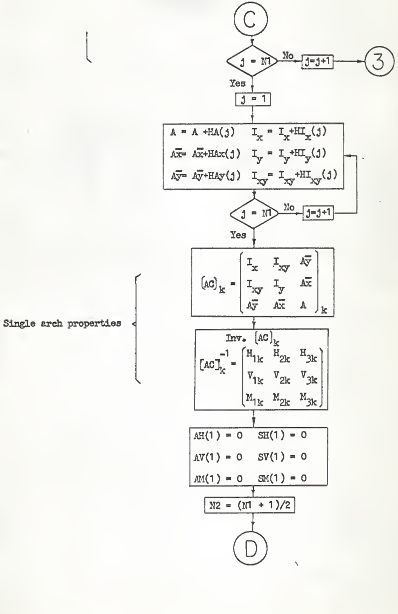
Constants of Gaussian 5-pt. Integration Formula

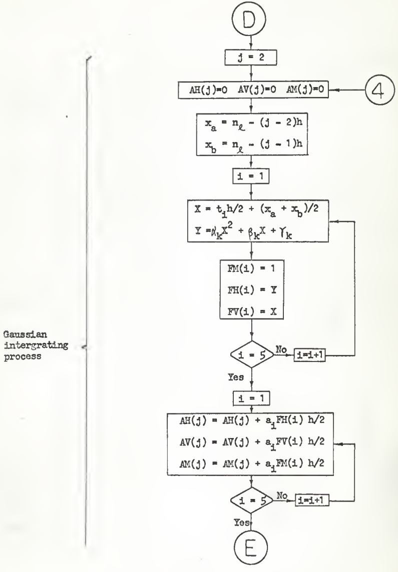
Arch span length

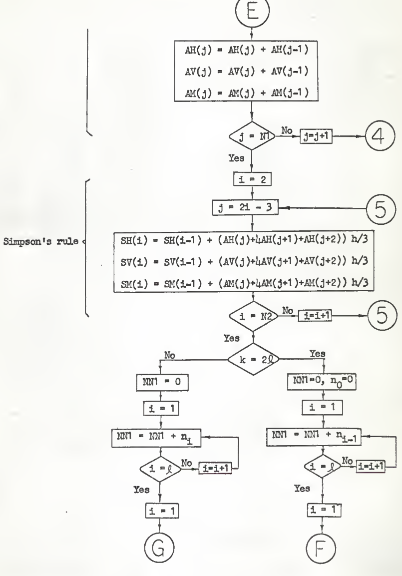


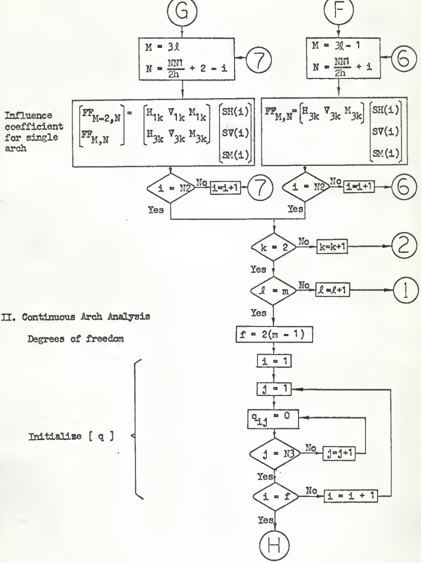

A
j = 1
Initialize [FF]
FF_ij = 0
j = N3
No
j=j+1
Yes
i = 3m
No
i=i+1
Yes
ℓ = 1
N1 = n_ℓ/h + 1
k = 2ℓ - 1
1
READ α_k, β_k, γ_k
2
A=0 I_x=0 HA(1)=0 HI_x(1)=0
Ax̄=0 I_y=0 HAx(1)=0 HI_y(1)=0
Aȳ=0 I_xy=0 HAy(1)=0 HI_xy(1)=0
j = 2
HA(j)=0 HAx(j)=0 HAy(j)=0
HI_x(j)=0 HI_y(j)=0 HI_xy(j)=0
3
B

