THE ANALYSIS AND CONTROL OF A PRESSURE PROCESS

by

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NOMENCLATURE

A_b, A_d	Areas of valve openings, ft^2.
A_o	Area of orifice, ft^2.
A_1, A_2	Cross-sectional areas, ft^2.
a_x, a_y, etc.	Constants, lbf-sec/ft^2.
B_v	Viscous damping of working fluid, lbf-sec/ft.
b_k, b_v, etc.	Constants, ft-sec.
C	Output function of illustrative system.
c_1	Constant, sec^{-1}.
c_k	Constant, sec^{-3}.
E	A function of P_1, ρ_1 and r.
G(s)	Forward-loop transfer function.
$G_k(s)$	Transfer function of controller.
g	Acceleration due to gravity, ft/sec^2.
H	Feedback-loop transfer function.
H_1, H_2	Nonlinear functions of P_v, x etc.
K	Gain of controller.
k	Ratio of constant pressure specific heat to constant volume specific heat.
k_m	A constant of proportionality, ft^2.
K_v	Spring constant, lbf/ft.
$\dot{M}_{in}, \dot{M}_b$, etc.	Mass-flow rate of fluid, lbf-sec/ft.
$\bar{\dot{M}}_{in}, \bar{\dot{M}}_b$	Nondimensional mass-flow rate of fluid.
m_v	Mass of the moving parts of the final control element, lbf-sec^2/ft.

N	Ratio of P_s to P_a.
n	Polytropic exponent of a perfect gas.
P, P_s, etc.	Fluid pressures, lbf/ft^2.
$\Delta P(s)$	Output function, lbf/ft^2.
$\Delta P_i(s)$	Input function, lbf/ft^2.
R	Input function of illustrative example.
R_g	Gas constant, ft^2/sec^2-^oR.
r	Pressure ratio.
s	Laplace transform operator, sec^{-1}.
T	Temperature, ^oR.
T, T_k	Time constants, sec.
t	Time, sec.
V_1, V_2	Velocities of fluid flow, ft/sec.
V_c, V_v	Volumes of tank and chamber, ft^3.
x,y	Displacements of control elements of valves, ft.
ρ_1, ρ_2, etc.	Densities of fluid, $lbf\text{-}sec^2/ft^4$.
θ	Angle of asymptote for root-locus plot, degrees.
ζ	Damping ratio.
ω_n	Undamped natural frequency, rad./sec.

CHAPTER I

INTRODUCTION

Fluid control systems are widely employed in the control of chemical and petroleum installations, aircraft, machine tools and farm machinery. New applications are continuously being made. This is especially true in the newest field of fluid control known as Fluidics.

The physical variable to be controlled by a fluid control system differs according to the function of the system. It may be one or more of the following:

1. speed,
2. temperature,
3. displacement,
4. velocity,
5. flow of fluid,
6. pressure,
7. humidity,

   etc.

According to whether a liquid or a gas is used as the working fluid, fluid control systems are classified into hydraulic systems or pneumatic systems (1) *. In general, pneumatic systems are cheaper to manufacture and less costly to operate than hydraulic systems. Because of these charac-

* Numbers in parentheses refer to items of references.

teristics, pneumatic control systems are widely used in chemical and petroleum industry and in other specialized branches of engineering.

This report presents an analysis of a simple pneumatic-type pressure-control-system which is shown schematically in Fig.1. The object of this pressure-control-system is to maintain the pressure of a compressible fluid in a tank (or a

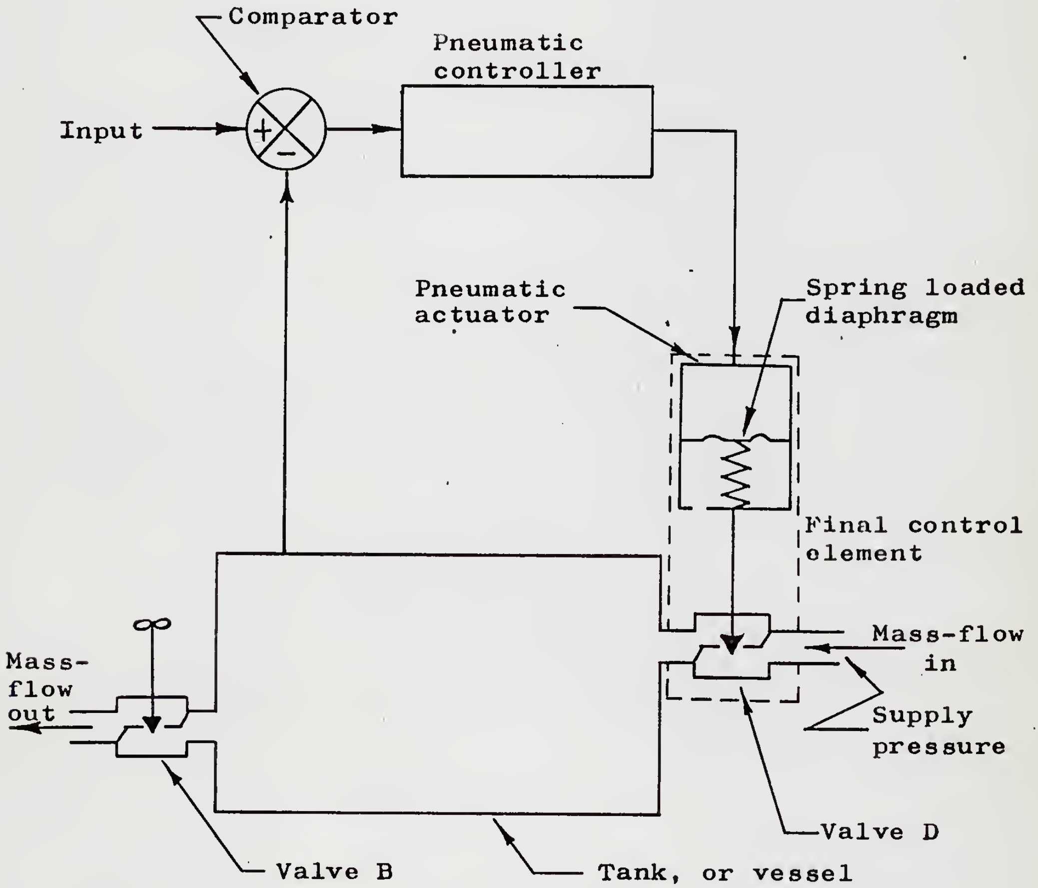

Fig.1. Schematic representation of the pressure-control-system considered in this report.

vessel) approximately constant as demand (or disturbance) flow through valve B varies. Such pressure-control problems can be found in the distillation columns of chemical process plants (2).

If a change, say a small decrease, in outflow occurs, the pressure in the tank is increased. This increase in pressure is "sensed" by the comparator which transmits an error signal to the pneumatic controller. The controller in turn transmits a pneumatic signal to the final control element by forcing air to flow into the upper chamber of the pneumatic actuator which is part of the final control element. Due to the increase in pressure, the spring loaded diaphragm sepera-ting the two chambers of the actuator is forced to move down-ward, thus reducing the valve opening of valve D. This reduc-tion of valve opening causes a decrease of fluid flow into the tank, thereby reducing the pressure in the tank.

If the change in outflow is an increase in magnitude, the control action is reversed.

This type of control system, where the input is fixed, is known as a regulating system. A control system in which the input is varied is termed a servo-system.

CHAPTER II

FLOW THROUGH A VALVE

Flow through an orifice is a very idealized form of fluid flow. However, many actual flow situations can be approximated by this kind of flow, a typical example is <u>flow through a valve</u>. Flow through a short line can also be considered as flow through an orifice without causing significant error.

Consider Fig.2 which shows an orifice through which a compressible fluid flows from station 1 to station 2. Let the pressure, density, velocity, elevation of the fluid at station 1 be P_1, ρ_1, V_1, Z_1 and that at station 2 be p_2, ρ_2, V_2, Z_2 respectively. Before an analysis is made, it is advantageous to make the following assumptions:

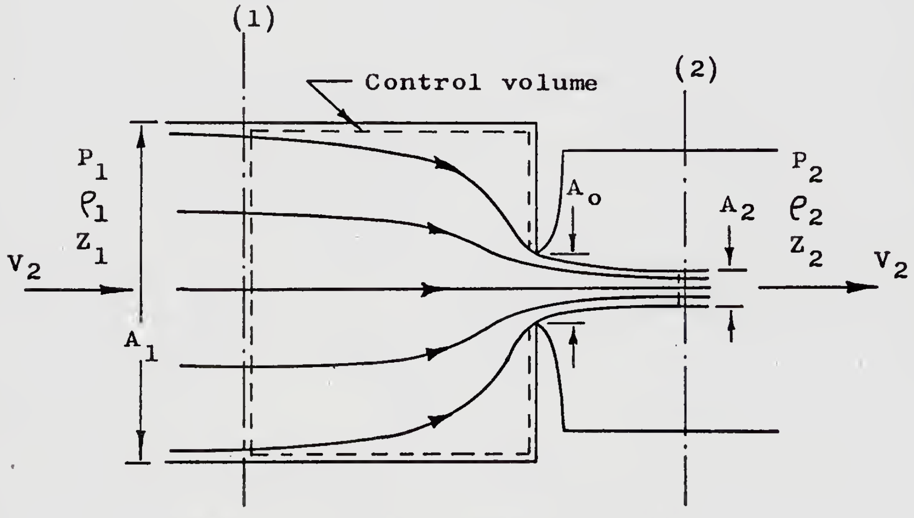

Fig. 2. Flow through orifice.

1. The fluid obeys the perfect gas law $P=\rho R_g T$, where P, ρ and T are the pressure, density and temperature respectively of the fluid and R_g the gas constant.
2. Pressure, velocity, density and temperature are uniform across stations 1 and 2.

A. Polytropic Flow Through an Orifce

In flowing from station 1 to station 2, the fluid in general undergoes a polytropic process in which

$$\frac{P}{\rho^n} = c, \text{ a constant.} \tag{2.1}$$

If the process is isothermal (very low velocity), $n=1$; if it is adiabatic (very high velocity), $n=k$, where k is the ratio of constant pressure specific heat to constant volume specific heat.

Equation (2.1) can be rewritten as

$$P = c\rho^n . \tag{2.2}$$

Differentiating gives

$$dP = nc\rho^{n-1}d\rho \quad .$$

When this is substituted into Bernoulli's equation for compressible fluid flow

$$\int\frac{dP}{\rho} + gZ + \frac{V^2}{2} = \text{constant (along a streamline)},$$

the result is

$$nc\int \rho^{n-2} d\rho + gZ + \frac{V^2}{2} = \text{constant.} \qquad (2.3)$$

When the first term is integrated, equation (2.3) becomes

$$\frac{nc}{n-1}\,\rho^{n-1} + gZ + \frac{V^2}{2} = \text{constant.}$$

This can be rewritten as

$$\frac{n}{n-1}\,\frac{P}{\rho} + gZ + \frac{V^2}{2} = \text{constant.}$$

Applying this into the orifice shown in Fig.1 results in

$$\frac{n}{n-1}\,\frac{P_1}{\rho_1} + gZ_1 + \frac{V_1^2}{2} = \frac{n}{n-1}\,\frac{P_2}{\rho_2} + gZ_2 + \frac{V_2^2}{2}\,.$$

After grouping and noticing that (Z_2-Z_1) is negligible, the above equation reduces to

$$\frac{V_1^2 - V_1^2}{2} = \frac{n}{n-1}\left(\frac{P_1}{\rho_1} - \frac{P_2}{\rho_2}\right). \qquad (2.4)$$

Assume the flow is steady, i.e., mass flow at station 1 equals mass flow at station 2, then

$$\frac{dM_1}{dt} = \frac{dM_2}{dt}\,.$$

Let A_1 and A_2 be the cross-sectional areas at station 1 and station 2 respectively, the above expression is equivalent to

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2\,.$$

Solving for V_1 gives

$$V_1 = \frac{\rho_2 A_2 V_2}{\rho_1 A_1}\,.$$

When this is substituted into equation (2.4), the result is

$$\frac{V_2^2}{2}\left[1 - \left(\frac{\rho_2 A_2}{\rho_1 A_1}\right)^2\right] = \frac{n}{n-1}\left(\frac{P_1}{\rho_1} - \frac{P_2}{\rho_2}\right) .$$

Solving for V_2 yields

$$V_2 = \sqrt{\frac{\frac{2n}{n-1}\left(\frac{P_1}{\rho_1} - \frac{P_2}{\rho_2}\right)}{\left[1 - \left(\frac{\rho_2 A_2}{\rho_1 A_1}\right)^2\right]}} \quad . \tag{2.5}$$

After replacing $\frac{\rho_2}{\rho_1}$ by $\left(\frac{P_2}{P_1}\right)^{1/n}$ and simplifying, equation (2.5) becomes

$$V_2 = \sqrt{\frac{\frac{2n}{n-1}\left(\frac{P_1}{\rho_1}\right)\left[1 - \left(\frac{P_2}{P_1}\right)^{1-1/n}\right]}{\left[1 - \left(\frac{A_2}{A_1}\right)^2\left(\frac{P_2}{P_1}\right)^{2/n}\right]}} \quad . \tag{2.6}$$

Expression (2.6) gives the theoretical velocity of the flowing fluid at station 2. In fact, the actual velocity is slightly less than the theoretical due to friction. Let a velocity coefficient K_v be introduced here, such that the actual velocity is

$$V_{2a} = K_v V_2$$

where

$$K_v = f(\text{Reynolds number}).$$

The cross-sectional area at station 2 would be difficult to measure, but A_o, the cross-sectional area of the orifice, is easily measured. Let K_c be the ratio of A_2 to A_o, i.e.,

$$A_2 = K_c A_o ,$$

where

$$K_c = f(\text{Reynolds number}).$$

K_c is known as the coefficient of contraction.

