A STUDY OF THE CONTROLLABILITY AND OBSERVABILITY OF LINEAR SYSTEMS

by

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INTRODUCTION

The concepts of controllability and observability of linear systems were introduced by Kalman [2], [7], [11]. These concepts play an important role in control problems.

The central problem in the study of the concept of controllability of linear systems is to determine whether or not every initial state of a linear system can be transferred to any desired state in some finite interval of time by some control. As will be shown, every initial state of a linear system can be transferred to any desired state in some finite interval of time by some control if the system is controllable, and conversely there is at least one state that cannot be transferred to some other state in a finite time interval if the system is not controllable. In the study of the concept of observability of linear systems, the central problem is to see whether or not every initial state of a linear system can be detected at the output of the system in some finite interval of time. As will be shown, every initial state of a linear system can be detected at the output of the system in some finite interval of time if the system is strictly observable, and there will be at least one state which cannot be detected at the output of the system in a finite interval of time if the system is not strictly observable.

In "Controllability of Linear Dynamical Systems" [2], Kalman, Ho and Narendra developed some theorems about the concept of controllability of linear systems. Most of these theorems will be interpreted in the first part of this report. Parallel with the concept of controllability of linear systems, some theorems about the concept of observability will be proved and interpreted in the second part of this report.

Gilbert [8] explored the controllability and observability of composite systems and the transfer-function matrix representation of linear systems. Gilbert considered only time-invariant systems whose $\underline{A}$ matrices have distinct eigenvalues. The last part of this report will deal with this work. The transfer-function matrix representation of linear systems will be emphasized.

THE CONCEPT OF CONTROLLABILITY

Definitions of Controllability

Consider a linear system of the form

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) \tag{1}$$

$$\underline{y}(t) = \underline{C}(t)\underline{x}(t) + \underline{D}(t)\underline{u}(t) \tag{2}$$

where the state vector $\underline{x}(t)$, input $\underline{u}(t)$ and output $\underline{y}(t)$ are n, p and q-vectors belonging to n-dimensional state space X, p-dimensional input space U and q-dimensional output space Y respectively; $\underline{A}(t)$, $\underline{B}(t)$, $\underline{C}(t)$ and $\underline{D}(t)$ are matrices with suitable order. All quantities in (1) and (2) are real. The matrices $\underline{A}(t)$, $\underline{B}(t)$, $\underline{C}(t)$ and $\underline{D}(t)$ and the input $\underline{u}(t)$ are at least piecewise continuous and are defined for all $-\infty < t < +\infty$.

The solution of (1) is [1].

$$\underline{x}(t) = \underline{\Phi}(t,t_o)\underline{x}(t_o) + \int_{t_o}^{t} \underline{\Phi}(t,\tau)\underline{B}(\tau)\underline{u}(\tau)d\tau \tag{3}$$

where $\underline{x}(t_o)$ is the initial state at t_o and $\underline{\Phi}(t,\tau)$ is the state transition matrix. The output is

$$\underline{y}(t) = \underline{C}(t)\underline{\Phi}(t,t_o)\underline{x}(t_o) + \int_{t_o}^{t} \underline{C}(t)\underline{\Phi}(t,\tau)\underline{B}(\tau)\underline{u}(\tau)d + \underline{D}(t)\underline{u}(t) \tag{4}$$

as can be seen from (2) and (3).

Definition 1. Consider a system described by (1) and (2). A state $\underline{x}(t_o)$ is said to be controllable if there exists some finite $t_1 > t_o$ such that $\underline{x}(t_o)$ can be transferred to $\underline{x}(t_1) = \underline{0}$ by some control $\underline{u}(t)$. If every state at t_o is controllable, then the system is said to be controllable at t_o. If the system is controllable at every t_o, then the system is said to be controllable.

Time-variant Systems

This section is an interpretation of Kalman, Ho and Narendra's work [2].

Theorem 1. A necessary and sufficient condition for a state $\underline{x}(t_o)$ to be controllable is that there exists some input $\underline{u}(t)$ such that

$$\underline{x}(t_o) = -\int_{t_o}^{t_1} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\underline{u}(\tau)d(\tau) \tag{5}$$

for some finite $t_1 > t_o$.

Proof. This theorem is an immediate consequence of Definition 1 and (3).

Example 1. Consider a system with the following state equation

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\underline{u}(t)$$

where $\underline{u}(t) = [u_1(t) \quad u_2(t)]'$. The prime means transpose.

The state transition matrix is

$$\underline{\phi}(t_o,\tau) = \begin{bmatrix} 1 & \varepsilon^{t_o-\tau}-1 \\ 0 & \varepsilon^{t_o-\tau} \end{bmatrix}$$

For any t_o

$$-\int_{t_o}^{t_1} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\underline{u}(\tau)d = -\int_{t_o}^{t_1} \begin{bmatrix} u_1(\tau)-u_2(\tau) \\ 0 \end{bmatrix} d\tau \qquad (6)$$

If $t_1 > t_o$, then any input which satisfies the equation

$$u_1(t) - u_2(t) = -\frac{3}{t_1-t_o}$$

will make

$$-\int_{t_o}^{t_1} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\underline{u}(\tau)d\tau = [3 \quad 0]'$$

By theorem 1, $[3 \quad 0]'$ is controllable at t_o.

Let the set of all states controllable at t be denoted by P(t) hereafter.

Theorem 2. The set $P(t_o)$ is a subspace of the state space X.

This theorem can be proved using Theorem 1 and the definition of a vector space.

Example 2. Consider the system in example 1. What is the subspace $P(t_o)$?

By Theorem 1 and (6), it is obvious that any state $[k_1 \;\; k_2]'$, where k_1 and k_2 are any real numbers and $k_2 \neq 0$, is not controllable at any t_o. But any state belonging to the subspace spanned by the vector $[1 \;\; 0]'$ is controllable at any t_o, since any state belonging to this subspace has the form $[k \;\; 0]'$, where k is some real number, and any input $\underline{u}(t)$ which satisfies the equation

$$u_1(t) - u_2(t) = -K/t_1-t_o$$

for $t_1 > t_o$ will make

$$-\int_{t_o}^{t_1} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\,\underline{u}(\tau)d\tau = [K \;\; 0]' \qquad (7)$$

for any t_o. This can be proved using (6). By Theorem 1 and (7), any state belonging to the subspace spanned by the vector $[1 \;\; 0]'$ is controllable at any t_o.

It is concluded that at any t_o, $P(t_o)$ is the subspace of X spanned by $[1 \;\; 0]'$.

Let $\{\underline{e}_i\}$ denote a basis set for $P(t_o)$, then there exists some finite $t_1(\underline{e}_i, t_o) > t_o$, where $t_1(\underline{e}_i, t_o)$ is a function of $\underline{e}_i$ and t_o, such that $\underline{e}_i$ can be transferred to the zero state at $t_1(\underline{e}_i, t_o)$ by some control $\underline{u}_i(t)$. Since $P(t_o)$ is finite dimensional, there exists some $t_1(t_o)$ that is the maximum of the set of all $t_1(\underline{e}_i, t_o)$'s.

Theorem 3. Every $\underline{x}(t_o) \varepsilon P(t_o)$ can be transferred to the origin at $t = t_1(t_o)$ by some control $\underline{u}(t)$.

Proof. Since $t_1(\underline{e}_i, t_o) \leq t_1(t_o)$, hence $\underline{e}_i$ can be transferred to the origin at $t_1(t_o)$ by some control

$$\underline{u}_i'(t) = \begin{cases} \underline{u}_i(t) & \text{for } t_o \leq t \leq t_1(\underline{e}_i, t_o) \\ \underline{0} & \text{for } t_1(\underline{e}_i, t_o) \leq t \leq t_1(t_o) \end{cases} \tag{8}$$

By (5)

$$\underline{e}_i = -\int_{t_o}^{t_1(t_o)} \underline{\Phi}(t_o,\tau)\underline{B}(\tau)\underline{u}_i'(\tau)d\tau \tag{9}$$

where $\underline{u}_i'(t)$ is defined in (8). Equation (9) implies that for any $\underline{x}(t_o)\varepsilon P(t_o)$

$$\underline{x}(t_o) = \sum_i K_i\underline{e}_i$$

$$= -\int_{t_o}^{t_1(t_o)} \underline{\Phi}(t_o,\tau)\underline{B}(\tau)\{\sum_i K_i\underline{u}_i'(\tau)\}d\tau$$

Thus the theorem has been proved.

Example 3. Consider the system in example 1.

Let $\underline{e}_1 = [1 \quad 0]'$ be a basis of $P(t_o)$. By (6), any input $\underline{u}(t)$ which satisfies the equation

$$u_1(t) - u_2(t) = \frac{-1}{t_1 - t_o}$$

for $t_1 > t_o$ will make

$$-\int_{t_o}^{t_1} \underline{\Phi}(t_o,\tau)\underline{B}(\tau)\underline{u}(\tau)d\tau = [1 \quad 0]'$$

This means that $[1 \quad 0]'$ can be transferred to the origin on any finite closed interval of time $[t_o, t_1]$, where $t_1 > t_o$. Hence $t_1(t_o)$ can be any time such that $t_1(t_o) > t_o$.

Any state in $P(t_o)$ has the form $[k \quad 0]'$, where k is some real number. By Theorem 1 and (6), any input $\underline{u}(t)$ which satisfies the equation

$$u_1(t) - u_2(t) = \frac{-K}{t_1(t_o)-t_o}$$

will make

$$-\int_{t_o}^{t_1(t_o)} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\underline{u}(\tau)d\tau = [K \quad 0]'$$

This implies that $[k \quad 0]'$ can be transferred to the origin at $t_1(t_o)$ by some control.

Consider (3): For every time t_o and for every initial state $\underline{x}(t_o)$ and for every input function $\underline{u}(t)$ defined on the interval $-\infty < t < +\infty$ there exists a motion $\underline{x}_{\underline{u}}(t;\ \underline{x}(t_o),\ t_o) \equiv \underline{x}(t)$ defined for all $-\infty < t < +\infty$ and $\underline{x}(t_o;\ \underline{x}(t_o),\ t_o) = \underline{x}(t_o)$.

Example 4. Consider a system with one dimensional state space. Two motions defined by $\underline{x}_a(t) \equiv \underline{x}_{\underline{u}1}(t;\ \underline{x}_1(t_o),\ t_o)$ and $\underline{x}_b(t) \equiv \underline{x}_{\underline{u}2}(t;\ \underline{x}_1(t_o),\ t_o)$ pass through the same point $(\underline{x}_1(t_o), t_o)$ as shown in Fig. 1.

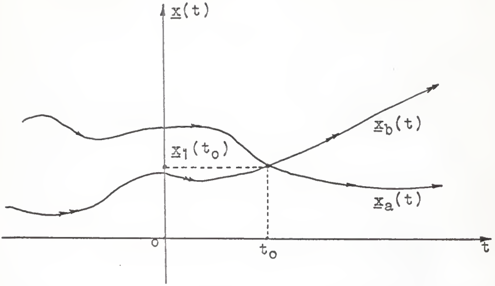

Fig. 1. Two motions defined by $\underline{x}_a(t) \equiv \underline{x}_{\underline{u}1}(t;\ \underline{x}_1(t_o),\ t_o)$ and $\underline{x}_b(t) \equiv \underline{x}_{\underline{u}2}(t;\ \underline{x}_1(t_o),\ t_o)$.

