OPTIMUM DESIGN OF A GEAR-TRAIN BY THE DISCRETE MAXIMUM PRINCIPLE

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INTRODUCTION

Multiple mesh gear-trains are very often used in automotive transmission systems. The most frequently used gear-train is probably the motor driven train which is simply connected and is composed only of spur gears. A schematical representation of the n-mesh gear-train system with the notations to be employed is shown in Fig. 1. The motor and the load are connected by the pinion-gear couplings which, in fact, are the vital part of the study.

Analysis of the system reveals that for a given motor torque, the accelerations (or torques) of all shafts in the system are entirely dependent upon the gear ratios. Hence it is desirable to determine the optimum gear ratios which give the system a maximum condition for a certain chosen criterion. The most important criterion for the motor driven system is the output acceleration. In servomechanisms, the acceleration of the motor shaft is frequently desired. It is to be noted that for each optimization problem, the train value or the overall gear ratios, E, can either be specified or unspecified. Therefore, the chosen criteria are: maximization of

(i) the acceleration of the motor shaft with E unspecified,

(ii) the acceleration of the motor shaft with E specified,

(iii) the acceleration of the load shaft with E unspecified, and

(iv) the acceleration of the load shaft with E specified.

The same problems have been worked out by the classical differential calculus method for less than three-mesh systems. Burgess [1] presented the solution of CASE (iii) for a one-mesh system. For the two-mesh system, he presented a set of two equations which are to be solved simultaneously to obtain the optimum solutions for CASES (iii) and (iv). For the n-mesh system, a recurrence equation which relates the optimum gear ratios of the successive meshes was obtained, and the solutions for CASES (iii) and (iv) were suggested.

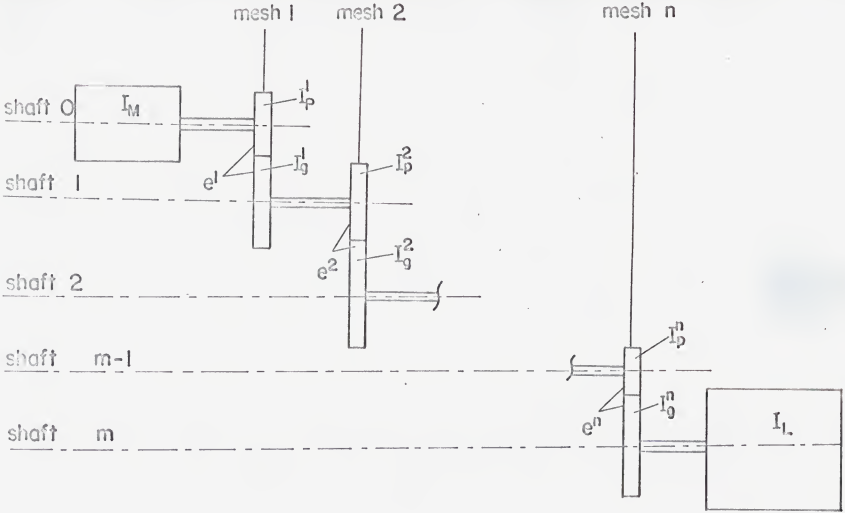

Fig. 1. An n-mesh gear-train system.

He also illustrated a method for determining the optimum number of meshes by means of charts. Mischke [2] presented the solutions for CASES (i) and (iii) for one-mesh and two-mesh systems. A comprehensive inertia torque analysis was also given by him.

The solutions of one-mesh systems obtained by the classical differential calculus method are quite simple. Whereas solutions of two-mesh, three-mesh systems are manageable, although a bit tedious, for a system of more than four-mesh, the corresponding procedures become so involved as to be practically impossible. Therefore, the more advanced optimization techniques are needed. Here, a discrete version of Pontryagin's maximum principle is used.

The maximum principle was first proposed in 1956 by Pontryagin and his associates [3] for individual types of time-optimizing continuous processes. The first attempt to extend the maximum principle to the optimization of stage-wise processes was made by Rozonoer [4], and various discrete versions of the maximum principle were proposed by Chang [5], Katz [6], Fan and Wang [7] and others.

The purpose of this paper is to obtain the general solution of n-mesh gear-train systems for CASES (i) and (ii) which have not been solved, to obtain a more practical solution for CASE (iii) and to demonstrate the advantages of the discrete maximum principle in applying to the optimum designs of the multi-stage engineering systems.

The governing equations are developed first. The algorithm of the discrete maximum principle is introduced, followed by the solutions of n-mesh systems by this method. A summary and two numerical examples are then given, followed by a conclusion.

GOVERNING EQUATIONS [2]

The definition of the gear ratio at the n-th stage and its relation to the accelerations of the corresponding shafts can be shown as (Fig. 1)

$$e^{n} = \frac{r_g^n}{r_p^n} = \frac{\alpha_{m-1}}{\alpha_m}, \qquad \text{where } m = n \tag{1}$$*

The inertia forces analysis of a one-mesh gear-train system can be visualized by drawing a free body diagram for each shaft (Fig. 2). For shaft 0, Euler's equation can be written as

$$T_M - r_p^1 F_t = (I_M + I_p^1)\alpha_0 \tag{2}$$

For shaft 1,

$$r_g^1 F_t = (I_g^1 + I_L)\alpha_1 \tag{3}$$

where the friction forces, external forces and inertia of the shafts are all assumed to be negligible. Eliminating F_t from equations (2) and (3) yields

$$T_M = \left[I_M + I_p^1 + \frac{r_p^1}{r_g^1}\frac{\alpha_1}{\alpha_0}(I_g^1 + I_L)\right]\alpha_0 \tag{4}$$

Since the diameters of the gears in mesh are closely associated with some acceptable contact ratios, it is often desirable that I_g^1 be replaced by some equivalent I_p^1. For this purpose, it is assumed that the gears in mesh are made of the same material and have the same thickness, and then the ratio of the moment of inertias of pinion and gear is

$$\frac{I_p^1}{I_g^1} = \frac{\frac{\pi}{4}(r_p^1)^4}{\frac{\pi}{4}(r_g^1)^4}$$

* The superscript, n, refers to the corresponding number of mesh counting from the motor. The subscript, m, refers to the corresponding number of shaft counting from the motor shaft.