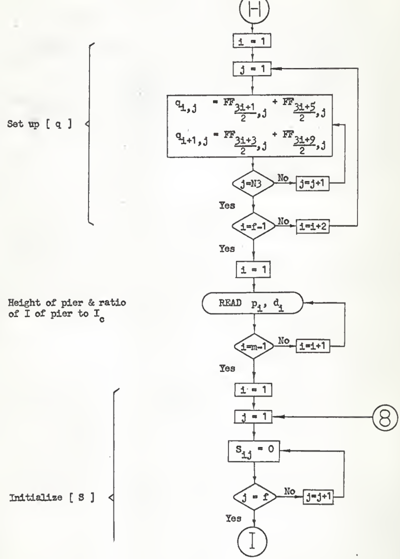

B
x_a = n_ℓ - (j - 2)h
x_b = n_ℓ - (j - 1)h
i = 1
X = t_i h/2 + (x_a + x_b)/2
Y = α_k X^2 + β_k X + γ_k
GA(i)=1. GI_x(i)=Y^2
GX(i)=X GI_y(i)=X^2
GY(i)=Y GI_xy(i)=XY
i = 5
No
i=i+1
Yes
i = 1
HA(j) = HA(j) + a_i GA(i)h/2
HAx(j) = HAx(j) + a_i GX(i)h/2
HAy(j) = HAy(j) + a_i GY(i)h/2
HI_x(j) = HI_x(j) + a_i GI_x(i)h/2
HI_y(j) = HI_y(j) + a_i GI_y(i)h/2
HI_xy(j)= HI_xy(j)+ a_i GI_xy(i)h/2
i = 5
No
i=i+1
Yes
C
Gaussian integrating process

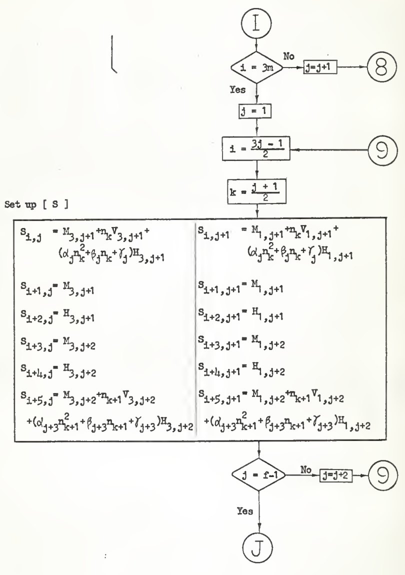

C
j = N1
No
j=j+1
3
Yes
j = 1
A = A +HA(j)
$I_x = I_x + HI_x(j)$
$A\bar{x} = A\bar{x} + HAx(j)$
$I_y = I_y + HI_y(j)$
$A\bar{y} = A\bar{y} + HAy(j)$
$I_{xy} = I_{xy} + HI_{xy}(j)$
j = N1
No
j=j+1
Yes
$[AC]_k = \begin{pmatrix} I_x & I_{xy} & A\bar{y} \\ I_{xy} & I_y & A\bar{x} \\ A\bar{y} & A\bar{x} & A \end{pmatrix}_k$
Single arch properties
$\text{Inv. } [AC]_k$
$[AC]_k^{-1} = \begin{pmatrix} H_{1k} & H_{2k} & H_{3k} \\ V_{1k} & V_{2k} & V_{3k} \\ M_{1k} & M_{2k} & M_{3k} \end{pmatrix}$
AH(1) = 0
SH(1) = 0
AV(1) = 0
SV(1) = 0
AM(1) = 0
SM(1) = 0
N2 = (N1 + 1)/2
D