Theorem 4. Every motion which passes through some $\underline{x}_1(t_o) \epsilon P(t_o)$ can be reached from every $\underline{x}(t_o) \epsilon P(t_o)$ at or before $t_1(t_o)$.

Proof. Since $\underline{x}(t_o) - \underline{x}_1(t_o)$ belongs be $P(t_o)$, it follows that

$$\underline{\phi}(t_1(t_o),t_o)[\underline{x}_1(t_o)-\underline{x}(t_o)] = \int_{t_o}^{t_1(t_o)} \underline{\phi}(t_1(t_o),\tau)\underline{B}(\tau)\underline{u}_o(\tau)d\tau \tag{10}$$

For any control $\underline{u}_1(t)$

$$\underline{x}_{\underline{u}1}(t_1(t_o);\ \underline{x}_1(t_o),t_o)$$

$$= \int_{t_o}^{t_1(t_o)} \underline{\phi}(t_1(t_o),\tau)\underline{B}(\tau)\underline{u}_1(\tau)d\tau+\underline{\phi}(t_1(t_o),t_o)\underline{x}_1(t_o) \tag{11}$$

By (10) and (11)

$$\underline{x}_{\underline{u}1}(t_1(t_o);\underline{x}_1(t_o),t_o)$$

$$= \underline{\Phi}(t_1(t_o),t_o)\underline{x}(t_o) + \int_{t_o}^{t_1(t_o)} \underline{\Phi}(t_1(t_o),\tau)\underline{B}(\tau)[\underline{u}_o(\tau)+\underline{u}_1(\tau)]d\tau$$

$$= \underline{x}_{\underline{u}o+\underline{u}1}(t_1(t_o);\underline{x}(t_o),t_o)$$

The theorem has been proved.

Example 5. Consider the system in example 1. Let some motion be defined by

$$\underline{x}_{\underline{u}1}(t;\underline{x}_1(t_{o1}),t_{o1}) = \underline{x}_{\underline{u}1}(t;[2 \quad 0]', t_{o1})$$

This motion passes through ($[2 \quad 0]'$, t_{o1}). Denote any state belonging to $P(t_{o1})$ except $[2 \quad 0]'$ by $[k \quad 0]'$, where k is some real number and $k \neq 2$. The point of the motion $\underline{x}_{\underline{u}1}(t;\ [2 \quad 0]',\ t_{o1})$ at $t_1(t_{01})$ is

$$\underline{x}_{\underline{u}1}(t_1(t_{o1});\ [2 \quad 0]',\ t_{o1})$$

$$= \underline{\Phi}(t_1(t_{o1}),\ t_{o1})[2 \quad 0]' + \int_{t_{o1}}^{t_1(t_{o1})} \underline{\Phi}(t_1(t_{o1}),\tau)\underline{B}(\tau)\underline{u}_1(\tau)d\tau$$

$$= \begin{bmatrix} 2 + \int_{t_{o1}}^{t_1(t_{o1})} [u_{11}(\tau)-u_{12}(\tau)]d\tau \\ 0 \end{bmatrix} \tag{12}$$

Define another motion by

$$\underline{x}_{\underline{u}o+\underline{u}1}(t;\underline{x}(t_{o1}),t_{o1}) = \underline{x}_{\underline{u}o+\underline{u}1}(t;[K \quad 0]',t_{o1})$$

The point of the motion $\underline{x}_{\underline{u}o+\underline{u}1}(t;\ [k \quad 0]',\ t_{o1})$ at $t_1(t_{o1})$ is

$$\underline{x}_{\underline{u}o+\underline{u}1}(t_1(t_{o1});\ [K\ \ 0]',\ t_{o1})$$

$$= \underline{\phi}(t_1(t_{o1}),t_{o1})[K\ \ 0]' + \int_{t_{o1}}^{t_1(t_{o1})} \underline{\phi}(t_1(t_{o1}),\tau)\underline{B}(\tau)[\underline{u}_o(\tau)+\underline{u}_1(\tau)]d\tau$$

$$= \begin{bmatrix} K \\ 0 \end{bmatrix} + \begin{bmatrix} \int_{t_{o1}}^{t_1(t_{o1})} [u_{o1}(\tau)-u_{o2}(\tau)+u_{11}(\tau)-u_{12}(\tau)]d\tau \\ 0 \end{bmatrix}$$

Choose

$$\underline{u}_o(t) = \frac{1}{t_1(t_{o1})-t_{o1}} \begin{bmatrix} 2 \\ K \end{bmatrix} \tag{13}$$

then

$$\underline{x}_{\underline{u}o+\underline{u}1}(t_1(t_{o1});[K\ \ 0]',t_{o1}) = \begin{bmatrix} 2+\int_{t_{o1}}^{t_1(t_{o1})} [u_{11}(\tau)-u_{12}(\tau)]d\tau \\ 0 \end{bmatrix} \tag{14}$$

By (12)and (14)

$$\underline{x}_{\underline{u}1}(t_1(t_{o1});[2\ \ 0]',t_{o1}) = \underline{x}_{\underline{u}o+\underline{u}1}(t_1(t_{o1});[K\ \ 0]',t_{o1}) \tag{15}$$

The input $\underline{u}_1(t)$ can range over U, and it is always possible to find a corresponding $\underline{u}_o(t)$ such that (15) is satisfied. Hence any motion which passes through $[2\ \ 0]' \epsilon P(t_{o1})$ can be reached from any state in $P(t_{o1})$.

Theorem 5. Any state in $P(t_o)$ can be transferred to any state in $P(t_2)$, where $t_2 \geq t_1(t_o)$. No motion can enter the subspace $P(t)$, if it starts from any state $\underline{x}(t_o) \not\epsilon P(t_o)$.

Proof. For free motion

$$\underline{x}(t_o) = \underline{\phi}(t_o,t)\underline{x}(t) \qquad t \geq t_o \tag{16}$$

If $\underline{x}(t)\varepsilon P(t)$, then $\underline{x}(t_o)$ in (16) must belong to $P(t_o)$. Hence

$$P(t_o) \supset \underline{\phi}(t_o,t)P(t) \qquad t \geq t_o \tag{17}$$

This implies that any $\underline{x}(t_2)\varepsilon P(t_2)$ can be reached from some $\underline{x}(t_o)\varepsilon P(t_o)$ through some free motion. Then, by Theorem 4, the first part of the theorem can be proved.

Assume that there is some $\underline{x}(t_o)\not\varepsilon P(t_o)$ which can enter $P(t)$ for some $t > t_o$, then $\underline{x}(t_o)$ is controllable. This is a contradiction and part 2 of the theorem has been proved.

Example 6. Consider the system in example 1.

It was shown in example 3 that $t_1(t_o)$ can be any real number such that $t_1(t_o) > t_o$. Assume that $t_2 > t_o$, then $t_1(t_o)$ can be chosen such that $t_2 \geq t_1(t_o) > t_o$. By example 2, $P(t_o)$ and $P(t_2)$ are the same subspace spanned by the vector $[1 \quad 0]'$.

By Theorem 5, any state $\underline{x}(t_o) = [k_1 \quad 0]'$ can be transferred to any state $\underline{x}(t_2) = [k_2 \quad 0]'$, where K_1 and K_2 can be any real numbers. Let $\underline{x}(t_o) = [K_1 \quad 0]'$, then by (3)

$$\underline{x}(t_2) = \underline{\phi}(t_2,t_o)\underline{x}(t_o) + \int_{t_o}^{t_2} \underline{\phi}(t_2,\tau)\underline{B}(\tau)\underline{u}(\tau)\,d\tau$$

$$= [K_1 \quad 0]' + \int_{t_o}^{t_2} \begin{bmatrix} u_1(\tau)-u_2(\tau) \\ 0 \end{bmatrix} d\tau$$

If $\underline{u}(t)$ is chosen such that

$$u_1(t)-u_2(t) = \frac{K_2}{t_2-t_o} - \frac{K_1}{t_2-t_o}$$

then

$$\underline{X}(t_2) = [K_2 \quad 0]'$$

This means that any state $\underline{x}(t_o) = [k_1 \quad 0]'$ can be transferred to any state $\underline{x}(t_2) = [k_2 \quad 0]'$.

If $\underline{x}(t_o) = [k_1 \quad k_2]$, where k_1 and k_2 can be any real numbers but $k_2 \neq 0$, then $\underline{x}(t_o) \notin P(t_o)$. By (3)

$$\underline{x}(t) = \underline{\phi}(t,t_o)[K_1 \quad K_2]' + \int_{t_o}^{t} \underline{\phi}(t,\tau)\underline{B}(\tau)\underline{u}(\tau)d\tau$$

$$= \begin{bmatrix} (K_1-K_2)+K_2\varepsilon^{t-t_o} \\ K_2\varepsilon^{t-t_o} \end{bmatrix} + \int_{t_o}^{t} \begin{bmatrix} u_1(\tau)-u_2(\tau) \\ 0 \end{bmatrix} d\tau$$

Since $k_2\varepsilon^{t-t_o} \neq 0$, $\underline{x}(t)$ will never belong to $P(t)$.

Let a linear transformation be defined by

$$\underline{V}(t_o,t_1) = \int_{t_o}^{t_1} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\underline{B}'(\tau)\underline{\phi}'(t_o,\tau)d\tau \tag{18}$$

It is a symmetrical non-negative definite matrix. The rank of $\underline{V}$ is bounded by n and non-decreasing with t_1.

Theorem 6. A necessary and sufficient condition for being able to transfer $\underline{x}_o(t_o)$ to $\underline{x}_1(t_1)$ is that $\underline{\phi}(t_o, t_1)\underline{x}_1(t_1) - \underline{x}_o(t_o)$ belongs to $R[\underline{V}(t_o, t_1)]$, the range of $\underline{V}(t_o, t_1)$.

Proof. Let

$$\underline{V}(t_o,t_1)\underline{y}(t_o) = \underline{\Phi}(t_o,t_1)\underline{x}_1(t_1)-\underline{x}_o(t_o)$$

and

$$\underline{u}(t) = \underline{B}'(t)\underline{\Phi}'(t_o,t)\underline{y}(t_o)$$

Then

$$\underline{x}(t_1) = \underline{\Phi}(t_1,t_o)\underline{x}_o(t_o) + \int_{t_o}^{t_1}\underline{\Phi}(t_1,\tau)\underline{B}(\tau)\underline{B}'(\tau)\underline{\Phi}'(t_o,\tau)\underline{y}(t_o)d\tau$$

$$= \underline{\Phi}(t_1,t_o)\underline{x}_o(t_o) + \underline{\Phi}(t_1,t_o)\int_{t_o}^{t_1}\underline{\Phi}(t_o,\tau)\underline{B}(\tau)\underline{B}'(\tau)\underline{\Phi}'(t_o,\tau)\underline{y}(t_o)d\tau$$

$$= \underline{\Phi}(t_1,t_o)\underline{x}_o(t_o) + \underline{\Phi}(t_1,t_o)[\underline{\Phi}(t_o,t_1)\underline{x}_1(t_1)-\underline{x}_o(t_o)]$$

$$= \underline{x}_1(t_1)$$

This implies that $\underline{x}_o(t_o)$ can be transferred to $\underline{x}_1(t_1)$. Thus the sufficiency has been proved.