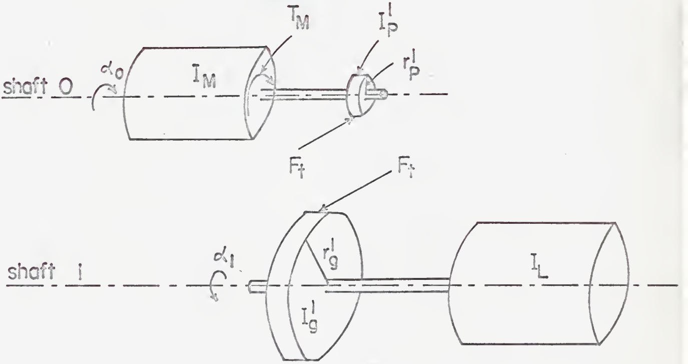

Fig. 2. A free body diagram of a one-mesh gear train system.

or

$$I_g^1 = (e^1)^4 I_p^1 \tag{5}$$

Substitution of this relation together with equation (1) into equation (4) results in

$$\begin{aligned} T_M &= (I_{eq}^1)_0 \alpha_0 \\ T_M &= (I_{eq}^1)_1 \alpha_1 = e^1 (I_{eq}^1)_0 \alpha_1 \end{aligned} \tag{6}$$

where

$$(I_{eq}^1)_0 = I_M + I_p^1 + \frac{(e^1)^4 I_p^1 + I_L}{(e^1)^2} \tag{7}$$

Equation (7) is regarded as the equivalent system inertia for a one-mesh system referred to the motor shaft.

Expanding the above analysis, it can be proven that the equivalent system inertia referred to the motor shaft for a two-mesh system is

$$(I_{eq}^2)_0 = \left[I_M + I_p^1 + \frac{(e^1)^4 I_p^1 + I_p^2}{(e^1)^2} + \frac{(e^2)^4 I_p^2 + I_L}{(e^1 e^2)^2} \right] \tag{8}$$

and for each shaft,

$$\begin{aligned} T_M &= (I_{eq}^2)_0 \alpha_0 \\ T_M &= (I_{eq}^2)_1 \alpha_1 = e^1 (I_{eq}^2)_0 \alpha_1 \\ T_M &= (I_{eq}^2)_2 \alpha_2 = e^1 e^2 (I_{eq}^2)_0 \alpha_2 \end{aligned} \tag{9}$$

In general, for an n-mesh system, the equivalent system inertia referred to the motor shaft is

$$(I_{eq}^{n})_0 = I_M + I_p^1 + \frac{(e^1)^4 I_p^1 + I_p^2}{(e^1)^2} + \frac{(e^2)^4 I_p^2 + I_p^3}{(e^1e^2)^2}$$

$$+ \frac{(e^3)^4 I_p^3 + I_p^4}{(e^1e^2e^3)^2} + \cdots + \frac{(e^n)^4 I_p^n + I_L}{(e^1e^2\ldots e^n)^2} \tag{10}$$

and for each shaft

$$\begin{aligned}
T_M &= (I_{eq}^{n})_0\alpha_0 \\
T_M &= (I_{eq}^{n})_1\alpha_1 = e^1(I_{eq}^{n})_0\alpha_1 \\
T_M &= (I_{eq}^{n})_2\alpha_2 = e^1e^2(I_{eq}^{n})_0\alpha_2 \\
&\vdots \\
T_M &= (I_{eq}^{n})_m\alpha_m = e^1e^2\ldots e^n(I_{eq}^{n})_0\alpha_m
\end{aligned} \tag{11}$$

ALGORITHM OF THE DISCRETE MAXIMUM PRINCIPLE [7,8]

A schematical representation of a simple multistage system is shown in Fig. 3. The system consists of N stage connected in series. The state of the system stream denoted by an s-dimensional vector, $x = (x_1, x_2, \cdots, x_s)$, is transformed at each stage according to an r-dimensional decision vector, $\theta = (\theta_1, \theta_2, \cdots, \theta_r)$, which represents the decision made at that stage. The transformation of the system stream at the n-th stage is described by a set of performance equations,

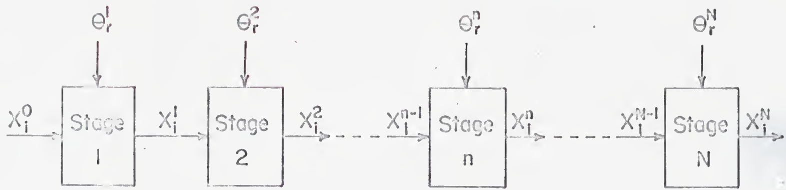

Fig. 3. Multistage decision system.