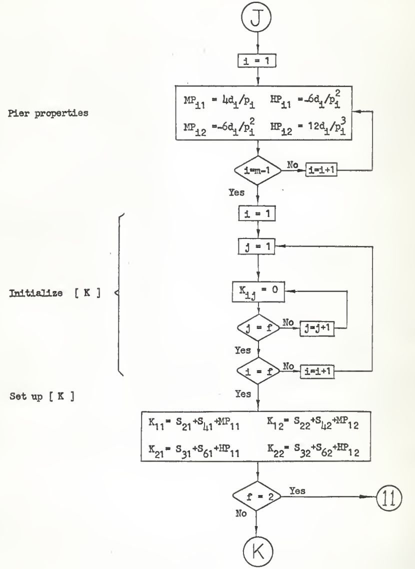

D
j = 2
AH(j)=0 AV(j)=0 AM(j)=0
4
$x_a = n_\ell - (j - 2)h$
$x_b = n_\ell - (j - 1)h$
i = 1
$X = t_i h/2 + (x_a + x_b)/2$
$Y = \alpha_k X^2 + \beta_k X + \gamma_k$
FM(i) = 1
FH(i) = Y
FV(i) = X
i = 5
No
i=i+1
Yes
i = 1
$AH(j) = AH(j) + a_i FH(i)\ h/2$
$AV(j) = AV(j) + a_i FV(i)\ h/2$
$AM(j) = AM(j) + a_i FM(i)\ h/2$
i = 5
No
i=i+1
Yes
E
Gaussian intergrating process

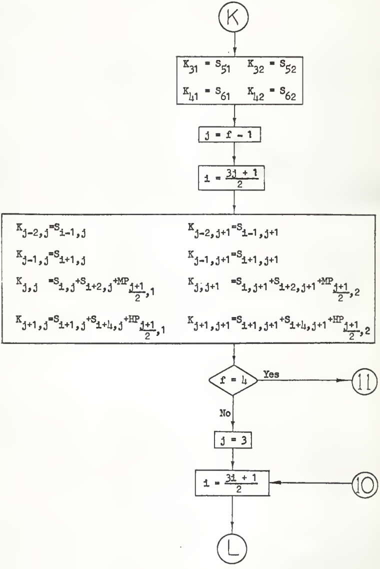

E
AH(j) = AH(j) + AH(j-1)
AV(j) = AV(j) + AV(j-1)
AM(j) = AM(j) + AM(j-1)
j = N1
No
j=j+1
4
Yes
i = 2
j = 2i - 3
5
Simpson's rule
SH(i) = SH(i-1) + (AH(j)+4AH(j+1)+AH(j+2)) h/3
SV(i) = SV(i-1) + (AV(j)+4AV(j+1)+AV(j+2)) h/3
SM(i) = SM(i-1) + (AM(j)+4AM(j+1)+AM(j+2)) h/3
i = N2
No
i=i+1
5
Yes
k = 2ℓ
No
Yes
NN1 = 0
NN1=0, n0=0
i = 1
i = 1
NN1 = NN1 + ni
NN1 = NN1 + ni-1
i = ℓ
No
i=i+1
i = ℓ
No
i=i+1
Yes
Yes
i = 1
i = 1
G
F

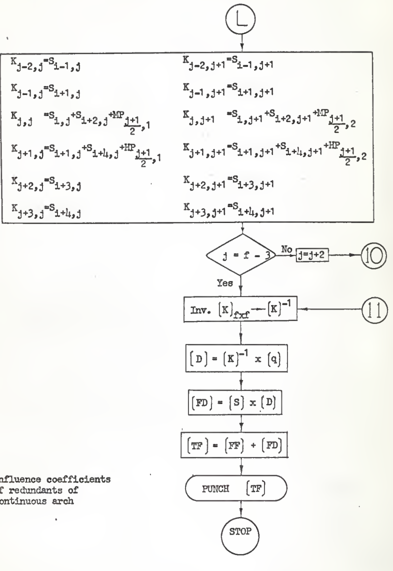

G
F
M = 3ℓ
N = NN1/2h + 2 − i
7
M = 3ℓ − 1
N = NN1/2h + i
6
Influence coefficient for single arch
FF_{M−2,N} ; FF_{M,N} = [H_{1k} V_{1k} M_{1k} ; H_{3k} V_{3k} M_{3k}] [SH(i) ; SV(i) ; SM(i)]
FF_{M,N} = [H_{3k} V_{3k} M_{3k}] [SH(i) ; SV(i) ; SM(i)]
i = N2
No
i=i+1
7
i = N2
No
i=i+1
6
Yes
Yes
k = 2
No
k=k+1
2
Yes
ℓ = m
No
ℓ = ℓ+1
1
Yes
II. Continuous Arch Analysis
Degrees of freedom
f = 2(m − 1)
i = 1
j = 1
Initialize [q]
q_{ij} = 0
j = N3
No
j=j+1
Yes
i = f
No
i = i + 1
Yes
H