The actual mass-rate of flow through the orifice is then

$$\frac{dM}{dt} = K_v K_c A_o \rho_2 V_2 . \qquad (2.7)$$

Substituting equation (2.6) into equation (2.7) and denoting $\frac{dM}{dt}$ by $\dot{M}$ results in

$$\dot{M} = K_v K_c A_o \sqrt{\frac{\frac{2n}{n-1}\left(\frac{P_1}{\rho_1}\right)\rho_2^2\left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}}\right]}{\left[1 - \left(\frac{K_c A_o}{A_1}\right)^2 \left(\frac{P_2}{P_1}\right)^{2/n}\right]}} . \qquad (2.8)$$

Define

$$K_d = \frac{K_v K_c}{\sqrt{1 - \left(\frac{K_c A_o}{A_1}\right)^2 \left(\frac{P_2}{P_1}\right)^{2/n}}} , \qquad (2.9)$$

where

$$K_d = f(\text{Reynolds number}, \frac{A_o}{A_1}).$$

Replace ρ_2^2 by $\rho_1^2 \left(\frac{P_2}{P_1}\right)^{2/n}$, equation (2.8) then becomes

$$\dot{M} = K_d A_o \sqrt{\frac{2n}{n-1} P_1 \rho_1 \left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}}\right] \left(\frac{P_2}{P_1}\right)^{2/n}} . \qquad (2.10)$$

K_d in expression (2.9) is known as the discharge coefficient. In many cases, it has constant values if $A_o \ll A_1$ and the flow Reynolds number is high. At very low velocity, it varies with the Reynolds number.

B. Critical and Subcritical Flow

For control valves the length of the high-velocity flow stream is quite short and experimental data has shown that the effects of heat transfer are insignificant (3). Therefore when applying equation (2.10), n=k can be used for control valves.

It should, however, be remembered that for very low velocities, i.e., when P_2 and P_1 are nearly equal, the flow through the valve is approaching an isothermal process.

The mass-flow rate as a function of pressure ratio for n=k=1.4 is shown in Fig.3, where P_1 and ρ_1 are assumed constant.

When P_2 approaches zero, equation (2.10) shows that $\dot{M}$ also approaches zero (curve ABO). This is an obvious absurdity. In order to make this clear, experiments were performed by scientists, and the resulting data revealed that the actual mass-flow rate curve was represented by curve ABC instead of ABO (4). In other words, the mass-flow rate remains constant after reaching its maximum value (point B). The pressure at which the maximum flow rate occurs is called critical pressure. The part of flow for which the pressure P_2 is higher

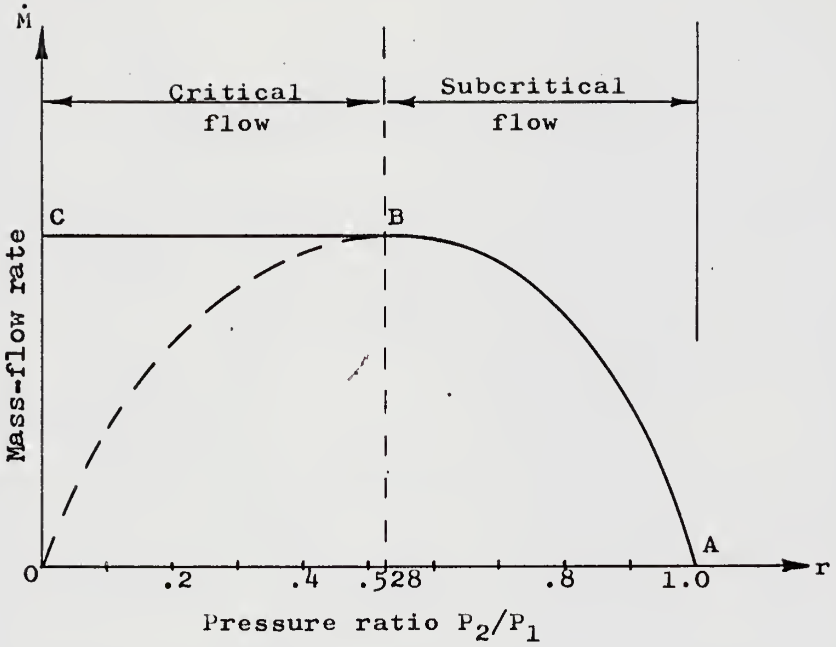

Fig.3. Mass-flow rate as a function of pressure ratio for adiabatic flow.. (n=k=1.4)

than the critical pressure (part AB) is known as subcritical (or subsonic) flow, the the other part (part BC) is termed critical (or sonic) flow. The velocity of critical flow is equal to the acoustic velocity in the fluid at critical pressure (4).

Let $r = \frac{P_2}{P_1}$ and replace n by k for adiabatic flow, equation (2.10) becomes

$$\dot{M} = K_d A_o \sqrt{\frac{2k}{k-1} P_1 \rho_1 \left[1 - r^{\frac{k-1}{k}} \right] r^{2/k}} \qquad (2.10a)$$

Define $$E = \frac{2k}{k-1} P_1 \rho_1 \left[1 - r^{\frac{k-1}{k}}\right] r^{\frac{2}{k}} .$$

It is clear that $\dot{M}$ is a maximum when E is a maximum. Differentiating the above expression with respect to r, considering K_d, A_o, ρ_1, P_1 and k as constants, yields

$$\frac{dE}{dr} = \frac{2k}{k-1} P_1 \rho_1 \left[\frac{2}{k} r^{\frac{2-k}{k}} - \frac{k+1}{k} r^{1/k}\right].$$

When $\frac{dE}{dt}$ is equated to zero and solved for r, which represents the critical pressure ratio, it results in

$$r_c = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{2.11}$$

where the subscript c denotes critical. Substituting this into equation (2.10) gives the critical mass-flow rate, denoted by $\dot{M}_c$,

$$* \; \dot{M}_c = K_d A_o \sqrt{k \rho_1 P_1 \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \quad . \tag{2.12}$$

The above equation shows that, for critical flow through an orifice, the mass-flow rate is independent of the downstream pressure P_2. If K_d and k are assumed constant, $\dot{M}_c$ is proportional to the square root of the product of P_1 and ρ_1, the pressure and density of the upstream fluid.

$$\dot{M}_c = K_m \sqrt{\rho_1 P_1} \tag{2.13}$$

* See Appendix A.

where K_m is a constant of proportionality and equal to

$$K_d A_o \sqrt{k\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \,.$$

In this chapter, characteristics of flow through an orifice is discussed. As mentioned at the beginning of this chapter, this idealized form of flow can be applied to flow through valves which will be treated in later chapters. Equations (2.10) and (2.12) will be used for describing subcritical and critical flow respectively through a valve.

CHAPTER III

STEADY STATE FLOW ANALYSIS FOR THE CONTROLLED SYSTEM

Figure 4 shows the tank of the system being considered. Fluid flows into it through a control valve D and discharges into the atmosphere through valve B. Let V_c be the volume of the tank, P the fluid pressure in the tank, and $\dot{M}_{in}$, $\dot{M}_b$ the mass-flow rates through valves D, B respectively. For steady fluid flow, the rate of mass flow is independent of time, and

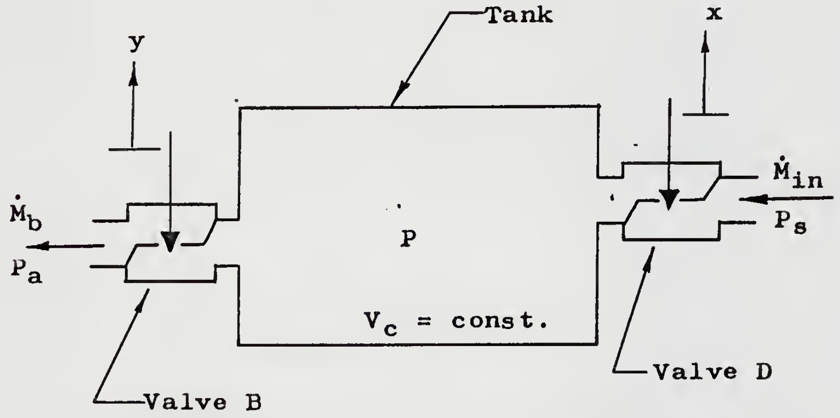

Fig.4. Fluid flow through valves and tank.

so is the pressure of the fluid in the tank. There is, therefore, no change in mass stored in the tank if P is fixed and the temperature inside the tank is assumed constant. If, however, P changes, the mass-flow rate changes also. Hence it is desirable to construct steady state curves to relate the mass-flow entering and leaving the tank and the fluid pressure in the tank. A method from which such curves can be constructed is discussed in the following paragraphs.

Before starting, let the following assumptions be made:

1. The working fluid is a perfect gas.
2. The outside pressure is atmospheric pressure.
3. The pressure P is uniform throughout the tank.
4. The flow entering and leaving is one dimensional.
5. The walls of the tank are infinitely rigid, i.e., V_c is constant.
6. The discharge coefficient K_d for valves B and D is constant. ($A_o \ll A_1$, Reynolds number high).
7. The supply pressure P_s and supply temperature T_s to valve D are constant.

Since the heat conductivity of a gas is low and the velocity for gas flow is usually high, the heat transfer between a unit mass of the gas and the surroundings is generally negligible. It is therefore reasonable to assume the flow adiabatic.

A. Steady Flow Through Valve D

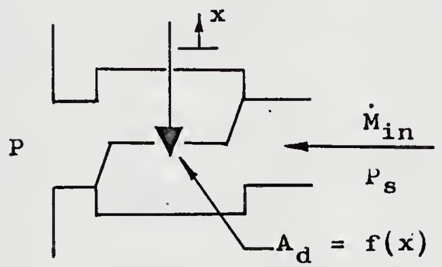

Fig.5. Flow through valve D.

There are two possible types of flow through valve D, viz., critical and subcritical. They are discussed separately as follows.

1. Critical flow. This happens when the pressure ratio

$$\frac{P}{P_s} \leq \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} .$$

The critical flow equation in this case is

$$\dot{M}_{inc} = K_d A_d \sqrt{k \rho_s P_s \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \tag{3.1}$$

where A_d is the area of the valve opening in valve D through which fluid flows. It is a variable quantity such that $A_d = A_d(x)$, (See Fig.5).

For convenience, it is desirable to make the steady state mass-flow-pressure characteristics nondimensional. This can be done by defining a maximum mass flow rate as follows,

$$\dot{M}_{max} = K_d A_{d_{max}} \sqrt{k \rho_s P_s \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} . \tag{3.2}$$

When equation (3.1) is divided by equation (3.2), the result is

$$\frac{\dot{M}_{inc}}{\dot{M}_{max}} = \frac{A_d}{A_{d_{max}}} . \tag{3.3}$$

Define

$$\bar{\dot{M}}_{inc} = \frac{\dot{M}_{inc}}{\dot{M}_{max}} ,$$

$$\bar{A}_d = \frac{A_d}{A_{d_{max}}} ,$$

equation (3.3) then becomes

$$\bar{\dot{M}}_{inc} = \bar{A}_d \, . \tag{3.4}$$

All quantities in the above equation are dimensionless. They are said to be normalized. $\bar{\dot{M}}_{inc}$ is termed normalized mass-flow rate and $\bar{A}_d$ normalized area, and equaation (3.4) is the normalized equation for critical flow through valve D.

2. Subcritical flow. This occurs in valve D when the pressure ratio

$$\frac{P}{P_s} > \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} .$$

The subcritical flow equation for valve D is

$$\dot{M}_{ins} = K_d A_d \sqrt{\left(\frac{2k}{k-1}\right) \rho_s P_s \left[1 - \left(\frac{P}{P_s}\right)^{\frac{k-1}{k}}\right] \left(\frac{P}{P_s}\right)^{2/k}} \, . \tag{3.5}$$

When this is divided by equation (3.2), the result is

$$\bar{\dot{M}}_{ins} = \bar{A}_d \sqrt{\frac{2}{k-1}\left(\frac{k+1}{2}\right)^{\frac{k+1}{k-1}} \left[1 - \bar{P}^{\frac{k-1}{k}}\right] \bar{P}^{\frac{2}{k}}} \tag{3.6}$$

where $\bar{P} = \frac{P}{P_s}$. Simplifying gives

$$\bar{\dot{M}}_{ins} = \bar{A}_d \bar{P}^{1/k} \sqrt{\frac{2}{k-1}\left(\frac{k+1}{2}\right)^{\frac{k+1}{k-1}} \left[1 - \bar{P}^{\frac{k-1}{k}}\right]} \, . \tag{3.7}$$

Equation (3.7) represents the normalized equation for subcritical flow through valve D.

B. Steady Flow Through Valve B

Like that for valve D, either critical or subcritical flow may occur in this valve according to whether the pressure ratio is greater than or less than the critical pressure ratio.

1. Critical flow. In this case, the pressure ratio must be

$$\frac{P_a}{P} \leqslant \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} ,$$

or

$$P \geqslant \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} P_a .$$

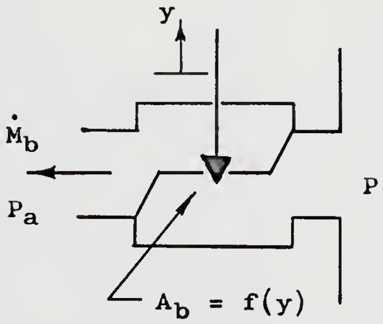

Fig.6. Flow through Valve B

Dividing both sides by the positive quantity P_s gives

$$\bar{P} \geqslant \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \frac{1}{N}$$

where

$$N = \frac{P_s}{P_a} .$$

The mass-flow rate for critical flow through valve B is

$$\dot{M}_{bc} = K_d A_b \sqrt{k \rho P \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \qquad (3.9)$$

where A_b is the area of the valve opening in valve B through which fluid flows. In general, $A_b = A_b(y)$. (See fig.6.)