$$x_i^n = T_i^n(x_1^{n-1}, x_2^{n-1}, \cdots, x_s^{n-1}; \theta_1^n, \theta_2^n, \cdots, \theta_r^n) ,$$
$$x_i = \alpha_i , \qquad i = 1,2,\cdots,s; \qquad n = 1,2,\cdots,N \tag{12}$$

or in vector form

$$x^n = T^n(x^{n-1}; \theta^n) , \qquad n = 1,2,\cdots,N$$
$$x^0 = \alpha . \tag{13}$$

A typical optimization problem associated with such a system is to find a sequence of θ^n, $n = 1,2,\cdots,N$, subject to constraints

$$\psi_i^n[\theta_1^n, \theta_2^n, \cdots, \theta_r^n] \leq 0 , \qquad n = 1,2,\cdots,N, \quad i = 1,2,\cdots,r \tag{14}$$

which makes a function of the state variable of the final stage

$$S = \sum_{i=1}^{s} c_i x_i^N , \qquad c_i = \text{constant} \tag{15}$$

an extremum when the initial condition $x^0 = \alpha$ is given. The function S which is to be maximized (or minimized), is the objective function of the system.

The procedure for solving such an optimization problem by a discrete version of the maximum principle is to introduce an s-dimensional adjoint vector z^n and a Hamiltonian function H^n which satisfy the following relations:

$$H^n = (z^n)^T x^n = \sum_{i=1}^{s} z_i^n T_i^n(x^{n-1}; \theta^n) , \qquad n = 1,2,\cdots,N \tag{16}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}} , \qquad i = 1,2,\cdots,s; \quad n = 1,2,\cdots,N \tag{17}$$

and

$$z_i^N = c_i , \qquad i = 1,2,\cdots,N . \tag{18}$$

If the optimal decision vector function, $\bar{\theta}^n$, which makes the objective function S an extremum (maximum or minimum), is interior to the set of admissible decision, θ^n, a necessary condition for S to be a (local) extremum with respect to θ^n is

$$\frac{\partial H^n}{\partial \theta^n} = 0 , \qquad n = 1,2,\cdots,N \tag{19}$$

If $\bar{\theta}^n$ is at the boundary of the set, it can be determined from the condition that H^n is (locally) extremum.

SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

The optimum solutions for the four chosen criteria can be obtained by formulating the equations derived previously to fit into the algorithm.

(i) Maximization of the acceleration of the motor shaft, α_0, with the train value, E, unspecified.

An N-mesh gear-train system is represented schematically in Fig. 4, where a stage represents a mesh and the n-th stage represents the n-th mesh counting from the motor. The decision variable θ^n and the state variables x_1^n and x_2^n are defined as follows:

θ^n = the gear ratio at the n-th stage, hence $\theta^n = e^n$,

x_1^n = the accumulated train value up to and including the n-th stage of N-stage system,

x_2^n = the equivalent system inertia referred to the motor shaft up to and including the n-th stage.

Thus for the first state variable, the performance equation is

$$\begin{aligned} x_1^n &= x_1^{n-1}\theta^n , \qquad n = 1,2,\cdots,N \\ x_1^0 &= 1 . \end{aligned} \tag{20}$$

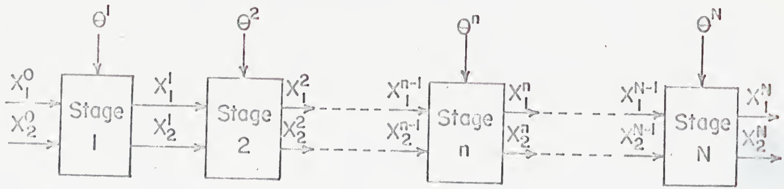

Fig. 4. Schematical representation of a multistage gear train system.

For the second variable; for n=1, from equation (7), x_2^1 can be written as

$$x_2^1 = I_M + I_P^1 + \frac{I_P^1(\theta^1)^4 + I_L}{(\theta^1)^2} \tag{21}$$

if x_2^0 is defined as

$$x_2^0 = I_M + I_L$$

this equation is substituted in equation (21) and results in

$$x_2^1 = x_2^0 + \frac{I_P^1(\theta^1)^4 + (I_P^1 - I_L)(\theta^1)^2 + I_L}{(\theta^1)^2} \tag{22}$$

For n=2, from equation (8), x_2^2 can be written as

$$x_2^2 = I_M + I_P^1 + \frac{I_P^1(\theta^1)^4 + I_P^2}{(\theta^1)^2} + \frac{I_P^2(\theta^2)^4 + I_L}{(\theta^1\theta^2)^2} \tag{23}$$

Substitution of equation (22) into (23) results in

$$x_2^2 = x_2^1 + \frac{I_P^2(\theta^2)^4 + (I_P^2 - I_L)(\theta^2)^2 + I_L}{(\theta^1\theta^2)^2} \tag{24}$$

and it can be shown that we can write x_2^n, in general, as

$$x_2^n = x_2^{n-1} + \frac{I_P^n(\theta^n)^4 + (I_P^n - I_L)(\theta^n)^2 + I_L}{(\theta^1\theta^2\cdots\theta^n)^2} \tag{25}$$