H
i = 1
j = 1
$q_{i,j} = FF_{\frac{3i+1}{2},j} + FF_{\frac{3i+5}{2},j}$
$q_{i+1,j} = FF_{\frac{3i+3}{2},j} + FF_{\frac{3i+9}{2},j}$
j=N3
No
j=j+1
Yes
i=f-1
No
i=i+2
Yes
i = 1
READ p_i, d_i
i=m-1
No
i=i+1
Yes
i = 1
j = 1
8
S_{ij} = 0
j = f
No
j=j+1
Yes
I
Set up [q]
Height of pier & ratio of I of pier to I_c
Initialize [S]


I
i = 3m
No
j=j+1
8
Yes
j = 1
i = (3j - 1)/2
9
k = (j + 1)/2
Set up [S]
$S_{i,j} = M_{3,j+1} + n_k V_{3,j+1} + (\alpha_j n_k^2 + \beta_j n_k + \gamma_j) H_{3,j+1}$
$S_{i,j+1} = M_{1,j+1} + n_k V_{1,j+1} + (\alpha_j n_k^2 + \beta_j n_k + \gamma_j) H_{1,j+1}$
$S_{i+1,j} = M_{3,j+1}$
$S_{i+1,j+1} = M_{1,j+1}$
$S_{i+2,j} = H_{3,j+1}$
$S_{i+2,j+1} = H_{1,j+1}$
$S_{i+3,j} = M_{3,j+2}$
$S_{i+3,j+1} = M_{1,j+2}$
$S_{i+4,j} = H_{3,j+2}$
$S_{i+4,j+1} = H_{1,j+2}$
$S_{i+5,j} = M_{3,j+2} + n_{k+1} V_{3,j+2} + (\alpha_{j+3} n_{k+1}^2 + \beta_{j+3} n_{k+1} + \gamma_{j+3}) H_{3,j+2}$
$S_{i+5,j+1} = M_{1,j+2} + n_{k+1} V_{1,j+2} + (\alpha_{j+3} n_{k+1}^2 + \beta_{j+3} n_{k+1} + \gamma_{j+3}) H_{1,j+2}$
j = f-1
No
j=j+2
9
Yes
J


J
i = 1
Pier properties
MP_{i1} = 4d_i/p_i   HP_{i1} = -6d_i/p_i^2
MP_{i2} = -6d_i/p_i^2   HP_{i2} = 12d_i/p_i^3
i = m-1
No
i=i+1
Yes
i = 1
j = 1
Initialize [K]
K_{ij} = 0
j = f
No
j=j+1
Yes
i = f
No
i=i+1
Set up [K]
Yes
K_{11} = S_{21}+S_{41}+MP_{11}   K_{12} = S_{22}+S_{42}+MP_{12}
K_{21} = S_{31}+S_{61}+HP_{11}   K_{22} = S_{32}+S_{62}+HP_{12}
f = 2
Yes
11
No
K


K
$K_{31} = S_{51}$ $K_{32} = S_{52}$
$K_{41} = S_{61}$ $K_{42} = S_{62}$
$j = f - 1$
$i = \frac{3j + 1}{2}$
$K_{j-2,j} = S_{i-1,j}$
$K_{j-2,j+1} = S_{i-1,j+1}$
$K_{j-1,j} = S_{i+1,j}$
$K_{j-1,j+1} = S_{i+1,j+1}$
$K_{j,j} = S_{i,j} + S_{i+2,j} + MP_{\frac{j+1}{2},1}$
$K_{j,j+1} = S_{i,j+1} + S_{i+2,j+1} + MP_{\frac{j+1}{2},2}$
$K_{j+1,j} = S_{i+1,j} + S_{i+4,j} + HP_{\frac{j+1}{2},1}$
$K_{j+1,j+1} = S_{i+1,j+1} + S_{i+4,j+1} + HP_{\frac{j+1}{2},2}$
$f = 4$
Yes
11
No
$j = 3$
$i = \frac{3i + 1}{2}$
10
L



Influence coefficients
of redundants of
continuous arch

APPENDIX B.

