THE MAXIMUM RESPONSE OF A SINGLE DEGREE OF FREEDOM SYSTEM TO DIFFERENT EXCITATIONS

by

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NOMENCLATURE

A, A_1, A_2	:	Constants.
C	:	Damping coefficient, lb-sec/in.
F_o	:	Excitation force, lb.
g	:	Gravity acceleration, ft/sec^2.
h	:	Height of free fall, in.
K	:	Spring constant, lb/in.
k	:	Magnification factor.
M	:	Mass, slug.
P	:	Rectangular pulse, lb.
r	:	Frequency ratio.
t	:	Time, second.
W	:	Excitation frequency, rad/sec.
W_d	:	Frequency of damped oscillation, rad/sec.
W_n	:	Natural frequency, rad/sec.
X	:	Displacement of the mass, in.
X_m	:	Maximum displacement, in.

INTRODUCTION

The purpose of this study was to investigate the relationship of various factors to the maximum response of a single degree of freedom system excited by six different disturbances. One relationship which was defined as shock spectrum[2] involved the maximum response and the natural frequency or the damping factor of the system.

These six disturbances were (1) harmonic force, (2) sinusodial displacement of the system's base, (3) unit step force, (4) delta function, (5) rectangular force and (6) free fall of the system.

The first part of this study deals with analytical work and the second part concerns experiments and comparisons.

* Superscript numbers refer to items in the References

PART I

ANALYSIS

1. HARMONIC FORCE EXCITATION[6]

The system to be considered is shown in Fig. 1. The mass M is given an excitation force $F_o \mathrm{Sin} Wt$ as shown in Fig. 2 and the general equation of motion is

$$M\ddot{X} + C\dot{X} + KX = F_o \mathrm{Sin} Wt \,, \tag{1}$$

assuming the system to be underdamped ($\zeta < 1$). The general solution X(t) is the sum of

$$X_c = Ae^{-\zeta W_n t} \mathrm{Sin}(W_d t + \psi) \,, \tag{2}$$

and

$$X_p = \frac{F}{K} k \,\mathrm{Sin}(Wt - \phi) \,, \tag{3}$$

where

$$\left.\begin{aligned} W_n &= \sqrt{\frac{K}{M}} \,, \\ \zeta &= \frac{C}{2\sqrt{KM}} \,, \\ W_d &= W_n\sqrt{1 - \zeta^2} \,, \end{aligned}\right. \tag{4}$$

$$r = \frac{W}{W_n} \,,$$

$$k = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \,, \tag{5}$$

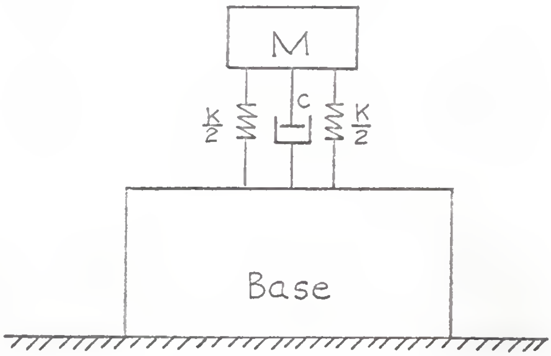


Fig. 1

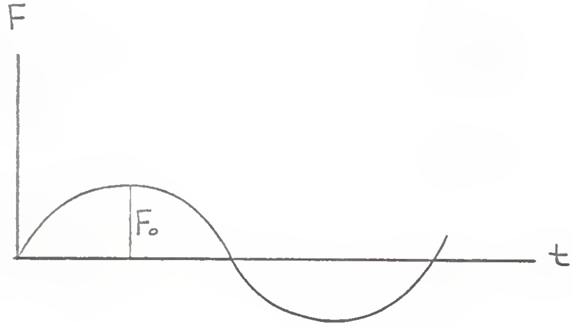


Fig. 2

$$\phi = \mathrm{Tan}^{-1} \frac{2\zeta r}{(1 - r^2)} . \tag{6}$$

Therefore, for $\zeta < 1$, the solution of Equation (1) is

$$X = Ae^{-\zeta W_n t} \mathrm{Sin}(W_d t + \psi) + \frac{F_o}{K} k \,\mathrm{Sin}(Wt - \phi) . \tag{7}$$

Substituting the intial condition, $X(0) = 0$, into Equation (7) gives

$$X(0) = A \,\mathrm{Sin}\,\psi - \frac{F_o}{K} k \,\mathrm{Sin}\phi = 0 ,$$

where

$$\mathrm{Sin}\phi = \frac{2\zeta r}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = k(2\zeta r) ,$$

$$A \,\mathrm{Sin}\,\psi = \frac{F_o}{K} 2\zeta r k^2 . \tag{8}$$

Taking the derivative of Equation (7) and substituting $\dot{X}(0) = 0$ into it gives

$$\dot{X}(0) = -\zeta W_n A \,\mathrm{Sin} \quad + AW_d \,\mathrm{Cos}\,\psi + \frac{F_o}{K} kW \,\mathrm{Cos}(-\phi) = 0 ,$$

$$\zeta W_n A \,\mathrm{Sin}\psi - \frac{F_o}{K} k^2 W (1 - r^2) = AW_d \,\mathrm{Cos}\,\psi . \tag{9}$$

Substituting Equation (8) into Equation (9) gives

$$\zeta W_n \frac{F_o}{K} 2\zeta r k^2 - \frac{F_o}{K} k^2 W (1 - r^2) = \frac{F_o}{K} 2\zeta r k^2 W_d \,\mathrm{Cot}(\psi) ,$$

and therefore,

$$\psi = \mathrm{Tan}^{-1} \frac{2\zeta r W_d}{2\zeta^2 r W_n - W(1 - r^2)} , \tag{10}$$

$$\mathrm{Sin}\,\psi = \frac{2\zeta r W_d}{\sqrt{(2\zeta^2 r W_n - W(1 - r^2))^2 + (2\zeta r W_d)^2}} . \tag{11}$$

Substituting Equation (11) into Equation (8) gives

$$A = \frac{F_o}{K}(2\zeta r k^2)\frac{\sqrt{(2\zeta^2 rW_n - W(1 - r^2))^2 + (2\zeta rW_d)^2}}{2r\zeta W_d}$$

$$= \frac{F_o}{K}k\left[\frac{k\sqrt{(2\zeta^2 rW_n - W(1 - r^2))^2 + (2\zeta rW_d)^2}}{W_d}\right]. \qquad (12)$$

Substituting Equation (12) into Equation (7) gives the response equation

$$X = \frac{F_o}{K}k\left[\frac{k\sqrt{(2\zeta^2 rW_n - W(1 - r^2))^2 + (2\zeta rW_d)^2}}{W_d}e^{-\zeta W_n t}\right.$$

$$\left.\mathrm{Sin}(W_d t + \psi) + \mathrm{Sin}(Wt - \phi)\right]. \qquad (13)$$

Substituting ϕ and ψ into Equation (13) gives

$$X = \frac{F_o}{K}k^2\left[\frac{e^{-\zeta W_n t}}{W_d}((2\zeta^2 rW_n - W(1 - r^2))\,\mathrm{Sin}W_d t\right.$$

$$\left. + 2\zeta rW_d t\,\mathrm{Cos}W_d t) + (1 - r^2)\,\mathrm{Sin}Wt - 2\zeta r\,\mathrm{Cos}Wt\right],$$

or

$$\left(\frac{K}{k^2 F_o}\right)X = \frac{e^{-\zeta W_n t}}{W_d}\left[(2\zeta^2 rW_n - W(1 - r^2))\,\mathrm{Sin}W_d t\right.$$

$$\left. + 2\zeta rW_d\,\mathrm{Cos}W_d t\right] + (1 - r^2)\mathrm{Sin}Wt - 2\zeta r\,\mathrm{Cos}Wt\,. \qquad (14)$$

Taking the derivative of Equation (14) with respect to t and equating it to zero gives

$$\frac{K\dot{X}}{k^2 F_o} = \frac{e^{-\zeta W_n t}}{W_d}(-\zeta W_n)((2\zeta^2 rW_n - W(1 - r^2))\mathrm{Sin}W_d t$$

$$+ 2\zeta rW_d\,\mathrm{Cos}W_d t) + \frac{e^{-\zeta W_n t}}{W_d}W_d\left[(2\zeta^2 rW_n - W(1 - r^2))\mathrm{Cos}W_d t\right.$$

$$\left. - 2\zeta rW_d\,\mathrm{Sin}W_d t\right] + W(1 - r^2)\mathrm{Cos}Wt + 2\zeta rW\,\mathrm{Sin}Wt = 0. \qquad (15)$$

Upon rearrangment, Equation(15) becomes

$$\frac{e^{-\zeta W_n t}}{\sqrt{1-\zeta^2}}\left[(\zeta W(1-r^2) - 2\zeta r W_n)\operatorname{Sin}W_d t - W(1-r^2)\sqrt{1-\zeta^2}\operatorname{Cos}W_d t\right]$$

$$+ W(1-r^2)\operatorname{Cos}Wt + 2\zeta r W \operatorname{Sin}Wt = 0 .$$

Dividing the above equation by W_n gives

$$\frac{e^{-\zeta W_n t}}{\sqrt{1-\zeta^2}}\left[(\zeta r(1-r^2) - 2\zeta r)\operatorname{Sin}\sqrt{1-\zeta^2}W_n t - (r(1-r^2)\sqrt{1-\zeta^2}\right.$$

$$\left.\operatorname{Cos}\sqrt{1-\zeta^2}W_n t)\right] + r(1-r^2)\operatorname{Cos}rW_n t + 2\zeta r^2 \operatorname{Sin}rW_n t = 0 . \quad (16)$$

Those $W_n t$ corresponding to peak responses are determined by a trial error method on Equation (16). Substituting them into Equation (14) and selecting the maximum absolute value gives the maximum peak response X_m and the corresponding $W_n t_m$. The nondimensional maximum response is

$$\frac{KX_m}{k^2 F_o} = \frac{e^{-\zeta W_n t_m}}{\sqrt{1-\zeta^2}}\left[(2\zeta r - r(1-r^2))\operatorname{Sin}(W_n\sqrt{1-\zeta^2}\,t_m)\right.$$

$$\left.+ 2\zeta r\sqrt{1-\zeta^2}\operatorname{Cos}(W_n\sqrt{1-\zeta^2}\,t_m)\right] + (1-r^2)\operatorname{Sin}(rW_n t_m)$$

$$- 2\zeta r \operatorname{Cos}(rW_n t_m) . \quad (17)$$

2. A SINUSODIAL DISPLACEMENT OF THE SYSTEM'S BASE

Assume that the system's base is given a displacement $Y_o \operatorname{Sin}Wt$ and the displacement of the base of the oscillator, measured upward from the floor, is y. The equation of motion is

$$M\ddot{X} + C\dot{X} + KX = C\dot{y} + Ky . \quad (18)$$

Let $Z = X - y$ be the relative displacement between the mass and the base. Upon rearrangement Equation (18) becomes

$$M\ddot{Z} + C\dot{Z} + KZ = My . \quad (19)$$

Substituting $\ddot{y} = -Y_o W^2 \,\mathrm{Sin}Wt$ into Equation (19) gives

$$M\ddot{Z} + C\dot{Z} + KZ = -(MY_oW^2)\,\mathrm{Sin}Wt \,. \tag{20}$$

By comparing Equation (20) with Equation (1), $(-MY_oW^2)$ in Equation (20) is seen to be equivalent to F in Equation (1). Similarily the maximum response of Equation (20) is

$$\frac{KX_m}{k^2(-MY_oW^2)} = \frac{e^{-\zeta W_n t_m}}{\sqrt{1-\zeta^2}}\left[(2\zeta r - r(1-r^2))\mathrm{Sin}(W_n\sqrt{1-\zeta^2}\,t_m)\right.$$
$$\left. + 2\zeta r\sqrt{1-\zeta^2}\,\mathrm{Cos}(W_n\sqrt{1-\zeta^2}\,t_m)\right] + (1-r^2)\,\mathrm{Sin}(rW_nt_m)$$
$$- 2\zeta r\,\mathrm{Cos}(rW_nt_m) \,. \tag{21}$$

The calculated data for nondimensional maximum response of Harmonic Force Excitation are given in Table 1 and the plots of nondimensional maximum response versus frequency ratio are shown in Fig. 3 and 4.

3. UNIT STEP FUNCTION EXCITATION[5]

The mass M is given a force P, as shown in Fig. 5. The motion equation of this unit step excitation is

$$M\ddot{X} + C\dot{X} + KX = P \,.$$

The general solution of the above equation is

$$X = Ae^{-\zeta W_n t}\,\mathrm{Cos}(W_d t - \phi) + \frac{P}{K} \,. \tag{22}$$

Substituting the initial condition, $X(0) = 0$, into Equation (22) gives

$$X(0) = A\,\mathrm{Cos}\phi + \frac{P}{K} = 0 \,,$$

TABLE 1*

NONDIMENSIONAL MAXIMUM RESPONSE OF HARMONIC FORCE EXCITATION

Frequency ratio (r)	Nondimensional maximum response			
	(ζ = 0.15)	(ζ = 0.20)	(ζ = 0.50)	(ζ = 0.75)
0.1	0.99	0.99	0.99	1.00
0.2	0.96	0.96	0.98	1.00
0.3	1.00	0.98	0.96	1.01
0.4	0.94	0.91	0.93	1.03
0.5	0.79	0.79	0.90	1.06
0.6	0.73	0.72	0.87	1.10
0.7	0.59	0.60	0.86	1.16
0.8	0.44	0.48	0.87	1.26
0.9	0.32	0.40	0.91	1.42
1.0	0.29	0.39	1.00	1.64
1.5	2.40	2.30	2.70	3.65
2.0	6.95	6.89	6.92	7.54
2.5	15.65	15.08	13.59	13.67
3.0	27.69	27.11	23.01	22.17
3.5	43.58	41.83	35.78	32.00

*Table 1 is also for nondimensional maximum response for the disturbance of sinusodial displacement of the system's base.

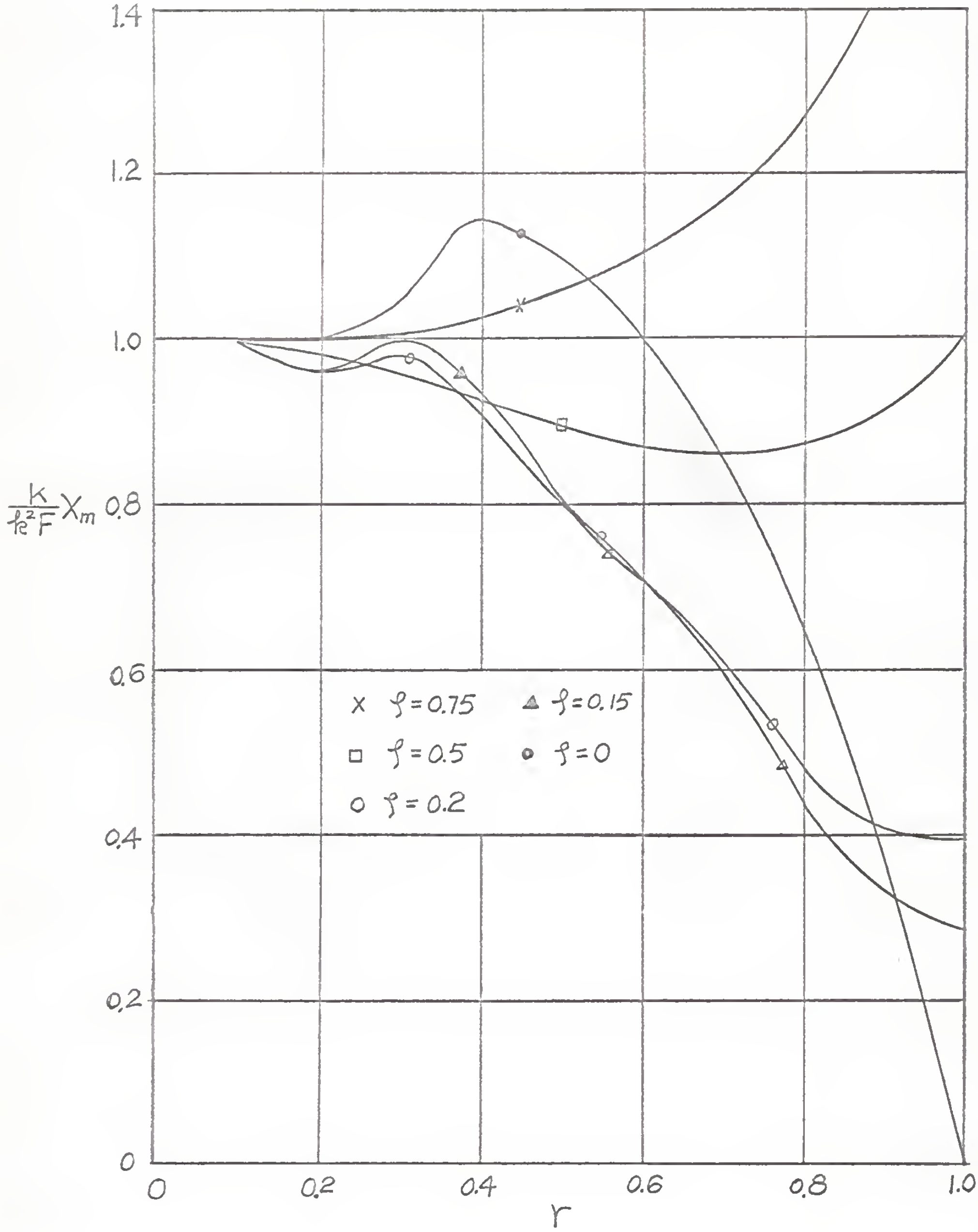


Fig. 3 Nondimensional Maximum Response to Frequency Ratio.