Suppose that $\underline{x}_o(t_o)$ can be transferred to $\underline{x}_1(t_1)$, then there is some $\underline{u}_1(t)$ such that

$$\underline{x}_1(t_1) = \underline{\Phi}(t_1,t_o)\underline{x}_o(t_o) + \int_{t_o}^{t_1}\underline{\Phi}(t_1,\tau)\underline{B}(\tau)\underline{u}_1(\tau)d\tau$$

$$\underline{\Phi}(t_o,t_1)\underline{x}_1(t_1) - \underline{x}_o(t_o) = \int_{t_o}^{t_1}\underline{\Phi}(t_o,\tau)\underline{B}(\tau)\underline{u}_1(\tau)d\tau \qquad (19)$$

If the state $\underline{\Phi}(t_o, t_1)\underline{x}_1(t_1) - \underline{x}_o(t_o)$ is an initial state at t_o, then

$$\underline{x}(t_1) = \underline{\phi}(t_1, t_o)[\underline{\phi}(t_o, t_1)\underline{x}_1(t_1) - \underline{x}_o(t_o)] + \int_{t_o}^{t_1} \underline{\phi}(t,\tau)\underline{B}(\tau)\underline{u}(\tau)\,d\tau$$

By (19)

$$\underline{x}(t_1) = \underline{\phi}(t_1, t_o)\int_{t_o}^{t_1} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\underline{u}_1(\tau)\,d\tau + \int_{t_o}^{t_1} \underline{\phi}(t,\tau)\underline{B}(\tau)\underline{u}(\tau)\,d\tau$$

$$= \int_{t_o}^{t_1} \underline{\phi}(t_1,\tau)\underline{B}(\tau)\underline{u}_1(\tau)\,d\tau + \int_{t_o}^{t_1} \underline{\phi}(t,\tau)\underline{B}(\tau)\underline{u}(\tau)\,d\tau$$

Let $\underline{u}(t) = -\underline{u}_1(t)$, then $\underline{x}(t_1) = \underline{0}$. This implies that $\underline{\phi}(t_o, t_1)\underline{x}_1(t_1) - \underline{x}_o(t_o)$ can be transferred to the origin in the closed time interval $[t_o, t_1]$ if $\underline{x}_o(t_o)$ can be transferred to $\underline{x}_1(t_1)$.

Let $N[\underline{V}(t_o, t_1)]$ denote the null space of the linear transformation $\underline{V}(t_o, t_1)$. Since $\underline{V}(t_o, t_1)$ is symmetrical, the orthogonal direct sum decomposition of the state space is [4]

$$X = R[\underline{V}(t_o, t_1)] \oplus N[\underline{V}(t_o, t_1)]$$

Suppose that $\underline{x}(t_o) \epsilon N[\underline{V}(t_o, t_1)]$ and $\underline{x}(t_o) \neq \underline{0}$, then

$$<\underline{x}(t_o) \quad \underline{V}(t_o, t_1)\underline{x}(t_o)> = \int_{t_o}^{t_1} ||\underline{B}'(\tau)\underline{\phi}'(t_o,\tau)\underline{x}(t_o)||^2 d\tau = 0$$

Hence

$$\underline{B}'(t)\underline{\phi}'(t_o, t)\underline{x}(t_o) \equiv \underline{0} \qquad \text{on } [t_o, t_1] \tag{20}$$

Assume that $\underline{x}(t_o)$ can be transferred to the origin in the closed time interval $[t_o, t_1]$. Then, by (5), there is some control $\underline{u}_2(t)$ such that

$$\underline{x}(t_o) = -\int_{t_o}^{t_1} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\underline{u}_2(\tau)d\tau$$

and

$$||\underline{x}(t_o)||^2 = <\underline{x}(t_o) \quad -\int_{t_o}^{t_1} \underline{\phi}(t_o,\tau)\underline{B}(\tau)\underline{u}_2(\tau)d\tau>$$

$$= -\int_{t_o}^{t_1} [\underline{B}'(\tau)\underline{\phi}'(t_o,\tau)\ \underline{x}(t_o)]'\underline{u}_2(\tau)d\tau \qquad (21)$$

By (20), the right hand side of (21) is equal to zero. But $||\underline{x}(t_o)||^2 > 0$, a contradiction. Hence there is no non-zero state in $N[\underline{V}(t_o,t_1)]$ that can be transferred to the origin in the closed time interval $[t_o, t_1]$.

Let $\underline{x}'(t_o) = \underline{\phi}(t_o,t_1)\underline{x}_1(t_1) - \underline{x}_o(t_o)$. Suppose that $\underline{x}_o(t_o)$ can be transferred to $\underline{x}_1(t_1)$ and $\underline{x}'(t_o) \not\in R[V(t_o,t_1)]$. Then $\underline{x}'(t_o)$ can be decomposed as

$$\underline{X}'(t_o) = \underline{X}_r'(t_o) + \underline{X}_n'(t_o)$$

where $\underline{x}_r'(t_o) \in R[\underline{V}(t_o,t_1)]$, $\underline{x}_n'(t_o) \in N[\underline{V}(t_o,t_1)]$ and $\underline{x}_n'(t\ ) \neq \underline{0}$. Since $\underline{x}_n'(t_o) \in N[\underline{V}(t_o,t_1)]$, hence $\underline{x}'(t_o)$ cannot be transferred to the origin in the closed interval of time $[t_o,t_1]$. This is a contradiction, and the theorem has been proved.

Example 7. Consider the system in example 1.

The matrix

$$\underline{V}(t_o,t_1) = \int_{t_o}^{t_1} \underline{\Phi}(t_o,\tau)\underline{B}(\tau)\underline{B}'(\tau)\underline{\Phi}(t_o,\tau)\,d\tau$$

$$= (t_1-t_o)\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

It's rank is one for $t_1 > t_o$ and $R[\underline{V}(t_o,t_1)]$ is the subspace of X spanned by the vector $[1 \quad 0]'$. Let $\underline{x}(t_o) = [k \quad 0]'$ and $\underline{x}(t_1) = [k' \quad 0]'$, where k and k' are any real numbers. Then

$$\underline{\Phi}(t_o,t_1)\underline{x}(t_1) - \underline{x}(t_o) = [K'-K \quad 0]'$$

hence $\underline{\Phi}(t_o,t_1)\underline{x}(t_1) - \underline{x}(t_o)$ belongs to $R[\underline{V}(t_o,t_1)]$. By Theorem 6, any state $\underline{x}(t_o) = [k \quad 0]'$ can be transferred to any state $\underline{x}(t_1) = [k' \quad 0]'$.

Now, let $\underline{x}(t_o) = [k_1 \quad k_2]'$ and $\underline{x}(t_1) = [k_3 \quad 0]'$, where k_1, k_2 and k_3 are any real numbers and $k_2 \neq 0$. Then

$$\underline{\Phi}(t_o,t_1)\underline{x}(t_1) - \underline{x}(t_o) = [K_3 - K_1 \qquad -K_2]'$$

which does not belong to the range of $\underline{V}(t_o,t_1)$, hence $\underline{x}(t_o)$ can not be transferred to $\underline{x}(t_1)$.

Corollary 6-1. The state $\underline{x}_o(t_o)$ can be transferred to $\underline{x}_1(t_1) = \underline{0}$ if and only if $\underline{x}_o(t_o) \varepsilon R[\underline{V}(t_o,t_1)]$.

This can be proved by setting $\underline{x}_1(t_1) = \underline{0}$ in Theorem 6.

Corollary 6-2. Let $t_1(t_o)$ be any value of t_1 such that the rank of $\underline{V}(t_o,t_1)$ is maximum, then $P(t_o) = R[\underline{V}(t_o,t_1(t_o))]$.

This can be proved by Corollary 6-1 and the fact that the rank of $\underline{V}(t_o,t_1)$ is non-decreasing with t_1 and bounded by n.

Corollary 6-3. A system is controllable at t_o if and only if $\underline{V}(t_o,t_1(t_o))$ is positive definite.

If $\underline{V}(t_o,t_1(t_o))$ is positive definite, then the rank of $\underline{V}(t_o,t_1(t_o))$ is n. Hence it's range is the state space X. Thus it follows from Corollary 6-1 that the system is controllable at t_o.

Time-invariant Systems

For time-invariant systems described by (1) and (2), the matrices $\underline{A}(t)$, $\underline{B}(t)$, $\underline{C}(t)$ and $\underline{D}(t)$ are constant. These matrices will be denoted by $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ hereafter, and (1) and (2) will be written as

$$\dot{\underline{x}}(t) = \underline{Ax}(t) + \underline{Bu}(t) \tag{22}$$

$$\underline{y}(t) = \underline{Cx}(t) + \underline{Du}(t) \tag{23}$$

If any state $\underline{x}_1$ is controllable at t_o, then, by (5), there is some input $\underline{u}_1(t)$ and some finite $t_1 > t_o$ such that

$$\underline{x}_1 = -\int_{t_o}^{t_1} \varepsilon^{\underline{A}(t_o-\tau)}\underline{B}\,\underline{u}_1(\tau)d\tau \tag{24}$$

Change the variable τ in (24) such that $\tau = \sigma - T$, where T is any positive real number, then

$$\underline{x}_1 = -\int_{t_o+T}^{t_1+T} \varepsilon^{\underline{A}[(t_o+T)-\sigma]}\underline{B}\,\underline{u}_1(\sigma-T)\,d\sigma$$

$$= -\int_{t_o+T}^{t_2} \varepsilon^{\underline{A}[(t_o+T)-\sigma]}\underline{B}\,\underline{u}_2(\sigma)\,d\sigma$$

By (5), this implies that if a state is controllable at t_o it will be controllable for all time. This also implies that if a state is uncontrollable at t_o it will be uncontrollable for all time. Thus one only needs to investigate the controllability of a time-invariant system at $t = 0$. It is obvious that a time-invariant system controllable at $t = 0$ is controllable.

The following is a criterion for controllability of a time-invariant system described by (22) and (23) in terms of the matrices $\underline{A}$ and $\underline{B}$ of the system.

Theorem 7. A time-invariant system described by (22) and (23) is controllable if and only if the column vectors of the matrix

$$\underline{Z}_c = [\underline{B}\ \ \underline{A}\,\underline{B}\ \ \underline{A}^2\underline{B} \cdots\cdots \underline{A}^{n-1}\underline{B}]$$

span the state space X of the system.

Proof. Suppose that the column vectors of $\underline{Z}_c$ span X, but there is some state which cannot be driven to the origin at some

$t = t_1$, where $t_1 > 0$. Then by Corollary 6-1, there is some non-zero state $\underline{x}_n(0) \epsilon N[\underline{V}(0, t_1)]$. And by (20)

$$\underline{B}'\underline{\phi}'(0,t)\underline{x}_n(0) = \underline{B}'\underline{\epsilon}^{-\underline{A}'t}\underline{x}_n(0) = \underline{0} \quad \forall\ 0 \leq t \leq t_1 \qquad (25)$$

Differentiate (25) and set $t = 0$

$$\left[\frac{d^k}{dt^k}\underline{B}'\underline{\epsilon}^{-\underline{A}'t}\underline{x}_n(0)\right]_{t=0} = \underline{B}'(-\underline{A}')^k\underline{x}_n(0) = \underline{0} \qquad (26)$$

where $k = 0, 1, \ldots\ldots, n - 1$

This implies that $\underline{x}_n(0)$ is orthogonal to every column vector of $\underline{Z}_c$. This contradicts the assumption that the column vectors of $\underline{Z}_c$ span the state space X.