Dividing equation (3.9) by equation (3.2) yields

$$\bar{\dot{M}}_{bc} = \bar{A}_b \sqrt{\bar{P} \frac{\rho}{\rho_s}} \qquad (3.10)$$

where

$$\bar{\dot{M}}_{bc} = \frac{\dot{M}_{bc}}{\dot{M}_{max}} ,$$

$$\bar{A}_b = \frac{A_b}{A_{d_{max}}} .$$

After replacing $\frac{\rho}{\rho_s}$ by $\bar{P}^{1/k}$, equation (3.10) becomes

$$\bar{\dot{M}}_{bc} = \bar{A}_b \bar{P}^{\frac{k+1}{2k}} , \tag{3.11}$$

which is the normalized equation for critical flow through valve B.

2. Subcritical flow. This occurs when

$$\bar{P} < \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \frac{1}{N} .$$

The flow equation in this case is

$$\dot{M}_{bs} = K_d A_b \sqrt{\frac{2k}{k-1} \rho P \left[1 - \left(\frac{P_a}{P}\right)^{\frac{k-1}{k}}\right] \left(\frac{P_a}{P}\right)^{2/k}} . \tag{3.12}$$

Normalizing as before and using $P_s = NP_a$ yields

$$\bar{\dot{M}}_{bs} = \bar{A}_b \sqrt{\frac{2}{k-1} \left(\frac{k+1}{2}\right)^{\frac{k+1}{k-1}} \bar{P}^{\frac{k+1}{k}} \left[1 - (N\bar{P})^{\frac{1-k}{k}}\right] (N\bar{P})^{-\frac{2}{k}}} . \tag{3.13}$$

With the normalized equations for critical and subcritical flow for valves D and B, it is possible to plot steady-state curves relating pressure and mass-flow rate with areas of valve openings as parameters. This is illustrated by the following example.

C. Illustrative Example

The working fluid is air with constant P_s of 100 psia. The flow is assumed adiabatic. Plot the flow curves for valves B and D.

For air, k=1.4. The critical pressure ratio for valve D is

$$\bar{P}_c = \left(\frac{2}{1.4+1}\right)^{\frac{1.4}{1.4-1}}$$

$$= 0.528.$$

When $0 < \bar{P} \leq 0.528$, the flow through valve D is critical, and the normalized flow equation is

$$\bar{\dot{M}}_{inc} = \bar{A}_d \ . \qquad (3.14)$$

When $0.528 < \bar{P} \leq 1$, the flow is subcritical. From equation (3.7), the normalized mass-flow rate can be shown to be

$$\bar{\dot{M}}_{ins} = 3.86\ \bar{A}_d \bar{P}^{0.715} \sqrt{1 - \bar{P}^{0.286}} \ . \qquad (3.15)$$

The ratio of supply pressure to atmospheric pressure

$$N = \frac{100}{14.7} = 6.8 \ .$$

Substituting this into equation (3.8) gives the critical pressure ratio for flow through valve B

$$\bar{P}_{cb} = \left(\frac{1.4+1}{2}\right)^{\frac{1.4}{1.4-1}} \left(\frac{1}{6.8}\right)$$

$$= 0.2766 \ .$$

It is clear that when $\bar{P} = 0.147$, there will be no flow through valve B, because the pressure in the tank is the same as the atmospheric pressure. This can be checked by substituting $\overline{P} = \frac{1}{N} = 0.147$ into equation (3.13), the result is $\overline{\dot{M}}_{bs} = 0$.

Subcritical flow through valve B occurs when $0.147 < \overline{P} < 0.2786$, and the normalized flow equation (3.13) reduces to

$$\overline{\dot{M}}_{bs} = 0.98\ \overline{A}_b \sqrt{\overline{P}^{0.286} - 0.578} \quad . \tag{3.16}$$

For $0.2786 \leq \overline{P} < 1$, the flow is critical. From equation (3.11), it is seen that the normalized mass-flow rate through valve B is

$$\overline{\dot{M}}_{bc} = \overline{A}_b \overline{P}^{0.857} \quad . \tag{3.17}$$

Flow curves can now be plotted using equations (3.14) through (3.17) for various values of A_d and A_b. These are shown in Fig.7.

A given desired steady state pressure $\overline{P}_o = \frac{P_o}{100}$ in the tank corresponds to a vertical line in Fig.7. Then for any given steady state outflow $\overline{\dot{M}}_b$, which is a function of $\overline{A}_b$, the inlet valve D must have an area $\overline{A}_d$ such that $\overline{\dot{M}}_b = \overline{\dot{M}}_{in}$, which is the necessary condition for steady state flow with pressure $\overline{P}_o$ in the tank.

It is seen that the steady state flow curves can be divided into 3 regions, depending on the steady state value

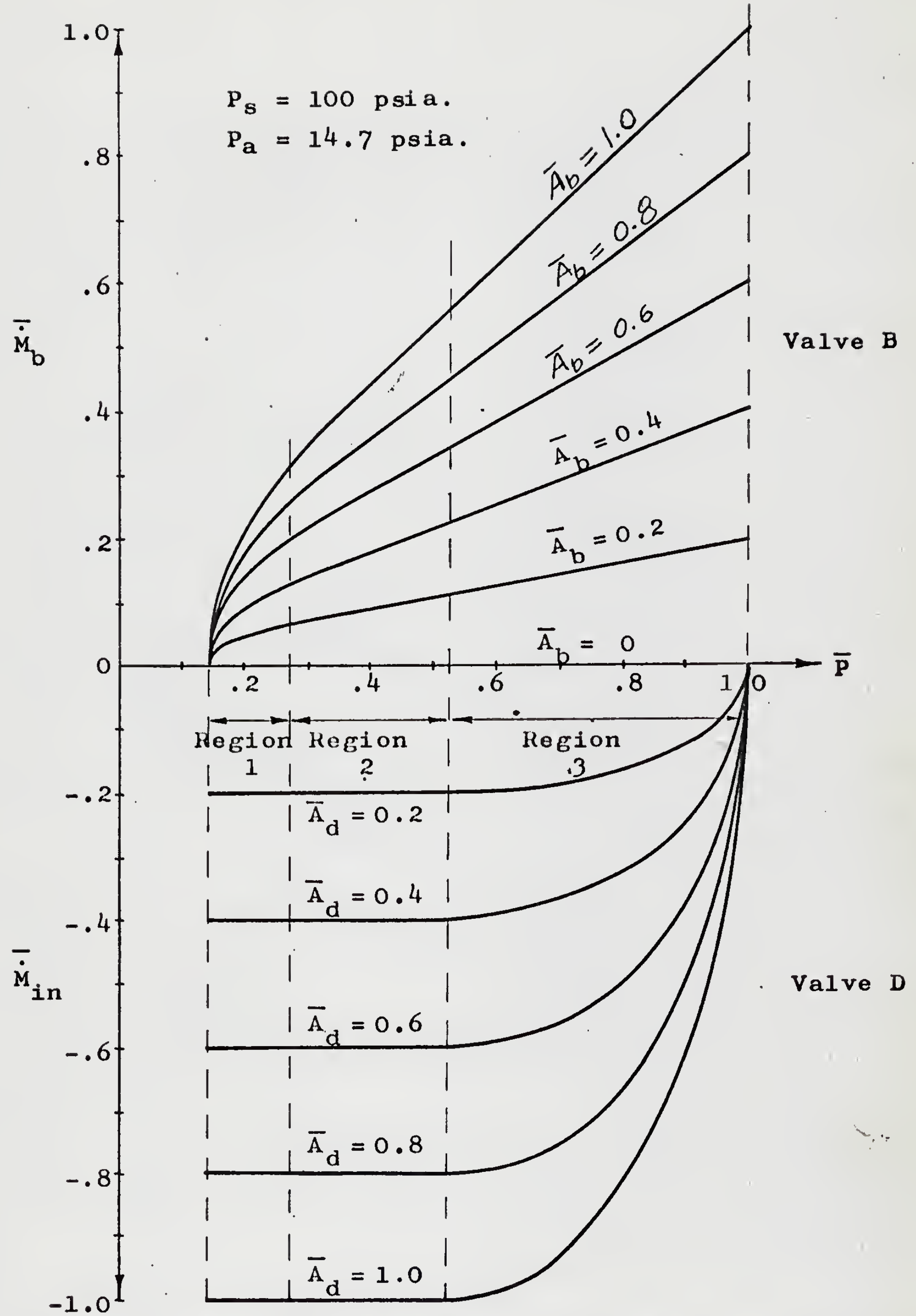

Fig.7. Dimensionless mass-flow rate as a function of pressure ratio for adiabatic flow.

P_o in the tank. In region 1, the flow through valve B is subcritical while that through valve D is critical; in region 2, flow through both valves are critical; in region 3, the flow through valve B is critical while that through valve D is subcritical.

If the supply pressure should be less than $3.59P_a$, regions 1 and 3 would overlap and the flow through both valves would be subcritical. Since the supply pressure in many actual cases will be higher than this, such a situation will not be considered in this analysis.

The steady state curves shown in Fig.7 are useful in deriving the dynamic relationship for P(t) when unsteady flow conditions exist. This will be done in Chapter IV.

CHAPTER IV

DERIVATION OF TRANSFER FUNCTION FOR THE CONTROLLED SYSTEM

As shown in Chapter III, the equations for the flow of a compressible fluid through valves can be either linear or nonlinear. In the derivation of a transfer function, it is usually the case to combine several equations together. Difficulties will arise when combining these equations in their original form if any are nonlinear. It is, therefore, desirable to use some technique to overcome these difficulties. In this chapter, the Taylor series expansion will be used to transform the nonlinear (exact) equations into linear (approximate) equations.

A. Unsteady Flow Through Valve D.

It is shown in Chapter III that the mass-flow rate through valve D is a function of the pressure in the tank and the displacement of the control element of the valve, if the supply pressure and temperature are assumed constant. This can be expressed mathematically as

$$\dot{M}_{in} = f(P,x) \tag{4.1}$$

where P and x are functions of time. Expanding the above in a Taylor series for two variables about an initial-steady-state-operating point $\dot{M}_{in_o}$, P_o and x_o gives

$$\dot{M}_{in} = \dot{M}_{in_o} + \left.\frac{\partial \dot{M}_{in}}{\partial P}\right|_{\substack{P_o \\ x_o}} (P - P_o) + \left.\frac{\partial \dot{M}_{in}}{\partial x}\right|_{\substack{P_o \\ x_o}} (x - x_o) + \text{higher order terms.} \tag{4.2}$$

where the partial derivatives are evaluated at an initial-steady-state-operating point $P = P_o$, and $x = x_o$. The partial derivatives for valve D can be graphically evaluated from the upper half of the steady-state-characteristic curves shown in Fig.7 in Chapter III, or from the analytical expression for $\dot{M}_{in}$. If a restriction is made that $(P - P_o)$ and $(x - x_o)$ are small, the higher order terms can be neglected, only the linear terms of the Taylor series remain.

Let

$$P - P_o = \Delta P \ ,$$

$$x - x_o = \Delta x \ ,$$

$$\dot{M}_{in} - \dot{M}_{in_o} = \Delta \dot{M}_{in} .$$

The "Δ variables" are functions of time. Equation (4.2) then reduces to

$$\Delta \dot{M}_{in} = \left.\frac{\partial \dot{M}_{in}}{\partial P}\right|_{\substack{P_o \\ x_o}} \Delta P + \left.\frac{\partial \dot{M}_{in}}{\partial x}\right|_{\substack{P_o \\ x_o}} \Delta x \ . \tag{4.3}$$

Define

$$a_{p1} = \left.\frac{\partial \dot{M}_{in}}{\partial P}\right|_{\substack{P_o \\ x_o}} ,$$

$$a_x = \left.\frac{\partial \dot{M}_{in}}{\partial x}\right|_{\substack{P_o \\ x_o}} ,$$

equation (4.3) then becomes

$$\Delta \dot{M}_{in} = a_{p1}\Delta P + a_x \Delta x \,. \tag{4.4}$$

Equation (4.4) is a linearized equation. The values of a_{p1} and a_x vary according to whether the flow is critical (regions 1 and 2 in Fig.7) or subcritical (region 3 in Fig.7). Both are defined analytically in Appendix B.

B. Unsteady Flow Through Valve B

By means of the same technique used in section A, the linearized equation for the unsteady mass-flow rate through valve B can be shown to be

$$\Delta \dot{M}_b = \left.\frac{\partial M_b}{\partial P}\right|_{\substack{P_o\\x_o}} \Delta P + \left.\frac{\partial M_b}{\partial y}\right|_{\substack{P_o\\x_o}} \Delta y \,. \tag{4.5}$$

Define

$$a_{p2} = \left.\frac{\partial M_b}{\partial P}\right|_{\substack{P_o\\x_o}} ,$$

$$a_y = \left.\frac{\partial M_b}{\partial P}\right|_{\substack{P_o\\x_o}} ,$$

equation (4.5) then becomes

$$\Delta \dot{M}_b = a_{p2}\,\Delta P + a_y\,\Delta y \,. \tag{4.6}$$

Again the values of a_{p2} and a_y vary according to whether the flow is critical (regions 2 and 3 in Fig.7) or subcritical (region 1 in Fig.7). Both are defined analytically in Appendix C.