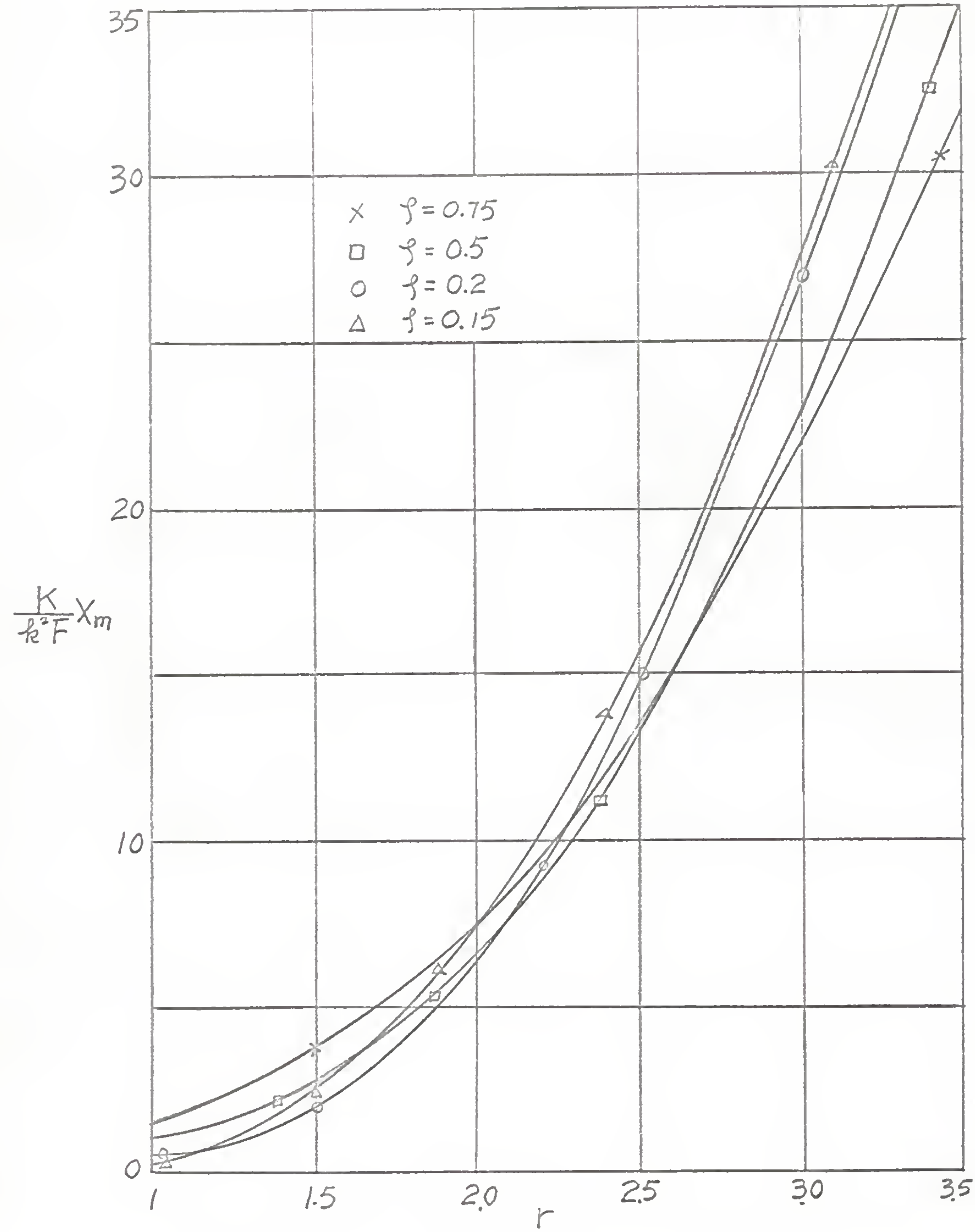


Fig. 4 Nondimensional Maximum Response to Frequency Ratio.

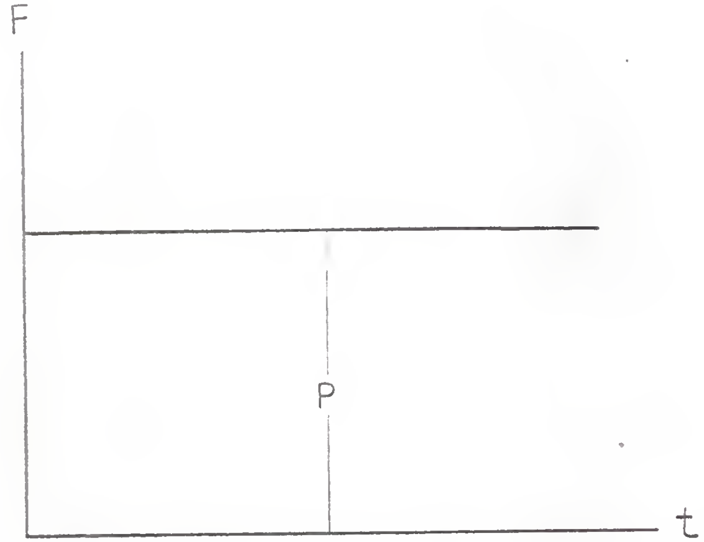


Fig. 5

$$A \cos\phi = -\frac{P}{K} . \qquad (23)$$

Taking the derivative of Equation (22) with respect to t and substituting the initial condition, $\dot{X}(0) = 0$, gives

$$\dot{X}(0) = A(-\zeta W_n \cos\phi + W_d \sin\phi) = 0 . \qquad (24)$$

Therefore,

$$\phi = \tan^{-1}\frac{W_n}{W_d} = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} . \qquad (25)$$

Substituting Equation (25) in (23) gives

$$A = -\frac{P}{K\sqrt{1-\zeta^2}} . \qquad (26)$$

Substituting Equation (25) and (26) into (22) gives

$$X = \frac{P}{K}\left[1 - \frac{e^{-\zeta W_n t}}{\sqrt{1-\zeta^2}} \cos(W_d t - \phi)\right] . \qquad (27)$$

Taking the derivative of Equation (27) gives

$$\frac{K}{P}\dot{X} = W_n \frac{e^{-\zeta W_n t}}{\sqrt{1-\zeta^2}} \cos(W_d t - \phi) + \frac{e^{-\zeta W_n t}}{\sqrt{1-\zeta^2}} W_d \sin(W_d t - \phi) . \qquad (28)$$

Upon rearrangement, it gives

$$\frac{K}{P}\dot{X} = \frac{W_n e^{-\zeta W_n t}}{\sqrt{1-\zeta^2}} \sin((W_d t - \phi) + \theta) , \qquad (28a)$$

where

$$\theta = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} = \phi .$$

Therefore, Equation (28a) becomes

$$\frac{K}{P} X = \frac{W_n e^{-\zeta W_n t}}{\sqrt{1-\zeta^2}} \operatorname{Sin}(W_d t) \,. \tag{29}$$

By setting Equation (29) equal to zero, it gives

$$\operatorname{Sin} W_d t = \operatorname{Sin}(W_n \sqrt{1-\zeta^2}\, t) = 0 \,. \tag{30}$$

That is

$$W_n t = \frac{\pi}{\sqrt{1-\zeta^2}} \,. \tag{30a}$$

Substituting Equation (30a) into (27) gives the peak response X_m in nondimensional form as

$$\frac{K}{P} X_m = 1 - \frac{e^{-\pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \operatorname{Cos}\left(\pi - \operatorname{Tan}^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$= 1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}} \,. \tag{31}$$

It is found in this case that the maximum peak response is independent of the natural frequency, W_n, of the system and depends only on the damping factor, ζ. Data for the time-response of step function excitation are given in Table 2 and the time response and maximum response in nondimensional form are shown in Figs. 6 and 7, respectively. Datas for the shock spectrum of step function excitation are given in Table 3.

TABLE 2

DATA FOR THE TIME-RESPONSE OF STEP FUNCTION EXCITATION

Time ($W_n t$)	Nondimensional response (K/P)X				
	(ζ = 0.05)	(ζ = 0.10)	(ζ = 0.25)	(ζ = 0.50)	(ζ = 0.90)
0.0	0.000	0.000	0.000	0.000	0.000
1.0	0.445	0.431	0.392	0.340	0.277
2.0	1.333	1.258	1.070	0.849	0.632
3.0	1.845	1.720	1.430	1.124	0.848
4.0	1.569	1.498	1.337	1.153	0.949
5.0	0.821	0.901	1.036	1.074	0.987
6.0	0.300	0.494	0.827	1.002	0.999
7.0	0.441	0.582	0.825	0.974	1.001
8.0	1.057	1.002	0.950	0.979	1.001
9.0	1.564	1.344	1.062	0.992	1.000
10.0	1.529	1.336	1.084	1.002	1.000
11.0	1.034	1.050	1.037	1.004	1.000
12.0	0.556	0.773	0.981	1.002	1.000
13.0	0.512	0.735	0.961	1.000	1.000
14.0	0.898	0.924	0.976	0.999	0.999
15.0	1.337	1.142	1.003	0.999	0.999
16.0	1.438	1.201	1.016	0.999	0.999
17.0	1.146	1.082	1.012	1.000	0.999
18.0	0.754	0.915	1.001	1.000	0.999
19.0	0.613	0.849	0.993	1.000	1.000
20.0	0.824	0.920	0.993	1.000	1.000

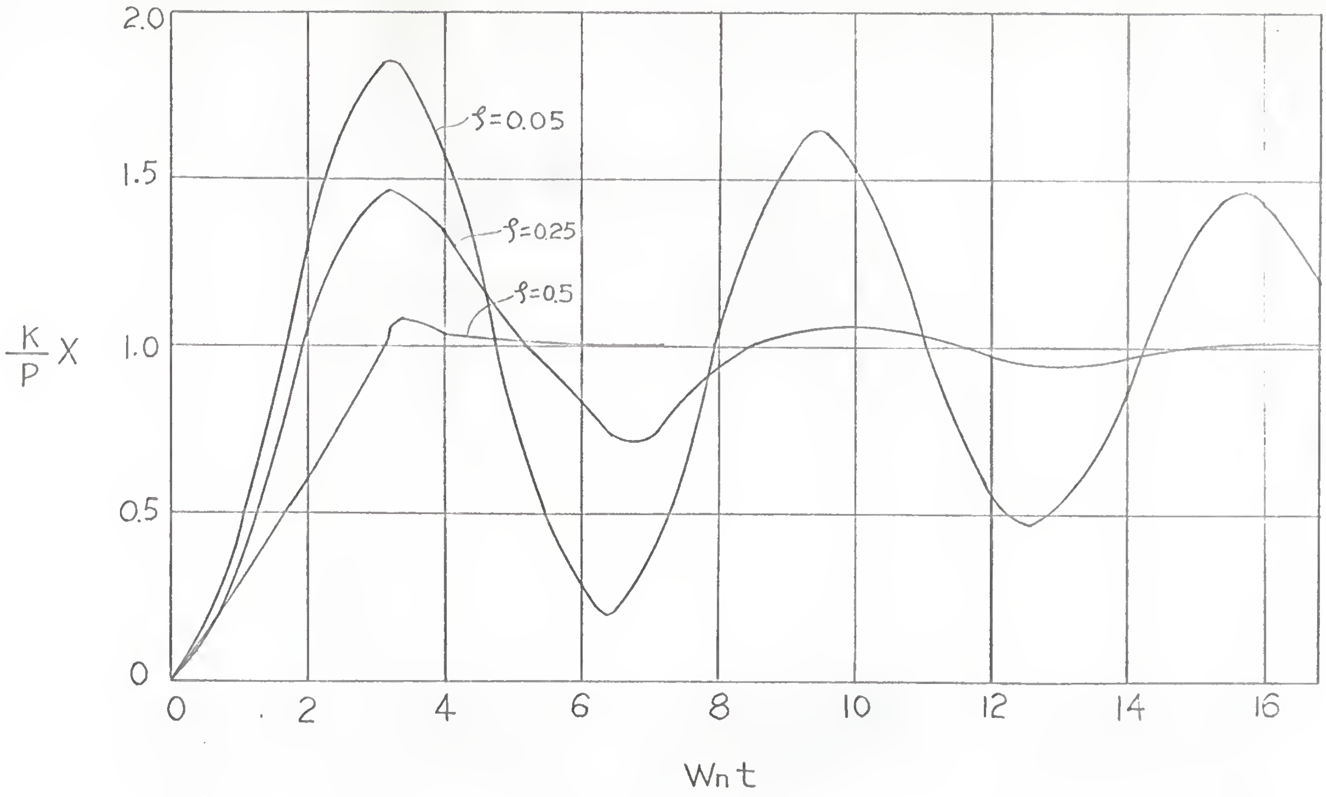


Fig. 6 Time Response by Unit Step Function.

TABLE 3

DATA FOR THE SHOCK SPECTRUM OF STEP FUNCTION EXCITATION

Damping factor (ζ)	Nondimensional maximum response (K/P)X	Damping factor (ζ)	Nondimensional maximum response (K/P)X
0.00	2.000	0.62	1.083
0.01	1.969	0.64	1.073
0.05	1.857	0.66	1.063
0.10	1.729	0.68	1.054
0.15	1.620	0.70	1.045
0.18	1.562	0.72	1.038
0.20	1.562	0.74	1.031
0.25	1.444	0.76	1.025
0.30	1.372	0.78	1.019
0.35	1.309	0.80	1.015
0.40	1.253	0.85	1.006
0.45	1.205	0.90	1.001
0.50	1.163	0.91	1.001
0.55	1.126	0.92	1.000
0.60	1.094	0.99	1.000

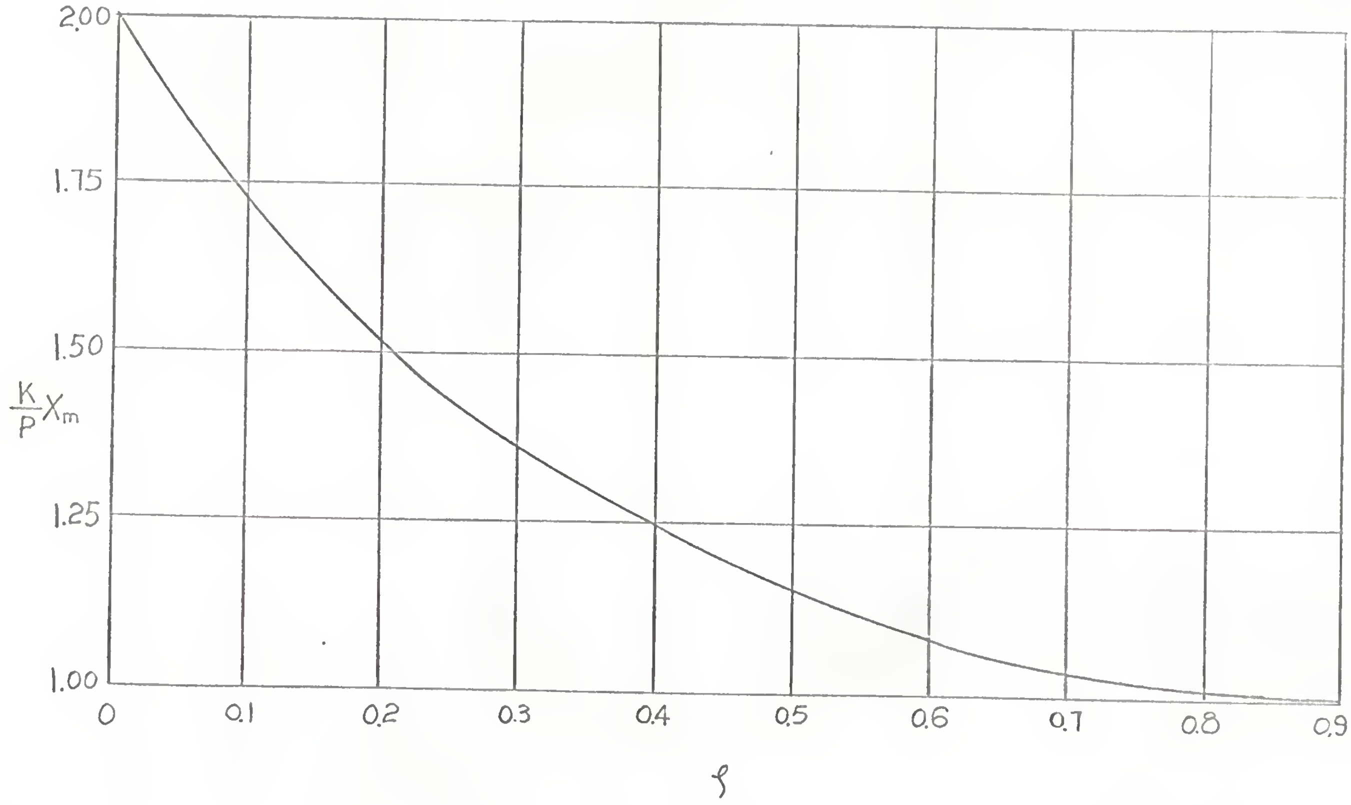


Fig. 7 Shock Spctrum of Unit Step Function Excitation.