Now, suppose that the system is controllable but the column vectors of $\underline{Z}_c$ do not span X, i.e. the range of $\underline{Z}_c$, $R(\underline{Z}_c)$, is a subspace of X. Then, there is some $\underline{x}_n(0) \epsilon R(\underline{Z}_c)^{\perp}$, the orthogonal complement of $R(\underline{Z}_c)$, such that

$$\underline{B}'(\underline{A}')^k\underline{x}_n(0) = \underline{0}$$

$$k = 0, 1, \ldots\ldots n - 1$$

By Cayley-Hamilton Theorem

$$\underline{B}'\epsilon^{-\underline{A}'t}\underline{x}_n(0) = \sum_{k=0}^{n-1} C_k(t)\underline{B}'(\underline{A}')^k\underline{x}_n(0) = \underline{0} \qquad (27)$$

By (18) and (27)

$$\underline{V}(0,t_1)\underline{x}_n(0) = \int_0^{t_1} \varepsilon^{-\underline{A}\tau}\underline{B}\,\underline{B}'\varepsilon^{-\underline{A}'\tau}\underline{x}_n(0)\,d\tau$$

$$= \underline{0} \qquad \forall t_1$$

This implies that $\underline{x}_n(0)\varepsilon N[\underline{V}(0,t_1)]$ for all t_1. By Corollary 6-1, $\underline{x}_n(0)$ is not controllable. This is a contradiction and the theorem has been proved.

Corollary 7-1. The initial state $\underline{x}(0)$ is controllable if and only if it belongs to the vector space $R(\underline{Z}_c)$.

The proof is similar to that of Theorem 7.

Corollary 7-2. If a time-invariant system described by (22) and (23) is controllable, then any state can be driven to the origin in any interval of time $[0,t_1]$, $t_1 > 0$.

Proof. Since the time t_1 was not restricted in the proof of the sufficient part of Theorem 7.

Example 8. Consider the following system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \underline{u}(t)$$

The matrix $\underline{Z}_c$ of the system is

$$\underline{Z}_c = [\underline{B}\ \underline{A}\,\underline{B}] = \begin{bmatrix} 0 & 1 & 1 & -2 \\ 1 & -2 & -3 & 4 \end{bmatrix}$$

It's column vectors span the state space of the system, hence the system is controllable.

Example 9. Consider the following system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \underline{u}(t)$$

The matrix $\underline{Z}_c$ of the system is

$$\underline{Z}_c = [\underline{B}\ \underline{A}\,\underline{B}] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of $\underline{Z}_c$ is 1, hence it's columns do not span the state space of the system and the system is not controllable. Only the states in the subspace spaned by $[1 \quad 0]'$ are controllable.

Consider the special case when the matrix $\underline{A}$ has distinct eigen values λ_i, $i = 1, 2, \ldots\ldots, n$. In this case, equations (22) and (23) can be represented in normal form

$$\underline{\dot{q}}(t) = \underline{\Lambda}\,\underline{q}(t) + \underline{B}_n\underline{u}(t) \tag{28}$$

$$\underline{y}(t) = \underline{C}_n\underline{q}(t) + \underline{D}\,\underline{u}(t) \tag{29}$$

where $\underline{q}(t) = \underline{\rho}^{-1}\underline{x}(t)$, $\underline{\Lambda} = \underline{\rho}^{-1}\underline{A}\,\underline{\rho}$, $\underline{B}_n = \underline{\rho}^{-1}\underline{B}$, $\underline{C}_n = \underline{C}\,\underline{\rho}$ and $\underline{\rho}$ is the modal matrix.

It has been proved that a system described by (28) and (29) is controllable if and only if there is no zero row in $\underline{B}_n$ [8].

Example 10. Consider the system in Example 8. This system's state equations in normal form are

$$\underline{\dot{q}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underline{q}(t) + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \underline{u}(t)$$

There is no zero row in $\underline{B}_n$, hence the system is controllable.

Example 11. Consider the system in Example 9. The normal form state equations are

$$\dot{\underline{q}}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{q}(t) + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \underline{u}(t)$$

The second row of $\underline{B}_n$ is zero vector, hence the system is not controllable.

THE CONCEPT OF OBSERVABILITY

Definitions of Observability

Definition 2. Consider a system described by (1) and (2). A state, $\underline{x}(t_o)$, of the system is said to be strictly observable if every coordinate of $\underline{x}(t_o)$ can be determined from a knowledge of the zero input response of the system over some finite closed interval of time $[t_o,t_1]$. A state, $\underline{x}(t_o)$, of the system is said to be strictly unobservable if no coordinate of $\underline{x}(t_o)$ can be determined from a knowledge of the zero input response of the system over some finite closed interval of time $[t_o,t_1]$. A state, $\underline{x}(t_o)$, of the system is said to be partially observable if it is neither strictly observable nor strictly unobservable. If every state of the system at t_o is strictly observable, then the system is said to be strictly observable at t_o. If every state of the system at t_o is strictly unobservable, then the system is said to be strictly unobservable at t_o. If the system is neither strictly observable nor strictly unobservable at t_o, then the system is said to be partially observable at t_o. If the system is strictly observable at all t_o, then the system is said to be strictly observable. If the system is strictly unobservable at all t_o, then the system is said to be strictly unobservable. If the system is neither strictly unobservable nor strictly observable, then the system is said to be partially observable.

Time-variant Systems

For any input $\underline{u}(t)$, the output of the system described by (1) and (2) is

$$\underline{y}(t) = \underline{C}(t)\left\{\underline{\Phi}(t,t_o)\underline{x}(t_o) + \int_{t_o}^{t} \underline{\Phi}(t,\tau)\underline{B}(\tau)\underline{u}(\tau)\,d\tau\right\} + \underline{D}(t)\underline{u}(t)$$

$$\underline{C}(t)\underline{\Phi}(t,t_o)\underline{x}(t_o) = \underline{y}(t) - \underline{C}(t)\int_{t_o}^{t} \underline{\Phi}(t,\tau)\underline{B}(\tau)\underline{u}(\tau)\,d\tau - \underline{D}(t)\underline{u}(t)$$

The forced response of the system can always be subtracted from the output. Thus the input has no effect on determining the coordinates of $\underline{x}(t_o)$. Hence in Definition 2, the system is assumed to be under free motion.

Consider the free motion case, the output of a system described by (1) and (2) will be

$$\underline{y}(t) = \underline{C}(t)\underline{\Phi}(t,t_o)\underline{x}(t_o) \tag{30}$$

Let $Q(\xi,\mu)$ be the set of initial states at ξ such that

$$\underline{y}(t) = \underline{C}(t)\underline{\Phi}(t,\xi)\underline{x}(\xi) \equiv \underline{0} \qquad \text{on } [\xi,\mu] \tag{31}$$

Obviously $Q(\xi,\mu)$ constitutes a vector space. Suppose that $t_o \leq t_2 \leq t_3$, then by (31)

$$Q(t_o,t_2) \supset Q(t_o,t_3) \tag{32}$$

Multiply both sides of (30) by $\underline{\Phi}'(t,t_o)\underline{C}'(t)$ and integrate the result from t_o to some finite time t_1, then

$$\int_{t_o}^{t_1} \underline{\Phi}'(\tau,t_o)\underline{C}'(\tau)\underline{y}(\tau)d\tau = \int_{t_o}^{t_1} \underline{\Phi}'(\tau,t_o)\underline{C}'(\tau)\underline{C}(\tau)\underline{\Phi}(\tau,t_o)d\tau\underline{x}(t_o) \tag{33}$$

Now, it will be shown that (33) can be used to determine $\underline{x}(t_o)$.

Let the linear transformation on the right hand side of (33) be denoted by $\underline{W}(t_o,t_1)$, that is

$$\underline{W}(t_o,t_1) = \int_{t_o}^{t_1} \underline{\Phi}'(\tau,t_o)\underline{C}'(\tau)\underline{C}(\tau)\underline{\Phi}(\tau,t_o)d\tau \tag{34}$$

As in the case of $\underline{V}(t_o,t_1)$ of equation (18), the linear transformation $\underline{W}(t_o,t_1)$ is symmetrical and non-negative definite matrix whose rank is non-decreasing with t_1 and bounded by n.

Let $t_1(t_o)$ be the value of t_1 such that $\underline{W}(t_o,t_1)$ has maximum rank, and let $R[\underline{W}(t_o,t_1(t_o))]$ and $N[\underline{W}(t_o,t_1(t_o))]$ be the range and the null space of $\underline{W}(t_o,t_1(t_o))$ respectively. Then, $\forall \underline{x}(t_o) \in N[\underline{W}(t_o,t_1(t_o))]$

$$\underline{W}(t_o,t_1(t_o))\underline{x}(t_o) = \underline{0}$$

and

$$\langle \underline{x}(t_o) \quad \underline{W}(t_o,t_1(t_o))\underline{x}(t_o) \rangle$$

$$= \langle \underline{x}(t_o) \cdot \int_{t_o}^{t_1(t_o)} \underline{\phi}'(\tau,t_o)\underline{C}'(\tau)\underline{C}(\tau)\underline{\phi}(\tau,t_o)\,d\tau\,\underline{x}(t_o)\rangle$$

$$= \int_{t_o}^{t_1(t_o)} \|\, \underline{C}(\tau)\underline{\phi}(\tau,t_o)\underline{x}(t_o)\|^2 d\tau$$

$$= 0$$

This implies that $\forall \underline{x}(t_o) \varepsilon N[\underline{W}(t_o,t_1(t_o))]$

$$\underline{C}(t)\underline{\phi}(t,t_o)\underline{x}(t_o) \equiv \underline{0} \qquad \text{on } [t_o,t_1(t_o)] \tag{35}$$

Equations (31) and (35) imply that $\forall \underline{x}(t_o) \varepsilon N[\underline{W}(t_o,t_1(t_o))]$

$$\underline{x}(t_o) \varepsilon Q(t_o,t_1(t_o)) \tag{36}$$

On the other hand, if $\underline{x}(t_o) \not\varepsilon N[\underline{W}(t_o,t_1(t_o))]$, then

$$\underline{W}(t_o,t_1(t_o))\underline{x}(t_o) = \int_{t_o}^{t_1(t_o)} \underline{\phi}'(\tau,t_o)\underline{C}'(\tau)\underline{C}(\tau)\underline{\phi}(\tau,t_o)\,d\tau\,\underline{x}(t_o)$$

$$\neq \underline{0}$$

This implies that $\forall \underline{x}(t_o) \not\varepsilon N[\underline{W}(t_o,t_1(t_o))]$

$$\underline{C}(t)\underline{\phi}(t,t_o)\underline{x}(t_o) \not\equiv \underline{0} \qquad \text{on } [t_o,t_1(t_o)]$$

Hence $\forall \underline{x}(t_o) \not\varepsilon N[\underline{W}(t_o,t_1(t_o))]$

$$\underline{x}(t_o) \not\varepsilon Q(t_o,t_1(t_o)) \tag{37}$$

Equations (36) and (37) imply that

$$Q(t_o, t_1(t_o)) = N[\underline{W}(t_o, t_1(t_o))] \qquad (38)$$

Now, suppose that $\sigma \geq t_1(t_o)$, then by (32) and (38)

$$N[\underline{W}(t_o, t_1(t_o))] = Q(t_o, t_1(t_o)) \supset Q(t_o, \sigma) = N[\underline{W}(t_o, \sigma)]$$

Since the rank of $\underline{W}(t_o, t_1)$ is non-decreasing with t_1 and $\underline{W}(t_o, t_1(t_o))$ has maximum rank, the rank of $\underline{W}(t_o, t_1(t_o))$ is equal to the rank of $\underline{W}(t_o, \sigma)$. Thus

$$N[\underline{W}(t_o, t_1(t_o))] = Q(t_o, t_1(t_o))$$

$$= Q(t_o, \sigma) = N[\underline{W}(t_o, \sigma)] \quad \forall \sigma \geq t_1(t_o) \qquad (39)$$

For convenience, denote $Q(t, t_1(t))$ by $Q(t)$ hereafter.