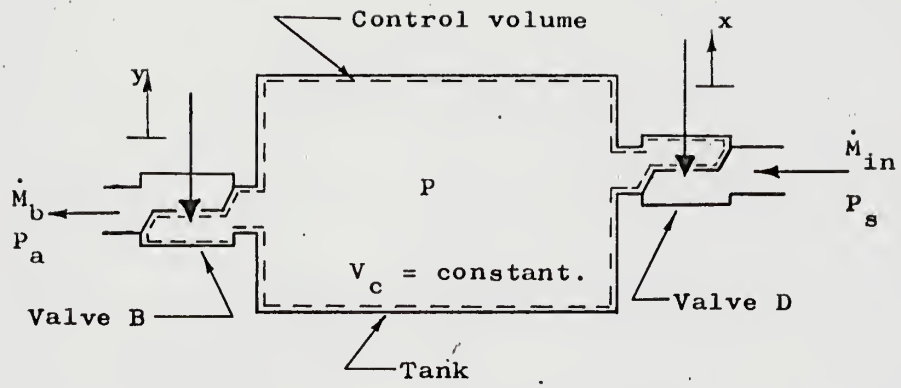

Fig.8. Control volume for unsteady flow analysis.

C. Derivation of the Transfer Function

In the analysis of unsteady flow, it is helpful to introduce to the controlled system a control volume, which is shown by dotted lines in Fig.8. Applying the equation of continuity for unsteady flow gives

$$-\sum(\text{mass-flow rate in}) + \sum(\text{mass-flow rate out}) + \frac{\partial}{\partial t}(\rho V_c) = 0.$$

This can be expressed for small changes in variables for the system of Fig.8 as

$$-\Delta\dot{M}_{in} + \Delta\dot{M}_b + V_c\Delta\dot{\rho} = 0. \qquad (4.7)$$

From equations (4.4) and (4.6), it is seen that the above equation can be expressed as

$$-(a_{p2} - a_{p1})\Delta P + a_x\Delta x - a_y\Delta\dot{y} = V_c\Delta\dot{\rho} . \qquad (4.8)$$

The values of the "a's" differ in different operating regions as defined in Fig.7, and will be discussed later.

From the perfect gas relation

$$\rho = \rho_s \left(\frac{P}{P_s}\right)^{1/k} ,$$

it can be shown that

$$\dot{\rho} = \frac{1}{k}\left(\frac{\rho_s}{P_s}\right)\left(\frac{P}{P_s}\right)^{\frac{1-k}{k}} \dot{P} \; . \qquad (4.9)$$

This is a nonlinear expression. After linearizing, the following expression is obtained.

$$\Delta\dot{\rho} = \frac{\partial\dot{\rho}}{\partial\dot{P}}\bigg|_{\substack{\dot{P}_o \\ P_o}} \Delta\dot{P} + \frac{\partial\dot{\rho}}{\partial P}\bigg|_{\substack{\dot{P}_o \\ P_o}} \Delta P \qquad (4.10)$$

where the changes in variables $\Delta P = (P - P_o)$, $\Delta\dot{P} = (\dot{P} - \dot{P}_o)$ and $\Delta\dot{\rho} = (\dot{\rho} - \dot{\rho}_o)$ are assumed saml1.

The analytical expression for the partial derivatives are

$$\frac{\partial\dot{\rho}}{\partial\dot{P}}\bigg|_{\substack{\dot{P}_o \\ P_o}} = \frac{1}{k}\left(\frac{\rho_s}{P_s}\right)\left(\frac{P_o}{P_s}\right)^{\frac{1-k}{k}} ,$$

$$\frac{\partial\dot{\rho}}{\partial P}\bigg|_{\substack{\dot{P}_o \\ P_o}} = \frac{1-k}{k^2}\left(\frac{\rho_s}{P_s}\right)\left(\frac{P_o}{P_s}\right)^{\frac{1-2k}{k}} \dot{P}_o \; .$$

The second expreesion is zero because P_o is a constant and therefore $\dot{P}_o = 0$. Substituting these into equation (4.10) gives

$$\Delta\dot{\rho} = \frac{1}{k}\left(\frac{\rho_s}{P_s}\right)\left(\frac{P_o}{P_s}\right)^{\frac{1-k}{k}} \Delta\dot{P} \; . \qquad (4.11)$$

Define

$$a_\rho = \frac{1}{k}\left(\frac{\rho_s}{P_s}\right)\left(\frac{P_o}{P_s}\right)^{\frac{1-k}{k}}, \tag{4.12}$$

expression (4.11) then becomes

$$\Delta\dot{\rho} = a_\rho \Delta\dot{P}$$

Substituting this into equation (4.8) results in

$$-(a_{p2} - a_{p1})\,\Delta P + a_x\,\Delta x - a_y\,\Delta y = a_\rho V_c\,\Delta\dot{P},$$

rearranging yields

$$a_\rho V_c\,\Delta\dot{P} + (a_{p2} - a_{p1})\,\Delta P = a_x\,\Delta x - a_y\,\Delta y. \tag{4.13}$$

This is a linearized differential equation relating the pressure change in the tank to the displacement of the control elements of valves D and B. It is valid only when the changes of all variables from a given initial-steady-state-operating point are small.

When equation (4.13) is Laplace-transformed into the s-domain, assuming zero initial conditions, and solved for $\Delta P(s)$, the result is

$$\Delta P(s) = \frac{a_x\,\Delta X(s) - a_y\,\Delta Y(s)}{a_\rho V_c s + (a_{p2} - a_{p1})}. \tag{4.14a}$$

Let

$$a_{xt} = \frac{a_x}{a_{p2} - a_{p1}}, \tag{4.15a}$$

$$a_{yt} = \frac{a_y}{a_{p2} - a_{p1}}, \tag{4.15b}$$

$$T = \frac{a_\varrho V_c}{a_{p2} - a_{p1}}, \tag{4.15c}$$

equation (4.14a) then becomes

$$\Delta P(s) = \frac{a_{xt}\,\Delta X(s) - a_{yt}\,\Delta Y(s)}{Ts + 1}. \tag{4.14b}$$

Equation (4.14b) is the transfer function for the controlled system shown in Fig.8. The output is the fluid pressure in the tank and the input is the displacement of the control element of valve D. The displacement of the control element of valve B serves as a disturbance to the system. The quantity T is known as the time constant for the system.

As can be seen from Fig.7 in Chapter III, $a_{p1} = \frac{\partial \dot{M}_{in}}{\partial P}$ is either negative or zero, while $a_{p2} = \frac{\partial \dot{M}_b}{\partial P}$ is always positive, therefore T is always positive. Since a_{xt}, a_{yt} and T are functions of the initial steady-state-operating point, they will take on different values depending upon the region in which P_o is located. Their analytical expressions are given in Appendix D.

The magnitude of T for each of the three possible operating regions defined in Fig.7 can be qualitatively evaluated as follows:

Region	a_{p2} has..	a_{p1} is or has..	$\therefore$ * T will have..
1	maximum values	0	smallest values
2	minimum values	0	largest values
3	minimum values	maximum values	smallest values

These qualitative results for T will be used to aid in qualitatively analyzing the transient response of the system to a step type disturbance in Chapter VII.

* T is evaluated qualitatively using equation (4.15c).

CHAPTER V

DERIVATION OF TRANSFER FUNCTIONS FOR THE FINAL CONTROL ELEMENT

In order to control the pressure in the tank shown in Fig.8, it is necessary to (1) measure the pressure in the tank, (2) compare this pressure with a signal which represents the desired pressure to form an error signal, (3) operate on this error signal with a controller to produce a manipulated output signal and (4) use this manipulated output signal to position valve D. This can better be explained by means of a block diagram which is shown in Fig.9.

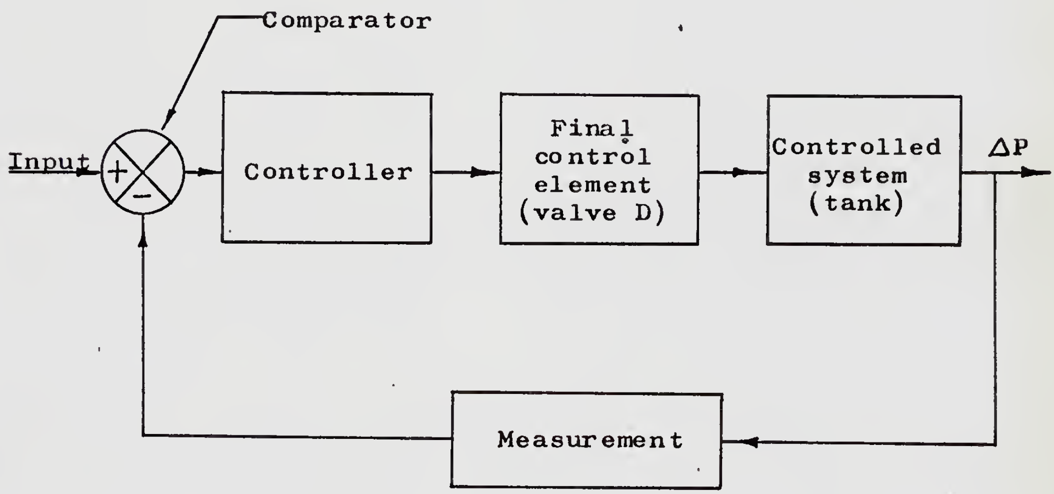

Fig.9. Illustrative block diagram of the system.

The linearized transfer function for the controlled system was derived in Chapter IV. In this chapter, the transfer function for the final control element will be derived.

Figure 10 shows a schematic representation of the final control element commonly used in pressure control systems. It is composed of two parts, the actuator and the control valve D. The actuator consists of two chambers separated by a spring loaded diaphragm. A pneumatic signal from the controller passes through a line into the upper chamber and so forces the diaphragm to move downward. This causes valve D to decrease its valve opening and therefore reduces the fluid flow into the tank. If, however, the pressure in the upper chamber is higher than that in the controller, reversed action results.

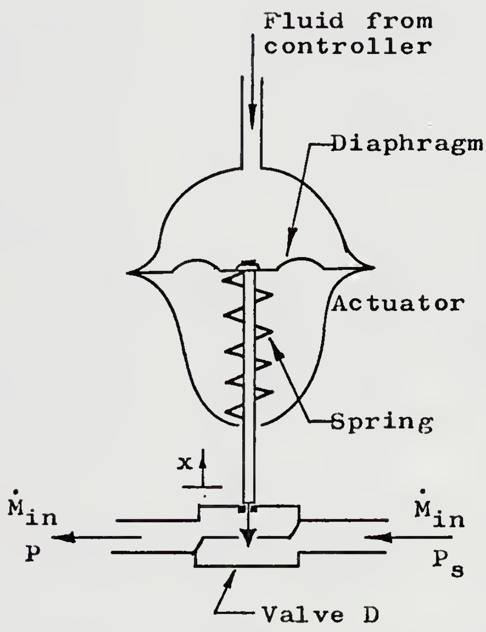

Fig.10. Schematic representation of the final control element.

Since the line connecting the controller with the actuator is "short", it may be assumed that the resistance of the line can be lumped into an orifice of area A_o, i.e., the pressure drop across the line is equivalent to that across an orifice.

Analyses of the upper and the lower portions of the final control element are discussed separately as follows.

A. Transfer Function for the Upper Chamber

For convenience, two separate diagrams are drawn for the final control element, so that the analyses can be made more clearly.

The upper portion of the final control element is shown in Fig.11. Let P_k be the pressure of the fluid from the controller, $\dot{M}_k$ the mass-rate of flow into the chamber, A_v the effective area of the diaphragm, P_v the fluid pressure in the chamber, V_v the instantaneous volume and V_{vo} its original volume when $x = 0$, i.e., $V_v(x=0)=V_{vo}$, then

$$V_v = V_{vo} - A_v x \qquad (5.1)$$

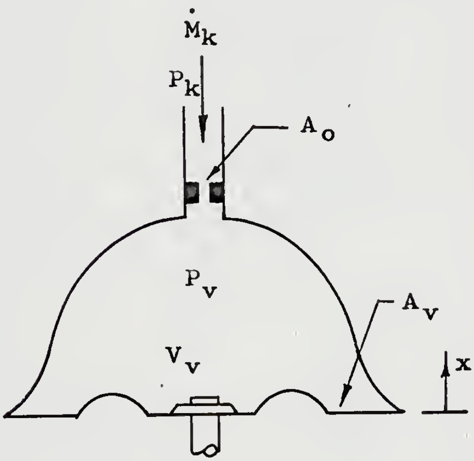

Fig.11. Schematic representation of the upper portion of the final control element.

Differentiating with respect to t gives

$$\dot{V}_v = -A_v\dot{x} \qquad (5.2)$$

Applying the equation of continuity for unsteady fluid flow into the chamber results in

$$-\dot{M}_k + \frac{\partial}{\partial t}(\rho_v V_v) = 0 \tag{5.3}$$

where ρ_v is the density of the fluid in the chamber.

Equation (5.3) can be rewritten as

$$-\dot{M}_k + \dot{\rho}_v V_v + \rho_v \dot{V}_v = 0 \ . \tag{5.4}$$

Substitution of expressions (5.1) and (5.2) into equation (5.4) yields

$$-\dot{M}_k + (V_o - A_v x)\dot{\rho}_v - A_v \rho_v \dot{x} = 0 \ . \tag{5.5}$$

From the relation

$$\rho_v = \rho_{max}\left(\frac{P_v}{P_{max}}\right)^{1/n} ,$$

it is clear that

$$\dot{\rho}_v = \frac{1}{n}\left(\frac{\rho_{max}}{P_{max}}\right)\left(\frac{P_v}{P_{max}}\right)^{\frac{1-n}{n}} \dot{P}_v \ , \tag{5.6}$$

where ρ_{max} and P_{max} are the density and pressure respectively of the fluid at the maximum controller output pressure, which is usually 15 psig, or approximately 29.7 psia.

