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ANCHORAGE ZONE STRESSES IN POST-TENSIONED
PRESTRESSED CONCRETE BEAMS

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VINUBHAI FULABHAI PATEL

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Approved by:

Vernon H. Rosebraugh
Major Professor

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SYNOPSIS

In many fields of Civil Engineering, the problem of transmission of heavy forces applied on a small area on the surface, through the elastic body is not unusual. Many authors and investigators have worked on such a problem, which can directly or indirectly be applied to evaluate the stress distribution in the anchorage zone of post-tensioned prestressed beams.

In this report three different theories of evaluating the stress distribution in an anchorage zone are included. These three theories are (1) Using the Airy stress function, (2) Magnel's theory and, (3) Guyon's approximate solutions. In the first theory, use is made of the Fourier series to represent the load distribution on either end of the end block. Using the Airy stress function, formulas for longitudinal stress, transverse stress and shear stress in the end block are derived in a general form for any kind of load distribution at the end of the beam, represented by the Fourier series. Only a transverse stress is of interest to us for design, so the formulas for it are generated for the following load cases.

1. Only a single axial load at mid depth of the beam.
2. Two symmetric axial loads.
3. One eccentric load.

Magnel's theory is based on the assumption of the variation of a transverse stress as a third degree polynomial. Using the known boundary conditions, the constants of the polynomial are evaluated. Then this polynomial represents the transverse stress at any point. From the equilibrium of the stresses, shear stress is determined and the longitudinal stresses

are found as those for a column loaded eccentrically.

Guyon's method explains how these difficult theories can be simplified using approximations. His two approximate procedures, (1) Partitioning method, and (2) symmetric prism method, are discussed. Most of the possible loading conditions are treated.

Practical rules of providing the reinforcement in an end block are given in brief. For the comparative study of different methods, a simple illustrative problem is solved.

INTRODUCTION

Scope of Study: Prestressed concrete is a step of progress in the field of the reinforced concrete. But it has been said about the prestressed concrete that while the design of the prestressed concrete is easy, its detailing is much more difficult. Engineers have to face this difficulty of detailing the prestressed concrete units. Since the origin of the prestressed concrete, the investigators have tried to simplify the procedures of prestressing. Initially there was no proper method of stretching the wires to apply compression on the concrete. People tried prestressing with medium strength steel and the ordinary available stretching units. But it was soon found that it could not serve the purpose as a limited amount of the prestressing force was soon lost due to high losses like shrinkage and creep loss. Although this difficulty was overcome with the use of high strength steel and better stretching units, the problem of prestressing didn't come to an end. Many other details are yet necessary to give the prestressing unit a perfect shape. One among these is the design of an end block. Once the designer starts to design the big prestressed girder and finally comes out with an appreciably high value of necessary initial prestressing force, he is required to design a proper end block to withstand the stresses produced by high prestressing forces applied at the end of the beam.

As for the other structures, as well as the end blocks, the design requires proportioning the section of concrete and determining the amount of steel reinforcement needed to resist the stresses developed by external forces. So the first step toward the design of an end block of the prestressed beam is to find out the stresses developed due to the prestressing force. But this is not so simple as for the simple beams where simple beam theory is applicable to find the stresses. The linear stress distribution along the depth of the beam obtained by the simple beam theory is not valid in the end block. Here now we are in a position to define the end block as a length of the end of the beam in which simple beam theory for finding the stress distribution is not valid. Using the St. Venant's principle and with the experimental verification, it has been found that beyond a certain distance from the end, approximately equal to the depth of the beam, the stresses are only longitudinal and can be determined by using the simple beam theory. This length of the end of the beam is known as the end block, which is named by Guyon as "lead-in-zone." There may exist, in this lead-in-zone, transverse stresses, the amount, nature (compressive or tensile) and location of which largely depend on the manner in which the prestressing forces are spread over the end surface.

Complexity of the problem of evaluating the stresses in the end block is due to its three-dimensional stress distribution. As in most of the cases where prestressing forces are distributed through steel plates, there is concentration of the load along the depth as well as the width of the beam, over a portion equal to the respective dimensions of the plate along the depth and the width. This causes stress variations in the end block in all three directions, that is, along the length of the block, along the depth, and along the width of the block. Up to now, no exact solution has been found for the three dimensional stress distribution, so most of the investigators solved this problem by reducing it to a two dimensional problem by making some approximations. They neglected the stress variation along the width by assuming the load spread across the width. Thus, the problem remaining is of evaluating the longitudinal and transverse direct stresses on any vertical plane of the end block and the shear stresses. This simple approximation has greatly reduced the complexity of the problem because theories are available for the solution of two-dimensional stress variations. We saw earlier that the stresses at the other end of the lead-in-zone are only longitudinal and linearly distributed. Most of the investigators looked upon the end block as a deep beam loaded by distributed loads (longitudinal stresses) at the end of the lead-in-zone, and supported on one or more supports at the other end, depending upon the number of prestressing forces applied at the end of the beam. Thus, the stresses in the end block are the corresponding stresses in the deep beam. The first analysis of continuous, deep beams was made by Dischinger[1] for uniform and concentrated load by using trigonometric series. His results were reproduced by the Portland Cement Association in a chart[2]. The initial work by Dischinger was meant for determining the stresses in masonary walls in which the weight of wall itself was important and, hence, it has no direct application to the stresses in the end block. However, it gives an indication of the stresses existing in such deep beams. In 1888 exact solutions of the stresses in semi-infinite elastic medium due to the local uniform pressure was developed by Boussinesq[3]. This is the same as our problem if it can be modified for the finite width of the semi-infinite medium, so that stress trajectories issuing from the load must have double curvature due to

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[1]F. Dischinger, "Contributions to the Theory of Wall-Like Girders," pub. Intern. Assoc. for Bridge and Structural Engg., 1932, Vol I, p. 69.

[2]Design of Deep Girders, Structural and Railway Bureau, Portland Cement Assoc.

[3]I. Todhunter and K. Pearson, "A History of the Theory of Elasticity," New York Cambridge Univ., Press, 1893 Vol. ii, p. 240.

the presence of a free boundary on either side of the block of finite width. This origin of trajectories for the end-block was intelligently explained by Guyon. He determined quantitatively the stresses in the end blocks with various combinations of loadings on opposite ends by means of the solution of a biharmonic equation[4]. This two dimensional problem is solved also by Geer[5], using finite differences. Magnel[6] solved the same problem by an approximation. He assumed transverse stress distribution inside the anchorage zone as a polynomial of the third degree. Dr. Iyengar[7] used multiple Fourier series to solve the problem; he obtained the formulas for stresses for the three-dimensional case also, but since it is very complicated, it has less practical importance.

Many other investigators also solved this stress distribution using different methods. A photoelastic study of the problem was also done by Tesar[8], which proves the correctness of many theoretical results. In this report three methods of analysis are discussed in detail.

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4 Y. Guyon, "Efforts aux Extremities des Pieces Prismatiques," Publ. Int. Assoc. for Bridges and Steuctural Engg. Vol. II, 1952, p.165.

5 E. Geer, "Stresses in Deep Beams," Jour. of Am. Conc. Inst., Jan.,1960, p.651.

6 Magnel, "Prestressed Concrete," p.69.

7 K.T.S.R. Iyengar and G. Pickett, "Stress Concentration in Post-Tensioned Prestressed Concrete Beams," Jour. of Tech.,India, Vol. I, No.2, 1956.

8 Tesar, V.,"Experimental Determination of Stresses at the Ends of the Prismatic Members with Imperfect Joints," Pub., Int., Assoc. of Bridges and Structural Engg., Vol. I, 1932.

PURPOSE

The purpose of this report is to emphasize the importance of the design of end blocks, making use of available theoretical knowledge of the stresses in the end block rather than arbitrarily designing end blocks of prestressed beams. Although exact solution for this problem is not yet in the hands of designers, many theories are proved to be sufficiently accurate by comparing their results with the results of photoelastic studies. So at least the available theories can be used as a guide to the designing of end blocks. It is also intended to compare the results obtained by some of the theories. A problem is solved at the end of the report to show that the application of the theories to a practical problem is not difficult.

ANCHORAGE ZONE STRESSES IN GENERAL

From the definition of lead-in-zone, we know that the stresses at the end of the lead-in-zone are linearly distributed through the depth of the beam whereas on the outside surface of the end block there are concentrated prestressing forces. So the longitudinal prestressing forces must pass progressively from discontinuous distribution inside the lead-in-zone to continuous distribution at the end. In fact this passage of stresses cannot occur except by giving rise to transverse stresses and shear stresses on both horizontal and vertical planes.

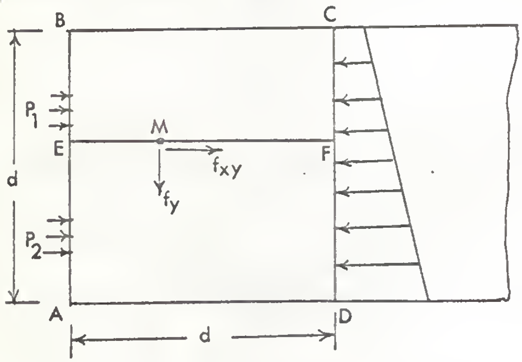

FIGURE 1

End-block loaded at CD and supported at AB like a beam.

Let us consider Figure 1 which shows the end of the beam. Segment ABCD is the end block and is in equilibrium under the forces on CD shown by the linear distribution and the forces on AB which act on small areas.

This segment can be looked upon as a beam supported by the reactions P_1 and P_2 and loaded along CD. It can be seen that a segment like EBCF in the lead-in-zone can be in equilibrium if, and only if, there exists some transverse stress f_y and shear stress f_{xy} as at M. Because f_y is produced along EF to balance the moment caused on EBCF by the forces on EB and CF, it depends upon the position of EF. The following conditions of equilibrium of the forces on EBCF must be satisfied.

1. As no external force acts on ABCD, the resultant of f_y must be zero. This implies that there should exist along EF zones of tension and zones of compression.
2. Sum of the moments of the stresses f_y about a point on EF should equal the algebric sum of the moments of the forces acting along EB and CF about the same point.
3. Resultant of f_{xy} should equal the resultant of the forces on EB and CF.

Unfortunately, the conditions mentioned above are not sufficient for determining the stress distribution in the anchorage zone. The variation of f_y along EF is not linear as in simple beams. Moreover, the stress f_y, as seen earlier depends upon the position of EF, not only along the depth of the beam but also along the width of the beam, because the prestressing forces are concentrated along the depth as well as the width. For the beam loaded as in Figure 2, the stresses on a horizontal plane EF and the stresses on the vertical plane GH are as shown in Figure 2.

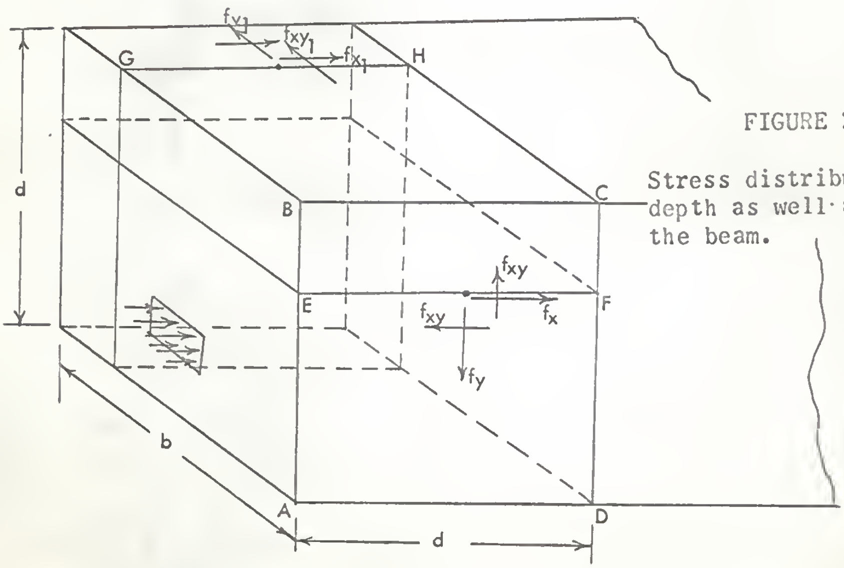

FIGURE 2

Stress distribution along the depth as well as the width of the beam.

Thus, this in reality is the problem of the stresses in an anchorage zone, involving three-dimensional stress variation. However, the problem can be reduced to two dimensions by assuming a uniform distribution of load along the width of the beam. In all the theories to follow, this assumption is made while examining the stress distribution in the anchor zone of the post-tensioned prestressed beam.

STRESS DISTRIBUTION IN ANCHORAGE ZONE BY AIRY STRESS FUNCTION

We know that the problem of stress distribution in an anchorage zone, which really is a problem of three dimensional stress distribution, can be treated, without introducing much error, as a two-dimensional stress distribution problem. This can be done by assuming a knife-edge load distributed throughout the width of the beam. This two-dimensional stress distribution problem has been evaluated by Pijush Kanti Som and Kalyanmay Ghosh[1], using the Airy stress function. To help in understanding the idea of the Airy stress function, the two-dimensional theory is described below in brief.

TWO-DIMENSIONAL THEORY AND AIRY STRESS FUNCTION

Let us consider a plate of unit thickness lying in the xy plane and acted upon by the forces as shown in Figure 3. The plate is acted upon by three types of varying stresses, f_x, f_y, and f_{xy}. Where f_x is direct stress in x direction, f_y is direct stress in y direction. The shear stress in the xy plane is f_{xy}. Equating the forces in the x direction, we get,

$$\left(f_x + \frac{\partial f_x}{\partial x} dx\right) dy - f_x dy + \left(f_{xy} + \frac{\partial f_{xy}}{\partial y} dy\right) dx - f_{xy} dx = 0$$

$$\therefore \quad \frac{\partial f_x}{\partial x} + \frac{\partial f_{xy}}{\partial y} = 0 \qquad (1)$$

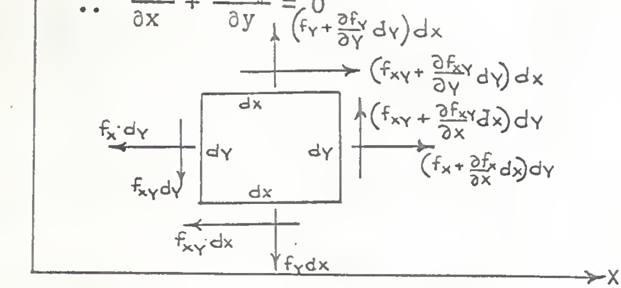

FIGURE 3

Equilibrium of an element dxdy of a plate in which stresses vary from point to point.

[1] Anchor Zone Stresses in prestressed concrete beams, proceedings of the ASCE, Vol. 90, No. ST4, August, 1964, p. 49.