CHECK RESULTS OF EXAMPLE USING ENERGY METHOD

The energy method may be divided into the following operations:

(1) Six redundants are treated as external loads acting on a statically determinate base structure. The energy of arch AB, U_1; of arch BC, U_2; and of pier BD, U_3; is then calculated. Thus,

$$U_1 = U_1(X_1,\ X_2,\ X_3),$$

$$U_2 = U_2(X_4,\ X_5,\ X_6),$$

$$U_3 = U_3(X_2,\ X_3,\ X_4,\ X_6).$$

The total energy is given by $U = U_1 + U_2 + U_3$. Thus,

$$\left\{\frac{\partial U}{\partial X_i}\right\} = [\ C\ ]\left\{X_i\right\} \qquad i = 1,\ 2,\ \ldots\ ,\ 6.$$

(2) $\left\{X_i\right\} = [\ C\ ]^{-1}\left\{\frac{\partial U}{\partial X_i}\right\}$ $\quad i = 1,2,\ \ldots\ ,\ 6.$ Let $[\ B\ ]$ denotes $[\ C\ ]^{-1}$, then

$$\left\{X_i\right\} = [\ B\ ]\left\{\frac{\partial U}{\partial X_i}\right\}.$$

(3) The influence line for X_j is the deflection curve caused by the $\left\{B_{ij}\right\}$, where $i = 1,2,\ldots,6.$

Step 1 Set up $\left\{\frac{\partial U}{\partial X_i}\right\} = [\,C\,]\left\{X_i\right\} \qquad i = 1, 2, \ldots, 6.$

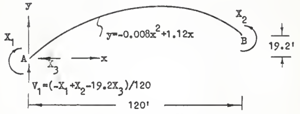


Fig. B1 Free body of arch AB.

$$U_1 = \int_0^L \frac{M^2}{2EI}\,ds = \int_0^{120} \frac{M^2}{2EI_c \sec\theta}\,\frac{dx}{\cos\theta}$$

$$= \frac{1}{EI_c}\int_0^{120} \frac{1}{2}\,(X_1 + V_1 x + X_3 y)^2\,dx$$

$$= \frac{1}{EI_c}\int_0^{120} \frac{1}{2}\,[X_1 + \frac{x}{120}(-X_1 + X_2 - 19.2X_3) + X_3(-0.008x^2 + 1.12x)]^2\,dx.$$

$$\frac{\partial U_1}{\partial X_1} = \frac{1}{EI_c}\,(40X_1 + 20X_2 + 1152X_3)\,,$$

$$\frac{\partial U_1}{\partial X_2} = \frac{1}{EI_c}\,(20X_1 + 40X_2 + 1152X_3)\,,$$

$$\frac{\partial U_1}{\partial X_3} = \frac{1}{EI_c}\,(1152X_1 + 1152X_2 + 53085.16X_3)\,.$$

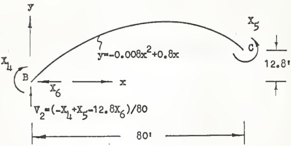


Fig. B2 Free body of arch BC.

$$U_2 = \int_0^L \frac{M^2}{2EI}\,ds = \int_0^{80} \frac{M^2}{2EI_c\sec\theta}\,\frac{dx}{\cos\theta}$$

$$= \frac{1}{EI_c}\int_0^{80} \frac{1}{2}\,(X_4 + V_2x + X_6y)^2\,dx$$

$$= \frac{1}{EI_c}\int_0^{80} \frac{1}{2}\,[X_4 + \frac{x}{80}(-X_4 + X_5 - 12.8X_6) + X_6(-0.008x^2 + 0.8x)]^2\,dx.$$

$$\frac{\partial U_2}{\partial X_4} = \frac{1}{EI_c}\,(26.66667X_4 + 13.33333X_5 + 341.33333X_6)\ ,$$

$$\frac{\partial U_2}{\partial X_5} = \frac{1}{EI_c}\,(13.33333X_4 + 26.66667X_5 + 341.33333X_6)\ ,$$

$$\frac{\partial U_2}{\partial X_6} = \frac{1}{EI_c}\,(341.33333X_4 + 341.33333X_5 + 6990.50667X_6)\ .$$