Theorem 8. A state $\underline{x}(t_o)$ is strictly unobservable if and only if it belongs to $N[\underline{W}(t_o, t_1(t_o))]$.

Proof. By (39), if $\underline{x}(t_o) \epsilon N[\underline{W}(t_o, t_1(t_o))]$, then $\underline{x}(t_o) \epsilon Q(t_o, \sigma)$ for every $\sigma \geq t_1(t_o)$. This means that

$$\underline{y}(t) = \underline{C}(t)\underline{\Phi}(t, t_o)\underline{x}(t_o) \equiv \underline{0} \quad \text{on } [t_o, \sigma]$$

where σ is any value such that $\sigma \geq t_1(t_o)$. Hence for every $\underline{x}(t_o) \epsilon N[\underline{W}(t_o, t_1(t_o))]$

$$\underline{y}(t) = \underline{C}(t)\underline{\Phi}(t, t_o)\underline{x}(t_o) \equiv \underline{0} \quad \forall t \geq t_o \qquad (40)$$

This implies that there is no coordinate of $\underline{x}(t_o)$ that can be determined from a knowledge of the zero input response over some finite interval of time $[t_o, t_1]$. Thus $\underline{x}(t_o)$ is strictly unobservable.

Now, suppose that $\underline{x}(t_o) \notin N[\underline{W}(t_o, t_1(t_o))]$, then

$$\underline{W}(t_o, t_1(t_o))\underline{x}(t_o) = \int_{t_o}^{t_1(t_o)} \underline{\Phi}'(\tau, t_o)\underline{C}'(\tau)\underline{y}(\tau)d\tau \neq \underline{0} \tag{41}$$

and $\underline{x}(t_o)$ can be decomposed as [4].

$$\underline{x}(t_o) = \underline{x}^r(t_o) + \underline{x}^n(t_o)$$

where $\underline{x}^r(t_o) \neq \underline{0}$ and

$$\underline{x}^r(t_o) \varepsilon R[\underline{W}'(t_o, t_1(t_o))] = R[\underline{W}(t_o, t_1(t_o))]$$

$$\underline{x}^n(t_o) \varepsilon N[\underline{W}'(t_o, t_1(t_o))] = N[\underline{W}(t_o, t_1(t_o))]$$

By the pseudo inverse [4] of $\underline{W}(t_o, t_1(t_o))$, $\underline{x}^r(t_o)$ can be determined, that is $\underline{x}^r(t_o)$ can be determined from a knowledge of the zero input response over some finite interval of time $[t_o, t_1(t_o)]$. Hence a strictly unobservable state at t_o must belong to $N[\underline{W}(t_o, t_1(t_o))]$.

The theorem has been proved.

Corollary 8-1. A system is strictly unobservable at t_o if and only if the matrix $\underline{W}(t_o, t_1(t_o))$ of the system is zero.

Proof. If the matrix $\underline{W}(t_o, t_1(t_o))$ is zero, then the dimension of $N[\underline{W}(t_o, t_1(t_o))]$ is n. By Theorem 8, the system is strictly unobservable at t_o.

Suppose that the system is strictly unobservable at t_o but the matrix $\underline{W}(t_o,t_1(t_o))$ is not zero. Then the dimension of $N[\underline{W}(t_o,t_1(t_o))]$ is less than n, hence the system is not strictly unobservable at t_o. This is a contradiction.

The corollary has been proved.

Corollary 8-2. A system is strictly observable at t_o if and only if the rank of the matrix $\underline{W}(t_o,t_1(t_o))$ is n.

Proof. If the rank of $\underline{W}(t_o,t_1(t_o))$ is n, then the inverse of $\underline{W}(t_o,t_1(t_o))$ exists. By (41), any $\underline{x}(t_o)$ can be uniquely determined. Hence the system is strictly observable at t_o.

Now suppose that the system is strictly observable at t_o but the rank of $\underline{W}(t_o,t_1(t_o))$ is less than n. Then the dimension of $N[\underline{W}(t_o,t_1(t_o))]$ is at least one, hence by Theorem 8, the system is not strictly observable at t_o. This is a contradiction.

The corollary has been proved.

Corollary 8-3. Let r be the rank of the matrix $\underline{W}(t_o,t_1(t_o))$ of a system, then the system is partially observable at t_o if and only if $0 < r < n$.

Corollary 8-3 is an immediate consequence of Corollary 8-1 and Corollary 8-2.

Example 12. Consider the following system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = [1 \quad 0]\underline{x}(t)$$

The matrix

$$\underline{W}(t_o,t_1) = \int_{t_o}^{t_1} \underline{\Phi}'(\tau,t_o)\underline{C}'(\tau)\underline{C}(\tau)\underline{\Phi}(\tau,t_o)d\tau$$

$$= \begin{bmatrix} t_1-t_o & \frac{1}{2}[t_1-t_o]^2 \\ \frac{1}{2}[t_1-t_o]^2 & \frac{1}{3}[t_1-t_o]^3 \end{bmatrix}$$

and

$$\det. \underline{W}(t_o t_1) = \frac{1}{12}(t_1-t_o)^4$$

This implies that for any t_o such that $t_1 > t_o$, $\underline{W}(t_o,t_1)$ is non-singular, i.e. $t_1(t_o)$ can be any value such that $t_1(t_o) > t_o$ and $\underline{W}(t_o,t_1(t_o))$ is non-singular for any t_o. This implies that the system is strictly observable at any t_o. Hence the system is strictly observable.

Example 13. Consider the following system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\underline{u}(t)$$

$$\underline{y}(t) = [0 \quad 1]\underline{x}(t)$$

The matrix

$$\underline{W}(t_o,t_1) = \int_{t_o}^{t_1} \underline{\Phi}'(\tau,t_o)\underline{C}'(\tau)\underline{C}(\tau)\underline{\Phi}(\tau,t_o)d\tau$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & t_1-t_o \end{bmatrix}$$

For any t_o, the rank of $\underline{W}(t_o,t_1)$ reaches it's maximum value at any $t_1 > t_o$, i.e. $t_1(t_o)$ can be any value such that $t_1(t_o) > t_o$ and $\underline{W}(t_o,t_1(t_o))$ is of rank one for any t_o. Hence the system is partially observable at any t_o by Corollary 8-3, and therefore it is partially observable.

The matrix

$$\underline{W}(t_o,t_1(t_o)) = \begin{bmatrix} 0 & 0 \\ 0 & t_1(t_o)-t_o \end{bmatrix}$$

is of rank one for any t_o. Thus $R[\underline{W}(t_o,t_1(t_o))]$ is spanned by the vector $[0 \quad 1]'$ for any t_o and $N[\underline{W}(t_o,t_1(t_o))]$ is spanned by $[1 \quad 0]'$ for all t_o. Any state belonging to the space spanned by $[1 \quad 0]'$ is strictly unobservable for all t_o and any state that does not belong to that space is partially observable for all t_o.

Time-invariant Systems

Consider a system described by (22) and (23). The output of a free motion starting at t_o with initial state $\underline{x}_o$ is

$$\underline{y}_1(t) = \underline{C}\ \varepsilon^{\underline{A}(t-t_o)}\underline{x}_o \tag{42}$$

If the free motion starts at any $t_o + T$ with the same initial state $\underline{x}_o$, then

$$\underline{y}_2(t) = \underline{C}\ \varepsilon^{\underline{A}[t-(t_o+T)]}\underline{x}_o \tag{43}$$

By changing the variable t to $t = \sigma + T$, equation (43) becomes

$$\underline{y}_2(\sigma+T) = \underline{C}\ \underline{\varepsilon}^{\underline{A}(\sigma-t_o)}\underline{x}_o \tag{44}$$

By (42) and (44)

$$\underline{y}_1(\sigma) = \underline{y}_2(\sigma+T) = \underline{C}\ \underline{\varepsilon}^{\underline{A}(\sigma-t_o)}\underline{x}_o \tag{45}$$

Equations (42) and (45) have the same form. Hence if $\underline{x}_o$ is strictly observable at t_o, then it is strictly observable for all time; if $\underline{x}_o$ is strictly unobservable at t_o, then it is strictly unobservable at all time; if $\underline{x}_o$ is partially observable at t_o, then it is partially observable at all time. Thus one only needs to investigate the observability aspects of a time-invariant system at $t = 0$. It is obvious that a time-invariant system strictly observable at $t = 0$ is strictly observable.

As will be shown in the following theorems, the observability characteristics of a time-invariant system are determined by the matrices $\underline{A}$ and $\underline{C}$ of the system.

Theorem 9 . A necessary and sufficient condition for a system described by (22) and (23) to be strictly observable is that the column vectors of the matrix

$$\underline{Z}_o = [\underline{C}'\ \underline{A}'\underline{C}'\ (\underline{A}')^2\underline{C}' \ldots\ldots. (\underline{A}')^{n-1}\underline{C}']$$

span the state space X.

Proof. Suppose that the system is strictly observable but the column vectors of $\underline{Z}_o$ do not span the state space X. Then there is some $\underline{x}_n(0) \varepsilon R(\underline{Z}_o)^{\perp}$, the orthogonal complement of $R(\underline{Z}_o)$, such that

$$[(\underline{A}')^k \underline{C}']' \underline{x}_n(0) = \underline{C}\ \underline{A}^k \underline{x}_n(0) = \underline{0}$$

$$k = 0, 1, \ldots\ldots, n-1$$

By the Cayley-Hamilton Theorem and in a manner similar to that used in the proof of Theorem 7

$$\underline{y}(t) = \underline{C}\varepsilon^{\underline{A}t} \underline{x}_n(0) = \underline{0} \qquad \forall t$$

This implies that $\underline{x}_n(0)$ cannot be determined from the knowledge of a zero input response of the system over any finite interval of time. This contradicts the assumption.