The performance equations for the problem are summarized as follows,

$$\begin{aligned} x_1^n &= x_1^{n-1}\theta^n, \qquad n = 1,2,\cdots,N, \\ x_1^0 &= 1 \end{aligned} \tag{26}$$

$$x_2^n = x_2^{n-1} + \frac{y^n}{(x_1^{n-1}\theta^n)^2}, \qquad n = 1,2,\cdots,N,$$

$$x_2^0 = I_M + I_L \tag{27}$$

where

$$y^n = I_p^n(\theta^n)^4 + (I_p^n - I_L)(\theta^n)^2 + I_L \tag{28}$$

subject to constraints

$$\theta^n \geqslant 1, \qquad n = 1,2,\cdots,N \tag{29}$$

The objective function is to minimize

$$S = \sum_{i=1}^{2} c_i x_i^N = x_2^N, \tag{30}$$

$$c_1 = 0, \qquad c_2 = 1$$

which is the total equivalent inertia referred to the motor shaft of the N stage system. The Hamiltonian function and the adjoint vectors are as follows,

$$H^n = z_1^n[x_1^{n-1}\theta^n] + z_2^n[x_2^{n-1} + \frac{y^n}{(x_1^{n-1}\theta^n)^2}], \qquad n = 1,2,\cdots,N \tag{31}$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n\theta^n - \frac{2(y^n)}{(x_1^{n-1})^3(\theta^n)^2}, \qquad n = 1,2,\cdots,N \tag{32}$$

$$z_1^N = c_1 = 0 \tag{33}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n, \qquad n = 1,2,\cdots,N \tag{34}$$

$$z_2^N = c_2 = 1 \tag{35}$$

From equations (34) and (35), it concludes that

$$z_2^n = 1, \qquad n = 1,2,\cdots,N \tag{36}$$

Hence the Hamiltonian function, equation (31), is reduced to

$$H^n = z_1^n x_1^{n-1} \theta^n + x_2^{n-1} + \frac{y^n}{(x_1^{n-1}\theta^n)^2}, \qquad n = 1,2,\cdots,N \tag{37}$$

Partially differentiating H^n with respect to θ^n gives

$$\frac{\partial H^n}{\partial \theta^n} = z_1^n x_1^{n-1} + \frac{2I_p^n(\theta^n)^4 - 2I_L}{(x_1^{n-1})^2(\theta^n)^3} = 0$$

or

$$z_1^n = \frac{-2[I_p^n(\theta^n)^4 - I_L]}{(x_1^n)^3}, \qquad n = 1,2,\cdots,N \tag{38}$$

Substituting equation (38) into both sides of equation (32) gives the recurrence relation

$$I_p^{n-1}(\theta^{n-1})^4 = I_p^n[2(\theta^n)^2 + 1], \qquad n = 2,3,...,N \tag{39}$$

which relates the optimum gear ratios of two successive gear meshes. Remembering that $z_1^N = 0$, equation (38) can be written for $n = N$, as

$$I_p^N(\theta^N)^4 - I_L = 0$$

or

$$\theta^N = \left[\frac{I_L}{I_p^N}\right]^{\frac{1}{4}} \tag{40}$$

Substituting θ^N into the recurrence relation, equation (39), θ^{N-1} can be obtained. Likewise θ^n, n = N-2, N-3, ..., 1 can be determined and the set of θ^n is the optimum solution.

For instance, for a one-mesh gear-train system (N=1), from equation (40),

$$\theta^1 = \left[\frac{I_L}{I_p^1}\right]^{\frac{1}{4}}$$

This agrees with the result obtained by Mischke [2]. For a two-mesh gear-train system (N=2),

$$\theta^2 = \left[\frac{I_L}{I_p^2}\right]^{\frac{1}{4}}$$

$$\theta^1 = \left[\frac{I_p^2 + 2\sqrt{I_p^2 I_L}}{I_p^1}\right]^{\frac{1}{4}}$$

The optimum train value can be calculated from equation (26). The minimum equivalent system inertia referred to the motor shaft, x_2^N, can be obtained by equation (27), and the maximum acceleration of the motor shaft, α_0, is to be determined from

$$T_M = x_2^N \alpha_0 \qquad (41)$$

(ii) Maximization of the acceleration of the motor shaft, α_0, with the train value, E, specified.

Since the train value, E, is specified, the problem is usually classified as a fixed end problem and the aim is to apportion E among the N stages so that α_0 is a maximum. The definitions of the decision variable θ^n and the state variables x_1^n and x_2^n remain the same as in case (i), while the performance equations, the constraints and the objective function are also remained unchanged except that

$$x_1^N = E \tag{42}$$

is to be added to equation (26). Hence instead of equation (33),

$$z_1^N \neq c_1 = 0 \tag{43}$$

By following the same procedures, it ends up with the same recurrence relation as for CASE (i). This relation is repeated as follows

$$I_p^{n-1}(\theta^{n-1})^4 = I_p^n[2(\theta^n)^2 + 1], \qquad n = 2,3,\cdots,N \tag{44}$$

But for this case, an equation equivalent to equation (40) of CASE (i) is not available. Nevertheless, the set of optimum θ^n can be uniquely determined by equations (44) and (42). The solution can be obtained numerically and will be illustrated in the next section. Once the set of θ^n is known, it is possible to calculate x_2^N, hence α_0 as mentioned in CASE (i).

(iii) Maximization of the acceleration of the load shaft, α_m, with the train value, E, unspecified.

The nature of the problem remains the same as in CASE (i). Instead of maximizing the acceleration, the equivalent system inertia referred to the load shaft, $(I_{eq}^n)_m$, shall be minimized. The decision and state variables are defined as follows

θ^n = the gear ratio at the n-th stage, hence $\theta^n = e^n$,

x_1^n = the accumulated train value up to and including the n-th stage of N stage system,

x_2^n = the equivalent system inertia referred to the load shaft up to and including the n-th stage.