Similarly equating forces in the y direction, we get,

$$\frac{\partial f_y}{\partial y} + \frac{\partial f_{xy}}{\partial x} = 0 \qquad (2)$$

From equations 1 and 2, it can be seen that the state of stress in the two-dimensional situation is described by the set of three independent variables f_x, f_y, and f_{xy}, and each in turn depends on two independent variables x and y. Thus, the problem of evaluating stresses in two dimensions is mathematically extremely complicated.

However, the complication is avoided to a great extent by a single function φ of x and y instead by three functions f_x, f_y, and f_{xy}. The stresses f_x, f_y, and f_{xy} then can be found as follows:

$$f_x = \frac{\partial^2\varphi}{\partial y^2} \; ; \; f_y = \frac{\partial^2\varphi}{\partial x^2} \text{ and } f_{xy} = -\frac{\partial^2\varphi}{\partial x \partial y} \qquad (3)$$

Thus, it can be seen that for any function φ of x and y, which is three times differentiable, the equilibrium equations 1 and 2 are satisfied with values of stresses given by equations 3. Besides satisfying the equilibrium conditions, the choice of the arbitrary function φ should be such that it will satisfy the compatibility[2] condition, which has been found to be,

$$\frac{\partial^4\varphi}{\partial x^4} + \frac{2\partial^4\varphi}{\partial x^2 \partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0 \qquad (4)$$

Thus, if one selects a function φ which satisfies the fourth order partial differential equation 4 of compatibility then equilibrium equations are satisfied automatically. These are represented by equations

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[2]For detail of compatibility condition, interested readers are referred to "Advanced Strength of Materials" by J.P. Den Hartog. McGraw-Hill Book Co., Inc., 1952.

1 and 2, the stresses f_x, f_y, and f_{xy} can be easily found using equations 3. We shall make use of Airy stress function to evaluate transverse stress f_y; longitudinal stress f_x and shear stress f_{xy} in the anchor zone. The problem of stresses in the anchorage zone by this method is attacked in general by imposing boundary conditions in a most general manner so as to include all possible loading conditions. The Airy stress function φ for our problem shall be chosen to satisfy equation 4 and the boundary conditions imposed on the edge of the beam, i.e. at $x = -\frac{d}{2}$ in Figure 4.

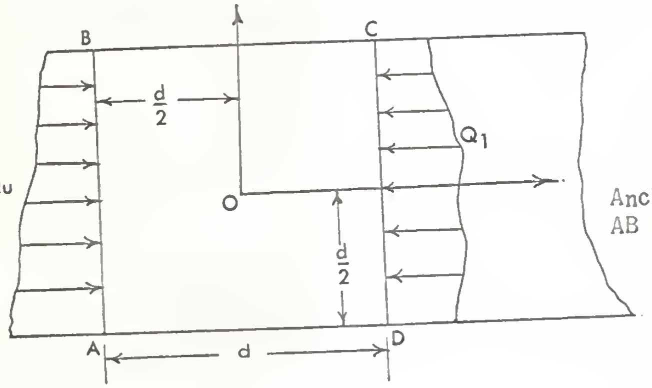

FIGURE 4

Anchor block loaded on boundary AB and CD by general loading.

Here the anchor block is assumed to be of a length equal to the depth of the beam d as usual and the axes are chosen as shown in Figure 4. Boundary conditions are imposed on the other end of the anchor block $(x = \frac{d}{2})$ also. Boundary conditions on either face AB and CD are in the form of the load intensity on the corresponding face. To cover all possible cases of normal loading on the edge of the beam, we assume the most general distribution of loading along two ends AB and CD of the anchor block. This can be represented by the following series.

$$Qu = \text{load intensity on AB} = A_o + \sum_{m=1}^{\infty} A_m \sin \frac{2m\pi y}{d} + \sum_{m=1}^{\infty} A_m^1 \cos \frac{2m\pi y}{d} \tag{5}$$

$$Q_1 = \text{stress distribution on CD} = B_o + \sum_{m=1}^{\infty} B_m \cdot \sin \frac{2m\pi y}{d} + \sum_{m=1}^{\infty} B_m^1 \cos \frac{2m\pi y}{d}$$

In the equations 5, A_o and B_o represent the uniform stress distribution on AB and CD respectively.

Now let us assume some arbitrary Airy stress function φ as $\varphi = \sum \sin \frac{m\pi y}{d/2} \cdot f_x$, where f_x, the function of x can be determined by imposing boundary conditions (5) and satisfying the compatibility condition 4. To satisfy the compatibility condition various derivatives of function φ are found below.

$$\frac{\partial^4 \varphi}{\partial x^4} = f_x^{IV} \cdot \sum \sin \alpha y, \text{ where } \alpha = \frac{2m\pi}{d}$$

$$\frac{\partial^4 \varphi}{\partial y^4} = \sum \alpha^4 \sin \alpha y \, f_x \text{ and}$$

(Superscript of f_x denotes order of derivative f_x w.r. to x.)

$$\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} = \sum - \alpha^2 \sin \alpha y \, f_x^{II}$$

Substituting these values in equation 4, we get,

$$f_x^{IV} \sum \sin \alpha y + (-2 \sum \alpha^2 \sin \alpha y \, f_x^{II}) + \sum \alpha^4 \sin \alpha y \, f_x = 0$$

$$\therefore f_x^{IV} - 2 \sum \alpha^2 f_x^{II} + \sum \alpha^4 f_x = 0.$$

This is a fourth order partial differential equation in f_x, whose solution is:

$$f_x = (a + bx) \sum e^{+\alpha x} + (c + dx) \sum e^{-\alpha x}$$

and representing the solution in trigometric form as follows:

$$f_x = (a + bx) \sum (\cosh \alpha x + \sinh \alpha x) + (c + dx) \sum (\cosh \alpha x - \sinh \alpha x)$$

$$= (a + c) \sum \cosh \alpha x + (a - c) \sum \sinh \alpha x + (b + d) x \sum \cosh \alpha x +$$

$$(b - d) \, x \sum \sinh \alpha \, x$$

replacing constants $(a + c)$ by C_1

$(a - c)$ by C_2

and $(b + d)$ by C_3

$(b - d)$ by C_4 we get,

$$f_x = C_1 \sum \cosh \alpha x + C_2 \sum \sinh \alpha x + C_3\, x \sum \cosh \alpha x + C_4\, x \sum \sinh \alpha x$$

Therefore, our stress function, satisfying the compatibility condition is:

$$\varphi = \sum \sin \alpha y \left[C_1 \cosh \alpha x + C_2 \sinh \alpha x + C_3 x \cosh \alpha x + C_4\, x \sinh \alpha x\right]$$

Now the set of stresses f_x, f_y, and f_{xy} can be found from this stress function by using equations 3.

According to equation 3, transverse stress $= f_y = \dfrac{\partial^2\varphi}{\partial x^2}$

Therefore, taking the double derivative of function φ, w.r. to x,

$$f_y = \sum \sin \alpha y \Big[C_1\, \alpha^2 \cosh \alpha x + C_2\, \alpha^2 \sinh \alpha x + C_3\, \alpha(2 \sinh \alpha x + \alpha x \cosh \alpha x)$$
$$+ C_4\, \alpha\, (2 \cosh \alpha x + \alpha\, x \sinh \alpha x)\Big] \qquad (6)$$

$$f_x = \frac{\partial^2\varphi}{\partial y^2} = -\sum \alpha^2 \sin \alpha y \Big[C_1 \cosh \alpha\, x + C_2 \sinh \alpha x + C_3\, x \cosh \alpha x +$$
$$C_4\, x \sinh \alpha x\Big] \qquad (7)$$

$$f_{xy} = -\frac{\partial^2\varphi}{\partial x \partial y} = \sum -\alpha \cos \alpha y \Big[C_1\, \alpha \sinh \alpha x + C_2 \alpha \cosh \alpha x + C_3\, (\cosh \alpha x + \alpha x$$
$$\sinh \alpha x) + C_4\, (\sinh \alpha x + x \cosh \alpha x)\Big] \qquad (8)$$

The above expressions for stresses are only partially determined because we have not yet evaluated constants C_1, C_2, C_3, and C_4. Moreover, boundary conditions are also not yet imposed on function φ. So by imposing known boundary conditions, we can determine constant C_1, C_2, C_3, and C_4 and consequently the stresses. As the general expression 5 for loading consists of three separate parts, we can evaluate constants using these boundary conditions with each part of it separately and then we can superimpose the results.

Terms A_o and B_o of equations 5 represent uniform loadings, for which the stresses f_x, f_y, and f_{xy} are obviously known. Therefore, let us first consider the loading represented by a sine series.

$$Q_u = \sum_{m=1}^{\infty} A_m \sin \frac{2m\pi y}{d}$$

where $\frac{2m\pi}{d} = \alpha$

$$Q_1 = \sum_{m=1}^{\infty} B_m \sin \frac{2m\pi y}{d}$$

Considering loading causing compression on the block as negative, the following are the boundary conditions for this case.

at $x = -\frac{d}{2}$, shear stress $f_{xy} = 0$ and $f_x = -\sum_{m=1}^{\infty} A_m \sin \alpha y$

at $x = +\frac{d}{2}$, shear stress $f_{xy} = 0$ and $f_x = -\sum_{m=1}^{\infty} B_m \sin \alpha y.$

Substituting shear boundary conditions in equation 8 for $x = -\frac{d}{2}$ and remembering that $\alpha = \frac{2m\pi}{d}$, we get,

$$C_1 \alpha \sinh(-m\pi) + C_2 \cosh(-m\pi) + C_3 \left[\cosh(-m\pi) - my \sinh(-m\pi)\right]$$
$$+ C_4 \left[\sinh(-m\pi) - m\pi \cosh(-m\pi)\right] = 0,$$
$$= -C_1 \alpha \sinh m\pi + C_2 \cosh m\pi + C_3 \left[\cosh m\pi + m\pi \sinh m\pi\right]$$
$$+ C_4 (-\sinh m\pi - m\pi \cosh m\pi) = 0 \qquad (9)$$

Similarly equating equation 8 to zero for $x = +\frac{d}{2}$,

$$C_1 \ \alpha \sinh (m\pi) + C_2 \cosh m\pi + C_3 (\cosh m\pi + m\pi \sinh m\pi)$$

$$+ C_4 \ (\sinh m\pi + m\pi \cosh m\pi) \qquad (10)$$

Eliminating C_2 and C_4 from equations 9 and 10, we get,

$$C_3 = -C_2 \frac{(\alpha \cosh m\pi)}{(\cosh m\pi + m\pi \sinh m\pi)}$$

Similarly eliminating C_3 and C_2,

$$C_4 = - C_1 \frac{\alpha \sinh m\pi}{(\sinh m\pi + m\pi \cosh m\pi)} \qquad (11)$$

Now imposing boundary conditions of f_x in equation 7, for $x = -\frac{d}{2}$,

$$\sum - \alpha^2 \sin \alpha y \left[C_1 \cosh m\pi - C_2 \sinh m\pi - C_3 \frac{d}{2} \cosh m\pi + C_4 \frac{d}{2} \sinh m\pi\right]$$

$$= - \sum A_m \sin \alpha y$$

$$\therefore \sum \alpha^2 \left[C_1 \cosh m\pi - C_2 \sinh m\pi - C_3 \frac{d}{2} \cosh m\pi + C_4 \frac{d}{2} \sinh m\pi\right] = \sum A_m \qquad (12)$$

Similarly for $x = \frac{d}{2}$, equating equation 7 to $\sum B_m \sin \alpha y$,

$$\sum + \alpha^2 \left[C_1 \cosh m\pi + C_2 \sinh m\pi + C_3 \frac{d}{2} \cosh m\pi + C_4 \frac{d}{2} \sinh m\pi\right] = \sum B_m \qquad (13)$$

By adding equations 12 and 13 and using equation 11 we get C_1 and C_4 as follows.

$$C_1 = \sum \frac{A_m + B_m}{\alpha^2} \ \frac{(\sinh m\pi + m\pi \cosh m\pi)}{\sinh 2m\pi + 2m\pi}$$

and

$$C_4 = \sum - \frac{A_m + B_m}{\alpha^2} \ \frac{\alpha \sinh m\pi}{\sinh 2m\pi + 2m\pi}$$

By subtracting equation 13 from 12 and using equation 11 we get C_2 and C_3 as follows.

$$C_2 = \sum \frac{A_m - B_m}{\alpha^2} \frac{\cosh m\pi + m\pi \sinh m\pi}{\sinh 2m\pi - 2m\pi}$$

and

$$C_3 = \sum \frac{A_m - B_m}{\alpha^2} \frac{\alpha \cosh m\pi}{\sinh 2m\pi - 2m\pi}$$

Now it has been shown by Timoshenko[1] and can obviously be seen also that for a load distribution represented by a cosine series, all the constants shall be multiplied by $\sum \frac{\cos \alpha y}{\sin \alpha y}$ and A_m and B_m shall be represented by A_m^1 and B_m^1. Substituting values of the constants C_1, C_2, C_3, and C_4 in equations for f_x, f_y, and f_{xy} for both sine and cosine series separately and then adding results algebraically, we get these stresses for load distribution at the anchor block end represented by a trigonometric series. These stresses are as given below.

$$f_y = \sum_{m=1}^{\infty} (A_m + B_m) A_1 (x) \sin \alpha y - \sum (A_m - B_m) B_1 (x) \sin \alpha y$$

$$+ \sum (A_m^1 + B_m^1) A_1 (x) \cos \alpha y - \sum (A_m^1 - B_m^1) B_1 (x) \cos \alpha y \qquad (14)$$

Where $A_1 (x)$ and $B_1 (x)$ are the functions of x and are given as follows:

$$A_1 (x) = \frac{(m\pi \cosh m\pi - \sinh m\pi) \cosh \alpha x - \alpha x \sinh \alpha x \sinh m\pi}{\sinh 2m\pi + 2m\pi}$$

and

$$B_1 (x) = \frac{(m\pi \sinh m\pi - \cosh m\pi) \sinh \alpha x - \alpha x \cosh \alpha x \cosh m\pi}{\sinh 2m\pi - 2m\pi}$$

$$f_x = - \sum_{m=1}^{\infty} (A_m + B_m) A_2 (x) \sin \alpha y + \sum (A_m - B_m) B_2 (x) \sin \alpha y$$

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[1] Timoshenko, S., "Theory of Elasticity," 1st Edition Mc-Graw Hill Book Co. Inc., New York, London 1934, p. 44-51.

$$-\sum (A_m^1 + B_m^1)\ A_2\ (x)\ \cos \alpha y + \sum (A_m^1 - B_m^1)\ B_2\ (x)\ \cos \alpha y \quad (15)$$

where

$$A_2\ (x) = \frac{(m\pi \cosh m\pi + \sinh m\pi) \cosh \alpha x - \alpha x \sinh \alpha x \sinh m\pi}{\sinh 2m\pi + 2m\pi}$$

and

$$B_2\ (x) = \frac{(m\pi \sinh m\pi + \cosh m\pi) \sinh \alpha x - \alpha x \cosh \alpha x \cosh m\pi}{\sinh 2m\pi - 2m\pi}$$

$$f_{xy} = -\sum_{m=1}^{\infty} (A_m + B_m)\ A_3\ (x)\ \cos \alpha y + \sum (A_m - B_m)\ B_3\ (x)\ \cos \alpha y$$

$$+ \sum (A_m^1 + B_m^1)\ A_3\ (x)\ \sin \alpha y - \sum (A_m^1 - B_m^1)\ B_3\ (x)\ \sin \alpha y \qquad (16)$$

where

$$A_3\ (x) = \frac{(m\pi \cdot \cosh m\pi \cdot \sinh \alpha x - \alpha\ x \cosh \alpha x \cdot \sinh m\pi)}{\sinh 2m\pi + 2m\pi}$$

and

$$B_3\ (x) = \frac{(m\pi \cdot \sinh m\pi \cdot \cosh \alpha x - \alpha x \sinh \alpha x \cdot \cosh m\pi)}{\sinh 2m\pi - 2m\pi}$$

In all the above equations, first two terms for stresses are for loading represented by sine series and second for loading represented by cosine series. Stresses due to the uniform load distribution at the ends of the anchor block should be obtained separately and added algebraically to the results obtained from equations 14, 15, and 16.