4. DELTA FUNCTION EXCITATION[5]

The mass M is given an impulsive force of magnitude $\frac{\overline{F}}{\epsilon}$ with a time duration of ϵ. As ϵ approaches zero, such a force tends to become infinite; however, the impulse defined by its time integral is $\overline{F}$. When $\overline{F}$ is equal to unity, such a force in the limiting case $\epsilon \to 0$ is called a delta function. The equation of motion becomes

$$M\ddot{X} + C\dot{X} + KX = 0 . \tag{32}$$

The general solution of Equation (32) is

$$X = Ae^{-\xi W_n t} \, \text{Sin}(W_d t + \psi) . \tag{33}$$

Taking the derivative of Equation (33) with respect to t gives

$$\dot{X} = -\xi W_n A e^{-\xi W_n t} \, \text{Sin}(W_d t + \psi) + Ae^{-\xi W_n t} \, W_d \, \text{Cos}(W_d t + \psi) . \tag{34}$$

Substituting the initial conditions, $X(0) = 0$ and $\dot{X}(0) = \frac{\overline{F}}{M}$, into Equations (33) and (34) gives

$$A \, \text{Sin} \, \psi = 0 ,$$

$$\therefore \psi = 0 , \tag{35}$$

and

$$\frac{\overline{F}}{M} = A W_d ,$$

$$A = \frac{\overline{F}}{M W_d} . \tag{36}$$

Substituting Equations (35) and (36) into (33) gives

$$X = \frac{\overline{F}}{M W_d} e^{-\xi W_n t} \, \text{Sin} W_d t ,$$

or

$$\frac{XM}{F} = \frac{e^{-\xi W_n t}}{W_n\sqrt{1-\xi^2}} \operatorname{Sin} W_d t \ . \qquad (37)$$

Upon rearrangement, Equation (37) becomes

$$\frac{X\sqrt{KM}}{F} = \frac{e^{-\xi W_n t}}{\sqrt{1-\xi^2}} \operatorname{Sin}(W_n\sqrt{1-\xi^2}\,t) \ . \qquad (38)$$

The derivative of Equation (38) with respect to t is

$$\frac{M}{F}\dot{X} = e^{-\xi W_n t}\left(-\frac{\xi}{\sqrt{1-\xi^2}} \operatorname{Sin} W_d t + \operatorname{Cos} W_d t\right) ,$$

or

$$\frac{M}{F}\dot{X} = \frac{e^{-\xi W_n t}}{\sqrt{1-\xi^2}} \operatorname{Cos}(W_d t + \phi) , \qquad (39)$$

where

$$\phi = \operatorname{Tan}^{-1}\frac{\xi}{\sqrt{1-\xi^2}} \ . \qquad (40)$$

Equation (39) is set equal to zero to find t_m corresponding to the maximum response, that is

$$W_d t + \phi = \frac{\pi}{2} ,$$

$$W_n\sqrt{1-\xi^2}\,t + \phi = \frac{\pi}{2} ,$$

$$W_n t = \frac{\frac{\pi}{2} - \phi}{\sqrt{1-\xi^2}} \ . \qquad (41)$$

Substituting Equation (41) into (38) gives

$$\frac{X_{\wedge}\sqrt{KM}}{\bar{F}} = \frac{e^{-\xi(\pi/2 - \phi)/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \operatorname{Sin}(\pi/2 - \phi)$$

$$= e^{-\xi(\pi/2 - \phi)/\sqrt{1-\xi^2}} .$$

Therefore the nondimensional form of the maximum peak response X_m is

$$\frac{\sqrt{KM}}{\bar{F}} X_m = e^{-\xi(\pi/2 - \phi)/\sqrt{1-\xi^2}} . \tag{42}$$

The calculated data for nondimensional maximum response for Delta Function and the time response for Delta Function Excitation are shown in Table 4 and 5, respectively. Also, the time response and the maximum response in nondimensional form are shown in Figs. 8 and 9, respectively.

5. RECTANGULAR PULSE EXCITATION[5]

It is assumed that a rectangular pulse height of F excites the oscillator in the time interval $0 < t < t_1$. When $t < t_1$, Equations (27) and (29) apply for this condition. The displacement and velocity equations are

$$\frac{K}{P} X = 1 - \frac{e^{-\xi W_n t}}{\sqrt{1-\xi^2}} \operatorname{Cos}(W_d t - \phi) , \tag{27}$$

$$\dot{X} = \frac{P}{K} \frac{W_n e^{-\xi W_n t}}{\sqrt{1-\xi^2}} \operatorname{Sin}(W_d t) , \tag{29}$$

where

$$\phi = \operatorname{Tan}^{-1} \frac{\xi}{\sqrt{1-\xi^2}} .$$

When $t = t_1$, the displacement and velocity equations will be

TABLE 4

NONDIMENSIONAL MAXIMUM RESPONSE FOR DELTA FUNCTION

Damping factor (ζ)	Nondimensional maximum response $\frac{\sqrt{KM}}{\bar{F}} X_m$	Damping factor (ζ)	Nondimensional maximum response $\frac{\sqrt{KM}}{\bar{F}} X_m$
0.00	1.000	0.50	0.546
0.05	0.927	0.55	0.522
0.10	0.863	0.60	0.499
0.15	0.806	0.65	0.478
0.20	0.756	0.70	0.495
0.25	0.712	0.75	0.440
0.30	0.672	0.80	0.424
0.35	0.636	0.85	0.408
0.40	0.603	0.90	0.394
0.45	0.573	0.95	0.380

TABLE 5

TIME-RESPONSE FOR DELTA FUNCTION EXCITATION

Time	Response (X)				
($W_n t$)	($\zeta = 0.05$)	($\zeta = 0.10$)	($\zeta = 0.25$)	($\zeta = 0.50$)	($\zeta = 0.90$)
0.0	0.000	0.000	0.000	0.000	0.000
1.0	0.800	0.763	0.663	0.534	0.316
2.0	0.824	0.752	0.585	0.419	0.394
3.0	0.124	0.116	0.114	0.133	0.290
4.0	-0.618	-0.500	-0.254	-0.049	0.149
5.0	-0.749	-0.589	-0.293	-0.088	0.062
6.0	-0.212	-0.170	-0.105	-0.050	0.021
7.0	0.459	0.314	0.085	-0.007	0.005
8.0	0.665	0.449	0.139	0.013	0.000
9.0	0.269	0.185	0.071	0.013	-0.000
10.0	-0.323	-0.185	-0.062	0.005	-0.000
11.0	-0.578	-0.334	-0.042	0.000	-0.000
12.0	-0.302	-0.177	0.000	-0.002	-0.000
13.0	0.212	0.987	0.026	-0.001	-0.000
14.0	0.491	0.243	0.022	-0.000	-0.000
15.0	0.314	0.158	0.004	0.000	-0.000
16.0	-0.121	-0.043	-0.010	0.000	0.000
17.0	-0.409	-0.171	-0.011	0.0002	0.000
18.0	-0.312	-0.134	-0.004	0.000	0.000
19.0	0.489	0.008	0.003	-0.000	0.000
20.0	0.332	0.118	0.010	-0.000	0.000

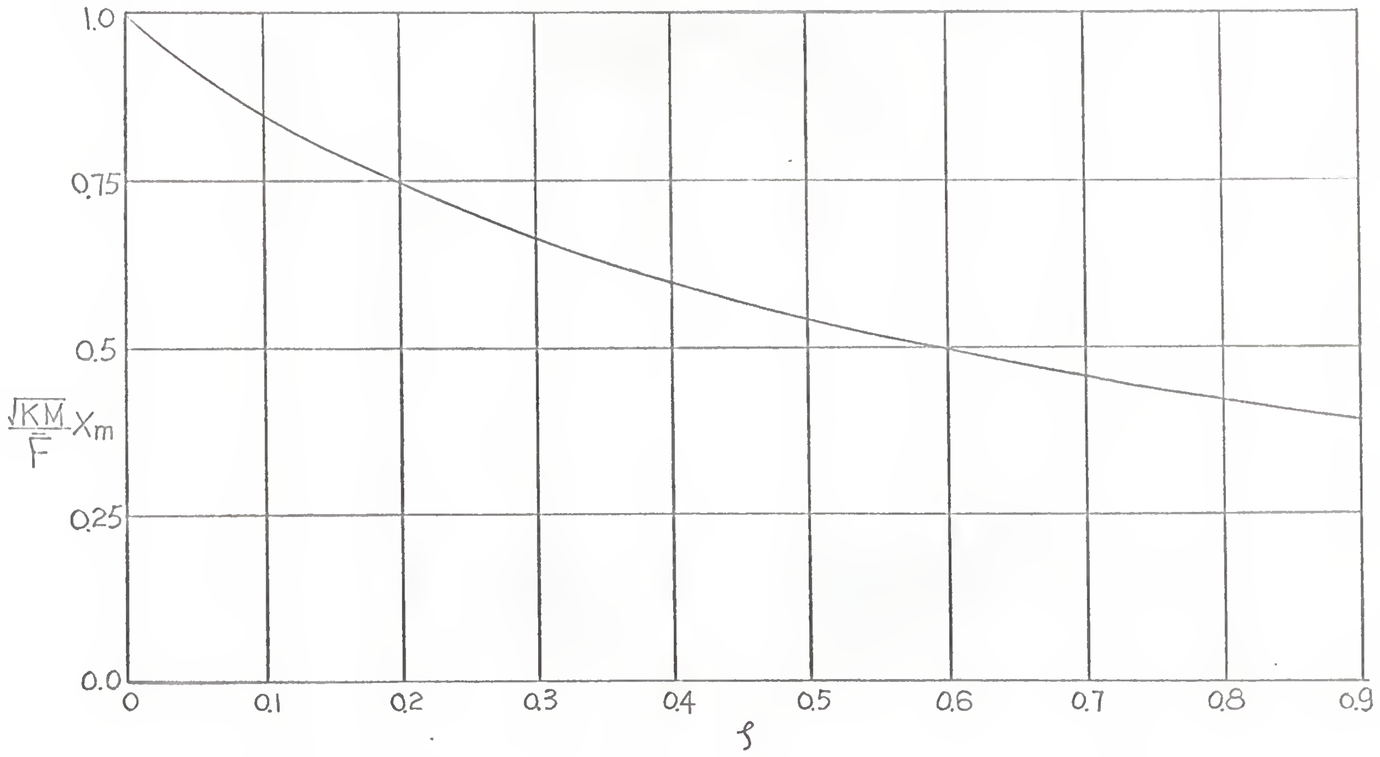


Fig.8 Shock Spectrum of Delta Function Excitation.

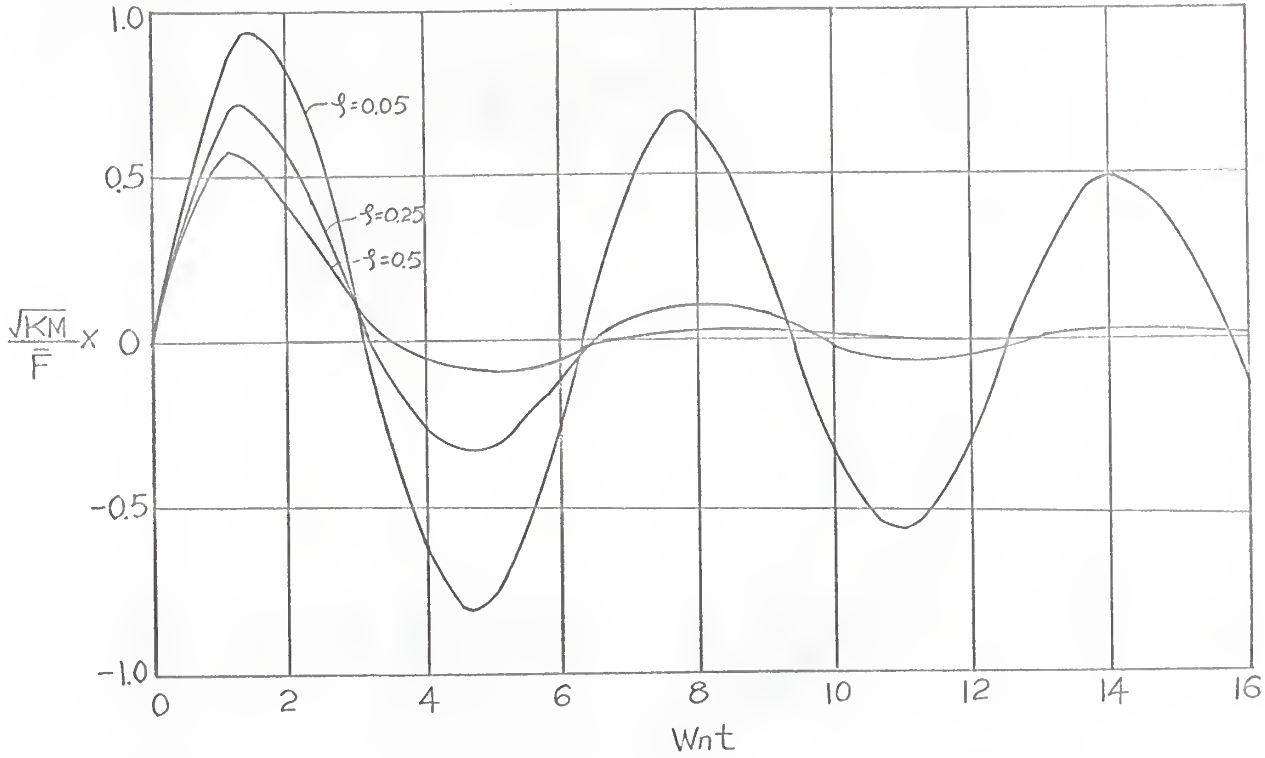


Fig. 9 Time Response for Delta Function.

$$X_1 = \frac{P}{K}\left(1 - \frac{e^{-\xi W_n t_1}}{\sqrt{1 - \xi^2}} \mathrm{Cos}(W_d t_1 - \phi)\right) , \tag{43}$$

$$\dot{X}_1 = \frac{P}{K} \frac{W_n e^{-\xi W_n t_1}}{\sqrt{1 - \xi^2}} \mathrm{Sin} W_d t_1 , \tag{44}$$

that is,

$$\alpha = \left(1 - \frac{e^{-\xi W_n t_1}}{\sqrt{1 - \xi^2}} \mathrm{Cos}(W_d t_1 - \phi)\right) , \tag{43a}$$

$$q = \frac{e^{-\xi W_n t_1}}{\sqrt{1 - \xi^2}} \mathrm{Sin} W_d t_1 . \tag{44a}$$

Therefore,

$$X_1 = \frac{P}{K}\alpha ,$$

and

$$\dot{X}_1 = \frac{P W_n q}{K} .$$

When $t > t_1$, there is no excitation force and the equation of motion is

$$M\ddot{X} + C\dot{X} + KX = 0 ,$$

and the solution of the above equation is

$$X = A e^{-\xi W_n t} \mathrm{Sin}(W_n t + \psi) . \tag{45}$$

Taking the derivative of Equation (45) with respect to t, it gives

$$\dot{X} = A e^{-\xi W_n t} W_n \left[\sqrt{1 - \xi^2} \mathrm{Cos}(W_d t + \psi) - \xi \mathrm{Sin}(W_d t + \psi)\right] . \tag{46}$$

Shifting t to t_1 and substituting the initial condition, $X(0) = X_1 = \frac{P}{K}\alpha$ and $\dot{X}(0) = \dot{X}_1 = P W_n q/K$, into Equation (45) and (46) gives

$$\frac{P}{K}\alpha = A\,\mathrm{Sin}\,\psi\,, \tag{47}$$

$$\frac{P}{K}W_n q = A W_n\,(\sqrt{1-\xi^2}\,\mathrm{Cos}\,\psi - \mathrm{Sin}\,\psi)\,. \tag{48}$$

From Equation (47) and (48) one obtain

$$A = \frac{P}{K}\frac{\sqrt{q^2 + 2q\alpha\xi + \alpha^2}}{\sqrt{1-\xi^2}}\,, \tag{49}$$

$$\psi = \mathrm{Tan}^{-1}\frac{(\sqrt{1-\xi^2})\,\alpha}{q + \xi\alpha}\,. \tag{50}$$

The equation of response at $t > t_1$ is therefore

$$X = \frac{P}{K}\frac{\sqrt{q^2 + 2q\alpha\xi + \alpha^2}}{\sqrt{1-\xi^2}}\,e^{-\xi W_n t}\,\mathrm{Sin}(W_d t + \psi)\,. \tag{51}$$

Substituting Equation (49) and (50) into (46) and equating it to zero gives t_m corresponding to the peak response X_m.

$$\dot{X} = \frac{P}{K}\frac{\sqrt{q^2 + 2q\alpha\xi + \alpha^2}}{\sqrt{1-\xi^2}}\,e^{-\xi W_n t}\,W_n\left[\sqrt{1-\xi^2}\,\mathrm{Cos}(W_d t + \psi) - \xi\,\mathrm{Sin}(W_d t + \psi)\right] = 0\,,$$

or

$$\dot{X} = \frac{PW_n}{K}\frac{\sqrt{q^2 + 2q\alpha\xi + \alpha^2}}{\sqrt{1-\xi^2}}\,e^{-\xi W_n t}\,\mathrm{Cos}(W_d t + \psi + \gamma) = 0\,, \tag{53}$$

where

$$\gamma = \mathrm{Tan}^{-1}\frac{\xi}{\sqrt{1-\xi^2}}\,. \tag{54}$$

Setting Equation (53) equal to zero gives

$$\mathrm{Cos}(W_d t + \psi + \gamma) = 0 ,$$

and therefore,

$$W_n t \sqrt{1 - \zeta^2} + \psi + \gamma = \pi/2 ,$$

$$W_n t = (\pi/2 - \psi - \gamma)/\sqrt{1 - \zeta^2} . \tag{55}$$

Substituting Equation (55) into (51) gives the maximum peak response X_m. That is

$$X_m = \frac{P}{K} \frac{\sqrt{q^2 + 2q\alpha\zeta + \alpha^2}}{\sqrt{1 - \zeta^2}} e^{-(\pi/2 - \psi - \gamma)/\sqrt{1 - \zeta^2}}$$

$$\mathrm{Sin}(\pi/2 - \psi - \gamma + \psi) . \tag{56}$$

Substituting Equation (54) into (56) gives X_m in nondimensional form. That is

$$\frac{K}{P} X_m = \sqrt{q^2 + 2q\alpha\zeta + \alpha^2}\, e^{-\zeta(\pi/2 - \psi - \gamma)/\sqrt{1 - \zeta^2}} . \tag{57}$$

In the special case with $\zeta = 0$, both ϕ and γ are equal to zero. Equations (43a) and (44a) simplify to

$$\alpha = 1 - \mathrm{Cos} W_n t_1 , \tag{58}$$

and

$$q = \mathrm{Sin} W_n t_1 . \tag{59}$$

Thus Equation (57) reduces to

$$\frac{K}{P} X_m = \sqrt{q^2 + \alpha^2} . \tag{60}$$

Substituting Equation (58) and (59) into (60) gives

$$\frac{K}{P} X_m = \sqrt{\sin^2 W_n t_1 + (1 - 2 \cos W_n t_1 + \cos^2 W_n t_1)} \ , \tag{61}$$

$$\frac{K}{P} X_m = \sqrt{2(1 - \cos W_n t_1)} \ . \tag{62}$$

The calculated data of nondimensional maximum response for the rectangular pulse time intervals are shown in Table 6. There are two plots of shock spectrum of rectangular pulse shown. The one without damping but with time duration t_1 as a parameter is shown in Fig. 10. The other one with different damping factors is shown in Fig.11.