Thus for the first state variable, the performance equation is

$$x_1^n = x_1^{n-1}\theta^n , \qquad n = 1,2,\cdots,N$$
$$x_1^0 = 1 \tag{45}$$

For the second state variable: for a one-stage system, n=1, x_2^1 can be written as

$$x_2^1 = \theta^1 \left[I_M + I_p^1 + \frac{I_p^1(\theta^1)^4 + I_L}{(\theta^1)^2} \right] \tag{46}$$

again defining

$$x_2^0 = I_M + I_L$$

then substituting this relation into equation (46) results in

$$x_2^1 = x_2^0\theta^1 + \frac{I_p^1(\theta^1)^4 + (I_p^1 - I_L)(\theta^1)^2 + I_L}{\theta^1} \tag{47}$$

For n=2, from equations (8) and (9), x_2^2 can be written as

$$x_2^2 = \theta^1\theta^2 \left[I_M + I_p^1 + \frac{I_p^1(\theta^1)^4 + I_p^2}{(\theta^1)^2} + \frac{I_p^2(\theta^2)^4 + I_L}{(\theta^1\theta^2)^2} \right] \tag{48}$$

Substituting equation (47) into (48) yields

$$x_2^2 = x_2^1\theta^2 + \frac{I_p^2(\theta^2)^4 + (I_p^2 - I_L)(\theta^2)^2 + I_L}{\theta^1\theta^2} \tag{49}$$

and, in general, x_2^n can be written as

$$x_2^n = x_2^{n-1}\theta^n + \frac{I_p^n(\theta^n)^4 + (I_p^n - I_L)(\theta^n)^2 + I_L}{\theta^1\theta^2\cdots\theta^n} , \tag{50}$$

$$n = 1,2,\cdots,N$$

The performance equations are summarized as follows

$$x_1^n = x_1^{n-1}\theta^n, \qquad n = 1,2,\cdots,N$$
$$x_1^0 = 1 \tag{51}$$

$$x_2^n = x_2^{n-1}\theta^n + \frac{y^n}{x_1^{n-1}\theta^n}, \qquad n = 1,2,\cdots,N$$
$$x_2^0 = I_M + I_L \tag{52}$$

subjects to constraints

$$\theta^n \geq 1, \qquad n = 1,2,\cdots,N \tag{53}$$

The objective function is to minimize the total equivalent system inertia referred to the load shaft

$$S = \sum_{i=1}^{2} c_i x_i^N = x_2^N$$
$$c_1 = 0, \qquad c_2 = 1 \tag{54}$$

The Hamiltonian function and the adjoint variables are

$$H^n = z_1^n(x_1^{n-1}\theta^n) + z_2^n\left[x_2^{n-1}\theta^n + \frac{y^n}{x_1^{n-1}\theta^n}\right], \quad n = 1,2,\cdots,N \tag{55}$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n\theta^n - z_2^n\left[\frac{y^n}{(x_1^{n-1})^2\theta^n}\right], \tag{56}$$
$$n = 1,2,\cdots,N$$

$$z_1^N = c_1 = 0 \tag{57}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n \theta^n , \qquad n = 1,2,\cdots,N \tag{58}$$

$$z_2^N = c_2 = 1 \tag{59}$$

Equations (58) and (59) gives

$$z_2^n = z_2^{n+1} \theta^{n+1} , \qquad n = 1,2,\cdots,N-1 \tag{60}$$

Taking the partial derivative of equation (55) with respect to θ^n yields

$$\frac{\partial H^n}{\partial \theta^n} = z_1^n x_1^{n-1} + z_2^n \left[x_2^{n-1} + \frac{3I_p^n(\theta^n)^4 + (I_p^n - I_L)(\theta^n)^2 - I_L}{(x_1^{n-1})(\theta^n)^2} \right] = 0$$

or

$$z_1^n = \frac{- z_2^n \left[3I_p^n(\theta^n)^4 + (x_2^{n-1} x_1^{n-1} + I_p^n - I_L)(\theta^n)^2 - I_L \right]}{(x_1^n)^2} , \tag{61}$$

$$n = 1,2,\cdots,N$$

Substitution of equation (61) into both sides of (56) and making use of equation (60) result in the following recurrence relation

$$I_p^{n-1}(\theta^{n-1})^4 = I_p^n \left[2(\theta^n)^2 + 1 \right] , \qquad n = 2,3,\cdots,N \tag{62}$$

which is identical to equations (39) and (44), the recurrence relations obtained for CASES (i) and (ii). For n=N, equation (60) becomes

$$3I_p^N(\theta^N)^4 + (x_1^{N-1} x_2^{N-1} + I_p^N - I_L)(\theta^N)^2 - I_L = 0 \tag{63}$$

Thus for a one-stage system, N=1, equation (63) becomes

$$3I_p^1(\theta^1)^4 + (I_M + I_p^1)(\theta^1)^2 - I_L = 0 \tag{64}$$

The solution of this equation gives

$$\theta^1 = \left[\frac{-(I_M + I_p^1) + \sqrt{(I_M + I_p^1)^2 + 12 I_p^1 I_L}}{6 I_p^1}\right]^{\frac{1}{2}} \tag{65}$$

which agrees with the solutions obtained by Burgess [1] and that by Mischke [2]. However, for N equals two or more, the set of optimum θ^n is to be determined by equations (62) and (63). The solution can be obtained numerically and will be illustrated in the next section. Once the set of θ^n is known, it is possible to calculate x_2^N, and hence α_m by

$$T_M = x_2^N \alpha_m \tag{66}$$

(iv) Maximization of the acceleration of the load shaft, α_m, with the train value, E, specified.