Now, suppose that the column vectors of $\underline{Z}_o$ span the state space X but the system is not strictly observable. Then, the system is not strictly observable at t = 0. By Corollary 8-2, the rank of the matrix $\underline{W}(0,t_1(0))$ is less than n, i.e. the dimension of $N[\underline{W}(0,t_1(0))]$ is at least one, hence there is some $\underline{x}_n(0) \varepsilon N[\underline{W}(0,t_1(0))]$. By (40) $\forall t \geq 0$

$$\underline{y}(t) = \underline{C}\ \underline{\Phi}(t,0)\underline{x}_n(0) = \underline{C}\varepsilon^{\underline{A}t} \underline{x}_n(0) = \underline{0} \tag{46}$$

Differentiate (46) and set t = 0

$$\underline{C}(\underline{A})^k \underline{x}_n(0) = [(\underline{A}')^k \underline{C}']' \underline{x}_n(0) = \underline{0}$$

$$k = 0, 1, 2, \ldots\ldots, n-1$$

This implies that $\underline{x}_n(0)$ is orthogonal to every column vector of $\underline{Z}_o$. This contradicts the assumption that the column vectors of $\underline{Z}_o$ span the state space X.

Theorem 10. If $R(\underline{Z}_o)$ is a subspace of X, then only the component of $\underline{x}(0)$ in $R(\underline{Z}_o)$ can be determined from a knowledge of the zero input response of the system over some finite interval of time $[0,t_1]$.

Proof. For any $\underline{x}_n \varepsilon R(\underline{Z}_o)^{\perp}$

$$[(\underline{A}')^k \underline{C}']' \underline{x}_n = \underline{C}(\underline{A})^k \underline{x}_n = \underline{0}$$

$$k = 0,\ 1, \ldots\ldots, n-1$$

By Cayley-Hamilton Theorem

$$\underline{y}(t) = \underline{C}\varepsilon^{\underline{A}t}\underline{x}_n = \sum_{k=0}^{n-1} \ell_k(t)\underline{C}(\underline{A})^k \underline{x}_n = 0 \qquad \forall t$$

and

$$\underline{W}(0,t_1(0))\underline{x}_n = \int_0^{t_1(0)} \varepsilon^{\underline{A}'\tau}\underline{C}'\underline{C}\varepsilon^{\underline{A}\tau}\underline{x}_n d\tau = \underline{0}$$

This implies that for any $\underline{x}_n \varepsilon R(\underline{Z}_o)^{\perp}$

$$\underline{x}_n \varepsilon N[\underline{W}(0,t_1(0))]$$

Hence

$$R(\underline{Z}_o)^{\perp} \varepsilon N[\underline{W}(0,t_1(0))] \tag{47}$$

Now, suppose that $\underline{x}_{n1} \varepsilon N[\underline{W}(0,t_1(0))]$, then

$$\int_0^{t_1(0)} \varepsilon^{\underline{A}'\tau} \underline{C}'\underline{C}\, \varepsilon^{\underline{A}\tau} \underline{x}_{n1} d\tau = \underline{0}$$

and

$$\underline{C}\varepsilon^{\underline{A}t} \underline{x}_{n1} \equiv \underline{0} \qquad \text{on } [0, t_1(0)]$$

Since $\underline{C}\varepsilon^{\underline{A}t}\underline{x}_{n1}$ is analytic, hence

$$\underline{C}\varepsilon^{\underline{A}t}\underline{x}_{n1} = 0 \qquad \forall t \tag{48}$$

Differentiate (48) and set t = 0, then

$$\underline{C}(\underline{A})^k \underline{x}_{n1} = [(\underline{A}')^k \underline{C}']' \underline{x}_{n1} = \underline{0}$$

$$k = 0, 1, \ldots\ldots, n-1$$

This implies that $\underline{x}_{n1} \varepsilon R(\underline{Z}_o)^{\perp}$. Therefore for every $\underline{x}_{n1} \varepsilon N[\underline{W}(0,t_1(0))]$, $\underline{x}_{n1} \varepsilon R(\underline{Z}_o)^{\perp}$. And

$$N[\underline{W}(0,t_1(0))] \varepsilon R(\underline{Z}_o)^{\perp} \tag{49}$$

Equations (47) and (49) imply that

$$R(\underline{Z}_o)^{\perp} = N[\underline{W}(0,t_1(0))] \tag{50}$$

and hence

$$R(\underline{Z}_o) = R[\underline{W}(0,t_1(0))] \tag{51}$$

It has been shown in Theorem 8 that for any state $\underline{x}(0)$, only the component of the state belonging to $\underline{R}[\underline{W}(0,t_1(0))]$ can be determined from a knowledge of the zero input response over some finite interval of time $[0,t_1]$. By (51), it is obvious that only the component of any state $\underline{x}(0)$ in $R(\underline{Z}_o)$ can be determined from a knowledge of the zero input response over some finite interval of time $[0,t_1]$.

The theorem has been proved.

The implication of this theorem is that $R(\underline{Z}_o)^{\perp}$ is the subspace of all the strictly unobservable states. Any state $\not\in R(\underline{Z}_o)^{\perp}$ is partially observable, since only the coordinates of $\underline{x}_r(0)$ can be determined, nothing is known about $\underline{x}_n(0)$ of $\underline{x}(0)$.

Example 14. Consider the following system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = [1 \quad 1]\underline{x}(t)$$

The matrix

$$\underline{Z}_o = [\underline{C}' \quad \underline{A}'\underline{C}'] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

It's column vectors span the state space of the system, hence the system is strictly observable.

Example 15. Consider the following system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = [0 \quad 1]\,\underline{x}(t)$$

The matrix

$$\underline{Z}_o = [\underline{C}' \quad \underline{A}'\underline{C}'] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

It's column vectors do not span the state space, hence the system is partially observable. The states belonging to the subspace spanned by $[1 \quad 0]'$ are strictly unobservable at any time.

When the matrix $\underline{A}$ of a system has distinct eigenvalues λ_i, $i = 1, 2, \ldots\ldots, n$, the equations (22) and (23) can be represented by (28) and (29).

It has been proved that a system described by (28) and (29) is strictly observable if and only if there is no zero column in $\underline{C}_n$ [8].

Example 16. Consider the system in example 14. The matrix $\underline{C}_n = [1 \quad 1]$, hence the system is strictly observable.

Example 17. Consider the system in example 15. The matrix $\underline{C}_n = [1 \quad 0]$, hence the system is not strictly observable.

TRANSFER-FUNCTION MATRIX OF LINEAR SYSTEMS

Here only the time-invariant systems are to be considered and the matrix $\underline{A}$ of any system is assumed to have distinct eigenvalues.

For this case, a system has a normal form representation described by (28) and (29). A coordinate $q_i(t)$ is decoupled from the input $\underline{u}(t)$ and cannot be influenced by it if the ith row of the matrix $\underline{B}_n$ is a zero vector; a coordinate $q_i(t)$ is decoupled from the output and is not detectible in the output if the ith column of the matrix $\underline{C}_n$ is a zero vector. Gilbert [8] suggested the following definitions.

Definition 3. A coordinate $q_i(t)$ is called controllable or uncontrollable according to whether the ith row of $\underline{B}_n$ is non-zero or zero vector.

Definition 4. A coordinate $q_i(t)$ is called observable or unobservable according to whether the ith column of $\underline{C}_n$ is non-zero or zero vector.

The following is an interpretation of Gilbert's work [8].

Decompositions of a Linear System

For the system described above, the matrix $\underline{\Lambda}$ is diagonal and the system can be decomposed according to the following theorem.

Theorem 11. It is always possible to partition a system S into four possible subsystems as shown in Fig. 2.

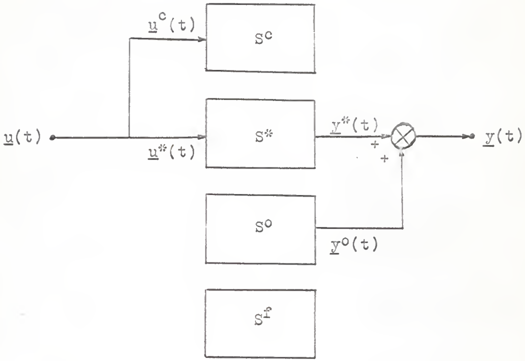

Fig. 2. A decomposition of a linear system

Part (1): A controllable and strictly observable subsystem S* which has a transmission matrix $\underline{D}$ is of order n*.

Part (2): A strictly observable and uncontrollable subsystem S^o with order n^o.

Part (3): A controllable and strictly unobservable subsystem S^c with order n^c.

Part (4): An uncontrollable and strictly unobservable subsystem S^f with order n^f.

Here S* and S^c have the same input $\underline{u}(t)$, i.e. $\underline{u}^*(t) = \underline{u}^c(t) = \underline{u}(t)$; the output of the system is equal to the sum of the outputs of S* and S^o, i.e. $\underline{y}(t) = \underline{y}^*(t) + \underline{y}^o(t)$; and the order of S is equal to the sum of the orders of it's subsystems, i.e. $n = n^* + n^o + n^c + n^f$.

Proof. By a normal form representation of the system described by (28) and (29), the coordinates of $\underline{q}(t)$ can be partitioned according to Part (1) through Part (4).

Example 18. Consider the following system

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 2 & \underline{0} & \\ & \underline{0} & 3 & \\ & & & 4 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}(t) \qquad (52)$$

$$\underline{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \underline{u}(t) \qquad (53)$$

By (52) and (53)

$$\dot{q}_1(t) = q_1(t) + [1 2]\underline{u}(t)$$

$$\dot{q}_2(t) = 2q_2(t)$$

$$\dot{q}_3(t) = 3q_3(t) + [1 \quad 0]\underline{u}(t)$$

$$\dot{q}_4(t) = 4q_4(t)$$

$$\underline{y}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_1(t) + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}_2(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} q_2(t)$$

Accordingly, the four subsystems are:

$$S^*: \begin{cases} \dot{q}_1(t) = q_1(t) + [1 \quad 2]\, \underline{u}(t) \\ \underline{y}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_1(t) + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \underline{u}(t) \end{cases}$$

$$S^o: \begin{cases} \dot{q}_2(t) = 2q_2(t) \\ \underline{y}_2(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} q_2(t) \end{cases}$$

$$S^c: \quad \dot{q}_3(t) = 3q_3(t) + [1 \quad 0]\underline{u}(t)$$

$$S^f: \quad \dot{q}_4(t) = 4q_4(t)$$

And

$$n = n^* + n^o + n^c + n^f$$

Transfer-function Representation of a Linear System

In the four sybsystems of a decomposition of a linear system, the subsystems S^c and S^f are strictly unobservable, they have no effect on the transfer-function. All the states of S^o are strictly observable but uncontrollable and the definition of a transfer-function is based on the assumption that the initial state is zero, hence S^o does not affect the transfer-function. Only S^*, which is controllable and strictly observable, is characterized by the transfer-function.