Use of these general equations will be made for some particular cases. The following cases shall be considered here.

1. Single axial load.
2. Two symmetric loads.
3. Single eccentric load.

In all the cases q will represent the intensity of load, and the width of the beam shall be taken as unity.

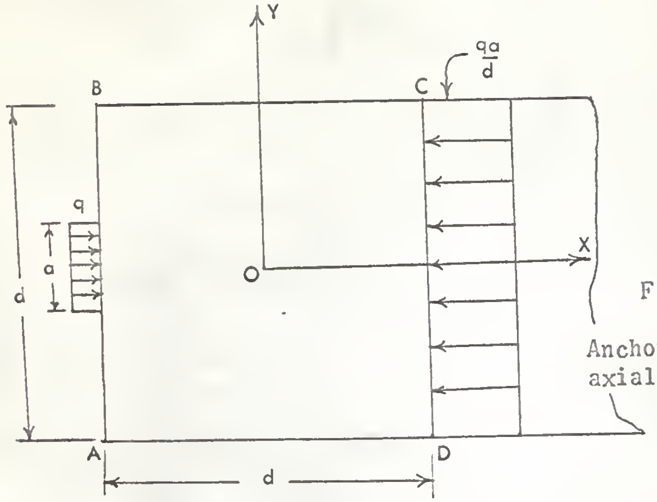


FIGURE 5

Anchor block loaded by single axial load.

1. Single Axial Load:

Let the single axial load be distributed over a depth $\pm \frac{a}{2}$ from the x axis, Figure 5. As the load is symmetric the stress distribution at the end of the anchor block, i.e. at CD, is uniform. Hence, Q_1 = stress distribution along CD can be represented by the constant term B_o and $B_m = B_m^{\,1} = 0$. Also the load along AB is symmetric, so along AB the stress is compressive throughout the depth, therefore, sine series, which shows stress of opposite sign for points below x-axis should vanish. So the load distribution along AB can be represented by,

$$Q_u = -\frac{qa}{d} - \sum A_m^{\,1} \cos \frac{2m\pi y}{d}$$

q = load intensity, and the thickness of the beam is assumed to be unity. Here $A_m^{\,1}$ is the fourier coefficient and is given by,

$$A_m^{\,1} = \frac{1}{d/2} \int_{-\frac{a}{2}}^{+\frac{a}{2}} q \cdot \cos \frac{2m\pi y}{d} \cdot dy$$

$$= \frac{2}{d} \left[\frac{q \cdot d}{2m\pi} \cdot \sin \frac{2m\pi y}{d} \right]_{-\frac{a}{2}}^{+\frac{a}{2}}$$

$$= \frac{2q}{m\pi} \sin \frac{m\pi a}{d}$$

Thus, for this particular case we have $A_o = -\frac{qa}{d}$, $A_m = 0$

$$A_m^1 = \frac{2q}{m\pi} \sin \frac{m\pi a}{d},\; B_o = -\frac{qa}{d} \text{ and } B_m^1 = 0$$

To find the transverse stress f_y which is important, we substitute the values of the constants as shown above. This substitution is made for terms due to cosine series only. Taking the first two terms of the Fourier series, i.e. m = 1 and 2, the transverse stress at y = 0, along the x-axis is obtained as follows. Putting $\alpha = \frac{2m\pi}{d}$ and the values of the hyperbolic sine and cosine in the last two terms of equation 14.

$$f_y = \frac{2q}{\pi} \sin \frac{\pi a}{d} \left\{ \left[\frac{24.925 \cosh \frac{2\pi x}{d} - 72.6 \frac{x}{d} \sinh \frac{2\pi x}{d}}{272.5} \right] - \left[\frac{24.925 \sinh \frac{2\pi x}{d} - 72.6 \frac{x}{d} \cosh \frac{2\pi x}{d}}{260} \right] \right\}$$
$$+ \frac{q}{\pi} \cdot \sin \frac{2\pi a}{d} \left\{ \left[\frac{1410 \cosh \frac{4\pi x}{d} - 3350 \frac{x}{d} \sinh \frac{4\pi x}{d}}{1.42 \times 10^5} \right] - \left[\frac{1410 \sinh \frac{\pi x}{d} - 3350 \frac{x}{d} \cosh 4\pi \frac{x}{d}}{1.42 \times 10^5} \right] \right\} \quad (17)$$

From equation 17, it can be seen that the terms in the bracket are independent of any case of load distribution at the ends of the anchor block but are dependent upon the various values of $\frac{x}{d}$. Giving the following designations to these terms they are evaluated for certain values of $\frac{x}{d}$ and tabulated as below to use for the other two cases also.

$$K_1 = \frac{24.925 \cosh 2\pi \frac{x}{d} - 72.6 \left(\frac{x}{d}\right) \sinh 2\pi \left(\frac{x}{d}\right)}{272.5}$$

$$K_2 = \frac{1410 \cosh 4\pi\ (\frac{x}{d}) - 3350\ (\frac{x}{d}) \sinh 4\pi\ (\frac{x}{d})}{1.42 \times 10^5}$$

$$N_1 = \frac{24.925 \sinh 2\pi\ (\frac{x}{d}) - 72.6\ (\frac{x}{d}) \cosh 2\pi\ (\frac{x}{d})}{260.0}$$

$$N_2 = \frac{1410 \sinh 4\pi\ (\frac{x}{d}) - 3350\ (\frac{x}{d}) \cosh 4\pi\ (\frac{x}{d})}{1.42 \times 10^5}$$

In the above designations, the subscript denotes the value of m taken in the Fourier series. These are calculated for $\frac{x}{d} = -\frac{1}{2}, -\frac{1}{4}, 0, +\frac{1}{4}$ and $+\frac{1}{2}$ and tabulated below, in Table 1.

x/d	K_1	K_2	N_1	N_2
$-\frac{1}{2}$	-0.4800	-0.4950	+0.5050	+0.4950
$-\frac{1}{4}$	+0.0765	+0.04690	-0.0452	-0.04690
0	+0.0915	+0.0100	0	0
$+\frac{1}{4}$	+0.0765	+0.04690	+0.0452	+0.04690
$+\frac{1}{2}$	-0.4800	-0.4950	-0.5050	-0.4950

TABLE 1 *

Values of Constants K_1, K_2, N_1, and N_2 for various values of x/d.

Thus, the equation for f_y for a single axial load reduces to the form,

$$f_y = \frac{2q}{\pi} \sin \frac{\pi a}{d} \left[K_1 - N_1\right] + \frac{q}{\pi} \sin \frac{2\pi a}{d} \left[K_2 - N_2\right]$$

This transverse stress has been found for a = 0.2d along the center of the beam, i.e., at y= 0 and at the distance along the x-axis as given in Table 1. These values are given in Table 2.

x/d	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$+\frac{1}{4}$	$+\frac{1}{2}$
f_y	-0.67q	+0.0639q	+0.03722q	+0.0117q	+0.00935q

TABLE 2

Transverse stress for various values of x/d for Case I.

---

* For $m \geqq 1$ sin hm is taken equal to cosh m.

Similar calculations can be made for longitudinal stress f_x and shear stress f_{xy}, but in the calculation of longitudinal stress f_x, care should be exercised to include the effect of uniform stress A_o and B_o. The terms f_x and f_{xy} being not so important as f_y in the design of the anchor block they are not calculated here.

2. Two Symmetric Loads:

Two loads, each acting over depth $\frac{a}{2}$ and having their centers at $\pm\frac{d}{6}$ from the x-axis are considered as shown in Figure 6.

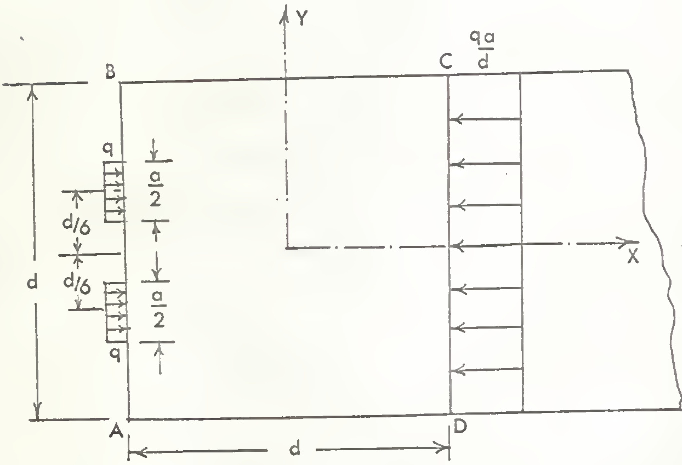


FIGURE 6

Anchor block loaded by two symmetric loads.

Again, under the same reasoning the sine series for load distribution should drop down and we will be left with the distribution Q_u along AB and Q_l along CD.

$$Q_u = -\frac{qa}{d} - \sum A_m^1 \cos\frac{2m\pi y}{d}$$

and

$$Q_l = -\frac{qa}{d} = B_o$$

Here $A_m{}^1$ can be evaluated as,

$$A_m{}^1 = \frac{2}{d/2}\int_{(d/6 - a/4)}^{(d/6 + a/4)} q\cdot \cos\frac{2m\pi y}{d}\cdot dy$$

$$= \frac{4q}{d}\left[\frac{d}{2m\pi}\ \sin\frac{2m\pi y}{d}\right]_{(d/6\ -\ a/4)}^{(d/6\ +\ a/4)}$$

$$= \frac{2q}{m\pi}\left[\ \sin\frac{2m\pi}{d}\left(\frac{d}{6}+\frac{a}{4}\right) - \sin\frac{2m\pi}{d}\left(\frac{d}{6}-\frac{a}{4}\right)\right]$$

expanding the bracket,

$$A_m^{\ 1} = \frac{4q}{m\pi}\ \cos\frac{m\pi}{3}\ \sin\frac{m\pi q}{2d}$$

To evaluate f_y, we have to substitute these values of constants in the last two terms of equation 14. Then for transverse stress f_y, the equation becomes,

$$f_y = \frac{4q}{\pi}\cos\frac{\pi}{3}\sin\frac{\pi a}{2d}\left[K_1 - N_1\right]\ \cos\frac{2\pi y}{d} + \frac{2q}{\pi}\cos\frac{2\pi}{3}\sin\frac{\pi a}{d}$$

$$\left[K_2 - N_2\right]\ \cos\frac{4\pi y}{d} \qquad (18)$$

Again, taking $a = 0.2d$, and using Table 1 for the values of K_1, K_2, N_1, and N_2, the stress f_y has been found at $y = 0$ and $y = \frac{d}{6}$ and the values are tabulated in Table 3.

x/d	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$+\frac{1}{4}$	$+\frac{1}{2}$
f_y at $y = 0$	-0.009 q	+0.0064q	+0.01613q	+0.00616q	+0.00492q
f_y at $y = \frac{d}{6}$	-0.1895q	+ 0.02074q	+0.0100q	+0.00308q	+0.00246q

TABLE 3

Transverse stress along x-axis and along the line of action of resultant of each load for various x/d values in Case II.

3. <u>One Eccentric Load:</u>

In this case a single normal load is distributed over a depth equal to $\frac{a}{2}$ and having its CG at $\frac{d}{6}$ from x-axis. Stress distribution at the edge CD is as shown in Figure 7.

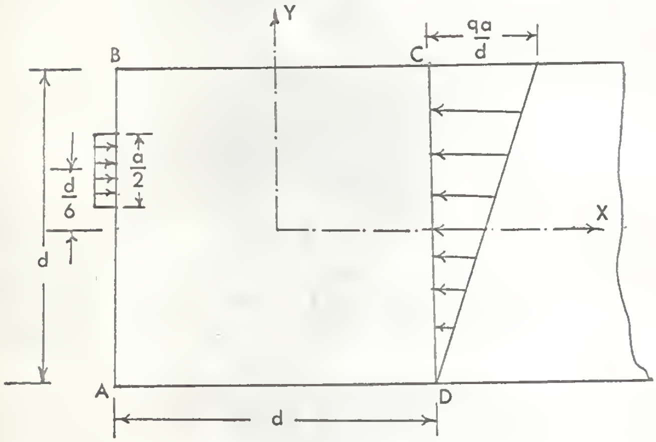


FIGURE 7

Anchor block loaded by single eccentric load.

The load at AB is eccentric and so the distribution of Qu along AB can be represented as,

$$Q_u = -\frac{qa}{2d} - \sum_{m=1}^{\infty} A_m \sin\frac{2m\pi y}{d} - \sum_{m=1}^{\infty} A_m^{1} \cos\frac{2m\pi y}{d}$$

where A_m is given by,

$$A_m = \frac{1}{d/2}\int_{(d/6 - a/4)}^{(d/6 + a/4)} q \cdot \sin\frac{2m\pi y}{d} \cdot dy$$

$$\therefore \quad A_m = \frac{2}{d}\left[-\frac{qd}{2m\pi}\cos\frac{2m\pi y}{d}\right]_{(\frac{d}{6} - \frac{a}{4})}^{(\frac{d}{6} + \frac{a}{4})}$$

substituting the limits of y and expanding,

$$A_m = \frac{2q}{m\pi}\ \sin\frac{m\pi}{3} \cdot \sin\frac{m\pi a}{2d}$$

Similarly, $A_m^{\,1} = \frac{2}{d}\int_{(\frac{d}{6}-\frac{a}{4})}^{(\frac{d}{6}+\frac{a}{4})} (q\cdot\cos\frac{2m\pi y}{d})\,dy$

This integration has been performed in Case II, from which

$$A_m^{\,1} = \frac{2q}{m\pi}\cos\frac{m\pi}{3}\sin\frac{m\pi a}{2d}$$

Boundary condition at $x = +\frac{d}{2}$, i.e., along CD, can be represented as

$$Q_1 = -\frac{qa}{2d} - \sum_{m=1}^{\infty} B_m \sin\frac{2m\pi y}{d}$$

As shown in Figure 8, distribution of stress along CD is represented by uniform stress $-\frac{qa}{2d}$ and the remaining stress, being negative above the x-axis and positive below thex-axis, can be represented by a sine series. The constant of the sine series B_m can be given by,

$$\frac{1}{d/2}\int_{-d/2}^{+d/2} \frac{qa}{2d}\cdot\sin\frac{2m\pi y}{d}\cdot dy$$

evaluating we get $B_m = -\frac{qa}{d}\cdot\frac{1}{m\pi}\cos m\pi$

So here we have $A_o = B_o = -\frac{qa}{2d}$, $A_m = \frac{2q}{m\pi}\sin\frac{m\pi}{3}\sin\frac{m\pi a}{2d}$,

$A_m^{\,1} = \frac{2q}{m\pi}\cos\frac{m\pi}{3}\sin\frac{m\pi a}{2d}$, $B_m = -\frac{qa}{dm\pi}\cdot\cos m\pi$

and $B_m^{\,1} = 0$.