6. FREE FALL EXCITATION[6]

Suppose the whole system falling from a height h, as shown in Fig. 12, with the assumption that:

(1) The mass of the base is large compared to the supported mass M, so that the free fall of the base is not influenced by the relative motion of the mass M.

(2) On striking the floor the base remains in contact with the floor.

Letting X be the displacement of M relative to the base, measured downward from the static equilibrium position, t is measured when the base strikes the floor. The equation of motion is

$$M\ddot{X} + C\dot{X} + KX = 0 \ . \tag{63}$$

The displacement and velocity equations are

$$X = e^{-\zeta W_n t}(A_1 \cos W_d t + A_2 \sin W_d t) \ , \tag{64}$$

and

TABLE 6

RECTANGULAR PULSE TIME INTERVAL AND NONDIMENSIONAL MAXIMUM RESPONSE

Time interval	Nondimensional maximum response ($\frac{K}{P}X_m$)				
($W_n t_1$)	(ζ = 0.00)	(ζ = 0.10)	(ζ = 0.30)	(ζ = 0.50)	(ζ = 0.75)
0.0	0.00	0.00	0.00	0.00	0.00
1.0	0.96	0.83	0.64	0.52	0.42
2.0	1.68	1.45	1.13	0.92	0.75
3.0	1.96	1.72	1.36	1.13	0.94
4.0	1.81	1.15	1.31	1.15	1.01
5.0	1.19	0.79	1.08	1.07	1.02
6.0	0.28	0.38	0.89	1.00	1.01
7.0	0.70	0.65	0.87	0.97	1.00
8.0	1.51	1.09	0.95	0.98	1.00
9.0	1.95	1.35	1.03	0.99	0.99
10.0	1.91	0.98	1.05	1.00	0.99
11.0	1.41	0.80	1.02	1.00	0.99
12.0	0.56	0.58	0.99	1.00	0.99
13.0	0.40	0.74	0.98	1.00	0.99
14.0	1.30	0.95	0.98	0.99	1.00
15.0	1.87	1.15	0.99	0.99	1.00
16.0	1.98	1.20	1.00	0.99	1.00
17.0	1.60	0.78	1.00	1.00	0.99
18.0	0.84	0.67	1.00	1.00	0.99
19.0	0.11	0.85	0.99	1.00	0.99
20.0	1.05	0.93	0.99	1.00	0.99

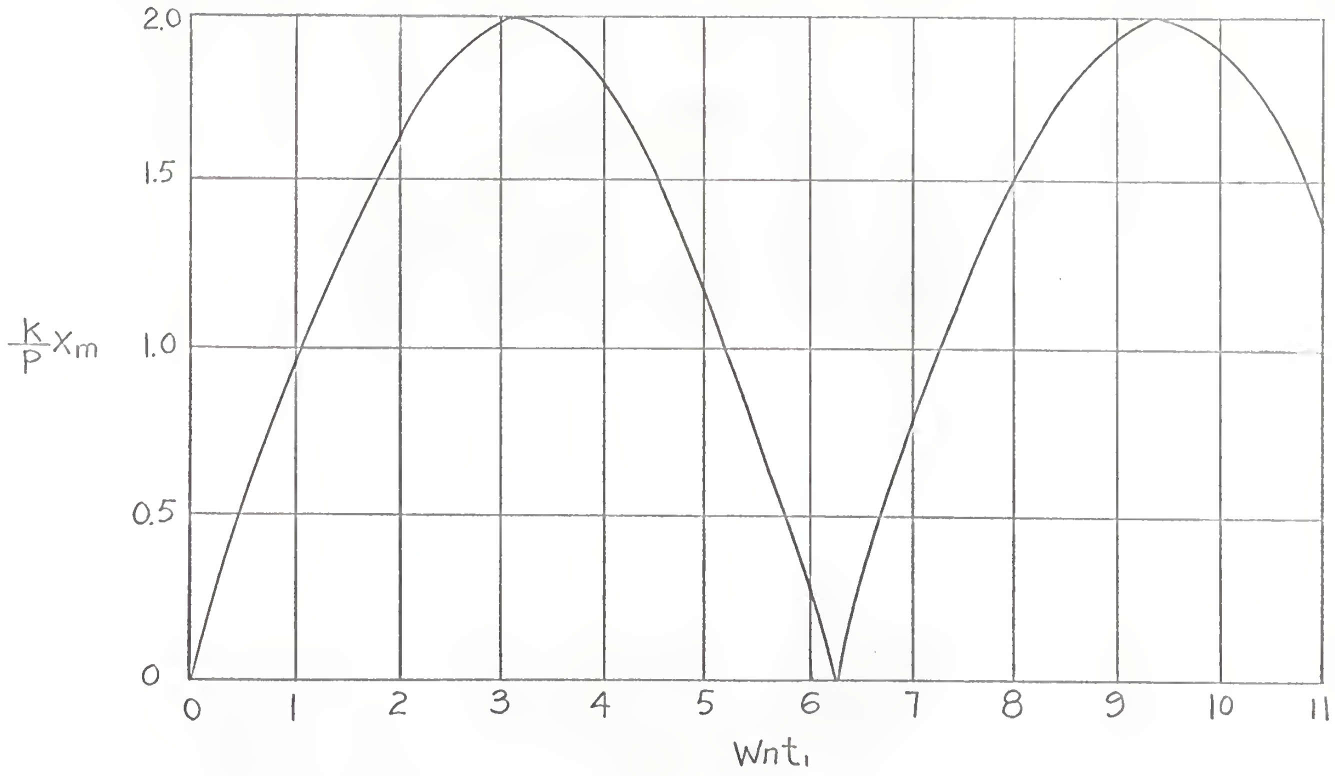


Fig. 10 Nondimensional Maximumn Response to Regtangular Pulse Interval.

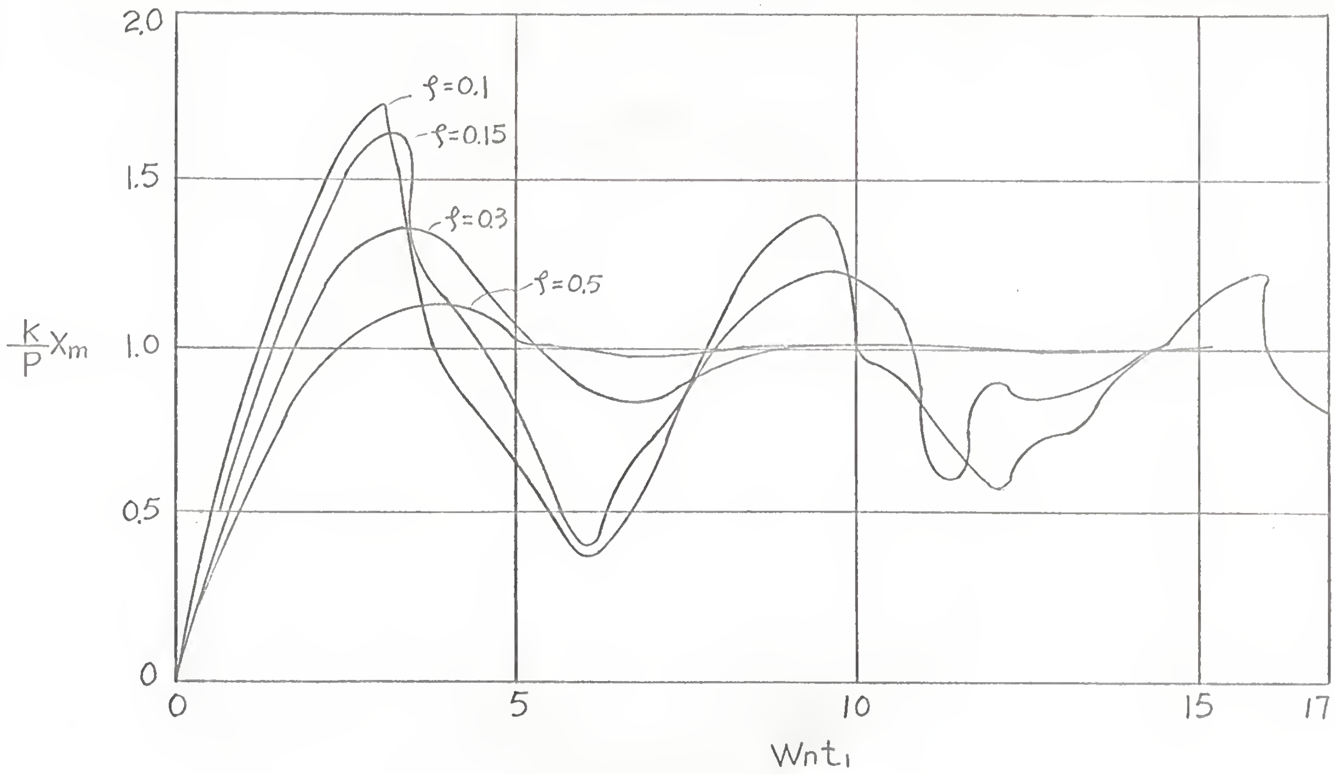


Fig. 11 Nondimensional Maximum Response to Rectangular Pulse Interval.

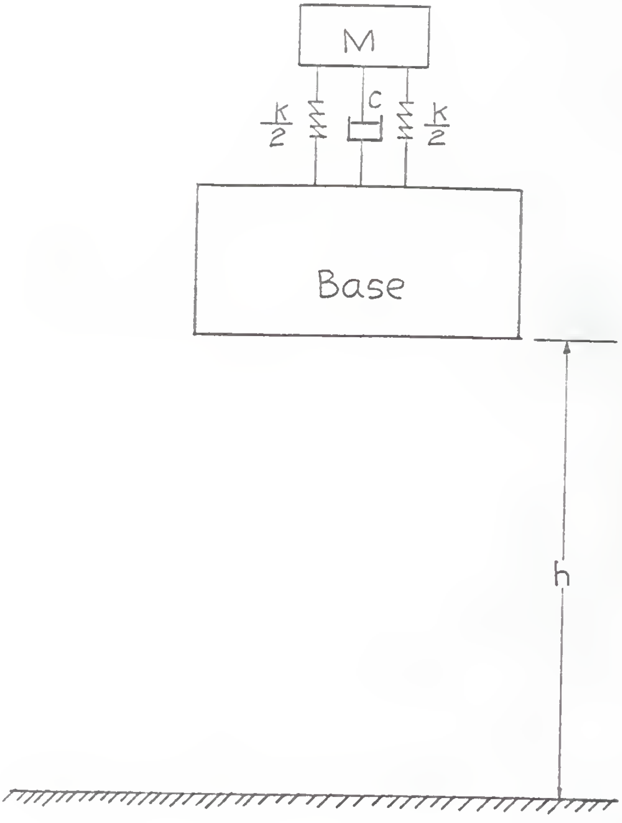


Fig. 12 Free Fall.

$$\dot{X} = e^{-\zeta W_n t}[-\zeta W_n(A_1 \mathrm{Cos} W_d t + A_2 \mathrm{Sin} W_d t) +$$

$$W_d(-A_1 \mathrm{Sin} W_d t + A_2 \mathrm{Cos} W_d t)] \; . \tag{65}$$

Substituting the initial condition, $X(0) = 0$ and $\dot{X}(0) = \sqrt{2gh}$, into Equations (64) and (65) gives

$$A_1 = 0 \; , \tag{66}$$

$$A_2 = \sqrt{2gh}/W_d \; , \tag{67}$$

where g is the gravity acceleration. Therefore, Equation (64) becomes

$$X = e^{-\zeta W_n t} \frac{\sqrt{2gh}}{W_d} \mathrm{Sin} W_d t \; . \tag{68}$$

Taking the derivative of Equation (68) with respect to t and equating to zero gives t_m corresponding to the maximum peak response. That is

$$\dot{X} = e^{-\zeta W_n t} \sqrt{2gh} \left(\frac{-\zeta}{\sqrt{1-\zeta^2}} \mathrm{Sin} W_d t + \mathrm{Cos} W_d t\right)$$

$$= e^{-\zeta W_n t} \frac{\sqrt{2gh}}{\sqrt{1-\zeta^2}} (-\zeta \mathrm{Sin} W_d t + \sqrt{1-\zeta^2} \mathrm{Cos} W_d t)$$

$$= e^{-\zeta W_n t} \mathrm{Cos}(W_d t + \gamma) = 0 \; , \tag{69}$$

$$W_d t + \gamma = \frac{\pi}{2} \; ,$$

where

$$\gamma = \mathrm{Tan}^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} \; . \tag{70}$$

Therefore it gives

$$t_m = (\pi/2 - \gamma)/W_d \;. \tag{71}$$

Substituting Equation (71) into (68) gives the maximum peak response X_m. That is

$$X_m = e^{-\zeta(\pi/2 - \gamma)/\sqrt{1 - \zeta^2}} \frac{\sqrt{2gh}}{W_d} \operatorname{Cos}\gamma$$

$$= e^{-\zeta(\pi/2 - \gamma)/\sqrt{1 - \zeta^2}} \frac{\sqrt{2gh}}{W_n} \;. \tag{72}$$

The calculated data of maximum response for a free fall excitation are shown in Table 7 and Table 8. Two plots of maximum response to natural frequency are shown in Fig. 13 and Fig. 14.

TABLE 7

MAXIMUM RESPONSE FOR FREE FALL OF A SINGLE DEGREE OF FREEDOM SYSTEM

h = 5 in.

Natural frequency (rad/sec)	Maximum response X_m (in) ($\zeta = 0.0$)	($\zeta = 0.2$)	($\zeta = 0.5$)	($\zeta = 0.8$)
2	31.00	23.49	16.97	13.17
10	6.21	4.7	3.39	2.63
20	3.1	2.35	1.7	1.32
30	2.07	1.57	1.13	0.98
40	1.55	1.17	0.85	0.66
50	1.24	0.94	0.68	0.53
60	1.03	0.78	0.56	0.44
70	0.89	0.67	0.48	0.38
80	0.78	0.59	0.42	0.33
90	0.69	0.52	0.37	0.29
100	0.62	0.47	0.34	0.26

TABLE 8

MAXIMUM RESPONSE FOR FREE FALL OF A SINGLE DEGREE OF FREEDOM SYSTEM

h = 10 in.