As in CASE (ii), the problem is classified as a fixed end problem and the aim is to apportion E among the N stages so that α_m is a maximum. The definitions of the decision variable θ^n and the state variables x_1^n and x_2^n remain the same as in CASE (iii), while the performance equations, the constraints and the objective function are also unchanged except that

$$x_1^N = E \tag{67}$$

is to be added to equation (51). Hence instead of equation (57),

$$z_1^N + c_1 = 0 \tag{68}$$

By following the same procedures, it ends up with the following recurrence relation

$$I_p^{n-1}(\theta^{n-1})^4 = I_p^n\left[2(\theta^n)^2 + 1\right], \qquad n = 2,3,\cdots,N \tag{69}$$

which is identical to equations (39), (44) and (62), the recurrence relation obtained for CASES (i), (ii) and (iii). The optimum solution θ^n can be determined by equations (67) and (69). It is to be noted that, in spite of the differences between the performance equations, the two equations which determine the optimum solution for CASE (iv) are identical to those of CASE (ii). Therefore CASE (ii) and CASE (iv) are called the dual problem, i.e., to maximize CASE (ii) is equivalent to maximize CASE (iv).

SUMMARY AND NUMERICAL EXAMPLES

In spite of the differences in the performance equations and the criteria chosen to be maximized, it has been found that the recurrence relations which relate the gear ratio of the successive gear mesh are identical for the four cases considered. This relation is repeated below.

$$I_p^{n-1}(\theta^{n-1})^4 = I_p^n[2(\theta^n)^2 + 1], \qquad n = 2,3,\cdots,N \tag{70}$$

which can be rewritten as

$$\theta^{n-1} = \left[\frac{I_p^n[2(\theta^n)^2 + 1]}{I_p^{n-1}}\right]^{\frac{1}{4}}, \qquad n = 2,3,\cdots,N \tag{70a}$$

or

$$\theta^n = \left[\frac{I_p^{n-1}(\theta^{n-1})^4 - I_p^n}{2I_p^n}\right]^{\frac{1}{2}}, \qquad n = 2,3,\cdots,N \tag{70b}$$

This recurrence relation consists of N unknowns but with only N-1 equations. An additional equation obtained for each of CASES (i) and (iii) and given for each of CASES (ii) and (iv) is rewritten here:

CASE (i)

$$\theta^N = \left[\frac{I_L}{I_p^N}\right]^{\frac{1}{2}} \tag{40}$$

CASE (ii)

$$x_1^N = E \quad \text{(given)} \tag{42}$$

CASE (iii)

$$3I_p^N(\theta^N)^4 + (x_1^{N-1}x_2^{N-1} + I_p^N - I_L)(\theta^N)^2 - I_L = 0 \tag{63}$$

CASE (iv)

$$x_1^N = E \quad \text{(given)} \tag{67}$$

Solution of CASE (i) is quite straightforward, while the solutions of CASES (ii), (iii) and (iv) are to be obtained by numerical iteration methods. A specific one, the Falsi method, has been applied to obtain the solution as described in the steps below.

CASE (ii)

Step 1. Assume two trial values for θ^1, i.e., $\theta^1(1)$, $\theta^1(2)$.

Step 2. Calculate θ^n, $n = 2,3,\cdots,N$ by means of equation (70b).

Step 3. Calculate x_1^n, $n = 1,2,\cdots,N$ by means of equation (26).

Step 4. Calculate

$$G = x_1^N - E$$

Step 5. Check the inequality

$$|G| - \epsilon \leq 0$$

where ϵ is the maximum allowable error. If it is satisfied, the set of θ^n is the optimum solution. Go to Step 10. Otherwise proceed to the next step.

Step 6. Calculate a new trial value $\theta^1(3)$ from the following equation,

$$\theta^1(3) = \frac{\theta^1(2)\,G(1) - \theta^1(1)\,G(2)}{G(1) - G(2)} \tag{71}$$

Step 7. Repeat Step 2 through Step 4 to obtain G(3).

Step 8. Check the inequality

$$|G(3)| - \epsilon \leq 0$$

If it is satisfied, the set of θ^n is the optimum solution. Go to Step 10. Otherwise, proceed to the next step.

Step 9. Replace $\theta^1(2)$ and $\theta^1(3)$ by $\theta^1(1)$ and $\theta^1(2)$, and G(2) and G(3) by G(1) and G(2), respectively. Repeat Step 6 through Step 8.

Step 10. Calculate x_2^n, $n = 1,2,\cdots,N$ by means of equation (27).

Step 11. Calculate α_0 by means of equation (41).

CASE (iii) The procedures given in CASE (ii) can be applied to this case by making the following modifications.

(1) Replace Step 4 by Steps 4a and 4b shown below,

Step 4a. Calculate x_2^n, $n = 1,2,\cdots,N$ by means of equation (52).

Step 4b. Calculate

$$G = 3I_p^N(\theta^N)^4 + (x_1^{N-1}x_2^{N-1} + I_p^N - I_L)(\theta^N)^2 - I_L$$

(2) Replace Step 10 by: Calculate α_m by means of equation (66).

(3) Skip Step 11.

CASE (iv) Since CASES (iv) and (ii) are dual problems, the procedures of solution are identical to that of CASE (ii).

The determination of the optimum number of meshes for a certain chosen criterion can be done by tabulating and comparing the optimum solutions obtained for $n = 1,2,\cdots,n$.