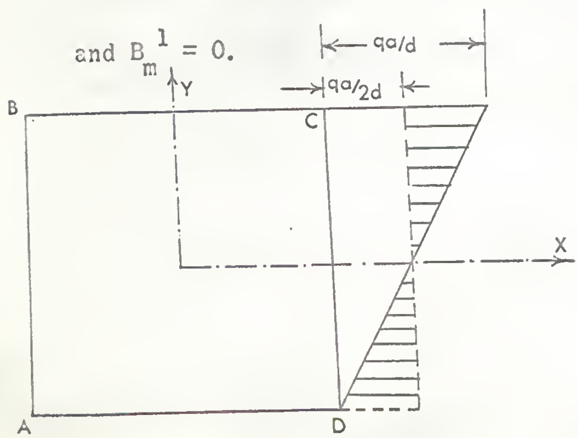


FIGURE 8

Boundary condition along CD.

∴ Substituting these constants in equation 14 for f_y we can find the transverse stress for this case of loading condition. For the first two terms of series, this can be written as,

$$f_y = \left[\left(\frac{2q}{\pi} \sin\frac{\pi}{3} \sin\frac{\pi a}{2d} - \frac{qa}{d\pi} \cos\pi\right)(K_1) - \left(\frac{2q}{\pi} \sin\frac{\pi}{3} \sin\frac{\pi a}{2d} + \frac{qa}{d\pi} \cos\pi\right)(N_1)\right] \cdot \sin\frac{2\pi y}{d}$$

$$+\left[\frac{2q}{\pi} \cos\frac{\pi}{3} \sin\frac{\pi a}{2d} (K_1 - N_1)\right] \cdot \cos\frac{2\pi y}{d} + \left[\left(\frac{q}{\pi} \sin\frac{2\pi}{3} \cdot \sin\frac{\pi a}{d} - \frac{qa}{2d\pi} \cos 2\pi\right)(K_2) - \left(\frac{q}{\pi} \sin\frac{2\pi}{3} \cdot \sin\frac{\pi a}{d} + \frac{qa}{2d\pi} \cos 2\pi\right)(N_2)\right] \sin\frac{4\pi y}{d}$$

$$+\left[\frac{q}{\pi} \cos\frac{2\pi}{3} \sin\frac{\pi a}{d} (K_2 - N_2)\right] \cdot \cos\frac{4\pi y}{d} \qquad (19)$$

If we seek f_y at $y = \frac{d}{6}$, i.e., along the line of the resultant prestressing force, by substituting in the above formulas $y = \frac{d}{6}$, we can reduce this equation to the following form.

$$f_{y\ y = d/6} = 0.251q \cdot K_1 - 0.141q \cdot N_1 + 0.1598q \cdot K_2 - 0.2148q \cdot N_2$$

Substituting the values K_1, N_1, K_2, and N_2 for various values of x/d from Table 1, the values of transverse stress have been evaluated and are tabulated in Table 4.

x/d	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$+\frac{1}{4}$	$+\frac{1}{2}$
f_y at $y = \frac{d}{6}$	-0.3786q	+0.04316q	+0.02455q	+0.0103q	-0.0212q

TABLE 4

Transverse stress f_y, at $y = \frac{d}{6}$ for various values of x/d for Case III.

MAGNEL'S THEORY

Theory:

The way of arriving at the stresses in the anchorage zone given by Magnel is apparently different from those given by the others. His analysis is somewhat empirical. Like other authors, he also assumes the end block of length equal to the depth of the beam. He figures out the end block as a deep beam of depth equal to the length of the end block and length equal to the depth of the end block as shown in Figure 9.

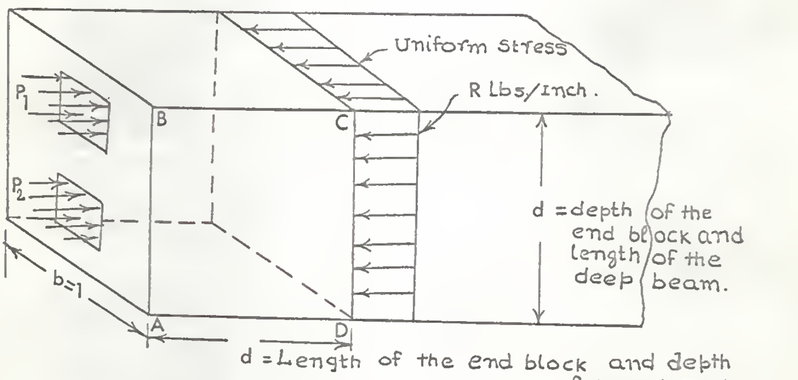


FIGURE 9

End block as deep beam.

But as the length of the end block is assumed to be equal to the depth of the beam, the length and the depth of the deep beam are equal. This deep beam at the end of the end block, i.e., at CD, is assumed to be loaded by a distributed load = R lbs. per inch, if the width is taken as unity (Figure 9). It is supported at AB by one or multiple supports having reactions equal to the applied prestressing forces, Figure 10. If the loads are eccentric, the corresponding stress diagram at CD, which is the load on the deep beam, will be varying U.D.L. as shown in Figure 10.

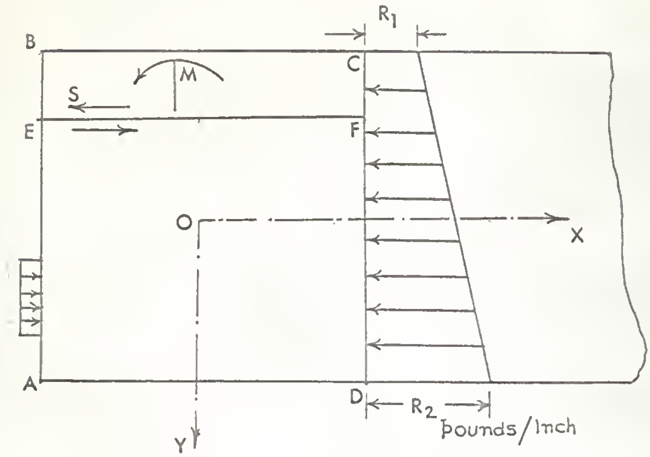


FIGURE 10

Deep beam loaded along CD and supported along AB.

Now in simple beam theory, the longitudinal stresses are assumed to be distributed linearly along the depth. In the case of our deep beam also under the action of the distributed loads, longitudinal stresses which act along the y-axis do act but are no more linearly distributed. Moreover, the exact variation of these stresses is also not known. At any plane along the depth of the deep beam, such as EF in Figure 10, the moment M and the shear force S due to the loads have to be resisted. It is assumed that the bending stresses produced along this plane EF have a variation which can reasonably be represented by a parabola of the third degree. The general shape is as shown in Figure 11.

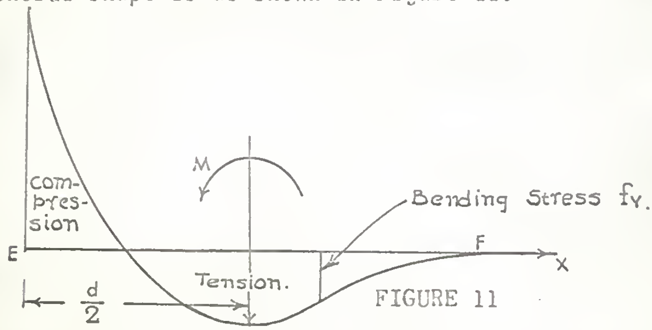


FIGURE 11

Bending stress distribution along the section EF of Figure 10.

Thus, if at section EF the bending moment = M, and the shear force = S, then the stress f_y at distance x from the origin (Figure 10) can be given by $f_y = A + Bx + Cx^2 + Dx^3$ (20)

This f_y is also the transverse stress in the end block in which we are interested. So if we can evaluate the four constants A, B, C, and D of equation 20, then we can find the variation of transverse stress f_y along the length of the end block and at any section as EF. Magnel suggests the following four boundary conditions to evaluate these four constants of equation 20.

(1) At $x = +\frac{d}{2}$; $f_y = 0$ - because we assume linear distribution of longitudinal stress at $x = +\frac{d}{2}$. Therefore, no transverse stress exists there.

(2) At $x = +\frac{d}{2}$; $\frac{df_y}{dx} = 0$; i.e., the stress curve of Figure 11 is horizontal at the end of the end block.

(3) As no external load acts on the beam (in this analysis we don't consider external loads on the prestressed beam but consider only the prestressing force), the summation of the total force due to transverse stress $f_y = 0$.

$$\therefore \int_{-\frac{d}{2}}^{+\frac{d}{2}} f_y b dx = 0$$

where b = width of the end block. The stress f_y is assumed to be uniform along the width of the beam.

(4) Summation of the moment about the origin of the force due to the transverse stresses is equal to the external moment M.

$$\therefore \int_{-\frac{d}{2}}^{+\frac{d}{2}} f_y \cdot bdx \cdot x = M$$

Now imposing condition 1 on equation 20, we get,

$$A + \frac{Bd}{2} + \frac{Cd^2}{4} + \frac{Dd^3}{8} = 0 \tag{21}$$

From the boundary condition, 2,

$$\frac{df_y}{dx} = B + 2Cx + 3Dx^2 = 0 \text{ for } x = + \frac{d}{2}$$

$$\therefore\ B + Cd + \frac{3Dd^2}{4} = 0$$

$$\therefore\ 4B + 4Cd + 3Dd^2 = 0 \tag{22}$$

Using the 3rd condition,

$$b \cdot \int_{-\frac{d}{2}}^{+\frac{d}{2}} (A + Bx + Cx^2 + Dx^3)\, dx = 0,$$

$$= \left[Ax + Bx^2/2 + Cx^3/3 + Dx^4/4 \right]_{-\frac{d}{2}}^{+\frac{d}{2}} = 0.$$

From which, $12\,Ad + Cd^3 = 0$ (23)

The fourth condition yields,

$$b\, x \int_{-\frac{d}{2}}^{+\frac{d}{2}} (Ax + Bx^2 + Cx^3 + Dx^4)\, dx = M$$

which gives,

$$\frac{20Bd^3}{3} + Dd^5 = \frac{80M}{b} \tag{24}$$

Solving these four equations 21, 22, 23, and 24 simultaneously, we get

$$A = \frac{5M}{bd^2}\ ;\ B = 0;\ C = -\frac{60M}{bd^4} \text{ and } D = \frac{80M}{bd^5}$$

Substituting the values of constants in equation 21,

$$f_y = \frac{5M}{bd^2} - \frac{60M}{bd^4}x^2 + \frac{80M}{bd^5}x^3$$

$$= \frac{5}{bd^2}\left[1 - \frac{12x^2}{d^2} + \frac{16x^3}{d^3}\right] x M$$

$$= \frac{KM}{bd^2} \qquad (25)$$

where $K = 5\left[1 - 12x^2/d^2 - 16x^3/d^3\right] \rightarrow (26)$ and M/bd^2 is a constant for a particular section where the bending moment = M. Thus, the variation of K along the length of the end block can represent to some scale the variation of the transverse stress at any section; the value of K has been found for various values of x/d and tabulated in Table 5.

x/d	-0.5	-0.4	-0.3	-0.25	-0.2	-0.1	0.0	+0.1	+0.2	+0.3	+0.4	+0.5
K	-20	-9.72	-2.56	0	+1.96	+4.32	+5	+4.48	+3.24	+1.76	+0.52	0

TABLE 5

Values of constant K along the length of the end block.

The results are plotted in Figure 12.

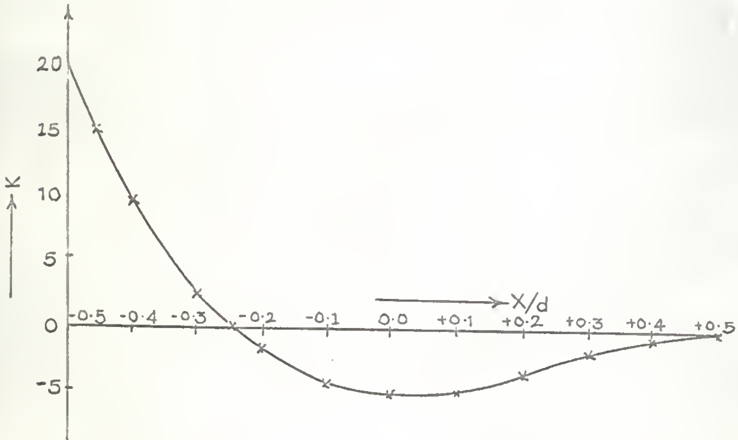


FIGURE 12

Variation of K along the length of the end block.

Once knowing the transverse stress distribution, the shear stress f_{xy}, which also varies along the length and the depth of the end block, can be found as follows.

Let M be the B·M (in the assumed deep beam) at a distance -y from the axis ox of Figure 13 and $M + \delta M$ be the bending moment at $(y+\delta y)$ from the x-axis. Let us consider the equilibrium of the forces acting on a portion δy thick and of the width between x=x and $x = \frac{d}{2}$ as shown in Figure 13. This portion of the end block is under equilibrium by the following vertical forces.

(1) Transverse force f_y acting on plane at -y and of some other magnitude on plane at $-(y + \delta y)$ from x-axis.