Natural frequency (rad/sec)	Maximum response X_m (in)			
	(ζ = 0.0)	(ζ = 0.2)	(ζ = 0.5)	(ζ = 0.8)
2	43.93	33.22	24.00	18.62
10	8.79	6.64	4.80	3.73
20	4.39	3.32	2.40	1.86
30	2.95	2.21	1.60	1.24
40	2.20	1.66	1.20	0.93
50	1.76	1.33	0.96	0.75
60	1.46	1.11	0.80	0.62
70	1.26	0.95	0.69	0.53
80	1.10	0.83	0.60	0.47
90	0.98	0.74	0.53	0.41
100	0.88	0.66	0.48	0.37

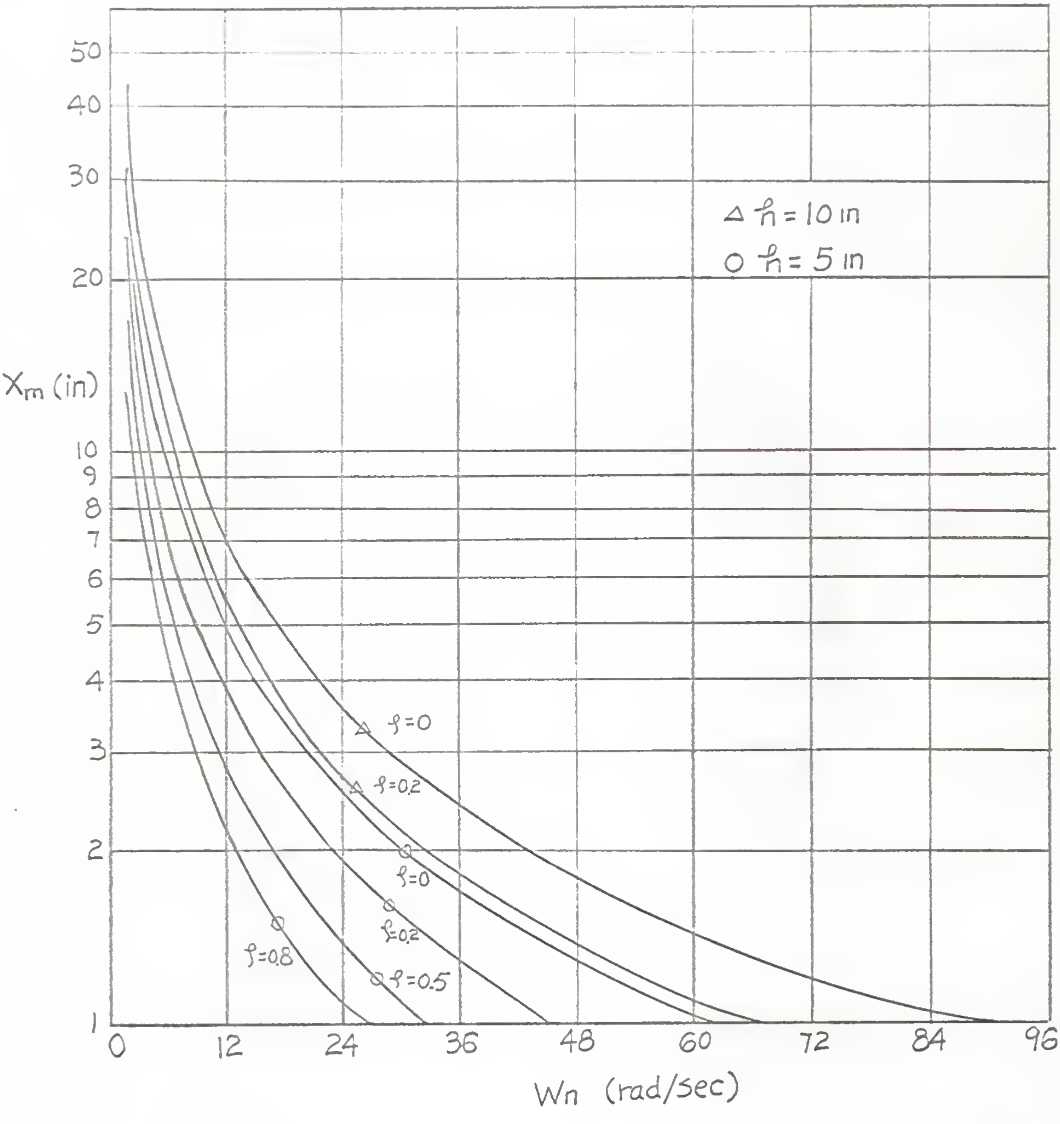


Fig. 13 Maximum Response for Free Fall.

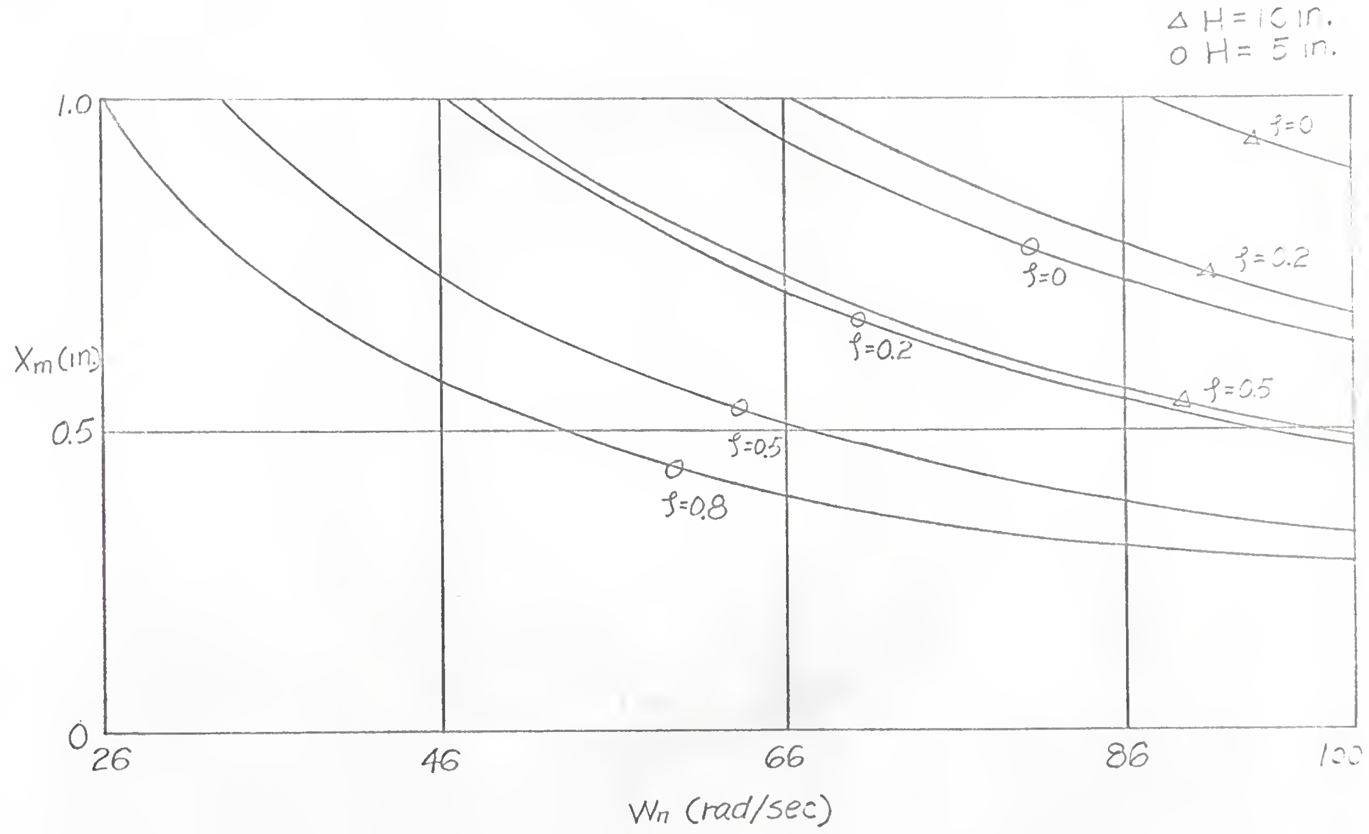


Fig. 14 Maximum Response for Free Fall.

PART II

EXPERIMENT

1. EXPERIMENTAL APPARATUS

(1) Single degree of freedom system: Using a cold rolled iron plate clamped on a rigid structure as a cantilever, the single degree of freedom system is made by setting the center of iron blocks on the free end of the cantilever which connects to the rod of a dashpot. The length of the cantilever, the weight of iron blocks and the dashpot are adjustable. The single degree of freedom is shown in Fig. 15.

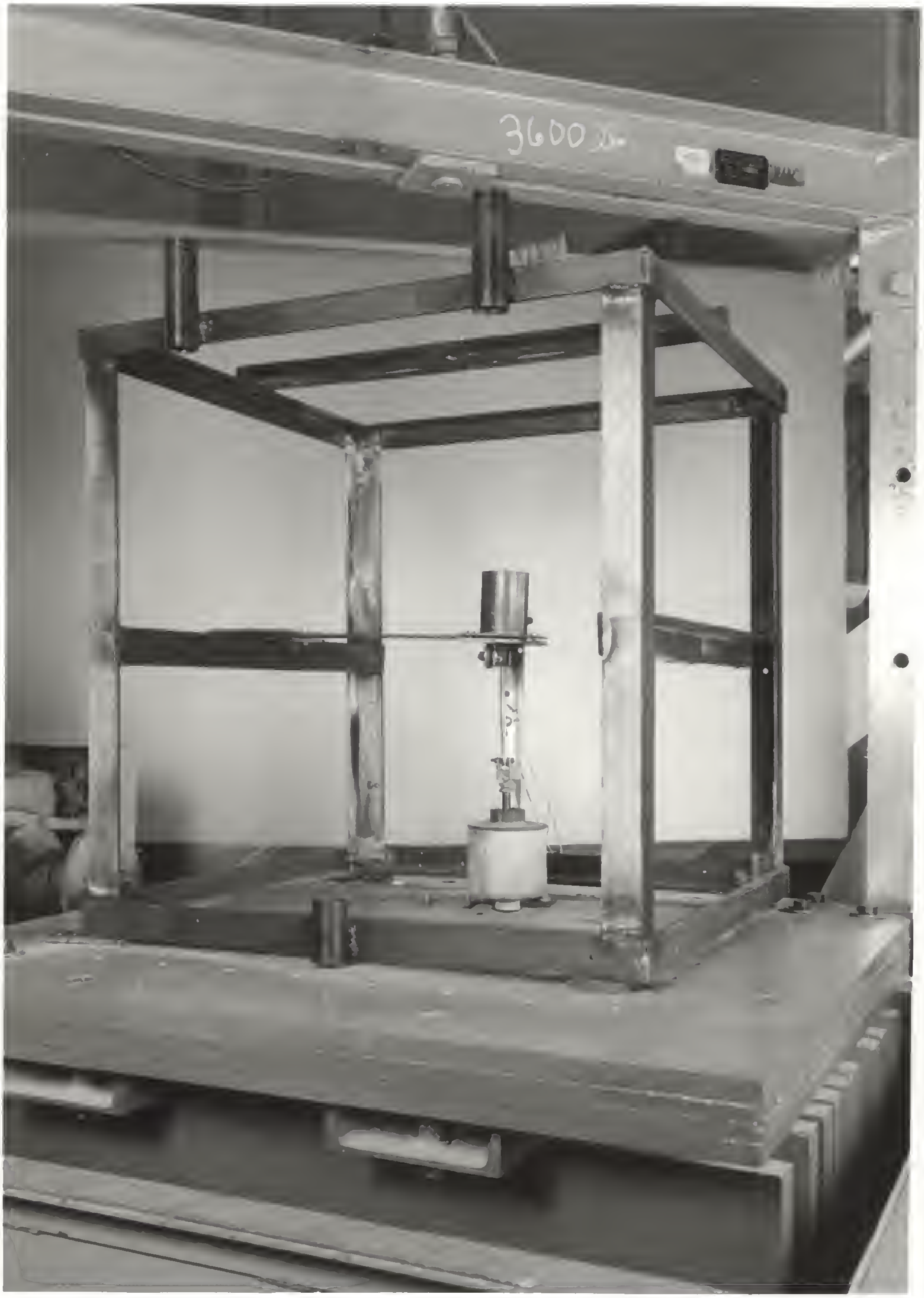
(2) Measuring instrument: A copper needle, fixed on the mass of the oscillator, is contacted with a high potential resistance wire fixed on the structure. As the mass vibrates relative to the structure, the base of the oscillator, the needle moves along the wire and changes the voltage recorded. The measuring system is shown in Fig. 16.

2. EXPERIMENTAL PROCEDURE

The experimental work is separated into five sections according to different types of excitation. Each section consists of the following items:

(1) Purpose: The aim of the experiment.

(2) Method: The method of applying the excitation to the oscillator is described and a table is given which gives the weight of the mass, the spring constant, the excitation force etc. in each test. Since the damping coefficient of the dashpot was difficult to determine, it is described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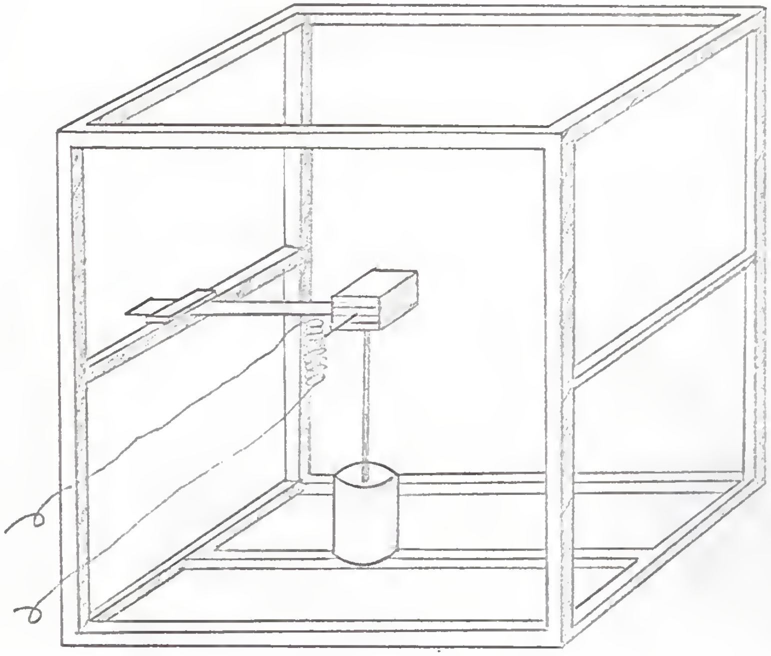

Fig. 15 Single Degree of Freedom System. 1/8" = 1'

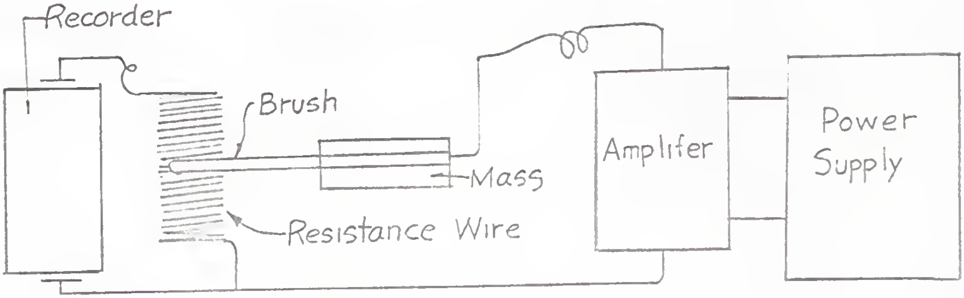

Fig. 16 Measuring System Diagram.

as 'Loose' or 'Tight' in 'Damping Consition' on the table. 'Loose' refers to the smallest damping coefficient of the dashpot obtained by adjusting the dashpot to its extreme loose position and 'Tight' refers to the greatest damping coefficient of the dashpot when the dashpot is adjusted to its extreme tight position.

(3) Record and Calculation: A table is given showing experimental results from the recorder and transferring them into the nondimensional maximum response.

(4) Comparison: The result of experiment and analytical work is compared in plots. The former is represented by dots and the latter is shown as a solid curve.

3. EXPERIMENT I. HARMONIC FORCE EXCITATION

(1) Purpose: To investigate the relation between the maximum response and the ratio of the harmonic forcing frequency to the natural frequency of the oscillate.

(2) Method: Attaching a small motor on the mass of the oscillator and fixing an eccentric mass of 0.5 oz. with an eccentricity 1.5 in. on the motor will give a harmonic force excitation to the mass when the motor turns. Table 9 gives the mass, spring constant and damping condition for each test.

(3) Record & Calculation: The amplititude of the harmonic excitation force is given by the equation

$$F = M_e r_e W^2 , \tag{73}$$

where

M_e : Eccentric mass

r_e : Eccentricity

TABLE 9

VARIOUS CONSTANTS USED FOR EXPERIMENT I

Test No.	Mass M (lb)	Spring constant K (lb/in)	Natural frequency W_n (rad/sec)	Motor frequency W (rad/sec)	Damping condition
1	1.14		48.6		
2	1.55		41.6		
3	1.83	7	38.2	18.84	Loose
4	2.11		35.7		
5	2.52		32.6		
6	3.03		30.0		
7	2.62		31.7		
8	2.34	7	33.8	18.84	Loose
9	2.06		36.1		
10	1.57		41.5		
11	1.57		28.6		
12	2.06		24.7		
13	2.34	3.31	23.4	18.8	Tight
14	2.62		22.2		
15	3.03		20.6		
16	2.52		22.6		
17	2.11		24.6		
18	1.83	3.31	26.4	18.8	Loose
19	1.55		28.8		
20	1.14		33.5		
21	1.14	1.8	24.7	18.8	Loose
22	1.55		21.1		
23	2.06	1.8	18.3	18.8	Tight
24	1.57		21.0		

Substituting $M_e = 0.5/16$, $r_e = 1.5$ and $W = 1.88$ into Equation (73) gives

$$F = 0.043 \text{ lb.}$$

Table 10 is given showing the experimental results from the recorder and transferring them into the nondimensional maximum response.

(4) Comparison: The result of experiment and analytical work is compared in Fig. 17. The former is represented by dots and the latter is shown as a solid curve.

4. EXPERIMENT II. A SINUSOIDIAL DISPLACEMENT OF THE SYSTEM'S BASE

(1) Purpose: To find the relation between the maximum response and the ratio of W to natural frequency by giving a displacement to the base of the oscillator.