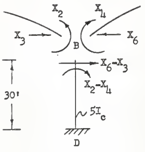


Fig. B3 Free body of pier BD.

$$U_3 = \int_0^L \frac{M^2}{2EI}\,dx$$

$$= \frac{1}{EI_c}\int_0^{30} \frac{1}{10}\,[X_2 - X_4 + (X_6 - X_3)x]^2\,dx\ .$$

$$\frac{\partial U_3}{\partial X_2} = \frac{1}{EI_c}\,(6X_2 - 90X_3 - 6X_4 + 90X_6)\ ,$$

$$\frac{\partial U_3}{\partial X_3} = \frac{1}{EI_c}\,(-90X_2 + 1800X_3 + 90X_4 - 1800X_6)\ ,$$

$$\frac{\partial U_3}{\partial X_4} = \frac{1}{EI_c}\,(-6X_2 + 90X_3 + 6X_4 - 90X_6)\ ,$$

$$\frac{\partial U_3}{\partial X_6} = \frac{1}{EI_c}\,(90X_2 - 1800X_3 - 90X_4 + 1800X_6)\ .$$

The total energy U is the sum of U_1, U_2, and U_3. Thus,

$$\frac{\partial U}{\partial X_1} = \frac{\partial U_1}{\partial X_1} + \frac{\partial U_2}{\partial X_1} + \frac{\partial U_3}{\partial X_1} = \frac{1}{EI_c}\,(40X_1 + 20X_2 + 1152X_3)\ ,$$

$$\frac{\partial U}{\partial X_2} = \frac{1}{EI_c}\,(20X_1 + 46X_2 + 1062X_3 - 90X_6)\ ,$$

$$\frac{\partial U}{\partial X_3} = \frac{1}{EI_c}\,(1152X_1 + 1062X_2 + 54885.16X_3 + 90X_4 - 1800X_6)\ ,$$

$$\frac{\partial U}{\partial X_4} = \frac{1}{EI_c}\,(-6X_2 + 90X_3 + 32.66667X_4 + 13.33333X_5 + 251.33333X_6)\ ,$$

$$\frac{\partial U}{\partial X_5} = \frac{1}{EI_c}\,(13.33333X_4 + 26.66667X_5 + 341.33333X_6)\ ,$$

$$\frac{\partial U}{\partial X_6} = \frac{1}{EI_c}\,(90X_2 - 1800X_3 + 251.33333X_4 + 341.33333X_5 + 8790.50667X_6)\ .$$

In matrix notation,

$$\begin{pmatrix} \frac{\partial U}{\partial X_1} \\ \frac{\partial U}{\partial X_2} \\ \frac{\partial U}{\partial X_3} \\ \frac{\partial U}{\partial X_4} \\ \frac{\partial U}{\partial X_5} \\ \frac{\partial U}{\partial X_6} \end{pmatrix} = \frac{1}{EI_c} \begin{pmatrix} 40 & 20 & 1152 & 0 & 0 & 0 \\ 20 & 46 & 1062 & -6 & 0 & 90 \\ 1152 & 1062 & 54885.16 & 90 & 0 & -1800 \\ 0 & -6 & 90 & 32.66667 & 13.33333 & 251.33333 \\ 0 & 0 & 0 & 13.33333 & 26.66667 & 341.33333 \\ 0 & 90 & -1800 & 251.33333 & 341.33333 & 8790.50667 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix}$$