(2) Shearing stress acting on area (b x dy). As the transverse stress varies between x = x and $x = \frac{d}{2}$, let us consider a strip of length dx between x = x and x = x + dx, and let us assume constant transverse stress $= f_y$ over dx on the horizontal plane at y from the axis. Similarly, a constant stress equal to $(f_y + \frac{\partial f_y}{\partial y} dy)$ is assumed to act over the length dx and on the horizontal plane at $-(y + dy)$ from axis ox. As the stress distribution is assumed to be two-dimensional, the variation of the stresses along the width of the beam is neglected, and hence the area over which the stresses f_y and $(f_y + \frac{\partial f_y}{\partial y})$ act is b·dx as shown in Figure 13. Considering vertical equilibrium of the forces acting on the portion of the end block between y = y and y = y + dy and between x = x and x = d/2, we can write

$$\int_x^{\frac{d}{2}} \left[(f_y + \frac{\partial f_y}{\partial y} dy)\, b \cdot dx - f_y \cdot b dx \right] = f_{xy} \cdot b dy$$

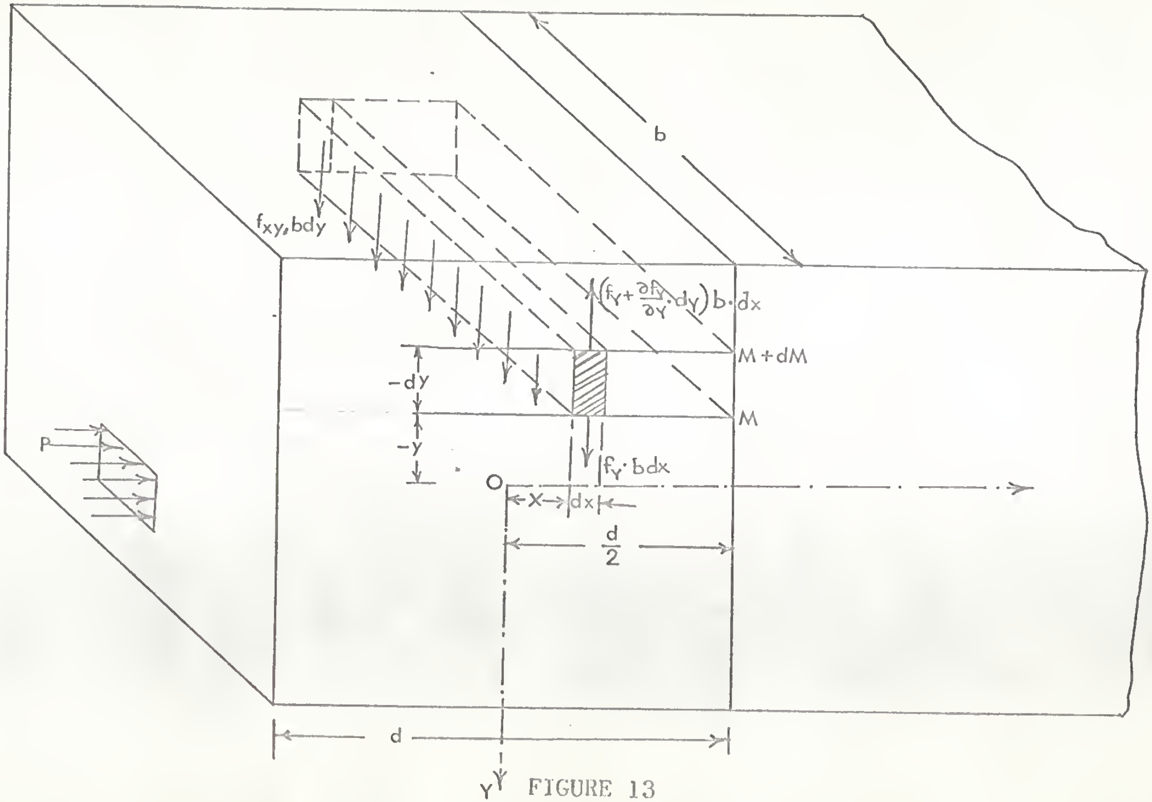


FIGURE 13

Equilibrium of forces on a strip dx • dy • b.

where f_{xy} = shearing stress at distance x and y from oy and ox axes, respectively. From the above equation,

$$\int_{x}^{+\frac{d}{2}} \frac{\partial f_y}{\partial y}\, dx = f_{xy}$$

b·dy is a constant area, therefore, can be cancelled on either side.

But it is proved that $f_y = \frac{KM}{bd^2}$

Therefore, $\frac{\partial f_y}{\partial y} = \frac{K}{bd^2} \; \frac{\partial M}{\partial y}$

As in simple beam theory, for the deep beam $\frac{\partial M}{\partial y}$ = S = shear force at y = y from ox in the assumed deep beam.

$$\therefore\; f_{xy} = \int_{x}^{\frac{d}{2}} \frac{K}{bd^2}\; S\cdot dx$$

Substituting the value of K from equation 26 and integrating, we get,

$$f_{xy} = \frac{5S}{bd}\left[\frac{1}{4} - \frac{x}{d} + \frac{4x^3}{d^3} - \frac{4x^4}{d^4}\right] \tag{27}$$

$$= \frac{K_1 S}{bd} \quad \text{where } K_1 = 5\left[\frac{1}{4} - \frac{x}{d} + \frac{4x^3}{d^3} - \frac{4x^4}{d^4}\right] \tag{28}$$

Here $\frac{S}{bd}$ is a constant for a particular section where the shear force is S. Thus, the variation of K_1 along the length of the end block can represent to some scale the variation of shear stress f_{xy} at any section. This value of K_1 has been found for various values of x/d and tabulated in Table 6.

x/d	-0.5	-0.4	-0.3	-0.25	-0.2	-0.1	0	+0.1	+0.2	+0.3	+0.4	+(
K_1	0	+1.458	+2.048	+2.109	+2.058	+1.728	+1.25	+0.768	+0.378	+0.128	0.018	

TABLE 6

Values of constant K_1 along the length of the end block.

The results are plotted in Figure 14.

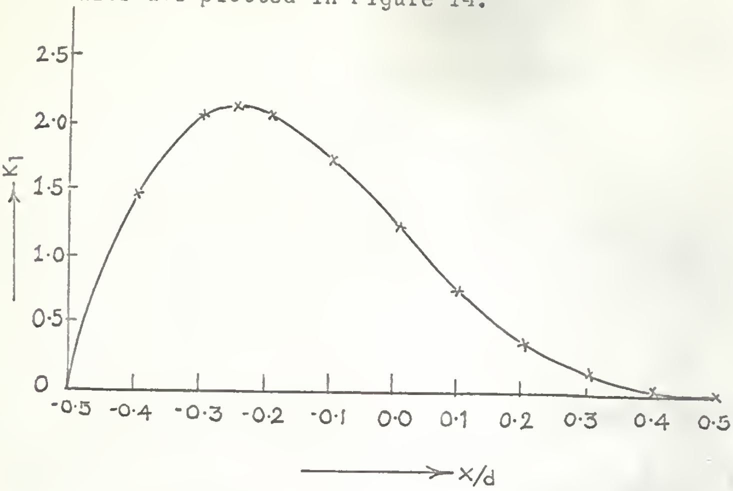


FIGURE 14

Variation of K_1 along the length of the end block.

Now at any point of the end block we know transverse stress f_y and shearing stress f_{xy}. To complete the analysis we need to know the longitudinal compressive stress f_x in the end block. This stress acts on the planes normal to ox. According to Magnel's theory an exact value of f_x cannot be found. Therefore, the following approximate procedure is adopted in this theory.

Let us assume that the prestressing forces disperse at 45^0 in the anchor block. Considering only the portion of the end block within the lines of dispersion as effective in taking the prestressing force, the simple law of eccentric compression is applied to each vertical plane of this portion to find the stress f_x. According to this empirical rule, as shown in Figure 15, the prestressing force acting over EF is spread through an area within lines EG and FN. Let us consider a vertical plane KL in the effective end block EGCDNF. The center of gravity of KL is M and hence the load P is eccentric on the plane KL by an eccentricity e.

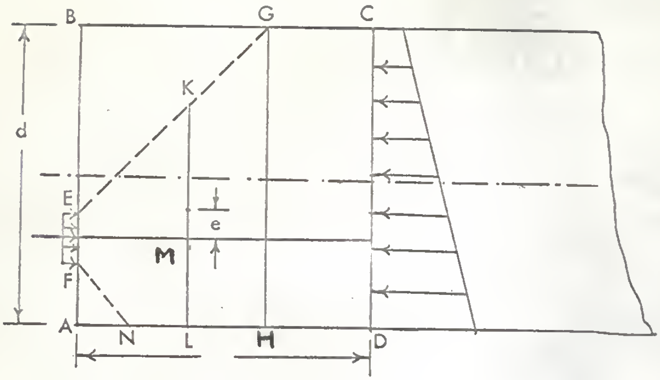


FIGURE 15

Dispersion of applied prestressing force.

Then the compressive stress f_x on plane KL is given by $f_x = -\frac{P}{A} + \frac{Pey}{I}$

negative sign shows compression, and A = area = b x KL,

y = distance of the point along KL where f_x is sought and is measured from the point M.

I = Moment of inertia of the area b x KL about the axis normal to KL and along the width of the beam.

Therefore, $I = \frac{b \times (KL)^3}{12}$

An important remark regarding f_x is that the distribution of f_x in the portion GHDC of the end block is the same and if the beam is rectangular throughout the length, then this stress distribution is same as that at any section of the beam except at the end blocks.

Magnel suggests that if the principal tensile stress exceeds the allowable tensile stress, then reinforcement should be provided. Hence, it is necessary to compute principal tensile stress from the known values of f_y, f_x, and f_{xy}.

Principal tensile stress = t is then given by the formula,

$$t = \frac{f_x + f_y}{2} - \sqrt{f_{xy}^{2} - \left(\frac{f_x - f_y}{2}\right)^{2}}$$

or the Mohr's Circle can be drawn to find t.

GUYON'S APPROXIMATE SOLUTIONS[1]

General: Complications of the formulas and difficulties in their application to the practical problems of determining the stress distribution in anchorage zones are obvious. Even if the tables supplying the values of f_x, f_y, and f_{xy} are readily available, (Guyon prepared such tables from his exact analysis) it is very difficult to examine the complete elastic state of the whole beam for maximum value of the stresses.

However, the work is much simplified if it is possible to apply the results of the theories to the problem without complications. Guyon pointed out that instead of examining the whole end block, it is sufficient to investigate the stress on only the most unfavorable planes. Thus, the calculations are limited to only a few planes. The problem is to identify these critical planes in advance by any means. This was achieved by Guyon using the approximate solutions. Moreover, as no high precision is necessary in evaluating the stresses, he suggested some empirical methods.

The following cases of the loading of the beam at the end by prestressing forces are considered here.

(1) Single axial load.

(2) Multiple symmetric forces (principle of partitioning).

(3) Single eccentric force (symmetric prism method).

(4) General case of loading, i.e., the case of irregularly distributed prestressing forces.

Let us treat each case separately.

[1] Guyon, "Prestressed Concrete," Contractors Record Ltd., London, p. 127-174.

(1) Case of Single Axial Force: The width of the beam is assumed to be unity. Only the axial load P is applied and is assumed to be uniformly distributed over a depth = a, along AB (Figure 16). As usual the lead-in-zone is assumed to be equal to the depth of the beam then, on CD edge the load is distributed uniformly.

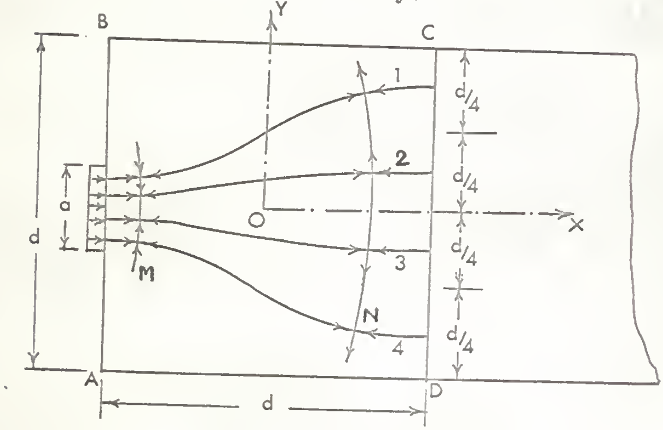


FIGURE 16

Stress trajectories of isostatics.

The block ABCD as usual can be assumed to be a spreader beam supported on a central reaction P at AB and loaded by U.D.L. at CD. But the distribution of stresses in this beam is not as simple as in shallow beams.

However, the forces can be looked upon as if passing from AB to CD along the trajectories, numbered as 1, 2, 3, and 4 in Figure 16. These are called isostatics by Guyon. As these trajectories represent the passage of longitudinal stresses, they should be parallel to the normal prestressing forces applied at AB and should be perpendicular to CD at that end. So if the depth over which the prestressing force applied is less than d, which is the general case in practice, these Isostatics must have double curvature as shown in Figure 16 to satisfy the requirement mentioned above. For the four isostatics shown in Figure 16 each should carry a load = $\frac{P}{4}$. As the isostatics are curved, they cannot take compressive load without exerting force in the direction normal to their own. This force exerted by isostatics is transverse force and it may be outward causing tension or inward causing compression, depending upon the curvature of istostatics.

Let us examine two sections M and N. At M the curvature of isostatics is convex inwards, so from Figure 16, it is obvious that it will exert compressive transverse stress at M. Whereas, due to the opposite curvature of isostatics at N, the transverse stress will be exerted away from the center, thus causing tensile transverse stress at N. Along the section N, each Isostatic exerts some transverse stress away from axis ox, so that the total sum of the transverse stress produced by each isostatic is maximum at the central axis ox. This value goes on decreasing along either edge BC and AD of the beam.

It is, therefore, known with the help of istostatics that the maximum transverse stress occurs along the central axis ox. It can also be seen that the transverse stress varies along the length of the beam, being compressive near the edge AB and tensile around the center of the end block. To know the actual variation of the transverse stress, we consider the average isostatics, one above and the other below the axis ox in Figure 17.

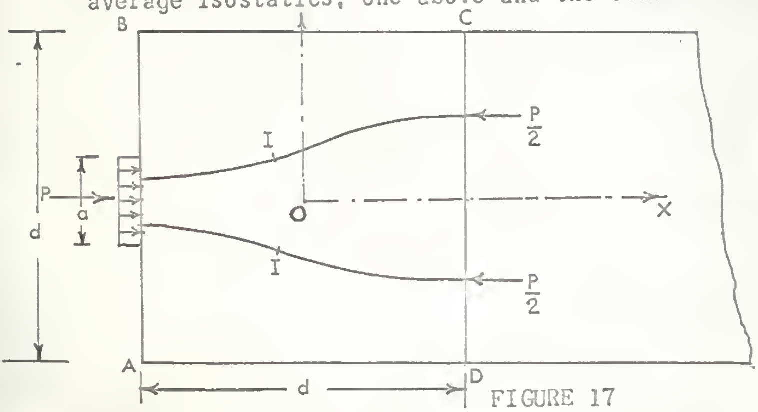


FIGURE 17

Average isostatics.

Transverse stress now is examined only along the x-axis as it is seen that maximum f_y occurs only along the x-axis. In Figure 17, each isostatic carries a load equal to $\frac{p}{2}$, each starting from the C.G. of the upper and lower half of the depth a at AB and meeting perpendicular to CD at the C.G. of the upper and lower half of CD respectively. From AB up to I each isostatic has the curvature convex towards the x-axis, and has curvature convex away from x-axis from I onwards up to near CD. If the radius of curvature at any point is R, then the transverse stress at that point can be found as follows.

Consider a small isolated length of one of the isostatics as in Figure 18.