(2) Method: The whole oscillator was hung by a cable which was connected to an eccentric cam rotated by a motor. The base would move up and down since the cable was fixed to it. The maximum displacement can be investigated and measured from the chart record. Table 11 gives the weight of the mass, spring constant and damping for each test.

(3) Record & Calculation: The amplitude of equivalent excitation force is given by the equation

$$F = -MY_oW^2 , \tag{74}$$

where

Y_o : Maximum displacement in inch (downward positive).

Substituting $Y_o = -0.5$, $W = 22.3$ into Equation (74) gives

$$F = -M(0.5)(22.3)^2 .$$

Table 12 is given showing the experimental results and transferring them

TABLE 10

EXPERIMENTAL RESULTS FOR HARMONIC FORCE EXCITATION

Test No.	Frequency ratio $r = (W/W_n)$	Damping factor	$\frac{K}{k^2F}$	Maximum response X_m(in.)	$\frac{K}{k^2F} X_m$
1	0.39	0.07	14.8	0.07	1.04
2	0.45	0.08	14.2	0.07	1.0
3	0.495	0.1	11.4	0.07	0.8
4	0.529	0.1	10.4	0.07	0.73
5	0.58	0.09	8.12	0.08	0.65
6	0.63	0.22	8.2	0.06	0.50
7	0.595	0.22	9.4	0.08	0.75
8	0.56	0.16	10.6	0.07	0.745
9	0.525	0.15	11.3	0.07	0.79
10	0.455	0.13	13.2	0.06	0.79
11	0.66	0.207	3.6	0.14	0.5
12	0.765	0.166	2.2	0.21	0.47
13	0.80	0.18	2.0	0.47	0.45
14	0.85	0.145	1.38	0.25	0.35
15	0.915	0.13	1.15	0.245	0.28
16	0.835	0.17	1.63	0.19	0.31
17	0.765	0.08	1.76	0.33	0.6
18	0.715	0.06	2.4	0.28	0.67
19	0.655	0.162	2.55	0.23	0.6
20	0.562	0.07	5.07	0.16	0.83
21	0.765	0.11	1.06	0.47	0.5
22	0.895	0.16	1.71	0.17	0.29
23	1.03	0.154	1.65	0.28	0.36
24	0.90	0.147	1.38	0.23	0.31

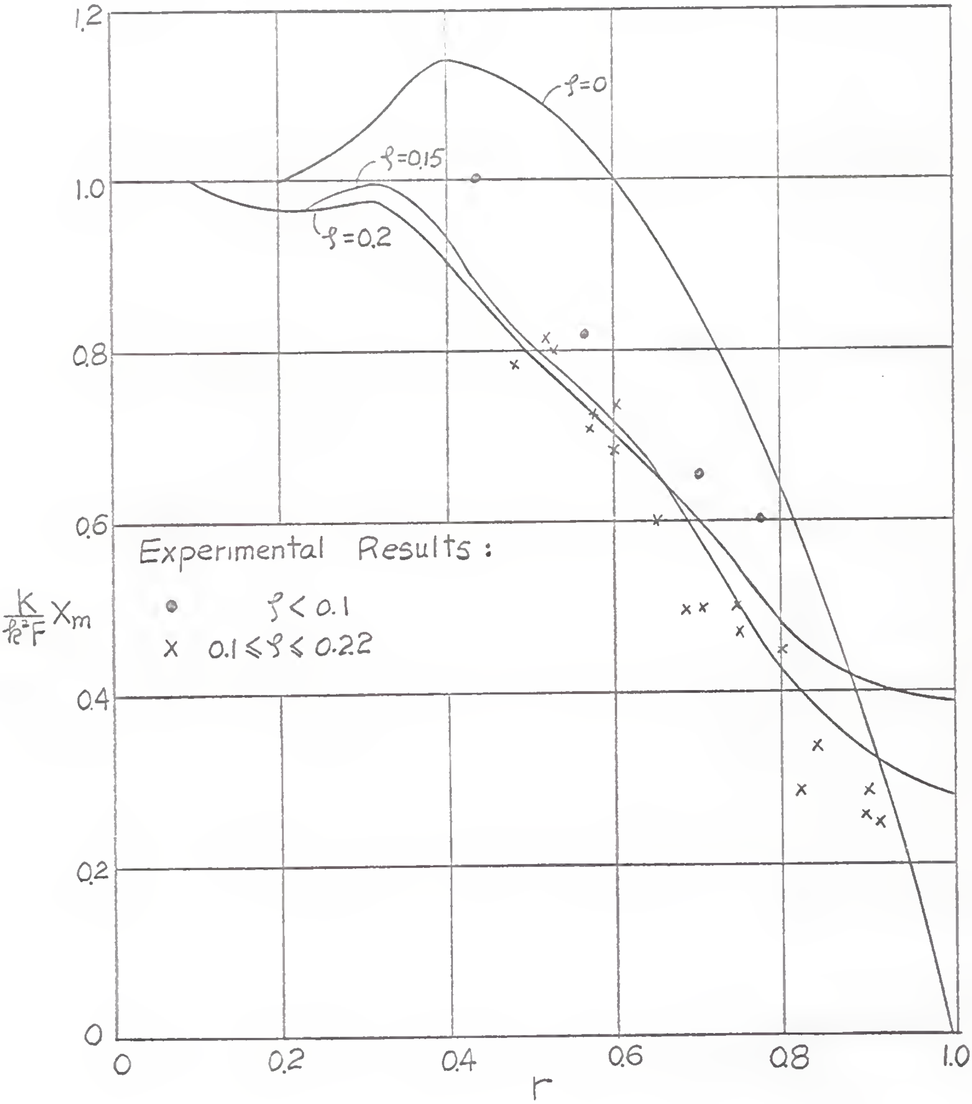

Fig. 17 Plot of Nondimensional Maximum Response to Frequency Ratio by Harmonis Force Excitation.

TABLE 11

VARIOUS CONSTANTS USED FOR EXPERIMENT II

Test No.	Mass M (lb)	Spring constant K (lb/in)	Natural frequency W_n (rad/sec)	Damping condition	Y_o (in)	W (rad/sec)
1	1.14		48.6			
2	1.55		41.6			
3	1.83	7	38.2	Loose	0.5	22.3
4	2.11		35.7			
5	2.52		32.6			
6	3.03		30.0			
7	2.62		31.7			
8	2.34	7	33.8	Tight	0.5	22.3
9	2.06		36.1			
10	1.57		41.5			
11	1.57		28.6			
12	2.06		24.7			
13	3.34	3.31	23.4	Tight	0.5	22.3
14	2.62		22.2			
15	3.03		20.6			
16	2.52		22.6			
17	2.11		24.6			
18	1.83	3.31	26.4	Loose	0.5	22.3
19	1.55		28.8			
20	1.14		33.5			
21	1.14		24.7			
22	1.55	1.8	21.1	Loose	0.5	22.3
23	1.55		18.3			
24	1.14	1.8	21.0	Tight	0.5	22.3

TABLE 12
EXPERIMENTAL RESULTS FOR A SINUSOIDIAL DISPLACEMENT OF THE SYSTEM'S BASE

Test No.	Frequency ratio $r = (W/W_n)$	Damping factor	F (lb)	$\frac{K}{k^2 F}$	Maximum displacement X_m (in.)	$\frac{K}{k^2 F} X_m$
1	0.39	0.01	0.9	5.9	0.18	1.08
2	0.45	0.09	1.0	4.9	0.21	1.05
3	0.495	0.11	1.17	3.42	0.22	0.78
4	0.53	0.1	1.36	2.7	0.28	0.76
5	0.58	0.1	1.62	1.74	0.4	0.70
6	0.63	0.22	1.95	1.47	0.36	0.52
7	0.60	0.20	1.69	1.95	0.35	0.70
8	0.56	0.16	1.51	2.45	0.32	0.79
9	0.525	0.15	1.33	3.25	0.25	0.82
10	0.455	0.13	1.0	4.62	0.2	0.93
11	0.765	0.2	1.0	1.25	0.41	0.51
12	0.765	0.17	1.33	0.57	0.97	0.53
13	0.80	0.08	1.51	0.47	0.95	0.45
14	0.85	0.15	1.69	0.28	1.25	0.35
15	0.915	0.13	1.95	0.20	1.5	0.30
16	0.835	0.17	1.62	0.35	0.94	0.33
17	0.765	0.08	1.36	0.445	1.35	0.6
18	0.715	0.07	1.17	0.71	0.96	0.68
19	0.655	0.16	1.0	0.88	0.91	0.80
20	0.562	0.07	0.9	1.96	0.25	0.5
21	0.765	0.11	0.9	0.47	0.85	0.40
22	0.895	0.16	1.0	0.59	0.51	0.30
23	1.03	0.15	1.0	0.57	0.68	0.39
24	0.90	0.15	0.9	0.53	0.55	0.29

into nondimensional maximum response.

(4) Comparison: The result of experiment and analytical work is compared in Fig. 18. The former is represented by dots and the latter is shown as a solid curve.

5. EXPERIMENT III. STEP FUNCTION EXCITATION

(1) Purpose: To find the relation between maximum displacement and various damping factors by step function excitation.

(2) Method: The mass of the single degree of freedom system is given a displacement of X_o from its equlibrium position and released with zero initial velocity. The first peak response measured from the released position will be the maximum displacement. The damping factor is changed either by adjusting the dashpot or by varying the mass and spring constant. Table 13 which shows the mass, spring constant and dashpot.

(3) Record & Calculation: The maximum responses in chart unit from the recorder are shown in Table 14. The force of step function is given by

$$P = \frac{X_o}{C} K ,$$

where

X_o : Displacement

C : One inch in chart unit

K : Spring constant

Table 15 is given showing the nondimensional maximum response.

(4) Comparison: The result of experiment and analytical work is compared in Fig. 19. The former is represented by dots and the latter is shown as a solid curve.

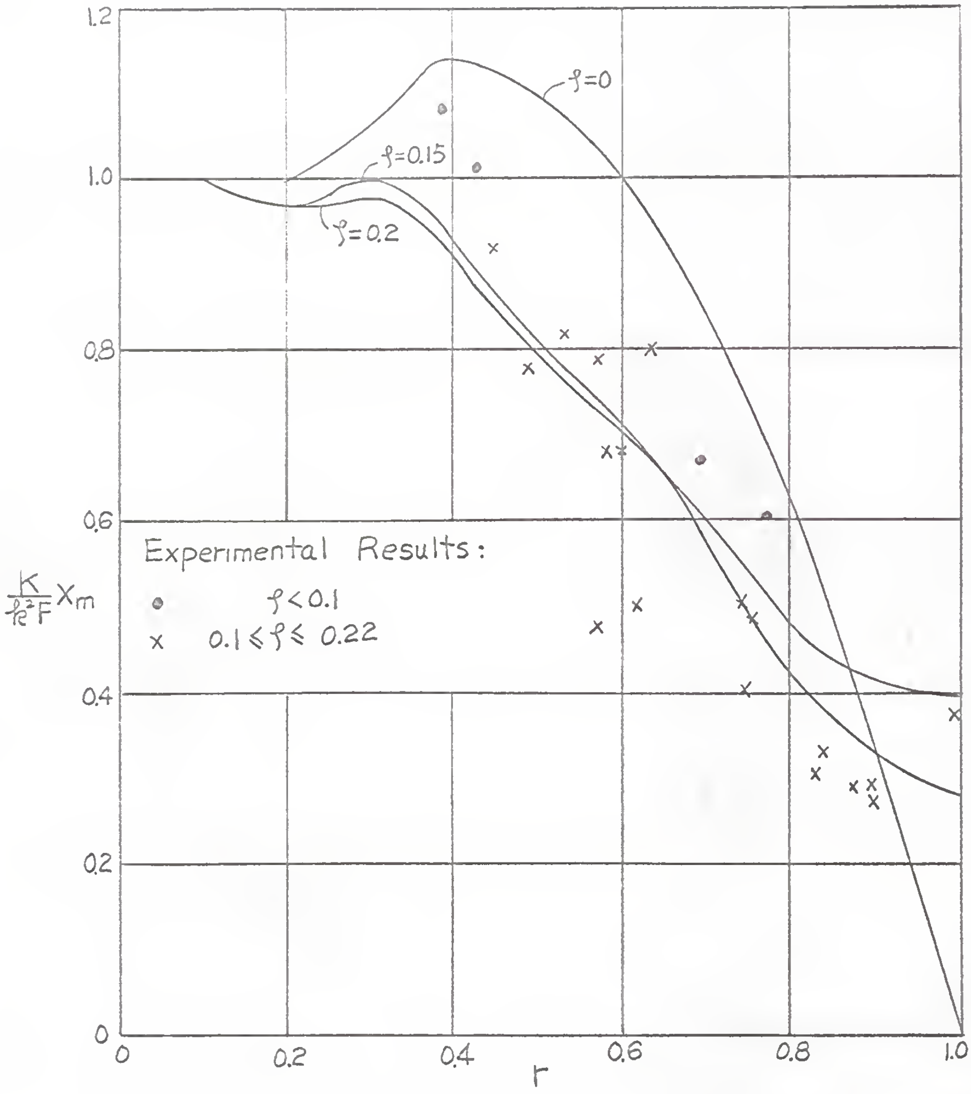

Fig. 18 Plot of Nondimensional Maximum Response to Frequency Ratio due to a Sinsoidal Displacement to the Base.

TABLE 13

VARIOUS CONSTANTS USED FOR EXPERIMENT III

Test No.	Mass (lb)	Spring constant (lb/in)	Dashpot adjusted position
1	W1 = 1.25	7	Loose
2	W2 = 1.74		
3	W3 = 2.02		
4	W4 = 2.3		
5	W5 = 2.71		
6	W5		Tight
7	W4		
8	W3		
9	W2		
10	W1		
11	W2	3.31	Tight
12	W3		
13	W4		
14	W5		
15	W5		Loose
16	W4		
17	W3		
18	W2		
19	W3	1.81	Loose
20	W4		
21	W5		
22	W5		Tight
23	W4		
24	W3		

TABLE 14

EXPERIMENTAL RESULTS IN CHART UNIT FOR STEP FUNCTION EXCITATION

Test No.	Maximum response X_m (Chart unit)	X_o (Chart unit)	$\delta = \frac{X_o}{X_m - X_o}$
1	13.5		2.86
2	14.2		2.38
3	15.25		1.92
4	15.4		1.85
5	15.2		1.92
6	15.2		1.92
7	13.9		2.57
8	13.4		2.94
9	12.8		3.56
10	11.6	10	6.25
11	15.5		1.82
12	15.9		1.7
13	17.8		1.82
14	15.9		1.7
15	17.8		1.28
16	16		1.66
17	16		1.66
18	16		1.66
19	18.1		1.23
20	18.2		1.23
21	12	7	1.4
22	12.5	8	1.77
23	17.2	10	1.39
24	16.4	10	1.56

TABLE 15

EXPERIMENTAL RESULTS IN NONDIMENSIONAL FORM FOR STEP FUNCTION EXCITATION

Test No.	P (lb)	$\frac{K}{P}X_m$	$\rho = e^{\pi\delta}$
1		1.35	0.33
2		1.42	0.27
3		1.52	0.2
4		1.54	0.19
5	$P = \frac{10}{17} \times 7$	1.52	0.2
6	$= 4.1$	1.52	0.2
7		1.39	0.31
8		1.34	0.34
9		1.28	0.4
10		1.16	0.58
11		1.55	0.19
12		1.59	0.16
13		1.55	0.18
14		1.59	0.19
15	$P = \frac{10}{20} \times 3.31$	1.78	0.08
16	$= 1.65$	1.6	0.16
17		1.6	0.16
18		1.6	0.16
19		1.81	0.04
20		1.82	0.04
21	0.7	1.73	0.107
22	0.8	1.53	0.18
23	4.1	1.72	0.1
24		1.64	0.14

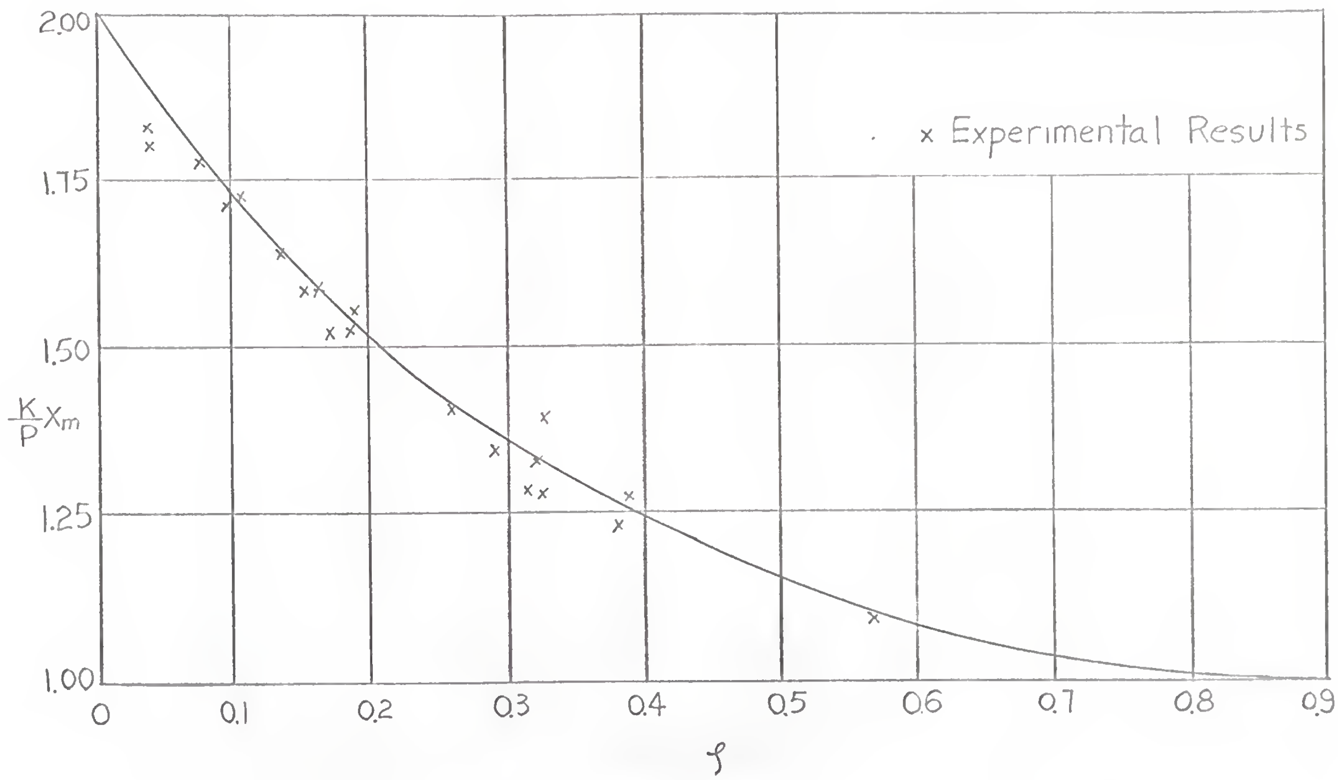

Fig. 19 Shock Spectrum of Step Function.