When these are substituted into equation (5.5), it results in

$$-\dot{M}_v + (V_{vo} - A_v x)\frac{1}{n}\,\frac{\rho_{max}}{P_{max}}\left(\frac{P_v}{P_{max}}\right)^{\frac{1-n}{n}} \dot{P}_v - A_v \rho_{max}\left(\frac{P_v}{P_{max}}\right)^{1/n} \dot{x} = 0 \ . \tag{5.7}$$

Define

$$H_1 = (V_{vo} - A_v x)^{\frac{1}{n}} \frac{\rho_{max}}{P_{max}} \left(\frac{P_v}{P_{max}}\right)^{\frac{1-n}{n}} \dot{P}_v , \qquad (5.8a)$$

$$H_2 = A_v \rho_{max} \left(\frac{P_v}{P_{max}}\right)^{1/n} \dot{x} , \qquad (5.8b)$$

equation (5.7) then becomes

$$-\dot{M}_k + H_1 - H_2 = 0 .$$

This equation can be redefined in terms of small change of the variables about an initial operating point as follows,

$$-\Delta\dot{M}_k + \Delta H_1 - \Delta H_2 = 0 . \qquad (5.9)$$

All quantities in the above equation are nonlinear functions. When n is assumed constant, they can be expressed by

$$\dot{M}_k = f(P_k, P_v),$$

$$H_1 = f(x, P_v, \dot{P}_v),$$

$$H_2 = f(\dot{x}, P_v).$$

In order to obtain a linear differential equation, they must be linearized using the same technique as before.

$$\Delta\dot{M}_k = \frac{\dot{M}_k}{P_k}\Bigg|_{\substack{P_{ko}\\P_{vo}}} \Delta P + \frac{\dot{M}_k}{P_v}\Bigg|_{\substack{P_{ko}\\P_{vo}}} \Delta P_v , \qquad (5.10)$$

$$\Delta H_1 = \frac{H_1}{x}\Bigg|_{\substack{x_o\\P_{vo}\\\dot{P}_{vo}}} \Delta x + \frac{H_1}{P}\Bigg|_{\substack{x_o\\P_{vo}\\\dot{P}_{vo}}} \Delta P_v + \frac{H_1}{\dot{P}}\Bigg|_{\substack{x_o\\P_{vo}\\\dot{P}_{vo}}} \Delta \dot{P}_v , \qquad (5.11)$$

$$\Delta H_2 = \left.\frac{H_2}{\dot{x}}\right|_{\substack{\dot{x}_o \\ P_{vo}}} \Delta \dot{x} + \left.\frac{H_2}{P_v}\right|_{\substack{\dot{x}_o \\ P_{vo}}} \Delta P_v \;, \tag{5.12}$$

where all the partial derivatives are evaluated at an initial-steady-state-operating point

$$P_v = P_{vo} \,,$$

$$\dot{P}_v = \dot{P}_{vo} \,,$$

$$P_k = P_{ko} \,,$$

$$x = x_o \,,$$

$$\dot{x} = \dot{x}_o \,.$$

$\Delta \dot{M}_k$, ΔH_1, ΔH_2 are defined as follows:

$$\Delta \dot{M}_k = \dot{M}_k - \dot{M}_{ko} \,,$$

$$\Delta H_1 = H_1 - H_{1o} \,,$$

$$\Delta H_2 = H_2 - H_{2o} \,.$$

Again the change in all "Δvariables" must be small for equations (5.10), (5.11) and (5.12) to be valid.

By letting

$$b_k = \left.\frac{\dot{M}_k}{P_k}\right|_{\substack{P_{ko} \\ P_{vo}}} \,,$$

$$-b_v = \left.\frac{\dot{M}_k}{P_v}\right|_{\substack{P_{ko} \\ P_{vo}}} \,,$$

expression (5.10) then becomes

$$\Delta \dot{M}_k = b_k \Delta P_k - b_v \Delta P_v \,. \tag{5.13}$$

The analytical expressions for b_k and b_v are given in Appendix E.

From equations (5.8a) and (5.8b), it is seen that

$\left.\frac{\partial H_1}{\partial x}\right|_{\substack{x_o \\ P_{vo} \\ \dot{P}_{vo}}}$, $\left.\frac{\partial H_1}{\partial P_v}\right|_{\substack{x_o \\ P_{vo} \\ \dot{P}_{vo}}}$, and $\left.\frac{\partial H_2}{\partial P_v}\right|_{\substack{\dot{x}_o \\ P_{vo}}}$ are zero because $\dot{x}_o$ and $\dot{P}_{vo}$ are zero at a steady-state-operating point.

By letting

$$b_{\dot{p}} = \left.\frac{\partial H_1}{\partial \dot{P}_v}\right|_{\substack{x_o \\ P_{vo} \\ \dot{P}_{vo}}},$$

$$b_{\dot{x}} = \left.\frac{\partial H_2}{\partial \dot{x}}\right|_{\substack{P_{vo} \\ \dot{x}_o}},$$

expressions (5.11) and (5.12) become

$$\Delta H_1 = b_{\dot{p}} \Delta \dot{P}_v , \tag{5.14}$$

$$\Delta H_2 = b_{\dot{x}} \Delta \dot{x} . \tag{5.15}$$

The analytical expressions for $b_{\dot{p}}$ and $b_{\dot{x}}$ are given in Appendix F.

Substituting expressions (5.13), (5.14) and (5.15) into equation (5.9) results in

$$-b_k \Delta P_k + b_v \Delta P_v + b_{\dot{p}} \Delta \dot{P}_v - b_{\dot{x}} \Delta \dot{x} = 0 .$$

This can be rewritten as

$$b_{\dot{p}} \Delta \dot{P}_v + b_v \Delta P_v = b_k \Delta P_k + b_{\dot{x}} \Delta \dot{x} . \tag{5.16}$$

Equation (5.16) is a linearized first order differential equation relating the fluid pressure in the chamber (shown in Fig.11) to the output pressure of the controller and the displacement of the control element in valve D. It must be remembered that it is valid only when the changes ΔP_v, $\Delta \dot{P}_v$, ΔP_k and $\Delta \dot{x}$ are small. When Laplace-transformed into the s domain, assuming zero initial conditions, equation (5.16) becomes

$$(b_{\dot{p}}s + b_v)\Delta P_v(s) = b_k \Delta P_k(s) + b_{\dot{x}} s \Delta X(s).$$

Solving for $\Delta P_v(s)$ yields

$$\Delta P_v(s) = \frac{b_k \Delta P_k(s) + b_{\dot{x}} s \Delta X(s)}{b_{\dot{p}} s + b_v} . \qquad (5.17)$$

The above equation represents the transfer function of the upper portion of the final control element, the output being $\Delta P_v(s)$. It will be shown later that the coefficient of $\Delta X(s)$ becomes an inner-loop feedback function in the overall closed-loop system.

B. Transfer Function For the Lower Portion of the Final Control Element

The lower portion of the final control element is shown schematically in Fig.12. Let K_v be the spring constant, B_v the viscous damping of the fluid, m_v the mass of the moving parts, F_f the flow forces acting on the valve and F_c the coulomb friction forces due to seals.

The upward force due to the pressure difference on the

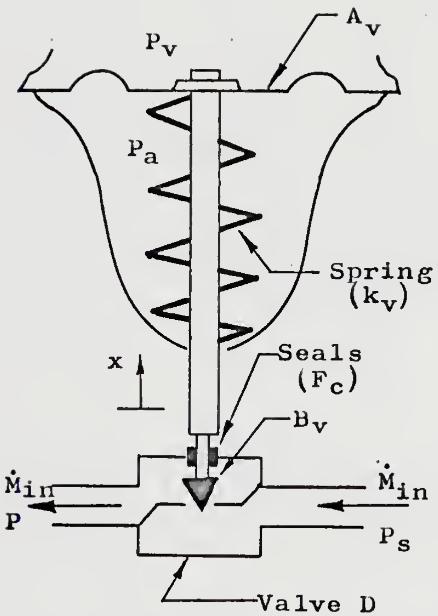

Fig.12. Schematic representation of the lower portion of the final control element.

diaphragm is

$$F_x = (P_a - P_v)A_v .$$

This force is balanced by the reactive forces which consists of the inertial force of the moving parts, the damping force of the fluid, the spring resistance, the coulomb friction forces due to seals and the flow force of the fluid on the valve, i.e.,

$$(P_a - P_v)A_v - m_v\ddot{x} - B_v\dot{x} - k_v x - F_c\frac{\dot{x}}{|\dot{x}|} - F_f = 0 . \quad (5.19)$$

The flow force F_f, in general, is a function of the rate of change of the momentum flux entering and leaving the valve and the pressure drop across the valve. If the working fluid is air or some other gas, the density is very small. Therefore the flow force can be considered insignificant.

If a double ported valve is used, the design is such that the net pressure force on the valve is approximately zero. In general the pressure forces will be small unless very high pressure drops exist across the valve.

In the general case, the coulomb friction forces from the seals may be significant. Figure 13 shows the coulomb friction

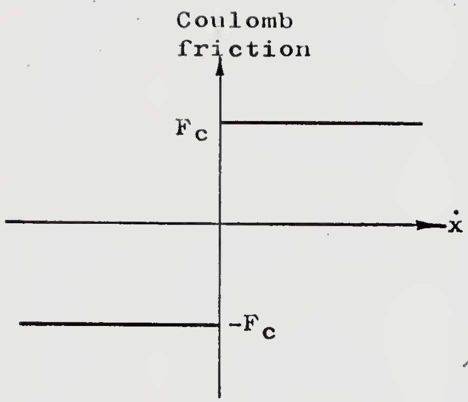

Fig.13. Coulomb friction force as a function of velocity.

force as a function of $\dot{x}$. It is seen that this function is discontinuous at $\dot{x} = 0$. Thus is is not possible to take into account the coulomb friction force in terms of a linearized transfer function.

Since the damping and inertial forces are small, it is clear that the spring resistance and the coulomb friction are the dominant forces in this case. Therefore it must be kept in mind that instability caused by friction may occur in some cases, especially for type 1 and type 2 systems (5). However, it will be shown in Chapter VI that this is a type 0 system, and therefore there is less danger of instability resulting from coulomb friction forces. Therefore it will be assumed that the coulomb friction force can be neglected as a factor which would cause instability in the system.

When the flow forces and coulomb friction force are neglected, equation (5.19) reduces to

$$(P_a - P_v)A_v - m_v\ddot{x} - B_v\dot{x} - k_v x = 0 . \qquad (5.20)$$

This linear equation can be redefined in terms of small changes of the variables about a given initial-steady-state-operating point as follows

$$-A_v \Delta P_v - m_v \Delta \ddot{x} - B_v \Delta \dot{x} - k_v \Delta x = 0 .$$

Taking the Laplace transform, assuming zero initial conditions, and solving for $\Delta X(s)$ yields

$$\Delta X(s) = \frac{-A_v \Delta P_v(s)}{m_v s^2 + B_v s + k_v} . \tag{5.21}$$

Equation (5.21) represents the transfer function for the lower portion of the final control element. It is valid for any magnitude change of the variables since equation (5.20) is itself a linear differential equation.

CHAPTER VI

ANALYSIS OF THE SYSTEM BY BLOCK DIAGRAM APPROACH

A block diagram is very helpful in system analysis and is used to obtain the open-loop and closed-loop transfer functions of a system (7). It consists of a series of blocks linked together by line segments. Each component of the system is represented by a block in the block diagram and the directions of information flow are represented by arrows on the line segments. The transfer function of each component is written in the appropriate block, thus indicating the relation between the signals entering and leaving the component.

The block diagram of the system discussed in this report is shown in Fig.14. It consists of two loops. The inner loop represents the final control element. For the outer feedback loop a first order lag has been included to account for the transmission line between the tank and the comparator in the same way as was done for the transmission line between the controller and the final control element.

In order to simplify the block diagram, it is useful to derive the expression for the closed-loop transfer function. Consider the system shown in Fig.15, the closed-loop transfer function can be obtained as follows.