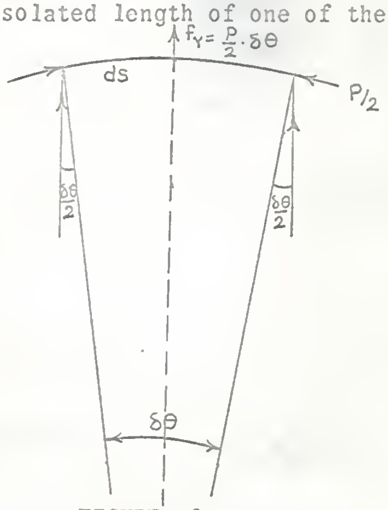


FIGURE 18

Component f_y of the forces carried by the isostatics.

From Figure 18, $f_y = 2\left[\frac{p}{2} \cdot \frac{\delta\theta}{2}\right]$, because $\delta\theta/2$ is a small angle, $f_y = \frac{p}{2} \cdot \delta\theta$. But $R\delta\theta = ds$. Therefore, $f_y = \frac{pds}{2R}$ where ds = a small length of an isostatic, which approximately equals the length along the x-axis. Therefore, f_y per unit length along the x-axis $= \frac{p}{2R}$.

Thus, if R is known at every point along the x-axis, f_y can be found. It is seen that f_y is compressive where R is negative, i.e, for curvature convex towards the x-axis and tensile where R is positive for curvature of an isostatic convex away from x-axis. R, at the point of inflection I, is infinite. Therefore, $f_y = 0$ there. The variation of f_y according to this analysis can be represented as shown in Figure 19.

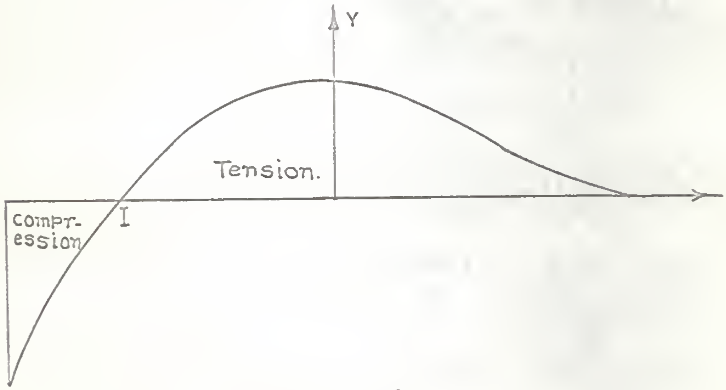


FIGURE 19

General shape of variation of f_y along x-axis.

This variation of f_y, depends largely on the ratio $\frac{a}{d}$. The variation of f_y along x-axis for various values of a/d is shown in Figure 20.

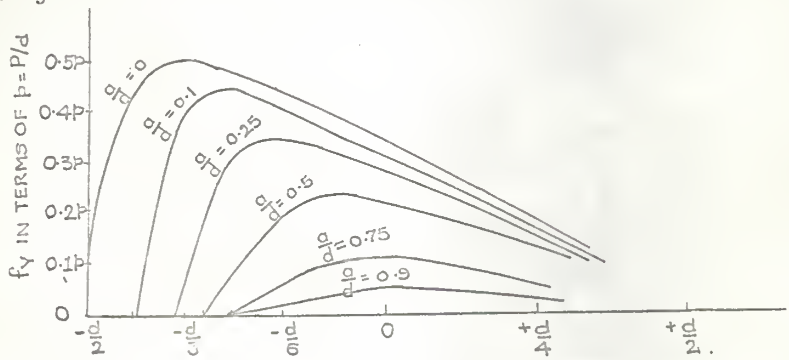


FIGURE 20

Variation of f_y for different values of a/d (from Guyon's Prestressed Concrete).

In a similar way, f_y can be found along any other line parallel to the x-axis. The curves joining the points of equal stress f_y in the whole end block are called the Isobars. The isobars for various values of a/d are shown in Appendix I (a).

From the isobars, it can be seen that there are two different zones of tension, one deep inside the beam called bursting zone and the other along the edge AB called the spalling zone. The value of stress f_y in the spalling zone is much higher than the bursting zone but the area on which this stress acts is smaller than the area of the bursting zone.

To compare the results, the values of f_y from Figure 20, a/d = 0.2 are found as follows.

Values of f_y in Figure 20 are in terms of $p = \frac{P}{d}$, if q = intensity of loading then P = q x a, therefore,

$p = q\,\frac{a}{d}$. and for $\frac{a}{d} = 0.2/p = 0.2q$.

With this modification, the values are tabulated below in Table 7.

x/d	$-\frac{1}{2}$	-1/3	$-\frac{1}{4}$	0	$+\frac{1}{4}$	$+\frac{1}{2}$
f_y in terms of p	--	+0.22p	+0.33p	+0.29p	+0.15p	--
f_y in terms of q	--	+0.044q	+0.066q	+ 0.058q	+ 0.03q	--

TABLE 7

Transverse stress for various values of x/d for end block loaded by single axial load.

(2) Case of two symmetrical forces, each acting at C.G. of half the depth of the beam.

The prestressing force P at the center is assumed to be divided into equal parts $\frac{P}{2}$, each applied symmetrically about axis ox at a distance $\pm \frac{d}{4}$ from x-axis Figure 21.

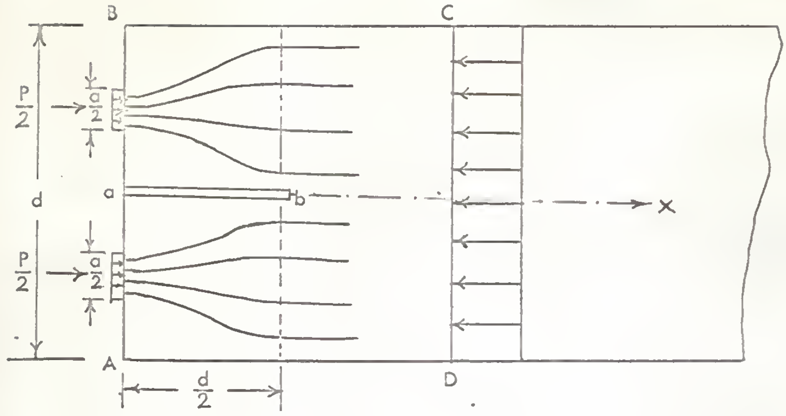
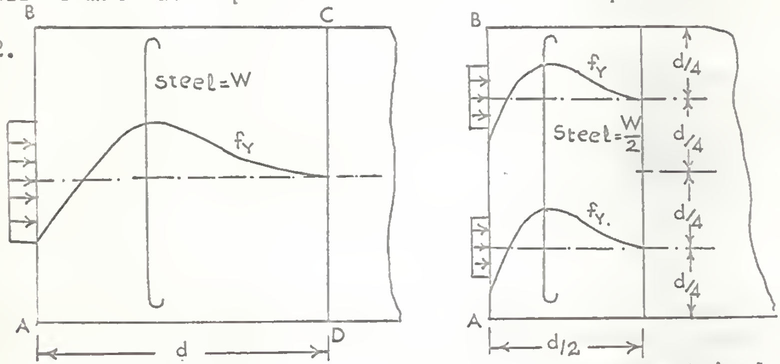


FIGURE 21

Isostatics for two symmetric loads.

For this case it is assumed that the end of the beam is cut along the x-axis for a length ab = $\frac{d}{2}$. Now we have two beam ends, each of depth $\frac{d}{2}$, with an axial load of $\frac{P}{2}$. If the lead-in-zone is again taken equal to the depth of the beam, its length for each part of the end is now $\frac{d}{2}$. This with the hypothesis of cutting the beam, reduces this case exactly to Case I. Therefore, as in Case I, the isostatics for each half the end block can be drawn as shown in Figure 21. The length of the isostatics in this case is only $\frac{d}{2}$ and in the remaining length $\frac{d}{2}$ of the end block they are assumed to be horizontal. Though these isostatics satisfy all the conditions imposed on AB and CD, these conditions are not sufficient for them to be correct. However, it is seen that they are very near to true isostatics.

As shown in Figure 21, the depth over which each force acts is $\frac{a}{2}$. Then these two forces shall cause the same bearing pressure as in Case I, and we can have the same isostatic for each part of this case as for Case I. But the length of the isostatics in this case is half as much as in the first case. Hence, the radius of curvature of the isostatics of this case is half of that for the first case. Therefore, the stresses f_y remain the same in both the cases, but the length over which these stresses are applied is half in this case and so the total resultant force is also half of that in case I. The reinforcement required for this case is also half as much as required for Case II. The comparison is shown in Figure 22.



single axial load FIGURE 22 two symmetric loads

Comparison of a beam loaded by single axial load and two symmetric loads.

This particular case of two loads can be extended for multiple symmetrical forces. If a single axial force is replaced by n equal symmetrical forces, such that the bearing pressure caused by all is the same as caused by a single axial load, then the theory which is called <u>Principle of Partitioning</u> can be applied. The maximum transverse stress f_y remains the same as in case of a single axial load, but the length of the lead-in-zone for n symmetrical forces is reduced to $\frac{d}{n}$. The total tensile force and the reinforcement required are respectively $\frac{F}{n}$ and $\frac{w}{n}$, where F and w are resultant transverse tensile force and amount of the steel required, respectively, for Case I.

Following are listed the advantages of dividing a single axial load into multiple loads, indicated by this method.

(1) Amount of steel required is reduced.

(2) No precision in position of placing the reinforcement is necessary.

(3) As the lead-in-zone is reduced, the support can be placed near the end of the beam without causing the effect of the reaction of the support to affect stress distribution into lead-in-zone.

Isobars of f_y for two symmetric forces are shown in Appendix I (b).

(3) Single Eccentric Force:

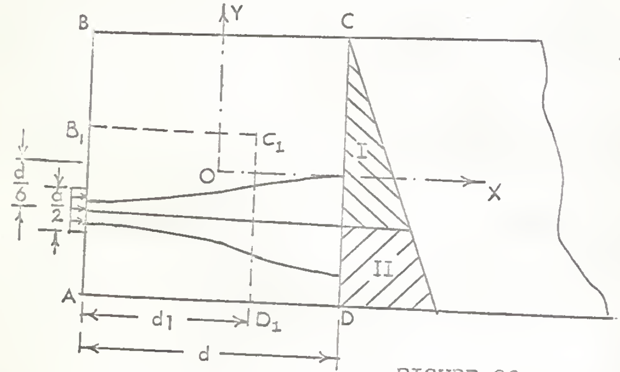


FIGURE 23

Isostatics drawn for the end block loaded by eccentric force (symmetric prism method).

Figure 23 shows an eccentric force p applied at $\frac{d}{6}$ from x-axis. The stress distribution along CD is triangular as shown.

Once again to reduce the case similar to Case I of the single axial force, the following argument can be made.

The force P can be said to be carried from AB to CD by the mean isostatics each representing force $\frac{p}{2}$ and arriving at the center of the areas of the triangle marked, I and II. The two isostatics are approximately symmetric about the line of force P. It is assumed that the line of separation of two equal areas I and II is not very far from the line of the prestressing force P, which is nearly true for practical prestressing. The other assumption made is that the C.G. of area II is almost at its mid-depth.

With the above assumptions, we can now say that the values of tensile stresses are not very different from those which would arise in an imaginary prism $AB_1C_1D_1$, of depth d_1 and length also of d_1, where $d_1/2$ is the distance of the nearest edge from the line of force. This is true because we assume the lower isostatic to pass through the middle of the distance between the line of P and edge AD, i.e., through middle of $d_1/2$. So once again we can find the tensile transverse stress for this case along the line of force using the results of Case I, Figure 20, with slight modification of the ratio $\frac{a}{d}$. For this case the ratio is $\frac{a}{d_1}$ instead of $\frac{a}{d}$. This method is called the <u>Symmetric Prism Method</u>.

As shown in Figure 20, the values of f_y are given in terms of $p = \frac{P}{d}$, so we have to get corresponding value of p for this case. For Figure 23, we have $\frac{d_1}{2} = \frac{d}{3}$, therefore, $d_1 = \frac{2d}{3}$. In the imaginary prism $AB_1C_1D_1$ of the depth $\frac{2d}{3}$ the average compression caused by load p will be,

$$p_1 = P/\ 2d/3 = \frac{3P}{2d} = \frac{3}{2}\ p$$

where p is the average compression caused by the action of P on the real prism. The values of f_y for this case can be found using Figure 20, in terms of q, the intensity of loading.

The values from the graphs are in terms of p_1 where,

$$p_1 = \frac{3}{2}p = \frac{3}{2}\frac{P}{d} \quad \text{where } P = \frac{qa}{2}$$

$$\therefore p_1 = \frac{3}{2} \times \frac{qa}{2d} = \frac{3}{4}\ \frac{aq}{d}, \text{ then for } a/d = 0.2,$$

$$p_1 = 0.15q$$

Also, for this case the ratio of loaded length to the depth of the beam = $\frac{a}{2d} = \frac{0.2}{2} = 0.1$. So we have to see the graph of f_y for $a/d = 0.1$ from Figure 20..

Moreover, these stresses will be along the lead-in-zone of length $= d_1 = \frac{2d}{3}$ only. With all these modifications some values are found and tabulated in Table 8.

x/d	$-\frac{8}{18}$	$-\frac{7}{18}$	$-\frac{1}{3}$	$-\frac{5}{18}$	$-\frac{1}{6}$	0	$+\frac{1}{6}$
f_y in terms of p_1	0	$+0.38p_1$	$+0.43p_1$	$+0.4p_1$	$+0.3p_1$	$+0.16p_1$	0
f_y in terms of q	0	+0.057q	+0.0645q	+0.06q	+0.045q	+0.024q	0

TABLE 8

Values of transverse stress for various values of x/d for the end block loaded by eccentric load.

Guyon has compared the results of this approximate "symmetric prism method" with the real values and has said that the results by the approximate method are not far from the correct values. This case can also be extended to a case of multiple axial loads. For the case of multiple eccentric loads the results can be obtained by combining the principle of partitioning and the symmetric prism method. The only requirement for applying these methods to this case is that having divided the trapezium of stress distribution along CD, Figure 24, into a number of trapzia equal to the number of loads, all of the equal areas should have their C.G. at the level of a corresponding prestressing force as shown in Figure 24. To the stress distribution at CD fulfilling this requirement Guyon calls "linear distribution."

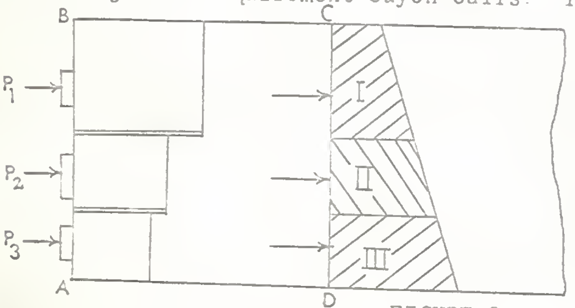


FIGURE 24

Beam loaded by forces with a linear distribution relationship and solved by principle of partitioning and symmetric prism method.