6. EXPERIMENT IV. DELTA FUNCTION EXCITATION

(1) Purpose: To find the relation between the maximum displacement and the damping factor for a single degree of freedom system excitated by delta function excitation.

(2) Method: Attach a velocity pickup on the mass of the oscillator and strike the mass with a hammer. The first peak response and initial velocity can be measured by the chart. The damping factor can be calculated by comparing the first positive peak response and the first negative peak response. Table 16 shows the conditions for each test.

(3) Record & Calculation: Maximum responses in chart unit from the recorder are shown in Table 17. Table 18 is given showing nondimensional maximum response.

(4) Comparison: The result of experiment and analytical work is compared in Fig. 20. The former is represented by dots and the latter is shown as a solid curve.

7. EXPERIMENT V. RECTANGULAR PULSE EXCITATION

(1) Purpose: To find the relation between maximum response and natural frequency and the rectangular pulse interval.

(2) Method: Fix the mass on its equilibrium position and add a iron block to the mass by an electromagnet. Release the mass with initial zero velocity and drop the iron bolck by turning off the electricity after a time interval t_1. The first peak response measured after t_1 will be the maximum response for a rectangular pulse excitation. The damping factor can be determined by comparing the first two opposite peak response. Table 19 gives the mass, spring constant and damping for each test.

TABLE 16

VARIOUS CONSTANTS USED FOR EXPERIMENT IV

Test No.	Mass M (lb)	Spring constant K (lb/in)	Dashpot adjusted position
1	W0 = 1.58		Loose
2	W1 = 2		
3	W1	1.8	Tight
4	W0		
5	W1		
6	W2 = 2.41		
7	W3 = 2.69		Tight
8	W4 = 3.07		
9	W5 = 3.48		
10	W5	3.31	
11	W4		
12	W3		Loose
13	W2		
14	W1		
15	W1		
16	W2		
17	W3		Loose
18	W4		
19	W5		
20	W5	7	
21	W4		
22	W3		Tight
23	W2		
24	W1		

TABLE 17
EXPERIMENTAL RESULTS IN CHART UNIT FOR DELTA FUNCTION EXCITATION

Test No.	Maximum response X_m (Chart unit)	Initial velocity V (Chart unit)		1 Chart unit in.	1 Chart unit in/sec
1	5.5	14	1.1		
2	2.5	6	1.25		
3	2.0	7	2.0		1.8
4	3.0	9	1.7	0.2	
5	3.3	12	1.65		
6	3.2	10	1.6		
7	4.0	12	1.33		
8	4.5	6	1.5		
9	5.0	8	1.43		
10	6.0	8	1.25		
11	5.1	7	1.37		
12	8.0	12	1.48		
13	9.0	14	1.5		
14	8.5	13	1.29		
15	6.5	15	1.18		
16	7.0	14	1.16	0.1	2
17	9.0	20	1.6		
18	9.7	21	1.62		
19	8.0	13	1.23		
20	4.0	10	2.25		
21	5.0	11	1.67		
22	6.5	17	2.1		
23	6.7	19	1.86		
24	6.7	20	2.16		

TABLE 18

EXPERIMENTAL RESULTS IN NONDIMENSIONAL FORM FOR DELTA FUNCTION EXCITATION

Test No.	Maximum response X_m (inch)	Initial velocity V (in/sec)	$\sqrt{K/M}$ (rad/sec)	$\frac{\sqrt{KM}}{MV} X_m$	Damping factor $\gamma = e^{\pi\delta}$
1	1.1	24.1	21	0.95	0.01
2	0.5	10.8	18.7	0.86	0.07
3	0.4	12.6	18.7	0.60	0.22
4	0.67	38	18.7	0.59	0.2
5	0.66	24	25.1	0.69	0.158
6	0.64	20.3	23	0.705	0.15
7	0.8	24	21.8	0.725	0.09
8	0.45	12	20.4	0.765	0.129
9	0.5	16	19.1	0.60	0.11
10	0.6	16	19.1	0.72	0.07
11	0.51	14	20.4	0.745	0.1
12	0.77	24	21.8	0.70	0.125
13	0.9	28	23	0.74	0.129
14	0.85	26	25.1	0.82	0.083
15	0.65	30	36.7	0.79	0.052
16	0.7	28	33.5	0.83	0.047
17	0.9	40	31.6	0.71	0.15
18	0.97	42	29.6	0.72	0.153
19	0.8	26	28	0.86	0.066
20	0.4	20	28	0.56	0.25
21	0.5	22	29.6	0.67	0.162
22	0.65	34	31.6	0.59	0.235
23	0.67	38	33.5	0.59	0.2
24	0.67	40	36.7	0.61	0.245

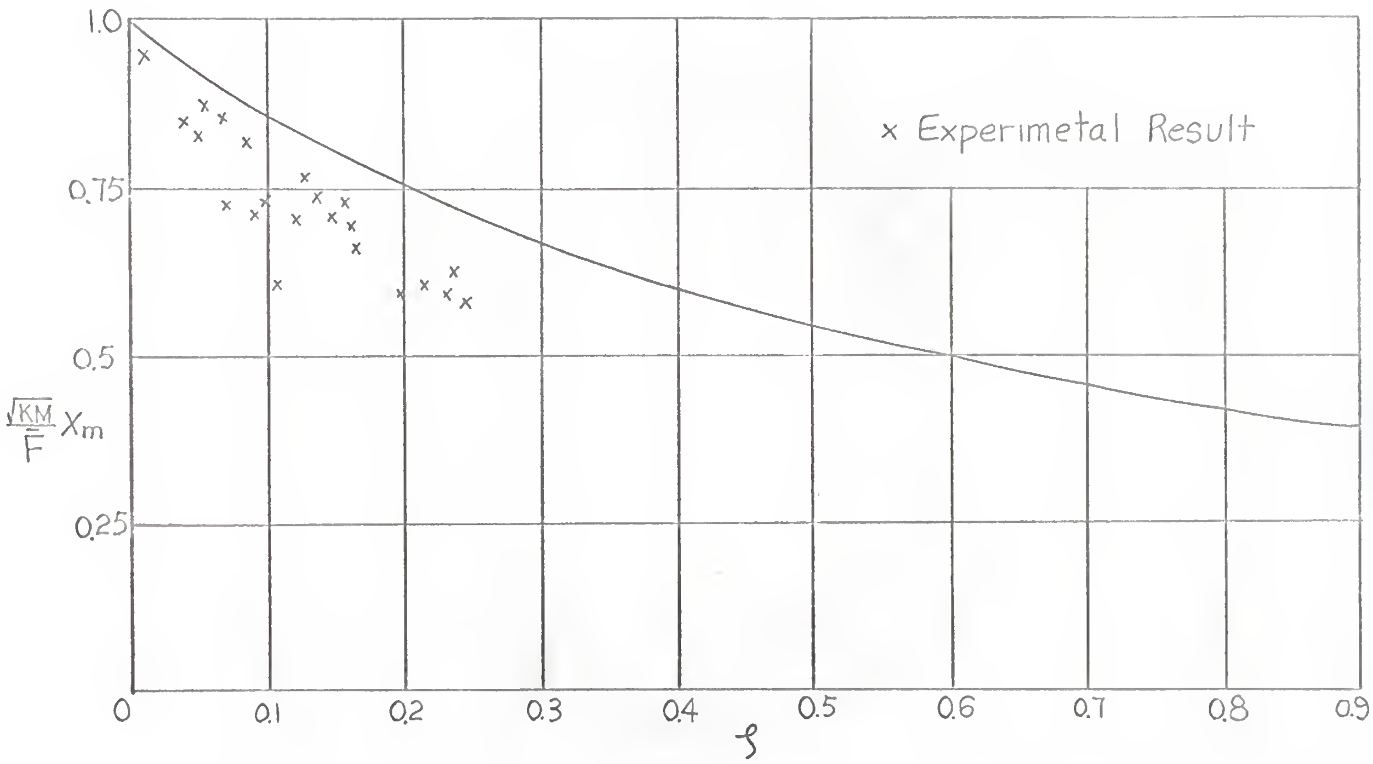

Fig.20 Shock Spectrum of Delta Function.

TABLE 19

VARIOUS CONSTANTS USED FOR EXPERIMENT V

Test No.	Mass M (lb)	Spring constant K (lb/in)	Rectangular pulse P (lb)	Damping
1				
	1.14	3.31	0.8	No
21				
22				
23	1.77			
24				
25				
26	2.06	3.31	0.67	Loose
27				
28	2.34			
29				
30	2.62			
31				
32	3.03			
33				
34	3.03			
35		3.31	0.67	Tight
36	2.62			
37				

TABLE 19 (Cont'd)

Test No.	Mass M (lb)	Spring constant K (lb/in)	Rectangular pulse P (lb)	Damping
38				
39	2.34			
40				
41		3.31	0.67	Tight
	2.06			
42				
43				
	1.77			
44				

(3) Record & Calculation: Table 20 and Table 21 show the experimental results from the recorder and transferring them into nondimensional maximum response.

(4) Comparison: The result of experiment and analytical work is compared in Fig. 21 and Fig. 22. The former is represented by dots and the latter is shown as a solid curve.

8. EXPERIMENT VI. FREE FALL

(1) Purpose: To find the relation between the maximum response and natural frequency by free fall of the oscillator from a certain height h.

(2) Method: By using a drop machine we let the whole structure free fall from heights of 5 in. and 10 in. The first peak response is the maximum displacement of the mass. Table 22 gives the weight of the mass, spring constant and damping for each test.

(3) Record & Calculation: Table 23 and Table 24 show the maximum responses for free fall from heights 10 inches and 5 inches, respectively.

(4) Comparison: The result of experiment and analytical work is compared in Fig. 23 and Fig. 24. The former is represented by dots and the latter is shown as a solid curve.

TABLE 20

EXPERIMENTAL RESULTS FOR RECTANGULAR PULSE EXCITATION WITHOUT DAMPING

Test No.	Maximum response X_m (inch)	Rectangular pulse interval t_1 (sec)	Natural frequency W_n (rad/sec)	$\frac{K}{P}X_m$	$W_n t_1$
1	0.46	1.98		1.91	66.1
2	0.35	2.2		1.45	73.3
3	0.34	2.0		1.4	67.3
4	0.05	2.1		0.24	69.1
5	0.32	1.93		1.32	64.9
6	0.33	2.1		1.36	70.9
7	0.31	2.2		1.27	73.6
8	0.25	2.04		1.03	68.2
9	0.44	2.15		1.8	72.2
10	0.21	2.04	33.5	0.865	68.3
11	0.31	2.2		1.29	70.9
12	0.115	2.18		0.475	73.4
13	0.105	2.1		0.45	70.4
14	0.13	2.21		0.535	74.9
15	0.34	1.72		1.4	67.5
16	0.34	2.2		1.4	73.9
17	0.12	2.21		0.495	74.9
18	0.45	2.17		1.81	72.7
19	0.42	2.13		1.43	71.5
20	0.29	1.98		1.2	64.6
21	0.405	1.97		1.71	66

TABLE 21

EXPERIMENTAL RESULTS FOR RECTANGULAR PULSE EXCITATION WITH DAMPING

Test No.	Natural frequency W_n (rad/sec)	Rectangular pulse interval t_1 (Sec)	$W_n t_1$	Maximum response X_m (Chart unit)	$\frac{K}{P} X_m$	δ	Damping factor $\zeta = e^{\pi\delta}$
22		0.94	25.4	1.4	0.99	1.6	0.15
23	27	0.8	21	1.36	0.96	1.57	0.145
24		1.43	38.5	1.51	1.0	2.0	0.22
25		1.0	25	1.39	0.95	1.64	0.157
26	25	1.02	26.2	1.4	0.99	1.5	0.129
27		0.84	21	1.1	0.78	2.08	0.23
28	23.4	1.1	25.8	1.3	0.92	2.0	0.22
29		1.02	23.9	1.53	1.07	2.0	0.22
30	21.9	1.0	21.9	1.6	1.11	1.7	0.16
31		1.31	28.7	1.2	0.84	2.0	0.22
32	20.6	1.27	25.5	1.7	1.19	1.6	0.153
33		1.3	26.8	1.31	0.93	2.0	0.22
34	20.6	1.1	21.6	1.3	0.9	2.0	0.22
35		1.3	24.8	1.2	0.85	4.0	0.44
36	21.9	1.05	23	1.3	0.9	1.7	0.17
37		1.0	21.9	1.5	1.06	2.0	0.22
38		0.94	22	1.4	0.99	3.6	0.41
39	23.4	1.11	26.0	1.5	1.06	2.0	0.22
40		1.2	29.0	1.37	0.98	2.0	0.22
41	25	0.9	22.5	2.1	1.48	1.5	0.129
42		1.1	27.5	1.2	0.85	2.0	0.22
43	27	1.05	28.1	1.37	0.97	1.3	0.08
44		0.9	24.3	1.22	0.86	2.0	0.22

A). From Test 1 to Test 21 without Damping.

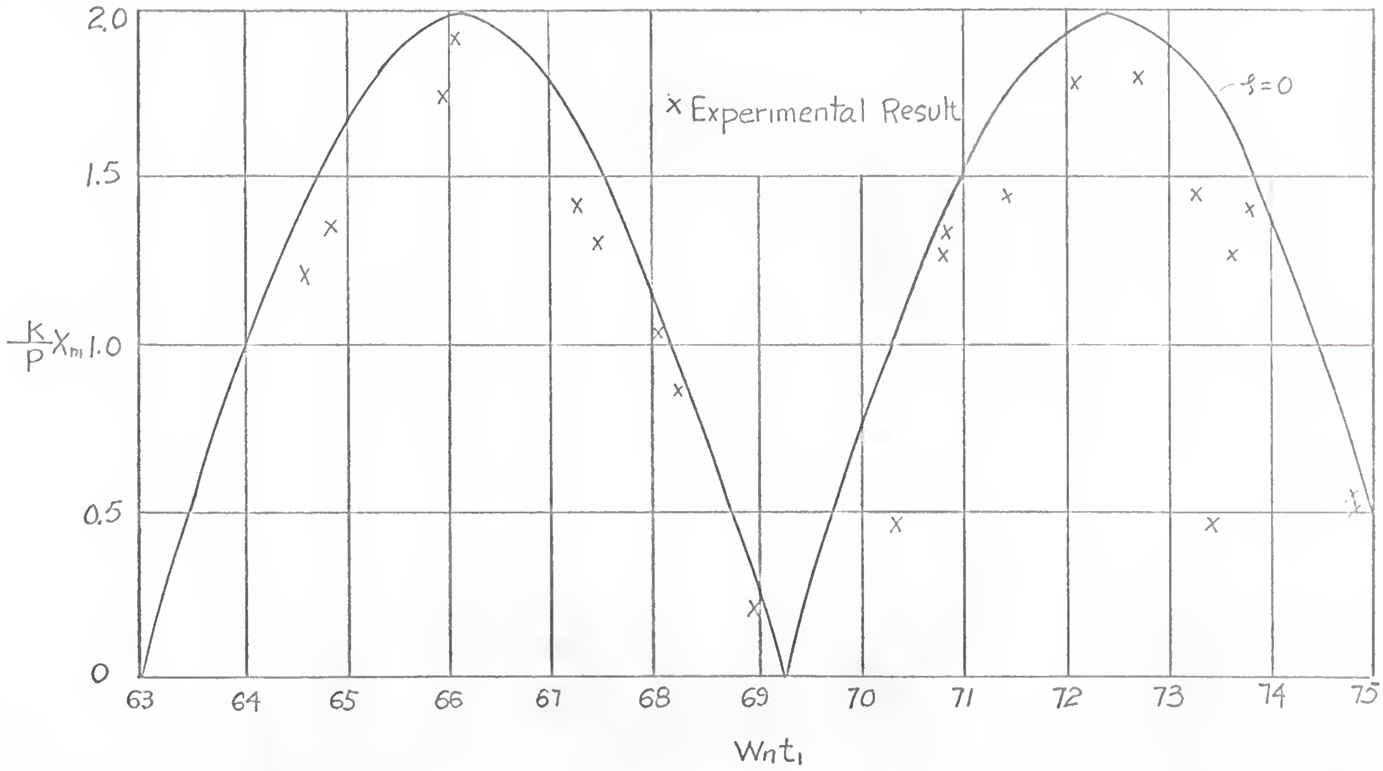

Fig 21 Shock Spectrum of Rectangular Pulse with t_1 as Parameter.