The output of the system is

$$C = G(R - HC).$$

Grouping yields $\quad C(1 + GH) = GR.$

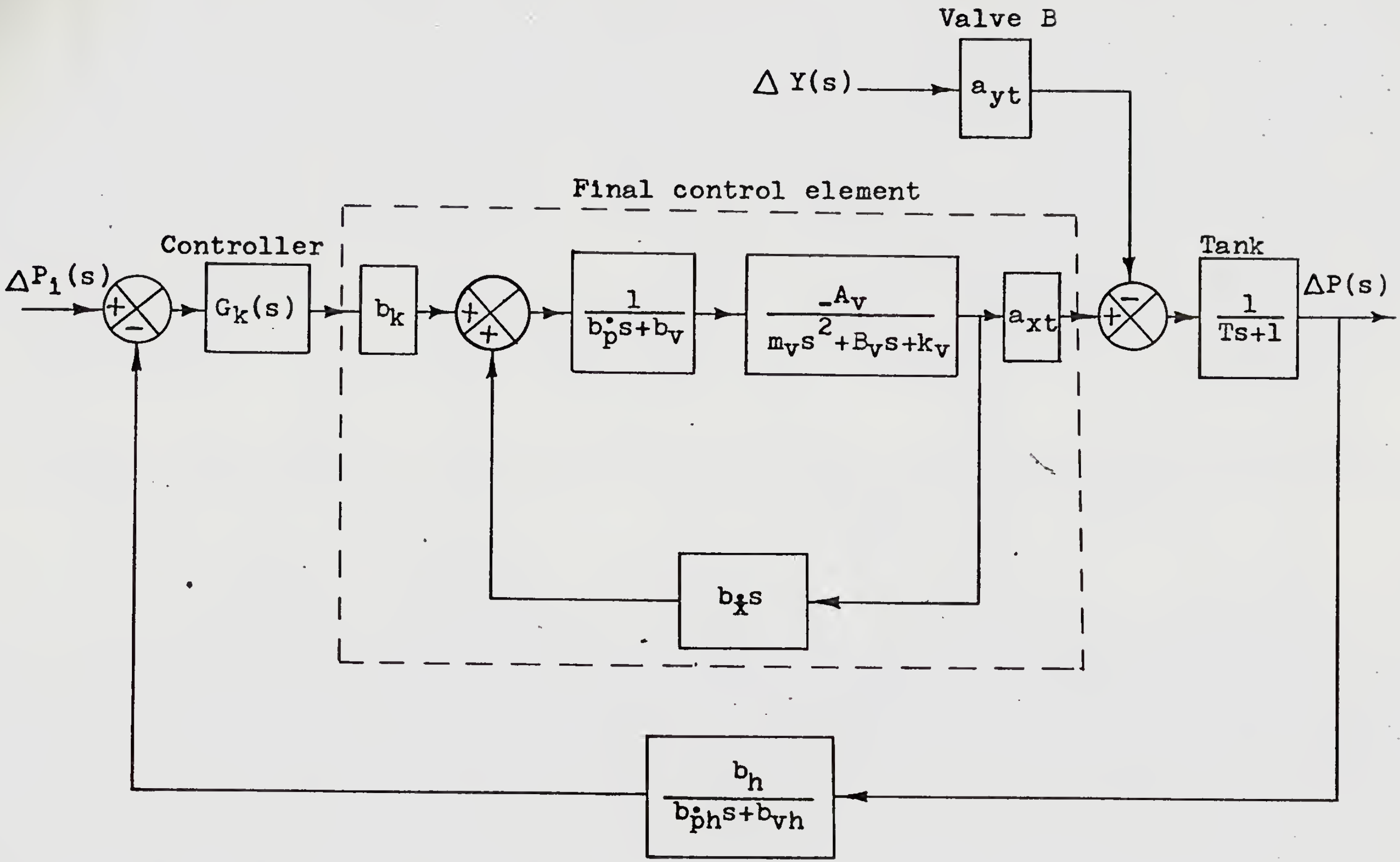

Fig.14. Block diagram of the whole system.

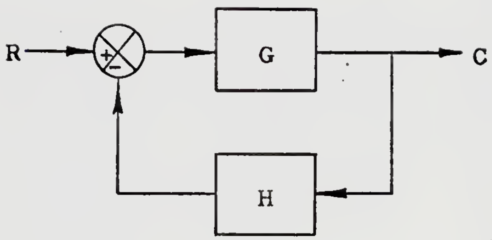

Fig.15. Illustrative control system for obtaining formula for closed-loop transfer function.

Solving for $\frac{C}{R}$ results in

$$\frac{C}{R} = \frac{G}{1 + GH} \,. \tag{6.1}$$

The above formula can be used to obtain the transfer function for any single-looped block diagram. Care must be taken that if the feedback is positive, the plus sign in expression (6.1) must be changed to minus.

Applying this to the inner loop of Fig.15, we can obtain the closed-loop transfer function of the final control element.

$$\frac{\Delta X(s)}{\Delta P_k(s)} = \frac{-A_v b_k a_{xt}}{(b_{\dot{p}}s+b_v)(m_v s^2+B_v s+k_v) + A_v b_{\dot{x}} s} \,. \tag{6.2}$$

The denominator on the right side of equation (6.2) can be multiplied out, like powers of s grouped together, and then refactored into the following form:

$$\frac{\Delta X(s)}{\Delta P_k(s)} = \frac{-A_v b_k a_{xt}}{b_{\dot{p}} m_v (s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)} \tag{6.3}$$

where c_1, ζ and ω_n have values such that equation (6.3) is equivalent to equation (6.2).

Let $c_k = \frac{-A_v b_k a_{xt}}{b_{\dot{p}} m_v}$, the block diagram of the system then reduces to that shown in Fig.16.

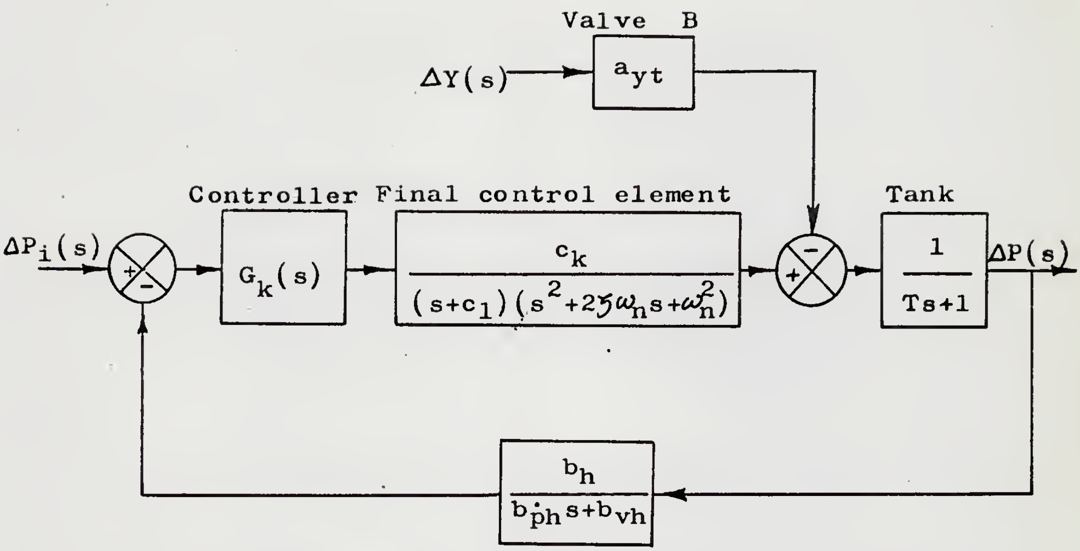

Fig.16. Block diagram of the system after simplifying.

Setting $\Delta Y(s)$ equal to zero, we can obtain the forward-loop transfer function (7),

$$G(s) = \frac{c_k G_k(s)}{(s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)(Ts+1)} , \qquad (6.4)$$

and the open-loop transfer function,

$$G(s)H(s) = \frac{b_h c_k G_k(s)}{(s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)(Ts+1)(b_{ph}s+b_{vh})} . \qquad (6.5)$$

Since there are no open-loop poles at the origin, this is a type 0 system.

Assume a proportional controller is used, such that $G_k(s) = K$. The closed-loop transfer function of the whole system can be obtained by applying formula (6.1) once more.

$$\frac{\Delta P(s)}{\Delta P_i(s)} = \frac{c_k K(b_{\dot{p}h}s+b_{vh})}{(s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)(Ts+1)(b_{\dot{p}h}s+b_{vh}) + c_k K b_h} \cdot (6.6)$$

In most cases a pressure control system will operate with $\Delta P_i(t)$ = constant and the disturbance ratio is of interest. The disturbance ratio is defined as

$$\frac{\Delta P(s)}{\Delta Y(s)} = \frac{-a_{yt}(s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)(b_{\dot{p}h}s+b_{vh})}{(s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)(Ts+1)(b_{\dot{p}h}s+b_{vh}) + c_k K b_h} \cdot (6.7)$$

In general, it is desired to minimize the steady state effect of any disturbances. That is, a disturbance $\Delta y(t)$ = step change should have minimum effect on $\Delta P(t)_{ss}$.

For a step change $\Delta y(t) = \Delta y_o$ $\left(\Delta Y(s) = \frac{\Delta y_o}{s}\right)$, the final value theorem can be applied to find $\Delta P(t)_{ss}$ as follows,

$$\Delta P(t)_{ss} = \lim_{s\to 0} s\,\Delta P(s) \ ,$$

$$\Delta P(t)_{ss} = \frac{-a_{yt}c_1\omega_n^2 b_{vh}\Delta y_o}{c_1\omega_n^2 b_{vh} + c_k K b_{vh}} \ . \qquad (6.8)$$

Therefore, in order to minimize the effect of disturbance Δy_o on $\Delta P(t)_{ss}$ when using a proportional controller, the gain of the controller K must be very high.

CHAPTER VII

USE OF ROOT LOCUS PLOT FOR TIME DOMAIN ANALYSIS

The open-loop transfer function for the system considered is given by equation (6.5). It is known that the root-locus diagram starts at the open-loop poles for $K = 0$ and for all finite values of K gives the location of the closed-loop poles as defined by equation (6.6).

Because T, the time constant of the tank, is directly proportional to V_c, the volume of the tank, it will in general be quite large. However, as indicated in Chapter III, the magnitude of T is also affected by whether the system is operating in region 1, 2, or 3 as defined in Fig.7. In regions 1 and 3, T will have its minimum values, while in region 2, T will have its largest values. But in general, regardless of the operating region, $1/T$ will be the smallest open-loop pole, while c_1, $b_{vh}/b_{\dot{p}h}$, and the poles for the quadratic factor $(s^2+2\zeta\omega_n s+\omega_n^2)$ will be larger in magnitude. Therefore the open-loop poles for $K = 0$ will be typically located as shown in Fig.17. The roots of $(s^2+2\zeta\omega_n s+\omega_n^2)$ have been assumed real ($\zeta > 1$, overdamped). They could be assumed complex ($\zeta < 1$, underdamped) without changing the characteristics of the root-locus plot.

The angles of asymptotes for the root-locus plot are (6)

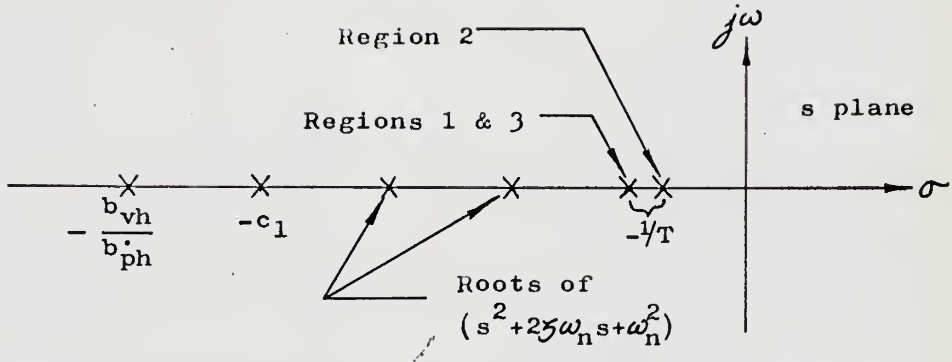

Fig.17. Typical location of the open-loop poles with proportional controller.

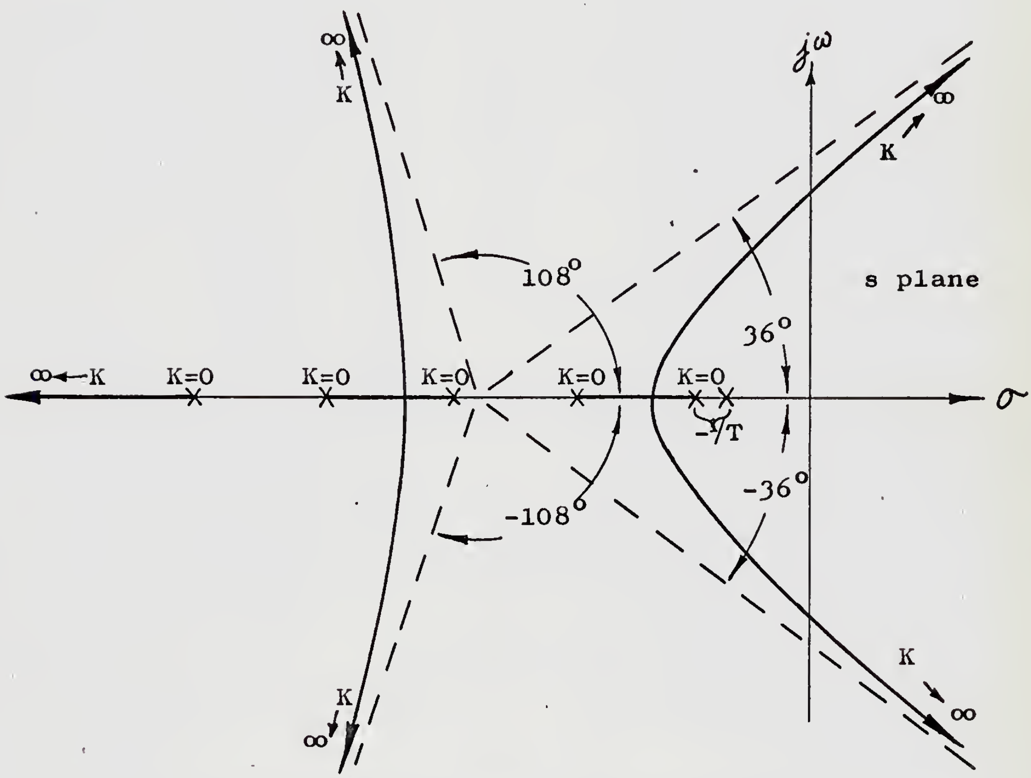

Fig.18. Typical root-locus plot with a proportional controller.

$$\theta = \frac{(1+2m)180^{\circ}}{5} = \pm 36^{\circ},\ \pm 108^{\circ},\ \pm 180^{\circ}.$$

Although the intersection of the asymptotes with the real axis cannot be determined quantitatively, it can be qualitatively estimated. The resultant root-locus plot is shown in Fig.18.