Theorem 12. The transfer-function matrix of a system described by (22) and (23) is

$$\underline{H}(s) = \underline{C}(\underline{I}S-\underline{A})^{-1}\underline{B} + \underline{D} = \underline{C}_n(\underline{I}S-\underline{\Lambda})^{-1}\underline{B}_n + \underline{D}$$

$$= \sum_{i=1}^{n^*} \frac{\underline{K}_i}{S-\lambda_i^*} + \underline{D} \tag{54}$$

where the ranks of $\underline{K}_i$'s are one and n^* and λ_i^*'s represent the order of S^* and the eigenvalues of $\underline{A}^*$ respectively.

Proof. The first expression of (54) is found by taking the Laplace transform of (22) and (23) and setting all the initial conditions to zero. The second expression of (54) is found in the same way from (28) and (29).

The matrix

$$\underline{C}_n(\underline{I}S-\underline{\Lambda})^{-1}\underline{B}_n$$

$$= \underline{C}_n \left\{ \frac{1}{\prod_{i=1}^{n}(S-\lambda_i)} \text{ adj.} \begin{bmatrix} S-\lambda_1 & & & \\ & S-\lambda_2 & & \\ & & \ddots & \\ & & & S-\lambda_n \end{bmatrix} \right\} \underline{B}_n$$

$$= \underline{C}_n \begin{bmatrix} \frac{1}{S-\lambda_1} & & \underline{0} \\ & \frac{1}{S-\lambda_2} & \\ \underline{0} & & \ddots & \\ & & & \frac{1}{S-\lambda_n} \end{bmatrix} \underline{B}_n$$

$$= \sum_{i=1}^{n} \frac{\underline{C}_{ni}\underline{B}_{ni}^{T}}{S-\lambda_i}$$

where $\underline{C}_{ni}$ and $\underline{B}_{ni}^{T}$ represent the ith column of $\underline{C}_n$ and the ith row of $\underline{B}_n$ respectively. If the ith coordinate is uncontrollable or unobservable, then $\underline{C}_{ni}\underline{B}_{ni} = \underline{0}$. In

$$\sum_{i=1}^{n} \frac{\underline{C}_{ni}\underline{B}_{ni}^{T}}{S-\lambda_i} ,$$

only those terms corresponding to the controllable and observable coordinates are retained. These retained terms correspond to those of the transfer-function of S*. Hence $\underline{H}(s)$ has the form of (54).

Since $\underline{K}_i = \underline{C}_{ni}\underline{B}_{ni}^{T}$, every column vector of $\underline{K}_i$ is a multiple of $\underline{C}_{ni}$, hence the rank of $\underline{K}_i$ is one.

Example 19. Consider the following system

$$\dot{\underline{q}}(t) = \begin{bmatrix} -1 & & & & \\ & -2 & & \underline{0} & \\ & & -3 & & \\ & \underline{0} & & -4 & \\ & & & & -5 \end{bmatrix} \underline{q}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \underline{q}(t)$$

The subsystem S* is

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

The transfer-function matrix of S is

$$\underline{H}(s) = \sum_{i=1}^{5} \frac{\underline{C}_{ni}\underline{B}_{ni}^T}{S-\lambda_i}$$

$$= \left\{ \frac{1}{S+1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \quad 0] + \frac{1}{S+2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [0 \quad 1] \right\}$$

$$+ \left\{ \frac{1}{S+3} \begin{bmatrix} 0 \\ 0 \end{bmatrix} [1 \quad 2] \right\} + \left\{ \frac{1}{S+4} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 0] \right\}$$

$$+ \left\{ \frac{1}{S+5} \begin{bmatrix} 0 \\ 0 \end{bmatrix} [0 \quad 0] \right\}$$

$$= \frac{1}{S+1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \frac{1}{S+2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \underline{H}^*(s)$$

The first term of the second expression of $\underline{H}(s)$ is the same as $\underline{H}^*(s)$; the second, third and fourth terms of the second expression of $\underline{H}(s)$ correspond to those of S^c, S^o and S^f respectively.

By Theorem 12, it is obvious that the transfer-function matrix of a system S represents only the subsystem S* of S. The second expression of (54) implies that the poles of S are the eigenvalues of $\underline{A}$. When the system is not controllable and strictly observable, then the poles which do not originate in S* are cancelled.

If one realizes a transfer-function matrix by a linear system which is not controllable and strictly observable, then it may happen that the system is unstable which cannot be detected by investigating the transfer-function matrix.

Example 20. Consider the system shown in Fig. 3.

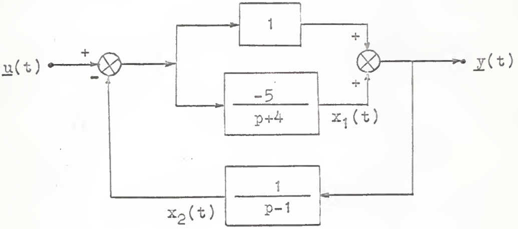

Fig. 3. The linear system of example 20

The state equations of the system are found to be

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = [1 \quad -1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

It's normal form representation is

$$\underline{\dot{q}}(t) = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix} \underline{q}(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = [6 \quad 0]\underline{q}(t)$$

The eigenvalues of $\underline{A}$ are $\lambda_1 = -5$ and $\lambda_2 = 1$. Since λ_2 is positive, the system is unstable [4].

The transfer-function is

$$\underline{H}(s) = \frac{(S-1)/(S+4)}{1 + (S-1)^{-1}(S-1)/(S+4)}$$

$$= \frac{(S-1)^2}{(S-1)(S+5)}$$

$$= \frac{S-1}{S+5}$$

Hence the transfer-function is stable.

The coordinate corresponding to λ_2 is uncontrollable and unobservable, hence it does not affect the transfer-function and the instability cannot be detected from the transfer-function.

Multivariable Feedback Systems

The following theorems relating the controllability and observability of composite systems to the controllability and observability of their subsystems were developed by Gilbert [8]. These theorems will be needed in the discussion of the transfer-function representation of multivariable feedback systems.

Here the matrix $\underline{A}$ of any subsystem of a composite system is assumed to have distinct eigenvalues which are different from all those of other subsystems of the composite system.

Theorem 13. Let a composite system S be formed by connecting subsystems S_a and S_b in parallel as shown in Fig. 4. Then:

(a) the order of S is equal to the sum of the orders of S_a and S_b, i.e. $n = n_a + n_b$;

(b) the eigenvalues of S are the totality of the eigenvalues of S_a and S_b, i.e. $\lambda_1,\ldots,\lambda_n = \lambda_{1a},\ldots,\lambda_{naa},\lambda_{1b},\ldots,\lambda_{nbb}$;

(c) the system is controllable (strictly observable) if and only if both S_a and S_b are controllable (strictly observable).

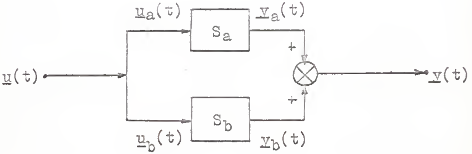

Fig. 4. The system of Theorem 13

Theorem 14. Let the composite system S be formed by connecting S_a and S_b in series as shown in Fig. 5., where S_a is followed by S_b. Then:

(a) the order of S is equal to the sum of the orders of S_a and S_b, i.e. $n = n_a + n_b$;

(b) the eigenvalues of S is the totality of the eigenvalues of S_a and S_b, i.e. $\lambda_1,\ldots,\lambda_n = \lambda_{1a},\ldots,\lambda_{naa},\lambda_{1b},\ldots,\lambda_{nbb}$;

(c) both S_a and S_b must be controllable (strictly observable) if S is to be controllable (strictly observable).

(d) any uncontrollable (unobservable) coordinates of S must originate in $S_b(S_a)$ if S_a and S_b are both controllable (strictly observable).

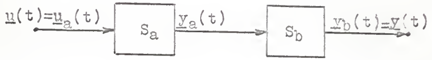

Fig. 5. The system of Theorem 14

Theorem 15. The feedback system shown in Fig. 6 is formed by connecting S_a and S_b as forward and return paths respectively. Denote the series connections of S_a followed by S_b and S_b followed by S_a as S_c and S_o respectively. And assume that $|\underline{I} + \underline{D}_a\underline{D}_b| \neq 0$. Then:

(a) the order of S is equal to the sum of the orders of S_a and S_b, i.e. $n = n_a + n_b$;

(b) the system S is controllable (strictly observable) if and only if $S_c(S_o)$ is controllable (strictly observable);

(c) both S_a and S_b must be controllable (strictly observable) if S is to be controllable (strictly observable);

(d) when S_a and S_b are both controllable (strictly observable) all of the uncontrollable (unobservable) coordinates of S originate in S_b and are uncontrollable (unobservable) coordinates of $S_c(S_o)$.

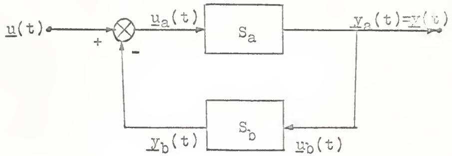

Fig. 6. The system of Theorem 15

According to Theorem 15, the closed-loop controllability and observability can be investigated without examining the closed-loop equations, the open-loop systems S_c and S_o will give all the informations about the controllability and observability of the closed-loop system.

Theorem 13, 14, and 15 can be applied to the composite systems consisting of many subsystems connected in parallel, series and feedback.

Let the combination of the subsystems S^o, S^c and S^f of a system S be denoted by S^u. Then S^u of a composite system contains S_a^u, S_b^u,..., since the coordinates of S_a^u, S_b^u,.... are uncontrollable or unobservable or uncontrollable and unobservable in the composite system S. The remainning coordinates of S_a, S_b, ... can be investigated by applying Theorem 13, 14 and 15 to the interconnection of the subsystems S_a^*, S_b^*,.... For example, S^u of the system of Theorem 15 consists of S_a^u, S_b^u and the coordinates of S_b^* which are uncontrollable in the system S_c^*, the system S_a^*

followed by S_b^*, and unobservable in the system S_o^*, the system S_b^* followed by S_a^*.

Since in this section, all the composite systems and their subsystems are assumed to have distinct eigenvalues, hence by Theorem 12, all the transfer-function matrices have simple poles of rank one, where the rank of a pole is defined as the rank of $\underline{K}_i$ in Theorem 12. Therefore, in what follows, all the transfer-function matrices are assumed to have simple poles of rank one.

Let the transfer-function matrices of S_a and S_b in Theorem 15 be represented by $\underline{H}_a$ and $\underline{H}_b$ respectively. Then,

$$\underline{U}_a = \underline{U}_b - \underline{V}_b = \underline{U} - \underline{H}_b\underline{V}$$

$$= \underline{U} - \underline{H}_b\underline{H}_a\underline{U}_a$$

$$\underline{V} = \underline{H}_a\underline{U}_a = \underline{H}_a(\underline{I} + \underline{H}_b\underline{H}_a)^{-1}\underline{U}$$

and

$$\underline{V} = \underline{H}_a(\underline{U} - \underline{V}_b) = \underline{H}_a\underline{U} - \underline{H}_a\underline{H}_b\underline{V}$$

$$= (\underline{I} + \underline{H}_a\underline{H}_b)^{-1}\underline{H}_a\underline{U}$$

The transfer-function matrix of the system is

$$\underline{H} = \underline{H}_a(\underline{I} + \underline{H}_b\underline{H}_a)^{-1} = (\underline{I} + \underline{H}_a\underline{H}_b)^{-1}\underline{H}_a$$

The transfer-function matrix $\underline{H}$ represents only the controllable and strictly observable part of S, S*. It gives no information about S_a^u and S_b^u, since they are not represented by $\underline{H}_a$ and $\underline{H}_b$.