Step 2 Find $[\,B\,] = [\,C\,]^{-1}$

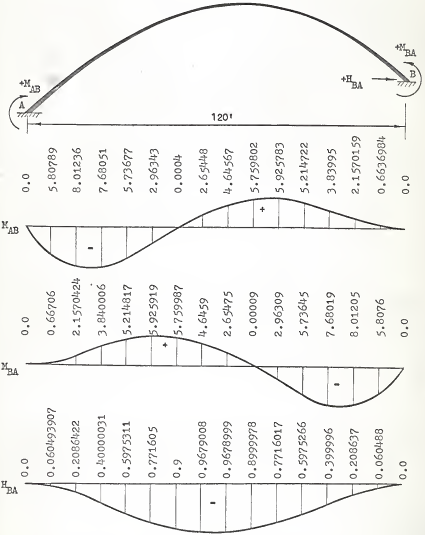
$$
[\,B\,] = \begin{bmatrix}
+0.071011512 & +0.017965618 & -0.0019095251 & +0.015160427 & +0.010591581 & -0.0014196713 \\
+0.017965618 & +0.061222623 & -0.0016866988 & +0.027521586 & +0.017407098 & -0.0024349914 \\
-0.0019095251 & -0.001686699 & +0.000095585934 & -0.0010042091 & -0.00066996984 & +0.000091556831 \\
+0.015160427 & +0.027521586 & -0.0010042091 & +0.054373016 & -0.002085218 & -0.0019610378 \\
+0.010591581 & +0.017407099 & -0.0006699698 & -0.002085218 & +0.083138013 & -0.003484017 \\
-0.0014196713 & -0.0024349915 & +0.000091568307 & -0.0019610378 & -0.003484017 & +0.00034879189
\end{bmatrix}
$$

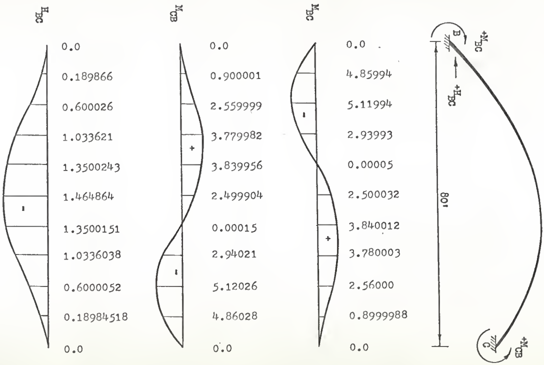
Step 3 Find influence lines for X_i, $i = 1, 2, \ldots, 6$.

The influence line for X_j is the deflection curve caused by $\{B_{ij}\}$, $i = 1, 2, \ldots, 6$. The results are given in Fig. 11 through Fig. 16.

APPENDIX C

SINGLE ARCH INFLUENCE LINES




$+M_{BC}$
B
$+H_{BC}$
80'
C
$+M_{CB}$
M_{BC}
0.0
4.85994
5.11994
2.93993
0.00005
2.500032
3.840012
3.780003
2.56000
0.8999988
0.0
-
+
M_{CB}
0.0
0.900001
2.559999
3.779982
3.839956
2.499904
0.00015
2.94021
5.12026
4.86028
0.0
+
-
H_{BC}
0.0
0.189866
0.600026
1.033621
1.3500243
1.464864
1.3500151
1.0336038
0.6000052
0.18984518
0.0
-

MATRIX ANALYSIS OF CONTINUOUS PARABOLIC ARCHES

by

CHAO-SHYONG HSU

Diploma, Taipei Institute of Technology, 1963

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1967

This report presents a matrix analysis of continuous arches on slender piers by obtaining influence lines for the redundants. The analysis is based on the displacement method.

The method consists of two main parts. First, each single arch is analyzed by applying single fixed arch theory. Second, continuous arches are solved by applying the displacement method.

The procedure is demonstrated by a detailed flow diagram, given in Appendix A. The flow diagram is determined based on the assumptions that (1) the arches are parabolic, the equations of the centroidal axis of each arch with the origin at each end are given, (2) the ratio of the moment of inertia at any section of the arch to the moment of inertia at the crown is $\sec\theta$, where θ is the angle between the tangent to the arch and the horizontal axis; the ratio of the moment of the inertia of the pier to that of the crown is constant, (3) the interior joints are on piers. The method applies to any number of arch spans.

A numerical example consisting of two parabolic arch spans with a central slender pier and two fixed ends is then presented. In addition to analyzing the example by the displacement method, the energy method is used to check the results. A comparison of the results obtained from the two methods shows excellent agreement.

The matrix method of analysis used in the report involves the application of the Müller-Breslau Principle and numerical double integration. In this study, the Gaussian 5-point Integration Formula is used for the first integration and Simpson's Rule is used for the second integration.