Since each prestressing force is in equilibrium with the resultant of the stress represented by the corresponding trapezium, we can divide the beam into different parts by making cuts along the line between each trapezium. Then the symmetric prism method can be applied for each separate block and the stresses f_y along the line of action of each prestressing force can be found.

(4) General Case:

Having gained the knowledge of the approximate methods discussed above we are now in a position to simplify a problem of the irregularly loaded prestressed beam. To solve the problem shown in Figure 25 by exact methods, or the two previously discussed methods is a difficult job.

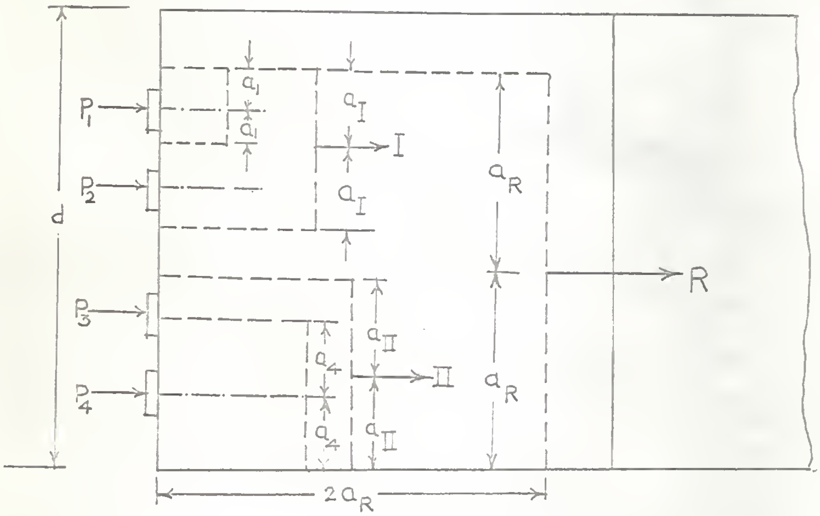


FIGURE 25

Symmetric prism method for irregularly spaced loads.

However, we know that the maximum transverse stress f_y usually occurs first on the axis of the individual forces, then on the axis of the resultant of the group of the forces and finally on the axis of the resultant of all the forces. With this knowledge we can find the maximum stresses for the case of Figure 25 by the following approach.

This approach consists in working backward. We may examine stresses along the line of the resultant of all the forces first, then along the resultant of the group of forces and at the end along the line of the individual force.

In Figure 25, let P_1, P_2, P_3, and P_4 be the four prestressing forces having resultant R. If a_R is the distance of the nearest edge of the beam from the resultant R then we can examine stress f_y along the action of R by considering an imaginary prism of depth and length each equal to $2a_R$, to be loaded by a single axial load R. If I is the resultant of P_1 and P_2 and has a distance a_I between it and nearest edge of an <u>Imaginary prism of R</u>, then the stress f_y along the action of resultant I can be found using an imaginary prism of depth and length = $2a_I$, and loaded by the single axial load I. Similar analysis can be made for the resultant of P_3 and P_4. Using the smaller imaginary prisms for individual forces P_1, P_2, P_3, and P_4, we can find the stress f_y along the line of action of each of these forces. While finding the stresses f_y account should be taken of the concentration factor, i.e., the ratio of length over which loads are distributed to the depth of the respective prisms.

It has been found that the values of f_y obtained by this approximate analysis are little bit higher than the correct values. The method yields conservative design criteria for reinforcement.

PROBLEM

Statement of Problem: The post-tensioned prestressed concrete beam 10" wide and 20" deep is prestressed with a prestressing force of 120 kips applied at the center of the beam. Design the bearing plate area and investigate the maximum transverse stresses along the centroidal axis by all three methods. Allowable bearing stress of concrete = $0.6f_c'$ psi and $f_c' = 5000$ psi.

SOLUTION

Prestressing force = 120 kips.

Allowable bearing stress = $0.6f_c'$

$= 3000$ psi.

Area of plate required $= \dfrac{120,000}{3000} = 40\ in^2$

providing the width of the plate equal to the width of the beam = 10". The depth of the plate required = 4" (this may not be a feasible depth but to illustrate the problem by the methods discussed, it is necessary.) In solving the problem by all the methods, the effect of duct hole on transverse stress is neglected.

(1) By Airy stress function method: The problem is as sketched in Figure 26.

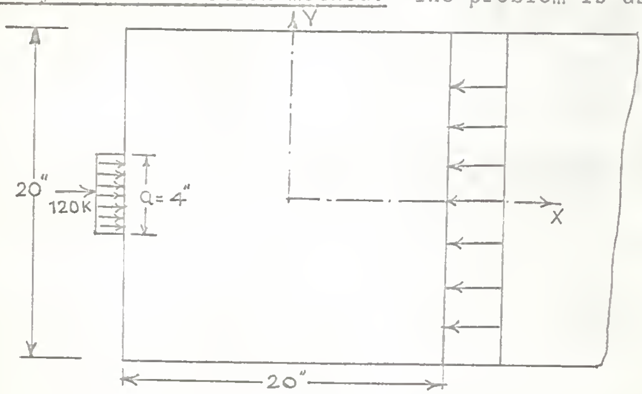


FIGURE 26

End block of the problem.

From Figure 26, $\frac{a}{d} = \frac{4}{20} = 0.2$, and $q = \frac{120}{4} = 30$ kips/inch. Now for the case of a single axial load the values of f_y have been obtained at different values of x/d in terms of q. So directly from Table 2, values of f_y can be found but these values will be for a unit width so to get the stress per in^2 we have to divide f_y by width = 10". A negative sign indicates compression and a positive sign indicates tension for f_y values in Table 9.

x	-10	-5	0	+5	+10
f_y in terms of q	-0.67q	+0.0639q	+0.03722q	+0.0117q	0.00935q
f_y in psi	-2010	+191.7	112.0	35.10	28.0

TABLE 9

Transverse stress along the length of the end block by method using Airy stress function.

(2) By Magnel's Method:

In this method the transverse stress is derived in terms of moment M in a deep beam. First we have to find M along the centroidal axis.

The uniform stress distribution at CD is, taking a unit width of beam, $= \frac{120,000}{20} = 6000$ pounds/inch. ABCD is a deep beam loaded by U.D.L. - 6000 pounds/inch at CD and supported by a central support at AB.

As we are interested in f_y, only the moment diagram is needed which is as drawn in Figure 27.

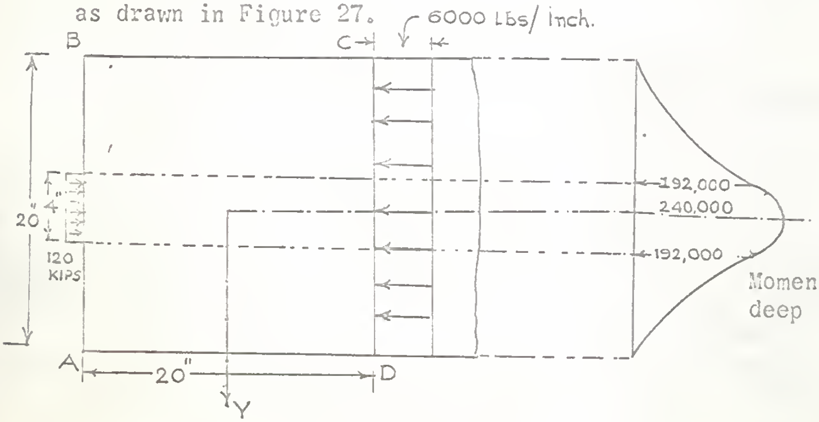


FIGURE 27

Moment diagram of the deep beam loaded as shown.

Moment at center = $600 \times 10 \times 5 - \frac{120,000}{4} \times 2 \times 1 = 300,000 - 60,000$ = 240,000 lb. inch. Then f_y, the transverse stress along centroidal axis, $= \frac{KM}{10} \cancel{\times}\ 20^2 = 60K$ lbs/in^2. For different values of K at different values of x, f_y is found and tabulated in Table 10.

x	K	f_y in psi	x	K	f_y in psi
-10 inch	-20	-1200	0	+5.00	+300
-8 inch	-9.72	-583.2	+2	+4.48	+268.8
-6 inch	-2.56	-153.6	+4	+3.24	+194.4
-5 inch	0.0	0.0	+6	+1.76	+105.6
-4 inch	-1.96	+117.06	+8	+0.52	+31.2
-2 inch	+4.32	+259.2	+10	0.0	0.0

TABLE 10

Transverse stress along length of the end block by Magnel's theory.

(3) By Guyon's method:

For this method also we have already tabulated values of transverse stress in terms of q for a unit width. So multiplying these values by q and dividing by b, the width of the beam, we get the following set of values for f_y along the centroidal axis.

x	-6.66	-5	0	+5	+10
f_y in terms of q	+0.44q	+0.066q	+0.058q	+0.03q	0
f_y in psi	+132	+198	+174	+90	0

TABLE 11

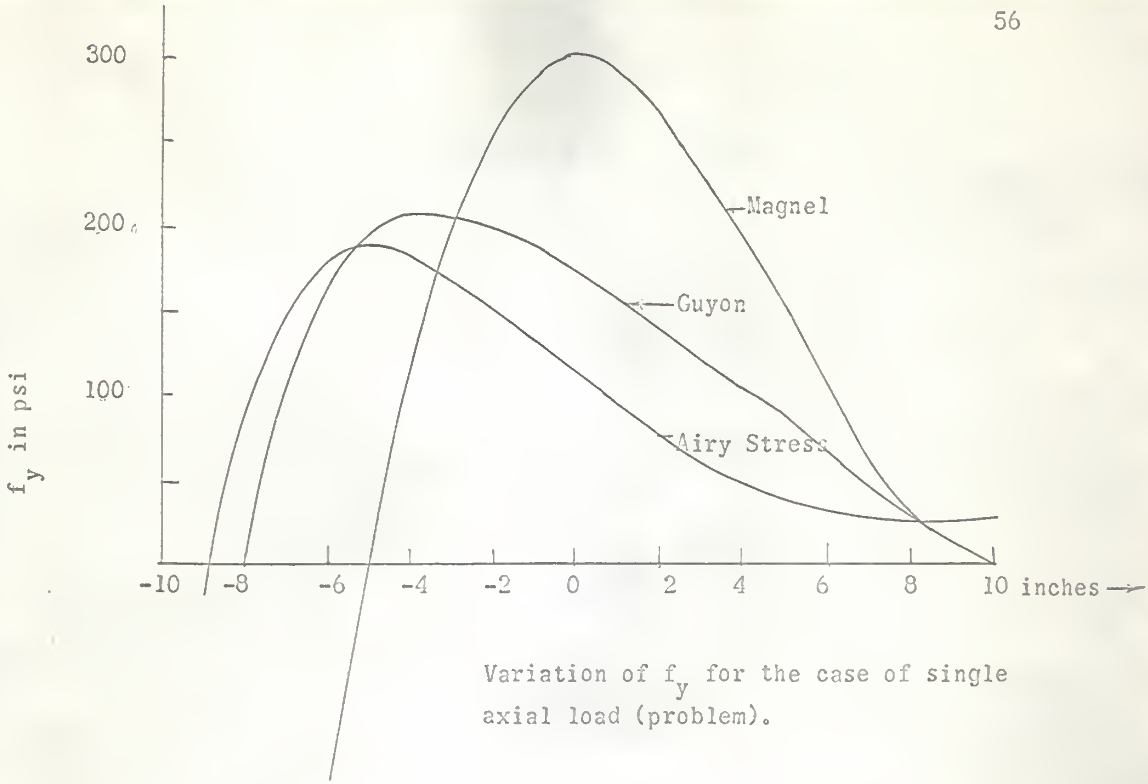
Transverse stress along length of the end block by Guyon's approximate solutions

The results obtained by all the methods will be critically examined in the next section.

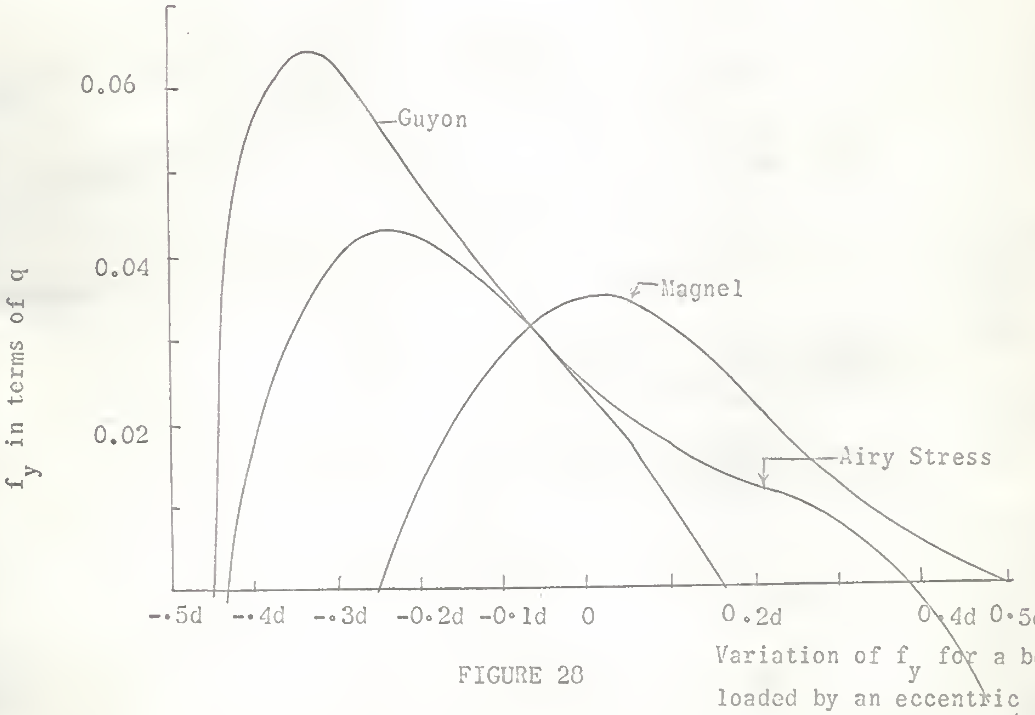
COMPARATIVE STUDY OF THE THEORIES

It is very interesting to examine critically all the theories discussed above. The values of Case I and CaseIII of the method using the Airy stress function are already calculated in all three methods and are plotted in Figure 28. From the figures, it can be seen that there is a wide variation of results obtained by three different theories. One reason for this is that none of the theories discussed here is exact. One general assumption that all the theories available to solve the problem of the distribution of stresses in the anchorage zone have made is that the load at the end of the beam is uniformly distributed over the whole width of the beam. However, results of the photoelastic study have shown that the error introduced due to this assumption is not serious. Further additional assumptions made by each of the theories discussed are briefly discussed below.