It was shown in Chapter VI that to minimize the effect of disturbances on $\Delta P(t)$, the gain K must be high if a proportional controller is used. However, from the root-locus diagram it can be seen that a high gain will cause two branches of the root locus to approach the $j\omega$ axis and the system will be oscillatory. For operation in region 2, the system will be least stable for a given K because the open-loop pole location $-1/T$ is closest to the imaginary axis.

Therefore it can be concluded that use of a proportional controller with K very large to minimize the effect of disturbances on the controlled variable $\Delta P(t)$ can result in an oscillatory system, or an unstable system if K is large enough.

The system can, however, be stabilized by means of compensation, thus improving its performance characteristics. Consider now the case when a derivative-plus-proportional controller with transfer function

$$G_k(s) = K(T_k s + 1)$$

is used. T_k is the time constant of this controller. Substituting this into equation (6.6) gives the new open-loop transfer function

$$G(s)H(s) = \frac{b_h c_k K(T_k s+1)}{(s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)(Ts+1)(b_{\dot{p}h}s+b_{vh})} . \quad (7.1)$$

The disturbance ratio in this case will be

$$\frac{\Delta P(s)}{\Delta Y(s)} = \frac{-a_{yt}(s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)(b_{\dot{p}h}s+b_{vh})}{(s+c_1)(s^2+2\zeta\omega_n s+\omega_n^2)(Ts+1)(b_{\dot{p}h}s+b_{vh}) + b_h c_k K(T_k s+1)} .$$

The final value theorem is applied here again for a step change $\Delta y(t) = \Delta y_o$, giving

$$\Delta P(t)_{ss} = \lim_{s \to o} s\,\Delta P(s),$$

$$\Delta P(t)_{ss} = \frac{-a_{yt}c_1\omega_n^2 b_{vh}\Delta y_o}{c_1\omega_n^2 b_{vh} + c_k b_{vh}K} . \quad (7.2)$$

This is exactly the same as equation (6.8), therefore high gain of the controller is again desirable to minimize the effect of disturbances.

The angles of asymptotes for the new root-locus plot are

$$\theta = \frac{(1 + 2m)180^o}{5-1} = \pm 45^o, \ \pm 135^o,$$

and the new root-locus plot is shown in Fig.19.

If the zero of the derivative-plus-proportional controller is located just to the left of the dominant pole $-1/T$, the zero will "cancel" this pole and the root-locus diagram is shifted farther to the left. The result is twofold: first, the overall stability of the system is increased for a given value of controller gain; second, the effect on the overall

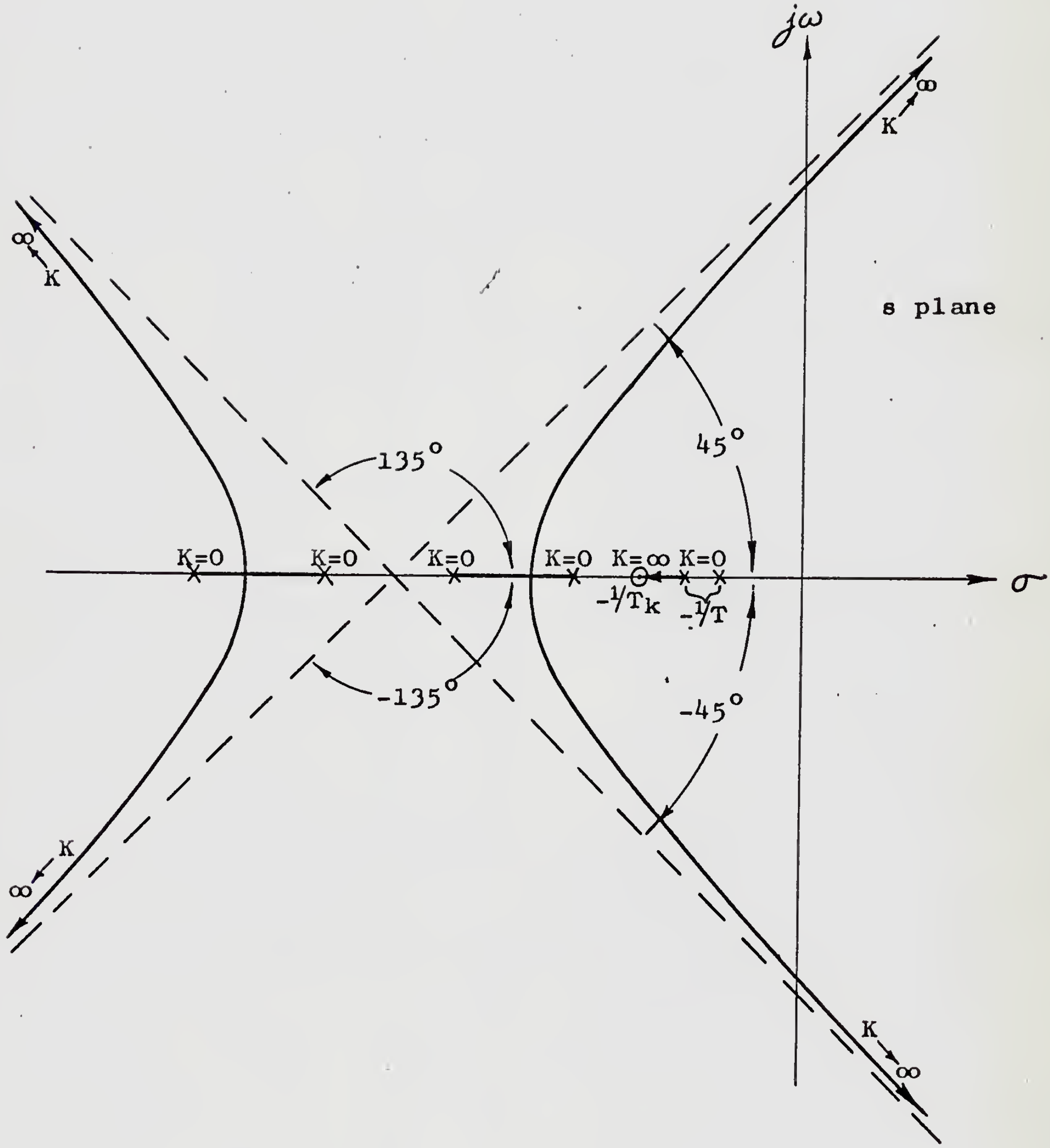

Fig.19. Typical root-locus plot with a derivative-plus-proportional controller.

system stability due to changes in location of $-1/T$ (for different operating regions as defined in Fig.7) is minimized.

The system performance can be further improved using a second or higher order controller. For example, if a controller which introduces 3 negative zeros, the values of which are such that intersection of asymptotes with the real axis is to the left of the imaginary axis, is chosen, the root locus diagram will lie entirely on the left half of the s-plane, because the angles of asymptotes are then $\pm 90^{\circ}$. This implies that no matter how high the controller gain is chosen, the system is still stable. On the other hand, increase in number of zeros of the controller will complicate its design and structure, thus making its manufacturing cost very high. Therefore, from the economic point of view, selection of a second or higher order controller cannot usually be justified in a pressure-control-system.

desired value, because of disturbances, will be kept small by the feedback action of the closed-loop system. Therefore the linearized analysis is especially applicable in this case.

The block diagram of the system is discussed in Chapter VI. It consists of two loops. The inner loop represents the final control element and the outer one the system feedback. From the block diagram, open-loop and closed-loop transfer functions are obtained. It turns out that the system analyzed is fifth order.

The disturbance ratio (the ratio of controlled variable to disturbance variable) is also obtained from the block diagram. The steady-state relation of the output pressure to a step disturbance reveals that in order to minimize the disturbance effect, the gain of the controller must be high. It is shown that the time constant associated with the tank is quite large. Therefore, using root-locus methods, it is shown that the time constant of the tank dominates the time domain response because its corresponding open-loop pole location is closest to the s-plane imaginary axis.

If a proportional controller is used, the system can become highly oscillatory at the high values of controller gain needed to minimize the effect of disturbances on the controlled variable. It is shown, however, that the system can be made more stable by use of a derivative-plus-proportional controller. Higher gain values can then be used and improved system performance obtained.

CHAPTER VIII

SUMMARY AND CONCLUSION

In this report the problem of controlling the pressure of a compressible fluid in a tank (or vessel) is considered. Examples of such pressure control problem can be found in the distillation columns in chemical process plants. The control of the pressure of a compressible fluid in a tank is of interest because the flow entering or leaving the tank through valves may be either critical or subcritical depending upon the ratio of downstream to upstream pressure at the valves.

It must be remembered that the analysis made in this report is theoretical and therefore some deviation in the response of such a system will be expected in an actual case. The coulomb friction force due to seals in the pneumatic actuator is disregarded because the system analyzed is type 0. In type 1 and higher order systems, neglecting the coulomb friction forces may create significant errors between theoretical and actual results.

In deriving the transfer functions for the system, nonlinear equations are linearized using the linear terms of Taylor series expansion. This places the restriction that the change of variable quantities from some initial-steady-state-operating point must be small. Because the system analyzed operates as a regulating system (input fixed), the change in the controlled variable (pressure in the tank) from its

If a second or higher order controller is used, the system performance can further be improved. But due to its complication in structure, and therefore high cost in manufacturing, use of such a controller is, in general, economically prohibitive.

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APPENDICES

APPENDIX A

CRITICAL MASS-FLOW RATE

Substitution of $r = (\frac{2}{k+1})^{\frac{k}{k-1}}$ into equation (2.10a) yields

$$\dot{M}_c = K_d A_o \sqrt{(\frac{2k}{k-1}) \rho_1 P_1 (1 - \frac{2}{k+1}) (\frac{2}{k+1})^{\frac{2}{k-1}}}$$

$$= K_d A_o \sqrt{(\frac{2k}{k-1}) \rho_1 P_1 (\frac{k-1}{k+1}) (\frac{2}{k+1})^{\frac{2}{k-1}}}$$

$$= K_d A_o \sqrt{k \rho_1 P_1 (\frac{2}{k+1})^{\frac{k+1}{k-1}}} \, .$$

where the subscript c denotes critical flow.

APPENDIX B

ANALYTICAL EXPRESSIONS FOR a_{pl} AND a_x FOR CRITICAL AND SUBCRITICAL FLOW THROUGH VALVE D

1. Critical flow. The flow equation in this case is

$$\dot{M}_{inc} = K_d A_d(x) \sqrt{k \rho_s P_s \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \quad . \tag{B.1}$$

Hence, it is clear that

$$a_{pl} = a_{plc} = 0 \, , \tag{B.2}$$

$$a_x = a_{xc} = K_d \sqrt{k \rho_s P_s \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \left. \frac{dA_d(x)}{dx} \right|_{x_o} \quad . \tag{B.3}$$

2. Subcritical flow. The flow equation for subcritical flow through valve D is

$$\dot{M}_{ins} = K_d A_d(x) \sqrt{\frac{2k}{k-1} \rho_s P_s \left[1 - \left(\frac{P}{P_s}\right)^{\frac{k-1}{k}}\right] \left(\frac{P}{P_s}\right)^{2/k}} \quad . \tag{B.4}$$

Differentiating with respect to P gives

$$\frac{\partial \dot{M}_{ins}}{\partial P} = K_d A_d(x) \sqrt{\frac{2k}{k-1} \rho_s P_s} \; \frac{\frac{1}{P_s}\left[\frac{2}{k}\left(\frac{P}{P_s}\right)^{\frac{2-k}{k}} - \frac{k+1}{k}\left(\frac{P}{P_s}\right)^{1/k}\right]}{2\sqrt{\left(\frac{P}{P_s}\right)^{2/k} - \left(\frac{P}{P_s}\right)^{\frac{k+1}{k}}}} \quad .$$

This can be rewritten as

$$\frac{\partial \dot{M}_{ins}}{\partial P} = \frac{K_d A_d(x) \sqrt{\frac{2k}{k-1} \rho_s P_s} \; (\frac{1}{kP_s})(\frac{P}{P_s})^{1/k} \left[2(\frac{P}{P_s})^{\frac{1-k}{k}} - (k+1)\right]}{2\sqrt{(\frac{P}{P_s})^{2/k}\left[1 - (\frac{P}{P_s})^{\frac{k-1}{k}}\right]}} .$$

After cancelling and simplifying, the above equation becomes

$$\frac{\partial \dot{M}_{ins}}{\partial P} = \frac{K_d A_d(x) \left[2(\frac{P}{P_s})^{\frac{1-k}{k}} - (k+1)\right]}{\sqrt{2k(k-1)(\frac{P_s}{\rho_s})\left[1 - (\frac{P}{P_s})^{\frac{k-1}{k}}\right]}} .$$

When equation (B.4) is differentiated with respect to x, the result is

$$\frac{\partial \dot{M}_{ins}}{\partial x} = K_d \sqrt{\frac{2k}{k-1} \rho_s P_s \left[1 - (\frac{P}{P_s})^{\frac{k-1}{k}}\right] (\frac{P}{P_s})^{2/k}} \; \frac{d\, A_d(x)}{dx}\bigg|_{x_o} .$$

Hence it follows

$$a_{p1} = a_{p1s} = \frac{\partial \dot{M}_{ins}}{\partial P}\bigg|_{\substack{P_o \\ x_o}}$$

$$= \frac{K_d A_d(x_o) \left[2(\frac{P_o}{P_s})^{\frac{1-k}{k}} - (k+1)\right]}{\sqrt{2k(k-1)\frac{P_s}{\rho_s}\left[1 - (\frac{P_o}{P_s})^{(k-1)/k}\right]}} , \qquad (B.5)$$

and

$$a_x = a_{xs} = \left.\frac{\partial \dot{M}_{ins}}{\partial x}\right|_{\substack{P_o \\ x_o}}$$

$$= K_d \sqrt{\frac{2k}{k-1} \rho_s P_s \left[1 - \left(\frac{P_o}{P_s}\right)^{\frac{k-1}{k}}\right] \left(\frac{P_o}{P_s}\right)^{\frac{2}{k}}} \left.\frac{d\, A_d(x)}{dx}\right|_{x_o} . \tag{B.6}$$