The following numerical examples are solved as an illustration.

Example 1. The optimum gear ratio of each mesh for maximizing the four chosen criteria for a three-mesh and a five-mesh gear-train system is obtained. The given parameters for these systems are shown in Tables 1 and 2.

Example 2. The parameters, I_M, I_L, T_M, I_p^1 and I_p^2 of an existing two-mesh gear-train system are given in Table 3. Additional pinions with $I_p = 0.3$ are available. The optimum additional number of gear meshes to be installed in the system for maximizing the acceleration of the load shaft with the train value unspecified is obtained.

The solutions of Examples 1 and 2 have been obtained by the use of a computer and are tabulated in Tables 1, 2 and 3. The results of simulations indicate that the solutions obtained are actually the optimum ones.

Referring to CASE (i) and CASE (ii) in Tables 1 and 2, for the maximization of the acceleration of motor shaft, α_0, with either the train value, E, specified or unspecified, the optimum gear ratios, θ^n, are allocated in such a way that the first N-1 gear ratios are close to one another, while the gear ratio at the last stage, θ^N, is much larger which, in fact, is the decisive factor for the system train value, E. The maximum acceleration, α_0, is almost independent of the train-value, E, in other words, no matter what train values are assigned to the system, the maximum motor shaft acceleration remains almost the same. For CASE (iii), the maximization of the acceleration of load shaft, α_m, with the train value, E, unspecified, the gear ratios, θ^n, are all close to one another, and there is no decisive factor for the train value. While for CASE (iv), with the train value, E, specified, the gear ratio of the last mesh, θ^N, is the decisive factor of the train value as appeared in CASES (i) and (ii).

Referring to Table 3, the results show that a four-mesh system gives the largest maximum load shaft acceleration, α_m. Thus, two more meshes are to be installed in the system. However, it must be pointed out that this optimal policy is for maximization of the load shaft acceleration as the criterion. It might not be the optimal policy for the system if cost factor is considered.

Table 1. The given parameters and the solutions obtained for Example 1 (N=3).

Given parameters	$N = 3$		$I_M = 12$	$I_p^1 = 1.0$
			$I_L = 50$	$I_p^2 = 0.8$
			$T_M = 200$	$I_p^3 = 0.4$
	CASE (i)	CASE (ii)	CASE (iii)	CASE (iv)
Optimum solutions obtained	$\theta^1 = 1.58$	$\theta^1 = 1.67$	$\theta^1 = 1.31$	$\theta^1 = 1.67$
	$\theta^2 = 1.85$	$\theta^2 = 2.09$	$\theta^2 = 1.16$	$\theta^2 = 2.09$
	$\theta^3 = 3.34$	$\theta^3 = 4.30$	$\theta^3 = 1.14$	$\theta^3 = 4.30$
	$E = 9.78$	$E = 15$ (given)	$E = 1.73$	$E = 15$ (given)
	$(I_{eq}^3)_0 = 18.0$	$(I_{eq}^3)_0 = 18.2$	$(I_{eq}^3)_3 = 56.9$	$(I_{eq}^3)_3 = 273$
	$\alpha_0 = 11.1$	$\alpha_0 = 11.0$	$\alpha_3 = 3.51$	$\alpha_3 = 0.73$
Corresponding values of I_{eq} & α	$(I_{eq}^3)_3 = 176$	$(I_{eq}^3)_3 = 273$	$(I_{eq}^3)_0 = 32.9$	$(I_{eq}^3)_0 = 18.2$
	$\alpha_3 = 1.137$	$\alpha_3 = 0.73$	$\alpha_0 = 6.08$	$\alpha_0 = 11.0$

Table 2. The given parameters and the solutions obtained for Example 1 (N=5).

Given parameters	$N = 5$	$I_M = 35$	$I_p^1 = 1.0$	$I_p^4 = 0.3$
		$I_L = 380$	$I_p^2 = 1.0$	$I_p^5 = 0.3$
		$T_M = 500$	$I_p^3 = 0.3$	
	CASE (i)	CASE (ii)	CASE (iii)	CASE (iv)
Optimum solutions obtained	$\theta^1 = 1.45$	$\theta^1 = 1.42$	$\theta^1 = 1.37$	$\theta^1 = 1.42$
	$\theta^2 = 1.30$	$\theta^2 = 1.23$	$\theta^2 = 1.12$	$\theta^2 = 1.23$
	$\theta^3 = 2.06$	$\theta^3 = 1.83$	$\theta^3 = 1.46$	$\theta^3 = 1.83$
	$\theta^4 = 2.92$	$\theta^4 = 2.25$	$\theta^4 = 1.33$	$\theta^4 = 2.25$
	$\theta^5 = 5.97$	$\theta^5 = 3.50$	$\theta^5 = 1.04$	$\theta^5 = 3.50$
	$E = 67.3$	$E = 25$ (given)	$E = 3.09$	$E = 25$ (given)
	$(I_{eq}^5)_0 = 40.2$	$(I_{eq}^5)_0 = 40.6$	$(I_{eq}^5)_5 = 245.7$	$(I_{eq}^5)_5 = 1014$
	$\alpha_0 = 12.4$	$\alpha_0 = 12.3$	$\alpha_5 = 2.04$	$\alpha_5 = 0.493$
Corresponding values of I_{eq} & α	$(I_{eq}^5)_5 = 2700$	$(I_{eq}^5)_5 = 1014$	$(I_{eq}^5)_0 = 79.4$	$(I_{eq}^5)_0 = 40.6$
	$\alpha_5 = 0.185$	$\alpha_5 = 0.493$	$\alpha_0 = 6.30$	$\alpha_0 = 12.3$

Table 3. Determination of the optimum number of meshes for Example 2.