B). From Test 22 to Test 44.

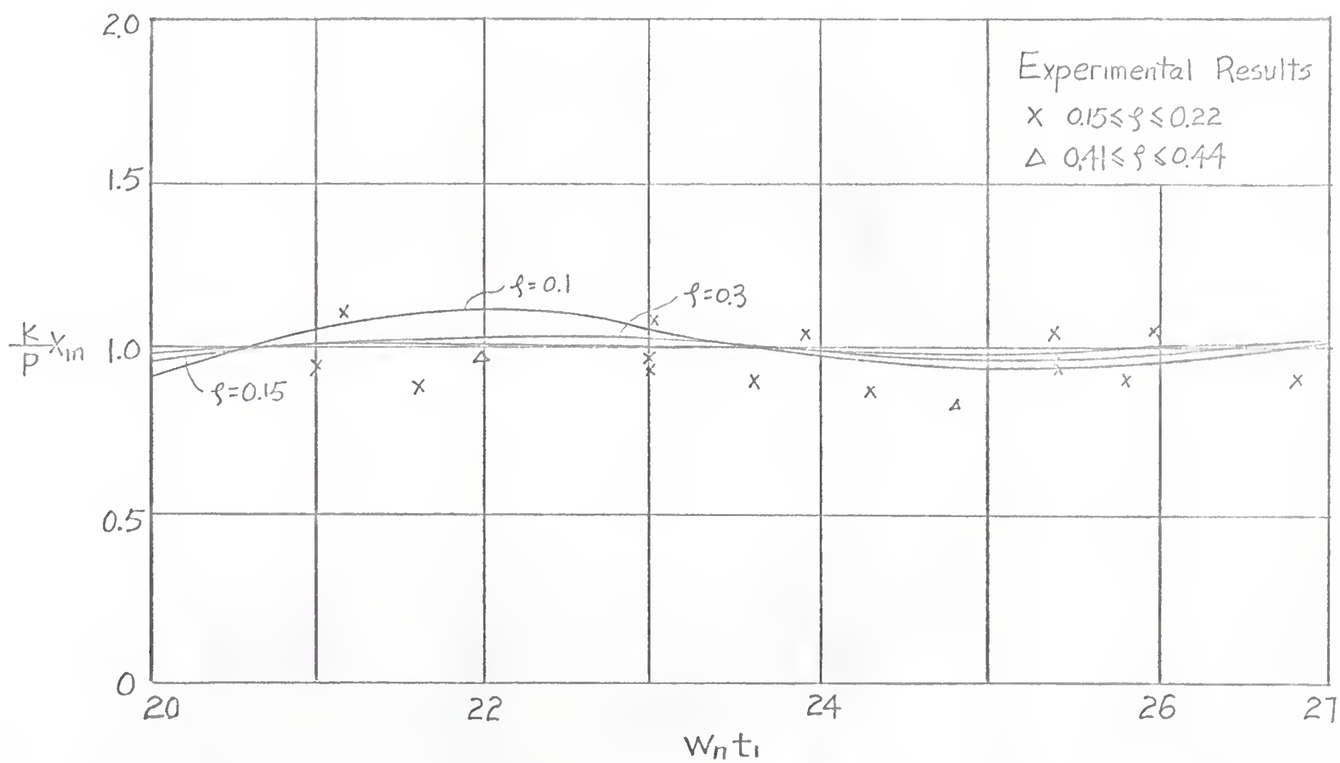

Fig.22 Shock Spectrum of Rectangular Pulse with t_1 as Parameter.

TABLE 22

VARIOUS CONSTANTS USED FOR EXPERIMENT VI

Test No.	Mass (lb)	Spring constant (lb/in)	Height (in)	Damping condition
1	2.28	40	10	No damping
2	1.87			
3	0.82			
4	2.28		5	
5	1.87			
6	0.82			
7	1.25	40	10	Loose
8	2.3			
9	2.71			
10	1.25		5	
11	2.3			
12	2.71			
13	2.28	20.5	10	No damping
14	1.87			
15	0.82			
16	2.28		5	
17	1.87			
18	0.82			

TABLE 22 (Cont'd)

Test No.	Mass (lb)	Spring constant (lb/in)	Height (in)	Damping condition
19	1.25	20.5	10	Loose
20	2.3			
21	2.71			
22	1.25		5	
23	2.3			
24	2.71			
25	2.71	14	10	Loose
26	2.3			
27	1.25			
28	2.71		5	
29	2.3			
30	1.25			
31	0.82	14	10	No damping
32	1.87			
33	2.28			
34	0.82		5	
35	1.87			
36	2.28			

TABLE 23

EXPERIMENTAL RESULT FOR FREE FALL

$h = 10$ in.

Test No.	Natural frequency W_n (rad/sec)	Maximum response X_m (in.)	Damping factor ζ
1	82.5	0.835	0.0
2	90.7	0.725	0.0
3	120	0.615	0.0
7	111	0.44	0.14
8	82	1.0	0.136
9	75.5	1.08	0.143
13	58.8	1.12	0.0
14	65	1.08	0.0
15	98	0.915	0.0
19	72.5	1.0	0.175
20	58.6	1.0	0.169
21	54.4	1.08	0.17
25	44.5	1.6	0.22
26	48.3	1.45	0.225
27	65.7	1.1	0.20
31	81	1.07	0.0
32	53.5	1.55	0.0

TABLE 24

EXPERIMENTAL RESULT FOR FREE FALL

h = 5 in.

Test No.	Natural frequency W_n (rad/sec)	Maximum response X_m (in.)	Damping factor ζ
4	82.5	0.532	0.0
5	90.7	0.425	0.0
6	124	0.30	0.0
10	111	0.33	0.14
11	82	0.5	0.17
12	75.5	0.67	0.13
16	58.8	0.94	0.0
17	65	0.833	0.0
18	98	0.46	0.0
22	72.5	0.625	0.207
23	58.6	0.79	0.163
24	54.4	0.963	0.157
28	44.5	1.14	0.155
29	48.3	1.04	0.2
30	65.7	0.79	0.171
34	81	0.73	0.0
35	53.5	1.3	0.0

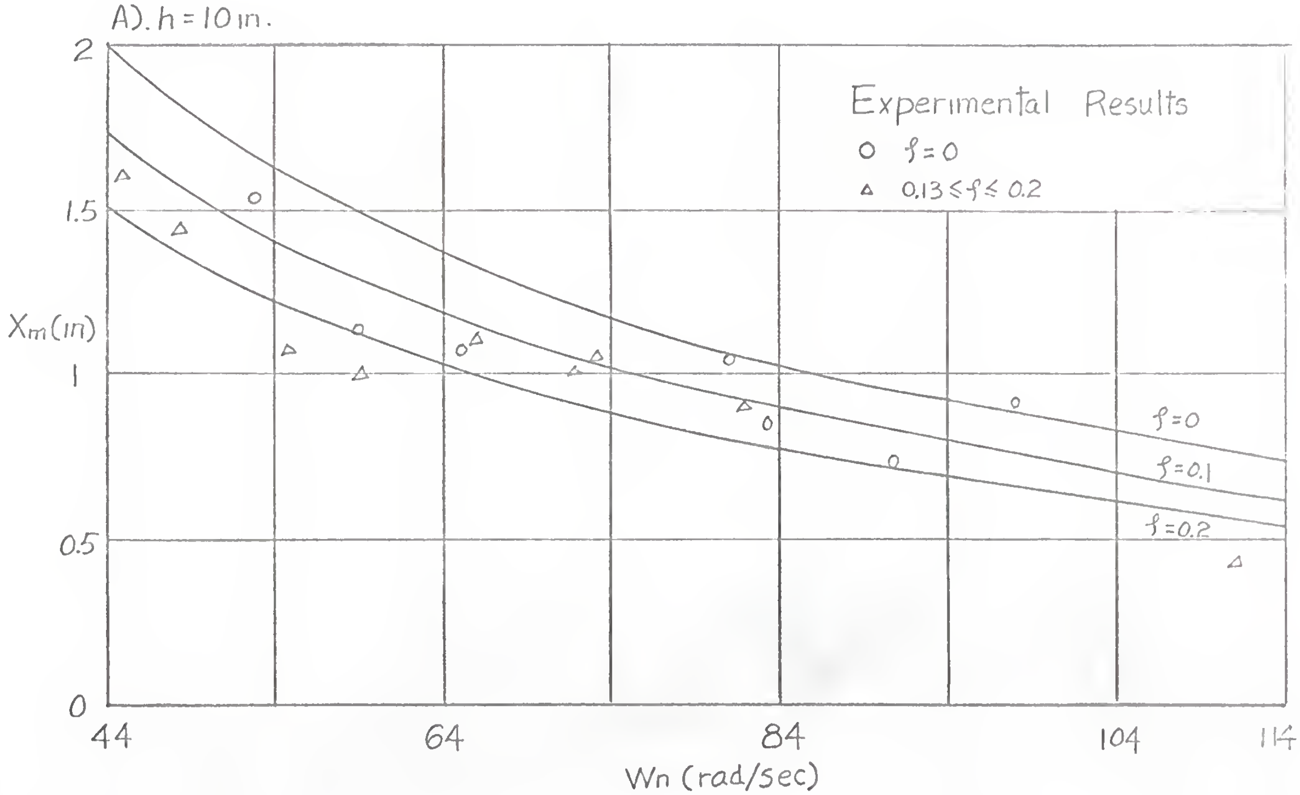

Fig. 23 Shock Spectrum of Free Fall.

B). $h = 5$ in.

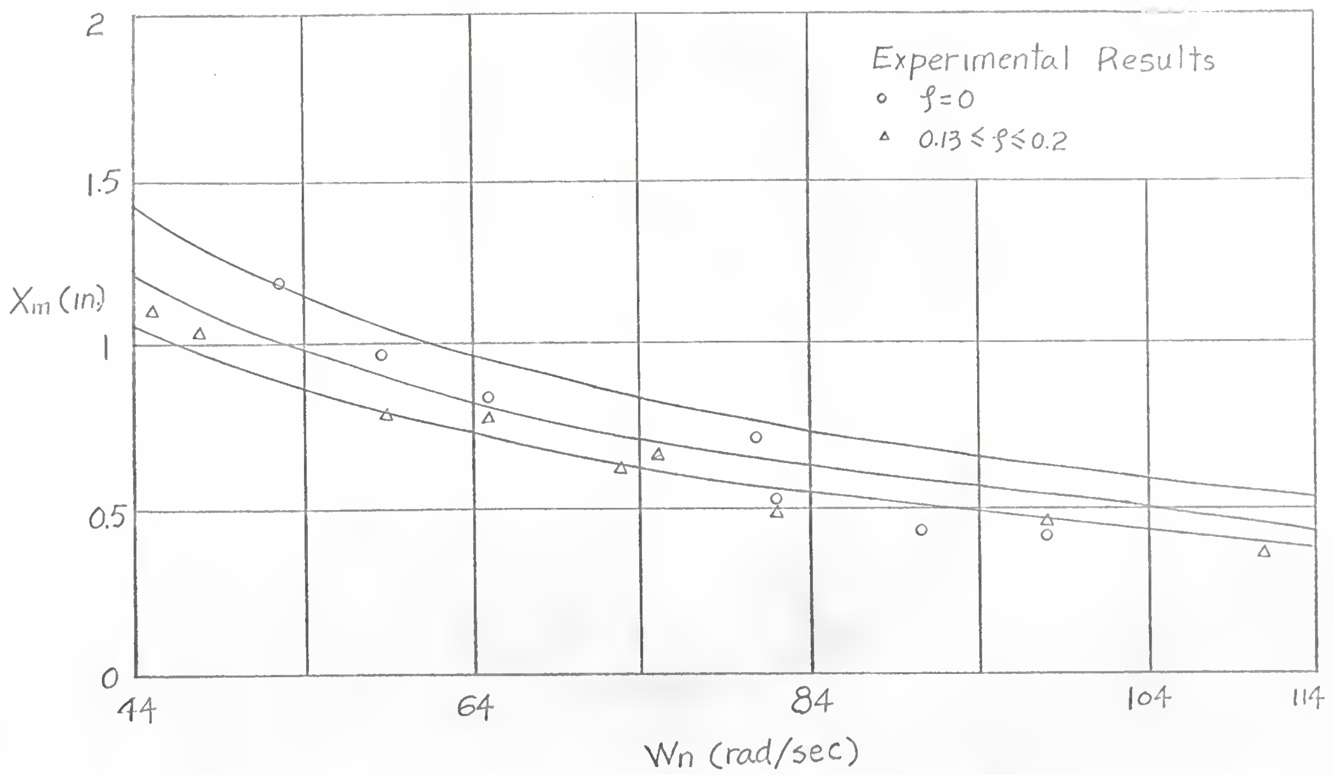

Fig. 24 Shock Spectrum of Free Fall.

DISCUSSION

Throughout the previous work, the experimental results seemed to support the analytical work. But there were some spurious factors which influenced the outcome of the experiment. Hence, the difference between the result of experiment and analysis was inevitable. These spurious factors were:

(1) The weight of the cantilever in analysis was neglected while in the experiment the ratio of the weight for beam and mass was extended to 0.25.

(2) The beam was bent when it was in the experiment of delta function and in the free fall test though a correction factor for this was applied in each test.

(3) The drag friction of the dashpot was obvious when the displacement of vibration became large. The connecting rod of the dashpot deviated from its perpendicular position when the beam vibrated in an arc position.

(4) The displacement of vibration on the chart was small and it was hard to determine its exact magnititude. For higher sensitivity the current noise influenced the result to a larger degree. Therefore the calculation of response and damping factors was influenced.

(5) The influences of the starting inertia of the motor for harmonic force excitation and the delay of release on the hook for free fall would reduce the effect of the excitation.

Curves showing theoretical nondimensional maximum response to the frequency ratio for harmonic force excitation and for a sinusoidal displacement to the base are given in Figs. 3 and 4. These curves in Fig. 4 show that the nondimensional maximum response increases as the frequency ratio increases from 1. Fig. 3 shows that the nondimensional maximum response decreases for a small damping factor but increases for greater damping factor as the frequency ratio increases from 0.4 to 1. Experiments of these two excitations are limited in that their frequency ratios are less than one but their results show that nondimensional maximum response decreases as the frequency ratio increases.

Shock spectra of unit step function and delta function, as shown in Figs. 7 and 8, are independent of the natural frequency. The ranges of the nondimensional maximum response are from 2 to 1 for unit step function and from 1 to 0.38 for delta function when their damping factors approach 1 from 0. The correlation of analysis and experiment for the unit step function is graphically given in Fig. 19. The nondimensional maximum response of the delta function is less in experiment than in analysis but has the same tendency as the damping factor increases.

The theoretical nondimensional maximum response to the natural frequency and rectangular pulse interval is given in Figs. 10 and 11. It can be seen in Fig. 10 that the nondimensional maximum response without damping is a curve of function $\sqrt{2(1-2\mathrm{Cos}W_n t_1)}$ from 2 to 1. These nondimensional maximum responses with different dampings, shown in Fig. 11, are irregular sinusodial curves which are convergent to 1 as the natural frequency and rectangular pulse interval becomes greater. The experiment of rectangular pulse excitation gives correlative results to the analysis.

The theoretical maximum response of the free fall, as shown in Figs. 13 and 14, decreases from a condition of small damping, higher free fall and lower natural frequency to a condition of greater damping, lower free fall and greater natural frequency. The slope of decrease is rapid at low natural frequencies and slow at higher natural frequencies. The results of this experiment supported the analysis.

Maximum responses are all first peak response except for these two excitations of harmonic force and a sinusoidal displacement of the system's base.

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THE MAXIMUM RESPONSE OF A SINGLE DEGREE OF FREEDOM SYSTEM TO DIFFERENT EXCITATIONS

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The maximum response of a single degree of freedom system to different excitations was investigated by both analytical work and experiment. These excitation were: (1) harmonic force, (2) sinusoidal displacement of the system's base, (3) unit step force, (4) delta function, (5) rectangular force and (6) free fall of the system.

A general equation of motion for each excitation was formed and the maximum response was derived from it. Transforming the maximum response into nondimensional form, the relationship of various factors to the maximum response was plotted.

Experiments involving these excitations were carried out and their results are shown in the tables. After transforming the results into nondimensional form, the maximum response was compared to that of the analytical work.

Throughout the comparison, the experimental results support the analytical work. It was found that the nondimensional maximum response increases as the frequency ratio increases for the first two excitations. The nondimensional maximum responses of the unit step function and the delta function are independent of the natural frequency and decreases as the damping factor increases. For the rectangular pulse, the nondimensional maximum responses are irregular sinusoidal curves. The maximum response of free fall is inversely proportional to the natural frequency.