The uncontrollable coordinates of S which originate in S_b^* correspond to the poles of $\underline{H}_b$ which are cancelled in $\underline{H}_b\underline{H}_a$, since the transfer-function matrix of the system S_a followed by S_b is $\underline{H}_b\underline{H}_a$; the unobservable coordinates of S which originate in S_b^* correspond to the poles of $\underline{H}_b$ which are cancelled in $\underline{H}_a\underline{H}_b$, since the transfer-function matrix of the system S_b followed by S_a is $\underline{H}_a\underline{H}_b$; the uncontrollable and unobservable coordinates of S which originate in S_b^* correspond to the poles of $\underline{H}_b$ which are cancelled in both $\underline{H}_b\underline{H}_a$ and $\underline{H}_a\underline{H}_b$.

Example 21. Consider the feedback system in Fig. 7.

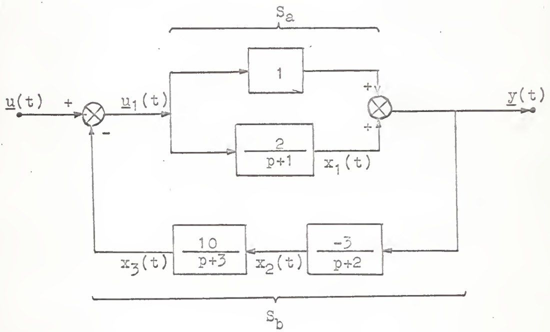

Fig. 7. The system of Example 21, S

The subsystem S_a is shown in Fig. 8.

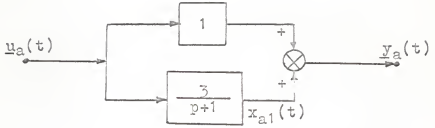

Fig. 8. The subsystem S_a

The state equations and transfer-function are

$$\dot{x}_{a1}(t) = -x_{a1}(t) + 2\underline{u}_a(t)$$

$$\underline{y}_a(t) = x_{a1}(t) + \underline{u}_a(t)$$

$$\underline{H}_a = \frac{S+3}{S+1}$$

Obviously S_a is controllable and strictly observable. There are no pole cancellations in this transfer-function.

The subsystem S_b is shown in Fig. 9.

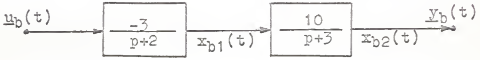

Fig. 9. The subsystem S_b

The state equations in normal form are

$$\underline{\dot{q}}_b(t) = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \underline{q}_b(t) + \begin{bmatrix} -3 \\ 30 \end{bmatrix} \underline{u}_b(t)$$

$$\underline{y}_b(t) = [10 \quad 1] \underline{q}_b(t)$$

The transfer-function is

$$\underline{H}_b = \frac{-30}{(S+2)(S+3)}$$

Hence S_b is controllable and strictly observable. There are no pole cancellations in this transfer-function.

Now, consider the feedback system S. The normal form state equations are

$$\underline{\dot{q}}(t) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 4 \end{bmatrix} \underline{q}(t) + \begin{bmatrix} 0 \\ \frac{2}{11} \\ \frac{3}{11} \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = [0 \quad -10 \quad 14] \underline{q}(t) + \underline{u}(t)$$

The coordinate $q_1(t)$ corresponding to the eigenvalue $\lambda_1 = -3$ is uncontrollable and unobservable. The transfer-function of the system S is

$$\underline{H} = \underline{H}_a(\underline{I} + \underline{H}_b\underline{H}_a)^{-1} = (\underline{I} + \underline{H}_a\underline{H}_b)^{-1}\underline{H}_b$$

$$= \frac{(S+3)/(S+1)}{1 + (\frac{S+3}{S+1})[(\frac{-3}{S+2})(\frac{10}{S+3})]}$$

$$= \frac{(S+3)^2(S+2)}{(S+3)(S+7)(S-4)}$$

$$= \frac{S+2}{(S+7)(S-4)}$$

The pole $s = -3$ is cancelled in the transfer-function, hence the coordinate corresponding to the eigenvalue equal to -3 is uncontrollable and unobservable.

For the system S_c, the system S_a followed by S_b, the block diagram is shown in Fig. 10.

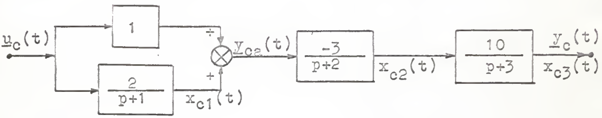

Fig. 10. The system S_c

The state equations of S_c in normal form are

$$\dot{\underline{q}}_c(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \underline{q}_c(t) + \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \underline{u}_c(t)$$

$$\underline{y}_c(t) = \begin{bmatrix} 10 & 15 & -10 \end{bmatrix} \underline{q}_c(t)$$

The coordinate $q_{c3}(t)$ corresponding to the eigenvalue -3 is uncontrollable. The transfer-function of S_c is

$$\underline{H}_c = \underline{H}_b \underline{H}_a = \frac{-30}{(S+2)(S+3)} \times \frac{S+3}{(S+1)} = \frac{-30}{(S+1)(S+2)}$$

The pole s = -3 is cancelled, hence the coordinate corresponding to the eigenvalue -3 is uncontrollable and unobservable.

Now, consider the system S_o, the system S_b followed by S_a, shown in Fig. 11.

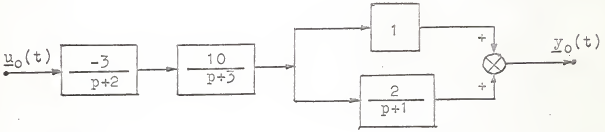

Fig. 11. The system S_o

The state equations of the system in normal form are

$$\dot{\underline{q}}_o(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \underline{q}_o(t) + \begin{bmatrix} -30 \\ -30 \\ 3 \end{bmatrix} \underline{u}_o(t)$$

$$\underline{y}_o(t) = \begin{bmatrix} 1 & 0 & 10 \end{bmatrix} \underline{q}_o(t)$$

The coordinate corresponding to the eigenvalue -3 is unobservable. The transfer-function of S_o is

$$\underline{H}_o = \frac{-30(S+3)}{(S+1)(S+2)(S+3)} = \frac{-30}{(S+1)(S+2)}$$

The pole S = -3 is cancelled, hence the coordinate corresponding to the eigenvalue -3 is uncontrollable and unobservable.

The above discussion is summed up as:

(a) The subsystems S_a and S_b are both controllable and strictly observable, and there is no pole cancellation in both $\underline{H}_a$ and $\underline{H}_b$.

(b) The system S_c is not controllable, the uncontrollable coordinate corresponds to the eigenvalue -3. The transfer-function $\underline{H}_c = \underline{H}_b\underline{H}_a$ have one pole, $S = -3$, cancelled which corresponds to the uncontrollable coordinate.

(c) The system S_o is not strictly observable. The unobservable coordinate corresponds to the eigenvalue -3. The transfer-function $\underline{H}_o = \underline{H}_a\underline{H}_b$ has one pole, $S = -3$, cancelled which corresponds to the unobservable coordinate.

(d) The system S is not controllable and strictly observable. The uncontrollable and unobservable coordinate corresponds to the eigenvalue -3. This coordinate is uncontrollable (unobservable) coordinate of $S_o(S_o)$ and originates in S_b. The transfer-function $\underline{H}$ has one pole, $S = -3$, cancelled which corresponds to the uncontrollable and unobservable coordinate, and it is cancelled in both $\underline{H}_b\underline{H}_a$ and $\underline{H}_a\underline{H}_b$.

SUMMARY

If a system is controllable, then any state can be transferred to any desired state in some finite interval of time by some control. If a system is not controllable, then only those states controllable at initial time t_o can be transferred to any state controllable at some $t_2 \geq t_1(t_o)$. For time-invariant systems, any initial state can be transferred to any state in any interval of time by some control if the system is controllable; if the system is not controllable, then the set of all the controllable states at any time will be the same and any controllable state can be transferred to any controllable state in any interval of time. The set of all the controllable states at t_o is the range of $\underline{V}(t_o,t_1(t_o))$. It depends on $\underline{A}(t)$, $\underline{B}(t)$ and t_o. For time-invariant systems, $\underline{V}(t_o,t_1(t_o))$ depends only on $\underline{A}$ and $\underline{B}$ and is the same as the range of $\underline{Z}_c$. It is difficult to test the controllability of a linear system if the order of the system is too large, for it is difficult to calculate the matrix $\underline{V}(t_o,t_1(t_o))$.

For a strictly observable system, any initial state at any time can be detected at the output of the system in a finite interval of time. If a system is partially observable, then only the component of any initial state belonging to the range of $\underline{W}(t_o,t_1(t_o))$ can be detected at the output of the system over some finite interval of time. For time-invariant systems, any initial state of the system can be detected from the output of the system over any finite interval of time if the system is strictly observable; if the system is partially observable, then the range

of $\underline{W}(t_o, t_1(t_o))$ will be the same for all initial time t_o and the component of any initial state belonging to this range can be detected at the output of the system over any interval of time. For time-variant systems, $\underline{W}(t_o, t_1(t_o))$ depends on $\underline{B}(t)$, $\underline{C}(t)$ and t_o; for time-invariant systems, $\underline{W}(t_o, t_1(t_o))$ depends only on $\underline{A}$ and $\underline{C}$, and it is same as the range of $\underline{Z}_o$. As in the case of controllability of a linear system, it is difficult to test the observability of a linear system if the order of the system is too large.

The transfer-function matrix of a linear system may not completely represent the system. Neglect of the controllability and observability characteristics of a linear system may result in an instability which cannot be detected by investigating only the transfer-function matrix of the system.

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A STUDY OF THE CONTROLLABILITY AND OBSERVABILITY OF LINEAR SYSTEMS

by

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AN ABSTRACT OF A MASTER'S REPORT

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This report mainly compiles some works of Kalman on the concepts of controllability and observability of linear systems and the work of Gilbert on the controllability and observability of composite systems and the transfer-function matrix representation of linear systems.

It begins with the concept of controllability. Some criteria for the study of the controllability characteristics of time-variant systems and the controllability characteristics of time-invariant systems are derived. All the criteria for the controllability characteristics of time-invariant systems can be derived from the time-variant case. For time-invariant systems whose $\underline{A}$ matrices have distinct eigenvalues, an alternative criterion for controllability is derived from a normal form representation.

The next part deals with the concepts of observability. This part parallels the discussion of the concept of controllability. Some criteria for the observability characteristics of time-variant systems and the observability characteristics of time-invariant systems are derived. All the criteria for the observability of time-invariant systems can be derived from the time-variant case. For time-invariant systems whose $\underline{A}$ matrices have distinct eigenvalues, an alternative criterion for observability is derived from a normal form representation.

Finally, the controllability and observability of composite systems and the transfer-function matrix are discussed. The transfer-function matrix representation of linear systems is emphasized. It is shown that the transfer-function matrix of a linear system represents only that part of the system that is controllable and strictly observable.