APPENDIX C

ANALYTICAL EXPRESSIONS FOR a_{p2} AND a_y FOR CRITICAL AND SUBCRITICAL FLOW THROUGH VALVE B

1. Critical flow. The flow equation in this case is

$$\dot{M}_{bc} = K_d A_b(y)\sqrt{k\rho P\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \; .$$

Substituting $\rho = \rho_s\left(\frac{P}{P_s}\right)^{1/k}$ into the above equation gives

$$\dot{M}_{bc} = K_d A_b(y)\sqrt{\frac{k\rho_s}{P_s^{1/k}}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}} P^{\frac{k+1}{k}}} \; . \qquad \text{(C.1)}$$

The analytical expressions of a_{p2} and a_y are obtained by differentiating the above equation with respect to P and y and substituting in the respective values at the initial-steady-state-operating point.

$$a_{p2} = a_{p2c} = \left.\frac{\partial \dot{M}_b}{\partial P}\right|_{\substack{P_o \\ y_o}}$$

$$= K_d A_b(y_o)\left(\frac{k+1}{2k}\right)P_o^{\frac{1-k}{2k}}\sqrt{\frac{k\rho_s}{P_s^{1/k}}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \; , \qquad \text{(C.2)}$$

$$a_y = a_{yc} = \left.\frac{\partial \dot{M}_b}{\partial y}\right|_{\substack{P_o \\ y_o}}$$

$$= K_d P_o^{\frac{k+1}{2k}} \sqrt{\frac{k\rho_s}{P_s^{1/k}}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \left.\frac{d\ A_b(y)}{dy}\right|_{y_o} . \qquad (C.3)$$

2. Subcritical flow. With the substitution of

$$\rho = \rho_s\left(\frac{P}{P_s}\right)^{1/k},$$

the flow equation in this case is

$$\dot{M}_{bs} = K_d A_b(y) \sqrt{\frac{2k}{k-1}\, \rho_s P_s \left(\frac{P}{P_s}\right)^{1/k} \left[\left(\frac{P_a}{P}\right)^{2/k} - \left(\frac{P_a}{P}\right)^{\frac{k+1}{k}}\right]} .$$

After simplifying, the above equation becomes

$$\dot{M}_{bs} = K_d A_b(y) \sqrt{\frac{2k}{k-1}\, \rho_s P_a \left(\frac{P_a}{P_s}\right)^{1/k} \left[\left(\frac{P}{P_a}\right)^{\frac{k-1}{k}} - 1\right]} . \qquad (C.4)$$

Differentiating with respect to P gives

$$\frac{\partial \dot{M}_{bs}}{\partial P} = K_d A_b(y) \sqrt{\frac{2k}{k-1}\, \rho_s P_a \left(\frac{P_a}{P_s}\right)^{1/k}} \cdot \frac{\frac{k-1}{k}\left(\frac{P}{P_a}\right)^{-1/k}}{2\sqrt{\left(\frac{P}{P_a}\right)^{\frac{k-1}{k}} - 1}} .$$

After simplifying and substituting in $P = P_o$, $y = y_o$, a_{p2} is obtained.

$$a_{p2} = a_{p2s} = \left.\frac{\partial \dot{M}_{bs}}{\partial P}\right|_{\substack{P_o \\ y_o}}$$

$$= K_d A_b(y_o) \sqrt{\frac{\left(\frac{k-1}{2k}\right)\left(\frac{P_a}{P_s}\right)^{1/k} \frac{\rho_s}{P_a}}{\left(\frac{P_o}{P_s}\right)^{2/k}\left[\left(\frac{P_o}{P_a}\right)^{\frac{k-1}{k}} - 1\right]}} \; . \tag{C.5}$$

In a similar manner, the analytical expression for a_y is easily shown to be

$$a_y = a_{ys} = \left.\frac{\partial \dot{M}_{bs}}{\partial y}\right|_{\substack{P_o \\ y_o}}$$

$$= K_d \sqrt{\left(\frac{2k}{k-1}\right) \rho_s P_a \left(\frac{P_a}{P_s}\right)^{1/k}\left[\left(\frac{P_o}{P_a}\right)^{\frac{k-1}{k}} - 1\right]} \; \left.\frac{d\, A_b(y)}{dy}\right|_{y_o} . \tag{C.6}$$

APPENDIX D

ANALYTICAL EXPRESSIONS FOR a_{xt}, a_{yt} AND T AT DIFFERENT OPERATING REGIONS DEFINED IN FIG.7

Region 1. Flow through valve B is subcritical while that through valve D is critical.

$$a_{xt} = \frac{a_{xc}}{a_{p2s}} ,$$

$$a_{yt} = \frac{a_{ys}}{a_{p2s}} ,$$

$$T = \frac{a_{\rho}V_c}{a_{p2s}} ,$$

where a_{xc}, a_{p2s} and a_{ys} are defined by equations (B.3), (C.5) and (C.6) respectively.

Region 2. Flow through both valves is critical.

$$a_{xt} = \frac{a_{xc}}{a_{p2c}} ,$$

$$a_{yt} = \frac{a_{yc}}{a_{p2c}} ,$$

$$T = \frac{a_{\rho}V_c}{a_{p2c}} ,$$

where a_{p2c} and a_{yc} are defined by equation (C.2) and (C.3) respectively.

Region 3. Flow through valve B is critical while that through valve D is subcritical.

$$a_{xt} = \frac{a_{xs}}{a_{p2c} - a_{p1s}} ,$$

$$a_{yt} = \frac{a_{yc}}{a_{p2c} - a_{p1s}} ,$$

$$T = \frac{a_{\rho} V_c}{a_{p2c} - a_{p1s}} ,$$

where a_{xs} is defined by equation (B.6).

APPENDIX E

EVALUATION OF b_k AND b_v

Since the difference between P_v and P_k (pressure drop across the line) is small, only subcritical flow needs to be considered. The mass-rate of flow into the chamber, after replacing ρ_k by $\rho_{max}(\frac{P_k}{P_{max}})^{1/n}$, is

$$\dot{M}_k = K_d A_o \sqrt{(\frac{2n}{n-1})\rho_{max} P_k (\frac{P_k}{P_{max}})^{1/n}\left[(\frac{P_v}{P_k})^{\frac{2}{n}} - (\frac{P_v}{P_k})^{\frac{n+1}{n}}\right]} . \quad \text{(E.1)}$$

Differentiating with respect to P_v gives

$$\frac{\partial \dot{M}_k}{\partial P_v} = \frac{K_d A_o \sqrt{(\frac{2n}{n-1})\rho_{max} P_k (\frac{P_k}{P_{max}})^{1/n}}}{2\sqrt{(\frac{P_v}{P_k})^{\frac{2}{n}} - (\frac{P_v}{P_k})^{\frac{n+1}{n}}}}\left[\frac{2}{n}(\frac{P_v}{P_k})^{\frac{2-n}{n}} - \frac{n+1}{n}(\frac{P_v}{P_k})^{1/n}\right]\frac{1}{P_k}$$

$$= \frac{K_d A_o \sqrt{(\frac{2n}{n-1})\rho_{max} P_k (\frac{P_k}{P_{max}})^{1/n}}}{2(\frac{P_v}{P_k})^{1/n}\sqrt{1 - (\frac{P_v}{P_k})^{\frac{n-1}{n}}}} \; \frac{1}{nP_k}(\frac{P_v}{P_k})^{1/n}\left[2(\frac{P_v}{P_k})^{\frac{1-n}{n}} - (n+1)\right].$$

After cancelling and simplifying, the above equation becomes

$$\frac{\partial \dot{M}_k}{\partial P_v} = K_d A_o \left[2(\frac{P_v}{P_k})^{\frac{1-n}{n}} - (n+1)\right]\sqrt{\frac{\frac{\rho_{max}}{P_k}(\frac{P_k}{P_{max}})^{1/n}}{2n(n-1)\left[1 - (\frac{P_v}{P_k})^{\frac{n-1}{n}}\right]}} \cdot \quad \text{(E.2)}$$

Equation (E.1) can be rewritten as

$$\dot{M}_k = K_d A_o \sqrt{\left(\frac{2n}{n-1}\right) \rho_{max} \frac{P_k^{\frac{1+n}{n}}}{P_{max}^{1/n}} \left(\frac{P_v}{P_k}\right)^{\frac{n+1}{n}} \left[\left(\frac{P_v}{P_k}\right)^{\frac{1-n}{n}} - 1\right]} \; .$$

Simplifying gives

$$\dot{M}_k = K_d A_o \sqrt{\left(\frac{2n}{n-1}\right) \rho_{max} P_v \left(\frac{P_v}{P_{max}}\right)^{1/n} \left[\left(\frac{P_k}{P_v}\right)^{\frac{n-1}{n}} - 1\right]} \; .$$

Differentiating with respect to P_k yields

$$\frac{\partial \dot{M}_k}{\partial P_k} = \frac{K_d A_o \sqrt{\left(\frac{2n}{n-1}\right) \rho_{max} P_v \left(\frac{P_v}{P_{max}}\right)^{1/n}}}{2\sqrt{\left(\frac{P_k}{P_v}\right)^{\frac{n-1}{n}} - 1}} \; \frac{n-1}{nP_k}\left(\frac{P_k}{P_v}\right)^{-1/n}$$

$$= \frac{K_d A_o}{P_k}\left(\frac{P_v}{P_k}\right)^{\frac{1}{n}} \sqrt{\frac{\left(\frac{n-1}{2n}\right) \rho_{max} P_v \left(\frac{P_v}{P_{max}}\right)^{1/n}}{\left(\frac{P_k}{P_v}\right)^{\frac{n-1}{n}} - 1}} \; . \qquad \text{(E.3)}$$

From equations (E.2) and (E.3), the analytical expressions for b_v and b_k are seen to be

$$-b_v = K_d A_o \left[2\left(\frac{P_{vo}}{P_{ko}}\right)^{\frac{1-n}{n}} - (n+1)\right] \sqrt{\frac{\frac{\rho_{max}}{P_{ko}} \left(\frac{P_{ko}}{P_{max}}\right)^{1/n}}{2n(n-1)\left[1 - \left(\frac{P_{vo}}{P_{ko}}\right)^{\frac{n-1}{n}}\right]}} \, ,$$

$$b_k = \frac{K_d A_o}{P_k} \left(\frac{P_{vo}}{P_{ko}}\right)^{\frac{1}{n}} \sqrt{\frac{\left(\frac{n-1}{n}\right) \rho_{max} P_{vo} \left(\frac{P_{vo}}{P_{max}}\right)^{1/n}}{\left[\left(\frac{P_{ko}}{P_{vo}}\right)^{\frac{n-1}{n}} - 1\right]}} \; .$$

APPENDIX F

EVALUATION OF $b_{\dot{p}}$ AND $b_{\dot{x}}$

Differentiating equation (5.8a) with respect to $\dot{P}_v$ yields

$$\frac{\partial H_1}{\partial \dot{P}_v} = \frac{1}{n}(V_{vo} - A_v x)\,\frac{\rho_{max}}{P_{max}}\left(\frac{P_v}{P_{max}}\right)^{\frac{1-n}{n}} .$$

Therefore

$$b_{\dot{p}} = \left.\frac{\partial H_1}{\partial \dot{P}_v}\right|_{\substack{x_o \\ P_{vo} \\ \dot{P}_{vo}}} = \frac{1}{n}(V_{vo} - A_v x_o)\,\frac{\rho_{max}}{P_{max}}\left(\frac{P_{vo}}{P_{max}}\right)^{\frac{1-n}{n}} .$$

Treating similarly for equation (5.8b) gives

$$b_{\dot{x}} = \left.\frac{\partial H_2}{\partial \dot{x}}\right|_{\substack{P_{vo} \\ \dot{x}_o}} = A_v \rho_{max}\left(\frac{P_{vo}}{P_{max}}\right)^{1/n} .$$

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THE ANALYSIS AND CONTROL OF A PRESSURE PROCESS

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This report presents an analysis of a pressure-control-system, which consists of a tank, a controller and a final control element. The object of this pressure-control-system is to keep the pressure of a compressible fluid in the tank approximately constant as disturbance flow varies.

The characteristics of compressible fluid flow through valves is discussed and flow equations for both critical and subcritical flow are derived. Nonlinear equations encountered are linearized using Taylor series expansion. Laplace transforms are used to obtain transfer functions for various components of the system.

From a steady state flow analysis, curves relating pressure in the tank and mass-flow rates to and from the tank are plotted. These curves are useful in understanding the unsteady operation of the system in critical and subcritical flow operating regions.

Block diagrams are constructed after transfer function for each component is derived. By means of these block diagrams, open-loop and closed-loop transfer functions are obtained. Also obtained is the disturbance ratio (the ratio of controlled variable to disturbance variable). This ratio indicates that in order to minimize the disturbance effect in the steady-state output pressure, high gain of the controller must be provided.

Use of root-locus plots shows that the gain of the controller must be restricted to some maximum value otherwise instability of the system can result. The use of compensation is discussed, and it is shown that if, instead of using a proportional controller, a derivative-plus-proportional controller is used, the performance characteristics of the system will be improved. From the economic point of view, use of second or higher order controller to make further improvement of the system performance is not justified.