Given parameters	$I_M = 35$	$I_p^1 = 1.0$	$(I_p^3 = 0.3)$	
	$I_L = 380$	$I_p^2 = 1.0$	$(I_p^4 = 0.3)$	
	$T_M = 500$		$(I_p^5 = 0.3)$	
	N = 2	N = 3	N = 4	N = 5
Optimum solutions obtained	$\theta^1 = 1.66$	$\theta^1 = 1.41$	$\theta^1 = 1.37$	$\theta^1 = 1.37$
	$\theta^2 = 1.80$	$\theta^2 = 1.22$	$\theta^2 = 1.13$	$\theta^2 = 1.12$
		$\theta^3 = 1.78$	$\theta^3 = 1.47$	$\theta^3 = 1.46$
			$\theta^4 = 1.36$	$\theta^4 = 1.33$
				$\theta^5 = 1.04$
	$E = 2.98$	$E = 3.07$	$E = 3.087$	$E = 3.090$
	$(I_{eq}^2)_2 = 247.57$	$(I_{eq}^3)_3 = 245.55$	$(I_{eq}^4)_4 = 245.50$	$(I_{eq}^5)_5 = 245.7$
	$\alpha_2 = 2.020$	$\alpha_3 = 2.036$	$\alpha_4 = 2.037$	$\alpha_5 = 2.035$
Corresponding values of $(I_{eq})_0$ & α_0	$(I_{eq}^2)_0 = 83.0$	$(I_{eq}^3)_0 = 79.9$	$(I_{eq}^4)_0 = 79.5$	$(I_{eq}^5)_0 = 79.4$
	$\alpha_0 = 6.02$	$\alpha_0 = 6.26$	$\alpha_0 = 6.29$	$\alpha_0 = 6.30$

The examples have been worked out merely to illustrate the procedures of solutions. Therefore, no dimension has been assigned, and the given parameters have been arbitrarily chosen. However, the inertia of the pinion chosen must always satisfy the following relation which is derived from equation (70b).

$$I_p^{n-1}(\theta^{n-1})^4 > I_p^n$$

CONCLUSION

By the use of the discrete maximum principle with the aid of modern computers, the practical optimum solutions of n-mesh gear-train systems for the four chosen criteria and the optimum number of meshes can be determined. The study reveals that the recurrence relations which relate the optimum gear-ratios of the gear-train systems are identical for all four cases. For optimization of the acceleration of any intermediate shaft between the motor and the load shafts when the train-value is specified, the recurrence relations are also identical to the one obtained, whereas for the case that the train value is unspecified, the recurrence relation remains to be investigated. The feature of the discrete maximum principle is that the n-stage system as a whole is treated, and hence the solutions obtained are valid for the n-stage system in general, and the computational procedures are simpler than the conventional methods. It shows that the discrete maximum principle is a powerful tool in dealing with the optimal designs of multistage systems. It can be applied to obtain the optimum solutions for many other design problems.

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NOMENCLATURE

e^n = gear ratio or step down ratio of nth stage

E = overall gear ratio or train value

F_t = transmission force

H = Hamiltonian function, defined by equation (16)

$(I^n_{eq})_m$ = equivalent system inertia of n-mesh system referred to mth shaft

I^n_g = inertia of gear of nth stage

I_L = inertia of load

I_M = inertia of motor

I^n_p = inertia of pinion of nth stage

N = number of stages of a multimesh gear-train

n = the nth stage of a multimesh gear-train defined in Fig. 3

r = number of decision variables

r^n_g = radius of gear of nth stage

r^n_p = radius of pinion of nth stage

S = objective function

s = number of state variables

T = transformation poerator, defined by equation (12)

T_M = motor torque

x = state vector

y = function defined by equation (28)

z = covariant vector of x introduced in maximum principle

α_m = acceleration of the mth shaft

ψ = constraint function

θ^n = control variable of nth stage

Superscript n = stage number

Subscript m = shaft number

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OPTIMUM DESIGN OF A GEAR-TRAIN BY THE
DISCRETE MAXIMUM PRINCIPLE

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The general algorithm of the discrete maximum principle is stated briefly. It is applied to obtain the optimum solutions of design problems involving the n-mesh gear-train system for four chosen criteria which are: the maximization of i) the acceleration of the motor shaft with the train value unspecified, ii) the acceleration of the motor shaft with the train value specified, iii) the acceleration of the load shaft with the train value unspecified and iv) the acceleration of the load shaft with the train value specified.

The purpose of this report is to obtain the general solution of n-mesh gear-train systems for the chosen criteria (i) and (ii) which have not been solved, to obtain a more practical solution for the chosen criterion (iii) and to demonstrate the advantages of the discrete maximum principle in applying to the optimum designs of the multistage engineering systems.

The study reveals that by the use of the discrete maximum principle with the aid of modern computers, the practical solutions of optimization problems involving design of n-mesh gear-train systems for the four chosen criteria and the optimum number of meshes can be determined. It also reveals that the recurrence relations which relate the optimum gear-ratios of the gear-train systems are identical for all four cases. The discrete maximum principle is shown to be a powerful tool in dealing with the optimal designs of multistage systems. It can be applied to obtain the optimum solutions for many other multistage design problems.