The first theory of using the Airy stress function introduced by Pijush Kanti Som and K. Ghosh has been tried fundamentally considering the exact nature of the problem. But to reduce the work or to simplify the theory, many assumptions have been made. This leads to results far from being true for some of the cases. The problem of the end block is solved as that of a finite strip with normal loadings on two opposite edges. As the end block is a zone in which St. Venant's principle is not applicable one must satisfy all the boundary conditions. In this theory the boundary conditions of the loading are satisfied on either boundary at $x = \pm d/2$ but no mention has been made regarding the boundary conditions to be satisfied along the edges $y = \pm d/2$. Thus, this theory always violates one of the boundary conditions.



Variation of f_y for the case of single axial load (problem).



Variation of f_y for a beam loaded by an eccentric force with eccentricity = d/6.

FIGURE 28

Also as we studied in the two dimensional theory by the Airy stress function the stress function selected should satisfy all the boundary conditions, so the stress function found in this theory is also not correct because it does not satisy the boundary conditions at $y = \pm d/2$. This is one reason why this theory is approximate and gives incorrect results. Moreover, only two terms of the Fourier series are taken to find values of transverse stress, so the convergence of the series is doubtful. It has been shown by Guyon by drawing isobars (appendix I (b)) for multiple symmetric loads that there always exists tensile transverse stress at the edge AB and between two loads. This is also proved by photoelastic studies and some exact theories. This theory gives (Table 3) compressive transverse stress at this section.

Magnel's theory as we found to be quite simple is not correct from the elastic theory viewpoint. First there is no sound reason behind the assumption of the variation of transverse stress at any longitudinal section as a polynomial of the third degree. It is seen from Figure 28 that Magnel's theory gives much higher values of f_y for the case of a single axial load.

It must be remembered that Guyon's theory discussed in this report is his approximate analysis. He has solved this problem by an exact analysis using two-dimensional elastic theory. He gave the solution for a semi-infinite strip under the loads on the narrow edge, satisfying boundary conditions on the narrow edge and afterwards improved the solution by introducing the approximations for boundary conditions on longitudinal boundaries. With this method he has derived many tables of values f_x, f_y, and f_{xy}. As we saw in his theory, we use the results of his exact theory to solve the problem by a simplified procedure discussed here. Most of the exact theories and the experimental results show that Guyon's exact analysis is nearly correct. Also, the approximate methods discussed herein are proved to be quite satisfactory for all practical purposes.

Thus, from this comparison the author suggests Guyon's approximate solutions for the application to practical problems. The results by this method are little higher so the designer will always be on the safe side.

PRACTICAL RULES OF PROVIDING REINFORCEMENT IN ANCHORAGE ZONE

Making an analysis of the end block, the next step is to design the reinforcement properly, if needed. Guyon suggested the following simple procedure.

Once knowing the variation of the transverse stress along the longitudinal planes which are the planes of maximum transverse stress, we have to plot the stress variation. For the purpose of illustration we shall design the reinforcement for the problem solved in this report. The stress variation for the problem is shown in Figure 29, from the results of Guyon's method.

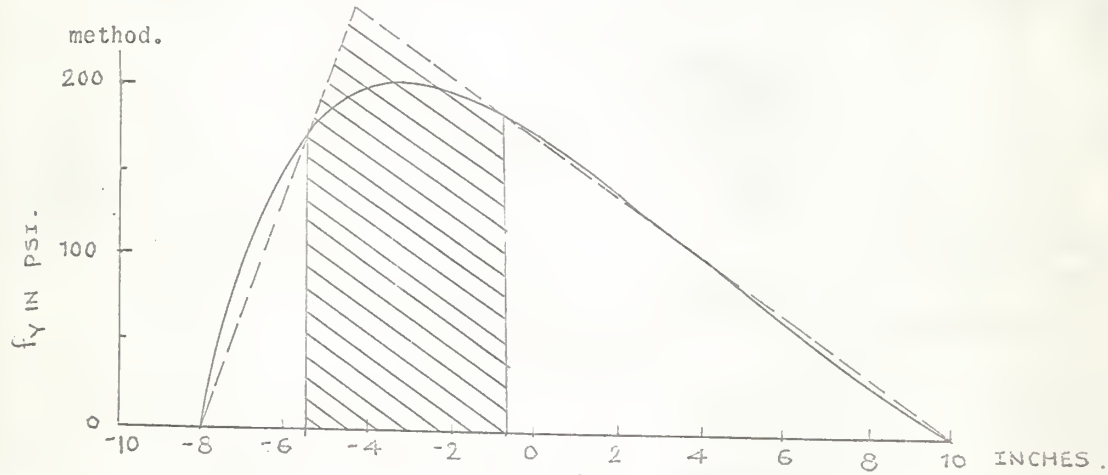


FIGURE 29

Variation of f_y along the centroidal axis of the end block.

Then we have to find an area under the curve. This can be done by approximating the curve as a triangle as shown. This area multiplied by the width of the beam gives total tension in the end block.

But it is not necessary to provide reinforcement for the entire tension, since concrete under tensile stress, below certain permissible stress, does not require reinforcement. There are many opinions as to the permissible value of the tensile stress resistance of concrete. Guyon suggested the value of ft = 180 psi. Accepting this value of f_t, we can draw a horizontal line representing a stress of 180 psi. So, as shown in Figure 29 the shaded area needs reinforcement. To find this area Guyon gives the following formula.

Area of shaded portion of curve =

$$s\left[1-\left(\frac{f_t}{f_{y\ max}}\right)^2\right] \tag{29}$$

where ft = 180 psi and f_y max. for the problem is 250 psi and s = area of the triangle x width of the beam. From Figure 29,

$s = \frac{18 \times 250}{2} \times 10$ where 10 = width of the beam

= 22500 lbs.

The force to be carried by the steel, using equation 29

$= 22500\left[1-\left(\frac{180}{250}\right)^2\right]$

$= 22500\,(1-0.518)$

$= 22500 \times 0.482$

= 10850 lbs.

Now if we select No. 3 bars and take the tensile strength of steel = 20,000 psi, the tension taken by one bar = 20,000 x 0.11 = 2200 lbs.

∴ the number of bars required in a 10" width of the beam = $\frac{10850}{2200}$ = 4.95

∴ provide 5 No. 3 bars in 10" width, i.e., at 2" center to center.

Now from Figure 29, it is obvious that the maximum f_y occurs at 5.5 inches from the edge of the beam. Therefore, provide the reinforcement calculated at 5.5 inch from the edge of the beam. This can be provided in the form of a mesh as shown in Figure 30.

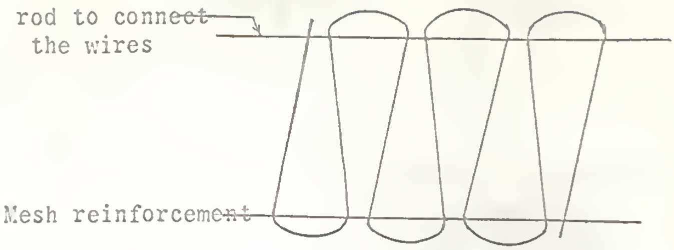


FIGURE 30
Mesh reinforcement.

Now it can be further seen from the isobars drawn by Guyon (Appendix 1 (a)) that there also exists tension along the edge of the beam. The tensile stress is very high in this spalling zone but the area of tension is less and so it requires less reinforcement. Guyon suggests surface binding for this zone. This is in the form of a group of ¼" diameter bars (continuous) at a spacing not greater than, say 2½", provided near to the surface of the beam.

CONCLUSIONS

From the study of the distribution of stresses in the anchorage zone one thing is obviously remarkable. The exact analysis of the stresses in anchorage zone is a problem of academic interest only. The application of the results of the theory in practice is another problem, because in practice we may find many complications, which hardly any of the available theories can overcome completely. ~~First,~~ Most of the available theories are based on the elastic theory, while concrete not being a perfectly elastic material, will act plastically, especially when a part of it is over stressed.

In most of the available theories known to the author, mention has not been made of the effects on the stress distribution by the duct holes, grouted or not grouted. One interesting fact regarding this is that there cannot exist any radial component of stress at the inner boundary of duct holes. This means that the transverse stress near the line of prestressing force should be zero. The results obtained by most of the acceptable theories are quite the reverse of this fact, i.e., showing the maximum transverse stress there. Moreover, in practice, many times the force is not applied on the end surface of the beam, which is the general assumption on which almost all the theories are based. If we take the example of the Freyssinet cone system, we know the cone is buried in the end block of the beam, so the force instead of being transferred through the base of the cone, as assumed, is also transformed partly through the lateral surface of the cone. So the point of conclusion is that even with the most exact theories, some approximations are necessary to apply the theory to practical problems. It is also realized that the more we try to get exact methods of analysis the more will the results be complicated, which can hardly be preferred for design purposes. This fact limits the preference of the exact and elaborate formulas for design purposes and simpler though approximate formulas should be sought.

The idea of the remarks made is not to lead the designers to disregard the available analysis of anchor blocks of post-tensioned prestressed beams. In fact, the approximate theories making use of the results of the exact analysis, as done by Guyon and discussed herein, are most welcome. Rather than arbitrarily designing the reinforcement for the end block it is preferred for the designer to analyze and design the end block by the approximate methods, which are the simplest, and to be sure of the safety of the structure.

ACKNOWLEDGMENT

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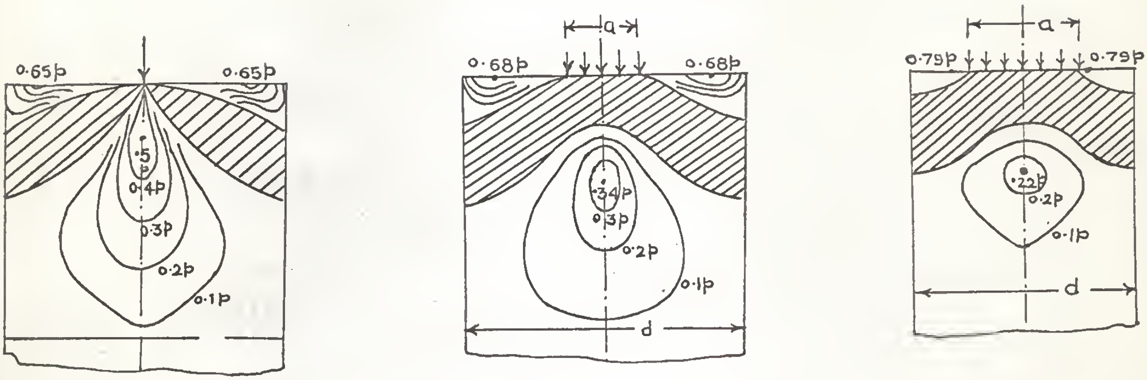
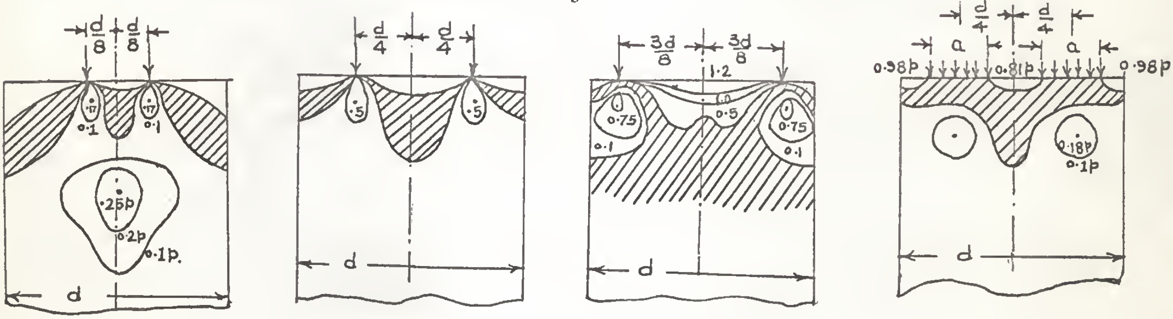
APPENDIX I

ISOBARS FOR VARIOUS LOAD CONDITIONS



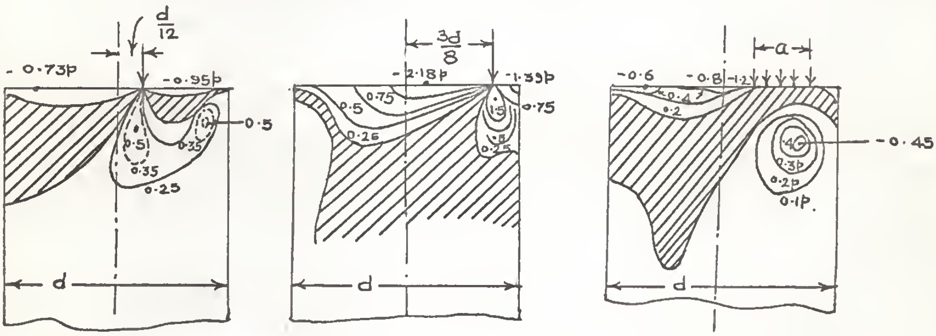
a/d=0 a/d = ¼ a/d = ½

(a) - Single Axial Load



a/d = 0 a/d = 0 a/d = 0 a/d = ¼

(b) = Two Symmetric Loads



a/d = 0 a/d = 0 a/d = ¼

(c) - Single Eccentric Load

Shaded area in figures represents compression.

Taken from "Prestressed Concrete," by Y. Guyon.

APPENDIX II

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ANCHORAGE ZONE STRESSES IN POST-TENSIONED
PRESTRESSED CONCRETE BEAMS

by

VINUBHAI FULABHAI PATEL

B. S., S. V. V. (University), Anand, 1964

AN ABSTRACT OF A MASTER'S REPORT

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Approved by:

Vernon H. Rosebraugh
Major Professor

In this report, an attempt is made to investigate the stress distribution in the anchorage zone of a post-tensioned prestressed concrete beam by three different theories. The three theories selected from among many available theories are:

(1) Using Airy stress function.

(2) Magnel's theory.

(3) Guyon's approximate solutions.

The purpose of this report is to compare these theories for their exactness and the amount of the work involved to investigate the stress disbribution in the anchorage zone and hence to find which is the most preferable theory for applying to practice.

The brief idea of how the problem is treated by each theory is given below.

In the first theory, use is made of the Fourier series to represent the load distribution on either side of the end block. Using the Airy stress function, formulas for longitudinal stress, transverse stress and shear stress in the end block are derived in general form for any kind of load distribution at the end of the beam, represented by the Fourier series. Only a transverse stress is of interest for design, so the formulas for it are generated for the following load conditions:

(1) Only a single axial load.

(2) Two symmetric loads.

(3) One eccentric load.

Magnel's theory is based on the assumption of the variation of a transverse stress as the third degree polynomial. Using the known boundary conditions, the constants of the polynomial are evaluated. Then this polynomial represents the transverse stress at any point. From the equilibrium of the stresses, shear stress is determined and the longitudinal stress is found as that for a column loaded by eccentric forces.

Guyon's method explains how these difficult theories can be simplified using approximations. His two approximate procedures are: (1) Partitioning method, and (2) Symmetric prism method.

It was realized that the approximate solutions given by Guyon are most welcome. These solutions, as compared to those by the other two theories described here, are better though not exact. Also the work involved in designing the end block is much less by this method.